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Presented by Asma Ladjeroud

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Dedication

I am dedicating this thesis to:

My parents, ***Houria*** and ***Abdelmadjid***. I express my heartfelt gratitude to you for your unwavering support. Throughout my journey, you have consistently been my pillars of strength. Mom, I extend my sincerest thanks for patiently answering countless phone calls, for soothing my anxieties, and for proofreading my work anytime and anywhere. Dad, I am immensely grateful for your boundless love and for consistently reminding me of the ultimate objective. Your support has been invaluable to me.

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ملخص

الهدف الرئيسي لهذه الدراسة هو تقديم برهان إنشائي منظم بدقة على وجود جاذب شبه زائدي ضمن فئة محددة من التطبيقات الخطية المتجزأة ثنائية الأبعاد. لتحقيق ذلك، نقترح تطبيق نظرية شيلنيكوف وجافريلوف، التي تتناول الخصائص الزائدية لأنواع معينة من التطبيقات السلسة. من خلال تطبيق هذه النظرية، نهدف إلى تقديم دليل شامل على وجود الجاذبية المذكورة. علاوة على ذلك، ستسهم هذه الطريقة في تحديد الظروف الدقيقة التي يمكن فيها تصنيف التطبيقات السلسة على أنها زائدية، مما يسمح بإظهار أشكال مختلفة من السلوكيات الفوضوية.

Résumé

L'objectif principal de cette thèse est de fournir une preuve rigoureuse de l'existence d'un attracteur quasi-hyperbolique dans une certaine famille d'applications linéaires morceau par morceau en deux dimensions. Pour cela, nous proposons d'appliquer le théorème de Shilnikov et Gavrilov (1976) concernant l'hyperbolicité de certains types d'applications lisses, afin d'établir une démonstration de l'existence de cet attracteur. Cette approche nous permettra d'établir les conditions sous lesquelles une application lisse peut être classée comme hyperbolique et, par conséquent, de présenter certains types de comportements chaotiques.

Abstract

The primary objective of this thesis is to provide a rigorous structured proof for the existence of a quasi-hyperbolic attractor within a specific class of two-dimensional piecewise linear maps. To accomplish this, we propose the application of the Shilnikov and Gavrilov theorem (1976), which addresses the hyperbolicity properties of particular types of smooth mappings. By employing this theorem, we aim to present a comprehensive demonstration of the existence of the aforementioned attractor. Moreover, this approach will facilitate the establishment of the precise conditions under which a smooth map can be classified as hyperbolic, thereby enabling the manifestation of various forms of chaotic behaviors.

Introduction

After the 1880s, dynamical systems became time-dependent and required initial conditions. Closed-form solutions are rare, so Henri Poincaré developed a qualitative approach, combining analysis and geometry to study system behaviors. Poincaré's method replaced the analytical approach, which was limited to mathematicians. A.M. Lyapunov and others later strengthened and required the qualitative method. Poincaré won a global contest with his work on the stability of the solar system. Poincaré's study of dynamical systems used the qualitative approach, leading to the mathematical theories for dynamical systems. A.M. Lyapunov's work on the stability of systems influenced this study, and the Lyapunov function and theorem were created.

Predator-prey populations have varying amplitudes in the Lotka-Volterra model, unlike the mathematical model developed by Holling and Tanner, which has constant amplitudes over time. Robert May and other scientists have developed population models to analyze dynamics. Unpredictability in natural and social phenomena has had a significant impact on human thought and scientific evolution, with Newtonian mechanics providing a deterministic view. The twentieth-century saw breakthroughs in the role of nonlinearity on dynamics, such as Cartwright and Littlewood proving a chaotic orbit for a forced van der Pol equation. Turbulence in fluid flows remains an unsolved problem, with Kolmogorov and his colleagues contributing to isotropic turbulence. Relaxation oscillation occurs in chemistry, as seen in the Belousov-Zhabotinsky reaction. The science of the unpredictable needs to be developed, with Lorenz's "Deterministic Non-periodic Flow" paper illustrating the butterfly effect and the sensitive dependence of a system's evolution on initial conditions. Chaotic solutions exist, as proven by Smale's "Differential Dynamical Systems" paper.

Mathematicians and physicists such as Lev D. Laudau, James Yorke, Robert May, Enrico Fermi, Stanislaw Ulam, J.G. Senai, Sarkovskii, Ruelle and Takens, A. Libchaber, and J. Maurer have made significant contributions to the development of nonlinear science and chaos theory. Mitchell Feigenbaum made a significant finding in the middle of the 1970s: the universality of period doubling bifurcation for unimodal maps. Fractal geometry, which seeks order in disorderly patterns and processes, was initially introduced in 1975 by Benoit Mandelbrot. Fractal objects are structures that are irregular and chaotic, yet self-similar, with patterns seen at larger scales recurring at increasingly smaller ones. Fractals are widely used in the natural and physical sciences and can be used to express chaotic orbit. The human anatomy has several instances of fractals, including many biological structures such as the heart and lungs. Fractals have also been used in medical research to prevent catastrophic illnesses.

Chaos is a reality occurrence that can be efficiently used for human welfare in addition to being destructive, as in the case of a tsunami, tornado, etc. The notion of chaos is being effectively used in video surveillance, safe data aggregation, digital watermarking, and computer security. After relativity and quantum physics, chaos has been ranked as the third-greatest discovery in 20th-century science and philosophy. Scientists and engineers have become aware of the possible applications of chaos in natural and technical sciences during the past 20 years.

The exploration and analysis of chaotic dynamics have captivated scientists and researchers for decades, as they offer a profound understanding of complex systems and their inherent unpredictability. Chaos theory provides a powerful framework for comprehending the intricate interplay of deterministic processes that exhibit sensitive dependence on initial conditions, resulting in seemingly random and unpredictable behavior.

This thesis is partitioned into two parts: The first part is concerned with the fascinating realm of chaotic dynamics, focusing on the examination of piecewise smooth maps which is organised into two chapters, while the main aim of second part is the study of the classification of strange attractors. Let us come more specifically to a quick overview of the tools, results, and publications of each chapter:

Chapter 1: Serves as the foundation for the subsequent chapters. This chapter provides a comprehensive survey of the theoretical of chaotic dynamics. We begin with an introduction to discrete dynamical systems, laying the groundwork for the subsequent discussion of routes to chaos, including period doubling, indeterminacy, and quasi-periodicity. The chapter then explores the notion of strange attractors and the various mathematical definitions and characterizations of chaotic attractors. Additionally, the concept of robust chaos is examined, providing valuable insights into the stability and persistence of chaotic behavior.

Chapter 2: Focuses on the examination of systems governed by piecewise smooth maps. We first give an introduction, setting the stage for the subsequent discussions. We then give the analysis of one-dimensional piecewise smooth maps, exploring their unique dynamics and properties. Moving forward, we extend the investigation to two-dimensional piecewise smooth maps, further unraveling the complexities and behaviors exhibited by these systems. A significant aspect addressed in this chapter is the examination of **border collision bifurcation scenario**s and their impact on the robustness of chaotic behavior. Specifically, the discussion encompasses the normal forms of one-dimensional and two-dimensional piecewise smooth maps, elucidating the bifurcation scenarios associated with border collisions.

Chapter 3: is devoted to the study of the the classification and characterization of various types of strange attractors. The chapter begins with an introduction, setting the stage for the subsequent discussions. We then focuses on the analysis of hyperbolic attractors, exploring their distinct properties and dynamics. Hyperbolic attractors play a crucial role in chaos theory and are known for their exponential sensitivity to initial conditions. It then focuses on the analysis of hyperbolic and Lorenz-type attractors, exploring their distinct properties and dynamics. Moving forward, we explores the concept of quasi-attractors introducing a new two-dimensional piecewise smooth map to define and study these attractors. The discussion delves into the notion of hyperbolicity, distinguishing between hyperbolic and non-hyperbolic regimes. We further explores the quasi-hyperbolic regime and the trapping region, elucidating the distinctive features and behaviors exhibited within this context and it takes part of a published work [28]. At the end, we generalize the findings to n-dimensional mappings, expanding the analysis to higher-dimensional systems.

By investigating and classifying different types of strange attractors, this chapter contributes to a deeper understanding of the many behaviors and properties observed within chaotic systems. It provides insights into the topological features, hyperbolicity, and trapping mechanisms associated with these attractors. The findings in this chapter further enrich the understanding of chaotic dynamics, laying the groundwork for future research and advancements in the field.

Chapter 1

Review of chaotic dynamics

In all chaos there is a cosmos, in all disorder a secret order

– Carl jumg

1.1 Introduction

Chaotic systems are deterministic systems that exhibit unpredictable behavior. Their sensitivity to initial conditions is a defining characteristic, as even slight variations in the starting conditions can result in significantly disparate outcomes. The study of chaos theory began in the early 20th century with the work of Henri Poincaré, who investigated the three-body problem in celestial mechanics. However, it was not until the development of computers in the latter half of the century that the theory of chaos was fully realized and its applications to a wide range of fields became apparent.

There are several known chaotic systems, such as the Lorenz, Rössler, Lozi, and Hénon attractors. These systems have been extensively studied and analyzed using techniques such as Lyapunov exponents and Poincaré maps. Researchers have also developed new chaotic systems based on memristive Hopfield networks and Chua's circuit.

