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Numerical Simulation for a Class of Fractional Differential Equations Using Reproducing Kernel Hilbert Space Method

Presented By: HADJI Bouthaina

Dissertation Committe:

Nouri Boumaza	MCA	Echahid Cheikh Larbi Tebessi University- Tebessa	President
Khelifa Bouaziz	MCA	Echahid Cheikh Larbi Tebessi Universiy - Tebessa	Examiner
Nadir Djeddi	MCB	Echahid Cheikh Larbi Tebessi University- Tebessa	Rapporteur

Dedication

In the name of Allah, all praise and thanks are due to Allah, the Lord of the worlds, by whose blessings righteous deeds are completed. All praise is due to Allah, by whose guidance and facilitation, I have completed my academic journey. I dedicate my graduation and success to my loving father, who has struggled for us and fought for our happiness, endured the hardships of life to serve us, tasted the colors of misery to raise us. He planted the seeds, and here you are reaping the fruits, a good generation full of goodness and giving, with Allah's permission. Father, I want to take a picture with you on my graduation day because it is an honor for me. I also dedicate the fruit of my effort to my dear mother, who gave me life, taught me the meaning of tenderness, giving, patience, strength, and love. May Allah protect her.

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To the grandson of the family, "Souhaib Fadlallah."

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To those who share the same blood, heart, and home: all the relatives.

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To all of these, I dedicate the fruit of my efforts.

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List of Abbreviations

DEs	Differential Equations
FDEs	Fractional Differential Equations
Eq./Eqs.	Equation/ Equations.
CF	Caputo-Fabrizio
RKHS	Reproducing Kernel Hilbert Space
RKA	Reproducing Kernel Algorithm
LDEs	Logistic Differential Equations
FLDEs	Fractional Logistic Differential Equations
FBDEs	Fractional Bernoulli Differential Equations
FRDEs	Fractional Riccati Differential Equations
IVP	Initial Value Problem
BVP	Boundary Value Problem
IC's	Initial Conditions
BC's	Boundary Conditions

List of Symbols

$\mathbb{C}$	Set of Complex Numbers
$\mathbb{R}$	Set of Real Numbers
$\mathbb{K}$	Set of Real or Complex Numbers
$L^p[a,b]$	Lebesgue Space
$\mathrm{W}_p^m[a,b]$	Sobolev Space
$\mathrm{R}_t(s)$	Reproducing Kernel Function
$\mathrm{K}_t(s)$	Reproducing Kernel Function
$\psi_i(t)$	Orthogonal Function System
$\widehat{\psi}_i(t)$	Orthonormal Function System
$\mathrm{L}v$	Linear Bounded Operator
$v(t)$	Analytical Solution
$v_n(t)$	Approximate Solution

ABSTRACT

In this work, we introduce an efficient numerical scheme, the reproducing kernel Hilbert space method, for providing numerical approximate solutions of a certain class of fractional differential equations in Caputo-Fabrizio sense within favorable aspects of the reproducing kernel Hilbert space. The algorithm methodology is based on generating an orthonormal basis from the reproducing kernel property to formulate the solution in the form of uniformly convergent series, in accordance with the constrain conditions in the space $W_p^m[a,b]$. Additionally, we present numerical experiments to test our hypothesis and confirm the design procedure of the proposed algorithm. Our results demonstrate that the the reproducing kernel Hilbert space method is a significant development tool for handling issues arising in computer science, physics, and engineering fields.

Introduction

Fractional calculus, the theory of derivatives and integrals of arbitrary order, has gained significant importance and popularity in the past three decades. Its wide range of applications in various fields of applied science and engineering is the primary reason behind this growing interest. Fractional calculus provides efficient tools for understanding and solving differential and integral equations, as well as complex systems involving special functions in mathematical physics and their extensions. The concept of fractional calculus originated from a question posed by Marquis de L'Hôpital to Gottfried Wilhelm Leibniz in the late seventeenth century, regarding the meaning of a non-integer derivative. Leibniz's response, stating that useful consequences would arise from this apparent paradox, has been realized in the present time (Miller and Ross, 1993).

Fractional differential equations (FDEs) have become a subject of great interest among researchers in various fields such as physics, bioengineering, signal processing, biomathematics, and epidemiology. The appeal of these fractional models lies in their ability to capture memory effects, allowing for a remarkable representation of complex and organized structures (Miller and Ross, 1993; Podlubny, 1999). Furthermore, solving these equations not only provides solutions for complex dynamic systems but also contributes to the exploration of known models within the research community.

Investigation of the closed-form solution of the FDEs is scarce. Not much work has been done for nonlinear models, and only a few numerical techniques have been proposed to solve nonlinear FDEs. Anyhow, applications have included classes of nonlinear problems with time-fractional derivatives, and this motivates us to develop a numerical technique for their solutions. However, several analytical and numerical techniques have been applied to solve ordinary differential equations, integral equations, and partial differential equations of physics and engineering significance. The most common numerical methods used are the Adomian decomposition method (Momani and Shawagfeh, 2006), Homotopy perturbation method (Abbasbandy, 2006), Modified homotopy perturbation method (Odibat and Momani, 2006), variational iteration method , Residual power series method (Al-Smadi, 2013), Legendre spectral method (Singh, et al., 2020), Lie-group shooting method (Abbasbandy, et al., 2011), Laplace decomposition method (Khuri, 2004), B-spline method (Caglara, et al., 2010).

Fractional calculus, unlike classical calculus, encompasses multiple fractional definitions with distinct integral operators and regularity properties. Notable definitions include Riemann-Liouville's, Caputo's, Grünwald-Letnikov's, Riesz's, and Weyl's. Fractional derivatives play a crucial role in explaining the behavior of complex dynamic systems. However, these definitions have limitations, such as the Riemann-Liouville fractional derivative not being equal to zero for constants, making it impractical for expressing initial conditions. Additionally, both Riemann-Liouville's and Caputo's definitions have weaknesses regarding genetic memory modeling due to singular kernels, impacting the realism of models.

To address these challenges, the authors proposed a novel fractional definition called the Caputo-Fabrizio (CF) fractional derivative, characterized by an exponential kernel that avoids singularity. This novel approach enhances the ability to describe nonlinear phenomena and dynamic systems with higher accuracy compared to previous fractional derivatives. For more detailed studies and recent results on the CF fractional derivative, readers are encouraged to refer to (Losada and Nieto, 2015; Abdeljawad and Baleanu, 2017; Loh et al., 2018) and the relevant references provided.

The theory of reproducing kernel was used for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions (Zaremba, 1907; Zaremba, 1908). The general theory of reproducing kernel Hilbert spaces was established simultaneously and independently by (Aronszajn, 1950; Bergman, 1950). The introduction of the reproducing kernel Hilbert spaces $W_2^m[a, b]$ by (Cui and Lin, 2009) in the 1980s led to an explosion in applications of reproducing kernel theory to many fields of mathematics where the reproducing kernel approach has been developed as an effective approximate and analytic algorithm in treating different type of ordinary differential equations (Geng, 2009; Al-Smadi, et al., 2016; Al-Smadi, 2018; Abu Arqub and Al-Smadi, 2018), partial differential equations (Abu Arqub, 2017; Wang, et al., 2013), integral equations (Gümah, et al., 2016), integro differential equations (Abu Arqub, et al., 2018; Abu Arqub and Al-Smadi, 2014; Al-Smadi, et al., 2013; Komashynska and Al-Smadi, 2014), fuzzy boundary value problems (Hasan, et al., 2017; Gumah, et al., 2018). Recently, the reproducing kernel Hilbert space (RKHS) method has been improved and successfully applied in obtaining approximations of solutions for many initial and boundary value problems arising in physics and engineering.

The general objective of this thesis is to demonstrate the universality of RKHS method for describing the analytical and numerical solutions for classes of nonlinear FDEs in the frame of CF fractional derivative. To achieve this goal, we set the following aims for us:

1. We are expanding the uses of the RKHS method based on the reproducing kernel theory in order to solve FDEs in the frame of CF fractional derivative, which can be used in diffusion problems, energy-conversion models, oil and gas dynamic models, thermal flow systems, and engineering systems.

2. We construct an efficient algorithm to confirm both theoretical and numerical action

in resolving required FDEs in the frame of CF fractional derivative.

3. We compare the results obtained by using the RKHS method with a set of results for existing methods that are known for their robustness and reliability.

This work has been structured into four chapters. Chapter One presents the basic notations, definitions, and preliminary based on fractional calculus theory and the related functional analysis. In Chapter Two, we introduce the description of a reproducing kernel Hilbert space as well as the reproducing kernel spaces are presented in order to construct their reproducing kernel functions. The representations of solutions and the actual theoretical results are also presented in the same chapter based on the reproducing kernel theory. While in Chapter three , we extend the application of RKHS method to provide a numerical approximate solution of the general Riccati and Bernoulli models within the CF fractional concepts. Accordingly, we are building modern operational algorithms that enable us to cover these models under modern fractional concepts. In addition, we support the theoretical side with numerical applications, as we included some meaningful numerical examples and compared their results with those of strong and well-known numerical methods, with the aim of demonstrating the feasibility and reliability of this approach under the qualitative effect of CF fractional derivative. This work ends in Chapter four with some concluding remarks.

CHAPTER 1

Basic Fundamentals

In this chapter, we present a concise introduction to fundamental notations, definitions, and preliminary concepts derived from fractional calculus theory and the associated functional analysis. These foundational elements will be utilized throughout the entirety of our thesis.

1.1 Functional analysis

In this section, we will explore introductory definitions and theorems pertaining to Hilbert space theory. Additionally, we will present the definitions of vector, norm, and pre-Hilbert spaces. Furthermore, a comprehensive overview of the standard function spaces will be provided. For a more detailed exploration, please refer to (Kreyszig, 1978) and the accompanying references.

Definition 1.1.1. *Let* H *be a vector space over* $\mathbb{K}$. *A function* $\|\cdot\| : \mathrm{H} \to \mathbb{R}^+$ *is said to be a norm on* H *if it fulfills the following:*

1. $\|f\| \geq 0$, *with equality iff* $f = 0$;
2. $\|\lambda f\| \leq |\lambda| \|f\|$;
3. $\|f + g\| \leq \|f\| + \|g\|$,

for all $f, g \in \mathrm{H}$ *and* $\lambda \in \mathbb{K}$.

Definition 1.1.2. *A normed vector space "normed space"* H *is a vector space with a norm defined on it.*

Definition 1.1.3. *A sequence* (f_n) *in a normed vector space* H *is convergent if there is* $f \in \mathrm{H}$ *such that*

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

Then, we can write $f_n \to f$ and call f is the limit of (f_n).

Definition 1.1.4. *A sequence (f_n) in a normed vector space* H *is a Cauchy sequence if*

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \|f_n - f_m\| < \epsilon, \ \forall n, m > N. \tag{1.1.1}$$

Definition 1.1.5. *Let* H *be a vector space over* $\mathbb{K}$*. A function $\langle \cdot, \cdot \rangle : \mathrm{H} \times \mathrm{H} \to \mathbb{K}$ is said to be an inner product on* H *if it fulfills the following:*

1. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$;
2. $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$;
3. $\langle f, g \rangle = \overline{\langle g, f \rangle}$;
4. $\langle f, f \rangle \geq 0$, *with equality iff* $f = 0$,

for all $f, g, h \in \mathrm{H}$ and $\lambda \in \mathbb{K}$.

Definition 1.1.6. *A pre-Hilbert space "inner product space"* H *is a vector space with an inner product defined on it.*

Definition 1.1.7. ***(Cauchy-Schwartz inequality).*** *Let* H *be a pre-Hilbert space. Then*

$$\forall f, g \in \mathrm{H}, \ |\langle f, g \rangle| \leq \|f\| \|g\|.$$

In the event that f and g are linearly dependent, the inequality becomes equal.

Definition 1.1.8. ***(Triangle inequality).*** *Let* H *be a pre-Hilbert space. Then*

$$\forall f, g \in \mathrm{H}, \ \|f + g\| \leq \|f\| + \|g\|.$$

If f and g are linearly dependent, the inequality is transformed into an equality.

Definition 1.1.9. *Let H_1 and H_2 be two vector space over the same field $\mathbb{K}$. An operator $\mathrm{L} : \mathfrak{D}(\mathrm{L}) \subseteq \mathrm{H}_1 \to \mathrm{H}_2$ is said to be linear if the following holds:*

1. $\mathrm{L}(f+g) = \mathrm{L}(f) + \mathrm{L}(g), \ \forall f, g \in \mathfrak{D}(\mathrm{L})$,
2. $\mathrm{L}(\lambda f) = \lambda \mathrm{L}(f), \ \forall \lambda \in \mathbb{K}$.

In which, $\mathfrak{D}(\mathrm{L})$ is called the domain of the operator L.

Definition 1.1.10. *Let H_1 and H_2 be two vector spaces over the same field $\mathbb{K}$ and $\mathrm{L} : \mathfrak{D}(\mathrm{L}) \subseteq \mathrm{H}_1 \to \mathrm{H}_2$ be any operator, not necessarily linear. Then L is said to be continuous at a single point $f_0 \in \mathfrak{D}(L)$, if for every $\epsilon > 0$ there exist $\delta > 0$ such that $\|\mathrm{L}f - \mathrm{L}f_0\| < \epsilon$ for all $f \in \mathfrak{D}(\mathrm{L})$ satisfying $\|f - f_0\| < \delta$.*

Definition 1.1.11. ***(Bounded linear operator).*** *Assume that H_1 and H_2 are two vector space over the same field $\mathbb{K}$. A linear operator $\mathrm{L} : \mathfrak{D}(\mathrm{L}) \subseteq \mathrm{H}_1 \to \mathrm{H}_2$ is said to be bounded if there is a constant $c \in \mathbb{R}^+$ such that $\|\mathrm{L}f\| \leq c\|f\|, \ \forall f \in \mathfrak{D}(\mathrm{L})$.*

Theorem 1.1.12. *Let $\mathrm{L} : \mathfrak{D}(\mathrm{L}) \subseteq \mathrm{H}_1 \to \mathrm{H}_2$ be linear operator, where H_1 and H_2 be two normed vector spaces over the same field $\mathbb{K}$. Then:*

1. *L is continuous if and only if L is bounded.*
2. *If L is continuous at a single point, it is continuous.*

Definition 1.1.13. *A linear functional "linear form" L is a linear operator with domain in vector space H and range in the scalar field $\mathbb{K}$; thus,*

$$\mathrm{L} : \mathfrak{D}(\mathrm{L}) \subseteq \mathrm{H} \to \mathbb{K}.$$

Definition 1.1.14. *Let* H *be a normed vector space. A bounded linear functional on* H *is a linear functional such that there is a positive real number* c *where* $|\mathrm{L}(f)| \leq c\|f\|, \ \forall f \in \mathrm{H}$.

Definition 1.1.15. *A complete normed space is called a Banach space.*

Definition 1.1.16. *A complete pre-Hilbert space* H *is called a Hilbert space. In other equivalent sense, if* f_n *is a Cauchy sequence in* H, *i.e.,* $\|f_n - f_m\| = 0$ *when* $n, m \to \infty$, *then there is* $f \in \mathrm{H}$ *such that* $\|f_n - f\| = 0$ *as* $n \to \infty$.

