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Theme

# Carleman estimates for parabolic equations: Applications to controllability problems

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بسم الله الرحمن الرحيم

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# Dedication

*I dedicate this modest work to:*

*I would like to thank second Mom for your love, support and constant encouragement to me for your patience and all your tips and guidance, all my words are not enough to thank you.*

*To my father, who was credited with what I am now, may allah prolong his life and gave him health and wellness.*

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*To the soul of my immaculate mother, god rest her soul.*

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## Abstract

The main goal of this memory is to show some Carlman estimates of parabolic PDEs and its application to controllability problems. We shall present two kinds: the first is for a heat equation and the second is the general case of PDEs as well as some applications to solve problems of controllability of parabolic equations. Actually, these inequalities are lead to the existence of solutions for controllability problems. Carleman estimates are weighted inequalities constructed via a very technical method.

**Key words**: *Carleman inequalities*, *parabolic equations, observability inequality, null controllability, weight function.*

## Résumé

Le but principal de ce mémoire est de montrer quelques estimations de Carlman pour les EDP paraboliques et son application aux problèmes de contrôlabilité. Nous présenterons deux types : le premier est pour une équation de chaleur et la deuxiéme est le cas général des EDP ainsi que quelques applications pour résoudre des problèmes de contrôlabilité des équations paraboliques. En fait, ces inégalités conduisent à l'existence de solutions aux problèmes de contrôlabilité. Les estimations de Carleman sont des inégalités pondérées construites par une méthode très technique.

**Mots clés:** *Estimation de Carleman, équations paraboliques, inégalité d'observabilité, controlabilité nulle,fonction de poids.*

## **ملخص**

الهدف الرئيسي لهذه الذاكرة هو إظهار بعض متراجحات كارلمان للمعادلات التفاضلية الجزئية المكافئة وتطبيقها على مسائل قابلية التحكم، وسنقدم نوعين: الأول يتعلق بمعادلة الحرارة والثاني هو الحالة العامة للمعادلات التفاضلية الجزئية المكافئة وكذلك بعض التطبيقات لحل مسائل قابلية التحكم في المعادلات المكافئة. في الواقع، تؤدي هذه التفاوتات إلى وجود حلول لمسائل قابلية على التحكم. متراجحات كارلمان هي متراجحات مثقلة تم إنشاؤها بطريقة تقنية للغاية.



**الكلمات المفتاحية:** *متراجحات كارلمان، المعادلات التفاضلية الجزئية المكافئة، عدم متراجحة الملاحظة، قابلية التحكم المعدوم، دالة الوزن(الثقل).*

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# Notations & abbreviations

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers. |
| $\Omega$ | An open set in $\mathbb{R}^n$ with boundary $\partial\Omega$. |
| $\partial\Omega$ | A boundary of $\Omega$. |
| $C_0^\infty(D)$ | The set of functions belonging to $C^\infty(D)$, compactly supported in $D$. |
| $C^1(Q)$ | Continuously differentiable function space in $Q$. |
| $L^2(D)$ | The space of measurable functions of summable squares in $D$. |
| $L^2(0,T;Y)$ | The bounded linear operator space. |
| $(.,.)_{L^2(Q)}$ | A inner product in space $L^2(Q)$ . |
| $H^{2,1}(Q)$ | Sobolev space. |
| $\overline{Q}$, $\overline{D}$ | Closure of $Q$ or $D$ respectively. |
| $\nabla = \left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)^T$ | The gradient operator. |
| $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ | The Laplacian operator. |
| $x \in \mathbb{R}^n$ | The spatial variables. |
| $\partial_j =$ | Derivative for $x$. |
| $t \geq 0$ | The time variables. |
| $\partial_t = \frac{\partial}{\partial t}$ | Derivative for $t$. |
| $\frac{\partial q}{\partial \eta} = \nabla q \cdot \eta$ | The normal derivative. |
| $P^*$ | The adjoint operator of $P$. |
| $\chi_\omega$ | Characteristic function of the set $\omega$. |
| $\omega_0$ | An open set such that $\overline{\omega_0} \subset \omega$. |
| $\overline{\Omega \backslash \omega_0}$ | Closure of $\Omega$ except $\omega_0$. |
| $suppu$ | Support of function $u$. |
| PDEs | Partial differential equations. |
| a. e | Almost every where. |
| $\lesssim$ | Smaller or equal to approximately. |
| $\sup_{x\in\Omega} ess\,\lvert u(x)\rvert$ | Sup essentail of function $u$. |
| $D(\Omega)$ | The space of the test functions. |
| $V$ | Open subset of $\Omega$. |

# Introduction

Carleman estimates were first presented by Swedish mathematician Torsten Carlman in the 1930s, exactly in 1932 [5] and [13]. The original motivation for these estimates was to study the behavior of partial differential equation (PDEs) solutions with variable transactions at exponential weights. These estimates were circulated and organised by L. Hörmander [9]and others for a large class of differential operators in arbitrary dimensions.

It was initially used to study PDEs, especially those that are difficult to solve using traditional methods. It also provides a way to estimate the behaviour of PDEs in areas where traditional methods fail. Carleman estimates address the characterisation of solutions for certain types of partial differential equations through indirect means.

The basic idea of Carleman estimates is to use an integrated identity that includes a PDE solution and weight function. The weight function is carefully selected to ensure that these estimates are found in good form.

Among the latter's uses in mathematics are some applications that include inverse problems [11] where they have been used for certain types of inverse problems of partial differential equations, where they seek to recover information about a factor or function unknown due to some data and knowledge of the differential equation that governs their behaviour as well as integrative equations. Carleman estimates were used in control theory (see for example [8] and [12]) to study the controllability and stabilisation of systems governed by partial differential equations. Spectral theory[15]to study the spectral characteristics of operators in Hilbert spaces. Descriptions of nonlinear analysis are used to study the behaviour of nonlinear systems, such as nonlinear partial differential equations. Without forgetting the engineering analysis to study the geometric properties of cubes and subfolds, such as bending and size.

Overall, Carleman estimates provide powerful tools for analyzing complex mathematical systems and have numerous applications across areas such as photography.

Control theory is a branch of engineering and mathematics that deals with the analysis and design of controllable systems to achieve the required behavior. This theory is concerned with creating systems capable of maintaining or changing their condition based on feedback, and involves using mathematical models to describe system behaviour and developing control algorithms to handle system input in order to achieve the desired output. The goal of control theory is to develop techniques and tools that can be used to design systems capable of achieving specific performance goals, such as stability, response, and accuracy, control theory uses all disciplines.

An example of steering a vehicle, driving an aircraft or satellite to a geostationary orbit, improving the flow of information into a network, coding and decoding a digital image or SMS, regulating the thermostat, refining oil, controlling the pH of chemical reactions or optimising profits from stock market flows... Control can also reduce pain and prolong life. For example, a blood pressure regulator is designed to maintain this pressure at a constant and appropriate level; we can also control an epidemic such as studying brain tumor treatment or performing laser surgery. Plus robots. It is also used in areas such as economics, biology, and social sciences to model and control complex systems.

Overall, control theory is a highly interdisciplinary field with broad-ranging applications.

This study aims to study the controllability of parabolic equations; the tool used to achieve this goal is the Carleman estimates.

So this work is organised as follows:

In the first chapter, we will provide a reminder of the basic spaces in functional analysis such as the Hilbert and $L^p$ and Sobolev spaces as well as part of the definitions needed for controllability, and finally we can consider some of the concepts of Carleman estimates including the particular concepts of creating these estimates and the difference between both types.

In the second chapter, we offer some ways to create Carleman estimates of the equivalent partial differential equations that we have put forward to study the problem of controllability of these equations from a suitable weight function.

The third chapter is devoted to study some control problems of parabolic equations to find observability inequality based on the demonstration of null controllability in Carleman estimates.

# Chapter 1

# Basics on the controllability of parabolic PDEs

In this chapter, we will present a reminder on the fundamental spaces in functional analysis as Hilbert spaces, $L^p$ spaces and Sobolev space as well as a part of definitions necessary on the controllability, and finally we can consider some concepts of Carleman estimates.

## 1.1 Functional spaces

### 1.1.1 Hilbert space

**Definition 1.1** *Let $H$ be a linear space over $\mathbb{R}$. We ask the inner product in $H$ to be a function*

$$(.,\ .) : H \times H \longrightarrow \mathbb{R},$$

with the following conditions met:

1. $\forall x \in H : (x,\ x) \geq 0$ and $(x,\ x) = 0 \Longleftrightarrow x = 0$ (positivity).
2. $\forall x, y \in H : (x,\ y) = (y,\ x)$ (symmetry).
3. $\forall x, y, z \in H : (\lambda x + \mu y,\ z) = \lambda (x,\ z) + \mu (y,\ z) \ \ \forall \lambda, \mu \in \mathbb{R}$ (bilinearity).

A linear space endowed with an inner product is called an **inner product space**.
An inner product induces a norm, given by

$$\|x\| = \sqrt{(x,\ x)}. \tag{1.1}$$

**Definition 1.2** *Let $H$ be an inner product space. We say that $H$ is a Hilbert space if it is complete concerning the norm* (1.1).

### 1.1.2 $L^p(\Omega)$ Spaces

**Definition 1.3** *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$. We call the space of Lebesgue $L^p(\Omega)$ space*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R},\, u \text{ measurable and } \int_\Omega |u(x)|^p\, dx < +\infty \right\},$$

equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left( \int_\Omega |u(x)|^p\, dx \right)^{\frac{1}{p}}. \tag{1.2}$$

- In the special case $p = 2$ :

$$L^2(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R},\, u \text{ measurable and } \int_\Omega |u(x)|^2\, dx < +\infty \right\},$$

is a Hilber space, its inner prduct is

$$\|u\|_{L^2(\Omega)} = \left( \int_\Omega |u(x)|^2\, dx \right)^{\frac{1}{2}}.$$

- In the case $p = \infty$ :

$$L^\infty(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R},\, u \text{ measurable and } \exists c > 0 : |u(x)| \leq c \text{ a.e on } \Omega \},$$

equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} ess\, |u(x)| = \inf \{ c > 0,\ \ |u(x)| \leq c \text{ a.e on } \Omega \}.$$

We can define in this space $L^2(\Omega)$ the inner product

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx, \forall u, v \in L^2(\Omega). \tag{1.3}$$

### 1.1.3 $L^p(0, T;\ X)$ Spaces

**Definition 1.4** *The space $L^p(0, T;\ X)$ is defined as follows:*

$$L^p(0, T;\ X) = \left\{ u(t) \text{ in } X,\, u \text{ measurable and } \int_0^T \|u(x)\|_X^p\, dt < +\infty \right\},$$

equipped with the norm

$$\|u\|_{L^p(0,T;\ X)} = \sup_{t\in[0,\ T]} ess\, \|u(x)\|_X.$$

### 1.1.4 Sobolev spaces

**The space $H^1(\Omega)$**

**Definition 1.5** *We denote by $H^1(\Omega)$ the linear subspace of $L^2(\Omega)$ or sobolev space order one on $\Omega$, the space*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega)\,;\ \frac{\partial u}{\partial x_i} \in L^2(\Omega)\,, 1 \leq i \leq n \right\}.$$

The inner product and the norm in $H^1(\Omega)$ are given respectively by

$$(u,\ v)_{H^1(\Omega)} = \int_\Omega uv\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx,$$

and

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega u^2\, dx + \int_\Omega |\nabla u\,|^2\, dx.$$

**Theorem 1.1** *$H^1(\Omega)$ is Hilbert space.*

**Definition 1.6** *$H^1_0(\Omega)$ is defined as the closure of $D(\Omega)$ in $H^1(\Omega)$ i. e.*

$$H^1_0(\Omega) = \overline{D(\Omega)}.$$

**Proposition 1.1** *$H^1_0(\Omega)$ equipped with the norm and inner product in $H^1(\Omega)$, respectively.*

**Theorem 1.2** ***(Rellich)*** *If $\Omega$ is an bounded open class $C^1$, then canonical injection of $H^1_0(\Omega)$ in $L^2(\Omega)$ is compact, i.e. any a bounded set of $H^1_0(\Omega)$ is relatively compact in $L^2(\Omega)$ . We write:*

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \ \text{ is compact } (\hookrightarrow \text{continuous injection}).$$

## 1.2 The controllability of parabolic PDEs

Control theory is mainly called dynamic improvement theory, which began in the 20th century and continued to evolve and adapt to the needs of mathematicians, physicists, mechanics, chemists, and biologists. However, it serves as a prelude to many issues of great practical importance, such as track planning and ecology.
The controllability in our field is the ability to find at least one control so that the controlled system can reach any final state beforehand and we also know that it is one of the structural characteristics that characterise the systems and may classify them through their geometric properties.

A control problem is of manipulating a system with an input-output space. The input is the control can be a function in a boundary condition, an initial condition, or a coefficient in a partial differential equation modelling the system, or any parameter in the equation, and the output is the state or the solution of the system or any information related to her as shown in the following chart:

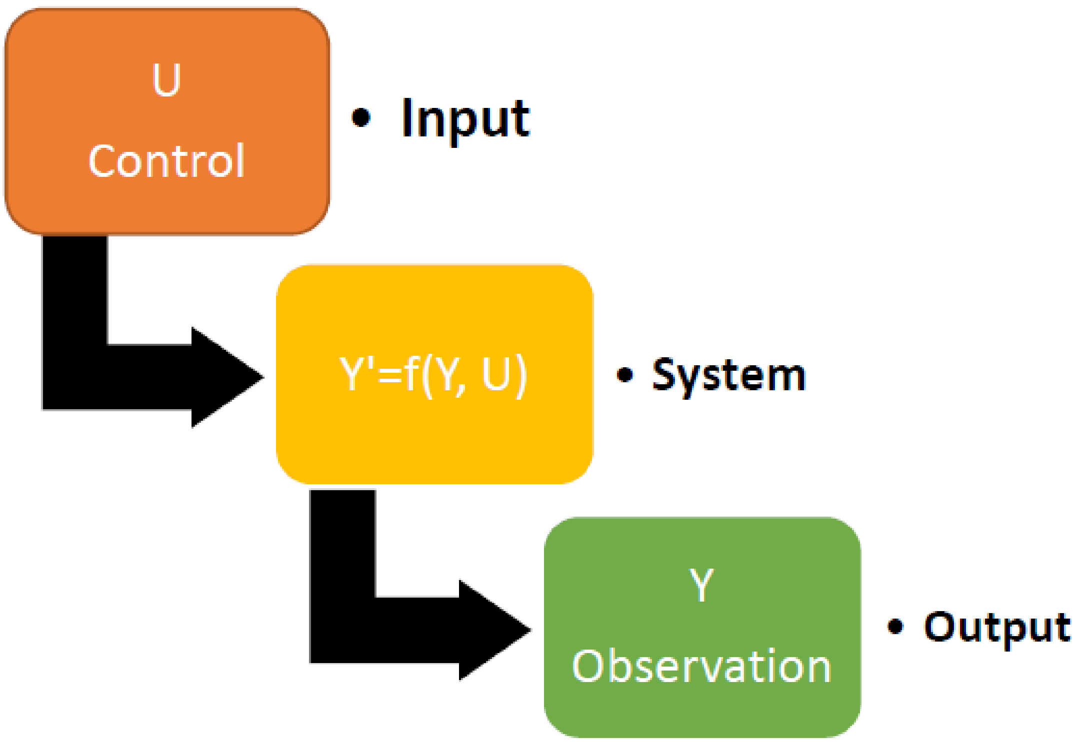


FIGURE 1: Control system

Consider the linear differential system

$$\begin{cases} y'(t) = Ay(t) + \mathbf{Bu}(t) & , \forall t \in (0,T) \quad \text{(control)} \\ z(t) = Cy, & \qquad\qquad\quad \text{(observation)} \\ y(0) = y_0 \in D(A) \subset Y, & \qquad\qquad\quad \text{(initial data)} \end{cases} \tag{1.4}$$

where

* $y(t) \in Y$ the state space, $\mathbf{u}(t) \in U$, and $z(t) \in Z$ are the spaces assumed to be a separable Hilbert of infinite dimension;

* $u$: $(0, T) \longrightarrow U$ is a locally integrable control (input) function;

* $z$: $(0, T) \longrightarrow Z$ is a locally integrable observation (output) function;

* $A$: $D(A) \subset Y \longrightarrow Y$ is an infintisimal generator of a $C_0$-semi-group $\{S(t)\}_{t\geq 0}$ on $Y$, with $A$ a parabolic operator;

* $B$: $U \rightarrow Y$ is a linear and bounded control operator;

* $C$: $Y \rightarrow Z$ is a linear and bounded observation operator.

