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Chaos in Fractional Order Systems

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السلام عليكم ورحمة الله وبركاته

شكر و تقدير

مصداقا لقوله تعالى بعد بسم الله الرحمان الرحيم

﴿ وَإِذْ تَأَذَّنَ رَبُّكُمْ لَئِن شَكَرْتُمْ لَأَزِيدَنَّكُمْ ۖ وَلَئِن كَفَرْتُمْ إِنَّ عَذَابِي لَشَدِيدٌ ﴾.

الحمد لله الذي وهبني عقلا مفكرا، ولسانا ناطقا وأنار دربي، ويسر أمري لإنهاء هذا العمل، والصلاة والسلام على رسول الله صلى الله عليه وسلم.

فاللهم لك الحمد والشكر في الأولى ولك الحمد والشكر في الأخرة ولك الحمد والشكر من قبل ولك الحمد والشكر من بعد وفي كل حين ودائماً وأبداً .

أتقدم بأسمى عبارات الشكر والتقدير إلى كل من علمني حرفا وكل من انار دربي إلى كل من علمني علما به انتفع وادبا به ارتفع .

شكر خاص للأستاذ المشرف " **حناشي فارح** " الذي أفادني بنصائحه وتوجيهاته طيلة إنجاز هذه المذكرة

كما أشكر أعضاء لجنة المناقشة التي شرفتني بقبولها مناقشة مذكرتي، أستاذنا القدير "**زراولية الحاج** "رئيسا والاستاذ " **نذير جدي**" ممتحنا.

اللذين لاشك أنهما سيفيضون لي بتوجيهاتهما القيمة وملاحظاتهما السديدة .

وفي الأخير أشكر كل من قدم لي يد العون والمساعدة سواء من قريب أو من بعيد ولو بكلمة طيبة أو بتوجيه أو حتى بدعوة في ظهر الغيب لهم جزيل الشكر والعرفان

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الى من قال فيهما الله جل و علا : "وقل ربي ارحمهما كما ربياني صغيرا ".

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الى أختاي الغاليتان " **وصال** " و " **أمينة** "

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إلى كل من تصفح هذه المذكرة وانتفع بها وتذكرني بدعائه.

Abstract

The purpose of this work is to study the existence of chaos in some fractional-order systems, by broving that the studied fractional system can display chaotic behavior in two cases. In the case commensurate using the minimal order systems and in the case incommensurate order using the caracteristic polynomial, then the corresponding simulation results are provided to demonstrate the effectiveness of the proposed method in Matlab.

Keywords:

Dynamical system, chaos, chaos theory , chaotic of Fractional-order, commensurate , incommensurate.

الملخص

الهدف من هذا العمل هو دراسة وجود الفوضى في بعض الانظمة الكسرية, من خلال اثبات ان النظام الكسري المدروس يمكنه عرض السلوك الفوضوي في الحالتين الحالة المتكافئة باستخدام الحد الادنى من معايير الترتيب المناسب والحالة الغير متكافئة باستخدام متعددة حدود متميزة ثم التحقق من النتائج باستخدام المحاكاة العددية من خلال برنامج الماتلاب.

الكلمات المفتاحية:

نظام ديناميكي، فوضى، جاذب فوضوي ، نظام فوضوي برتب كسرية.

Contents

1 Preliminaries **6**

1.1 Introduction 6

1.2 Dynamical systems 6

1.2.1 Continuous dynamic systems 6

1.2.2 Discrete dynamic systems 7

1.2.3 Phase space of a dynamical system 7

1.2.4 Equilibrium point 7

1.3 Chaos theory 8

1.3.1 Definition of chaotic systems 8

1.3.2 Some characteristics of chaotic systems 8

1.3.3 Chaos theory applications 9

1.4 Fractional calculus 10

1.4.1 Useful Mathematical Functions 10

1.4.2 Grünwald-Letnikov derivative 10

1.4.3 The Riemann-Liouville derivative 11

1.4.4 Caputo fractional derivative 12

1.4.5 Relation between Riemann-Liouville and Caputo fractional derivatives . 12

1.4.6 Stability of Fractional Order Systems 12

1.4.7 Numerical method for solving fractional order systems 13

1.5 Conclusion 14

2 Examples of fractional-order chaotic systems **15**

2.1 Introduction 15

2.2 Fractional-order chaotic systems 15

2.2.1 Fractional-order Genesio–Tesi system 15

2.2.2 The fractional-order simplified Lorenz system 17

2.2.3 The Rabinovich–Fabrikant chaotic system 18

2.2.4 3D Fractional-Order Chaotic System 19

2.2.5 Fractional-Order Rössler System 20

2.3 Conclusion 22

3 Existence of Chaos in a fractional order System **24**

3.1 Introduction 24

3.2 Description of the chaotic system 24

3.2.1 Chaotic Dynamics of Fractional chaotic system 25

3.2.2 Equilibrium points and stability 26

3.2.3 Minimal order for chaos 27

3.2.4 Chaos by using Lyapunov exponents test 29

3.3 Conclusion 31

List of Figures

2.1 Chaotic attractor of system (2.1) in $x - z$ plane 16
2.2 Chaotic attractor of system (2.1) in $y - z$ plane. 16
2.3 Chaotic attractor of system (2.1) in $x - y - z$ space. 17
2.4 Chaotic attractor of system (2.1) in $x - y$ plane. 17
2.5 Chaotic attractor of system (2.2) in $x - z$ plane. 18
2.6 Chaotic attractor of system (2.2) in $y - z$ plane. 18
2.7 Chaotic attractor of system (2.2) in $x - y - z$ space. 19
2.8 Chaotic attractor of system (2.3) in $x - y$ plane. 20
2.9 Chaotic attractor of system (2.3) in $x - z$ plan. 20
2.10 Chaotic attractor of system (2.3) in $x - y - z$ space. 21
2.11 Chaotic attractor of system (2.4) in $x - y$ plan. 21
2.12 Chaotic attractor of system (2.4) in $x - z$ plane. 22
2.13 Chaotic attractor of system (2.4) in $y - z$ plane. 22
2.14 Chaotic attractor of system (2.4) in $x - y - z$ space. 23
2.15 Chaotic attractor of system (2.5) in $y - z$ plane. 23
2.16 Chaotic attractor of system (2.5) in $x - y - z$ space. 23

3.1 Lyapunov exponents spectrum of system (4.1) for $\alpha = 0.95$ 30
3.2 Lyapunov exponents spectrum of system (4.1) for $\alpha_1 = 0.95, \alpha_2 = \alpha_1 = 0.98.$ . . . 30

General Introduction

Dynamical systems are part of life, Quite often it has been studied as an abstract concept in mathematics, chaos is one of the few concepts in mathematics which cannot usually be defined in a word or statement[12]. Most dynamical systems are considered chaotic depending on either the topological or metric properties of the system [15]. Chaotic systems have been a focal point of renewed interest for researchers in recent decades and such nonlinear systems can occur in various natural and man-made systems [16] .