Definition 1.1.17. *A set* M *of a Hilbert space* H *is said to be an orthogonal set if its element paire wise orthogonal. Also, an orthonormal set is an orthogonal sub set* $M \subset \mathrm{H}$ *whose elements satisfies the following:*

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & if \ \ i = j, \\ 0 & if \ \ i \neq j. \end{cases}$$

Lemma 1.1.18. *An orthonormal set is linearly independent.*

Definition 1.1.19. *An orthonormal system in a Hilbert space* H *is said to be a complete orthonormal system if there is no* $f \neq 0$ *in* H *such that* f *is orthogonal to every element* $\{e_i\}$, *that is, if* $\langle f, e_i \rangle = 0, \ \ i = 1, 2, \ldots;$ *implies that* $f = 0$.

Theorem 1.1.20. ***(Riesz representation theorem).*** *Every bounded linear functional* φ *on a Hilbert space* H *can be represented in terms of the inner product, namely,*

$$\varphi(f) = \langle f, g_\varphi \rangle,$$

where g *depends on* φ, *is uniquely determined by* φ *and has norm*

$$\|g_\varphi\| = \|\varphi\|.$$

Definition 1.1.21. *Let* $\mathrm{L} : \mathrm{H}_1 \to \mathrm{H}_2$ *be a continuous linear operator, where* H_1 *and* H_2 *are Hilbert spaces. Then, the Hilbert-adjoint operator* $L^\star$ *of* L *is the operator*

$$\mathrm{L}^\star : \mathrm{H}_2 \to \mathrm{H}_1,$$

such that

$$\langle \mathrm{L}f, g\rangle = \langle f, \mathrm{L}^\star g\rangle,$$

for all $f \in \mathrm{H}_1$ *and* $g \in \mathrm{H}_2$.

Definition 1.1.22. *Let* $\mathrm{L} : \mathrm{H} \to \mathrm{H}$ *be a continuous linear operator, where* H *is a Hilbert space. Then, the operator* L *is said to be self-adjoint (or Hermitian) if* $\mathrm{L} = \mathrm{L}^\star$.

Definition 1.1.23. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $p \geq 1$. *We denote by* $L^p(\Omega)$ *the Banach space consisting of all measurable function* $v : \Omega \to \mathbb{R}$, *such that*

$$\|v\|_{L^p} = \left(\int_\Omega |v(t)|^p dt\right)^{\frac{1}{p}} < \infty, \ \ for\ 1 \leq p < \infty,$$

and

$$\|v\|_{L^\infty} = esssup_{t\in\Omega}|v(t)|, \ \ for\ p = \infty.$$

Remark 1.1.24. *For* $p = 2$; *the space* $L^2(\Omega) = \left\{v : \int_\Omega v^2(t)dt < \infty\right\}$ *is a Hilbert space among* $L^p(\Omega)$ *with respect to the inner product*

$$\langle v_1, v_2\rangle = \int_\Omega v_1 v_2 dt.$$

Now, we recall the definition of absolute continuity for function of one variable.

Definition 1.1.25. *Let* $[a, b]$ *be an interval in* $\mathbb{R}$. *A function* $v : [a, b] \to \mathbb{R}$ *is called* *absolutely continuous if for every positive ϵ there is a positive δ such that for any finite set of disjoint sub-intervals $\{(s_j, t_j)\}_{j=1}^k \subset [a,b]$ satisfies $\sum_{j=1}^k |t_j - s_j| < \delta$, then*

$$\sum_{j=1}^{k} |v(t_j) - v(s_j)| < \epsilon.$$

Theorem 1.1.26. ***(Fundamental theorem of Lebesgue integral calculus).*** *A function $v_1 : [a,b] \to \mathbb{R}$ is set of all absolutely continuous functions if and only if there is a function $v_2 \in L^1[a,b]$ such that*

$$v_1(t) = v_1(a) + \int_a^t v_2(s)ds, \ \ \forall t \in [a,b].$$

Definition 1.1.27. *The Sobolev space $\mathrm{W}_2^m[a,b]$ is defined as follow:*

$$\mathrm{W}_2^m[a,b] = \{v(t) \mid \ \ v^{(j)}(t)\, is\, absolutely\; continous\; on\; [a,b] \ \ \ j = 1,2,\dots,m-1 \\ , \ \ v^{(m)}(t) \in L^2([a,b]),\ t \in [a,b]\}.$$

The standard inner product and norm in the Sobolev space $W_2^m[a,b]$ are defined, respectively, as follows:

$$\langle v_1, v_2 \rangle_{\mathrm{W}_2^m} = \sum_{i=0}^{m-1} v_1^{(i)}(a) v_2^{(i)}(a) + \int_a^b v_1^{(m)}(t) v_2^{(m)}(t)dt, \tag{1.1.2}$$

and

$$\|v_1\|^2_{\mathrm{W}_2^m} = \langle v_1, v_1 \rangle_{\mathrm{W}_2^m}.$$

The Sobolev space $\mathrm{W}_2^m[a,b]$ is a complete inner product space, i.e., $\mathrm{W}_2^m[a,b]$ is a Hilbert space.

For $\mathrm{W}_2^m[a,b]$ another notation $\mathrm{H}^m[a,b]$ is often used: $\mathrm{W}_2^m[a,b] = \mathrm{H}^m[a,b]$.

1.2 Fundamentals of Fractional Calculus

1.2.1 Special Mathematical Functions

Prior to exploring the definitions of fractional differentiation and integration, it is essential to review several significant mathematical tools that are intricately associated with fractional calculus. These tools include the Euler Gamma and Beta functions, as well as the Mittag-Leffler function.

- **Euler Gamma function**. The Euler Gamma function, which is defined by $\Gamma(t) = \int_0^\infty e^{-s}s^{t-1}ds,\ t \in \mathbb{R}^+$, is a generalization of the factorial function $k!$. That is $\Gamma(k+1) = k!$ for $k \in \mathbb{N}$.

Consequently, the Euler Gamma function possess the following properties:

1. $\Gamma(k+1) = k\Gamma(k),\ \ k \in \mathbb{N}$.
2. $\Gamma(k) = (k-1)!,\ \ k \in \mathbb{N}$.
3. $\Gamma(k)\Gamma(1-k) = \frac{\pi}{\sin \pi k}$.
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
5. $\Gamma(k+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^k}(2k-1)!,\ \ k \in \mathbb{N}$.

Next, we will defined the definition of the Euler Beta function, which is an important for the computation of fractional derivatives of the power function.

- **Euler Beta function**. The Euler Beta function, which is denoted by $\beta(s,t)$, is defined by

$$\beta(s,t) = \int_0^1 z^{s-1}(1-z)^{t-1}dz,\ \ t, s \in \mathbb{R}^+.$$

The Euler Beta function can be expressed in term of Euler Gamma function as follows:

$$\beta(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \forall s,t \in \mathbb{N}. \tag{1.2.1}$$

From relationship (1.2.1), we can conclude that the function is symmetric, that is,

$$\beta(s,t) = \beta(t,s).$$

- **Mittag-Leffler Function**.The Mittag-Leffler function, which was introduced by Podlubny (1999), is a significant component of fractional calculus, as it is a direct extension of the exponential function. Its one and two-parameter representations can be defined using a power series.

$$\begin{aligned} E_\alpha(t) &= \textstyle\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \ \alpha > 0, \\ E_{\alpha,\beta}(t) &= \textstyle\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \ \alpha > 0, \ \beta > 0. \end{aligned}$$

Some of its most crucial characteristics are as follows:

1. $E_{\alpha,1}(t) = E_\alpha(t)$.

2. $E_{1,1}(t) = \exp(t)$.

3. $E_{2,1}(t^2) = \cosh(t)$.

4. $E_{2,2}(t^2) = \frac{1}{t}\sinh(t)$.

1.2.2 Fractional Differentiation and Integration

There are many different definitions of fractional derivatives. All of them include integral operators with different regularity properties, and some have both singular and non-singular kernels. Here, we briefly touch on the most important characteristics of the CF fractional derivative adopted throughout this work.

Definition 1.2.1. ***(Kilbas, et al., 2006)*** *The Riemann-Liouville fractional integral for a function $v(t) \in L^1[a,b]$ of order $\alpha > 0$ is given as follows:*

$$J_{a^+}^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} v(s) ds, \;\; t > a. \tag{1.2.2}$$

If $\alpha = 0$, then $J_{a^+}^0 := I$ is the identity operator.

Definition 1.2.2. ***(Kilbas, et al., 2006)*** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is given as follows:*

$$\begin{aligned} {}^{RL}D_{a^+}^{\alpha} v(t) &= D^m J_{a^+}^{m-\alpha} v(t) \\ &= \tfrac{1}{\Gamma(m-\alpha)} \tfrac{d^n}{dt^n} \int_a^t (t-s)^{m-\alpha-1} v(s) ds, \;\; t > a, \end{aligned} \tag{1.2.3}$$

where $m \in \mathbb{N}^* (car \alpha > 0)$;$m = 0(-1 < \alpha \leq 0)$.

In order to avoid some of the difficulties observed in employing the Riemann-Liouville fractional derivatives. In 1967, Michelle Caputo introduced the Caputo fractional derivatives, as follows:

Definition 1.2.3. ***(Kilbas, et al., 2006)*** *The Caputo fractional derivative of order $\alpha > 0$ is given as follows:*

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha} v(t) &= J_{a^+}^{m-\alpha} (D^n v(t)) \\ &= \tfrac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} v^{(m)}(s) ds, \;\; t > a, \end{aligned} \tag{1.2.4}$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$.

Theorem 1.2.4. ***(Podlubny, 1999)*** *The relation between Riemann-Liouville and Caputo fractional derivatives with singular kernels given as:*

$$ {}^{C}D^{\alpha}_{a^+}v(t) = {}^{RL}D^{\alpha}_{a^+}v(t) - \sum_{k=0}^{m-1} \frac{v^{(k)}(a)}{\Gamma(k-a+1)}(t-a)^{k-a}. \tag{1.2.5} $$

Therefore,

$$ \text{If } v(a) = v'(a) = \cdots = v^{(n-1)}(a) = 0, \text{ then } {}^{C}D^{\alpha}_{a^+}v(t) = {}^{RL}D^{\alpha}_{a^+}v(t). \tag{1.2.6} $$

More recently, in (Caputo and Fabrizio, 2015), the authors proposed a novel derivative of fractional order with a non-singular kernel. This derivative has several interesting properties that are useful for modeling in many branches of sciences.

Definition 1.2.5. ***(Caputo and Fabrizio, 2015)*** *Let $\alpha \in (0,1)$ and $v(t)$ be a function in the usual Sobolev space $H^1[a,b]$. Then, the CF fractional derivative operator of order α is described by:*

$$ {}^{CF}D^{\alpha}_{a^+}v(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t v'(s)\exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]ds, \tag{1.2.7} $$

where, $M(\alpha)$ is a normalization function satisfies $M(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$ such that $M(0) = M(1) = 1$. However, if $v(t) \notin H^1[a,b]$ then the CF fractional derivative operator is defined as:

$$ {}^{CF}D^{\alpha}_{a^+}v(t) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (v(t)-v(s))\exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]ds. \tag{1.2.8} $$

Remark 1.2.6. *The formula (1.2.7) is obtained by changing the power law kernel $(t-s)^{-\alpha}$ in (1.2.4) with the exponential kernel $\exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$.*

Definition 1.2.7. ***(Abdeljawad and Baleanu, 2017)*** *Let $\alpha \in (0,1)$ and $v(t)$ be a function in the Sobolev space $H^1[a,b]$. Then, the CF fractional derivative operator of order α in Riemann-Liouville sense is described by:*

$$^{CFR}D^{\alpha}_{a^+}v(t) = \frac{M(\alpha)}{1-\alpha}\left(\frac{d}{dt}\right)\int_a^t v(s)\exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]ds, \tag{1.2.9}$$

Like to the formula (1.2.5), we find in (Abdeljawad and Baleanu, 2017) a relation between the CF fractional derivative in Caputo sense and the CF fractional derivative in Riemann-Liouville sense.

Theorem 1.2.8. *The relation between the CF fractional derivative in Caputo sense and the CF fractional derivative in Riemann-Liouville sense given as:*

$$^{CF}D^{\alpha}_{a^+}v(t) = ^{CFR}D^{\alpha}_{a^+}v(t) - \frac{M(\alpha)}{1-\alpha}v(a)\exp\left[\frac{-\alpha(t-a)}{1-\alpha}\right]. \tag{1.2.10}$$

Proof : See (Abdeljawad and Baleanu, 2017) □

Theorem 1.2.9. ***(Caputo and Fabrizio, 2015)*** *Let $\alpha \in [0,1]$ and $v(t) \in \mathrm{H}^1(\nabla)$. Then, the following holds:*

1. $\lim_{\alpha\to 1} {}^{CF}D^{\alpha}_{a^+}v(t) = \frac{d}{dt}v(t)$.

2. $\lim_{\alpha\to 0} {}^{CF}D^{\alpha}_{a^+}v(t) = v(t) - v(a)$.

Example 1.2.10. *Let $v(t) = \sin(\omega t)$, $\alpha = 0.66$, $a = -8$ and $\omega = 1$. Then, the CF fractional derivative is*

$$^{CF}D^{0.66}_{(-8)^+}\sin(t) = \frac{M(0.66)}{1-0.66}\int_{(-8)}^t \cos(s)\exp\left[\frac{-066(t-s)}{1-0.66}\right]ds,$$

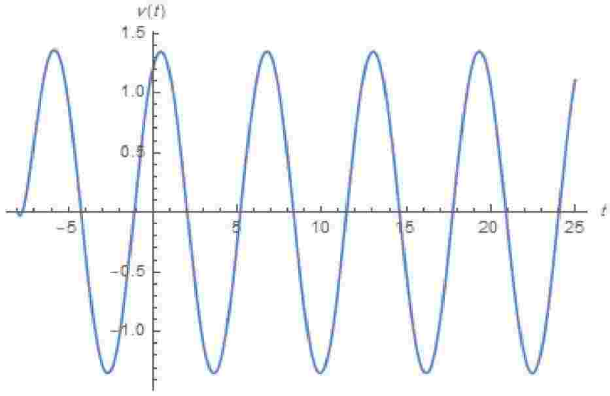


Figure 1.1: Simulation of CF fractional derivative with $\alpha = 0.66$ in the interval $[-8, 25]$.

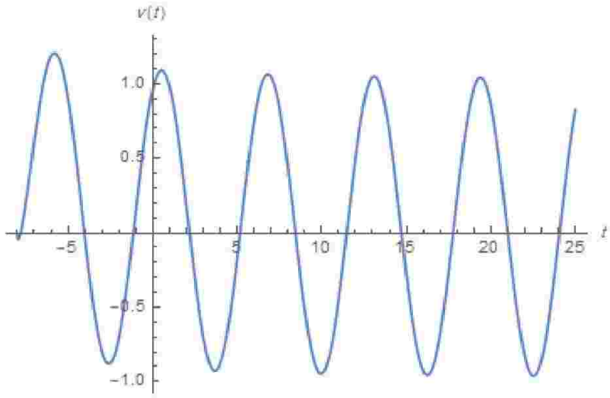


Figure 1.2: Simulation of Caputo fractional derivative with $\alpha = 0.66$ in the interval $[-8, 25]$.