We will note this by

- $(A,\ B,\ C)$ the system $(1.4)$.
- $(A,\ B)$ the system $(1.4)$ regardless of output$(C = 0)$.
- $(A,\ C)$ the system $(1.4)$ regardless of input$(B = 0)$.

The solution of system $(1.4)$ is given by

$$y(t, \mathbf{u}, \mathbf{y}_0) = S(t)\, y_0 + \int_0^t S(t-s)\, \mathbf{Bu}(t)\, ds. \tag{1.5}$$

### 1.2.1 The operator of controllability

For system$(1.4)$, consider $\mathcal{L}_T$ is the bounded linear operator defined by

$$\mathcal{L}_T : \begin{cases} L^2(0, T; U) \longrightarrow Y \\ u(.) \longrightarrow \int_0^t S(t-s)\, \mathbf{Bu}(t)\, ds, \end{cases}$$

and the adjoint operator is given by

$$\mathcal{L}_T^* : \begin{cases} Y \longrightarrow L^2(0, T; U) \\ x \longrightarrow \mathcal{L}_T^* x = u. \end{cases}$$

**Proof** given in [4].

### 1.2.2 Different concepts of controllability

#### Exact Controllability

**Definition 1.7** *The system* $(1.4)$ *or the pair* $(A,\ B)$ *is* ***exactly controllable*** *in* $Y$ *on* $[0,\ T]$*, if given any initial and desired data* $y_0$*,* $y_d \in Y$ *there exists a control* $\mathbf{u} \in L^2(0,\ T;\ U)$ *such that the solution of* $(1.5)$ *satisfies* $y(T) = y_d$*. In other words*

$$\forall y_d \in Y, \exists \mathbf{u} \in L^2(0, T; U) : y(T) = y_d. \tag{1.6}$$

**Proposition 1.2** *The system* $(1.4)$ *is* ***exactly controllable*** *if and only if:*

$$\operatorname{Im}(\mathcal{L}_T) = Y, \qquad (\mathcal{L}_T \text{ surjective}).$$

**Remark 1.1** *We recall that* $\operatorname{Im}\mathcal{L}_T = \{y \in Y / \exists u \in L^2(0, T; U) : y(t, \mathbf{u}, \mathbf{y}_0) = y\}$.

**Proposition 1.3** (***Observability inequality***)*The system* $(1.4)$ *is* ***exactly controllable*** *if and only if:*

$$\exists \gamma > 0, \forall y \in Y : \int_0^T \|B^* S^*(t) y\|_U^2 \geq \gamma \|y\|_Y^2. \tag{1.7}$$

This inequality is called "**Observability inequality**".

#### Approximate controllability

**Definition 1.8** *The system* $(1.4)$ *or the pair* $(A,\ B)$ *is* ***approximate*** *(or weakly)* ***controllable*** *in* $Y$ *on* $[0,\ T]$ *if:*

$$\forall y_d \in Y, \forall \varepsilon > 0, \exists \mathbf{u} \in L^2(0, T; U) : \|y(T) - y_d\|_Y < \varepsilon. \tag{1.8}$$

**Proposition 1.4** *The system* $(1.4)$ *is* ***approximate controllable*** *if and only if:*

$$\overline{\operatorname{Im}(\mathcal{L}_T)} = Y, \qquad (\text{the image of } \mathcal{L}_T \text{ is dense in } Y).$$

**Proposition 1.5** *The system* $(1.4)$ *is* ***approximate controllable*** *if and only if* $\mathcal{L}_T^*$ *is injective.*

#### Null Controllability

**Definition 1.9** *The system* $(1.4)$ *or the pair* $(A,\ B)$ *is* ***null*** *(or zero)* ***controllable*** *in* $Y$ *on* $[0,\ T]$ *if:*

$$\forall y_d \in Y, \exists \mathbf{u} \in L^2(0, T; U) : y(T) = 0. \tag{1.9}$$

**Proposition 1.6** *The system* $(1.4)$ *or the pair* $(A,\ B)$ *is* ***null*** *(or zero)* ***controllable*** *in* $Y$ *on* $[0,\ T]$ *if and only if:*

**i)** $\exists C > 0, \forall y \in Y : \int_0^T \|B^* S^*(t) y\|_U^2 \geq C \|S^*(t) y\|_Y^2$.

**ii)** $\operatorname{Im} S(t) \subset \operatorname{Im} \mathcal{L}_T$.

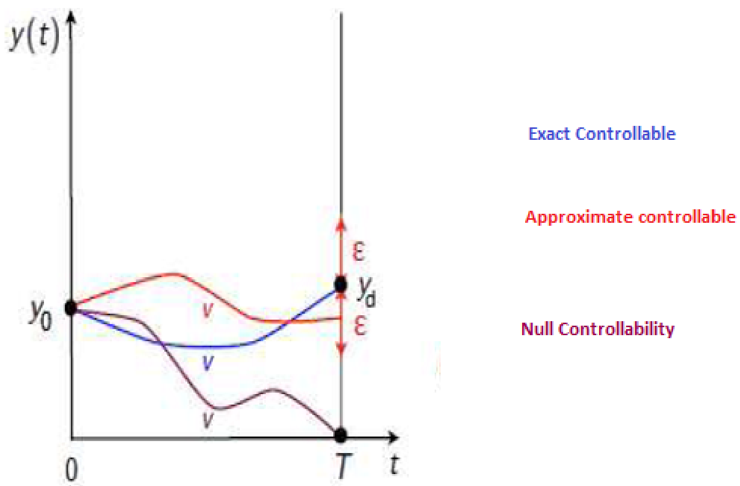


FIGURE 2 : Different concepts of controllability

## 1.3 Weight function

To develop Calman estimates, we need a special function called **the weight function**, and this function has conditions to choose summarised in the following Lemma.

**Lemma 1.1** *Let $\omega_0 \subset \omega$ be an arbitrarily fixed subdomain of $\Omega$ such that $\overline{\omega_0} \subset \omega \subset \Omega$. Then, there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\left\{ \begin{array}{cc} \psi(x) > 0, & \text{for every } x \in \Omega, \\ \psi(x) = 0, & \text{for all } x \in \partial\Omega, \\ |\nabla\psi(x)| > 0, & \text{for all } x \in \overline{\Omega \backslash \omega_0}. \end{array} \right\}$$

- The form of this function is given by

$$\psi(x,\ t) = d(x) - \beta(t - t_0)^2 + c_0,\ t_0 \in (0,\ T);\ \beta,\ c_0 > 0,$$

with $d \in C^2(\overline{D})$, $|\nabla d| \neq 0$ on $\overline{D}$, and $\inf_{(x,\ t) \in Q} \psi(x,\ t) > 0$.

**Proof.** The proof of this lemma is given in [18], [10]. ■

## 1.4 Carleman estimates

Carleman estimates were presented in 1939 by the Swedish mathematician Torsten Carlman. Carleman estimates found different applications in mathematical analysis branches, in their beginnings used to show the results of uniqueness in solutions to elliptical partial differential equations. Estimates of the Carlman type were also found in the fields of new applications, especially partial differential equation control theory and spectrum theory, as well as in the study of reverse problems.

Carleman estimates are a powerful tool in the study of parabolic equations, providing a way to estimate the solution of the equation in terms of its initial data and limits. This makes it useful to examine problems of control, as one is interested in creating the necessary conditions for controlling the solution of the parabolic equation, and this is the goal of this memory.

In particular, Carleman estimates can be used to show that if certain requirements are met for equation transactions and boundary data, the solution can be controlled by appropriate border controls. This is done by showing that if these requirements are met, the solution can be assessed in terms of its initial data and border controls.

Carleman estimates were also used to study control problems in other types of partial differential equations, such as hyperbolic equations and equation systems. In addition, they have been used to study reverse problems, where one is interested in recovering information about transactions or raw data from solution notes.

A typical Carleman inequality in the form below

$$\|e^{s\varphi}u\| \lesssim \|e^{s\varphi}Pu\|,$$

this formula for parabolic equation see [2]

$$\left\|h^{\frac{1}{2}}e^{\frac{\varphi}{h}}u\right\|_{L^2}^2 + \left\|h^{\frac{3}{2}}e^{\frac{\varphi}{h}}\nabla_x u\right\|_{L^2}^2 \leq C\left\|h^2 e^{\frac{\varphi}{h}}Pu\right\|_{L^2}^2,$$

where $P$ is a differential operator; $u$ a function; $\varphi$ a function called a weight function; $h > 0$ a small parameter; $s > 0$ a real parameter and $C > 0$.

### 1.4.1 Types of Carleman estimates

Carleman estimates are a powerful tool in the study of partial differential equations (PDEs). They provide a way to estimate the solution of a PDE from above or below by using a weight function that satisfies certain properties. In particular, Carleman estimates can be used to prove the uniqueness and stability results for solutions of PDEs.

Actually, there are two types of Carleman estimates

### Global Carleman estimates

Global Carleman estimates for parabolic equations are estimates that hold uniformly over the entire domain of the equation. These estimates are useful for proving global uniqueness and stability results for solutions of parabolic equations.

**Theorem 1.3** ***(Global Carleman estimate)***. *Let $\varphi$ be a function that satisfies Assumption 7.7* $($*see[14]* $)$. *Then there exist* $\delta_4 > 0$ *and* $C \geq 0$ *such that*

$$\left\|h^{\frac{1}{2}} e^{\frac{\varphi}{h}} u\right\|_{L^2(Q)}^2 + \left\|h^{\frac{3}{2}} e^{\frac{\varphi}{h}} \nabla_x u\right\|_{L^2(Q)}^2 \leq C \left( \left\|h^2 e^{\frac{\varphi}{h}} P u\right\|_{L^2(Q)}^2 + \left\|h^{\frac{1}{2}} e^{\frac{\varphi}{h}} u\right\|_{L^2((0,\,T)\times\omega)}^2 \right),$$

for $0 < (T + T^2)\varepsilon \leq \delta_4$, $h = \varepsilon t\,(T - t)$ and $u \in \mathcal{C}^\infty\left([0,\,T] \times \overline{\Omega}\right)$ such that $u|_{(0,\,T)\times\partial\Omega} = 0$.

**Proof.** see [[14], Theorem 7.8]. ■

### Local Carleman estimates

Local Carleman estimates, on the other hand, hold only in the neighbourhood of a given point in the domain. These estimates are useful for proving local uniqueness and stability results for solutions of parabolic equations.
There are different types of Carleman estimates depending on the type of PDE being studied and the properties of the weight function used. Examples include exponential Carleman estimates, polynomial Carleman estimates, and logarithmic Carleman estimates.

**Theorem 1.4** ***(Local Carleman estimate away from the boundary)***. *Let $K$ be a compact set of and $V$an open subset of $\Omega$ that is a neighbourhood of $K$. Let $\varphi$ be a weight function that satisfies Assumption 7.1*[*see [14]*] *in* $\overline{V}$.

Then there exist $C > 0$ and $\delta_2 > 0$ such that

$$\left\|h^{\frac{1}{2}} e^{\frac{\varphi}{h}} u\right\|_{L^2}^2 + \left\|h^{\frac{3}{2}} e^{\frac{\varphi}{h}} \nabla_x u\right\|_{L^2}^2 \leq C \left\|h^2 e^{\frac{\varphi}{h}} P u\right\|_{L^2}^2,$$

for $u \in C^\infty\left([0,\,T] \times \Omega\right)$, with $u \in C_c^\infty\left(K\right)$ for all $t \in [0,\,T]$ and $0 < (T + T^2)\varepsilon \leq \delta_2$.

**Proof.** see [[14], Theorem 7.3]. ■

# Chapter 2

# Construction of Carleman estimates of parabolic PDEs

In this chapter, we will present some techniques for creating Carleman estimates of parabolic partial differential equations in the case of Dirichlet boundary conditions. These estimates were presented in 1922 by Swedish mathematician Torsten Carlman who called them Carleman inequality .

## 2.1 Carleman estimates for the heat equation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded spatial domain with smooth boundary $\partial\Omega$. For a given time $T > 0$, we denote $Q = \Omega \times (0, T)$.

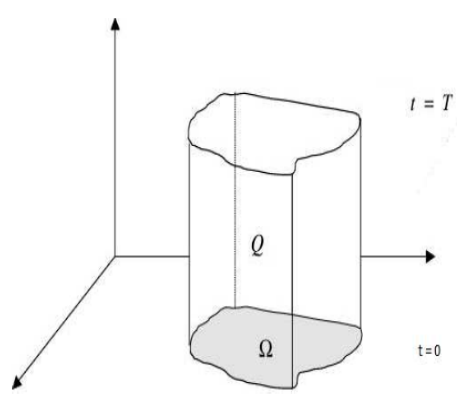


FIGURE 3: The space-time cylinder

Let $D \subset Q$ be a bounded with smooth boundary $\partial D$.

Consider the following heat equation with $u \in C_0^\infty(D)$

$$\partial_t u(x,t) = \Delta u(x,t) + f(x,t), \qquad \text{in } D. \tag{2.1}$$

Our aim is to find an estimate of the previous equation in the following form see [20]

$$\int_D s(|\nabla u(x,t)|^2 + |u(x,t)|^2)e^{2s\varphi(x,t)}dxdt \leq C \int_D |f\ (x,t)|^2 e^{2s\varphi(x,t)}dxdt, \tag{2.2}$$

where $s > 0$ is a large parameter in some domain, $\varphi(x,t)$ is a suitable weight function, and constant $C > 0$ .

This estimate (2.2) is called a Carleman estimate. We note that the estimate is valid uniformly for all parameters $s > 0$, i.e., $s \geq s_0$: a constant. Therefore, the constant $C > 0$ should be independent of $s > s_0$ and $u \in C_0^\infty(D)$.