The study of chaos in dynamical systems has revolutionized our understanding of complex and unpredictable behavior in various scientific disciplines [13]. In the late 20th century, the subject of research known as "chaos theory" began to take shape. Since then, it has had a significant influence on several fields [22], including mathematics, physics, biology, and many more [23]. Henri Poincaré, a French mathematician, made substantial contributions to the discipline in the early 19^{th} century [11], which can be seen as the beginning of the history of chaos in dynamical systems [14]. There are solutions that are quite sensitive to the beginning circumstances, as Poincaré's work on the three-body problem in celestial mechanics showed, the Butterfly Effect is a concept derived from chaos theory, in which this term refers to the concept that a tiny change in one location and time can cause significant, unforeseen effects in another location and time [21]. The idea of sensitive dependency on beginning conditions serves as the foundation for this, where small changes in the starting conditions of a system can lead to vastly different outcomes over time [18].

Indeed, the idea of chaos extends beyond integer-order systems to fractional-order systems, and fractional calculus offers a mathematical foundation for analyzing and simulating such systems since it works with derivatives and integrals of non-integer order [19]. Systems with fractional order have intricate dynamics, which may involve chaotic behavior [17]. The study of chaos in fractional-order systems is an active research area [20], and it has implications for understanding and modelling complex phenomena with memory effects and long-range interactions. Fractional calculus provides a powerful tool for analyzing and predicting the behavior of such systems, allowing for a more comprehensive understanding of their dynamics and potential applications in various fields.

In the first chapter, we recall some basic notions of dynamical systems and the theory of chaos, also basic definitions and properties of fractional derivatives are provided with numerical methods for solving fractional-order systems.

In the second chapter, we present some examples of fractional-order chaotic systems.

Finally, the last chapter is devoted to the study of the existence of chaos in a novel fractional-order system, in the first part, we describe the fractional-order system and we study the equilibrium points and the stability, and in the last part, we provide evidence that the system exhibits chaotic behavior once it reaches a certain threshold of minimum commensurate order.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we introduce some preliminaries about dynamical systems and chaos theory, also basic definitions and properties of fractional derivative are given with numerical method for solving fractional-order systems, and study its stability.

1.2 Dynamical systems

Dynamical systems refer to systems that change over time. Systems like this may be found in a number of disciplines, including physics, engineering, biology, economics, and social sciences. Dynamic systems can be either linear or nonlinear, its classified into two categories:

1.2.1 Continuous dynamic systems

Definition 1.1 *A continuous dynamic system is a system where its state changes continuously over time, and it is represented by the form:*

$$\dot{x_t} = F(x,t); \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+,$$

with $F : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ denotes the dynamics of the system.

1.2.2 Discrete dynamic systems

Definition 1.2 *A discrete-time dynamic system is a mathematical model that describes how a system evolves over time, where time is treated as a sequence of discrete points in time* [1] *, and it is represented by a fine difference equation as follows:*

$$x(k+1) = F(x(k), k), \tag{1.1}$$

with $x(k) \in \mathbb{R}^n$, $k \in \mathbb{N}$ *and* $F : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$.

1.2.3 Phase space of a dynamical system

Definition 1.3 *The phase space of a dynamical system is typically represented as a multidimensional space, where each dimension is a direct representation of a phase variable. A point in this space represents the system's state at any given moment, and a trajectory in the phase space depicts the system's mobility through time.*

1.2.4 Equilibrium point

Definition 1.4 *In mathematics, specifically in differential equations, an equilibrium point is a constant solution to a differential equation.*

The point $\overset{*}{x} \in \mathbb{R}^n$ *is an equilibrium point* [2] *for the differential equation:*

$$\tfrac{dx}{dt} = f(t, x) \quad \text{if } f\left(t, \overset{*}{x}\right) = 0 \quad \text{for all } t.$$

Similarly, the point $\overset{*}{x} \in \mathbb{R}^n$ *is an equilibrium point (or fixed point) for the difference equation.*

$$x_{k+1} = f(k, x_k),$$

if

$$f\left(k, \overset{*}{x}_k\right) = \overset{*}{x} \quad \text{for} \quad k = 0, 1, 2...$$

In the study of dynamic systems, equilibrium points are crucial because they provide details about the behavior and stability of the system. They may also be used to create control systems to manage the behavior of the system and examine how the system behaves close to its equilibrium point.

1.3 Chaos theory

1.3.1 Definition of chaotic systems

In dynamic systems, "**chaos**" refers to a complex behavior that appears to be random or unpredictable. Because chaotic systems are sensitive to the beginning conditions, tiny changes in the early circumstances can have a significant impact on the course of events. This makes it exceedingly difficult, if not impossible, to anticipate the long-term behavior of chaotic systems.

Definition 1.5 *Let V be a set. $f : V \to V$ is said to be chaotic on V if f has the following three properties:*

1· *f has sensitive dependence on initial conditions.*

2 · *f is topologically transitive.*

3· *The periodic points of f are dense in V.*

1.3.2 Some characteristics of chaotic systems

Chaotic systems have several properties that distinguish them from other types of dynamical systems:

- **Sensitive to initial conditions:** This means that small changes in the initial conditions of the system can lead to large differences in the behavior of the system over time.

- **topologically transitive:** In mathematics, If a point in the phase space has an orbit that is dense in the phase space, the dynamical system is said to be topologically transitive. This means that any point in the phase space is arbitrarily near to the system's trajectory, which it follows.

- **dense periodic orbits:** Since they offer a means of approximating the behavior of chaotic systems, dense periodic orbits are significant in the study of dynamical systems.

- **Lyapunov exponent:** The Lyapunov exponent is a way to gauge how quickly close paths in a dynamical system diverge. It is named after the Russian mathematician Alexander

Lyapunov, who created the concept in the late 19th century. The Lyapunov exponent is a measurement of the stability of a dynamical system. On the other hand, if the Lyapunov exponent is positive, the system is unstable and behaves chaotically because neighboring paths tend to diverge over time. Nearby paths in the system tend to converge over time if the Lyapunov exponent is negative, demonstrating that the system is stable. It is frequently used to investigate the behavior of chaotic systems, in which neighboring paths are subject to sudden and unpredictable divergence.

1.3.3 Chaos theory applications

Chaos theory has many applications in various fields. Here are some examples:

1. **Physics:** Chaos theory has been applied in physics to comprehend the behavior of complex systems like celestial mechanics, nonlinear optics, and fluid dynamics.

2. **Engineering:** Chaos theory has been applied in engineering to enhance the planning and management of intricate systems including power plants, chemical reactors, and communication networks.