A fundamental notion in chaos theory is the concept of a strange attractor, which serves as a mathematical entity that portrays the prolonged dynamics of a chaotic system. Strange attractors have fractal geometry and are characterized by their sensitivity to initial conditions, *which means that small changes in the initial state of the system can lead to vastly different outcomes.*

In this chapter, we will explore the mathematical theory of chaos in more details starting with giving some of the basics concerning the discrete dynamical systems in Sec 1.2. In Sec 1.3, we will take a look on the road to chaos and mention what conditions make a dynamical system chaotic. And finally in Sec 1.4, we will cite the most known definitions of chaos theory and its characteristics with giving an entry to the notion of robust chaos.

1.2 Preliminaries about discrete dynamical systems

Definition 1.2.1 *A* ***discrete dynamical system*** *is every system of recurrent algebraic equations defined by:*

$$X(k+1) = F(X(k), \mu), \quad k \in \mathbb{Z} \tag{1.1}$$

where, F is the recurrence vector function, $X(k) \in U \subseteq \mathbb{R}^n$ *is the state vector and* $\mu \in V \subseteq \mathbb{R}^n$ *is the parameters vector.*

Definition 1.2.2 *Consider a point X and a map*[1] *F. The set of points formed by repeatedly applying F to X, denoted as the* ***orbit****, can be expressed as* $\{X, F(X), F^2(X), ...\}$*. The initial point of the orbit, which is the starting point X, is referred to as the* ***initial value*** *of the orbit.*

Definition 1.2.3 *[3]* p *is a* ***fixed point (equilibrium point)*** *of the map* $X_{k+1} = F(X_k)$ *if it is verifying* $F(p) = p$*.*

[1] A function whose domain (input) space and range (output) are the same is called a **map** [3]

Definition 1.2.4 *The fixed point p of $F : U \to U \subset \mathbb{R}^n$, is* ***an attracting (stable)*** *fixed point if for every $X_0 \in U$*

$$\forall \epsilon > 0, \exists \delta > 0, |X_0 - p| < \delta \Rightarrow |F^k(X_0) - p| < \epsilon, \forall k \in \mathbb{N}^*$$

p is said to be locally asymptotically stable if it is stable and there exist $N_\delta(p)$ a neighborhood of p which is the set $\{v \in \mathbb{R}^n, |v - p| < \delta\}$ such that $\forall X_0 \in N_\delta(p)$, $\lim_{k\to\infty} F^k(X_0) = p$.

Definition 1.2.5 *The fixed point p of $F : U \to U \subset \mathbb{R}^n$, is said to be* ***unstable*** *if*

$$\exists \epsilon > 0, \forall \delta > 0, \exists X_0 \in N_\delta(p) \Rightarrow |F^k(X_0) - p| > \epsilon, \forall k \in \mathbb{N}^*$$

Theorem 1.2.1 *[3] Let p be a fixed point of F and suppose that $F \in \mathbb{R}^n$ is a smooth[2] map. Let λ_k be the eigenvalues of the jacobian matrix $DF(p)$, then*

1. *if $|\lambda_k| < 1$, then p is an asymptotically stable fixed point of F,*
2. *if $|\lambda_k| > 1$, then p is not a Lyapunov stable[3] fixed point of F.*

Definition 1.2.6 *[3] Let F be a map on $\mathbb{R}^n$, a point p is said to be periodic with least period n if $F^k(p) = p$ for $k = n$, $\forall k, n \in \mathbb{N}$, while this is not avalaible for any smaller value of k, and we call it* ***period-k point****. Since $F^n(p) = p$, there are only n distinct points that constitute the orbit of p, and we call it* ***period-k orbit****.*

Definition 1.2.7 *[3] Let F be a map and assume that p is a period-k point. The period-k orbit of p is said to be attractive if p is an attracting point for the map F^k. The orbit of p is a periodic repulsive orbit if p is a repulsive point for the map F^k.*

The study of the asymptotic behavior of a dynamical system governed by a set of non-linear differential equations often reveals the notion of an attractor, which is defined as the compact set of the phase space that is invariant under the system and towards which all trajectories of the system converge. In general, an attractor is defined as a closed subset of the phase space that "**attracts**" all other orbits towards it. An "**attractor**" is a term used to describe the region within the phase space where the trajectories of a dissipative dynamical system converge. Attractors represent geometric configurations that define the enduring behavior and evolution of dynamical systems over extended periods of time.

Definition 1.2.8 *Let $(X, \phi(t)_{t\in\mathbb{R}})$ be a dynamical system and A be a compact, closed set of the phase space. A is called an* ***attractor*** *if it satisfies the following four conditions :*

[2] A function for which derivatives of all orders exist and are continuous functions.

[3] We say that the set A is Lyapunov stable if for every neighborhood U of A, there exists a neighborhood V of A such that every solution $X(X_0, t) = \phi_t(X_0)$ will remain in U if $X_0 \in V$.

1. *A is invariant under the action of the flow, i.e.,* $\phi_t(A) \subset A, \forall t \in \mathbb{R}$,

2. *A is Lyapunov stable,*

3. *There exists a dense*[4] *orbit in A,*

4. $\bigcap_{t\geq 0} \phi_t(V) = A$.

There are four distinct types of attractors [45]:

• The point, reflecting a stationary state, is the simplest attractor.

• The periodic attractor: (or the limit cycle) is a closed trajectory that is a solution of the system.

• The quasi-periodic attractor: Torus T_n, $(n \geq 2)$ corresponds to a quasi-periodic regime with n independent frequencies.

• The strange attractor: details about the characteristics of this type will be provided in the next section.

Let's first investigate more closely the set of points whose orbits converge to anattracting fixed point or periodic point, called the **basin of attraction** or just **basin** of the attracting fixed point.

Definition 1.2.9 *Consider* F *as a mapping on* $\mathbb{R}^n$*, with* p *denoting an attracting fixed point or a periodic point. In this context, the term "basin of attraction" refers to the collection of points, denoted as X, which satisfy the following condition:*

$$|F^k(X) - F^k(p)| \to 0, \quad \text{as} \quad k \to \infty.$$

i.e.,

$$B(p) = \{X : \lim_{k\to\infty} F^k(X) = p\}$$

then, from this concept, we can say that while A is an attractor, the set

$$B(A) = \bigcup_{t\geq 0} \phi_t(V)$$

is called the **bassin of attraction** of A.

Definition 1.2.10 *In the realm of dynamical systems, a bifurcation pertains to a significant alteration in the qualitative characteristics of the solution X of the system (1.1), brought about by the modification of the control parameter* μ*. Specifically, such modifications may result in the disappearance or destabilization of existing solutions, or the emergence of entirely new solutions.*

[4] Suppose that X is a set and Y is a subset of X in X. Y is dense in X if $\overline{Y} = X$ i.e., $\forall x \in X$ we can find a set $\{y_n\} \in Y$ converges to x

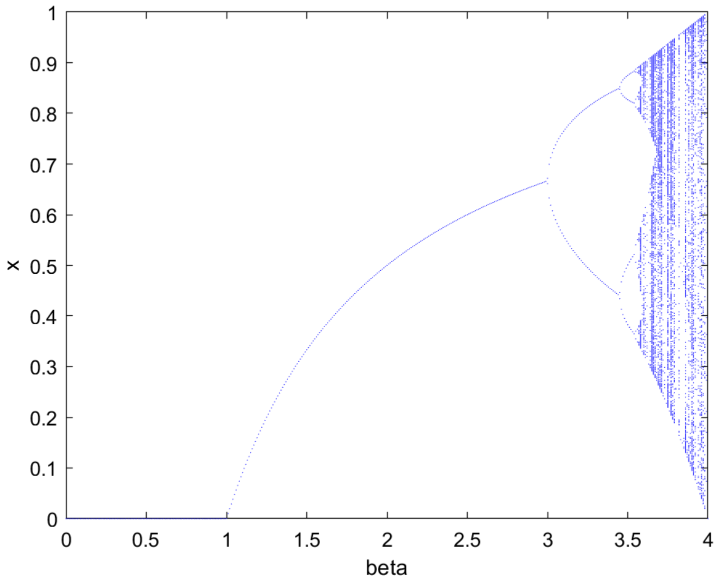

Figure 1.1: The bifurcation diagram of the logistic map $X_{n+1} = rX_n(1 - X_n)$.

So, the bifurcation process is often characterized by a sudden change in the system's behavior. This change is usually caused by small change in the system's parameters. It can cause a system to move from one state of equilibrium to another, resulting in a new behavior or pattern.

Definition 1.2.11 *A bifurcation diagram is a portion of the parameter space on which all bifurcation points are represented. (See Figure 1.1).*

There are several types of bifurcations depending on the properties of the second derivatives of the family of functions $F(X(k), \mu)$. Each type has its own unique characteristics and can produce different results. Also, each of these bifurcations is characterized by a normal form, which is the typical general equation of this type of bifurcation. Among the different types of bifurcations for discrete dynamical systems, we find:

Saddle-node bifurcation (or tangent, fold): This bifurcation occurs when one of the two eigenvalues of is equal to $+1$. On the bifurcation diagram, a curve of fixed points is observed to be tangent to the vertical straight line in this case. Two equilibrium points exist (one stable and one unstable) before the bifurcation. After the bifurcation, no equilibrium exists anymore.