To prove this estimate, we assume that the weight function $\varphi(x,t)$ verifies

$$\begin{cases} \varphi(x) > 0 & \text{for every } x \in \Omega, \\ \varphi(x)|_{\partial\Omega} = 0, & \\ |\nabla\varphi(x)| > 0 & \text{for all } x \in \overline{\Omega\backslash\omega_0}. \end{cases} \tag{2.3}$$

The first step is multiplying equation (2.1) by $e^{s\varphi(x,t)}$, to get

$$[\partial_t u(x,t) - \Delta u(x,t)]\, e^{s\varphi(x,t)} = f(x,t)e^{s\varphi(x,t)}.$$

Then

$$[\partial_t - \Delta]\, u(x,t)e^{s\varphi(x,t)} = f(x,t)e^{s\varphi(x,t)}.$$

Suppose that

$$\omega(x,t) = e^{s\varphi(x,t)}u(x,t),$$

and

$$P\omega(x,t) = e^{s\varphi(x,t)}(\partial_t - \Delta)(e^{-s\varphi(x,t)}\omega).$$

The second step is to calculate the norm in $L^2(D)$ space, attached

$$\int_D f^2 e^{2s\varphi(x,t)}dx\ dt = \int_D |P\omega(x,t)|^2\, dx\ dt.$$

Now we seek to find a lower estimate for $\|P\omega(x,t)\|^2_{L^2(D)}$.
We set

$$\begin{aligned} P\omega(x,t) &= e^{s\varphi(x,t)}(\partial_t - \Delta)(e^{-s\varphi(x,t)}\omega) \\ &= e^{s\varphi(x,t)}\left[(\partial_t(e^{-s\varphi(x,t)}\omega(x,t)) - \Delta(e^{-s\varphi(x,t)}\omega(x,t))\right]. \end{aligned}$$

We obtain

$$\begin{aligned}\partial_t(e^{-s\varphi(x,t)}\omega(x,t) &= \omega(x,t)\partial_t e^{-s\varphi(x,t)} + e^{-s\varphi(x,t)}\partial_t\omega(x,t) \\ &= e^{-s\varphi(x,t)}\partial_t\omega(x,t) - s\omega(x,t)\partial_t\varphi e^{-s\varphi(x,t)}.\end{aligned} \tag{2.4}$$

Besides that

$$\Delta\omega(x,t) = -se^{-s\varphi(x,t)}\omega(x,t)\Delta\varphi + s^2e^{-s\varphi(x,t)}\omega\left(|\nabla\varphi|\right)^2 - 2se^{-s\varphi(x,t)}\nabla\omega\cdot\nabla\varphi + \Delta\omega \tag{2.5}$$

result from

$$\begin{aligned}\frac{\partial(e^{-s\varphi(x,t)}\omega(x,t))}{\partial x_i} &= -s\frac{\partial\varphi}{\partial x_i}e^{-s\varphi(x,t)}\omega(x,t) + e^{-s\varphi(x,t)}\frac{\partial(\omega(x,t))}{\partial x_i}, \text{ and} \\ \frac{\partial^2(e^{-s\varphi(x,t)}\omega(x,t))}{\partial x_i^2} &= -s\left[\frac{\partial}{\partial x_i}(\frac{\partial\varphi}{\partial x_i}(e^{-s\varphi(x,t)}\omega(x,t))\right] + \frac{\partial}{\partial x_i}(e^{-s\varphi(x,t)}\frac{\partial(\omega(x,t))}{\partial x_i}) \\ &= -s\left[\frac{\partial^2\varphi}{\partial x_i^2}e^{-s\varphi(x,t)}\omega(x,t) + \frac{\partial\varphi}{\partial x_i}\frac{\partial}{\partial x_i}(e^{-s\varphi(x,t)}\omega(x,t)\right] \\ &\quad + \frac{\partial(\omega(x,t))}{\partial x_i}\frac{\partial}{\partial x_i}(e^{-s\varphi(x,t)}) + \frac{\partial^2\omega(x,t)}{\partial x_i^2}e^{-s\varphi(x,t)} \\ &= -s\left[\frac{\partial^2\varphi}{\partial x_i^2}e^{-s\varphi(x,t)}\omega(x,t)\right] + s^2\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}e^{-s\varphi(x,t)}\omega(x,t) \\ &\quad -s\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega(x,t)}{\partial x_i}e^{-s\varphi(x,t)} - s\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega(x,t)}{\partial x_i}e^{-s\varphi(x,t)} \\ &= -se^{-s\varphi(x,t)}\omega(x,t)\frac{\partial^2\varphi}{\partial x_i^2} + s^2(\frac{\partial\varphi}{\partial x_i})^2e^{-s\varphi(x,t)}\omega(x,t) - 2se^{-s\varphi(x,t)}\frac{\partial\varphi}{\partial x_i}\frac{\partial\omega(x,t)}{\partial x_i}.\end{aligned}$$

Then, by (2.4) and (2.5), we have this result

$$\begin{aligned}P\omega(x,t) &= e^{s\varphi(x,t)}[e^{-s\varphi(x,t)}\partial_t\omega(x,t) - s\omega(x,t)\partial_t\varphi e^{-s\varphi(x,t)}e^{-s\varphi(x,t)}\partial_t\omega(x,t) - s\omega(x,t)\partial_t\varphi e^{-s\varphi(x,t)} \\ &\quad - \left(-se^{-s\varphi(x,t)}\omega(x,t)\Delta\varphi + s^2e^{-s\varphi(x,t)}\omega(x,t)(|\nabla\varphi|)^2 - 2se^{-s\varphi(x,t)}\nabla\omega\cdot\nabla\varphi + \Delta\omega\right)] \\ &= \partial_t\omega(x,t) - s\omega(x,t)\partial_t\varphi + s\omega(x,t)\Delta\varphi - s^2\omega(|\nabla\varphi|)^2 - 2s\nabla\omega\cdot\nabla\varphi - \Delta\omega,\end{aligned}$$

and, we can write

$$P\omega(x,t) = \partial_t\omega(x,t) - \Delta\omega - 2s\nabla\omega\cdot\nabla\varphi + (-s\partial_t\varphi - s^2(|\nabla\varphi|)^2 + s\Delta\varphi)\omega(x,t). \tag{2.6}$$

In this case, we decompose the operator $P$ into the symmetric part $P_+$ and the antisymmetric part $P_-$ (see appendix **Proposition 4.3** [A.5])

$$P\omega = P_+\omega + P_-\omega.$$

The third step is a calculus of the adjoint $P^*$of $P$ with

$$(P\omega, \upsilon)_{L^2(D)} = (\omega, P^*\upsilon)_{L^2(D)} \quad \upsilon, \omega \in C_0^\infty(D).$$

We have, for all $\upsilon, \omega \in C_0^\infty(D)$

$$(\partial_t\omega, \upsilon)_{L^2(D)} = -(\omega, \partial_t\upsilon)_{L^2(D)}$$

and

$$(-\Delta\omega, \upsilon)_{L^2(D)} = (\omega, -\Delta\upsilon)_{L^2(D)}$$

by integration by parts (see appendix **Theorem 4.1**), and the Green theorem (see appendix **Theorem 4.2**). We get

$$P^*\omega = -\partial_t\omega(x,t) - \Delta\omega + 2s\nabla\omega\cdot\nabla\varphi - (s\partial_t\varphi + s^2(|\nabla\varphi|)^2 - s\Delta\varphi)\omega(x,t). \tag{2.7}$$

On the other hand, we define $P_+$ and $P_-$ (see appendix **Theorem 4.3** $[A.6]$, $[A.7]$) by

$$P_+ = \tfrac{1}{2}(P + P^*), \quad P_- = \tfrac{1}{2}(P - P^*).$$

Use the equation of $P$ and $P^*$, to get

$$\begin{aligned} P_+\omega &= \frac{1}{2}(P + P^*)\,\omega \\ &= -\Delta\omega(x,t) - (s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega \end{aligned} \tag{*}$$

and

$$\begin{aligned} P_-\omega &= \frac{1}{2}(P - P^*)\,\omega \\ &= \partial_t\omega + (s\Delta\varphi)\,\omega + 2s\nabla\varphi\cdot\nabla\omega. \end{aligned} \tag{**}$$

We have

$$\int_D f^2 e^{2s\varphi(x,t)} dx\, dt = \int_D |P\omega(x,t)|^2\, dx\, dt.$$

By $(*)$ and $(**)$, we get

$$\begin{aligned} \int_D f^2 e^{2s\varphi(x,t)} dx\ dt &= \int_D |P_+\omega + P_-\omega|^2\, dx\ dt \\ &= \|P_+\omega + P_-\omega\|^2_{L^2(D)}. \end{aligned}$$

We know that

$$\|P_+\omega + P_-\omega\|^2_{L^2(D)} = \|P_+\omega\|^2_{L^2(D)} + \|P_-\omega\|^2_{L^2(D)} + 2\left(P_+\omega\ , P_-\omega\right)_{L^2(D)} \geq 2\left(P_+\omega\ , P_-\omega\right)_{L^2(D)}.$$

Therefore, we will find an estimate of the right side as follows

$$\begin{aligned} 2\left(P_+\omega\ , P_-\omega\right)_{L^2(D)} &= 2\left(-\Delta\omega(x,t) - (s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega\ , \partial_t\omega + (s\Delta\varphi)\,\omega + 2s\nabla\varphi\cdot\nabla\omega\right)_{L^2(D)} \\ &= 2\left(-\Delta\omega(x,t), \partial_t\omega\right)_{L^2(D)} + 2\left(-\Delta\omega(x,t), 2s\nabla\varphi\cdot\nabla\omega\right)_{L^2(D)} \\ &\quad +2\left(-\Delta\omega(x,t), (s\Delta\varphi)\,\omega\right)_{L^2(D)} - 2\left((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega, \partial_t\omega\right) \\ &\quad -2\left((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega, 2s\nabla\varphi\cdot\nabla\omega\right)_{L^2(D)} \\ &\quad -2\left((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega, (s\Delta\varphi)\,\omega\right)_{L^2(D)}, \end{aligned}$$

where $\omega \in C_0^\infty(D)$, we will reduce the orders of derivatives of $\omega$, and the constants $C_i > 0$, $i = \overline{1,\ 4}$ denote constants independent of $s$.
Let us make some calculations

$$\begin{aligned} -2\left((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega, 2s\nabla\varphi\cdot\nabla\omega\right)_{L^2(D)} &= -4s\int_D \left(((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega)\nabla\varphi\cdot\nabla\omega\right) dx\ dt \\ &= -4s\sum_{i=1}^n \int_D \left\{(s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega\right\}\partial_i\varphi\cdot\partial_i\omega\ dx\ dt \\ &= 2s\sum_{i=1}^n \int_D \left\{(s\partial_t\varphi + s^2|\nabla\varphi|^2)\right\}\partial_i\varphi\cdot\partial_i\left(\omega^2\right)\ dx\ dt, \end{aligned}$$

since

$$2\omega\partial_i\omega = \partial_i\left(\omega^2\right) \Longleftrightarrow \partial_i\left(\omega^2\right) = \omega\partial_i\omega.$$

We have

$$\begin{aligned} -2\left((s\partial_t\varphi + s^2|\nabla\varphi|^2)\omega, 2s\nabla\varphi\cdot\nabla\omega\right)_{L^2(D)} &= 2s\sum_{i=1}^n \int_D \partial_i\left\{(s\partial_t\varphi + s^2|\nabla\varphi|^2)\partial_i\varphi\right\}\left(\omega^2\right)\ dx\ dt \\ &= 2s\int_D \nabla\left\{(s\partial_t\varphi + s^2|\nabla\varphi|^2)\nabla\varphi\right\}\left(\omega^2\right) dx\ dt \\ &= 2s\int_D \left\{\nabla(s\partial_t\varphi + s^2|\nabla\varphi|^2)\nabla\varphi\right\}\left(\omega^2\right) \\ &\quad +2s\int_D \left\{(s\partial_t\varphi + s^2|\nabla\varphi|^2)\Delta\varphi\right\}\left(\omega^2\right) dx\ dt. \end{aligned}$$

Using the Green formula (see appendix **Theorem 4.2**), we get

$$\begin{aligned} 2\left(-\Delta\omega,(s\Delta\varphi)\,\omega\right)_{L^2(D)} &= -2s\int_D \Delta\omega(x,t)\cdot((\Delta\varphi)\,\omega)dx\,dt \\ &= 2s\int_D \nabla\omega\cdot\nabla((\Delta\varphi)\,\omega)dx\,dt \\ &= 2s\int_D (\Delta\varphi)\cdot|\nabla\omega|^2\,\,dx\,dt+2s\int_D \nabla\left(\Delta\varphi\right)\cdot\omega\nabla\omega\,dx\,dt, \end{aligned}$$

and

$$\left|s\int_D \nabla\left(\Delta\varphi\right)^2\omega\nabla\omega\,dx\,dt\right|\leq\,C_1 s\int_D |\omega|\,|\nabla\omega|\,\,dx\,dt.$$

Hence, we get

$$2\left(-\Delta\omega(x,t),(s\Delta\varphi)\,\omega\right)\geq 2s\int_D (\Delta\varphi)\cdot|\nabla\omega|^2\,\,dx\,dt-C_1 s\int_D |\omega|\,|\nabla\omega|\,\,dx\,dt.$$

The last coefficient, noting $2(\partial_k\omega)(\partial_k\partial_j\omega)=\partial_j(|\partial_k\omega|^2)$, and integration by parts (see appendix **Theorem4.1**), we have

$$\begin{aligned} 2\left(-\Delta\omega(x,t),2s\nabla\varphi.\nabla\omega\right)_{L^2(D)} &= 2\sum_{j,k=1}^{n}\left(-\partial_k^2\omega,2s(\partial_j\varphi)\,.\partial_j\omega\right)_{L^2(D)} \\ &= 2\sum_{j,k=1}^{n}\left(\partial_k\omega,2s\partial_k\left[\partial_j\varphi.\partial_j\omega\right]\right)_{L^2(D)} \\ &= 2\sum_{j,k=1}^{n}\left(\partial_k\omega,2s(\partial_j\omega)(\partial_k\partial_j\varphi)\right)_{L^2(D)} \\ &\quad +\left(\partial_k\omega,2s\left(\partial_j\varphi\right)\left(\partial_k\partial_j\omega\right)\right)_{L^2(D)} \\ &= 4s\sum_{j,k=1}^{n}\int_D \partial_k\omega(\partial_j\omega)(\partial_k\partial_j\varphi)\,dx\,dt \\ &\quad +2s\sum_{j,k=1}^{n}\int_D (\partial_j\varphi)\,2\partial_k\omega(\partial_k\partial_j\omega)dx\,dt \end{aligned}$$

$$\begin{aligned} &= 4s\sum_{j,k=1}^{n}\int_D \partial_j\omega(\partial_k\omega)(\partial_k\partial_j\varphi)\,dx\,dt \\ &\quad +2s\sum_{j,k=1}^{n}\int_D (\partial_j\varphi)\,\partial_j\left(|\partial_k\omega|^2\right)dx\,dt. \\ &= 4s\sum_{j,k=1}^{n}\int_D \partial_j\omega(\partial_k\omega)(\partial_k\partial_j\varphi)\,dx\,dt \\ &\quad -2s\sum_{j,k=1}^{n}\int_D \left(\partial_j^2\varphi\right)|\partial_k\omega|^2\,dx\,dt. \end{aligned}$$

By note, we find that the maximum order terms $\omega^2$and $|\nabla\omega|^2$ are $s^3$, and $s$, respectively, we have

$$\begin{aligned}\frac{1}{2}\left(P_+\omega\ ,P_-\omega\right)_{L^2(D)} &\geq s^3\int_D\{\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi\}\omega^2dxdt\\ &+2s\sum_{j,k=1}^{n}\int_D\partial_j\omega(\partial_k\omega)(\partial_k\partial_j\varphi)\ dx\ dt\\ &-C_2\int_D s^2\omega^2dx\ dt-C_1 s\int_D|\omega|\,|\nabla\omega|\ dx\ dt\end{aligned}$$

$$\geq\ s^3\int_D\{\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi\}\omega^2dxdt+2s\sum_{j,k=1}^{n}\int_D\partial_j\omega(\partial_k\omega)(\partial_k\partial_j\varphi)\ dx\ dt$$
$$-C_3\int_D(s^2\omega^2-|\nabla\omega|)\ dx\ dt.$$

We used to write better the last inequality (see Appendix **Proposition 4.1** $[A.1]$)

$$s|\nabla\omega||\omega|\leq\frac{1}{2}s^2\left|\omega\right|^2+\frac{1}{2}\left|\nabla\omega\right|^2.$$

Other terms are non-principal because they are a lower estimate.

**Remark 2.1** *:Estimates are specific to variables $\omega^2$ and $|\nabla\omega|^2$.*

The fourth step is to take $s>0$ and large, and if $\varphi(x,t)$ satisfies $\{\partial_i\partial_j\varphi\}_{1\leq i,j\leq n}$ is positive definite, and $\exists\alpha>0,\alpha$ a constant such that

$$\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi\geq\alpha\ \text{ on }\overline{D}\ .$$

Then there exist constants $C_4>0$ and $s>0$ such that

$$\int_D\left(s\left|\nabla\omega\right|^2+s^3\left|\omega\right|^2\right)dx\ dt\leq C_4\int_D f^2e^{2s\varphi}dx\ dt,$$

for all $s\geq s_0$ and all $\omega\in C_0^\infty(D)$, with $\omega(x,t)=e^{s\varphi(x,t)}u(x,t)$, we rewrite in terms of $u$, we find

$$\int_D\left(s\left|\nabla u\right|^2+s^3\left|u\right|^2\right)e^{2s\varphi}dx\ dt\leq C_4\int_D f^2e^{2s\varphi}dx\ dt,$$

for all $s\geq s_0$ and all $u\in C_0^\infty(D)$.

## 2.2 Carleman estimates for a general of the second-order parabolic equation

Let $D \subset Q$ be a bounded domain whose boundary $\partial D$ composed a finite number of smooth surfaces.