3. **Biology:** Chaos theory has been applied in biology to comprehend the functioning of biological systems including ecological systems, brain networks, and heart cycles.

4. **Finance:** Chaos theory has been applied in finance to understand the behavior of financial markets and to develop models that can predict market fluctuations.

5. **Computer Science:** Chaos theory has been applied in computer science to develop algorithms for optimization and data analysis.

6. **Music and Art:** Chaos theory has been applied in music and art to create new forms of expression and to explore the relationship between randomness and creativity.

Overall, chaos theory has developed into a potent tool for comprehending the behavior of complex systems in a wide range of disciplines, and it continues to stimulate new research and applications in science, engineering, and the arts.

1.4 Fractional calculus

Over the years, many mathematicians using their own notation and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann Liouville definition.

1.4.1 Useful Mathematical Functions

We first explore several essential mathematical notions that are intrinsically linked to fractional calculus and will frequently be encountered before looking at the formulation of the Riemann-Liouville Fractional and caputo derivatives. The beta function and the gamma function are examples of these.

The Gamma Function

Definition 1.6 *The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers* [3]. *Its definition is given by*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in \mathbb{R}^+. \tag{1.2}$$

The Beta Function

Definition 1.7 *Like the Gamma function, the Beta function is defined by a definite integral* [3]. *Its definition is given by*

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in \mathbb{R}^+. \tag{1.3}$$

The Beta function can also be defined in terms of the Gamma function:

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y \in \mathbb{R}^+. \tag{1.4}$$

1.4.2 Grünwald-Letnikov derivative

The Grünwald-Letnikov derivative [9] is a method used to approximate the derivative of a function. It is a numerical approach that is particularly useful for functions that are not easily differentiable or for situations where analytical differentiation is not feasible.

Let us consider the continuous function $f(t)$. Its first derivative can be expressed as

$$\frac{d}{dt}f(t) \equiv f'(t) = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h} \tag{1.5}$$

By using $Eq.(1.5)$ twice, we obtain a second derivative of the function $f(t)$ in the form

$$\begin{aligned}\frac{d^2}{dt^2}f(t) &\equiv f''(t) = \lim_{h\to 0}\frac{f'(t)-f'(t-h)}{h} \\ &= \lim_{h\to 0}\frac{f(t)-2f(t-h)+f(t-2h)}{h^2}\end{aligned} \tag{1.6}$$

With (1.5) and (1.6) we can get a third derivative of the function $f(t)$ as

$$\begin{aligned}\frac{d^3}{dt^3}f(t) &\equiv f'''(t) = \lim_{h\to 0}\frac{f''(t)-f''(t-h)}{h} \\ &= \lim_{h\to 0}\frac{f(t)-3f(t-h)+3f(t-2h)-f(t-3h)}{h^3}\end{aligned}$$

The Grünwald-Letnikov derivative provides an alternative way to approximate derivatives, especially for functions that do not have a simple algebraic expression for their derivatives or for problems where numerical methods are more suitable. However, it's important to note that the convergence of the method depends on the properties of the function being differentiated and the choice of the time step Δt.

1.4.3 The Riemann-Liouville derivative

Definition 1.8 *The Riemann-Liouville derivative of fractional order α of function $x(t)$, [3]is given as*

$$\begin{aligned}{}^{RL}D_{0,t}^{\alpha}\, v(t) &= \frac{d^m}{dt^m}D_{0,t}^{-(m-\alpha)}\, v(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{\alpha-1}\, v(s)\, ds\end{aligned} \tag{1.7}$$

where $m-1 \leqslant \alpha < m \in \mathbb{Z}^+$.

This derivative was induced by the Riemann-Liouville derivative and is useful inphysics.

1.4.4 Caputo fractional derivative

Definition 1.9 *The Caputo fractional derivative of $v(t)$ is given as:*

$$ {}^{C}D_{x}^{\alpha}v\left(t\right)=\frac{1}{\Gamma\left(m-\alpha\right)}\int_{a}^{t}\frac{f^{(m)}\left(s\right)}{\left(t-m\right)^{\alpha-m+1}}ds, \tag{1.8}$$

where $m-1\leqslant\alpha<m\in\mathbb{Z}^{+}$.

1.4.5 Relation between Riemann-Liouville and Caputo fractional derivatives

The relation between Riemann-Liouville and Caputo fractional derivatives with singularekernels given as:

$$ {}^{C}D_{x}^{\alpha}v\left(t\right)={}^{RL}D_{0,t}^{\alpha}\,v\left(t\right)-\sum_{k=0}^{m-1}\frac{v^{(k)}\left(\alpha\right)}{\Gamma\left(k-\alpha+1\right)}\left(t-a\right)^{k-a} \tag{1.9}$$

There fore,

$$\text{If } v\left(a\right)=v'\left(a\right)=\cdots=v^{(n-1)}\left(a\right)=0,\ \text{then } {}^{C}D_{x}^{\alpha}v\left(t\right)={}^{RL}D_{0,t}^{\alpha}\,v\left(t\right). \tag{1.10}$$

1.4.6 Stability of Fractional Order Systems

Stability analysis of fractional order systems, which is of main interest in control theory $[4]$. We take into consideration the fractional order system in n dimensions below.

$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}}=h_1\left(x_1,x_2,...,x_n\right), \\ \frac{d^{q_2}x_2}{dt^{q_2}}=h_2\left(x_1,x_2,...,x_n\right), \\ \quad\vdots \\ \frac{d^{q_n}x_n}{dt^{q_n}}=h_n\left(x_1,x_2,...,x_n\right), \end{cases} \tag{1.11}$$

Where q_i are equal to real number or rational num bers between 0 and 1 and $\frac{d^{q_i}}{dt^{q_i}}$ is the Caputo frac tional derivative of order q_i,for $i=1,2,...,n$. If function f_i has second continuous partial deriva tives in a ball centered at an equilibrium point $P^{*}=(x_1^{*},x_2^{*},...,x_n^{*})$, that is $f_i\left(x_1^{*},x_2^{*},...,x_n^{*}\right)=0$ for $i=1,2,...,n$, then we have the following results.

- **Case commensurate** if $q_1 = q_2 = \cdots = q_n = q$ then the equilibrium point x* of system (1.11) is asymptotically stable if $|\arg(\text{spec}(J|_{x^*}))| > q\pi/2$, where the matrix J is the Jacobian matrix of the system (1.11) that is defined as $J = \left[\frac{\partial f_i}{\partial x_i}\right]_{i,j=1}^{n}$.