Period-doubling bifurcation (or flip): Occurs when one of the two eigenvalues of $DF(X(k), \mu)$ is equal to -1. A stable fixed point of order 1 becomes unstable at the same time as the appearance of a stable cycle of order 2, i.e., it is a type of bifurcation where the period of oscillation doubles.

Transcritical bifurcation: On the bifurcation diagram, this is represented by two different branches of fixed points that intersect at a point and by the change of stability of both branches at the intersection point. It is a type of bifurcation in which two equilibrium points collide and split into two new equilibrium points, but one of the new equilibrium points is unstable.

Neimark-Sacker bifurcation: Occurs when a stable periodic solution of a system loses stability, resulting in the creation of a stable invariant torus. In other words, $DF(X(k), \mu)$ has two complex eigenvalues equal to $e^{\pm i\theta}$.

Bifurcations leading to chaos: Occurs when a chaotic attractor emerges from a periodic or quasi-periodic attractor as a parameter is varied.

Definition 1.2.12 *The equilibrium point p of system (1.1) is stable if there exists a function (with respect to X) that is positively defined in a region around the logical* $V(X) : U \to \mathbb{R}$, *continuously differentiable, and possessing the following properties:*

1. U *is an open set of* $\mathbb{R}^n$ *and* $p \in U$,
2. $V(p) = 0$ *and* $V(X) > V(p), \forall X \neq p$ *in* U,
3. $\Delta V(X) = \sum_{j=0}^{n} \frac{\partial V}{\partial X_j} F_j(X) \leq 0, \forall X \neq p$ *in* U.

If $\forall X \neq p$ *in* U, $\Delta V(X)$ *is strictly negative, i.e.,* $\Delta V(X) < 0$, *then p is said to be asymptotically stable in the sens of Lyapunov.*

If we further assume that V *tends to infinity as* X *tends to infinity (in norm), then all trajectories, even those starting far from* p, *tend towards* p *(we say that* p *is globally asymptotically stable).*

1.3 Routes to chaos

At present, it is not known under what conditions a system becomes chaotic. However, there are several possible types of evolution for a regular dynamic system towards chaos. Let's suppose that the studied dynamics depend on a control parameter [13], [27],and [25]. When this parameter is varied, the system can makes a transition from a stationary state to a periodic state, and then beyond a certain threshold, follow a transition scenario and become chaotic. There are several scenarios that describe the transition from a fixed point to chaos.

In general, the evolution from a fixed point to chaos is not progressive but marked by discontinuous changes called bifurcations. A bifurcation marks the sudden transition from one dynamic regime to another, qualitatively different. All these scenarios have been predicted by theory and observed in many experiments. In physics, the Rayleigh-Bénard thermal convection, in which a layer of fluid located between two horizontal plates is subjected to a vertical temperature gradient, served originally as a model system for the study of chaos. Since then, chaos has been demonstrated in many other fields. We will briefly present three possible types of evolution.

1.3.1 Period doubling

This scenario of transition to chaos is probably the best known. By increasing the control parameter of the experiment, the frequency of the periodic regime doubles, then is multiplied by 4, by 8, by 16, etc. The doublings become increasingly closer, tending towards an accumulation point at which one would hypothetically obtain an infinite frequency. It is at this point that the system becomes chaotic.

It has been studied in particular in population dynamics by Robert May [35] on the logistic map, $X_{n+1} = rX_n(1-X_n)$. Depending on the value of the parameter r, the sequence converges either to a fixed point or not. As soon as a is greater than 3, the system bifurcates, meaning that it oscillates between two values around the fixed point. This is called an attractor cycle of period-2. By continuing to increase r, these two attractors move away from the fixed point until a new bifurcation occurs. Each point splits and we obtain an attractor cycle of period-4 (See Figure 1.1). This is called a doubling of period. It is from this example that Feigenbaum realized the existence of a form of universality in this transition to chaos in the form of a cascade of period doubling.

1.3.2 Indeterminacy

This scenario, via indeterminacy, is characterized by the erratic appearance of chaotic bursts in a system that oscillates regularly. The system maintains a periodic or nearly periodic regime for a certain amount of time, meaning a certain "regularity," and then destabilizes abruptly to give rise to a sort of chaotic explosion. It then stabilizes again to give rise to a new burst later on. It has been observed that the frequency and duration of the chaotic phases tend to increase the further one moves away from the critical value of the constraint that led to their appearance. In this case the limit cycle (corresponding to the periodic state from which this transition phenomenon arises) bifurcates sub-critically and that there is no attractor nearby. This is what is observed in the Rössler system [[30],[21]].

1.3.3 Quasi-periodicity

The scenario via quasi-periodicity was highlighted in the theoretical work of Ruelle and Takens (1971) [45], illustrated for example on the Lorenz model (1963) [33]. This scenario has been confirmed by numerous experiments, including the famous ones in thermo-hydrodynamic convection of Rayleigh-Bénard in a small box, and in chemistry such as the Belousov-Zhabotinsky reaction, among others. This route to chaos results from the "competition" of different frequencies in the dynamical system. In a system with periodic behavior at a single frequency, if we change a parameter, a second frequency appears. If the ratio between the two frequencies is rational, the behavior is periodic. But if the ratio is irrational, the behavior is quasi-periodic, and in this case, the trajectories cover the surface of a torus. Then, we change the parameter again and a third frequency appears, and so on until chaos. There are also systems that directly transition from two frequencies to chaos.

Definition 1.3.1 *The function $X_{t+T} = \lambda X_t$ describes a quasi-periodic solution, where T is the period of the cycle and T is minimal.*

1.4 Strange attractors

A strange attractor has a complex and fractal geometric shape that exists within a finite space and has a dimension that is not a decimal number. Its trajectory is intricate and almost all trajectories on the attractor never pass through the same point twice, meaning that each trajectory is aperiodic. This strange nature arises from two apparently conflicting tendencies: the attraction of trajectories towards the attractor and their divergence on it. Attraction is due to the dissipative nature of real systems, where trajectories tend to converge towards the attractor due to frictional forces. However, divergence comes from sensitivity to initial conditions. While the exponential divergence of two trajectories is a local phenomenon, attractors with finite dimensions cannot diverge infinitely, so they must fold back onto themselves. Thus, the strange attractor is the result of three simultaneous operations: contraction, stretching, and folding, which give rise to a characteristic horseshoe-shaped structure that is flattened, stretched, and folded. Due to its fractal geometry, these attractors are called strange attractors and are the signature of chaos. This signature allows for the authentication of chaotic behavior and its quantitative characterization within the basin of attraction. A "definition" of a strange attractor can be formulated as follow:

Definition 1.4.1 *A bounded subset as A within the phase space can be considered as a peculiar attractor for a transformation T of the space, under the condition that a neighborhood U of A exists. In simpler terms, for every point present within A, it is possible to identify a ball-shaped region that encompasses that specific point and is contained within the real number system $\mathbb{R}$, while adhering to the following set of properties:*

Attraction: *U is an absorbing zone, which means that every orbit whose initial point is in U is entirely contained in U. Additionally, every orbit of this type becomes and remains as close to A as desired.*

Sensitivity: *The orbits whose initial point is in U are extremely sensitive to initial conditions.*

Fractal shape: *A is a fractal object as it is shown in the Figure 1.2.*

Mixing property: *For any point in A, there exist orbits starting in U that pass as close as desired to this point.*

Definition 1.4.2 *Suppose $A \subset \mathbb{R}^n$ is an attractor. Then, we say that A is a* ***strange attractor*** *if it is chaotic.*

Some examples of strange attractors of discrete-time are illustrated bellow [see Figures 1.3, 1.4, 1.5, 1.6, and 1.7]:

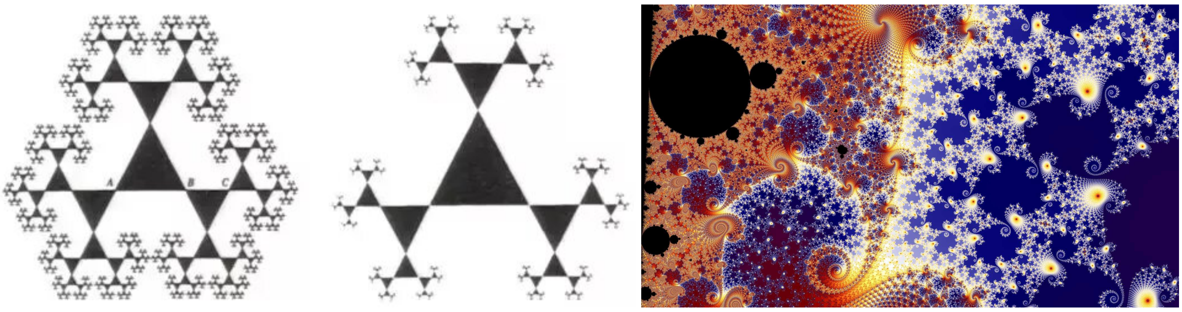

Figure 1.2: Fractal patterns of strange attractors.