In this part, we will prove Carleman's estimates in the general case of parabolic equations using another technique that is different from the first case. First, we recognise the parabolic equations in their general space. There are two types of parabolic equations :

$$\rho(x,t)\,\partial_t u(x,t) - \sum_{i,j=1}^{n} \partial_i\left(\widetilde{a}_{i,j}(x,t)\,\partial_j u(x,t)\right) - \sum_{k=1}^{n} \widetilde{b}_k(x,t)\,\partial_k u(x,t) - \widetilde{c}(x,t)\,u(x,t) = \widetilde{f}(x,t), \tag{2.8}$$

and

$$\partial_t u(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x,t)\,\partial_i\partial_j u(x,t) - \sum_{k=1}^{n} b_k(x,t)\,\partial_k u(x,t) - c(x,t)\,u(x,t) = f(x,t). \tag{2.9}$$

With $\rho(x,t) \in C^1\left(\overline{D}\right)$, $\rho > 0$ on $\overline{D}$ , and $b_k$, $\widetilde{b}_k$, $c$, $\widetilde{c} \in L^\infty(D)$ ,$1 \le k \le n$, with

$$\begin{cases} \widetilde{a}_{i,j} \in C^1\left(\overline{D}\right) & \widetilde{a}_{i,j} = \widetilde{a}_{j,i},\ 1 \le i,j \le n, \\ \sum_{i,j=1}^{n} \widetilde{a}_{i,j}(x,t)\,\xi_i\xi_j \ge \delta_1 \sum_{i=1}^{n} \xi_i^2 & (x,t) \in \overline{D}\ , \xi_1, \dots, \xi_n \in \mathbb{R}. \end{cases}$$

Let us set

$$Lu(x,t) = \partial_t u(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x,t)\,\partial_i\partial_j u(x,t) - \sum_{k=1}^{n} b_k(x,t)\,\partial_k u(x,t) - c(x,t)\,u(x,t) \quad \text{in } Q,$$

with

$$\begin{cases} a_{i,j} = \frac{\widetilde{a}_{i,j}}{\rho},\ a_{i,j} \in C^1\left(\overline{Q}\right) \quad a_{i,j} \in C^1\left(\overline{Q}\right),\ a_{i,j} = a_{j,i},\ 1 \le i,j \le n, \\ \sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \ge \sigma_1 \sum_{i=1}^{n} \xi_i^2,\ (x,t) \in \overline{Q}\ , \xi_1, \dots, \xi_n \in \mathbb{R},\ \sigma_1 > 0 \\ b_k = \frac{1}{\rho}\left(b_k + \sum_{i,j=1}^{n} \partial_i \widetilde{a}_{i,j}\right), \\ c = \frac{\widetilde{c}}{\rho}, \\ b_k, c \in C^1(Q), \end{cases} \tag{2.10}$$

and we set

$$L_0 u(x,t) = \partial_t u(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x,t)\,\partial_i\partial_j u(x,t).$$

**Theorem 2.1**

For $u \in H^{2,1}(Q)$, there exist three positive constants $C$, $\lambda$, and $s$ with $s \geq s_0$ such that $Lu = f$, and satisfies :

$$\int_D \left\{ \frac{1}{s\varphi} \left( |\partial_t u|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 \right) + s\lambda^2 \varphi |\nabla u|^2 + s^3 \lambda^4 \varphi^3 u^2 \right\} e^{2s\varphi} dx\, dt \leq C \int_D |Lu|^2 e^{2s\varphi} dx\, dt, \tag{2.11}$$

where $suppu \in D$, and verifies $(2.10)$.

**Proof.** To demonstrate the previous theorem, we first demonstrate the Carleman estimate for the operator $L_0$. Since $|L_0 u|^2 \leq 2|Lu|^2 + 2\sum_{k=1}^{n} |b_k \partial_k u + cu|^2$ in $Q$.

Let

$$L_0 u(x,t) = f \qquad \text{in } Q$$

with $(2.10)$.

We set

$$\partial_t u(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x,t)\, \partial_i \partial_j u(x,t) = f$$

$$\Longleftrightarrow \left[ \partial_t - \sum_{i,j=1}^{n} a_{i,j}(x,t)\, \partial_i \partial_j \right] u(x,t) = f. \tag{2.12}$$

We assume the weight function $\varphi(x,t)$ in the form of $e^{\lambda\psi}$, for $\lambda > 0$. It is a good form to guarantee the positivity of the coefficients of $|\omega|^2$and $|\nabla \omega|$ in estimating$\|P\omega\|^2_{L^2(D)}$.

On the other hand, let $d \in C^2(\overline{D})$ ,and $|\nabla d| \neq 0$ on $\overline{D}$, let us set

$$\psi(x,t) = d(x) - \beta(t-t_0)^2 - c_0 \tag{2.13}$$

with $t_0 \in (0,T)$, and $\beta$, $c_0 > 0$, such that

$$\varphi(x,t) = e^{\lambda\psi(x,t)}. \tag{2.14}$$

First, we assume that

$$u \in C_0^\infty(D).$$

We further set

$$\sigma(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t)\,(\partial_i d)(x)\,(\partial_j d)(x) \quad (x,t)\ \overline{Q}. \tag{2.15}$$

Multiplying the equation $(2.12)$ by $e^{s\varphi(x,t)}$, we have

$$\left[ \partial_t - \sum_{i,j=1}^{n} a_{i,j}(x,t)\, \partial_i \partial_j \right] e^{s\varphi(x,t)} u(x,t) = f e^{s\varphi(x,t)}.$$

We denote

$$\omega(x,t) = e^{s\varphi(x,t)}u(x,t)$$

$$\left[\partial_t - \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\right]\omega(x,t) = fe^{s\varphi(x,t)}. \tag{2.16}$$

By multiplying the equation $(2.16)$ respectively in $e^{s\varphi(x,t)}$ and $e^{-s\varphi(x,t)}$, we find

$$e^{-s\varphi(x,t)}\left[\partial_t - \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\right]e^{s\varphi(x,t)}\omega(x,t) = fe^{s\varphi(x,t)}.$$

We define an operator $P$ by

$$P\omega\left(x,t\right) = e^{s\varphi(x,t)}L_0\left(e^{-s\varphi(x,t)}\omega(x,t)\right). \tag{2.17}$$

Hence

$$P\omega\left(x,t\right) = fe^{s\varphi(x,t)}.$$

The operator $P$ has the form:

$$\begin{aligned} P\omega\left(x,t\right) &= \partial_t\omega - \sum_{i,j=1}^{n} a_{ij}\left(x,t\right)\partial_i\partial_j\omega + 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega \\ &\quad -s^2\lambda^2\varphi^2\sigma\omega + s\lambda^2\varphi\sigma\omega + s\lambda\varphi\omega\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d - s\lambda\varphi\omega\left(\partial_t\psi\right) \end{aligned}$$

From $(2.17)$, we have

$$\begin{aligned} P\omega\left(x,t\right) &= e^{s\varphi(x,t)}L_0\left(e^{-s\varphi(x,t)}\omega(x,t)\right) \\ &= e^{s\varphi(x,t)}\left[\partial_t - \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\right]\left(e^{-s\varphi(x,t)}\omega(x,t)\right) \\ &= e^{s\varphi(x,t)}\left[\partial_t\left(e^{-s\varphi(x,t)}\omega(x,t)\right) - \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\left(e^{-s\varphi(x,t)}\omega(x,t)\right)\right] \\ &= e^{s\varphi(x,t)}\left[I - J\right] \end{aligned}$$

with

$$I = \partial_t\left(e^{-s\varphi(x,t)}\omega(x,t)\right)$$

and

$$J = \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\left(e^{-s\varphi(x,t)}\omega(x,t)\right).$$

We simplify each part alone by integration by parts (see appendix **Theorem4.1**) and (2.13) and (2.15), we put

$$\begin{aligned} I &= \partial_t \left(e^{-s\varphi(x,t)}\omega(x,t)\right) \\ &= \omega\partial_t \left(e^{-s\varphi(x,t)}\right) + e^{-s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) \\ &= \omega\left(-se^{-s\varphi(x,t)}\partial_t\varphi\left(x,t\right)\right) + e^{-s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) \end{aligned}$$

$$\begin{aligned} &= e^{-s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) - s\omega e^{-s\varphi(x,t)}\left(\partial_t\varphi\left(x,t\right)\right) \\ &= e^{-s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) - s\omega e^{-s\varphi(x,t)}\left(\partial_t e^{\lambda\psi(x,t)}\right) \\ &= e^{-s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) - s\lambda\omega e^{-s\varphi(x,t)}\left(\partial_t\psi\left(x,t\right)e^{\lambda\psi(x,t)}\right) \\ I &= e^{s\varphi(x,t)}\partial_t \left(\omega\left(x,t\right)\right) - s\lambda\varphi\omega e^{-s\varphi(x,t)}\partial_t\psi\left(x,t\right), \end{aligned}$$

since

$$\partial_t e^{\lambda\psi(x,t)} = \lambda e^{\lambda\psi(x,t)}\partial_t\psi\left(x,t\right).$$

The second part

$$\begin{aligned} J &= \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[\partial_j\left(e^{-s\varphi(x,t)}\omega(x,t)\right)\right] \\ &= \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[\omega(x,t)\partial_j e^{-s\varphi(x,t)} + e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right] \\ &= \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[-s\lambda\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\left(x\right) + e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right] \end{aligned}$$

because

$$\begin{aligned} \partial_j e^{-s\varphi(x,t)} &= -se^{-s\varphi(x,t)}\partial_j\varphi \\ &= -se^{-s\varphi(x,t)}\partial_j e^{\lambda\psi(x,t)} \\ &= -s\lambda e^{-s\varphi(x,t)}e^{\lambda\psi(x,t)}\partial_j\psi\left(x,t\right) \\ &= -s\lambda e^{-s\varphi(x,t)}e^{\lambda\psi(x,t)}\partial_j d\left(x\right) \\ &= -s\lambda\varphi e^{-s\varphi(x,t)}\partial_j d\left(x\right). \end{aligned}$$

Then

$$\begin{aligned} J &= \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[-s\lambda\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\left(x\right) + e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right] \\ &= -s\lambda\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\left(x\right)\right] + \sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\left[e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right]. \end{aligned}$$

We denote

$$\begin{cases} X_1 = \omega(x,t)e^{-s\varphi(x,t)} \\ Y_1 = \varphi\partial_j d\,(x)\,. \end{cases}$$

Then

$$J = -s\lambda \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\,\partial_i\,[X_1Y_1] + \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\,\partial_i\left[e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right].$$

We simplify the first term of $J$ and find

$$\begin{aligned}
\partial_i\,[X_1Y_1] &= Y_1\partial_i X_1 + X_1\partial_i Y_1 \\
&= \varphi\partial_j d\,(x)\,\partial_i\left(\omega(x,t)e^{-s\varphi(x,t)}\right) + \omega(x,t)e^{-s\varphi(x,t)}\partial_i\left(\varphi\partial_j d\,(x)\right) \\
&= \varphi\partial_j d\,(x)\left[e^{-s\varphi(x,t)}\partial_i\omega(x,t) + \omega(x,t)\partial_i e^{-s\varphi(x,t)}\right] + \omega(x,t)e^{-s\varphi(x,t)} \\
&\quad \left[\varphi\partial_i\partial_j d\,(x) + \partial_j d\,(x)\,\partial_i\varphi\right] \\
&= \varphi\partial_j d\,(x)\left[e^{-s\varphi(x,t)}\partial_i\omega(x,t) - s\omega(x,t)e^{-s\varphi(x,t)}\partial_i\varphi\right] + \omega(x,t)e^{-s\varphi(x,t)} \\
&\quad \left[\varphi\partial_i\partial_j d\,(x) + \partial_j d\,(x)\,\partial_i e^{\lambda\psi(x,t)}\right]
\end{aligned}$$

$$\begin{aligned}
&= \varphi\partial_j d\,(x)\left[e^{-s\varphi(x,t)}\partial_i\omega(x,t) - s\lambda\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_i d\,(x)\right] + \omega(x,t)e^{-s\varphi(x,t)} \\
&\quad \left[\varphi\partial_i\partial_j d\,(x) + \lambda\partial_j d\,(x)\,\varphi\partial_i d\,(x)\right] \\
&= \varphi e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i\omega(x,t) - s\lambda\varphi^2\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \\
&\quad + \omega(x,t)\varphi e^{-s\varphi(x,t)}\partial_i\partial_j d\,(x) + \lambda\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x)\,.
\end{aligned}$$

Then

$$\begin{aligned}
J &= -s\lambda\sum_{i,j=1}^{n} a_{i,j}\,(x,t)\left\{\begin{array}{l} \varphi e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i\omega(x,t) - s\lambda\varphi^2\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \\ +\omega(x,t)\varphi e^{-s\varphi(x,t)}\partial_i\partial_j d\,(x) + \lambda\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \end{array}\right\} \\
&\quad + \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\,\partial_i\left[e^{-s\varphi(x,t)}\partial_j\omega(x,t)\right] \\
&= \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\left\{\begin{array}{l} -s\lambda\varphi e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i\omega(x,t) + s^2\lambda^2\varphi^2\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \\ -s\lambda\omega(x,t)\varphi e^{-s\varphi(x,t)}\partial_i\partial_j d\,(x) - s\lambda^2\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \end{array}\right\} \\
&\quad + \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\left[\partial_j\omega(x,t)\partial_i e^{-s\varphi(x,t)} + e^{-s\varphi(x,t)}\partial_i\partial_j\omega(x,t)\right] \\
&= \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\left\{\begin{array}{l} -s\lambda\varphi e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i\omega(x,t) + s^2\lambda^2\varphi^2\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \\ -s\lambda\omega(x,t)\varphi e^{-s\varphi(x,t)}\partial_i\partial_j d\,(x) - s\lambda^2\varphi\omega(x,t)e^{-s\varphi(x,t)}\partial_j d\,(x)\,\partial_i d\,(x) \end{array}\right\} \\
&\quad - \sum_{i,j=1}^{n} s\lambda a_{i,j}\,(x,t)\,\varphi e^{-s\varphi(x,t)}\partial_j\omega(x,t)\partial_i d\,(x) + \sum_{i,j=1}^{n} a_{i,j}\,(x,t)\,e^{-s\varphi(x,t)}\partial_i\partial_j\omega(x,t)
\end{aligned}$$

$$\begin{aligned} = & -s\lambda\varphi e^{-s\varphi(x,t)}\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i\omega(x,t) \\ & +s^2\lambda^2\varphi^2\omega(x,t)e^{-s\varphi(x,t)}\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i d\left(x\right) \\ & -s\lambda\omega(x,t)\varphi e^{-s\varphi(x,t)}\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j d\left(x\right) \end{aligned}$$

$$\begin{aligned} & -s\lambda^2\varphi\omega(x,t)e^{-s\varphi(x,t)}\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i d\left(x\right) \\ & -\sum_{i,j=1}^{n} s\lambda a_{i,j}\left(x,t\right)\varphi e^{-s\varphi(x,t)}\partial_j\omega(x,t)\partial_i d\left(x\right)+\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)e^{-s\varphi(x,t)}\partial_i\partial_j\omega(x,t). \end{aligned}$$

Therefore

$$\begin{aligned} P\omega\left(x,t\right) = & \ \partial_t\left(\omega\left(x,t\right)\right)-s\lambda\varphi\omega\partial_t\psi\left(x,t\right)+s\lambda\varphi\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i\omega(x,t) \\ & -s^2\lambda^2\varphi^2\omega(x,t)\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i d\left(x\right)+s\lambda\omega(x,t)\varphi\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j d\left(x\right) \\ & +s\lambda^2\varphi\omega(x,t)\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j d\left(x\right)\partial_i d\left(x\right)+s\lambda\varphi\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_j\omega(x,t)\partial_i d\left(x\right) \\ & -\sum_{i,j=1}^{n} a_{i,j}\left(x,t\right)\partial_i\partial_j\omega(x,t). \end{aligned}$$

Note that $\partial_j\omega(x,t)\partial_i d\left(x\right)=\partial_j d\left(x\right)\partial_i\omega(x,t)$ with (2.14), we have

$$\begin{aligned} P\omega\left(x,t\right) = & \ \partial_t\omega-\sum_{i,j=1}^{n} a_{ij}\left(x,t\right)\partial_i\partial_j\omega+2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega \\ & -s^2\lambda^2\varphi^2\sigma\omega+s\lambda^2\varphi\sigma\omega+s\lambda\varphi\omega\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d-s\lambda\varphi\omega\left(\partial_t\psi\right). \end{aligned}$$

As we said at the beginning, we will use a method different from the first case, but we will not overlook the similarity that is in the decomposition of operator $P$, but the latter is not the same and not the same transactions, but the decomposition is $P_1$ and $P_2$, where $P_1$ is composed of second-order and zeroth-order terms in $x$, and $P_2$ comprises first-order terms in $t$ and first-order terms in $x$. Then, we calculate the norm in space$L^2(D)$ and find an estimate for the following workers $\int_D\left(\left|P_2\omega\right|^2+2\left(P_1\omega\right)\left(P_2\omega\right)\right)dx\ dt$.