- **Case Incommensurate** A fractional-order system's stability is typically influenced by where its poles and zeros are situated on the complex plane. If all the poles lie in the left half of the complex plane, the system is said to be asymptotically stable. If some poles lie on the imaginary axis, the system may exhibit oscillations. If any pole lies in the right half of the complex plane, the system is unstable. If q_i are rational numbers between 0 and 1 such that $\alpha_i = l_i/m_i$, $(l_i, m_i) = 1$, l_i, $m_i \in \mathbb{N}$ for $i = 1, 2, \cdots, n$, then the equilibrium point X^* of system (1.11) is asymptotically stable if all roots λ of the equation $\det(\text{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \cdots, \lambda^{m\alpha_n}) - J|_{x^*}) = 0$. satisfy $|\arg(\lambda)| > q\pi/2$, where $q = 1/m$ and m be the least common multiple of the denominators m_i of α_i.

1.4.7 Numerical method for solving fractional order systems

Numerical methods for solving fractional-order dynamic systems have become increasingly important in recent years due to their wide range of applications in physics, engineering, finance, and other fields. Among these methods we introduce Adams-Bashforth-Moulton algorithm.

Adams-Bashforth-Moulton algorithm

Consider for $\alpha \in (m-1, m]$ the following initial value problem (IVP)

$$D^{\alpha} y(t) = f(t, y(t)), \qquad 0 \le t \le T, \tag{1.12}$$

$$y^{(k)}(0) = y_0^{(k)} \qquad k = 0, 1, ..., m-1. \tag{1.13}$$

The IVP $(1.12)$and (1.13) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau. \tag{1.14}$$

Consider the uniform grid $\{t_n = nh/n = 0, 1, ..., N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j), j = 1, 2, ..., n$ and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^p(t_{n+1})\right) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_n(t_j)\right), \quad (1.15)$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if} \quad j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if} \quad 1 \le j \le n, \\ 1, & \text{if} \quad j = n+1 \end{cases} . \quad (1.16)$$

The preliminary approximation $y_h^p(t_{n+1})$ is called predictor and is given by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, y_n(t_j)\right)$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right). \quad (1.17)$$

The error in this method is given by

$$\max_{j=0,1,...N} |y(t_j) - y_h(t_j)| = O(h^p), \quad (1.18)$$

where $p = \min(2, 1+\alpha)$.

It involves predicting the solution at the next time step using the Adams-Bashforth method and then correcting the solution, both of which are modified to handle fractional derivatives. The method is accurate and efficient, but it can be computationally expensive for high order dynamical systems.

1.5 Conclusion

This Chapter contain some prelimanaries about dynamical systems and chaos theory, Also,Basic definitions and properties of fractional derivative are given with numerical method for solving fractional differential equations.

Chapter 2

Examples of fractional-order chaotic systems

2.1 Introduction

Chaos in fractional order systems refers to the study of complex dynamical behavior in systems involving fractional derivatives or integrals. Unlike traditional integer-order systems, fractional order systems exhibit unique characteristics such as sensitivity to initial conditions, aperiodic long-term behavior, and the presence of strange attractors in phase space. This area of research provides insights into the intricate dynamics of physical phenomena.

2.2 Fractional-order chaotic systems

2.2.1 Fractional-order Genesio–Tesi system

The Genesio-Tesi system is a $3D$ dynamical system that was introduced by Raffaele Genesio and Alberto Tesi in 1985 $[5]$. This system has been studied extensively in the literature and has found applications in various fields, such as secure communication, image encryption, and chaos synchronization. The fractional-order Genesio-Tesi system can be used as a benchmark system for testing new fractional-order chaos detection and control algorithms, also this system can be used as a platform for investigating the effect of fractional-order derivatives on the dynamics of nonlinear systems". The fractional form of the Genesio-Tesi systemis is described as follows

$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = x_3, \\ D^q x_3 = -ax_1 - bx_2 - cx_3 + mx_1^2, \end{cases} \tag{2.1}$$

where x_1, x_2, x_3 are state variables, q is the fractional-order satisfying $0 < q \leq 1$, $q = 0.97$ and for the parameters $a = 6$, $b = 2.92$, $c = 1.2$, and $m = 1$, the system can display choatic attractor, and numerical simlulations of Genesio-Tesi system is depicted in Figure 2.1, 2.2, 2.3, and 2.4.

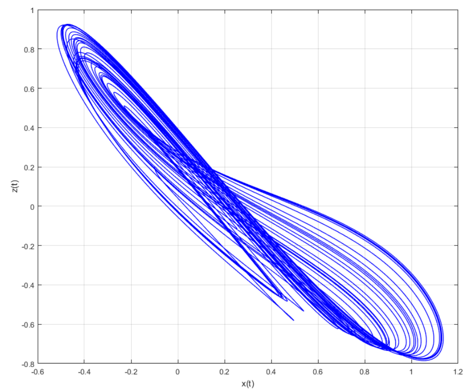

Figure 2.1: Chaotic attractor of system (2.1) in $x - z$ plane

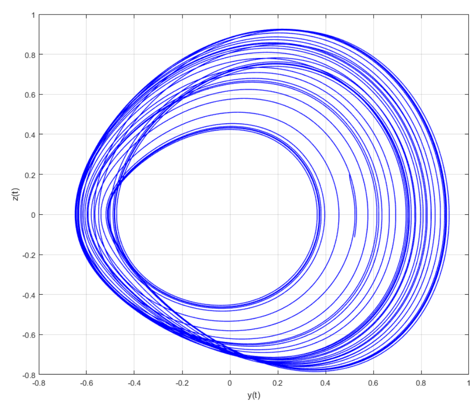

Figure 2.2: Chaotic attractor of system (2.1) in $y - z$ plane.

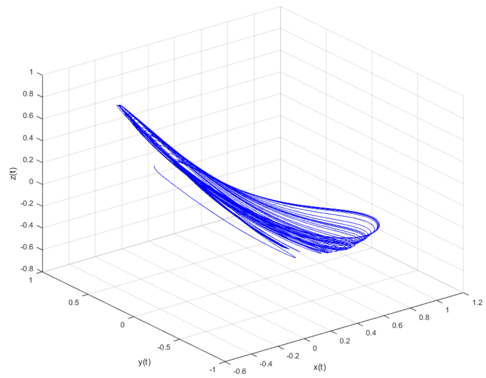

Figure 2.3: Chaotic attractor of system (2.1) in $x - y - z$ space.

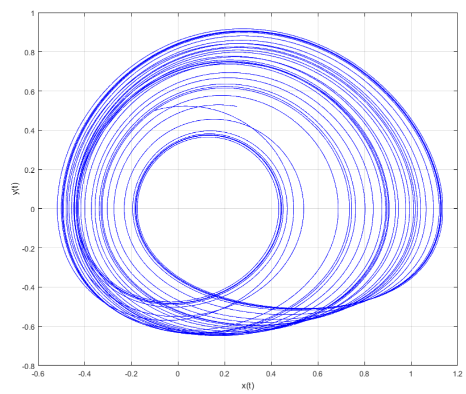

Figure 2.4: Chaotic attractor of system (2.1) in $x - y$ plane.