Hénon attractor (1976):

The Hénon system, proposed by mathematician Michel Hénon in 1976 [[23], [24]], is a model of dynamical systems that exhibit chaotic behavior. The Hénon attractor is a type of fractal belonging to the category of odd attractors, and its structure can be characterized in terms of unstable periodic orbits within the attractor. The attractor can be conceptualized as a sheet that is repeatedly folded and stretched, introducing iterations in the plane. While the Hénon attractor has a smooth structure in one direction, it exhibits a Cantor set in the other direction. It can be modeled using two generic equations defined by the following relations:

$$\begin{cases} X_{k+1} = 1 - aX_k^2 + Y_k \\ Y_{k+1} = bX_k \end{cases}$$

while a and b are parameters.

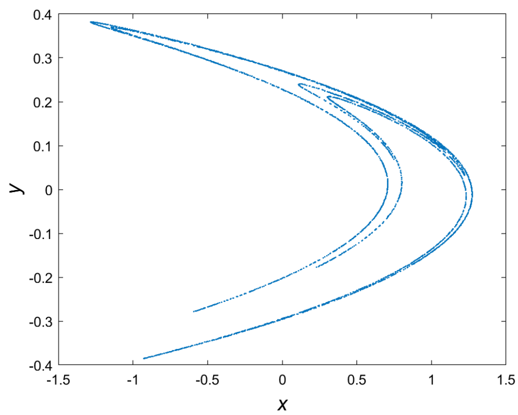

Figure 1.3: The Hénon attractor for a = 1.4 and b = 0.3.

Lozi attractor (1993):

The Lozi attractor is a two-dimensional nonlinear dynamical system that exhibits chaotic behavior. It is a simplification of the Hénon map, another well-known chaotic system was named after its creator, R. Lozi. It is defined by the following set of equations:

$$\begin{cases} X_{k+1} = 1 - a|X_k| + bY_k \\ Y_{k+1} = X_k \end{cases}$$

where a and b are parameters that control the nonlinear behavior of the system.

The Lozi attractor is characterized by its fractal geometry and its sensitive dependence on initial conditions, which are two hallmarks of chaotic systems. It has been used as a model for a variety of physical and biological phenomena, including the dynamics of neurons, the spread of forest fires, and the behavior of financial markets.

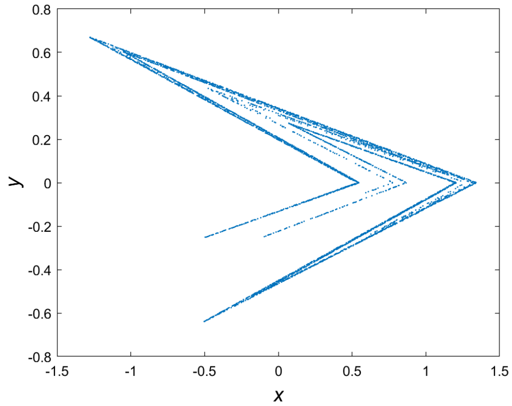

Figure 1.4: The Lozi attractor for a = 1.7 and b = 0.5.

It is a relatively simple system that is often used as a benchmark for testing numerical methods and algorithms for solving differential equations. Its simplicity and accessibility have made it a popular example for both theoretical and applied research in the field of nonlinear dynamics.

Zeraoulia-Sprott attractor (2011):
The Zeraoulia-Sprott attractor is a type of chaotic dynamical system that was discovered by Elhadj Zeraoulia and Julien Clinton Sprott in [52]. In their research, Zeraoulia and Sprott investigated a two-dimensional rational discrete mapping defined by the following system:

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = A \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \frac{aX_kY_k^2}{1+Y_k^2} \\ 0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} -a & 0 \\ 1 & b \end{pmatrix}$$

and a, b are parameters that influence the behavior of the system.

This simple 2-D map is characterized by the existence of only one rational fraction with no vanishing denominator. By analyzing the dynamics of this mapping, Zeraoulia and Sprott discovered a chaotic attractor with interesting geometric properties. The Zeraoulia-Sprott attractor is characterized by its spiral-like structure, with self-replicating patterns and intricate details.

The Zeraoulia-Sprott attractor has since gained attention in the field of chaos theory and has been studied for its unique properties and potential applications. It has been used

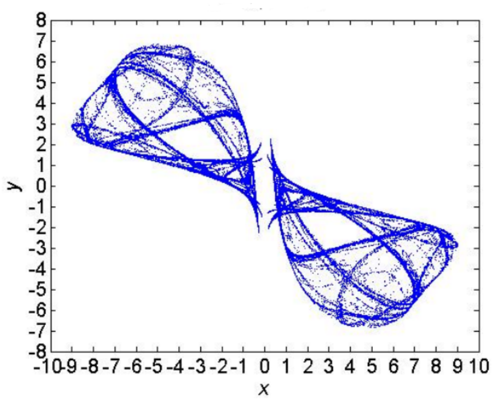

Figure 1.5: The Zeraoulia-sprott attractor for a = 3.9, b = 0.8.

in various fields, including mathematics, physics, and engineering, as a model system for studying chaos, bifurcations, and synchronization phenomena. It represents a fascinating example of a chaotic system that exhibits intricate and complex behavior. Its discovery has contributed to our understanding of nonlinear dynamics and the rich phenomena that can arise from simple mathematical mappings.

Duffing attractor (1918):
The Duffing attractor was first defined by the German mathematician Georg Duffing in 1918 in his paper [16], where he studied the dynamics of a nonlinear oscillator with a variable spring stiffness. Duffing's original system was a one-dimensional equation, but the Duffing attractor is typically associated with a two-dimensional system that was introduced later, in the 1960s, by several researchers independently. The Duffing attractor was further studied and popularized in the 1970s and 1980s, particularly by the work of Mitchell Feigenbaum and James Yorke in the field of chaos theory. It is know by the following recursive system:

$$\begin{cases} X_{k+1} = Y_k \\ Y_{k+1} = -bX_k + aY_k - Y_k^3 \end{cases}$$

where a and b represent the parameters.

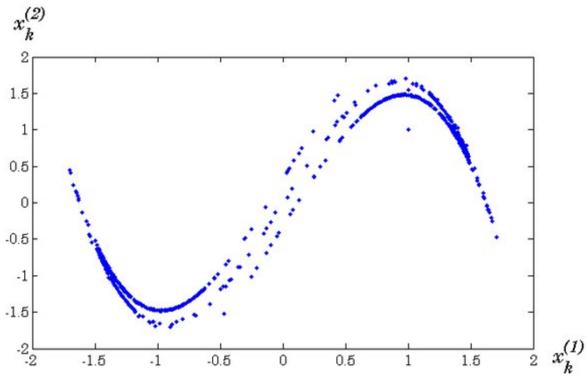

Figure 1.6: The Duffing attractor for a = 2.75 and b = 0.2.

Ikeda attractor (1979):
This recurrence was first proposed by by the Japanese physicist Kenichi Ikeda in 1979 as a model, under some simplifying assumptions, for the propagation of light in a nonlinear optical resonator (the type of cell that might be used in an optical computer):

$$\begin{cases} X_{k+1} = 1 + a(X_k cos(T_k) - Y_k sin(T_k)) \\ Y_{k+1} = a(X_k sin(T_k) + Y_k cos(T_k)) \end{cases}$$

where $T_{k+1} = b - \frac{c}{1+X_k^2+Y_k^2}$ and a, b, c are real parameters that control the nonlinear behavior of the system, and T is a phase variable.

The Ikeda attractor is characterized by its fractal nature and unpredictable patterns. When visualized, it often exhibits swirling, spiraling structures that evolve over time. These structures have both local and global symmetries, creating intricate and mesmerizing patterns.

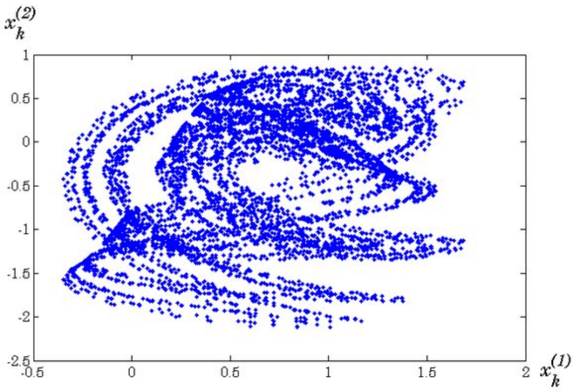

Figure 1.7: The Ikeda attractor for a = 0.9.

The dynamics of the Ikeda attractor are highly sensitive to initial conditions and parameter values. Even a small change in either can lead to dramatically different trajectories. This sensitivity to initial conditions is a hallmark of chaotic systems. It is a fascinating example of a chaotic system that showcases the beauty and complexity of nonlinear dynamics.

1.5 Notions of chaos theory

The concept of chaos was introduced in the study of discrete dynamical systems by Li and Yorke in 1975 [31]. Until the late 1980s, the study of chaotic dynamics was primarily limited to research publications. Devaney's book (An Introduction to Chaotic Dynamical Systems) in 1986 [14] marked the point where chaos (as a mathematical concept) became popular and began to enter university textbooks such as Holmgren's [26].

A dynamical system is described by a pair (X, F), where $F : X \to X$ represents a map from a topological or metric space X into itself. If F is a homeomorphism, meaning it has a continuous inverse function, the system (X, F) is considered reversible.