The simulation of this derivative produces the following pictures:

From these two simulations with $\alpha = 0.66$*, the behavior of the CF fractional derivative appears to be very similar to that of the Caputo fractional derivative.*

In (Caputo and Fabrizio, 2015), it had been proved that the CF fractional derivative satisfies the following properties:

1. The CF fractional derivative is zero when $v(t)$ is constant like in the usual Caputo fractional derivative, i.e., ${}^{CF}D^{\alpha}_{a^+}v(t) = 0$ if $v(t) = C, \ \forall \ C \in \mathbb{R}$.

2. If $m \geq 1$ and $\alpha \in [0, 1]$ then the fractional operator ${}^{CF}D^{\alpha+m}_{a^+}v(t)$ of order $m + \alpha$ is defined by:

$$ {}^{CF}D^{\alpha+m}_{a^+}v(t) = {}^{CF}D^{\alpha}_{a^+}(D^m v(t)). \tag{1.2.11}$$

3. The fractional operator ${}^{CF}D^{\alpha}_{a+}v(t)$ is linear, that is,

$${}^{CF}D^{\alpha}_{a+}(\mu_1 v_1(t)+\mu_2 v_2(t)) = \mu_1 \, {}^{CF}D^{\alpha}_{a+}v_1(t)+\mu_2 \, {}^{CF}D^{\alpha}_{a+}v_2(t), \;\; \forall \mu_1, \mu_2 \in \mathbb{R} \text{ and } \alpha > 0. \tag{1.2.12}$$

Theorem 1.2.11. ***(Caputo and Fabrizio, 2015)*** *For the CF fractional derivative, if the function $v(t)$ is such that:*

$$v^{(i)}(a) = 0, \;\; i = 1, 2, \ldots, m.$$

Then, we have

$${}^{CF}D^{\alpha}_{a+}(D^m v(t)) ={}^{CF} D^{m}_{a+}(D^{\alpha} v(t)).$$

Theorem 1.2.12. ***(Abdeljawad and Baleanu, 2017)*** *The relation between the CF fractional derivative and the corresponding integral is given by:*

$${}^{CF}I^{\alpha}_{a}({}^{CF}D^{\alpha}_{a}v(t)) = v(t) - v(a). \tag{1.2.13}$$

Example 1.2.13. *Let $m < \alpha < m+1$, for a given integer $k \geq \lceil \alpha \rceil$, the CF fractional derivative of order $\alpha \neq \lceil \alpha \rceil$ of $v(t) = t^k$, is given as:*

$${}^{CF}D^{\alpha}_{a}t^k = \frac{M(\alpha)\Gamma(1+k)}{\lceil \alpha \rceil - \alpha}\Big[\sum_{i=0}^{k-m-1} \frac{(-1)^i t^{k-m-i-1}}{\Gamma(k-m-i)\big(\frac{-\alpha}{\lceil\alpha\rceil-\alpha}\big)^i} + \frac{(-1)^{k-m}}{\big(\frac{-\alpha}{\lceil\alpha\rceil-\alpha}\big)^{k-m}} \exp\big(\frac{-\alpha t}{\lceil \alpha \rceil - \alpha}\big)\Big], \tag{1.2.14}$$

where, $\lceil\cdot\rceil$ to denote the ceiling function.

Proof : Using the definition of CF fractional derivative ${}^{CF}D^{\alpha}_{a}t^k = 0$ for $k < \lceil \alpha \rceil$, we have

$$
\begin{aligned}
{}^{CF}D_a^{\alpha}t^k &= \frac{M(\alpha)}{\lceil\alpha\rceil-\alpha}\int_0^t D^{(m+1)}s^k \exp\left[\frac{-\alpha}{\lceil\alpha\rceil-\alpha}(t-s)\right]ds \\
&= \frac{M(\alpha)}{\lceil\alpha\rceil-\alpha}\int_0^t \frac{\Gamma(k+1)}{\Gamma(k-m)}s^{k-m-1} \exp\left[\frac{-\alpha}{\lceil\alpha\rceil-\alpha}(t-s)\right]ds \\
&= \frac{M(\alpha)}{\lceil\alpha\rceil-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-m)} \exp\left[\frac{-\alpha}{\lceil\alpha\rceil-\alpha}t\right]\int_0^t s^{k-m-1}\exp\left[\frac{\alpha}{\lceil\alpha\rceil-\alpha}s\right]ds \\
&= \frac{M(\alpha)}{\lceil\alpha\rceil-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-m)} \exp\left[\frac{-\alpha}{\lceil\alpha\rceil-\alpha}t\right]\times\left[exp\left[\frac{\alpha}{\lceil\alpha\rceil-\alpha}t\right]\sum_{i=0}^{k-m-1}(-1)^i\right. \\
&\left.\times\frac{\Gamma(k-m)s^{k-m-i-1}}{\Gamma(k-m-i)\left(\frac{\alpha}{1-\alpha}\right)^{i+1}}-\frac{(-1)^{k-m-1}\Gamma(k-m)}{\left(\frac{\alpha}{1-\alpha}\right)^{k-m}}\right] \\
&= \frac{M(\alpha)\Gamma(k+1)}{\lceil\alpha\rceil-\alpha}\left[\sum_{i=0}^{k-m-1}\frac{(-1)^i s^{k-m-i-1}}{\Gamma(k-m-i)\left(\frac{\alpha}{1-\alpha}\right)^{i+1}}-\frac{(-1)^{k-m-1}}{(\frac{\alpha}{1-\alpha})^{k-m}}\exp\left(\frac{-\alpha}{\lceil\alpha\rceil-\alpha}t\right)\right].
\end{aligned}
$$

□

CHAPTER 2

Reproducing Kernel Hilbert Space

This chapter is divided into four sections. The first section provides an introduction to the fundamental concepts, definitions, and theorems related to reproducing kernel theory, focusing on the aspects relevant to this thesis. The second section focuses on the examination of a specific type of RKHSs denoted as $\mathrm{W}_2^m[a,b]$, which will be utilized in this study. In the third section, we determine the expressions of reproducing kernel functions on the space $\mathrm{W}_2^m[a,b]$, which can be represented by piecewise polynomials of degree $2m+1$. The fourth section presents an iterative scheme for constructing and determining the reproducing kernel function for the proposed m^{th}-order boundary value problem (BVP). We derive the formula for calculation and representation of the analytic solution in the RKHS $\mathrm{W}_2^m[a,b]$. Additionally, we provide an approximation of the solution by truncating the analytic solution after n-terms.

2.1 Concept of Reproducing Kernel Hilbert Space

The main purpose of this section is to survey all the information, concepts and symbols used and necessary to understand the reproducing kernel theory well.

Definition 2.1.1. ***(Aronszajn, 1950)*** *Let* H *be a Hilbert space of functions defined on a non empty set* Θ*. A function* $\mathrm{K}:\Theta\times\Theta\to\mathbb{R}$ *of* H *satisfies the following:*

1. $\mathrm{K}_t(\cdot)=\mathrm{K}(\cdot,t)\in\mathrm{H}$, $\forall t\in\Theta$,

2. $\langle f(\cdot),\mathrm{K}(\cdot,t)\rangle_{\mathrm{H}}=f(t)$, $\forall t\in\Theta$ *and* $\forall f\in\mathrm{H}$,

is called reproducing kernel function of H.

The last condition is knows as "the reproducing property", which reproduce the value of the function $f(\cdot)$ at the point t by the inner product of $f(\cdot)$ with $\mathrm{K}_t(\cdot)$.

Let $\mathrm{K}_t(\cdot)$ be a reproducing kernel function. Applying the reproducing property to $\mathrm{K}_t(\cdot)$ at s, we get

$$\mathrm{K}_t(s) = \mathrm{K}(s,t) = \langle \mathrm{K}_t(\cdot), \mathrm{K}_s(\cdot)\rangle, \ \ \forall s,t \in \mathrm{H}.$$

So, that

$$\|\mathrm{K}_t(\cdot)\| = \langle \mathrm{K}_t(\cdot), \mathrm{K}_t(\cdot)\rangle^{\frac{1}{2}} = \mathrm{K}(t,t)^{\frac{1}{2}}, \ \ \forall t \in \mathrm{H}.$$

Definition 2.1.2. ***(Aronszajn, 1950)*** *A Hilbert space* H *of functions on a non empty set* Θ *is called a RKHS if there exists a reproducing kernel function* $\mathrm{K}_t(\cdot)$ *of* H.

Theorem 2.1.3. ***(Aronszajn, 1950)*** *If a Hilbert space* H *of functions on a non empty set* Θ *possess a reproducing kernel function* $\mathrm{K}_t(\cdot)$*, then this reproducing kernel* K *is unique.*

Proof : Suppose $\mathrm{K}_t(\cdot)$ and $\mathrm{R}_t(\cdot)$ are both reproducing kernels of H such that $\mathrm{K}_t(\cdot) \neq \mathrm{R}_t(\cdot)$. Then, for all $t \in \Theta$, applying the reproducing property for $\mathrm{K}_t(\cdot)$ and $\mathrm{R}_t(\cdot)$ we get: $\|\mathrm{K}_t(s) - \mathrm{R}_t(s)\|^2 = \langle \mathrm{K}_t(s) - \mathrm{R}_t(s), \mathrm{K}_t(s) - \mathrm{R}_t(s)\rangle = \langle \mathrm{K}_t(s) - \mathrm{R}_t(s), \mathrm{K}_t(s)\rangle - \langle \mathrm{K}_t(s) - \mathrm{R}_t(s), \mathrm{R}_t(s)\rangle = (\mathrm{K}_t(s) - \mathrm{R}_t(s)) - (\mathrm{K}_t(s) - \mathrm{R}_t(s)) = 0$ for all $s \in \Theta$. This means that $\mathrm{K}_t(s) = \mathrm{R}_t(s)$. The proof is complete. □

We could equivalently define another definition of a RKHS as follow: let H be a Hilbert function space over a nonempty set Θ. For an element $t \in \Theta$, a Dirac functional "δ-functional" is a function such that for all $f \in \mathrm{H}$, we have $\delta_t(f) = f(t)$. Then H is a RKHS if all δ-functional in H are bounded (continuous), that is, RKHS is a Hilbert function space with the property that every point evaluation functional is bounded and linear. For example, the Hilbert space $L^2[a,b]$ is not a RKHS because the δ-function which has the reproducing property

$$f(t) = \int_a^b \delta(t-s) f(s) ds$$

does not satisfy the square integrable condition, that is,

$$\int_a^b \delta^2(s)ds \not< \infty,$$

thus the δ-function is not in $L^2[a, b]$.

Definition 2.1.4. ***(Aronszajn, 1950)*** *Let Θ be a nonempty set. A function $K : \Theta \times \Theta \to \mathbb{C}$ is called:*

1. *Hermitian kernel on Θ if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j \mathrm{K}(t_i, t_j) \in \mathbb{R}.$$

2. *Positive-definite (p.d.) kernel on Θ if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j \mathrm{K}(t_i, t_j) \geq 0.$$

holds for any finite set of points $\{t_1, t_2, \ldots, t_n\} \subseteq \Theta$, $n \in \mathbb{N}$, and $c_1, c_2, \ldots, c_n \in \mathbb{C}$.

Theorem 2.1.5. *The reproducing kernel function $K_t(s)$ of a RKHS* H *is a p.d. kernel.*

Proof : See (Aronszajn, 1950). □

Propertie 2.1.6. ***(Aronszajn, 1950)*** *Let* H *be a RKHS and its kernel $\mathrm{K}_t(s)$ on* H*, then for all $s, t \in \Theta$ we have:*

(i) $\mathrm{K}(s, s) \geq 0$.

(ii) $\mathrm{K}(s, t) = \overline{\mathrm{K}(t, s)}$.

(iii) $|\mathrm{K}(s, t)| \leq \mathrm{K}(s, s)\mathrm{K}(s, s)$ *(Schwarz inequality).*

(iv) *Let* $t_0 \in \Theta$. *Then the following statements are equivalent:*

a. $\mathrm{K}(t_0, t_0) = 0$;

b. $\mathrm{K}(s, t_0) = 0, \ \forall s \in \Theta$;

c. $f(t_0) = 0, \ \forall f \in \mathrm{H}$.

Theorem 2.1.7. ***(Aronszajn, 1950)*** *For any p.d. kernel* $\mathrm{K}_t(s)$ *on a nonempty* Θ, *there exists a unique Hilbert space* $\mathrm{H_K}$ *of functions on* Θ *with reproducing kernel function* $\mathrm{K}_t(s)$.

Theorem 2.1.8. ***(Aronszajn, 1950)*** *A Hilbert function space* H *is a RKHS if and only if for any fixed* $t \in \Theta$ *the linear functional* $\mathbf{I}(f) = f(t)$ *is bounded.*

Theorem 2.1.9. *Every sequence of functions* $(f)_{n\geq 1}$ *which converges strongly to a function* f *in* $\mathrm{H_K}(\Theta)$ *converges also in the pointwise sense, i.e., for any point* $t \in \Theta$,

$$\lim_{n\to\infty} f_n(t) = f(t).$$

Further, this convergence is uniform on every subset of Θ *on which* $t \to \mathrm{K}(t,t)$ *is bounded.*

2.2 Reproducing Kernel Hilbert Space $\mathrm{W}_2^m[a,b]$

Theorem 2.2.1. *The Hilbert space* $\mathrm{W}_2^m[a,b]$ *is a RKHS.*

Proof : Suppose that $\mathbf{I}(v) = v(t)$, $t \in [a,b]$ is a linear functional of $\mathrm{W}_2^m[a,b]$ and $v(t) \in \mathrm{W}_2^m[a,b]$. We have

$$v^{(m-1)}(t) = v^{(m-1)}(a) + \int_a^t v^{(m)}(s)ds,$$

and

$$\begin{aligned} |v^{(m-1)}(t)| &\leq |v^{(m-1)}(a)| + \int_a^t |v^{(m)}(s)|ds, \\ &\leq |v^{(m-1)}(a)| + \int_a^b |v^{(m)}(s)|ds. \end{aligned}$$

Note that

$$\begin{aligned} \int_a^b |v^{(m)}(t)|dt &\leq \left[(b-a)\int_a^b |v^{(m)}(t)|^2 dt\right]^{\frac{1}{2}} \\ &\leq \kappa_0 \left[\sum_{i=0}^{m-1} |v^{(i)}(a)|^2 + \int_a^b |v^{(m)}(s)|^2 ds\right]^{\frac{1}{2}} \\ &\leq \kappa_0 \|v\|_{\mathrm{W}_2^m}, \end{aligned}$$

and for any $0 \leq i \leq m-1$, we have

$$\begin{aligned} |v^{(i)}(a)| &\leq \left[\sum_{i=0}^{m-1} |v^{(i)}(a)|^2 + \int_a^b |v^{(m)}(s)|^2 ds\right]^{\frac{1}{2}} \\ &= \|v\|_{\mathrm{W}_2^m}. \end{aligned} \tag{2.2.1}$$

Therefore

$$|v^{(m-1)}(t)| \leq \kappa_1 \|v\|_{\mathrm{W}_2^m}. \tag{2.2.2}$$

Noting that

$$\begin{aligned} |v^{(m-2)}(t)| &\leq |v^{(m-2)}(a)| + \int_a^t |v^{(m-1)}(s)|ds \\ &\leq |v^{(m-2)}(a)| + \int_a^b |v^{(m-1)}(s)|ds. \end{aligned}$$

From Eqs. (2.2.1) and (2.2.2), we have

$$\begin{aligned} |v^{(m-2)}(t)| &\leq \|v\|_{\mathrm{W}_2^m} + (b-a)\kappa_1 \|v\|_{\mathrm{W}_2^m} \\ &= \kappa_2 \|v\|_{\mathrm{W}_2^m}. \end{aligned} \tag{2.2.3}$$

Similarly, we have

$$|\mathbf{I}(v)| = |v(t)| \leq \kappa_m \|v\|_{\mathrm{W}_2^m}.$$

Consequently, $\mathbf{I}$ is bounded linear functional in $\mathrm{W}_2^m[a,b]$. Thus, $\mathrm{W}_2^m[a,b]$ is RKHS . □

Now, let us find out the expression form of the reproducing kernel function $\mathrm{K}_t(s)$ in the RKHS $\mathrm{W}_2^m[a,b]$.