Therefore, we have

$$\int_D f^2 e^{2s\varphi(x,t)} dx\ dt = \int_D |P\omega(x,t)|^2\, dx\ dt.$$

First, we find estimation for $\|P\omega(x,t)\|^2_{L^2(D)}$by decomposition the operator $P$ into the parts $P_1$ and $P_2$with

$$\begin{cases} P_1 = -\sum_{i,j=1}^n a_{ij}(x,t)\,\partial_i\partial_j\omega - s^2\lambda^2\varphi^2\sigma\omega + A\omega \\ P_2 = \partial_t\omega + 2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_i d)\,\partial_j\omega \end{cases}$$

and $A = s\lambda^2\varphi\sigma\omega + s\lambda\varphi\omega\sum_{i,j=1}^n a_{ij}\partial_i\partial_j d - s\lambda\varphi\omega(\partial_t\psi) \equiv s\lambda^2\varphi a(x,t;s,\lambda)$ with $|a(x,t;s,\lambda)| \le C$ , then

$$\int_D f^2 e^{2s\varphi(x,t)} dx\ dt = \|P_1\omega + P_2\omega\|^2_{L^2(D)},$$

and we have

$$\int_D \left(|P_2\omega|^2 + 2(P_1\omega)(P_2\omega)\right) dx\ dt \le \int_D f^2 e^{2s\varphi(x,t)} dx\ dt. \tag{2.18}$$

Also, we have

$$\begin{aligned} \int_D |P_2\omega|^2\, dx\ dt &= \int_D \left|\partial_t\omega + 2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_j d)\,\partial_i\omega\right|^2 dx\ dt \\ &\ge \int_D \frac{1}{s\varphi}\left|\partial_t\omega + 2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_j d)\,\partial_i\omega\right|^2 dx\ dt \end{aligned}$$

by inequality$[A.2]$ , and for any $\varepsilon > 0$,we obtain

$$\begin{aligned} \int_D |P_2\omega|^2\, dx\ dt &\ge \int_D \frac{1}{s\varphi}|\partial_t\omega|^2\, dx\ dt - 4\int_D s\lambda^2\varphi\left|\sum_{i,j=1}^n a_{ij}(\partial_j d)\,\partial_i\omega\right|^2 dx\ dt \\ &\ge \varepsilon\int_D \frac{1}{s\varphi}|\partial_t\omega|^2\, dx\ dt - C_1\varepsilon\int_D s\lambda^2\varphi\,|\nabla\omega|^2\, dx\ dt, \end{aligned}$$

then

$$\varepsilon\int_D \frac{1}{s\varphi}|\partial_t\omega|^2\, dx\ dt \le C_2\int_D |P_2\omega|^2\, dx\ dt + C_1\varepsilon\int_D s\lambda^2\varphi\,|\nabla\omega|^2\, dx\ dt. \tag{2.19}$$

Moreover, for all large $s > 0$, we can give an estimation for $2\int_D (P_1\omega)(P_2\omega)\, dx\ dt$ with

$$2\,(P_1\omega, P_2\omega)_{L^2(D)} = 2\int_D (P_1\omega)(P_2\omega)\, dx\ dt.$$

We simplify this part $(P_1\omega)(P_2\omega)$, we have

$$\begin{aligned}(P_1\omega)(P_2\omega) &= -\sum_{i,j=1}^{n} a_{ij}(x,t)\,\partial_i\partial_j\omega\partial_t\omega - s^2\lambda^2\varphi^2\sigma\omega(\partial_t\omega) + A\omega(\partial_t\omega)\\ &\quad -2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}(x,t)\,\partial_i\partial_j\omega\sum_{k,l=1}^{n} a_{kl}(\partial_k d)\,\partial_l\omega\\ &\quad -2s^3\lambda^3\varphi^3\sigma\omega\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\,\partial_j\omega + 2s\lambda\varphi A\omega\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\,\partial_j\omega.\end{aligned}$$

After entering the integration, we find

$$\begin{aligned}\int_D (P_1\omega)(P_2\omega)\,dx\,dt &= -\sum_{i,j=1}^{n}\int_D a_{ij}(x,t)(\partial_i\partial_j\omega)(\partial_t\omega)\,dx\,dt\\ &\quad -\sum_{i,j=1}^{n}\int_D a_{ij}(x,t)\,\partial_i\partial_j\omega\;2s\lambda\varphi\sum_{k,l=1}^{n} a_{kl}(\partial_k d)\,\partial_l\omega dx\,dt\\ &\quad -\int_D s^2\lambda^2\varphi^2\sigma\omega(\partial_t\omega)\,dx\,dt - \int_D 2s^3\lambda^3\varphi^3\sigma\omega\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\,\partial_j\omega dx\,dt\\ &\quad +\int_D (A\omega)(\partial_t\omega)\,dx\,dt + \int_D (A\omega)\,2s\lambda\varphi\\ &\quad \sum_{i,j=1}^{n} a_{ij}(\partial_i d)\,\partial_j\omega dx\,dt\\ &\equiv \sum_{k=1}^{6} I_k.\end{aligned}$$

Now, applying the integration by parts, with $u \in C_0^\infty(D)$, $a_{i,j} = a_{j,\,i}$, and assuming that $\lambda > 1$ and $s > 1$ are sufficiently large, we reduce all the derivatives of $\omega$ to $\omega$, $\partial_i\omega$, $\partial_t\omega$. We obtain the estimation of $I_k$, $k = 1, ..., 6$.

The first term

$$\begin{aligned}|I_1| &= \left|-\sum_{i,j=1}^{n}\int_D a_{ij}(x,t)(\partial_i\partial_j\omega)(\partial_t\omega)\,dx\,dt\right|\\ &= \left|\sum_{i,j=1}^{n}\int_D [\partial_i a_{ij}(x,t)(\partial_t\omega) + a_{ij}(x,t)\,\partial_i(\partial_t\omega)]\,\partial_j\omega dx\,dt\right|\\ &= \left|\sum_{i,j=1}^{n}\int_D \partial_i a_{ij}(x,t)(\partial_t\omega)\,\partial_j\omega dx\,dt + \sum_{i,j=1}^{n}\int_D a_{ij}(x,t)(\partial_i\partial_t\omega)\,\partial_j\omega dx\,dt\right|\end{aligned}$$

with $i = j$ and $i > j$, we have

$$I_1 = \left| \begin{array}{c} \sum_{i,j=1}^{n} \int_D \partial_i a_{ij}(x,t)(\partial_t \omega)\partial_j \omega dx\, dt \\ + \left( \begin{array}{c} \sum_{i=j=1}^{n} \int_D a_{ii}(x,t)(\partial_i \partial_t \omega)(\partial_i \omega)\, dx\, dt \\ + \sum_{i>j=1}^{n} \int_D a_{ij}(x,t)\left[\partial_j \omega (\partial_i \partial_t \omega) + (\partial_i \omega)(\partial_j \partial_t \omega)\right] dx\, dt \end{array} \right) \end{array} \right|$$

$$\leq \left| \sum_{i,j=1}^{n} \int_D \partial_i a_{ij}(x,t)(\partial_t \omega)\partial_j \omega dx\, dt \right| + \frac{1}{2} \left| \sum_{i,j=1}^{n} \int_D (\partial_t a_{ij}(x,t))(\partial_i \omega)(\partial_j \omega)\, dx\, dt \right|.$$

Here, we used

$$\left( \sum_{i=j=1}^{n} \int_D a_{ii}(x,t)(\partial_i \partial_t \omega)(\partial_i \omega)\, dx\, dt + \sum_{i>j=1}^{n} \int_D a_{ij}(x,t)\left[\partial_j \omega (\partial_i \partial_t \omega) + (\partial_i \omega)(\partial_j \partial_t \omega)\right] dx\, dt \right)$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} \int_D (\partial_t a_{ij}(x,t))(\partial_i \omega)(\partial_j \omega)\, dx\, dt.$$

Then

$$I_1 \leq C_3 \int_D |\nabla \omega| |\partial_t \omega|\, dx\, dt + C_4 \int_D |\nabla \omega|^2\, dx\, dt. \tag{2.20}$$

The second term

$$\begin{aligned} I_2 &= -\sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \int_D 2s\lambda \varphi a_{ij}(x,t) a_{kl}(x,t)(\partial_k d)(\partial_l \omega)(\partial_i \partial_j \omega)\, dx\, dt \\ &= -2s\lambda\varphi \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \int_D a_{ij}(x,t) a_{kl}(x,t)(\partial_k d)(\partial_l \omega)(\partial_i \partial_j \omega)\, dx\, dt \end{aligned}$$

$$\begin{aligned} &= 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \lambda(\partial_i d)\varphi a_{ij} a_{kl}(\partial_k d)(\partial_l \omega)(\partial_j \omega)\, dx\, dt \\ &\quad + 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \varphi \partial_i (a_{ij} a_{kl}(\partial_k d))(\partial_l \omega)(\partial_j \omega)\, dx\, dt \\ &\quad + 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} (\partial_i \partial_l \omega)(\varphi a_{ij} a_{kl}(\partial_k d))(\partial_j \omega)\, dx\, dt. \end{aligned}$$

We have

$$\begin{aligned} & 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \lambda(\partial_i d)\varphi a_{ij} a_{kl}(\partial_k d)(\partial_l \omega)(\partial_j \omega)\, dx\, dt \\ &= 2s\lambda^2 \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \varphi (a_{ij} a_{kl} \partial_i d \partial_k d)(\partial_l \partial_j \omega)\, dx\, dt \end{aligned}$$

$$
\begin{aligned}
&= 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^{n} a_{ij}\partial_i d \partial_j \omega \right|^2 dx\ dt \qquad ; i,j = k,l \text{ resp.} \\
&= 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^{n} a_{ij}\partial_i d \partial_j \omega \right|^2 dx\ dt \qquad \geq 0
\end{aligned}
$$

where $s > 0$, $\lambda > 0$.

Similar to $I_1$, we can estimate

$$
\begin{aligned}
& 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} (\partial_i \partial_l \omega) (\varphi a_{ij} a_{kl} (\partial_k d)) (\partial_j \omega)\, dx\ dt \\
= {}& 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ij} a_{kl} (\partial_k d)\, \varphi \partial_l (\partial_i \omega) (\partial_j \omega)\, dx\ dt \\
= {}& s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ij} a_{kl} \varphi (\partial_k d)\, \partial_l (\partial_i \omega) (\partial_j \omega)\, dx\ dt + s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \varphi (a_{ij} a_{kl} \partial_k d)\, \partial_l (\partial_i \omega \partial_j \omega)\, dx\ dt.
\end{aligned}
$$

We put

$$
\begin{cases}
X_2 = s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ij} a_{kl} \varphi (\partial_k d)\, \partial_l (\partial_i \omega) (\partial_j \omega)\, dx\ dt \\
Y_2 = s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \varphi (a_{ij} a_{kl} \partial_k d)\, \partial_l (\partial_i \omega \partial_j \omega)\, dx\ dt.
\end{cases}
$$

Using the integration by party here too, we find

$$
\begin{aligned}
X_2 &= s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ij} a_{kl} \varphi (\partial_k d)\, \partial_l (\partial_i \omega) (\partial_j \omega)\, dx\ dt \\
&= -s\lambda^2 \int_D \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ij} a_{kl} \varphi (\partial_l d) (\partial_k d) (\partial_i \omega \partial_j \omega)\, dx\ dt \\
&= -s\lambda^2 \int_D \sum_{i,j=1}^{n} a_{ij} \varphi \left[ \sum_{k,l=1}^{n} a_{kl} (\partial_l d)(\partial_k d) \right] (\partial_i \omega \partial_j \omega)\, dx\ dt \\
&= -s\lambda^2 \int_D \sum_{i,j=1}^{n} a_{ij} \varphi \sigma \partial_i \omega \partial_j \omega dx\ dt \\
&= -s\lambda^2 \int_D \varphi \sigma \sum_{i,j=1}^{n} a_{ij} \partial_i \omega \partial_j \omega dx\ dt.
\end{aligned}
$$

And

$$\begin{aligned}
Y_2 &= s\lambda \int_D \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \varphi\left(a_{ij}a_{kl}\partial_k d\right)\partial_l\left(\partial_i\omega\partial_j\omega\right)dx\ dt\\
&= -s\lambda \int_D \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \varphi\partial_l\left(a_{ij}a_{kl}\partial_k d\right)\left(\partial_i\omega\partial_j\omega\right)dx\ dt\\
&= -s\lambda \int_D \varphi\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \partial_l\left(a_{ij}a_{kl}\partial_k d\right)\left(\partial_i\omega\partial_j\omega\right)dx\ dt.
\end{aligned}$$

Then

$$X_2+Y_2 = -s\lambda^2\int_D \varphi\sigma\sum_{i,j=1}^{n} a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right)dx\ dt - s\lambda\int_D \varphi\sum_{i,j=1}^{n}\sum_{k,l=1}^{n}\partial_l\left(a_{ij}a_{kl}\partial_k d\right)\left(\partial_i\omega\partial_j\omega\right)dx\ dt.$$

Hence

$$\begin{aligned}
I_2 &= 2s\lambda^2\int_D\varphi\left|\sum_{i,j=1}^{n}a_{ij}\partial_i d\partial_j\omega\right|^2 dx\ dt \ + 2s\lambda\int_D\sum_{i,j=1}^{n}\sum_{k,l=1}^{n}\varphi\partial_i\left(a_{ij}a_{kl}\left(\partial_k d\right)\right)\left(\partial_l\omega\right)\left(\partial_j\omega\right)dx\\
&\quad -s\lambda^2\int_D\varphi\sigma\sum_{i,j=1}^{n}a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right)dx\ dt - s\lambda\int_D\varphi\sum_{i,j=1}^{n}\sum_{k,l=1}^{n}\partial_l\left(a_{ij}a_{kl}\partial_k d\right)\left(\partial_i\omega\partial_j\omega\right)dx\ dt\\
&\geq -s\lambda^2\int_D\varphi\sigma\sum_{i,j=1}^{n}a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right)dx\ dt + 2s\lambda^2\int_D\varphi\left|\sum_{i,j=1}^{n}a_{ij}\partial_i d\partial_j\omega\right|^2 dx\ dt\\
&\quad -C_4\int_D s\lambda\varphi\left|\nabla\omega\right|^2 dx\ dt.
\end{aligned}$$

$$I_2 \geq -C_5 s\lambda^2\int_D\varphi\sigma\sum_{i,j=1}^{n}a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right)dx\ dt - C_4\int_D s\lambda\varphi\left|\nabla\omega\right|^2 dx\ dt. \tag{2.21}$$

The third term

$$\begin{aligned}
|I_3| &= \left|-\int_D s^2\lambda^2\varphi^2\sigma\omega\left(\partial_t\omega\right)dx\ dt\right|\\
&= \left|-\int_D \frac{1}{2}s^2\lambda^2\varphi^2\sigma\partial_t\left(\omega^2\right)dx\ dt\right|\\
&= \left|-\int_D s^2\lambda^2\sigma\left(\lambda\varphi^2\partial_t\left(\psi\right)\right)\omega^2 dx\ dt + \frac{1}{2}\int_D s^2\lambda^2\varphi^2\left(\partial_t\sigma\right)\omega^2 dx\ dt\right|
\end{aligned}$$

$$|I_3| \leq C_6\int_D s^2\lambda^3\varphi^2\omega^2 dx\ dt. \tag{2.22}$$

The fourth term

$$\begin{aligned}
I_4 &= -\int_D 2s^3\lambda^3\varphi^3\sigma\omega \sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega dx\, dt \\
&= -\int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^{n} a_{ij}\sigma\left(\partial_i d\right)\partial_j\left(\omega^2\right) dx\, dt \\
&= -\int_D s^3\lambda^3 \sum_{i,j=1}^{n} \varphi^3\partial_j\left(\omega^2\right) a_{ij}\sigma\left(\partial_i d\right) dx\, dt \\
&= \int_D s^3\lambda^3 \sum_{i,j=1}^{n} 3\varphi^2\left\{\lambda\left(\partial_j d\right)\varphi\right\} a_{ij}\sigma\left(\partial_i d\right)\omega^2 dx\, dt + \int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^{n} \partial_j\left(a_{ij}\sigma\left(\partial_i d\right)\right)\omega^2 dx\, dt
\end{aligned}$$

$$I_4 \geq \int_D 3s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\, dt - C_7\int_D s^3\lambda^3\varphi^3\omega^2 dx\, dt. \tag{2.23}$$