One of the fundamental properties in the mathematical theory of chaos is transitivity. So, let us get an eye to this property before citing the different definitions of chaos.

1.5.1 Mathematical definitions of chaotic attractors

Definition 1.5.1 *Let X be a metric space and G is a map. We say that G is* ***topologically transitive*** *if for any pair of open sets $U, W \in X$ there exists an element $X_0 \in U$ and a natural number $n \in \mathbb{N}$ such that $G^{(n)}(X_0) \in W$ i.e., $G^{(n)}(U) \cap W \neq \emptyset$.*

This definition is not always easy to verify in practice. Here is a theorem.

Theorem 1.5.1 *Let G be a map on X. Suppose there is a point $X_0 \in X$ such that its orbit*

$$\varphi(X_0) = G^n(X_0), n \in \mathbb{N}$$

is dense, then $G(X)$ is ***topologically transitive****.*

Definition 1.5.2 *(Sensitive dependence) Let (X,d) be a metric space and G a continuous map on X. We say that the topological dynamical system (X,G) has the property of* ***sensitivity to initial conditions*** *if there is a constant $c > 0$ such that:*

$$\forall x \in X, \forall \epsilon > 0, \exists y \in X, \exists n \in \mathbb{N} : d(x, y) < \epsilon$$

and

$$d(G^n(x), G^n(y)) > c.$$

Remark 1 *This property characterizes the part of the unpredictable behavior of a dynamical system. Indeed, even if the initial conditions of two orbits are very close, they move away from each other after a while.*

We are now ready to announce the definitions of chaos. Starting with chaos in the sense of **Lorenz (1960)** [33]:

Definition 1.5.3 *A system agitated by forces where only exist three independent frequencies, can become destabilized, its movements then becoming totally irregular and erratic.*

Chaos in the sense of **May (1970)** [35]:

Definition 1.5.4 *Chaos theory is the study of nonlinear dynamic systems that exhibit sensitive dependence on initial conditions.*

Chaos in the sense of **Ruelle Takens (1970s-80s)** [45]:

Definition 1.5.5 *Chaos theory is the study of strange attractors and their associated dynamics, including chaotic transients, intermittency, and bifurcations.*

Definition 1.5.6 *(**Sharkovskii Order** [47]): We define the order of Sharkovski on the nonzero natural numbers as follows:*

$$3 \triangleright 5 \triangleright 7 \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \ldots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \ldots$$
$$\ldots \triangleright 2^n \cdot 3 \triangleright 2^n \cdot 5 \triangleright 2^n \cdot 7 \triangleright \ldots \triangleright 2^n \triangleright \ldots \triangleright 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$

Theorem 1.5.2 *(See [3], page 135) Suppose a continuous map $F : [a, b] \to \mathbb{R}$ has a period-n orbit. If the Sharkovskii order holds such that $n \triangleright k$, then the map F also possesses a period-k orbit.*

James Alan Yorke and his doctoral student Tien-Yien Li published a paper titled "Period 3 Implies Chaos" in 1975, wherein they presented a noteworthy theorem. The theorem states that if a continuous function defined on a closed interval in the real line possesses a point of period 3, then it must also have points of all periods.

Lemma 1.5.1 *(See [29]) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous map, and let $I = [a, b]$ be an interval such that $F(I) \subseteq I$, then F has a fixed point in I. (in the book of Kathleen it is mentioned as the point fixe theorem with the proof in page 125)*

Lemma 1.5.2 *(See [29]) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous map, and I and J be two closed intervals such that $F(I) \subseteq J$. Then there exists a closed interval $K \subseteq I$ such that $F(K) = J$.*

Theorem 1.5.3 *(See [29]) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous map defined on some interval $X \subseteq \mathbb{R}$. If F has a periodic point of period 3, then it has a periodic point of period n for each n $\in$ N.*

Theorem 1.5.4 *(See [47]) Suppose $F : \mathbb{R} \to \mathbb{R}$ is a continuous map. If F has an orbit of period 3, then F is **chaotic**.*

Chaos in the sense of **Li and Yorke (1975)** [31]:

Definition 1.5.7 *Let G be a continuous map on a compact metric space (X,d). We say that G is **chaotic** in the sense of Li and Yorke if there exists a countable subset S of X satisfying the following properties:*

1. $\limsup_{n\to\infty} d(G^n(x), G^n(y)) > 0, \forall\, x, y \in S, x \neq y,$

2. $\liminf_{n\to\infty} d(G^n(x), G^n(y)) = 0$, $\forall$ *x*, *y* $\in$ *S*, *x* $\neq$ *y*,

3. $\limsup_{n\to\infty} d(G^n(x), G^n(y)) > 0$, $\forall$ *x* $\in$ *S*, *p* $\in$ *X*, *p periodic.*

Chaos in the sense of **Devaney (1989)** [[14], [12]]

Definition 1.5.8 *Let G be an application. Suppose the corresponding dynamical system has an attractor A. This system is said to be **chaotic** on its attractor if*

1. *G: A $\to$ A is topologically transitive,*
2. *The set of the periodic points of G is dense in A,*
3. *G has a sensitive dependence on the initial conditions.*

The following theorem (quite recent, it dates from 1992) shows that, with the exception of a few specific cases, the third condition can be excluded because it is a consequence of the first two:

Theorem 1.5.5 *(See [12]) Let G be a map defined in A. Suppose that :*

- *G is topologically transitive,*
- *The set of the periodic points of G is dense in A,*

*Then, if A contains an infinite number of points, the dynamical system defined by G is **chaotic** in A.*

Propositon 1.5.1 *When a continuous map $F : X \to X$ exhibits **chaotic** behavior on an infinite-dimensional metric space X, it demonstrates sensitive dependence on initial conditions.*

Propositon 1.5.2 *If a continuous map $F : X \to X$ satisfies the condition that for any two open subsets U and W of X, there exists a periodic point $p \in U$ and a positive integer n such that $F^n(p) \in W$, then it is considered to be **chaotic**.*

Propositon 1.5.3 *If $G : X \to X$ is conjugate to $F : X \to X$, then G is **chaotic** on Y if and only if F is **chaotic** on X.*

There is no universal and accepted definition of chaotic attractor and all the previous definitions are restrictive and reflect some properties observed in numerical simulations.

1.5.2 Characterization of chaotic attractors

Chaos refers to the phenomenon of seemingly random and unpredictable behavior in a deterministic system. There are several characteristics of chaos, including:

1. **Sensitivity to initial conditions:** This refers to the fact that a chaotic system is highly sensitive to its initial conditions. Even a small change in the starting conditions of the system can lead to vastly different outcomes over time. This sensitivity to initial conditions is often referred to as the "**butterfly effect**" because small changes in one part of the system can have large and unpredictable effects on other parts of the system.

2. **Phase space:** A mathematical construction that allows us to visualize the behavior of a chaotic system over time. In phase space, each point represents the state of the system at a particular moment of time. As time passes, the system traces out a path in phase space. The shape of this path can tell us a lot about the behavior of the system, including whether it is chaotic or not.

3. **Fractal dimension:** A measure of the complexity of a chaotic system. A fractal is a geometric object that has self-similar patterns at different scales. In a chaotic system, the behavior of the system may also exhibit self-similar patterns at different scales, and we can measure the fractal dimension of these patterns to quantify the system's complexity.

4. **Positive Lyapunov exponents:** A mathematical value derived from an application and divided into three categories that reveal the nature of the solution. An algorithm can calculate this exponent, and the results are interpreted as follows: If $LE < 0$, then the solution is a fixed point. If $LE = 0$, then the solution is cyclical. Finally, if $LE > 0$, the solution is both chaotic and bounded. For example, the **Lorenz system** is known to exhibit chaotic behavior for certain parameter values. One set of parameter values that leads to chaotic behavior is $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. In this case, the Lyapunov exponent has been calculated to be approximately 0.9.

5. **Lyapunov dimension:** A another measure of the complexity of a chaotic system. It is based on the concept of Lyapunov exponents, which describe how quickly nearby trajectories in phase space diverge from each other. The Lyapunov dimension is a measure of the dimensionality of the space that contains the chaotic attractor.

6. **Strange attractors:** As we have mentioned in the previous section, they are complex, non-repeating patterns that chaotic systems can exhibit. These patterns are often fractal in nature, and they are called "strange" because they are not found in simple periodic systems.

7. **Capacity dimension (Kolmogorov):** A measure of the amount of information needed to describe the behavior of a chaotic system. It is based on the concept of entropy, which is a measure of the amount of disorder in a system. The capacity dimension is a way to quantify the amount of information needed to specify the system's behavior. For example, The **Koch Snowflake** which is a fractal curve that is created through

an iterative process. Start with an equilateral triangle, and then replace each straight line segment with a smaller equilateral triangle. After each iteration, it becomes more intricate and complex. The capacity dimension of the Koch Snowflake is given by: $D = log(4)/log(3)$, which is approximately 1.2619.

1.5.3 Robust chaos

Robust chaos is a concept that describes chaotic systems that remain chaotic despite small perturbations. It is a type of chaos that is highly persists to external disturbances, making it a powerful tool for investigating complex systems.