Assume that $\mathrm{K}_t(s)$ is the reproducing kernel function of $\mathrm{W}_2^m[a,b]$, then for any $t \in [a,b]$ and any $v(t) \in \mathrm{W}_2^m[a,b]$. $\mathrm{K}_t(s)$ must satisfy the reproducing property

$$\langle v(s), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m} = v(t). \tag{2.2.4}$$

Using (1.1.2), we have

$$\langle v(s), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m} = \sum_{i=0}^{m-1} v^{(i)}(a)\partial_s^i \mathrm{K}_t(s) + \int_a^b v^{(m)}(s)\partial_s^m \mathrm{K}_t(s)dt.$$

Now, we apply the integration by parts for the second term of the right-hand of the above equation as:

$$\begin{aligned}
\textstyle\int_a^b v^{(m)}(s)\partial_s^m \mathrm{K}_t(s)ds &= \textstyle\sum_{i=0}^{m-1}(-1)^i v^{(m-i-1)}(s)\partial_s^{m+i}\mathrm{K}_t(s)\Big|_{s=a}^{s=b} \\
&\quad \textstyle+(-1)^m \int_a^b v(t)\partial_s^{2m}\mathrm{K}_t(s)ds, \\
&= \textstyle\sum_{i=0}^{m-1}(-1)^{m-i-1} v^{(i)}(t)\partial_s^{2m-i-1}\mathrm{K}_t(s)\Big|_{s=a}^{s=b} \\
&\quad \textstyle+(-1)^m \int_a^b v(s)\partial_s^{2m}\mathrm{K}_t(s)ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle v(s), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m} &= \textstyle\sum_{i=0}^{m-1} v^{(i)}(a)\left[\partial_s^i \mathrm{K}_t(a) - (-1)^{m-i-1}\partial_s^{2m-i-1}\mathrm{K}_t(a)\right] \\
&\quad \textstyle+ \sum_{i=0}^{m-1}(-1)^{m-i-1} v^{(i)}(b)\partial_s^{2m-i-1}\mathrm{K}_t(b) + (-1)^m \int_a^b v(s)\partial_s^{2m}\mathrm{K}_t(s)ds.
\end{aligned}$$

Therefore, $\mathrm{K}_t(s)$ is the solution of the generalized differential equations (DEs) described as follows:

$$\begin{cases} (-1)^m \partial_s^{2m} \mathrm{K}_t(s) = \delta(t-s), \\ \partial_s^i \mathrm{K}_t(a) - (-1)^{m-i-1} \partial_s^{2m-i-1} \mathrm{K}_t(a) = 0, \\ \partial_s^{2m-i-1} \mathrm{K}_t(b) = 0, \;\; i = 0, 1, \ldots, m-1. \end{cases} \tag{2.2.5}$$

While, $t \neq s$

$$(-1)^m \partial_s^{2m} \mathrm{K}_t(s) = 0, \tag{2.2.6}$$

subject to the boundary conditions (BC's):

$$\partial_s^i \mathrm{K}_t(a) - (-1)^{m-i-1} \partial_s^{2m-i-1} \mathrm{K}_t(a) = 0, \;\; \partial_s^{2m-i-1} \mathrm{K}_t(b) = 0, \;\; i = 0, 1, \ldots, m-1. \tag{2.2.7}$$

The characteristic equation of Eq. (2.2.6) is $r^{2m} = 0$, and their characteristic values are $r = 0$ with $2m$ multiple roots. Hence, the general solution of Eq. (2.2.6) is obtain as follows:

$$\mathrm{K}_t(s) = \begin{cases} \sum_{i=0}^{m-1} a_i(t) s^i, & s \leq t, \\ \sum_{i=0}^{m-1} b_i(t) s^i, & s > t. \end{cases} \tag{2.2.8}$$

Otherwise, since $(-1)^m \partial_s^{2m} \mathrm{K}_t(s) = \delta(t-s)$, we have

$$\lim_{s \to t^+} \partial_s^{2m-1} \mathrm{K}_t(s) = \lim_{s \to t^-} \partial_s^{2m-1} \mathrm{K}_t(s), \;\; \text{for } \; i = 0, 1, \ldots, 2m-2. \tag{2.2.9}$$

Integrating $(-1)^m \partial_s^{2m} \mathrm{K}_t(s) = \delta(t-s)$ from $t - \epsilon$ to $t + \epsilon$ with respect to s and let $\epsilon \to 0$, we have the jump degree of $\partial_s^{2m-1} \mathrm{K}_t(s)$ at $s = t$ given by

$$\lim_{s \to t^+} \partial_s^{2m-1} \mathrm{K}_t(s) - \lim_{s \to t^-} \partial_s^{2m-1} \mathrm{K}_t(s) = (-1)^{-m}. \tag{2.2.10}$$

The Eqs. (2.2.9) and (2.2.10) provided $2m$ conditions for determine the coefficients $a_i(t)$

and $b_i(t)$, $i = 0, 1, \ldots, 2m-1$, in Eq. (2.2.8). Moreover, through Eq. (2.2.7) provided $2m$ BC's. Thus, we possess $4m$ equations.

Note that these $4m$ equations are linear equations with coefficient $a_i(t)$ and $b_i(t)$ such that the unknown coefficient $a_i(t)$ and $b_i(t)$ of Eq. (2.2.8) could be determined out by several techniques such as Green's function method or by using Mathematica 12 software package. As long as the coefficients $a_i(t)$ and $b_i(t)$ are known, the accurate formula of the reproducing kernel function $\mathrm{K}_t(s)$ could be computed from Eq. (2.2.8). The expression of $\mathrm{K}_t(s)$ is a piecewise polynomial with $2m-1$ degrees.

Theorem 2.2.2. ***(Cui and Lin, 2008)*** *Let $\mathrm{K}_t(s)$ be the reproducing kernel function of the RKHS $\mathrm{W}_2^m[a,b]$. Then*

$$\frac{\partial^{i+j}}{\partial s^i \partial t^j}\mathrm{K}_t(s) \in \mathrm{W}_2^m[a,b], \;\; i+j = m-1,$$

with respect to s or t.

Theorem 2.2.3. ***(Cui and Lin, 2008)*** *If $v_n(t)$ converges to $v(t)$ in the sense of norm $\|\cdot\|_{\mathrm{W}_2^m}$, then $v_n^{(k)}(t)$ uniformly converges to $v^{(k)}(t)$ for $0 \leq k \leq m-1$.*

Proof : Let $v_n(t), v(t) \in \mathrm{W}_2^m[a,b]$, for $n = 1, 2, \ldots$ and let $\mathrm{K}_t(s)$ be the reproducing kernel function of $\mathrm{W}_2^m[a,b]$. By the reproducing property of $\mathrm{K}_t(s)$ in $\mathrm{W}_2^m[a,b]$, we have $v_n(t) - v(t) = \langle v_n(s) - v(s), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m}$. Thus

$$\begin{aligned} |v_n^{(k)}(t) - v^{(k)}(t)| &= |\langle v_n^{(k)}(s) - v^{(k)}(s), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m}|, \\ &= |\langle v_n(s) - v(s), \partial_s^k \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^m}| \\ &\leq \|v_n(s) - v(s)\|_{\mathrm{W}_2^m} \|\partial_s^k \mathrm{K}_t(s)\|_{\mathrm{W}_2^m}. \end{aligned}$$

Because $\|\partial_s^k \mathrm{K}_t(s)\|_{\mathrm{W}_2^m}$ is continuous, then

$$|v_n^{(k)}(t) - v^{(k)}(t)| \leq \kappa \|v_n(s) - v(s)\|_{\mathrm{W}_2^m} \rightharpoonup 0 \ \ as \ \ n \to \infty,$$

where $\kappa > 0$. □

2.3 Expressions of the Reproducing Kernel Function

Before we set to present the expressions of such reproducing kernel functions $\mathrm{K}_t(s)$ of $\mathrm{W}_2^m[a,b]$, we think it is essential to take the following notes:

1. The RKHS $\mathrm{W}_2^m[a,b]$ may contain various different types of homogeneous BC's. Of course, we will obtain different RKHS and harmonious reproducing kernel functions $\mathrm{K}_t(s)$. Further, occasionally several non-classical homogeneous BC's can be also enjoined on the RKHS $\mathrm{W}_2^m[a,b]$.

2. It should be noted that there are several equivalents expression formulas for the reproducing kernel function $\mathrm{K}_t(s)$ of $\mathrm{W}_2^m[a,b]$. This is not conflicting with the fact that the reproducing kernel function is unique. But the change is due to redefine the inner product expressed in Eq. (1.1.2) of $\mathrm{W}_2^m[a,b]$.

We are ready to provide some expressions of reproducing kernel function $\mathrm{K}_t(s)$ in the RKHS $\mathrm{W}_2^m[a,b]$ in the selected interval $[a,b] = [0,1]$, by using the method suggested above. Each time we re-define the standards inner product expressed in Eq. (1.1.2) and notice the changes output in the reproducing kernel function $\mathrm{K}_t(s)$.

1. The RKHS $\mathrm{W}_2^1[0,1]$ is defined as:

$$\mathrm{W}_2^1[0,1] = \{v(t) : v(t) \ is \ absolutely \ continuous \ on \ [0,1], v'(t) \in L^2[0,1]\}. \quad (2.3.1)$$

(a) The standards inner product and norm in $W_2^1[0,1]$ are equipped, respectively, by:

$$\langle v_1, v_2\rangle_{W_2^1} = v_1(0)v_2(0) + \int_0^1 v_1'(t)v_2'(t)dt, \tag{2.3.2}$$

and

$$\|v_1\|^2_{W_2^1} = \langle v_1, v_1\rangle_{W_2^1}.$$

In (Yao, 2008), the unique representation of the reproducing kernel function $K_t(s)$ of $W_2^1[0,1]$ equipped with (2.3.2) is described by:

$$K_t(s) = \begin{cases} 1+s, & s \le t, \\ 1+t, & s > t. \end{cases}$$

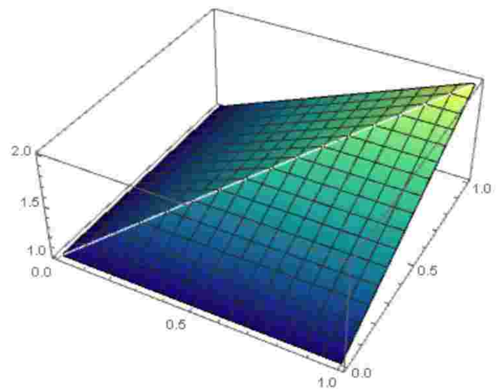


Figure 2.1: The 3D simulation of the reproducing kernel function $K_t(s)$ of $W_2^1[0,1]$.

(b) Define a new inner product in the RKHS $W_2^1[0,1]$ by:

$$\langle v_1, v_2\rangle_{W_2^1} = \int_0^1 (v_1(t)v_2(t) + v_1'(t)v_2'(t))dt, \tag{2.3.3}$$

and considering the same norm

$$\|v_1\|^2_{W_2^1} = \langle v_1, v_1\rangle_{W_2^1}.$$

In (Li and Cui, 2003), the unique representation of the reproducing kernel function $\mathrm{R}_t(s)$ of $\mathrm{W}_2^1[0,1]$ equipped with (2.3.3) is described by:

$$\mathrm{R}_t(s) = \frac{1}{2\sinh(1)}\big[\cosh(t+s-1)+\cosh(|t-s|-1)\big].$$

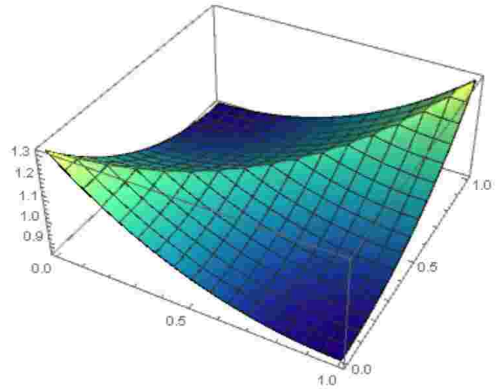


Figure 2.2: The 3D simulation of the reproducing kernel function $\mathrm{R}_t(s)$ of $\mathrm{W}_2^1[0,1]$.

2. The RKHS $\mathrm{W}_2^1[0,1]$ is defined as:

$$\mathrm{W}_2^1[0,1] = \{v(t) : v(t) \textit{ is Absolutely Continuous on } [0,1],\ v'(t)\in L^2[0,1],\ \text{ and } v(0)=v(1)=0\}.$$

The inner product and norm in $\mathrm{W}_2^1[0,1]$ are equipped, respectively, by:

$$\langle v_1, v_2\rangle_{\mathrm{W}_2^1} = \int_0^1 v_1'(t)v_2'(t)dt.$$

And

$$\|v_1\|^2_{\mathrm{W}_2^1} = \langle v_1, v_1\rangle_{\mathrm{W}_2^1}.$$

In (Paulsen, 2009), the expression of the reproducing kernel function $\mathrm{T}_t(s)$ of

$W_2^1[0,1]$ is described by:

$$T_t(s) = \begin{cases} (1-t)s, & s \le t, \\ (1-s)t, & s > t. \end{cases}$$

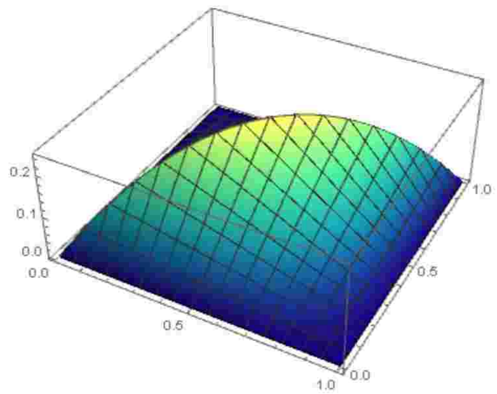


Figure 2.3: The 3D simulation of the reproducing kernel function $T_t(s)$ of $W_2^1[0,1]$.