The fifth term

$$\begin{aligned}
|I_5| &= \left|\int_D (A\omega)\left(\partial_t\omega\right) dx\, dt\right| \\
&= \left|\int_D s\lambda^2\varphi a\omega\left(\partial_t\omega\right) dx\, dt\right| \\
&= \left|\frac{1}{2}\int_D s\lambda^2\varphi a\left(\partial_t\left(\omega^2\right)\right) dx\, dt\right| \\
&= \frac{1}{2}\left|\int_D s\lambda^3\varphi\left(\partial_t\psi\right) a\omega^2 dx\, dt + \int_D s\lambda^2\varphi\left(\partial_t a\right)\omega^2 dx\, dt\right|
\end{aligned}$$

$$|I_5| \leq C_8\int_D s\lambda^3\varphi\omega^2 dx\, dt. \tag{2.24}$$

The last term

$$\begin{aligned}
|I_6| &= \left|\int_D (A\omega)\, 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega dx\, dt\right| \\
&= \left|\int_D s\lambda^2\varphi a \times 2s\lambda\varphi\omega \sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega dx\, dt\right| \\
&= \left|\int_D 2as^2\lambda^3\varphi^2 \sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\omega\left(\partial_j\omega\right) dx\, dt\right| \\
&= \left|\int_D as^2\lambda^3\varphi^2 \sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\left(\omega^2\right) dx\, dt\right| \\
&= \left|\int_D -\sum_{i,j=1}^{n} \partial_j\left(as^2\lambda^3\varphi^2 a_{ij}\left(\partial_i d\right)\right)\omega^2 dx\, dt\right|
\end{aligned}$$

$$|I_6| \leq C_9 \int_D s^2\lambda^4\varphi^2\omega^2 dx\, dt. \tag{2.25}$$

Hence, by $(2.20) - (2.25)$, we obtain

$$\begin{aligned}
\int_D (P_1\omega)(P_2\omega)\, dx\, dt \geq\ & 3\int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\, dt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i\omega)(\partial_j\omega)\, dx\, dt \\
& -C_4\int_D s\lambda\varphi\, |\nabla\omega|^2\, dx\, dt - C_{10}\int_D \left(s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2\right)\omega^2 dx\, dt \\
& -C_3\int_D |\nabla\omega|\, |\partial_t\omega|\, dx\, dt.
\end{aligned} \tag{2.26}$$

**Remark 2.2** *To find an appropriate estimate, we use only our factories of* $\omega^2$ *and* $|\nabla\omega|^2$*. The rest is unimportant.*

Consequently, from $(2.26)$, we get

$$\begin{aligned}
& 3\int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\, dt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i\omega)(\partial_j\omega)\, dx\, dt \\
\leq\ & \int_D (P_1\omega)(P_2\omega)\, dx\, dt + C_4\int_D s\lambda\varphi\, |\nabla\omega|^2\, dx\, dt \\
& +C_{10}\int_D \left(s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2\right)\omega^2 dx\, dt + C_3\int_D |\nabla\omega|\, |\partial_t\omega|\, dx\, dt.
\end{aligned} \tag{2.27}$$

Hence by $(2.18) - (2.19)$, we have for all $\varepsilon > 0$

$$\begin{aligned}
& 3\int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\, dt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i\omega)(\partial_j\omega)\, dx\, dt + \varepsilon\int_D \frac{1}{s\varphi}|\partial_t\omega|^2\, dx\, dt \\
\leq\ & C_{11}\int_D f^2 e^{2s\varphi(x,t)} dx\, dt + C_4\int_D s\lambda\varphi\, |\nabla\omega|^2\, dx\, dt + C_1\varepsilon\int_D s\lambda^2\varphi\, |\nabla\omega|^2\, dx\, dt \\
& +C_{10}\int_D \left(s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2\right)\omega^2 dx\, dt + C_3\int_D |\nabla\omega|\, |\partial_t\omega|\, dx\, dt.
\end{aligned} \tag{2.28}$$

As we said earlier in the note, we will take only $\omega^2$ and $|\nabla\omega|^2$ transactions plus $|\partial_t\omega|^2$, we also must take the maximal order in $s, \lambda, \varphi$.

Here, since the Cauchy–Schwarz inequality (see appendix **Proposition4.1**) implies that

$$\begin{aligned}
|\nabla\omega|\, |\partial_t\omega| &= \frac{1}{(s\lambda\varphi)^{\frac{1}{2}}}|\partial_t\omega|\,(s\lambda\varphi)^{\frac{1}{2}}|\nabla\omega| \\
&\leq \frac{1}{2}\left(\frac{1}{(s\lambda\varphi)^{\frac{1}{2}}}\right)^2 |\partial_t\omega|^2 + \frac{1}{2}\left((s\lambda\varphi)^{\frac{1}{2}}\right)^2 |\nabla\omega|^2 \\
&\leq \frac{1}{2}\frac{1}{s\lambda\varphi}|\partial_t\omega|^2 + \frac{1}{2}s\lambda\varphi\, |\nabla\omega|^2.
\end{aligned}$$

Then (2.28) becomes

$$\begin{aligned}
&\int_D 3s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\ dt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right) dx\ dt \\
&+\left(\varepsilon - \frac{C_{12}}{\lambda}\right)\int_D \frac{1}{s\varphi}\left|\partial_t\omega\right|^2 dx\ dt \\
\leq\ & C_{11}\int_D f^2 e^{2s\varphi(x,t)} dx\ dt + C_{12}\int_D s\lambda\varphi\left|\nabla\omega\right|^2 dx\ dt + C_1\varepsilon\int_D s\lambda^2\varphi\left|\nabla\omega\right|^2 dx\ dt \\
&+C_{10}\int_D \left(s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2\right)\omega^2 dx\ dt.
\end{aligned} \tag{2.29}$$

We will find other estimates of the first and second terms. Thus, we will execute

$$\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}\left(\partial_i\omega\right)\left(\partial_j\omega\right) dx\ dt$$

by using of

$$\int_D \left(P_1\omega + P_2\omega\right)\times\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt.$$

We selected this factor $\left(s\lambda^2\varphi\sigma\omega\right)$ to get $\left|\nabla\omega\right|^2$. We multiply this equation

$$\partial_t\omega + 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\partial_j\omega - \sum_{i,j=1}^{n} a_{ij}\left(x,t\right)\partial_i\partial_j\omega - s^2\lambda^2\varphi^2\sigma\omega + A\omega = fe^{s\varphi}$$

by $\left(s\lambda^2\varphi\sigma\omega\right)$, we have

$$\begin{aligned}
&\int_D \partial_t\omega\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt + \int_D 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\left(\partial_i d\right)\left(\partial_j\omega\right)\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt \\
&-\int_D\left(\sum_{i,j=1}^{n} a_{ij}\left(x,t\right)\partial_i\partial_j\omega\right) s\lambda^2\varphi\sigma\omega dx\ dt - \int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\ dt \\
&+\int_D\left(A\omega\right)\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt \\
\equiv\ & \sum_{k=1}^{5} J_k = \int_D fe^{s\varphi}\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt.
\end{aligned} \tag{2.30}$$

Now from $\omega \in C_0^2\left(D\right)$, and integration by parts (see appendix **Theorem 4.1**). Noting that $\left|\partial_t\varphi\right| = \left|\lambda\left(\partial_t\psi\right)\varphi\right| \leq C_{13}\lambda\varphi$ and $\partial_i\varphi = \lambda\left(\partial_i d\right)\varphi$, we find estimates of terms, respectively.

$$\left|J_1\right| = \left|\int_D \partial_t\omega\left(s\lambda^2\varphi\sigma\omega\right) dx\ dt\right|$$

$$
\begin{aligned}
&= \left| \int_D \frac{1}{2} s\lambda^2 \varphi\sigma \partial_t \left(\omega^2\right) dx\, dt \right| \\
&= \left| \int_D \frac{1}{2} s\lambda^3 \varphi\sigma \left(\partial_t \psi\right) \omega^2 dx\, dt \right| \\
&\le C_{14} \int_D s\lambda^3 \varphi \omega^2 dx\, dt. \qquad (2.31)
\end{aligned}
$$

And

$$
\begin{aligned}
|J_2| &= \left| \int_D 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij} \left(\partial_i d\right)\left(\partial_j \omega\right)\left(s\lambda^2 \varphi\sigma\omega\right) dx\, dt \right| \\
&= \left| \int_D 2s\lambda\varphi \left(s\lambda^2 \varphi\sigma\omega\right) \sum_{i,j=1}^{n} a_{ij} \left(\partial_i d\right)\left(\partial_j \omega\right) dx\, dt \right| \\
&= \left| \int_D s^2\lambda^3 \varphi^2 \sigma \sum_{i,j=1}^{n} a_{ij} \left(\partial_i d\right) \partial_j \left(\omega^2\right) dx\, dt \right| \\
&= \left| - \int_D \sum_{i,j=1}^{n} s^2\lambda^3\varphi^2 \partial_j \left\{\sigma a_{ij} \left(\partial_i d\right)\right\} \omega^2 dx\, dt - \int_D \sum_{i,j=1}^{n} s^2\lambda^3 \left\{2\lambda \left(\partial_i d\right) \varphi^2\right\} \sigma a_{ij} \left(\partial_i d\right) \omega^2 dx\, dt \right| \\
&\le C_{15} \int_D s^2\lambda^4\varphi^2\omega^2 dx\, dt. \qquad (2.32)
\end{aligned}
$$

And

$$
\begin{aligned}
J_3 &= -\int_D s\lambda^2 \varphi\sigma\omega \int_D \sum_{i,j=1}^{n} a_{ij}(x,t)\, \partial_i \partial_j \omega dx\, dt \\
&= -\int_D s\lambda^2 \int_D \sum_{i,j=1}^{n} \left[a_{ij}(x,t)\, \varphi\sigma\omega\right] \partial_i \partial_j \omega dx\, dt \\
&= \int_D s\lambda^2 \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x,t)\, \varphi\sigma\right) \omega \partial_j \omega dx\, dt + \int_D s\lambda^2 \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\, \varphi\sigma \partial_i \omega\right) \partial_j \omega dx\, dt \\
&\ge \int_D s\lambda^2 \varphi\sigma \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\, \partial_i \omega\right) \partial_j \omega dx\, dt - C_{16} \int_D s\lambda^3 \varphi \left|\nabla\omega\right| \left|\omega\right| dx\, dt. \qquad (2.33)
\end{aligned}
$$

Next

$$
J_4 = -\int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\, dt. \qquad (2.34)
$$

And the last

$$
\begin{aligned}
|J_5| &= \left| \int_D \left(A\omega\right)\left(s\lambda^2\varphi\sigma\omega\right) dx\, dt \right| \\
&= \left| \left(s\lambda^2\varphi a\omega\right)\left(s\lambda^2\varphi\sigma\omega\right) dx\, dt \right| \\
&= \left| s^2\lambda^4\varphi^2\sigma\omega^2 a dx\, dt \right|
\end{aligned}
$$

$$\leq C_{17}\int_D s^2\lambda^4\varphi^2\omega^2 dx\ dt. \tag{2.35}$$

Hence, by$(2.31)-(2.35)$, we obtain

$$\begin{aligned}
&\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^{n}\left(a_{ij}\left(x,t\right)\partial_i\omega\right)\partial_j\omega dx\ dt-\int_D s^3\lambda^4\varphi^3\sigma^2\omega^2 dx\ dt\\
\leq\ & C_{18}\int_D\left|fe^{s\varphi}\left(s\lambda^2\varphi\sigma\omega\right)\right|dx\ dt+C_{19}\int_D s^2\lambda^4\varphi^2\omega^2 dx\ dt+C_{16}\int_D s\lambda^3\varphi\left|\nabla\omega\right|\left|\omega\right|dx\ dt\\
&\leq C_{18}\int_D f^2e^{2s\varphi}dx\ dt+C_{20}\int_D s^2\lambda^4\varphi^2\omega^2 dx\ dt+C_{21}\int_D\lambda^2\left|\nabla\omega\right|^2 dx\ dt.
\end{aligned}\tag{2.36}$$

In the last inequality (see appendix **Proposition**$4.$**1**), we argue as follows by

$$\begin{aligned}
s\lambda^3\varphi\left|\nabla\omega\right|\left|\omega\right| &= s\lambda^2\varphi\left|\omega\right|\lambda\left|\nabla\omega\right|\\
&\leq \frac{1}{2}s^2\lambda^4\varphi^2\left|\omega\right|^2+\frac{1}{2}\lambda^2\left|\nabla\omega\right|^2
\end{aligned}$$

we get

$$\int_D s\lambda^3\varphi\left|\nabla\omega\right|\left|\omega\right|dx\ dt\leq\frac{1}{2}\int_D s^2\lambda^4\varphi^2\left|\omega\right|^2+\lambda^2\left|\nabla\omega\right|^2.$$

Furthermore

$$\begin{aligned}
\left|fe^{s\varphi}\left(s\lambda^2\varphi\sigma\omega\right)\right| &\leq \frac{1}{2}f^2e^{2s\varphi}dx\ dt+\frac{1}{2}s^2\lambda^4\varphi^2\omega^2\\
&\leq \frac{1}{2}f^2e^{2s\varphi}+C_{19}s^2\lambda^4\varphi^2\omega^2.
\end{aligned}$$

Finally, we consider $2\times(2.36)+(2.29)$. Using $\left(\sum_{i,j=1}^{n}a_{i,j}\xi_i\xi_j\geq\sigma_1\sum_{i=1}^{n}\xi_i^2\right)$ and $\sigma_0\equiv\inf_{(x,t)\in Q}\sigma\left(x,t\right)>0$, we obtain

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3\sigma_0^2\omega^2 dx\ dt-\left(\sigma_0\sigma_1-C_{22}\varepsilon\right)\int_D s\lambda^2\varphi\left|\nabla\omega\right|^2 dx\ dt\\
&+\left(\varepsilon-\frac{C_{12}}{\lambda}\right)\int_D\frac{1}{s\varphi}\left|\partial_t\omega\right|^2 dx\ dt+\left(\varepsilon-\frac{C_{12}}{\lambda}\right)\int_D\frac{1}{s\varphi}\left|\partial_t\omega\right|^2 dx\ dt\\
\leq\ & C_{18}\int_D f^2e^{2s\varphi}dx\ dt+C_{10}\int_D\left(s^3\lambda^3\varphi^3+s^2\lambda^4\varphi^2\right)\omega^2 dx\ dt\\
&+C_{23}\int_D\left(s\lambda\varphi+\lambda^2\right)\left|\nabla\omega\right|^2 dx\ dt
\end{aligned}\tag{2.37}$$

Therefore, first choosing $\varepsilon>0$ sufficiently small such that $\sigma_0\sigma_1-C_{22}\varepsilon>0$ and then taking $\lambda>0$ sufficiently large such that $\varepsilon-\frac{C_{12}}{\lambda}>0$.

By simplifying certain terms of $(2.37)$, we get

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3\omega^2 dx\ dt+\int_D s\lambda^2\varphi\left|\nabla\omega\right|^2 dx\ dt\\
+\int_D\frac{1}{s\varphi}\left|\partial_t\omega\right|^2 dx\ dt\ &\leq\ C_{24}\int_D f^2e^{2s\varphi}dx\ dt.
\end{aligned}\tag{2.38}$$

Replacing $\omega = ue^{s\varphi}$, we have

$$\int_D \left( \frac{1}{s\varphi} |\partial_t u|^2 + s\lambda^2\varphi |\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right) e^{2s\varphi} dx\ dt \leq C_{24} \int_D f^2 e^{2s\varphi} dx\ dt. \tag{2.39}$$

Finally, we obtain an estimation of operator $L_0$ with

$$\int_D \left( \frac{1}{s\varphi} |\partial_t u|^2 + s\lambda^2\varphi |\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right) e^{2s\varphi} dx\ dt \leq C_{24} \int_D f^2 e^{2s\varphi} dx\ dt.$$

Moreover, we have

$$\left| \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j \omega \right|^2 \leq C_{25} \left( |\partial_t \omega|^2 + s^2\lambda^2\varphi^2 |\nabla \omega|^2 + s^4\lambda^4\varphi^4 \omega^2 + |f|^2 e^{s\varphi} \right) \qquad \text{in } Q.$$

Hence, by (2.39) we find

$$\begin{aligned} \int_D \frac{1}{s\varphi} \left| \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j \omega \right|^2 dx\ dt \ \ \leq \ \ & C_{25} \int_D \left( \frac{1}{s\varphi} |\partial_t \omega|^2 + s\lambda^2\varphi |\nabla \omega|^2 + s^3\lambda^3\varphi^3 \omega^2 \right) dx\ dt \\ & + C_{26} \int_D f^2 e^{s\varphi} dx\ dt. \end{aligned} \tag{2.40}$$

for all large $s > 0$ and $\lambda > 0$.