Robust chaos is characterized by a high degree of unpredictability and sensitivity to initial conditions. It is also known for its ability to generate highly complex patterns and behaviors. It is closely related to the field of piecewise nonlinear dynamics, which studies the behavior of systems that are not governed by linear equations [53], [55]. So, it is an important concept in Nonlinear dynamics and complexity theory.

Definition 1.5.9 ***Robust chaos*** *is characterized by the absence of periodic windows and the presence of coexisting attractors within a certain neighborhood in the bifurcation parameter space of a dynamical system. This definition highlights the persistent chaotic behavior that remains unaffected by small variations in the system's parameters.*

The presence of periodic windows in certain chaotic regions implies that slight parameter variations can disrupt the chaotic behavior, indicating the fragility of this type of chaos. On the other hand, a system is considered robust when it can effectively adapt to changes in its operating environment without significant loss of functionality. In essence, robustness refers to the system's capacity to maintain proper operation across a broad range of operating conditions.

A less conventional example of robust chaos is the chaotic behavior of piecewise smooth maps which is the object of the next chapter. This type of chaos occurs when a system is perturbed by a discontinuity, such as a sharp change in slope. The resulting behavior of the system is highly sensitive to small changes in the initial conditions, and this can lead to a wide range of chaotic behaviors. Such chaos can be used to study the behavior of complex systems, as well as the impacts of discontinuities on chaotic systems.

Chapter 2

Chaotic behavior of piecewise smooth maps

Chaos often breeds life, when order breeds habit.

– Henry Adams

2.1 Introduction

A piecewise smooth map is a concept that serves as a representation of a function, over a specific domain. It consists of pieces or regions which seamlessly merge at their boundaries. This allows for the depiction of functions that cannot be easily expressed by an equation.

These maps can represent a wide array of functions ranging from straightforward linear equations to more intricate ones like sinusoidal curves and fractals. They find application in fields such as computer graphics, numerical analysis and game theory. In computer graphics they are employed to generate landscapes. In analysis they aid in solving equations efficiently. Moreover in game theory they serve as models for illustrating interactions among players. For instance different regions within a game may have rules or players might adopt varying strategies based on their location, within the game world.

Here in this chapter, we will focus on the various methods used for proving the chaotic behavior of piecewise mappings. Earlier, there was no systematic study categorize the possible bifurcations in piecewise smooth maps as there was no physical system with these characteristics occur in the real science. But, thanks to Banerjee and Grebogi [11], [7], many researches has been investigated on the bifurcations in the switching circuits used in power electronics and large classes of engineering systems. In this type of systems, the analysis of their chaotic dynamics revolves around examining the affinity between the respective normal forms of fixed points located at boundaries. The investigation of the behavior of fixed and periodic points in different scenarios is contingent upon the variation of the bifurcation parameter.

In Sec 2.2 and 2.3, we will talk about the piecewise smooth maps in 1-D and 2-D, respectively. In Sec 2.4, we will focus on the theory of the robust chaos and we will discuss the essential points of the chaotic behavior and bifurcation scenarios in the piecewise smooth mappings.

2.2 One-dimentional piecewise smooth maps

Piecewise smooth maps (PWS maps) are typically used to study the chaotic behavior of complex systems. These maps are defined by a series of conditions that must be met, such as the number of elements in the domain. By examining how these conditions interact with each other, it is possible to gain insight into the behavior of the system. This is especially true when the conditions are varied, as it allows for the study of the chaos theory in more details.

PWS maps are another type of chaotic system that can be studied using the Banerjee's normal form of border collision. They have discontinuities in their derivatives at certain points. This leads to a chaotic behavior that is not easily predicted, making it a useful tool for exploring the behavior of complex systems. This type of chaos has been used to study the dynamics of predator-prey models and the motion of celestial bodies.

Let's start by considering the following one-dimentional (1-D) PWS map, F is defined on $\mathbb{R}$ and depends smoothly on the bifurcation parameter μ :

$$X_{k+1} = F(X_k, \mu) = \begin{cases} f(X, \mu), X \leq X_b \\ g(X, \mu), X \geq X_b \end{cases} \tag{2.1}$$

where X_b is the border line which devides the state space in two regions R_L and R_R as follow:

$$R_L = \{X : X \leq X_b\}$$

and

$$R_R = \{X : X \geq X_b\}$$

Assuming that $X_0(\mu)$ is a possible path of fixed points of F which depends continuously on μ, while the fixed point hits the boundary at a critical parameter value μ_b ,i.e., $X_0(\mu_b) = X_b$.

Theorem 2.2.1 *The piecewise smooth one-dimensional map's normal form is provided by [Yuan (1997), Banerjee, et al. (2000)] as follow :*

$$G(X, \mu) = \begin{cases} aX + \mu, X \leq 0 \\ bX + \mu, X \geq 0 \end{cases} \tag{2.2}$$

Proof 1 *A piecewise affine approximation of the map near the border point X_b is the normal form (2.1) at a fixed point on the border. The way such a form is derived is as follows: If we assume that $\bar{X} = X - X_b$ and $\bar{\mu} = \mu - \mu_b$, then Eq. (2.1) becomes:*

$$\bar{F}(\bar{X}, \bar{\mu}) = \begin{cases} f(\bar{X} + X_b, \bar{\mu} + \mu_b), & \bar{X} \leq 0 \\ g(\bar{X} + X_b, \bar{\mu} + \mu_b), & \bar{X} \geq 0 \end{cases} \tag{2.3}$$

Consequently, the fixed point of map (2.3) is at the border for the parameter value $\bar{X} = 0$, and the state space is divided into two halves, $R_{-} = (\infty, 0]$ and $R_{+} = [0, +\infty)$. The result of expanding $\bar{F}$ to the first order near (0,0) is:

$$\bar{F}(\bar{X}, \bar{\mu}) = \begin{cases} a\bar{X} + \bar{\mu}\vartheta + O(\bar{X}, \bar{\mu}), & \bar{X} \leq 0 \\ b\bar{X} + \bar{\mu}\vartheta + O(\bar{X}, \bar{\mu}), & \bar{X} \geq 0 \end{cases} \tag{2.4}$$

where

$$\begin{cases} a = \lim\limits_{X \to 0_{-}} \frac{\partial}{\partial X} \bar{F}(\bar{X}, \bar{\mu}) \\ b = \lim\limits_{X \to 0_{+}} \frac{\partial}{\partial X} \bar{F}(\bar{X}, \bar{\mu}) \\ \vartheta = \lim\limits_{X \to 0^{+}_{-}} \frac{\partial}{\partial \mu} \bar{F}(\bar{X}, \bar{\mu}) \end{cases}$$

Due to the smoothness of F in μ, it should be noted that the final limit in (2.4) is independent of the direction in which 0 is approached by x. The non-linear terms are insignificant close to *the border if $\vartheta \neq 0$, $|a| \neq 1$, and $|b| \neq 1$. The 1-D normal form is given by defining a new parameter $\mu^* = \mu\vartheta$ and eliminating the upper order terms as in [49] and [7].*

$$G(\bar{X}, \bar{\mu}) = \begin{cases} a\bar{X} + \mu^*, \bar{X} \leq 0 \\ b\bar{X} + \mu^*, \bar{X} \geq 0 \end{cases} \tag{2.5}$$

Which is similar to the form (2.1).

2.3 Two-dimentional piecewise smooth maps

Power electronics is a field that deals with the efficient conversion of electrical power from one form to another. This area has practical applications and involves power converters that exhibit several nonlinear phenomena, including border-collision bifurcations, coexisting attractors, and chaos, which are created by switching elements. Piecewise-smooth systems can also exhibit border-collision bifurcations, which produce a discontinuous change in the Jacobian matrix's elements when the bifurcation parameter is varied. Several researchers have studied border-collision bifurcations in power electronics, and these phenomena have been reported in the works of the Roberts and Banerjee [44], [7], [11].

Let us consider the following 2-D piecewise smooth map given by:

$$G(X, Y, \mu) = \begin{cases} G_1(X, Y, \mu), & \text{if } X < S(Y, \mu) \\ G_2(X, Y, \mu), & \text{if } X \geq S(Y, \mu) \end{cases} \tag{2.6}$$

where

$$G_1(X, Y, \mu) = \begin{pmatrix} f_1(X, Y, \mu) \\ f_2(X, Y, \mu) \end{pmatrix}$$

and

$$G_2(X, Y, \mu) = \begin{pmatrix} g_1(X, Y, \mu) \\ g_2(X, Y, \mu) \end{pmatrix}$$

and μ denotes the bifurcation parameter.

The phase plane is partitioned into two distinct regions, denoted as R_L and R_R, by the smooth curve defined by $X = S(Y, \mu)$. This curve serves as a dividing line, separating the phase space into the left region R_L and the right region R_R given as follows:

$$R_L = \{(X, Y) \in \mathbb{R}^2, \ \ X < S(Y, \mu)\}$$

and

$$R_R = \{(X, Y) \in \mathbb{R}^2, \ \ X \geq S(Y, \mu)\}$$

and the boundary between them is defined by:

$$\Sigma = \{(X,Y) \in \mathbb{R}^2, \;\; X = S(Y,\mu)\}$$

A 2-D PWS map $G(X,Y,\mu)$ is defined by satisfying the following conditions:

1. The map G is continuous, but its derivative exhibits a discontinuity at the boundary given by $X = S(Y,\mu)$.
2. The functions G_1 and G_2 are both continuous and possess continuous derivatives.
3. The one-sided partial derivatives at the boundary are well-defined and finite in each subregion R_L and R_R.
4. The map $G(X,Y,\mu)$ has a single fixed point in R_L and another fixed point in R_R for a specific value μ_* of the parameter μ.