2.2.2 The fractional-order simplified Lorenz system

The fractional order Lorenz system[6] is a generalization of the classical Lorenz system. The system has been studied extensively in the literature and has found applications in various fields, such as chaos-based cryptography, secure communication, and image encryption and it is given by the following from:

$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}} = 10\left(y - x\right), \\ \frac{d^{q_2}x_2}{dt^{q_2}} = -xz + \left(24 - 4c\right)x + cy, \\ \frac{d^{q_3}x_3}{dt^{q_3}} = xy - \frac{8}{3}z, \end{cases} \tag{2.2}$$

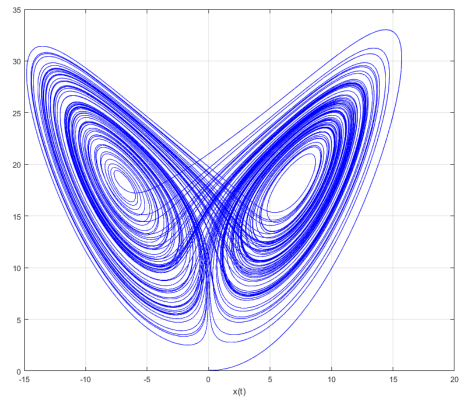

Figure 2.5: Chaotic attractor of system (2.2) in $x - z$ plane.

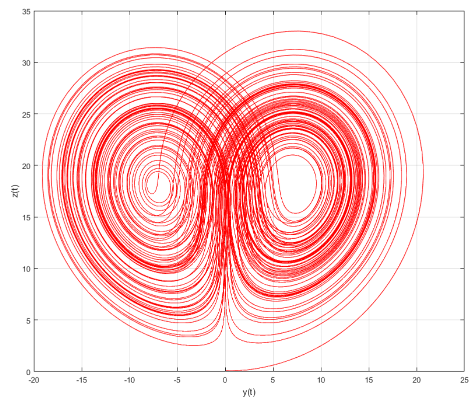

Figure 2.6: Chaotic attractor of system (2.2) in $y - z$ plane.

where x, y, z are the state variables, and $0 < q_i \leq 1, i = \overline{1,3}$ determine the fractional order of the system. For $q_1 = q_2 = q_3 = 1$ and for $c \in [2.6, 7.4]$, the system can display choatic attractor, and numerical simlulations of the Lorenz system is depicted in Figure 2.5, 2.6 and 2.7

2.2.3 The Rabinovich–Fabrikant chaotic system

The Rabinovich-Fabrikant chaotic system [7] is a $3D$ dynamical system that was introduced by Michael M. Sushchik and Leonid Fabrikant in 1979. This system has found applications in various fields, such as chaos-based cryptography, secure communications, and nonlinear control. It is also used as a benchmark system for testing new chaos detection and control

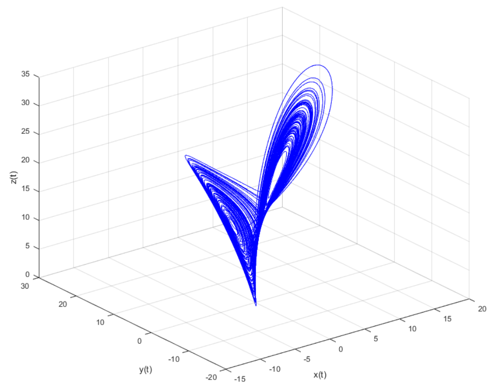

Figure 2.7: Chaotic attractor of system (2.2) in $x - y - z$ space.

algorithms.

This system is described by the following set of differential equations

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y\left(z - 1 + x^2\right) + ax, \\ \frac{d^\alpha y}{dt^\alpha} = x\left(3z + 1 - x^2\right) + ay, \\ \frac{d^\alpha z}{dt^\alpha} = -2z\left(b + xy\right), 0 < \alpha < 1 \end{cases} \tag{2.3}$$

where x, y, z are the state variables, $0 < \alpha \leq 1$ is the fractional-order derivative, and for the parameters $a = 0.87$, $b = 1.1$, and for $\alpha = 0.99$, the system can display choatic attractor, and numerical simlulations of Rabinovich-Fabrikant fractional-order system for the initial conditions $[-1, 0, 0.5]$ is depicted in Figure 2.8, 2.9 and 2.10.

2.2.4 3D Fractional-Order Chaotic System

The fractional-order $3D$ chaotic system $[8]$ is constructed, which is described as follows:

$$\begin{cases} D^q x = y, \\ D^q y = -x - yz, \\ D^q z = a\left|x\right| + xy - b, \end{cases} \tag{2.4}$$

where x, y, z are the state variables, $0 < q \leq 1$ is the fractional-order satisfying, and for the parameters $a = 2.5, b = 1.35$, and $q = 0.9$, the system can display choatic attractor and numerical

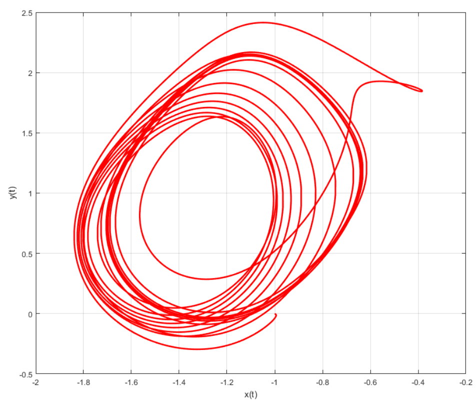

Figure 2.8: Chaotic attractor of system (2.3) in $x - y$ plane.

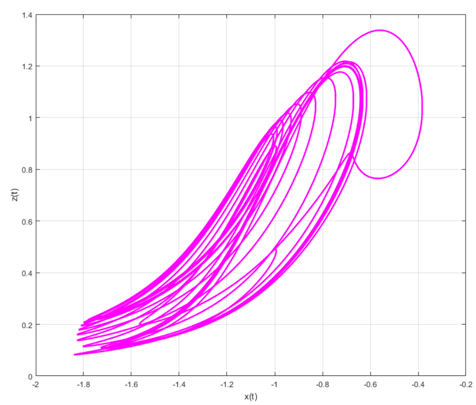

Figure 2.9: Chaotic attractor of system (2.3) in $x - z$ plan.

simlulations of the fractional-order $3D$ chaotic system is depicted in Figure $2.11,\ 2.12,\ 2.13$ and 2.14.

2.2.5 Fractional-Order Rössler System

The fractional order Rössler system is a generalization of the well-known Rössler system, which is a system of ordinary differential equations (ODEs) that exhibits chaotic behavior. The fractional order Rössler system extends the concept by introducing fractional derivatives instead of ordinary derivatives.