In the end, the estimation of $L$ is

$$\int_D \left\{ \frac{1}{s\varphi} \left( |\partial_t u|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 \right) + s\lambda^2\varphi |\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dx\ dt \leq C \int_D |Lu|^2 e^{2s\varphi} dx\ dt.$$

where $s$ is a large parameter. ■

# Chapter 3

# Applications in the controllability of parabolic PDEs

This chapter is devoted to the prove the null controllablility of some parabolic equations, this is related to create observation inequality using the Carleman estimates already presented in chapter 2.

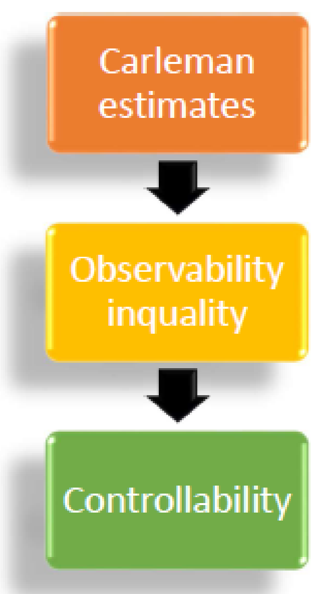


FIGURE 4: Objective of the chapter

## 3.1 Null controllability of linear heat equation with mixed boundary conditions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded spatial domain with smooth boundary $\Gamma$ of class $C^2$ and let $T > 0$. We denote $Q = \Omega \times ]0,\ T[$, and $\Sigma = \Gamma \times ]0,\ T[$ where $\Sigma_1$ is a piece of the boundary $\Sigma$ and $\Sigma_2 = \Sigma \backslash \Sigma_1$. Consider the following parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + a_0 u = h_0 \chi_{\mathcal{O}} + w \chi_\omega & \text{in } Q, \\ u = 0 & \text{on } \Sigma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_2, \\ u\,(x, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

with $a_0 \in L^\infty\,(Q)$, and $\omega$ be an open and non-empty subset of $\Omega$.
Where $\chi_\omega$ denote the characteristic functions of $\omega$. Then problem $(3.1)$admits a unique solution $u$ that satisfies $u \in C([0, T];\ L^2(\Omega)) \cap L^2(0, T;\ H_0^1(\Omega))$ (see [17]).
Here's a presentation for the sub-cylinder $\omega \times [0, T]$.

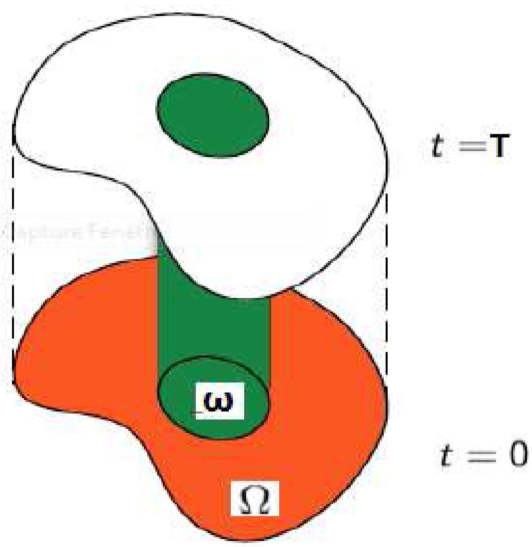


FIGURE 5:The sub-cylinder $\omega \times [0, T]$

**Theorem 3.1** *Let $u \in \mathcal{V}$ say defined in* $(3.3)$, *then there exists a positive constant* $C = C(\Omega; \omega; O; T; a_0)$ *such that*

$$\int_Q \frac{1}{\rho^2} |u|^2 \, dxdt \leq C \left[ \int_Q |Lu|^2 \, dxdt + \int_0^T \int_\omega |u|^2 \, dxdt \right], \tag{3.2}$$

where $\rho \in C^2(Q)$ positive with $\frac{1}{\rho}$ bounded.
This inequality is called observability inequality resulting from Carleman simplification estimates (see[16]) .
Carleman simplification estimates can be found in [[16], Corollary 3.5]. We will provide a little glimpse of how to simplify before launching into the previous theorem proof.
The first step is to set $L = \partial_t - \Delta + a_0 I$, and its formal adjoint $L^* = -\partial_t - \Delta + a_0 I$, where $I$ is the identity operator.
And this space within which we will work is as follows:

$$\mathcal{V} = \left\{ v \in C^\infty(Q) \text{ such that : } v|_{\Sigma_1} = \left.\frac{\partial v}{\partial t}\right|_{\Sigma_1} = 0 \text{ and } \left.\frac{\partial v}{\partial \nu}\right|_{\Sigma_2} = 0 \right\}. \tag{3.3}$$

The second step is to take [[16], Proposition 3.3] in the following form, which represents the Carleman estimate

$$\begin{aligned} &2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |u|^2 \, dxdt + 4s^2\lambda \int_{\Sigma_2} \varphi \frac{\partial \eta}{\partial t}\frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt \\ &-4s^3\lambda^3 \int_{\Sigma_2} \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt - 4s^2\lambda^3 \int_{\Sigma_2} \varphi^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt \\ &-2s\lambda \int_{\Sigma_2} \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} \partial_t u \, u d\gamma dt - 4s\lambda \int_{\Sigma_1} \varphi \nabla \psi e^{-2s\eta} \nabla u \frac{\partial u}{\partial \nu} d\gamma dt \\ &+2s\lambda \int_{\Sigma} \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |\nabla u|^2 \, d\gamma dt \\ \leq\ & C \left( \int_Q e^{-2s\eta} |\partial_t u - \Delta u|^2 \, dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 \, dxdt \right). \end{aligned} \tag{3.4}$$

On the other hand, we have the second proposition [[16], Proposition 3.4]

$$\begin{aligned} &2s^3\lambda^4 \int_Q \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}} |u|^2 \, dxdt - 4s^2\lambda \int_{\Sigma_2} \widetilde{\varphi} \frac{\partial \widetilde{\eta}}{\partial t}\frac{\partial \psi}{\partial \nu} e^{-2s\widetilde{\eta}} |u|^2 \, d\gamma dt \\ &+4s^3\lambda^3 \int_{\Sigma_2} \widetilde{\varphi}^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\widetilde{\eta}} |u|^2 \, d\gamma dt + 4s^2\lambda^3 \int_{\Sigma_2} \widetilde{\varphi}^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\widetilde{\eta}} |u|^2 \, d\gamma dt \\ &+2s\lambda \int_{\Sigma_2} \widetilde{\varphi} \frac{\partial \psi}{\partial \nu} e^{-2s\widetilde{\eta}} \partial_t u \, u d\gamma dt + 4s\lambda \int_{\Sigma_1} \widetilde{\varphi} \nabla \psi e^{-2s\widetilde{\eta}} \nabla u \frac{\partial u}{\partial \nu} d\gamma dt \\ &-2s\lambda \int_{\Sigma} \widetilde{\varphi} \frac{\partial \psi}{\partial \nu} e^{-2s\widetilde{\eta}} |\nabla u|^2 \, d\gamma dt \\ \leq\ & C \left( \int_Q e^{-2s\widetilde{\eta}} |\partial_t u - \Delta u|^2 \, dxdt + s^3\lambda^4 \int_0^T \int_\omega \widetilde{\varphi} e^{-2s\widetilde{\eta}} |u|^2 \, dxdt \right). \end{aligned} \tag{3.5}$$

We collect the two results $[(3.4)$ plus $(3.5)]$, we obtain

$$
\begin{aligned}
&2s^3\lambda^4\int_Q\left(\varphi^3e^{-2s\eta}+\widetilde{\varphi}^3e^{-2s\widetilde{\eta}}\right)|u|^2\,dxdt\\
&-4s^2\lambda\int_{\Sigma_2}\left(\varphi\frac{\partial\eta}{\partial t}e^{-2s\eta}-\widetilde{\varphi}\frac{\partial\widetilde{\eta}}{\partial t}e^{-2s\widetilde{\eta}}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}\,|u|^2\,d\gamma dt\\
&-4s^3\lambda^3\int_{\Sigma_2}\left(\varphi^3|\nabla\psi|^2e^{-2s\eta}-\widetilde{\varphi}^3|\nabla\psi|^2e^{-2s\widetilde{\eta}}\right)\frac{\partial\psi}{\partial\nu}|u|^2\,d\gamma dt
\end{aligned}
$$

$$
\begin{aligned}
&-4s^2\lambda^3\int_{\Sigma_2}\left(\varphi^2e^{-2s\eta}-\widetilde{\varphi}^2e^{-2s\widetilde{\eta}}\right)|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|u|^2\,d\gamma dt\\
&-2s\lambda\int_{\Sigma_2}\left(\varphi e^{-2s\eta}-\widetilde{\varphi}e^{-2s\widetilde{\eta}}\right)\frac{\partial\psi}{\partial\nu}\partial_t u\;ud\gamma dt\\
&-4s\lambda\int_{\Sigma_1}\left(\varphi e^{-2s\eta}-\widetilde{\varphi}e^{-2s\widetilde{\eta}}\right)\nabla\psi\nabla u\frac{\partial u}{\partial\nu}d\gamma dt
\end{aligned}
$$

$$
\begin{aligned}
&+2s\lambda\int_{\Sigma}\left(\varphi e^{-2s\eta}-\widetilde{\varphi}e^{-2s\widetilde{\eta}}\right)\frac{\partial\psi}{\partial\nu}|\nabla u|^2\,d\gamma dt\\
\leq\ & C[\int_Q\left(e^{-2s\eta}+e^{-2s\widetilde{\eta}}\right)|\partial_t u-\Delta u|^2\,dxdt\\
&+s^3\lambda^4\int_0^T\int_\omega\left(\varphi e^{-2s\eta}+\widetilde{\varphi}e^{-2s\widetilde{\eta}}\right)|u|^2\,dxdt].
\end{aligned}
$$

Now, it suffices to notice that $\varphi=\widetilde{\varphi}$ and $\eta=\widetilde{\eta}$ on $\Sigma$. Then, we find $\frac{1}{\rho^2}=\varphi^3e^{-2s\eta}+\widetilde{\varphi}^3e^{-2s\widetilde{\eta}}$. Therefore

$$
\begin{aligned}
&2s^3\lambda^4\int_Q 2\varphi^3e^{-2s\eta}|u|^2\,dxdt\\
\leq\ & C[\int_Q 2e^{-2s\eta}|\partial_t u-\Delta u|^2\,dxdt+s^3\lambda^4\int_0^T\int_\omega 2\varphi e^{-2s\eta}|u|^2\,dxdt]\\
\Longleftrightarrow\ & 2s^3\lambda^4\int_Q\varphi^3|u|^2\,dxdt\leq C[\int_Q|\partial_t u-\Delta u|^2\,dxdt\\
&+s^3\lambda^4\int_0^T\int_\omega\varphi|u|^2\,dxdt].
\end{aligned}
$$

We take

$$
\int_Q|\partial_t u-\Delta u|^2\,dxdt=\int_Q|Lu|^2\,dxdt
$$

$$
s^3\lambda^4\int_0^T\int_\omega\varphi^3|u|^2\,dxdt=\int_0^T\int_\omega\varphi^3|u|^2\,dxdt
$$

$$2s^3\lambda^4 \int_Q \varphi^3 |u|^2 \, dxdt = \int_Q \varphi^3 |u|^2 \, dxdt,$$

because $s \geq s_0 > 0$, and $\lambda \geq \lambda_0 > 0$.
Then

$$\int_Q \varphi^3 |u|^2 \, dxdt \leq C \int_Q |Lu|^2 \, dxdt + \int_0^T \int_\omega \varphi^3 |u|^2 \, dxdt.$$

In the third step, we will prove the theorem $(3.1)$.
At the beginning of the work, we start by finding the adjoint equation of the equation in the system $(3.1)$ by $q(x,t)$ we find this equation now

$$\begin{cases} -\partial_t q - \Delta q + a_0 q = h_0 \chi_{\mathcal{O}} + w\chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma_1, \\ \frac{\partial q}{\partial \nu} = 0 & \text{on } \Sigma_2, \\ q(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.6}$$

or

$$L^* q = h_0 \chi_{\mathcal{O}} + w\chi_\omega \qquad \text{in } Q. \tag{3.7}$$

In our case, we write the weak formulation of the following equation $(3.7)$, by multiplying the previous equation in the function $\upsilon \in \mathcal{V}$, and integrating on $Q$, we obtain

$$\int_0^T \int_\Omega L^* q \cdot \upsilon \, dxdt = \int_0^T \int_\Omega (h_0 \chi_{\mathcal{O}} + w\chi_\omega)\, \upsilon \, dxdt \qquad \text{in } Q. \tag{3.8}$$

This is all for proof of the controllability property by condition

$$q(x, T) = 0 \quad , \text{in } \omega. \tag{3.9}$$

Using integration by parts and the Green formula in $(3.8)$, we have

$$\int_0^T \int_\Omega (-\partial_t q - \Delta q + a_0 q\,)\, \upsilon \, dxdt = \int_0^T \int_\Omega h_0 \chi_O \upsilon \, dxdt + \int_0^T \int_\omega w\chi_\omega \upsilon \, dxdt$$

$$\int_0^T \int_\Omega -\partial_t q \, \upsilon \, dxdt - \int_0^T \int_\Omega \Delta q \, \upsilon \, dxdt + \int_0^T \int_\Omega a_0 q \, \upsilon \, dxdt = \int_0^T \int_\Omega h_0 \chi_O \upsilon \, dxdt + \int_0^T \int_\omega w\upsilon \, dxdt$$

$$\int_0^T \int_\Omega q \, \partial_t \upsilon \, dxdt - \int_0^T \int_\Omega q \, \Delta \upsilon \, dxdt + \int_0^T \int_\Omega a_0 q \, \upsilon \, dxdt - \int_\Omega q(x,\ T)\, \upsilon(x,\ T)\, dxdt$$
$$= \int_0^T \int_\Omega h_0 \chi_O \upsilon \, dxdt + \int_0^T \int_\omega w\upsilon \, dxdt.$$

Then

$$\int_0^T \int_\Omega q\,(\partial_t \upsilon - \ \Delta \upsilon \ + a_0 \upsilon\,)\, dxdt - \int_\Omega q(x,\ T)\, \upsilon(x,\ T)\, dxdt = \int_0^T \int_\Omega h_0 \chi_O \upsilon \, dxdt + \int_0^T \int_\omega w\upsilon \, dxdt$$

$$\int_0^T \int_\Omega q\,(\partial_t - \ \Delta \ + a_0\ )\, v\ dxdt - \int_\Omega q\,(x,\ T)\, v\,(x,\ T)\, dxdt \ = \ \int_0^T \int_\Omega h_0 \chi_O v\ dxdt + \int_0^T \int_\omega wv\ dxdt$$
$$\int_0^T \int_\Omega qLv\ dxdt - \int_\Omega q\,(x,\ T)\, v\,(x,\ T)\, dxdt \ = \ \int_0^T \int_\Omega h_0 \chi_O v\ dxdt + \int_0^T \int_\omega wv\ dxdt.$$

We assume that $q = Lu$, and $w = -u\chi_\omega$ with $u \in \mathcal{V}$, we obtain

$$\int_0^T \int_\Omega LuLv\ dxdt + \int_0^T \int_\omega u\chi_\omega v\ dxdt - \int_\Omega q\,(x,\ T)\, v\,(x,\ T)\, dxdt = \int_0^T \int_\Omega h_0 v\ dxdt.$$

We further set

$$a\,(u,\ v) = \int_0^T \int_\Omega LuLv\ dxdt + \int_0^T \int_\omega u\chi_\omega v\ dxdt,$$

and

$$l\,(v) = \int_0^T \int_\Omega h_0 \chi_O v\ dxdt.$$

Our null controllability problem becomes

$$a\,(u,\ v) - \int_\Omega q\,(x,\ T)\, v\,(x,\ T)\, dxdt = l\,(v) \qquad \text{for all } v \text{ in } V. \tag{3.10}$$

We consider the following subspace $V$ to be a Hilbert space for the scalar product $a(u; v)$ and the associated norm

$$v \longrightarrow \|v\|_V = \sqrt{a(v; v)},$$

and $V$ be the completion of $\mathcal{V}$.