Additionally, it is possible to consider $S(Y,\mu) = 0$ in equation (2.6) through a change of variables, as demonstrated in [7].

Let $Z = (X,Y)^t$, the $BCNF_{RC}$ in two-dimentional piecewise mapping given in [7] is the continuous map:

$$Z_{n+1} = (X_{n+1}, Y_{n+1})^t = \begin{cases} J_L Z_n + m, & \text{if } X_n < 0 \\ J_R Z_n + m, & \text{if } X_n \geq 0 \end{cases} \tag{2.7}$$

where

$$J_k = \begin{pmatrix} \tau_k & 1 \\ -\delta_k & 0 \end{pmatrix}, \; k = L \text{ or } R, \text{ and } \; \mathrm{m} = \begin{pmatrix} \mu \\ 0 \end{pmatrix}.$$

The parameters τ_k and δ_k are considered fixed, while μ represents the bifurcation parameter.

The fixed points of the system on both the right and left sides are defined as follows:

$$p_L = \left(\frac{-1}{\tau_L - 1 - \delta_L}, \frac{\delta_L}{\tau_L - 1 - \delta_L}\right), \; p_R = \left(\frac{1}{1 + \delta_R - \tau_R}, \frac{-\delta_R}{1 + \delta_R - \tau_R}\right). \tag{2.8}$$

Their stability is determined by the eigenvalues of the corresponding Jacobian matrix, i.e.,

$$\lambda_{1,2} = \frac{1}{2}(\tau_k \pm \sqrt{\tau_k^2 - 4\delta_k}) \tag{2.9}$$

The map (2.7) is smooth and the boundary between the two regions R_L and R_R is given by:

$$\Sigma = \{(X,Y) \in \mathbb{R}^2, X = 0, Y \in \mathbb{R}\}$$

The stability analysis of fixed points in a 2-D piecewise smooth map is determined by examining the eigenvalues of the corresponding Jacobian matrices, as described in previous studies such as [17], [15]. Furthermore, the investigation of period-1 and period-2 orbits occurring before and after the border collision has been explored, including a classification of border-collision bifurcations in n-dimensional piecewise smooth systems. This classification takes into account different scenarios based on the number of real eigenvalues greater than 1 or less than -1. In this context, we will present the established sufficient conditions for potential bifurcation phenomena in the normal form (2.7).

2.4 Border Collision Bifurcation Scenarios and Robustness

Border collision bifurcation (BCB) is a phenomena in which two equilibrium points collide and split into two new equilibrium points. This process is known as "bifurcation" and is most commonly seen in dynamical systems. So, it is a mathematical representation of a specific type of bifurcation that occurs when a dynamic system encounters a boundary or a border.

The border collision normal form specifically focuses on bifurcations that arise when a system operating in a stable state on one side of a border experiences a sudden transition or collision with the boundary. This collision can generate complex and unpredictable behavior, including the emergence of chaos. It can be used to explain the behavior of complex systems and can have a wide range of applications, from population dynamics to the behavior of lasers.

It has been used also to study the behavior of chaotic systems which found in weather, chemical reactions and biological systems, as well as to study the effects of small perturbations in a system's parameters. In control theory, it can be used to design controllers for complex systems. In Robotics, it can be used to design some controllers for robotic arms.

And, finally in electrical engineering, particularly in switching circuits is used in power electronics as it was provided in the works of Banerjee and Grebogi concerning this theory [7], [11]. The theory has been studied extensively, and powerful techniques such as renormalization have been applied to reveal previously undescribed bifurcation structures while the normal form is a basic tool used in this theory.

2.4.1 The normal form of one-dimentional PWS maps

Let's discuss some BCB scenarios of (2.1) for X_b when μ near to μ_b. (provided in [49], [8] and [55])

Let

$$\begin{cases} p_R = \frac{\mu}{1-b} \geq 0, \text{ if } b < 1 \\ p_L = \frac{\mu}{1-b} \leq 0, \text{ if } a < 1 \end{cases} \tag{2.10}$$

be the possible fixed points of the system (2.2) near the border to the right and to the left, respectively.

1. **Scenario A: Persistent fixed point (non-bifurcation)**

   As μ is varied, we can obtain different scenarios of BCB by various combinations of the parameters $a \geq b$. Since the normal form (2.2) is invariant under some specific transformation, we can see two situations of this scenario shown in Figure 2.1:

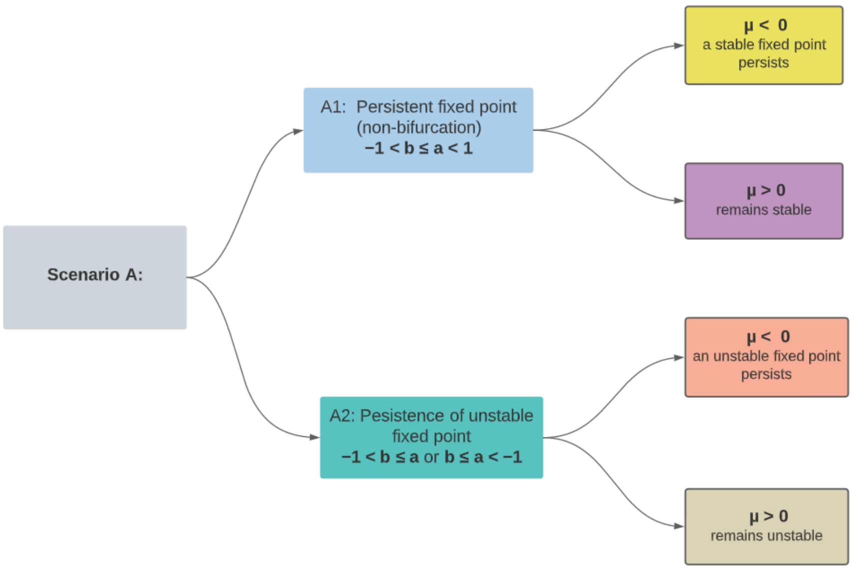

Figure 2.1: The different cases of Scenario A as μ varies.

   We have two cases, A1 and A2:

   In the case A1, the fixed point p varies continuously as a function of the bifurcation parameter, but the eigenvalue associated with the system linearization at p varies discontinuously from a to b at $\mu = 0$.

   While in case A2, there exist one unstable fixed point depends continuously on μ, but no local attractors exist while the system trajectory diverges for all initial conditions.

2. **Scenario B: Border collision pair bifurcation**

   For values of parameters a and b as specified in Scenarios A1 and A2, the two main types of BCB are observed:

   (a) Border collision pair bifurcation: which bears resemblance to saddle-node bifurcation (or tangent bifurcation) in smooth systems.

   (b) Border crossing bifurcation: which shares some similarities with period-doubling bifurcation in smooth maps (supercritical period doubling bifurcation in smooth maps, with one distinction).

Specifically in border collision pair bifurcation, the map (2.1) has two fixed points, one on each side of the border, with positive (negative) values of p resulting in one fixed point on each side, and negative (positive) values of J leading to no fixed points. Therefore, the border collision pair bifurcation occurs when

$$b < 1 < a \tag{2.11}$$

and then we find three situations leads to this scenario which we illustrate in the Figure 2.2:

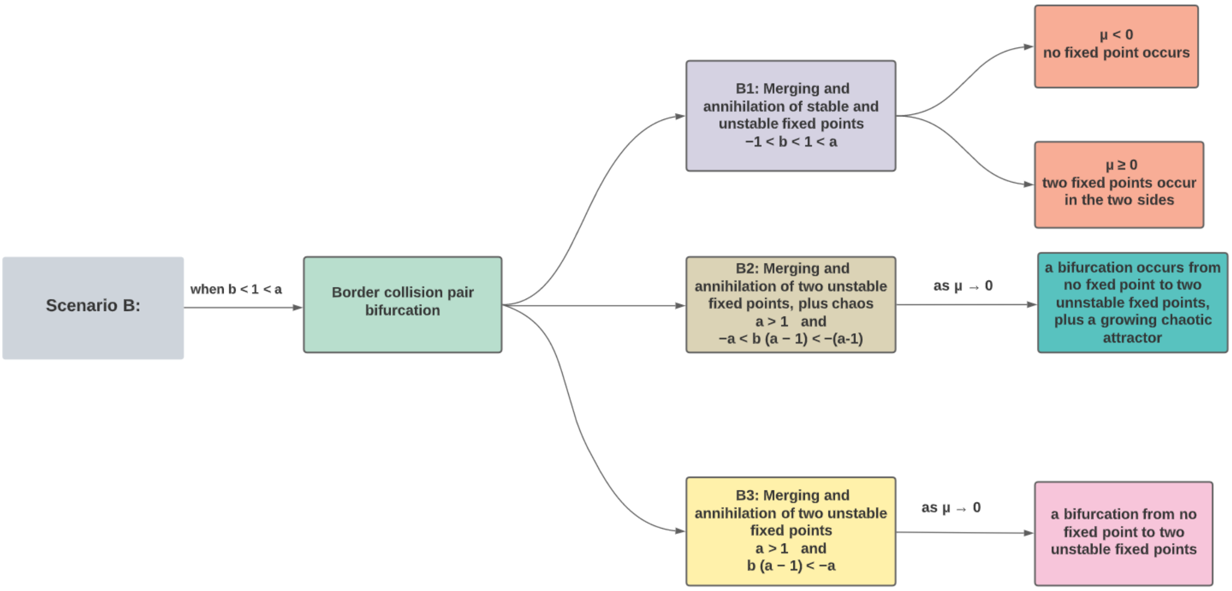

Figure 2.2: The different scenarios leading to scenario B.