2.4 Solution Methodology of Reproducing Kernel Algorithm

Reproducing kernel theory has important implementations in numerical analysis, biology, machine learning, finance, probability and statistics, learning theory, and so on. The reproducing kernels are geared to deal with a variety of linear and nonlinear problems such as nonlinear BVPs, integral equations, integro-DEs, singular perturbation DEs, fuzzy DEs, chaotic systems, partial differential equations (PDEs), etc.

In this section, we will establish an iterative scheme of RKA to construct and calculate the analytic and the approximate solutions for a general class of m^{th}-order BVP. The representation of the analytic solution is expressed in a convergent series given in the RKHS $W_2^m[a,b]$ that can be elegantly computed. Its approximate solution is obtained by amputating the n-term of the analytic solution. At any rate, we consider the general class of m^{th}-order BVP of the following form:

$$\begin{cases} \mathrm{L}v(t) = \mathrm{H}(t, v(t)), \;\; t \in [a, b], \\ \mathrm{B}v = \gamma_k, \;\; k = 0, 1, \dots, m-1. \end{cases} \tag{2.4.1}$$

Herein, L is a linear differential operator of order m of the form:

$$\mathrm{L}v(t) = \sum_{k=1}^{m} p_k(t) v^{(k)}(t), \tag{2.4.2}$$

while, B is typical Dirichlet, Neumann, or mixed BC's. Assume that $p_k(t), k = 0, 1, \dots, m$ are continuous real-valued functions, $\gamma_k, \;\; k = 0, 1, \dots, m-1$, are real finite constants; $v(t)$ is unknown function to be determined in $\mathrm{W}_2^{m+1}[a, b]$, and $\mathrm{H}(t, v(t))$ is a linear or nonlinear function depending on the problem discussed.

In order to find a solution to the m^{th}-order BVP (2.4.1), we must first create a RKHS $\mathrm{W}_2^{m+1}[a, b]$ whose reproducing kernel function satisfy the BC's related to Eq. (2.4.1). Also, let us assume that $\mathrm{K}_t(s)$ and $\mathrm{R}_t(s)$ be two reproducing kernel function of both RKHSs $\mathrm{W}_2^{m+1}[a, b]$ and $\mathrm{W}_2^{1}[a, b]$, respectively. Now, we will define a linear differential operator as follow:

$$\mathrm{L} : \mathrm{W}_2^{m+1}[a, b] \to \mathrm{W}_2^{1}[a, b],$$

such that

$$\mathrm{L}v(t) = \sum_{k=1}^{m} p_k(t) v^{(k)}(t).$$

After that, we homogenize the BC's of Eq. (2.4.1) using a simple transformation to obtain the following equivalent form:

$$\begin{cases} \mathrm{L}v(t) = \mathrm{H}(t, v(t)), \quad t \in [a, b], \\ \mathrm{B}v = 0, \end{cases} \tag{2.4.3}$$

where, $v(t) \in \mathrm{W}_2^{m+1}[a, b]$ and $\mathrm{H}(t, v(t)) \in \mathrm{W}_2^1[a, b]$. Note that its easy to show that the operator L is linear and bounded. The next step is how to create an orthogonal function system of $\mathrm{W}_2^{m+1}[a, b]$. To see this. we put $\varphi_i(t) = \mathrm{R}_{t_i}(t)$ and $\psi_i(t) = \mathrm{L}^\star\varphi_i(t)$, i=1,2,..., such that $\{t_i\}_{i=0}^{\infty}$ is dense countable set in $[a, b]$ and $\mathrm{L}^\star$ is the adjoint operator of L.

Look this, from the properties of reproducing kernel $\mathrm{R}_t(s)$, for every $v(t) \in \mathrm{W}_2^1[a, b]$, it follows that $\langle v(t), \varphi_i(t)\rangle\rangle_{\mathrm{W}_2^1} = \langle v(t), \mathrm{R}_{t_i}(t)\rangle_{\mathrm{W}_2^1} = v(t_i)$. Additionally, In terms of the properties of reproducing kernel $\mathrm{K}_t(s)$, one obtains

$$\langle v(t), \psi_i(t)\rangle_{\mathrm{W}_2^{m+1}} = \langle v(t), \mathrm{L}^\star\varphi_i(t)\rangle_{\mathrm{W}_2^{m+1}} = \langle \mathrm{L}v(t), \varphi_i(t)\rangle_{\mathrm{W}_2^1} = \mathrm{L}v(t_i), \; i = 1, 2, \ldots.$$

Lemma 2.4.1. *$\psi_i(t)$ can be formulated as follow $\psi_i(t) = \mathrm{L}\mathrm{K}_t(s)|_{s=t_i}$. In which, the subscript s by the operator* L *refer that* L *applies to the function of s.*

Proof : It is clear that,

$$\begin{aligned} \psi_i(t) &= \mathrm{L}^\star\varphi_i(t) = \langle \mathrm{L}^\star\varphi_i(t), \mathrm{K}_t(s)\rangle_{\mathrm{W}_2^{m+1}} \\ &= \langle \varphi_i(t), \mathrm{L}\mathrm{K}_t(s)\rangle_{\mathrm{W}_2^1} = \mathrm{L}\mathrm{K}_t(s)|_{s=t_i}. \end{aligned}$$

□

Lemma 2.4.2. $\psi_i(a) = \psi_i(b) = 0$, *for* $i = 1, 2, \ldots$.

Proof : $\psi_i(a) = \langle \psi_i(s), \mathrm{K}_a(s)\rangle_{\mathrm{W}_2^{m+1}} = \langle \mathrm{L}^\star\varphi_i(t), \mathrm{K}_a(s)\rangle_{\mathrm{W}_2^{m+1}} = \langle \varphi_i(t), \mathrm{L}\mathrm{K}_a(s)\rangle_{\mathrm{W}_2^1}$. From the symmetry property of $\mathrm{K}_t(s)$; we can get at $\mathrm{K}_a(s) = \mathrm{K}_s(a) = 0$, thus $\psi_i(a) = 0$ for

$i = 1, 2, \ldots$. Similarly, we can show $\psi_i(b) = 0$, for $i = 1, 2, \ldots$. □

To derive an orthonormal function system $\{\widehat{\psi}_i(t)\}_{i=1}^{\infty}$ of the RKHS $\mathrm{W}_2^{m+1}([a,b])$ from $\{\psi_i(t)\}_{i=1}^{\infty}$. we need to use the Gram-Schmidt orthogonalization process, which can be used as follows:

$$\widehat{\psi}_i(t) = \sum_{k=1}^{i} \varrho_{ik} \psi_k(t), \;\; i = 1, 2, \ldots, \tag{2.4.4}$$

where, ϱ_{ik} is the orthogonalization coefficients of $\{\psi_i(t)\}_{i=1}^{\infty}$ and are obtained as:

$$\begin{aligned}
\varrho_{11} &= (\|\psi_1(t)\|_{\mathrm{W}_2^{m+1}})^{-1}, \\
\varrho_{ij} &= \left(\|\psi_i(t)\|^2_{\mathrm{W}_2^{m+1}} - \textstyle\sum_{k=1}^{i-1} c_{ik}^2\right)^{-\frac{1}{2}}, \;\; i = j \neq 1, \\
\varrho_{ij} &= -\frac{1}{\sqrt{\|\psi_i(t)\|^2_{\mathrm{W}_2^{m+1}} - \sum_{k=1}^{i-1} c_{ik}^2}} \textstyle\sum_{k=j}^{i-1} c_{ik} \varrho_{kj}, \;\; i < j,
\end{aligned} \tag{2.4.5}$$

in which, $c_{ik} = \langle \psi_i(t), \widehat{\psi}_k(t) \rangle_{\mathrm{W}_2^{m+1}}$.

Theorem 2.4.3. *Under the assumption when the inverse operator* L^{-1} *in Eq.* (2.4.3) *exists. If* $\{t_i\}_{i=1}^{\infty}$ *is dense subset on* $[a,b]$*, then the sequence* $\{\psi_i(t)\}_{i=1}^{\infty}$ *is complete on* $\mathrm{W}_2^{m+1}([a,b])$.

Proof : For each $v(t) \in \mathrm{W}_2^{m+1}([a,b])$, and if $\langle v(t), \psi_i(t) \rangle_{\mathrm{W}_2^{m+1}} = 0$, $i = 1, 2, \ldots$, then

$$\begin{aligned}
\langle v(t), \psi_i(t) \rangle_{\mathrm{W}_2^{m+1}} &= \langle v(t), \mathrm{L}^{\star} \varphi_i(t) \rangle_{\mathrm{W}_2^{m+1}} \\
&= \langle \mathrm{L} v(t), \varphi_i(t) \rangle_{\mathrm{W}_2^{m+1}} \\
&= \mathrm{L} v(t_i) = 0.
\end{aligned} \tag{2.4.6}$$

Because, $\{t_i\}_{i=1}^{\infty}$ is dense sub set on $[a,b]$, we can obtain $\mathrm{L}v(t) = 0$. Subsequently, the existence of L^{-1} yields $v(t) = 0$. The proof is complete. □

Lemma 2.4.4. $\{\widehat{\psi}_i(t)\}_{i=1}^{n} \in \mathrm{W}_2^{m+1}([a,b])$ *is a linearly independent sequence.*

Proof : Let $\{\widehat{\psi}_i(t)\}_{i=1}^{n}$ be an orthonormal functions sequence in $\mathrm{W}_2^{m+1}([a,b])$ and consider $\{\gamma_i\}_{i=1}^{n}$ satisfies

$$\sum_{i=1}^{n}\gamma_i\widehat{\psi}_i(t)=0. \tag{2.4.7}$$

Multiply (2.4.7) by a stationary $\widehat{\psi}_j(t)$, we get

$$\langle\sum_{i=1}^{n}\gamma_i\widehat{\psi}_i(t),\widehat{\psi}_j(t)\rangle_{\mathrm{W}_2^{m+1}}=\sum_{i=1}^{n}\gamma_i\langle\widehat{\psi}_i(t),\widehat{\psi}_j(t)\rangle_{\mathrm{W}_2^{m+1}}=\gamma_j=0, j=1,2,\ldots,n.$$

This proof also implies if the obtained orthonormal sequence is infinite. □

Definition 2.4.5. *If $v(t)$ is a continuous function and $\{\widehat{\psi}_i(t)\}_{i=1}^{\infty}$ an orthonormal functions system, then $\langle v(t),\widehat{\psi}_i(t)\rangle_{\mathrm{W}_2^{m+1}}$ are called the Fourier coefficients of $v(t)$ with respect to the system $\{\widehat{\psi}_i(t)\}_{i=1}^{\infty}$ and $v(t)=\sum_{i=1}^{\infty}\langle v(t),\widehat{\psi}_i(t)\rangle_{\mathrm{W}_2^{m+1}}\widehat{\psi}_i(t)$ is called its Fourier series expansion.*

Theorem 2.4.6. *If $\{t_i\}_{i=1}^{\infty}$ is dense sub set on $[a,b]$ and the solution of Eq.(2.4.3) exist and unique. Then the following achieved:*

- *Whenever $n\to\infty$ the analytic solution fulfills:*

$$v(t)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}H(t_k,v(t_k))\widehat{\psi}_i(t), \tag{2.4.8}$$

- *The n-term approximate solution fulfills:*

$$v_n(t)=\sum_{i=1}^{n}\sum_{k=1}^{i}\varrho_{ik}H(t_k,v(t_k))\widehat{\psi}_i(t), \tag{2.4.9}$$

Proof : Using the Theorem (2.4.3), it is easy to show that $\{\widehat{\psi}_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal sequence of the RKHS $\mathrm{W}_2^{m+1}[a,b]$. Thus, the solution of Eq. (2.4.3) can be formulated as the following Fourier series expansion $v(t) = \sum_{i=1}^{\infty}\langle v(t), \widehat{\psi}_i(t)\rangle_{W_2^{m+1}}\widehat{\psi}_i(t)$. By the completeness of $W_2^{m+1}[a,b]$, the series $\sum_{i=1}^{\infty}\langle v(t), \widehat{\psi}_i(t)\rangle_{W_2^{m+1}}\widehat{\psi}_i(t)$ is convergent in sense of the norm in $W_2^{m+1}[a,b]$. On the other hand, we have

$$\begin{aligned} v(t) &= \sum_{i=1}^{\infty}\langle v(t), \widehat{\psi}_i(t)\rangle_{W_2^{m+1}}\widehat{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle v(t), \psi_i(t)\rangle_{W_2^{m+1}}\widehat{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle v(t), L^{\star}\varphi_k(t)\rangle_{W_2^{m+1}}\widehat{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle Lv(t), \varphi_k(t)\rangle_{W_2^{1}}\widehat{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle H(t, v(t)), \varphi_k(t)\rangle_{W_2^{1}}\widehat{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}H(t_k, v(t_k))\widehat{\psi}_i(t). \end{aligned} \tag{2.4.10}$$

With respect to finding the n-term approximate solution of Eq. (2.4.3). The following two cases show how to obtain it:

Case 1: If Eq. (2.4.3) is linear, i.e., $H(t, v(t)) = h(t)$. Then, the n-term approximation solution to Eq. (2.4.3) can be given directly by truncating n-term of Eq. (2.4.8). We share the n-term approximate solution to $v(t)$ by

$$v_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\varrho_{ik}H(t_k, v(t_k))\widehat{\psi}_i(t),$$

Case 2: If Eq. (2.4.3) is nonlinear, i.e. $H(t, v(t))$ is a non-linear term. Then, the n-term approximation solution to Eq. (2.4.3) can be given using the following iteration process.

From Eq. (2.4.8), the representation of the analytic solution of Eq. (2.4.3) can be shared by

$$v(t) = \sum_{i=1}^{\infty} \Pi_i \widehat{\psi}_i(t), \tag{2.4.11}$$

in which, $\Pi_i = \sum_{k=1}^{i} \varrho_{ik} \mathrm{H}(t_k, v(t_k))$. Here, It should be mentioned that the coefficients $\Pi_i, \ i = 1, \ldots,$ are unknown. In order to obtain its values, we will approximate $\Pi_i, \ i = 1, \ldots,$ using known coefficients Λ_i.

Let $v(t_1) = 0$, which implies that $\mathrm{H}(t_1, v(t_1))$ is known. So, for a numerical calculation, we set $v_0(t_1) = v(t_1)$ and define the n-term approximation to $v(t)$ by:

$$v_n(t) = \sum_{i=1}^{n} \Lambda_i \widehat{\psi}_i(t), \tag{2.4.12}$$

where the coefficients Λ_i, $i = 1, 2, \ldots, n$, are obtained as

$$\begin{aligned}
&\Lambda_1 = \varrho_{11} H(t_1, v_0(t_1)) && \Rightarrow v_1(t) = \Lambda_1 \widehat{\psi}_1(t), \\
&\Lambda_2 = \textstyle\sum_{k=1}^{2} \varrho_{2k} \mathrm{H}(t_k, v_{k-1}(t_k)) && \Rightarrow v_2(t) = \Lambda_1 \widehat{\psi}_1(t) + \Lambda_2 \widehat{\psi}_2(t), \\
& && . \\
& && . \\
&\Lambda_N = \textstyle\sum_{k=1}^{N} \varrho_{Nk} \mathrm{H}(t_k, v_{k-1}(t_k)) && \Rightarrow v_N(t) = \textstyle\sum_{i=1}^{N} \Lambda_i \widehat{\psi}_i(t).
\end{aligned} \tag{2.4.13}$$

□

Remark 2.4.7. *The iterative sequence* (3.2.10) *satisfies the BC's of Eq.* (2.4.3).