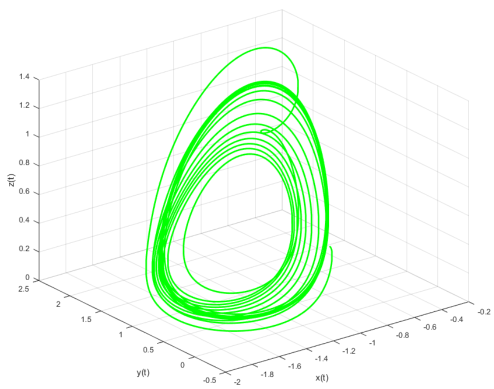

Figure 2.10: Chaotic attractor of system (2.3) in $x - y - z$ space.

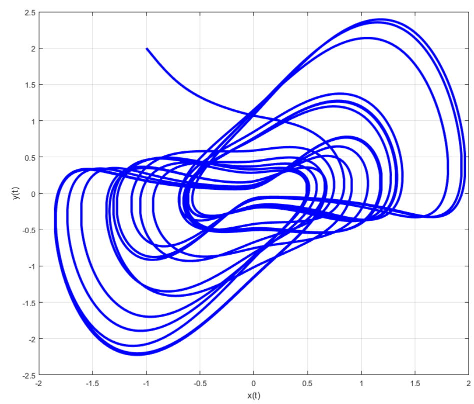

Figure 2.11: Chaotic attractor of system (2.4) in $x - y$ plan.

The fractional order Rössler system [9] is given by the following nonlinear equations:

$$\begin{cases} D^{q_1}x = -y - z, \\ D^{q_2}y = x + ay, \\ D^{q_3}z = bx - cz + xz. \end{cases} \tag{2.5}$$

where x, y, z are the state variables, a, b and c are parameters, and q_i, $i = \overline{1, 3}$ are the fractional-order derativative. For $q_1 = 0.9$, $q_2 = 0.85, q_3 = 0, 95$, and for the parameters $(a; b; c) = (0.5, 0.2, 10)$ and ICs $(x_0, y_0, z_0) = (0.5, 1.5, 0.1)$, the system can display choatic attractor, and numerical simlulations of the Rössler system is depicted in Figure 2.15 and 2.16.

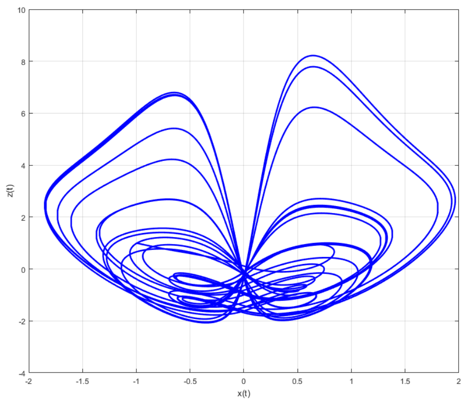

Figure 2.12: Chaotic attractor of system (2.4) in $x - z$ plane.

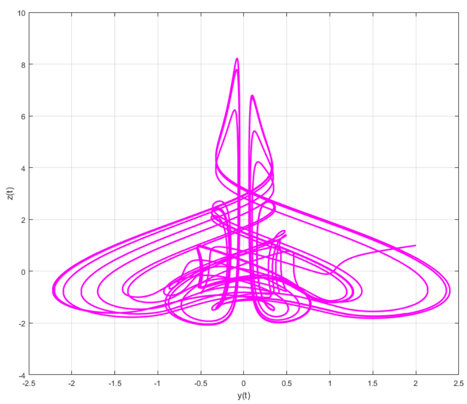

Figure 2.13: Chaotic attractor of system (2.4) in $y - z$ plane.

2.3 Conclusion

In this chapter, we have presented some examples of fractional-order $3D$ chaotic systems. There are many other examples of such systems, each with its own unique behavior and characteristics.

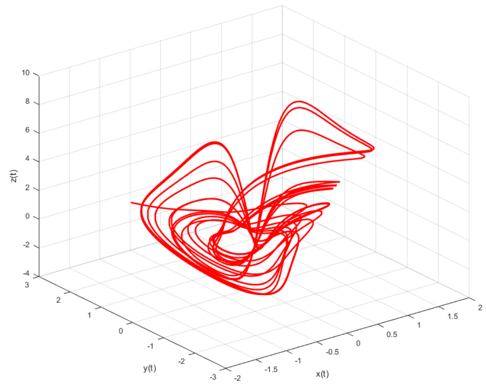

Figure 2.14: Chaotic attractor of system (2.4) in $x - y - z$ space.

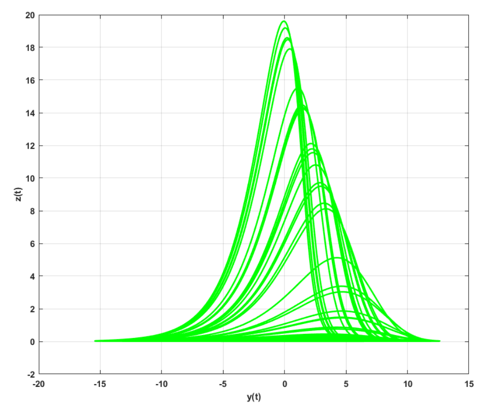

Figure 2.15: Chaotic attractor of system (2.5) in $y - z$ plane.

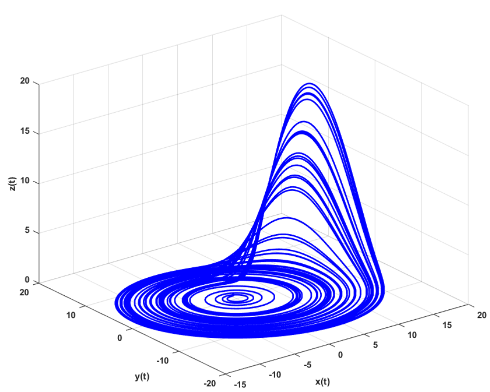

Figure 2.16: Chaotic attractor of system (2.5) in $x - y - z$ space.