**Remark 3.1** *We can precise the structure of the elements of $V$, let $L^2_\rho(Q)$ be the weighted Hilbert space defined by*

$$L^2_\rho(Q) = \left\{ v \in L^2\,(Q) \text{ such that } \int_Q \frac{1}{\rho^2}\,|v|^2\, dxdt \right\},$$

endowed with the norm

$$\|v\|_{L^2_\rho(Q)} = \left( \int_Q \frac{1}{\rho^2}\,|v|^2\, dxdt \right)^{\frac{1}{2}}.$$

This shows that $V$ is embedded continuously because of the inequality $(3.2)$ we have

$$\exists C > 0 : \|v\|_{L^2_\rho(Q)} \leq C\,\|v\|_V \qquad \text{for every } v \in V. \tag{3.11}$$

By the boundedness of $\frac{1}{\rho^2}$ on $Q$, we also see that $L^2(Q)$ is continuously embedded in $L^2_\rho(Q)$.

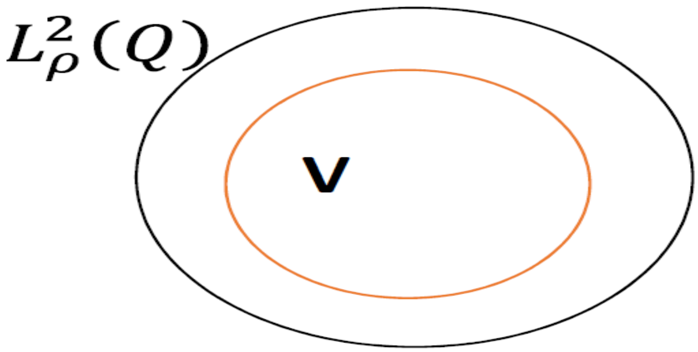


FIGURE 6: $V$ is embedded continuously in $L^2_\rho(Q)$

We apply the Lax-Milgram Theorem(see Appendix **Theorem3.5**) in the form $(3.10)$, we get
1. $\forall u, v \in V\ , \exists\ M > 0 : |a\,(u,\ v)| \leq M\,\|u\|_V\,\|v\|_V$

$$\begin{aligned}
|a\,(u,\ v)| &= \left|\int_0^T \int_\Omega LuLv\ dxdt + \int_0^T \int_\omega uv\ dxdt\right| \\
&= \left|\int_0^T \int_\Omega LuLv\ dxdt\right| + \left|\int_0^T \int_\omega uv\ dxdt\right| \\
&= (u,\ v)_V \leq^{Cauch-shwartz} \|u\|_V\,\|v\|_V\,.
\end{aligned}$$

With
$$(u,\ v)_V = \int_0^T \int_\Omega LuLv\ dxdt + \int_0^T \int_\omega uv\ dxdt$$

the inner product in space $V$. Then $a\,(u,\ v)$ is continu and $M =\ 1$.
2. $\forall u \in V, \exists\ \alpha > 0 : |a\,(u,\ u)| \geq \alpha\,\|u\|_V^2$

$$\begin{aligned}
|a\,(u,\ u)| &= \left|\int_0^T \int_\Omega (Lu)^2\,dxdt + \int_0^T \int_\omega u^2\ dxdt\right| \\
&= \int_0^T \int_\Omega |Lu|^2\ dxdt + \int_0^T \int_\omega |u|^2\ dxdt.
\end{aligned}$$

We have the norm in space $V$ in the form below

$$\|v\|_V^2 = \int_0^T \int_\Omega |Lu|^2\ dxdt + \int_0^T \int_\omega |u|^2\ dxdt.$$

Then $a(u,\ u)$ is coercive.
Therefore, based on theorem $(3.2)$ (Carleman estimates) and characteristic $(3.11)$, the condition of continuity and coercivity is an investigator, with $l(v)$ is continuous linear form. Based on the Lax-Milgram theorem, equation $(3.10)$ accepts a weak solution.
Finally, equation $(3.1)$ is **null controllable** by the Carleman estimate.

## 3.2 Null controllability of linear heat equation with Dirichlet boundary conditions and distributed controls

Let $\Omega \in \mathbb{R}^n$, $n \geq 1$ is a bounded domain with smooth boundary $\Gamma$ of class $C^2$. Let us consider the simplest case of the linear heat equation with Dirichlet boundary conditions and distributed controls:

$$\begin{cases} \partial_t y - \Delta y = v\chi_{\mathcal{O}} & \text{in } Q = \Omega \times ]0,\ T[, \\ y = 0 & \text{on } \Sigma = \partial\Omega \times ]0,\ T[, \\ y(x,0) = y^0 & \text{in } \Omega. \end{cases} \tag{3.12}$$

For $\mathcal{O} \subset \Omega$ is a nonempty open subset, $\chi_{\mathcal{O}}$is the characteristic function of $\mathcal{O}$, and $T$ is a given positive time. We assume that the initial state $y^0$ is given in $L^2(\Omega)$ and try to find a control $v \in L^2(\mathcal{O} \times (0,T))$ such that the associated state $y = y(x,t)$ possesses a desired behaviour at time $t = T$.
System $(3.12)$ has a unique weak solution $y$ (see [7])satisfy

$$y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$$

that depends continuously on $y^0$ and $v$.

**Theorem 3.2** *The observability inequality to the equation* $(3.12)$ *is*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \int\int_{\mathcal{O}\times(0,\ T)} |\varphi|^2\, dxdt, \tag{3.13}$$

with $C > 0$ constant.
The formula $(3.13)$ implies the null controllability of equation $(3.12)$.
**Proof.** see [7]. ■
The observability inequality consequence by simplification the Carleman estimate say defined in $(3.14)$ where $\rho$ is bounded.
The goal is to find the observability inequality for equation $(3.12)$from Carleman estimates, but in this case, the observability inequality implies the null controllability of $(3.12)$.

Before everything, we will prove that the null controllability of $(3.12)$is for every $y^0 \in L^2(\Omega)$, and control $\upsilon \in L^2(\mathcal{O},\ (0,\ T))$, we find the null controllability with condition $y(T) = 0$ in $\Omega$ from Carleman estimates (see [7] ), and those estimates are in the following form:

$$\int\int_{\Omega\times(0,\ T)} \rho^2 |\varphi|^2 \, dxdt \leq C \int\int_{\mathcal{O}\times(0,\ T)} \rho^2 |\varphi|^2 \, dxdt, \tag{3.14}$$

where $\rho = \rho(x,\ t)$ is a continuous and strictly positive weight function, and $C > 0$.
Therefore, we put the adjoint of the equation $(3.12)$ with the function $\varphi = \varphi(x,\ t)$ with $\varphi^0 \in L^2(\Omega)$

$$\begin{cases} -\partial_t\varphi - \Delta\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ y(x, T) = \varphi^0 & \text{in } \Omega. \end{cases} \tag{3.15}$$

In this case, we build a sequence of controls $\upsilon_\varepsilon \in L^2(\mathcal{O},\ (0,\ T))$ with $\varepsilon > 0$ that provides the **approximate controllability** of $(3.12)$ with

$$\|y_\varepsilon(T)\|_{L^2(\Omega)} \leq \varepsilon. \tag{3.16}$$

Let us introduce the functional $J_\varepsilon$, use equation $(3.12)$ and multiply with $\varphi$ and integrate by $Q$, we get

$$\int\int_Q (\partial_t y - \Delta y)\, \varphi dxdt = \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt,$$

by the Green formula and integration by parts, we obtain

$$\int\int_Q y\,(\partial_t\varphi - \Delta\varphi)\, dxdt + \int_\Omega y\varphi|_0^T \, dx = \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt.$$

By equations $(3.15)$ and $(3.16)$; we have

$$\begin{aligned} \int_\Omega y\varphi|_0^T \, dx &= \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt \\ \int_\Omega [y(T)\,\varphi(T) - y(0)\,\varphi(0)]\, dx &= \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt \\ \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt + \int_\Omega y^0\varphi(0)\, dx &= 0 \end{aligned}$$

Then, we add this value $\varepsilon \int_\Omega \varphi^0 dx$ that represents

$$\varepsilon \int_\Omega \varphi^0 dx + \int\int_{\mathcal{O}\times[0,\ T]} \upsilon\varphi\chi_{\mathcal{O}} dxdt + \int_\Omega y^0\varphi(0)\, dx = 0$$

we put $\upsilon\chi_{\mathcal{O}} = \varphi$

$$\varepsilon \int_{\Omega} \varphi^0 dx + \int\int_{\mathcal{O}\times[0,\,T]} \varphi^2 dxdt + \int_{\Omega} y^0 \varphi(0)\, dx = 0$$

Thus

$$J_\varepsilon\left(\varphi^0\right) = \frac{1}{2}\left[\int\int_{\mathcal{O}\times[0,\,T]} \varphi^2 dxdt + \varepsilon \int_{\Omega} \varphi^0 dx + \left(\varphi(0), y^0\right)_{L^2(\Omega)}\right], \tag{3.17}$$

for every $\varphi^0 \in L^2(\Omega)$. Here, $\varphi$ is the solution of $(3.15)$ associated with the initial condition $\varphi^0$. Using $(3.12)$, it is not difficult to check that $J$ is strictly convex, continuous, and coercive in $L^2(\Omega)$, so it possesses a unique minimum $\varphi^0_\varepsilon \in L^2(\Omega)$, whose associated solution is denoted by $\varphi_\varepsilon$. Let us the control $\upsilon_\varepsilon = \varphi_\varepsilon\chi_{\mathcal{O}}$ and denote by $y_\varepsilon$ the solution of $(3.12)$ associated to$\upsilon_\varepsilon$.
Let $y_1 = y(T)$ be the final state of the solution to $(3.12)$ with vanishing control.
We are now making a derivative of $J_\varepsilon$ with the derivate of Gataux in the direction $\varphi_\varepsilon$, we obtain

$$\int\int_{\mathcal{O}\times[0,\,T]} \varphi_\varepsilon \varphi dxdt + \varepsilon\left(\frac{\varphi^0_\varepsilon}{\|\varphi^0_\varepsilon\|_{L^2(\Omega)}}, \varphi^0\right)_{L^2(\Omega)} + \left(\varphi(0), y^0\right)_{L^2(\Omega)} = 0, \tag{3.18}$$

for $\varphi^0 = \varphi^0_\varepsilon$ using $(3.18)$ and $(3.13)$ we obtain $\|\upsilon_\varepsilon\|_{L^2(\Omega)} \leq \sqrt{C}\, \|y^0\|_{L^2(\Omega)}$where $C$ is the observability constant of $(3.13)$.
Since systems$(3.12)$and $(3.15)$ are in duality, we have

$$\int\int_{\mathcal{O}\times[0,\,T]} \varphi_\varepsilon \varphi dxdt = \left(y_\varepsilon, \varphi^0\right)_{L^2(\Omega)} + \left(\varphi(0), y^0\right)_{L^2(\Omega)},$$

which combined with $(3.17)$yields $(3.16)$.
Here we calculate the limit when $\varepsilon$ tends to zero. Since the sequence $\{\upsilon_\varepsilon\}$ is bounded in $L^2(\mathcal{O}$, $(0, T))$, it has a weak convergence with $\upsilon \in L^2(\mathcal{O}, (0, T))$. Then

$$y_\varepsilon \longrightarrow y \text{ weakly in } L^2(0, T; H^1_0(\Omega)) \cap C^0([0, T]; L^2(\Omega)), \tag{3.19}$$

where $y$ is the solution of$(3.12)$with control. In particular, this gives weak convergence for $\{y_\varepsilon\}\,(t \in [0; T])$ in $L^2(\Omega)$, so we have $y(T) = 0$.
Finally, the system $(3.12)$ is null controllable, and your observability inequality is the form $(3.13)$.

## Biography of Carleman

**Born** 8 July, 1892 Visseltofta, Sweden.

**Died** 11 January 1949 Stockholm, Sweden.

**Birth name** Tage Gillis Torsten Carleman.

**Nationality** Swedish.

**Formation** Katedralskolan (en) (until 1910).
Uppsala University (1910-1916).

**Activities** Mathematician, university professor.

**Relatives** Eric Lemming (father-in-law).

**Other information**

**Worked at** University of Stockholm (1924-1948).
Lund University (1923-1924).
Uppsala University (1917-1923).

**Member of** Royal Society of Physiography in Lund (en) (1924).
Royal Swedish Academy of Sciences (1926).
Academy of Sciences of Saxony (1934).

**Thesis director** Erik Albert Holmgren (EN).

**Distinctions** Cours Peccot (1922).
The Bear (1941).

**Principal works**

Carleman inequality, Carleman matrix, Denjoy Carleman–Ahlfors theorem, Carleman equation, Carleman condition.

FIGURE 7 :T. Carleman, 1892-1949

# Chapter 4

# Appendix

**Proposition 4.1** ***(Cauchy inequality)***

Let $a$, $b$ be any real numbers and $p,\_q$ are real numbers connected by the relationship $\frac{1}{p}+\frac{1}{q}=1$. Then, we have the Cauchy–Schwarz inequality

$$ab \leq \frac{1}{2}\left(a^2+b^2\right). \tag{A.1}$$

and

$$|\alpha+\beta| \geq \frac{1}{2}|\alpha|^2-|\beta|^2. \tag{A.2}$$

**Theorem 4.1** ***(Integration by parts)***

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $\nu$ denotes the outward normal unit vector to $\partial\Omega$. Let $u$ and $v$ be any two differentiable functions of variable, we have

$$\int_\Omega v\nabla u dx=\int_{\partial\Omega} uv\cdot\nu d\Gamma-\int_\Omega \nabla v\cdot u dx. \tag{A.3}$$

**Theorem 4.2** ***(Green Formula )***

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $\nu$ denotes the outward normal unit vector to $\partial\Omega$. And $\varphi$,$\psi$ be real functions of class $C^1\left(\overline{\Omega}\right)$, we have

$$\int_\Omega \psi\Delta\varphi dx=\int_{\partial\Omega}\psi\partial_\nu\varphi d\Gamma-\int_\Omega \nabla\varphi\cdot\nabla\psi dx. \tag{A.4}$$

**Theorem 4.3** ***(Cartesian Decomposition)***

Let $\mathcal{H}$ be a Hilbert space and let $T \in H$.Then there are unique self-adjoint operators, $T_+$, $T_-$ such that

$$T=T_+ + T_-, \tag{A.5}$$

with

$$T_{+} = \frac{T + T^{*}}{2} \qquad \text{, symetrique part of } T, \tag{A.6}$$

and

$$T- = \frac{T - T^{*}}{2} \qquad \text{, antsymetrique part of } T. \tag{A.7}$$

**Theorem 4.4** ***(The Lax-Milgram Theorem)***

Let $H$ be a Hilbert space with **the scalar product** $(\cdot,\cdot)$ and **norm** $\|.\|$. Let $a$: $H \times H \longrightarrow \mathbb{R}$ be a bilinear form in $H$. Assume that there exist two constants $M < \infty$, $\alpha > 0$ such that:

1. $|a(u,\ v)| \leq M \|u\| \|v\|$ for all $(u, v) \in H \times H$ (continuity);
2. $|a(u,\ u)| \geq \alpha \|u\|^2$ for all $u \in H$ (coercivity);

and $l(v)$ continuous linear form i.e $|l(v)| \leq C \|v\|, v \in H, C > 0$.
Then, $\forall f \in H^*$ (the dual space of $H$), there exists $u \in H$ unique, such that $a(u, v) = (f, v)$ for all $u \in H$.

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