Actuality, the solution of Eq. (2.4.3) is considered as the fixed point of the following functional under the suitable choice of the initial term $v_0(t)$:

$$v_{n+1}(t) \quad = \mathrm{L}^{-1} \mathrm{H}(t, v_n(t)) = \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \varrho_{ik} \mathrm{H}(t_k, v_k(t_k)) \widehat{\psi}_i(t).$$

Theorem 2.4.8. ***(Kreyszig,1989)(Banachs fixed point theorem.)*** *Suppose that* E *is a Banach space and let* $T : \mathrm{E} \to \mathrm{E}$ *be a contraction map on* E *satisfies*

$$\|\mathrm{T}[v_1] - \mathrm{T}[v_2]\| \leq \mathfrak{K}\|v_1 - v_2\|, \;\; \forall v_1, v_2 \in \mathrm{E},$$

for some $\mathfrak{K} < 1$*. Then,* T *has precisely unique fixed point. Moreover, the iterative sequence*

$$v_{n+1}(t) = \mathrm{T}[v_n],$$

with an arbitrary choice of $v_0 \in \mathrm{E}$ *converges to the fixed point of* T.

According to above Theorem, for the nonlinear mapping

$$v_{n+1}(t) \;\; = \mathrm{T}[v_n] = \mathrm{L}^{-1}\mathrm{H}(t, v_n(t)) = \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \varrho_{ik}\mathrm{H}(t_k, v_k(t_k))\widehat{\psi}_i(t).$$

a sufficient condition for convergence of the present iteration process is strictly contraction of T. Furthermore, the iterative sequence (3.2.10) converges to the precisely fixed point of T which is also the solution of Eq. (2.4.3). However, the approximate solution $v_n^N(t)$ can be obtained by taking finitely many terms in the series representation of $v_n(t)$ and

$$v_n^N(t) = \sum_{i=1}^{N}\sum_{k=1}^{i} \varrho_{ik}\mathrm{H}(t_k, v_{k-1}(t_k))\widehat{\psi}_i(t).$$

Based on all previous information, we will introduce an effective algorithm to deal with all nonlinear problems at the end of this section.

Algorithm 1 To approximate the solution of the BVP (2.4.3) based on RKA method :

procedure :

Input: n collocation points; independent domain $[a, b]$; reproducing kernel functions $l\mathrm{K}_t(s) = \sum_{i=0}^{m-1} a_i(t)s^i$ and $r\mathrm{K}_t(s) = \sum_{i=0}^{m-1} b_i(t)s^i$; operator $\mathrm{L}v(t)$; inner product $\langle v_1, v_2\rangle_{\mathrm{W}_2^{m+1}}$.

Output: Approximate solution $v_n(t)$ to $v(t)$ at n value of t.

`Step I:`

for $i = 1$ to n **do**

Set $t_i = \frac{i-1}{n-1}$.

Set **If** $s < t$ **then** $\mathrm{K}_t(s) = l\mathrm{K}_t(s)$ **else** $\mathrm{K}_t(s) = r\mathrm{K}_t(s)$.

Set $\psi_i(t) = \mathrm{L}_s\mathrm{K}_t(s) \mid_{s=t_i}$,

`Output I:` the orthogonal function system $\psi_i(t)$.

`Step II:`

for $i = 2$ to n and $k = 1$ to $i - 1$ **do**,

$\varrho_{ik} \mid_{i=k=1} = \frac{1}{\|\psi_i(t)\|_{\mathrm{W}_2^{m+1}}}$,

$\varrho_{ik} \mid_{i=k\neq 1} = \frac{1}{\|\psi_i(t)\|_{\mathrm{W}_2^{m+1}} - \sum_{p=1}^{i-1}\langle \psi_i(t), \widehat{\psi}_p(t)\rangle_{\mathrm{W}_2^{m+1}}}$,

$\varrho_{ik} \mid_{i>k} = -\frac{1}{\sqrt{\|\psi_i(t)\|_{\mathrm{W}_2^{m+1}} - \sum_{p=1}^{i-1}\langle \psi_i(t), \overline{\psi_i}_p(t)\rangle^2_{\mathrm{W}_2^{m+1}}}} \sum_{p=k}^{i-1}\langle \psi_i(t), \overline{\psi_i}_p(t)\rangle_{\mathrm{W}_2^{m+1}}\varrho_{pk}$

`Output II:` the orthogonalization coefficients ϱ_{ik}.

`Step III:`

for $i = 1$ to n **do**

Set $\widehat{\psi}_i(t) = \sum_{k=1}^{i} \varrho_{ik}\psi_i(t)$,

`Output III:` the orthonormal function system.

`Step IV: et \,$ v_{0}(t_{1})=0$, then do the following sub-steps:`

for $i = 1$ to n **do**,

Set $v(t_i) = v_{i-1}(t_i)$;

Set $\Lambda_i = \sum_{k=1}^{i} \varrho_{ik}\mathrm{H}(t_k, v_{k-1}(t_k))$;

Set $v_n(t) = \sum_{i=1}^{n} \Lambda_i\widehat{\psi}_i(t)$.

`Output IV:` The approximate solution $v_n(t)$ is obtained.

CHAPTER 3

RKHS method for Solving Caputo-Fabrizio Fractional Riccati and Bernoulli Models

This chapter aims to study numerical solutions for general Riccati and Bernoulli-type models of fractional order, considering favorable initial constraint conditions. The CF fractional operator is handled using RKHS method. Based on this approach, modern operational algorithms are constructed and discussed to address such models. Numerical applications are included in this context to demonstrate the feasibility and reliability of this approach, taking into account the qualitative impact of the CF fractional derivative. Additionally, numerical comparisons are conducted between the obtained results and results obtained using powerful numerical methods. From a numerical perspective, the results obtained through the proposed method exhibit a high level of accuracy and exceptional technical proficiency. These findings qualify the method to handle a wide range of fractional models emerging in various mathematical and physical disciplines, employing the CF fractional derivative.

3.1 Introduction

Let us examine the general form provided here, which comprehensively expresses the FRDEs and FBDEs in the frame of CF fractional derivative. (Note: when $\beta = 2$, the FRDE models are obtained).

$$^{CF}D_a^{\alpha} v(t) + a(t)v(t) + b(t)v^{\beta}(t) = c(t), \tag{3.1.1}$$

subject to the IC

$$v(a) = v_0. \tag{3.1.2}$$

In order to guarantee the proper well-posedness of the aforementioned equations, we must be standing for the following: $t \in [a, b]$; $v_0, \beta \in \mathbb{R}$; $a(t), b(t),$ and $c(\tau)$ are known continuous functions on $[a, b]$. Here, $v(t)$ is an unknown function to be determined in $\mathrm{W}_2^2[a, b]$, while ${}^{CF}D_a^{\alpha}$ indicates the CF fractional derivative of order α such that $[a, b] \in [0, 1]$. We assume that the Eqs.(3.1.1)and (3.1.2) have a unique solution.

The RKHS method offers several advantages as:

1. The conditions for determining the solution in Eqs (3.1.1) and (3.1.2) can be imposed on the RKHS, and therefore the reproducing kernel satisfying the conditions for determining the solution can be calculated.

2. The iterative sequence $v_n(t)$ of approximate solutions converges uniformly in C^1 to the exact solution $v(t)$.

3. It is possible to pick any point in the integration interval.

To utilize the RKHS method for solving Eqs. (3.1.1), it is necessary to redefine two reproducing kernel functions within the RKHS's $\mathrm{W}_2^1[a, b]$ and $\mathrm{W}_2^2[a, b]$, respectively. These functions should satisfy the IC's (3.1.2).

To begin with, the space $\mathrm{W}_2^1[a, b]$ is defined as follows:

$$\mathrm{W}_2^1[a, b] = \{v(t) : v \ is \ Absolutely \ Continuous \ on \ [a, b], \ \ v' \in L^2[a, b], \ \ t \in [a, b]\}.$$

The inner product and norm of the RKHS $\mathrm{W}_2^1[a, b]$ are attached, respectively, by

$$\langle v_1, v_2 \rangle_{\mathrm{W}_2^1} = v_1(a)v_2(a) + \int_a^b v_1'(t)v_2'(t)dt, \ \ v_1, v_2 \in \mathrm{W}_2^1[a, b], \tag{3.1.3}$$

and

$$\|v\|_{\mathrm{W}_2^1} = \sqrt{\langle v, v \rangle_{\mathrm{W}_2^1}}, \ \ v \in \mathrm{W}_2^1[a, b]. \tag{3.1.4}$$

In (Yao, 2008), it had been demonstrated that the space $\mathrm{W}_2^1[a, b]$ is a complete RKHS and its reproducing kernel function $T_t(s)$ is given by

$$\mathrm{T}_t(s) = \begin{cases} 1-a+s, & s \le t, \\ 1-a+t, & s > t. \end{cases}$$

Secondly, the space $\mathrm{W}_2^2[a,b]$ is defined as:

$$\mathrm{W}_2^2[a,b] = \{v(t) : v, v' \textit{ is Absolutely Continuous on } [a,b],\ \ v'' \in L^2[a,b],\ \ t \in [a,b],\ \ v(a) = 0\}.$$

Let us re-define the standard inner product described by the following:

$$\langle v_1, v_2 \rangle_{\mathrm{W}_2^2} = v_1(a)v_2(a) + v_1(b)v_2(b) + \int_a^b v_1^{(2)}(t)v_2^{(2)}(t)dt,\ \ v_1, v_2 \in \mathrm{W}_2^2[a,b], \qquad (3.1.5)$$

and the norm associated by:

$$\|v\|_{\mathrm{W}_2^2} = \sqrt{\langle v, v \rangle_{\mathrm{W}_2^2}},\ \ v \in \mathrm{W}_2^2[a,b]. \qquad (3.1.6)$$

It's easy to show that the Eq.(3.1.5) satisfies the requirements of the inner product as well as:

(i) For any $u, v, w \in \mathrm{W}_2^2[a,b]$, and for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \langle u+v, w \rangle_{\mathrm{W}_2^2} &= (u+v)(a)w(a) + (u+v)(b)w(b) + \textstyle\int_a^b (u+v)^{(2)}(t)w^{(2)}(t)dt \\ &= \langle u, w \rangle_{\mathrm{W}_2^2} + \langle v, w \rangle_{\mathrm{W}_2^2}; \end{aligned}$$

(ii) $\langle \lambda u, v \rangle_{\mathrm{W}_2^2} = \lambda \langle u, v \rangle_{\mathrm{W}_2^2}$;

(iii) $\langle u, v \rangle_{\mathrm{W}_2^2} = \langle v, u \rangle_{\mathrm{W}_2^2}$;

(iv) $\langle v, v\rangle_{W_2^2} = v^2(a) + v^2(b) + \int_a^b (v^{(2)}(t))^2 dt \geq 0$, with equality if $v = 0$.

The space $W_2^2[a, b]$ is a complete RKHS. Which mean, for each fixed $t \in [a, b]$ there exists a reproducing kernel function $K_t(s) \in W_2^2[a, b]$ such that $\langle v(s), K_t(s)\rangle_{W_2^2} = v(t)$, for any $v(t) \in W_2^2[a, b]$. The reproducing kernel function $K_t(s) \in W_2^2[a, b]$ can be written as follows:

$$K_t(s) = \begin{cases} \sum_{i=1}^{4} a_i(t)s^{i-1}, & s < t, \\ \sum_{i=1}^{4} b_i(t)s^{i-1}, & t \leq s. \end{cases} \tag{3.1.7}$$

The steps for obtaining $K_t(s)$ are well described in Section (2.1). Therefore, the unique representation of $K_t(s)$ is provided by:

$$K_t(s) = \begin{cases} \frac{1}{6(a-b)^2}\begin{pmatrix} -2a^3(b-t)(b-s) + a^2(6 + 2b^3 + t^3 + 3ts^2 - 3b(t^2 \\ +s^2)) + s(-3b^2t^2 + bt^3 - b^2s^2 + t(6 + 2b^3 + bs^2)) \\ -a((-3bt^2 + t^3)(b + s) + s(6 + 2b^3 - 3b^2s - bs^2) \\ +t(6 + 2b^3 + 3bs^2 + s^3)) \end{pmatrix}, s \leq t, \\ \frac{1}{6(a-b)^2}\begin{pmatrix} -2a^3(b-s)(b-t) + a^2(6 + 2b^3 + s^3 + 3st^2 - 3b(s^2 \\ +t^2)) + t(-3b^2s^2 + bs^3 - b^2t^2 + s(6 + 2b^3 + bt^2)) \\ -a((-3bs^2 + s^3)(b + t) + t(6 + 2b^3 - 3b^2t - bt^2) \\ +s(6 + 2b^3 + 3bt^2 + t^3) \end{pmatrix}, s > t. \end{cases} \tag{3.1.8}$$

3.2 Solutions shape of FRDEs and FBDEs

To go forward with the proposed RKHS method implementation to solve FRDEs and FBDEs within CF fractional derivatives. Let us first define a linear deferential operator as:

$$\begin{aligned} &\mathrm{L}: \mathrm{W}_2^2[a,b] \to \mathrm{W}_2^1[a,b] \\ &\mathrm{L}v(t) =^{CF} D_a^{\alpha} v(t) + a(t)v(t). \end{aligned} \tag{3.2.1}$$

The replacement $v(t) :\to v(t) - v_0$ converts (3.1.1) and (3.1.2) into the equivalent form as follows:

$$\mathrm{L}v(t) = c(t) - b(t)v^{\beta}(t). \tag{3.2.2}$$

subject to the IC

$$v(a) = 0. \tag{3.2.3}$$

In which, $a(t), b(t), and\ c(t)$ are continuous smooth functions on $[a,b]$, and $v(t) \in \mathrm{W}_2^2[a,b]$, It is clear that L is a bounded linear operator.

Currently. we construct an orthogonal function system of $\mathrm{W}_2^2[a,b]$. Putting $\varphi_i(t) = \mathrm{T}_{t_i}(t)$ and $\psi_i(t) = \mathrm{L}^{\star}\varphi_i(t)$, $i = 1, 2, \ldots$, such that $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$. The Gram-Schmidt orthogonalization process is used to create the orthonormal functions system $\{\widehat{\psi}_i(t)\}_{i=1}^{\infty}$ in $\mathrm{W}_2^2([a,b])$ as follows:

$$\widehat{\psi}_i(t) = \sum_{k=1}^{i} \varrho_{ik}\psi_k(t), \ \ \varrho_{ii} > 0, \ \ i = 1, 2, \ldots \tag{3.2.4}$$

where, ϱ_{ik} is the orthogonalization coefficients.