Chapter 3

Existence of Chaos in a fractional order System

3.1 Introduction

In this chapter, we study the existence of chaos in a fractional order system

3.2 Description of the chaotic system

The chaotic system [10] is described by the independent nonlinear system of differential equations that follows:

$$\begin{cases} \dot{x} = ay - x, \\ \dot{y} = -bx - z, \\ \dot{z} = cz + xy^2 - x, \end{cases} \tag{3.1}$$

where $x,\ y$ and z are the states and $a,\ b,\ c$ are constant, positive, parameters of the system. The new system (3.1) has totally seven terms on the right-hand side with a cubic nonlinearity. The parameters' typical values are:

$$a = 1,\ b = 0.46,\ c = 0.46. \tag{3.2}$$

3.2.1 Chaotic Dynamics of Fractional chaotic system

In this section, we study the chaotic dynamics of fractional novel chaotic system It is obtained from the classical system, described in (3.1), by replacing the first time derivative $\frac{d}{dt}$ by a fractional derivative $\frac{d^\alpha}{dt^\alpha}$, where the last denotes the differential operator in the sense of Caputo. The fractional version of novel chaotic system reads as

$$\begin{cases} \frac{d^{\alpha_1}x}{dt^{\alpha_1}} = ay - x, \\ \frac{d^{\alpha_2}y}{dt^{\alpha_2}} = -bx - z, \\ \frac{d^{\alpha_3}z}{dt^{\alpha_3}} = cz + xy^2 - x, \end{cases} \tag{3.3}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is subject to $0 < \alpha_1, \alpha_2, \alpha_3 \leqslant 1$.

The state space of the system (3.3) is three-dimensional, The right-hand side of the system (3.3) vector field is defined by

$$v(x, y, z) = \begin{bmatrix} v_1(x) \\ v_2(y) \\ v_3(z) \end{bmatrix} = \begin{bmatrix} y - x \\ -0.46x - z \\ 0.46z + xy^2 - x \end{bmatrix} \tag{3.4}$$

The divergence of the vector field v is easily calculated as

$$\operatorname{div} v(x) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = -1 + 0 + 0.46 = -0.54 < 0. \tag{3.5}$$

A necessary and sufficient condition for system (3.3) to be dissipative is that the divergence of the vector field v is negative. In view of Eq (3.5), it is immediate that system (3.3) is dissipative if and only if $c > 1$ with an exponential rate $\frac{dv}{dt} = e^{-0.54}$.

Thus, in the dynamical system (3.3), a volume element V_0 is apparently contracted by the flow into a volume element $V_0 e^{-0.54t}$ in time t. This means that each volume containing the trajectories of this dynamical system shrinks to zero as $t \to \infty$ at an exponential rate. So, all the orbits of the dynamical system (3.3) will be eventually confined to a special subset that has zero volume, and the asymptotic motion of system (3.3) will settle onto an attractor of the system.

3.2.2 Equilibrium points and stability

For the values of parameters (3.2), the system (3.3) has three equilibrium points, given by

$$\begin{aligned} E_1 &: (0,0,0), \\ E_2 &: (1.100727032,\ 1.100727032,\ -0.5063344349), \\ E_3 &: (-1.100727032,\ -1.100727032,\ 0.5063344349). \end{aligned}$$

Clearly, E_1 is an equilibrium of the system (3.3) for all values of the parameters a, b, and c. The equilibrium points $E_2,\ E_3$ of system (3.3) are real only when $cb \geq 1$. When $cb < 1$, E_1 is the only real equilibrium of (3.3).

The Jacobian matrix of the system (3.3) evaluated at the equilibrium point $E^* = (x^*, y^*, z^*)$ is:

$$J(E^*) = \begin{pmatrix} -1 & 1 & 0 \\ -0.46 & 0 & -1 \\ -1 + y^2 & 2xy & 0.46 \end{pmatrix}$$

For E_1:

$$J(E_1) = \begin{pmatrix} -1-\lambda & 1 & 0 \\ -0.46 & -\lambda & -1 \\ -1 & 0 & 0.46-\lambda \end{pmatrix}$$

Determinant:

$$P_\lambda(E_1) = -\lambda^3 - 0.54\lambda^2 + 1.2116.$$

So, we obtain the eigenvalues

$$\begin{aligned} \lambda_1 &= 0.9131216591, \\ \lambda_2 &= -0.7265608295 + 0.8938602918i, \\ \lambda_3 &= -0.7265608295 - 0.8938602918i. \end{aligned} \tag{3.6}$$

For E_2:

With the same method, the eigenvalues of the Jacobian at E_2 are

$$J(E_2) = \begin{pmatrix} -1-\lambda & 1 & 0 \\ -0.46 & -\lambda & -1 \\ 0.211599999 & 2.423199998 & 0.46-\lambda \end{pmatrix}$$

Determinant:

$$P_{\lambda}(E_2) = -\lambda^3 - 0.54\lambda^2 - 2.4232\lambda - 2.4232.$$

which has the eigenvalues

$$\lambda_1 = -0.887212, \ \lambda_2 = 0.173606 - 1.64351i, \ \lambda_3 = 0.173606 + 1.64351i \tag{3.7}$$

For E_3 :

With the same method, the eigenvalues of the Jacobian at E_3 are

$$J(E_3) = \begin{pmatrix} -1-\lambda & 1 & 0 \\ -0.46 & -\lambda & -1 \\ 0.211599999 & 2.423199998 & 0.46-\lambda \end{pmatrix}$$

Determinant:

$$P_{\lambda}(E_3) = -\lambda^3 - 0.54\lambda^2 - 2.4232\lambda - 2.4232.$$

which has the eigenvalues

$$\lambda_1 = -0.887212, \ \lambda_2 = 0.173606 - 1.64351i, \ \lambda_3 = 0.173606 + 1.64351i \tag{3.8}$$

Since the linearization matrices $J(E_1)$, $J(E_2)$, and $J(E_3)$ have eigenvalues with positive real parts, it follows from Lyapunov stability theory [17] that the equilibrium points E_1, E_2, and E_3 are unstable, and this implies chaos in the dissipative system (3.3). So, the trajectories of the system (3.3) diverge from the three equilibrium points and orbit onto the strange attractor of the system (3.3).

3.2.3 Minimal order for chaos

Commensurate case

In the case of the comensurate-order system, we have $a = 1, \ b = 0.46$ and $c = 0.46$, where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ a necessary condition for the fractional-order nonlinear system (3.3) to be chaotic is:

$$\alpha > \frac{2}{\pi} \arctan\left(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)}\right),$$

For E_1:

$$\begin{aligned}\alpha &> \frac{2}{\pi}\arctan\left(\frac{|\mathrm{Im}\,(\lambda_{2,3})|}{\mathrm{Re}\,(\lambda_{2,3})}\right)\\ &\simeq \frac{2}{\pi}\arctan\left(\frac{0.8938602918}{0.7265608295}\right)\\ &\simeq \frac{2}{\pi}(0.8882780847)\\ &\simeq 0.5654953921.\end{aligned}$$

For $E_2,\ E_3$:

$$\begin{aligned}\alpha &> \frac{2}{\pi}\arctan\left(\frac{|\mathrm{Im}\,(\lambda_{2,3})|}{\mathrm{Re}\,(\lambda_{2,3})}\right)\\ &\simeq \frac{2}{\pi}\arctan\left(\frac{1.64351}{0.173606}\right)\\ &\simeq \frac{2}{\pi}(1.465555353)\\ &\simeq 0.9330015154.\end{aligned}$$

Thus, the necessary condition of existence chaos in fractional-order system (3.3) is:

$$\alpha > 0.9330015154.$$

Incommensurate case

In the case of the incomensurate-order system where $\alpha_1 \neq \alpha_2 \neq \alpha_3$ If α_1, α_2 and α_3. are rational numbers between zero and one, which are not necessarily equal, The necessary condition for the system (3.3) to exhibit chaotic oscillations in the incommensurate case is :

$$\tfrac{\pi}{2M} - \min_i\left(\left|\arg\left(\lambda_i\left(J_E\right)\right)\right|\right) > 0, \qquad i = 1, 2, 3$$

Where $\lambda_i\left(J_E\right),\ i = 1, 2, 3$, are the eigenvalues of the Jacobian matrix J_E of the system (3.3) at the equilibrium E, M is the LCM of the fractional orders.