Theorem 3.2.1. *If $\{t_i\}_{i=1}^{\infty}$ is dense set on $[a,b]$ and the inverse operator L^{-1} exists, then $\{\psi_i(t)\}_{i=1}^{\infty}$ is complete on $\mathrm{W}_2^2[a,b]$.*

Proof : $\psi_i(t) = \mathrm{L}^{\star}\varphi_i(t)$ assures that $\psi_i(t) \in \mathrm{W}_2^2[a,b]$. For each $v(t) \in \mathrm{W}_2^2[a,b]$, if

$\langle v(t), \psi_i(t)\rangle_{W_2^2} = 0$, $i = 1, 2, \dots,$ then

$$\begin{aligned} \langle v(t), \psi_i(t)\rangle_{W_2^2} &= \langle v(t), L^{\star}\varphi_i(t)\rangle_{W_2^2} \\ &= \langle Lv(t), \varphi_i(t)\rangle_{W_2^2} \\ &= Lv(t_i) = 0. \end{aligned} \tag{3.2.5}$$

By the density of $\{t_i\}_{i=1}^{\infty}$ on $[a, b]$, we have $Lv(t) = 0$. The existence of L^{-1} yields $v(t) = 0$. Hence, $\{\psi_i(t)\}_{i=1}^{\infty}$ is complete on $W_2^2[a, b]$. □

Now, we will described the representation of the analytic-approximate solution of Eqs. (3.2.3) and (3.2.3) in the RKHS $W_2^2[a, b]$. Afterward, we investigate the convergence of approximate solution $v_n(t)$ to the analytic solution $v(t)$.

Theorem 3.2.2. *If $\{t_i\}_{i=1}^{\infty}$ is dense set on $[a, b]$ and the solution of Eqs.* (3.2.2) *and* (3.2.3) *exist and unique. then the represented formula of the analytic solution $v(t)$ expressed as follows:*

$$v(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \varrho_{ik}(c(t_k) - b(t_k)v^{\beta}(t_k))\widehat{\psi}_i(t). \tag{3.2.6}$$

Proof : $\{\widehat{\psi}_i(\tau)\}_{i=1}^{N}$ is a complete orthonormal sequence of the RKHS $W_2^2([a, b])$. Thus, the solution of Eq. (3.2.2) can be formulated as the following Fourier series expansion $v(t) = \sum_{i=1}^{\infty}\langle v(t), \widehat{\psi}_i(t)\rangle_{W_2^2}\widehat{\psi}_i(t)$. By the completeness of $W_2^2[a, b]$, the series $\sum_{i=1}^{\infty}\langle v(t), \widehat{\psi}_i(t)\rangle_{W_2^2}\widehat{\psi}_i(t)$ is uniformly convergent on $W_2^2[a, b]$. On the other hand, we have

$$\begin{aligned} v(t) &= \textstyle\sum_{i=1}^{\infty}\langle v(t),\widehat{\psi}_i(t)\rangle_{W_2^2}\widehat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle v(t),\psi_i(t)\rangle_{W_2^2}\widehat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle v(t),L^{\star}\varphi_k(t)\rangle_{W_2^2}\widehat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}\langle Lv(t),\varphi_k(t)\rangle_{W_2^1}\widehat{\psi}_i(t) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\varrho_{ik}(c(t_k)-b(t_k)v^{\beta}(t_k))\widehat{\psi}_i(t). \end{aligned} \tag{3.2.7}$$

□

Lemma 3.2.3. *If $v(t) \in W_2^2[a,b]$. Then the following relation holds*

$$|v^{(k)}(t)| \leq \mathfrak{K} \parallel v(t) \parallel_{W_2^2}, \;\; k=0,1, \;\; \mathfrak{K}>0. \tag{3.2.8}$$

Proof : For all $t \in [a,b]$ and $v(t) \in W_2^2[a,b]$, we have

$$|v^{(k)}(t)| = |\langle v(\cdot), \partial_t^k K_t(\cdot)\rangle_{W_2^2}| \leq \mathfrak{K} \parallel v(t) \parallel_{W_2^2},$$

such that $\mathfrak{K} = \max_{t\in[a,b]} \|\partial_t^k K_t(\cdot)\|_{W_2^2}$. □

Corollary 3.2.4. *The approximate solution $v_n(t)$ and its derivative $v_n'(t)$, are uniformly convergent at $n \to \infty$.*

Proof : The proof is given by the direct application of Lemma (3.2.3). □

Remark 3.2.5. *With respect to find the n-term approximate solution of Eq. (3.2.2). The following two cases show how to obtain it:*

Case 1: *If Eq.(3.2.2) is linear, i.e., $\beta = 0$ or $\beta = 1$. Then, the n-term approximation solution to Eq.(3.2.2) can be given directly by truncating n-term of Eq.(3.2.6).*

We share the n-term approximate solution to $v(t)$ by

$$v_n(t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \varrho_{ik} \left(c(t_k) - b(t_k) v^{\beta}(t_k) \right) \widehat{\psi}_i(t),$$

Case 2: *If Eq. (3.2.2) is nonlinear, i.e. $\beta \neq 0$ and $\beta \neq 1$. Then, the n-term approximation solution to Eq. (3.2.2) can be given using the following iteration process.*

From Eq.(3.2.6), the representation of the analytic solution of Eq.(3.2.2) can be shared by

$$v(t) = \sum_{i=1}^{\infty} \Pi_i \widehat{\psi}_i(t), \tag{3.2.9}$$

in which, $\Pi_i = \sum_{k=1}^{i} \varrho_{ik} \left(c(t_k) - b(t_k) v^{\beta}(t_k) \right)$. Here, It should be mentioned that the coefficients $\Pi_i, \;\; i = 1, \ldots,$ are unknown. In order to obtain its values, we will approximate $\Pi_i, \;\; i = 1, \ldots,$ using known coefficients Λ_i.

Let $v(t_1) = 0$, which implies that $c(t_1) - b(t_1) v^{\beta}(t_1) = c(t_1)$ is known. So, for a numerical calculation, we set $v_0(t_1) = v(t_1)$ and define the n-term approximation to $v(t)$ by:

$$v_n(t) = \sum_{i=1}^{n} \Lambda_i \widehat{\psi}_i(t), \tag{3.2.10}$$

where the coefficients Λ_i, $i = 1, 2, \ldots, n$, are obtained as

$$\begin{aligned}
&\Lambda_1 = \varrho_{11} c(t_1) && \Rightarrow v_1(t) = \Lambda_1 \overline{\psi}_1(t), \\
&\Lambda_2 = \textstyle\sum_{k=1}^{2} \varrho_{2k} \left(c(t_k) - b(t_k) v_{k-1}^{\beta}(t_k) \right) && \Rightarrow v_2(t) = \Lambda_1 \overline{\psi}_1(t) + \Lambda_2 \overline{\psi}_2(t), \\
& && . \\
& && . \\
&\Lambda_n = \textstyle\sum_{k=1}^{n} \varrho_{nk} \left(c(t_k) - b(t_k) v_{k-1}^{\beta}(t_k) \right) && \Rightarrow v_n(t) = \textstyle\sum_{i=1}^{n} \Lambda_i \overline{\psi}_i(t).
\end{aligned} \tag{3.2.11}$$

The iterative sequence (3.2.10) guarantees that the approximatation $v_N(t)$ satisfies the suitable choice of the initial condition $v_0(t)$. Furthermore, the approximate solution $v_n^N(t)$ can be provided by taking finitely many terms in the series representation of $v_n(t)$ and

$$v_n^N(t) \quad = \sum_{i=1}^{N} \sum_{k=1}^{i} \varrho_{ik} \left(c(t_k) - b(t_k) v_{k-1}^{\beta}(t_k) \right) \widehat{\psi}_i(t).$$

3.3 Exploratory Computational Studies

In this section, we aim to assess the performance, behavior, and suitability of our proposed method. To achieve this, we explore the solutions of FRDEs and FBDEs within the frame of CF fractional derivatives. Through numerical simulations, the results demonstrate that the RKHS method exhibits exceptional quality and can be readily applied to diverse fractional models using the selected fractional operator.

Example 3.3.1. *Let us consider the following FRDE with variable coefficients:*

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) + 2t^4 v(t) - t^3 v^2(t) = t^5 + 1, \;\; 0 < \alpha \leq 1, \\ v(0) = 0. \end{cases} \tag{3.3.1}$$

The exact solution of Eq. (3.3.1) at $\alpha = 1$, is given as: $v(t) = t$. Indeed, its exact solution at fractional value of α is not available. By using the RKHS method to Eq. (3.3.1), we explain the relationship between the exact and approximate solution by computing the error analysis attached to it at the chosen grid points with fixed step size 0.16*, and $\alpha = 1$. The results are summarized in Table (3.1). Moreover, Table (3.2) shows that the derivative of the appoximate solution converges numerically toward the derivative of the exact solution when $n = 26$ with step size* 0.16 *and $\alpha = 1$. Also, Table (3.3) shows a comparison of the* *numerical results of Eq. (3.3.1) with different values of the CF fractional order* α *when* $n = 26$*. On the other hand, Figure (3.1) illustrates the behavior curve of approximate solutions at different values of CF fractional order* α *for* $n = 26$*.*

In order to demonstrate the advantages and characteristics of the suggested approach utilizing the CF fractional derivative, and to verify the reliability of the acquired data, Table (3.4) presents a numerical assessment comparing the absolute errors of a set of resilient numerical techniques with the absolute error of our proposed method when $n = 26$*.*

Table 3.1: Numerical results for $v_n(t)$ at $\alpha = 1$, Example (3.3.1).

t_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.16	0.16	0.16	0.	0.
0.32	0.32	0.32	1.11022×10^{-16}	3.46944×10^{-16}
0.48	0.48	0.48	1.11022×10^{-16}	2.31296×10^{-16}
0.64	0.64	0.64	2.22044×10^{-16}	3.46944×10^{-16}
0.80	0.80	0.80	3.33066×10^{-16}	4.16333×10^{-16}
0.96	0.96	0.96	4.44089×10^{-16}	4.62592×10^{-16}

Table 3.2: Numerical results for $v_n'(t)$ at $\alpha = 1$, Example (3.3.1).

t_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.16	1.	1.	4.44089×10^{-16}	4.44089×10^{-16}
0.32	1.	1.	4.44089×10^{-16}	4.44089×10^{-16}
0.48	1.	1.	6.66133×10^{-16}	6.66133×10^{-16}
0.64	1.	1.	7.77156×10^{-16}	7.77156×10^{-16}
0.80	1.	1.	9.99200×10^{-16}	9.99200×10^{-16}
0.96	1.	1.	7.77156×10^{-16}	7.77156×10^{-16}

Table 3.3: Comparison of approximate solutions at different values of α for Example (3.3.1).

t_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	α=0.7
0.1	0.1	0.175169	0.264737	0.354483
0.2	0.2	0.266720	0.348061	0.429607
0.3	0.3	0.356707	0.428072	0.499737
0.4	0.4	0.446722	0.508175	0.570061
0.5	0.5	0.536739	0.588294	0.640414
0.6	0.6	0.626752	0.668389	0.710679
0.7	0.7	0.716758	0.748430	0.780756
0.8	0.8	0.806756	0.828404	0.850612
0.9	0.9	0.896755	0.908345	0.920353

Table 3.4: Comparison of absolute error corresponding to Example (3.3.1) from RKHS method and other methods.

t_i	Method in[?]	Method in[?]	Method in[34]	Method in [?]	Present method
0.1	3.20×10^{-5}	1.98×10^{-8}	3.58×10^{-5}	7.45×10^{-7}	0.
0.2	2.90×10^{-4}	1.03×10^{-6}	7.58×10^{-5}	8.51×10^{-7}	2.77×10^{-17}
0.3	1.10×10^{-3}	8.85×10^{-6}	1.20×10^{-4}	9.30×10^{-7}	1.11×10^{-16}
0.4	2.50×10^{-3}	3.33×10^{-5}	1.66×10^{-4}	1.08×10^{-6}	1.11×10^{-16}
0.5	4.40×10^{-3}	7.26×10^{-5}	2.12×10^{-4}	1.14×10^{-6}	1.11×10^{-16}
0.6	5.50×10^{-3}	9.98×10^{-5}	2.52×10^{-4}	1.14×10^{-6}	2.22×10^{-16}
0.7	5.50×10^{-3}	8.84×10^{-5}	2.87×10^{-4}	1.21×10^{-6}	2.22×10^{-16}
0.8	3.80×10^{-3}	1.54×10^{-5}	3.40×10^{-4}	1.04×10^{-6}	3.33×10^{-16}
0.9	3.20×10^{-3}	4.99×10^{-4}	4.90×10^{-4}	1.13×10^{-6}	4.44×10^{-16}
1.	3.40×10^{-3}	3.47×10^{-3}	9.22×10^{-4}	4.84×10^{-7}	4.44×10^{-16}

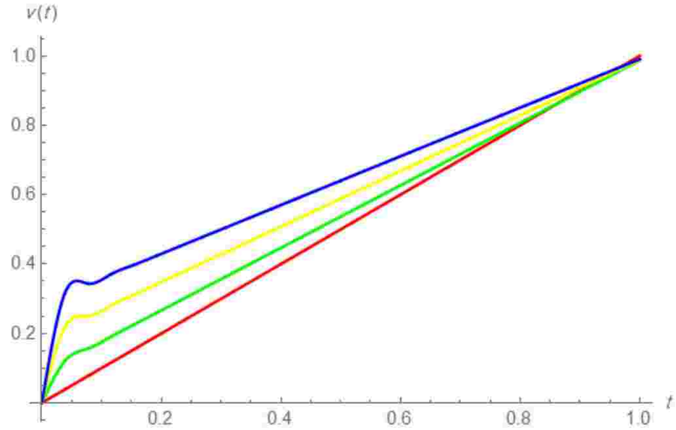


Figure 3.1: Graphs of the solution trajectories for Example (3.3.1) for different values of α when $n = 26$: Red line $\alpha = 1$, Green line $\alpha = 0.9$, Yellow line $\alpha = 0.8$, and blue $\alpha = 0.7$.

Example 3.3.2. *Consider the following FRDE with constant coefficients:*

$$\begin{cases} {}^{CF}D_0^{\alpha}v(t) + v(t) - v^2(t) = 0, \;\; 0 < \alpha \leq 1, \\ v(0) = \frac{1}{2}. \end{cases} \tag{3.3.2}$$

The exact solution of Eq. (3.3.2) at $\alpha = 1$, is given as: $v(t) = \frac{e^{-t}}{e^{-t}+1}$. Indeed, its exact solution at fractional value of α is not available.

By using the RKHS method to Eq. (3.3.2). Table (3.5) shows the errors analysis at the picked grid points with step size 0.16 and $n = 51$ for $\alpha = 1$. Table (3.6) shows a comparison of the numerical results with different values of fractional order α when $n = 26$. Moreover, Figure (3.2) the behavior of the absolute error function of Example

(3.3.2) *for* $n = 26$ *and* $n = 51$. *This illustrates that the accuracy of the approximate solution will be getting best and best as n increases. Figure* (3.3) *illustrates the behavior curve of approximate solutions at different fractional order* α *and* $n = 26$.

In contrast, we conducted a deliberate analysis of the absolute error at selected grid points with a step size of 0.2 *and* $t \in [0, 1]$ *for three different values of* $n = 11, 21$ *and* $n = 51$. *The findings presented in Table* (3.7) *reveal that by adjusting the value of n, we can effectively manage the error value.*

Table 3.5: Numerical results for $v_n(t)$ at $\alpha = 1$, Example (3.3.2).

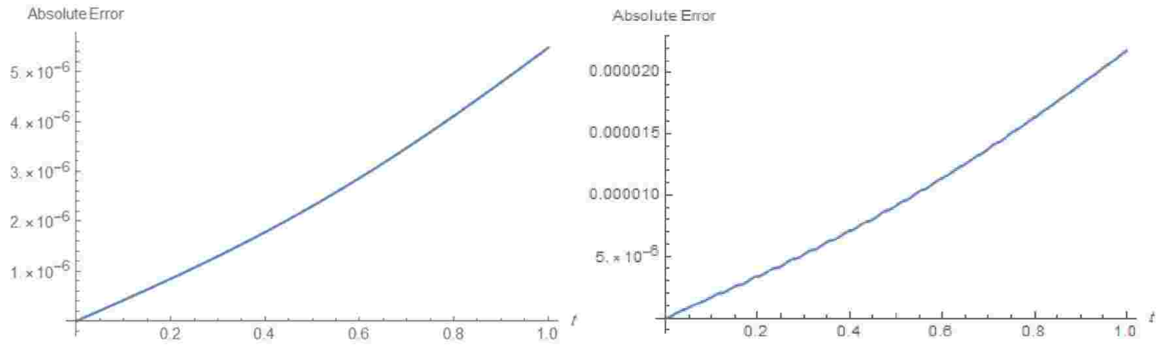
t_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.16	0.460085	0.460086	6.736483×10^{-7}	1.46418×10^{-6}
0.32	0.420676	0.420677	1.392826×10^{-6}	3.31092×10^{-6}
0.48	0.382252	0.382254	2.196113×10^{-6}	5.74519×10^{-6}
0.64	0.345247	0.345250	3.103683×10^{-6}	8.98976×10^{-6}
0.80	0.310026	0.310030	4.113212×10^{-6}	1.32673×10^{-5}
0.96	0.276878	0.276883	5.200572×10^{-6}	1.87829×10^{-5}

Table 3.6: Comparison of approximate solutions at different values of α for Example (3.3.2).

t_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	α=0.7
0.1	0.475022	0.456584	0.434660	0.413727
0.2	0.450169	0.434133	0.414729	0.396272
0.3	0.425563	0.412514	0.396081	0.380669
0.4	0.401319	0.391302	0.377850	0.365411
0.5	0.377550	0.370577	0.360091	0.350555
0.6	0.354355	0.350398	0.342837	0.336115
0.7	0.331826	0.330816	0.326112	0.322099
0.8	0.310042	0.311873	0.309937	0.308515
0.9	0.289070	0.293603	0.294327	0.295365

Table 3.7: Absolute errors of Example (3.3.2) for $\alpha = 1$.

t_i	$n = 11$	$n = 21$	$n = 51$	Method in[**?**]
0.2	2.10×10^{-5}	5.27×10^{-6}	8.47×10^{-7}	7.21×10^{-4}
0.4	4.38×10^{-5}	1.10×10^{-5}	1.78×10^{-6}	1.72×10^{-3}
0.6	7.00×10^{-5}	1.77×10^{-5}	2.86×10^{-6}	2.71×10^{-3}
0.8	1.00×10^{-4}	2.54×10^{-5}	4.11×10^{-6}	3.39×10^{-3}
1.	1.33×10^{-4}	3.39×10^{-5}	5.48×10^{-6}	3.61×10^{-3}



(a) n=51 **(b)** n=26

Figure 3.2: The absolute error function corresponding to Example (3.3.2).

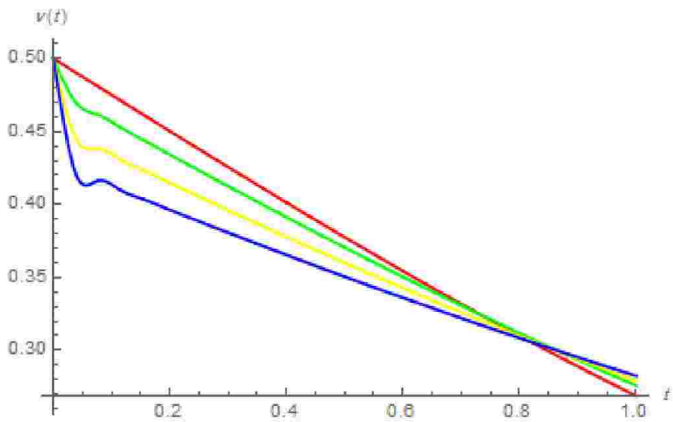


Figure 3.3: Graphs of the approximate solutions for Example (3.3.2) for different values of α when $n = 26$:: Red line $\alpha = 1$, Green line $\alpha = 0.9$, Yellow line $\alpha = 0.8$, and Blue line $\alpha = 0.7$.

Example 3.3.3. *Consider the following FBDE with constant coefficients:*

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) - v(t) - \frac{1}{v^3(t)} = 0, \;\; 0 < \alpha \leq 1, \\ v(0) = 1. \end{cases} \tag{3.3.3}$$

The exact solution of Eq. (3.3.3) *at* $\alpha = 1$, *is given as:* $v(t) = \sqrt[4]{2e^{4t} - 1}$.

The numerical results are summarized as follows: Table (3.3.3) *shows the errors analysis at the picked grid points with step size* 0.1 *and* $n = 51$. *Table Table2B1 shows a comparison of the numerical results with different values of* α *when* $n = 26$. *Moreover. Figure* (3.4) *illustrates the behavior curve of approximate solutions at different fractional order* α *and* $n = 26$.

Table 3.8: Numerical results for $v_n(t)$ at $\alpha = 1$, Example (3.3.3).

t_i	Exact solution	Approximate solution	Absolute error	Relative error
0.1	1.18677	1.18682	5.18228×10^{-5}	4.36671×10^{-5}
0.2	1.36298	1.36306	8.64279×10^{-5}	6.34111×10^{-5}
0.3	1.54108	1.54118	1.06674×10^{-4}	6.92207×10^{-5}
0.4	1.72751	1.72763	1.15328×10^{-4}	6.67593×10^{-5}
0.5	1.92663	1.92674	1.14501×10^{-4}	5.94307×10^{-5}
0.6	2.14188	2.14198	1.05159×10^{-4}	4.90965×10^{-5}
0.7	2.37635	2.37644	8.75047×10^{-5}	3.68231×10^{-5}
0.8	2.63304	2.63310	6.12383×10^{-5}	2.32577×10^{-5}
0.9	2.91494	2.91496	2.56880×10^{-5}	8.81255×10^{-6}
1.	3.22517	3.22515	2.01243×10^{-5}	6.23976×10^{-6}

Table 3.9: Comparison of approximate solutions at different values of α for Example (3.3.3).

t_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	α=0.7
0.1	1.18700	1.31548	1.47338	1.66158
0.2	1.36335	1.48012	1.64028	1.83576
0.3	1.54153	1.64971	1.81277	2.01929
0.4	1.72801	1.83248	2.00261	2.22365
0.5	1.92712	2.03118	2.21200	2.45107
0.6	2.14234	2.24861	2.44340	2.70408
0.7	2.37675	2.48759	2.69951	2.98550
0.8	2.63334	2.75110	2.98332	3.29856
0.9	2.91510	3.04233	3.29815	3.64688

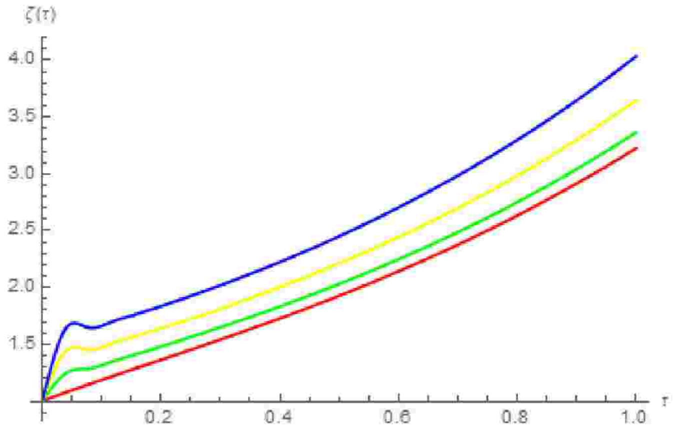


Figure 3.4: Graphs of the RKA solution trajectories for Example (3.3.3) for different values of α when $n = 26$: Red line $\alpha = 1$, Green line $\alpha = 0.9$, Yellow line $\alpha = 0.8$, and Blue line $\alpha = 0.7$.

Example 3.3.4. *Consider the following FBDE with variable coefficients:*

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) + tv(t) + tv^3(t) = 0, \quad 0 < \alpha \leq 1, \\ v(0) = 1. \end{cases} \tag{3.3.4}$$

The exact solution of Eq. (3.3.4) *at* $\alpha = 1$, *is given as:* $v(t) = \frac{1}{\sqrt{2e^{t^2} - 1}}$.

Table 3.10: Numerical results for $v_n(t)$ at $\alpha = 1$, Example (3.3.4).

t_i	Exact solution	Approximate solution	Absolute error	Relative error
0.16	0.975036	0.975027	9.08943×10^{-6}	9.32215×10^{-6}
0.32	0.906975	0.906950	2.45920×10^{-5}	2.71144×10^{-5}
0.48	0.811586	0.811557	2.85564×10^{-5}	3.51859×10^{-5}
0.64	0.704920	0.704902	1.79050×10^{-5}	2.54000×10^{-5}
0.80	0.598367	0.598367	1.05218×10^{-7}	1.75842×10^{-7}
0.96	0.498345	0.498363	1.85737×10^{-5}	3.72708×10^{-5}

Table 3.11: Numerical results for $v_n'(t)$ at $\alpha = 1$, Example (3.3.4).

t_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.16	-0.30432	-0.304530	2.09851×10^{-4}	6.89574×10^{-4}
0.32	-0.528978	-0.529197	2.18929×10^{-4}	4.13872×10^{-4}
0.48	-0.646154	-0.646244	8.99298×10^{-5}	1.39177×10^{-4}
0.64	-0.675329	-0.675301	2.86693×10^{-5}	4.24524×10^{-5}
0.80	-0.650086	-0.650002	8.46167×10^{-5}	1.30162×10^{-4}
0.96	-0.597223	-0.597129	9.41590×10^{-5}	1.57661×10^{-4}

By applying RKHS method to Eq. (3.3.4). *Table* (3.10) *shows the errors analysis at the picked grid points with step size* 0.16 *and* $n = 51$. *Table* (3.11) *shows that the derivative of the approximate solution converges numerically toward the derivative of the exact solution when* $n = 51$. *Table* (3.12) *shows a comparison of the numerical results with different fractional order* α *with step size* 0.1 *and* $n = 26$. *Moreover, Figure* (3.5) *illustrates the behavior curve of approximate solutions at different fractional order* α *and* $n = 26$.

Table 3.12: Comparison of approximate solutions at different values of α for Example (3.3.4).

t_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	α=0.7
0.1	0.990084	0.983976	0.971817	0.959928
0.2	0.961475	0.945912	0.920789	0.897817
0.3	0.917239	0.896804	0.865661	0.838089
0.4	0.861493	0.840808	0.809091	0.781218
0.5	0.798461	0.781173	0.752581	0.727101
0.6	0.731829	0.720322	0.697108	0.675637
0.7	0.664451	0.659964	0.643303	0.626759
0.8	0.598355	0.601253	0.591573	0.580424
0.9	0.534878	0.544945	0.542174	0.536429

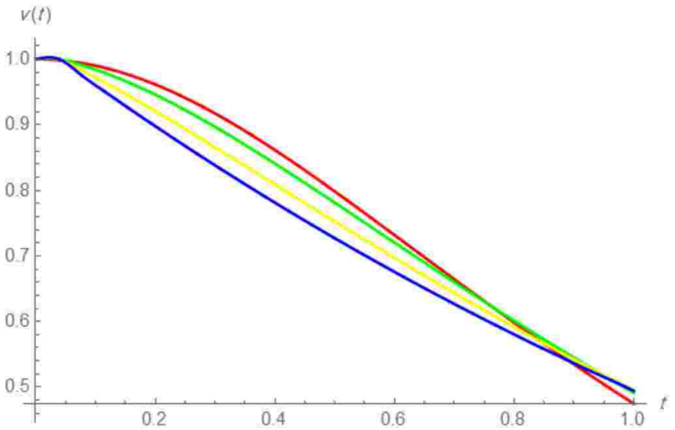


Figure 3.5: Graphs of the solution trajectories for Example (3.3.4) for different values of α when $n = 26$: Red line $\alpha = 1$, Green line $\alpha = 0.9$, Yellow line $\alpha = 0.8$, and Blue line $\alpha = 0.7$.

CHAPTER 4

Conclusions

4.0.1 Conclusions

In this work, we developed a numerical approach using reproducing kernel theory to investigate fractional models in applied sciences and engineering. Our method creates an orthogonal basis from reproducing kernel functions and constructs an orthonormal basis for efficiently convergent approximate series solutions. We utilized the RKHS $W_2^m[a, b]$, a special case of Hilbert spaces, with the reproducing property to adjust model properties and find solutions. Several numerical examples demonstrate the reliability and performance of our method under CF fractional derivative influence, showing the superiority of the RKHS method with minimal time and effort. Our algorithm provides an alternative tool for analyzing behavior of nonlinear fractional differential equations in engineering, physics, and science.

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محاكاة عددية لفئة من المعادلات التفاضلية الجزئية باستخدام طريقة إعادة إنتاج نواة هيلبرت الفضائية

اعداد

بثينة حاجي

المشرف

الدكتور نذير جدي

الملخص

في هذا العمل، نقدم مخططا عدديا جديدا، طريقة استنساخ النواة هيلبرت الفضائية، لتوفير حلول عددية تقريبية لفئة معينة من المعادلات التفاضلية الكسرية في معنى كابتوا فابريزيوا ضمن الجوانب المواتية لاستنساخ نواة هيلبرت الفضاء. تعتمد منهجية الخوارزمية على إنشاء أساس متعامد من خاصية إعادة انتاج النواة لصياغة الحل في شكل سلسلة متقاربة بشكل موحد، وفقا لظروف التقيد في الفضاء$W_2^m[a,b]$. بالإضافة إلى ذلك نقدم تجارب عددية لاختبار فرضيتنا وتأكيد إجراء تصميم الخوارزمية المقترحة. توضح نتائجنا أن طريقة استنساخ النواة هيلبرت الفضائية هي أداة تطوير مهمة للتعامل مع المشكلات الناشئة في علوم الحاسوب والفيزياء والمجلات الهندسية .