For example, if $\alpha_1 = 1,\ \alpha_2 = 0.95,\ \alpha_3 = 0.975$, then we have $l_1 = 40$, $l_2 = 38$, $l_3 = 39$ and $M = 40$. The characteristic equation of the system evaluated at the equilibrium E_i is :

$$\det\left(\mathrm{diag}\left[\lambda^{M_{\alpha_1}}, \lambda^{M_{\alpha_2}}, \lambda^{M_{\alpha_3}}\right] - J_{E_i}\right) = 0$$

$$\det\left(\mathrm{diag}\left[\lambda^{40}, \lambda^{38}, \lambda^{39}\right] - J_{E_i}\right) = 0 \qquad i = 1, 2, 3.$$

For E_1 :

$$\det\left(\operatorname{diag}\left[\lambda^{40},\lambda^{38},\lambda^{39}\right]-J_{E_1}\right)=0,$$

$$\det\left(\begin{pmatrix}\lambda^{40}&0&0\\0&\lambda^{38}&0\\0&0&\lambda^{39}\end{pmatrix}-\begin{pmatrix}-1&1&0\\-0.46&0&-1\\-1&0&0.46\end{pmatrix}\right)=0$$

$$\lambda^{117}-0.46\lambda^{78}+\lambda^{77}+0.46\lambda^{39}-0.46\lambda^{38}-1.2116=0$$

For $E_{2,3}$:

$$\det\left(\operatorname{diag}\left[\lambda^{40},\lambda^{38},\lambda^{39}\right]-J_{E_{2,3}}\right)=0,$$

$$\det\left(\begin{pmatrix}\lambda^{40}&0&0\\0&\lambda^{38}&0\\0&0&\lambda^{39}\end{pmatrix}-\begin{pmatrix}-1&1&0\\-0.46&0&-1\\0.2111599999&2.423199998&0.46\end{pmatrix}\right)=0$$

$$\lambda^{117}-0.46\lambda^{78}+\lambda^{77}+2.4232\lambda^{40}+0.46\lambda^{39}-0.46\lambda^{38}+2.42276=0$$

From the roots of the above equations, we find $\lambda=0.997665$. whose argument is zero which is the minimum argument,

$$\min_i\left(\left|\arg\left(\lambda_i\left(J_E\right)\right)\right|\right)=0,$$

and hence the necesary stability condition is holds because

$$\frac{\pi}{80}-0>0,$$

3.2.4 Chaos by using Lyapunov exponents test

The Lyapunov exponent is a way to gauge how quickly close paths in a dynamical system diverge. It is named after the Russian mathematician Alexander Lyapunov, who created the concept in the late $19th$ century. The Lyapunov exponent is a measurement of the stability of a dynamical system. Nearby paths in the system tend to converge over time if the Lyapunov exponent is negative, demonstrating that the system is stable. On the other hand, if the Lyapunov exponent is positive, the system is unstable and behaves chaotically because neighboring paths tend to diverge over time. It is frequently used to investigate the behavior of chaotic systems,

in which neighboring paths are subject to sudden and unpredictable divergence.
The Lyapunov exponent where computed using Matlab in $10^3 s$, and the Lyapunov spectrum is shown in figure 3.1 and 3.2 for two cases with commensurate and incommensurate respectively.

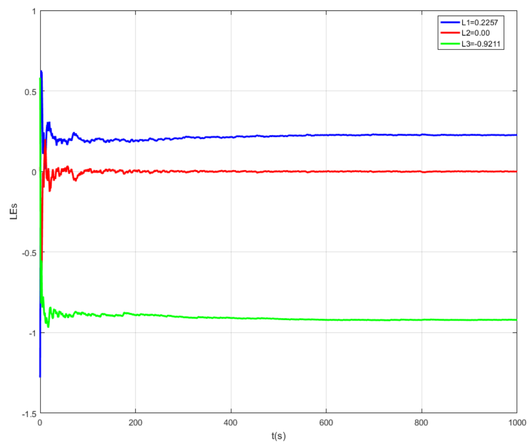

Figure 3.1: Lyapunov exponents spectrum of system (4.1) for $\alpha = 0.95$

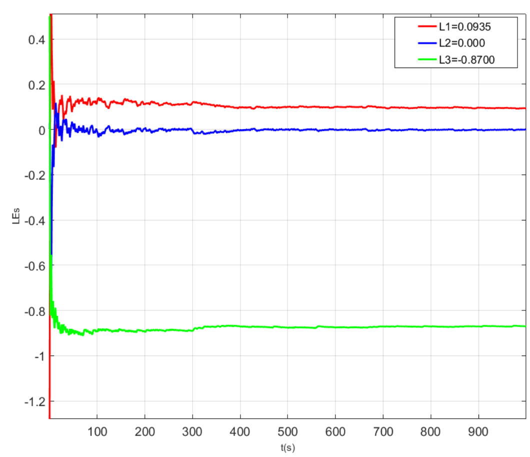

Figure 3.2: Lyapunov exponents spectrum of system (4.1) for $\alpha_1 = 0.95, \alpha_2 = \alpha_1 = 0.98$.

3.3 Conclusion

In this chapter, we and we provided evidence that these systems exhibit chaotic behavior once it reaches a certain threshold of minimum commensurate order.

General conclusion

In this work, we have studied the existence of chaos in fractional order systems in a novel chaotic system, in the first chapter we have mentioned some basic concepts of dynamical systems and chaos theory, and also basic definitions and properties of fractional derivatives with numerical methods to solve fractional-order systems, and in the next chapter we have provided some examples of chaotic fractional order systems with numerical simulation. In the last chapter we have a novel $3D$ fractional order system is introduced and its basic properties have been studied. Moreover, the two necessary conditions of the existence of chaos in commensurate order and incommensurate order are given. Also, the Lyapunov exponents are calculated using Matlab to prove that the proposed system exhibits chaotic behavior.

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