3. **Scenario C: Border crossing bifurcation**

   The equilibrium remains and crosses the border as it changed through zero, and other attractors or repellers arise or disappear as a result of the bifurcation. It does occur if

$$a > -1, \tag{2.12}$$

   and

$$b < -1 \tag{2.13}$$

   and we get three situations illustrated in the Figure 2.3:

   This situation is determined by the pair (a, b). Nusse and York [40] provide further informations.

2.4.2 The normal form of two-dimentional PWS maps

Subsequently, we provide a comprehensive overview of sufficient conditions that contribute to the potential occurrence of bifurcation phenomena in the normal form (2.7). The investigation of border collision bifurcations initially stemmed from an exploration of the

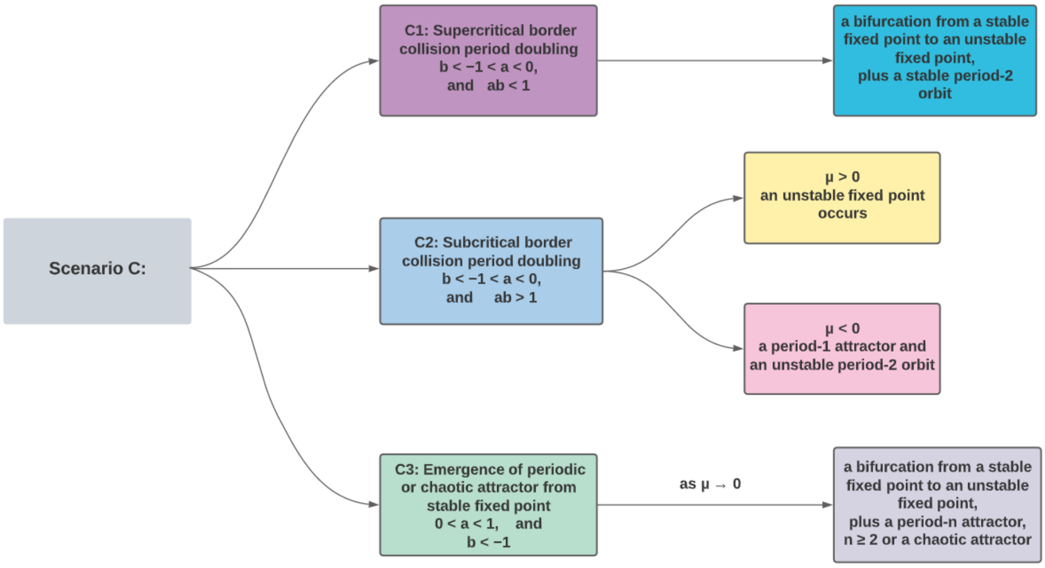

Figure 2.3: The different scenarios leading to scenario C.

dynamics of piecewise smooth maps, as noted in Feigin (1970) [17], and Nusse and Yorke (1992) [39]. Subsequently, it was observed that discrete-time models of numerous power electronic circuits exhibit piecewise smooth maps, which frequently manifest non-smooth bifurcations, as demonstrated in Yuan et al. (1998) [50].

Further insights were provided in [17] and [15], elucidating that for a two-dimensional piecewise smooth (PWS) map, the stability of its fixed points is related to the eigenvalues of the corresponding Jacobian matrices. Additionally, an investigation of the existence of period-1 and period-2 orbits pre-border and post-border collision was conducted, accompanied by a classification of border-collision bifurcations in n-dimensional piecewise smooth systems. This classification is depending upon the number of real eigenvalues surpassing 1 or falling below -1.

Alternative classification methodologies can be found in [7] and [8]. Their approaches involve the consideration of asymptotically stable orbits, including chaotic orbits, both before and after border collision, under the assumption that $|\delta_L| < 1$ and $|\delta_R| < 1$. However, it has been proven that attractors can indeed exist when the magnitude of the determinant on one side exceeds unity, while the determinant on the other side is smaller than unity. This particular scenario, occurs $|\delta_L| > 1$ and $\delta_R = 0$, as encountered in certain classes of power electronic systems, Parui and Banerjee (2002) [41].

Let us show the possible types of fixed points of the normal form (2.7) in the figures 2.4 and 2.5 bellow while we have two cases, the first when the determinant is positive and the second when it is negative:

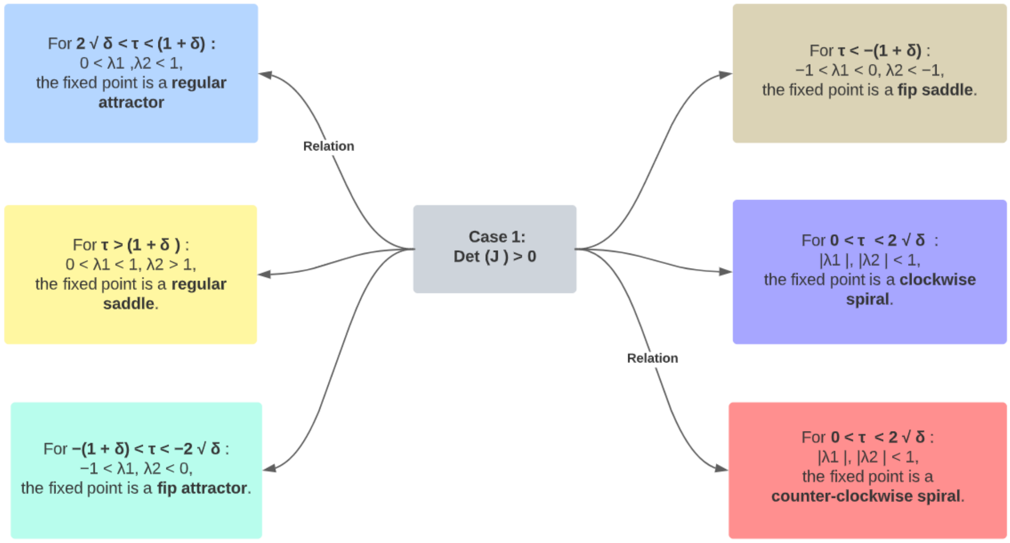

Figure 2.4: Fixed point types for the case 1: $Det(J) > 0$.

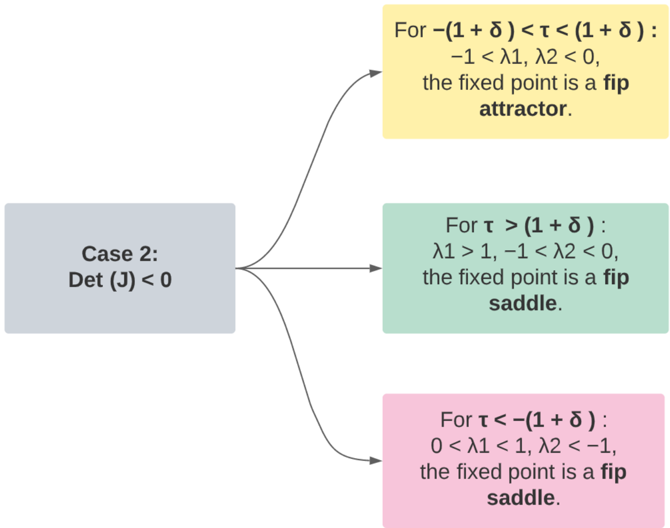

Figure 2.5: Fixed point types for the case 2: $Det(J) < 0$.

The instances in which a locally unique fixed point before the border gives rise to either a new locally unique fixed point or a locally unique period-2 attractor after the border, with no occurrence of chaos. These sets are called non chaos regions which we are going to discuss in the next section.

2.4.3 Regions leading to non-chaos

It depends on the sign of the determinants on both sides of the regions of system (2.7), so, the fourth cases will be illustrated using schemas where we have six essential scenarios, some of those scenarios have secondary scenarios.

Scenario A: As the bifurcation parameter μ is incrementally increased or decreased through zero, a stable fixed point remains persistent. In this particular scenario, the transition of a stable fixed point across the border results in the presence or absence of extraneous periodic orbits originating from the critical point. This occurrence is contingent upon the following conditions:

(i) A transformation from a regular attractor to a flip attractor,

(ii) The persistance of a regular attractor,

(iii) A transition from a flip attractor to a regular attractor,

(vi) The persistence of a flip attractor.

For further in-depth information, please refer to the details provided in [55].

Scenario B: This particular case can be subdivided into two distinct secondary scenarios, denoted as **B1** and **B2**. These scenarios arise due to the presence of two distinct regions within the parameter space where period doubling border collision bifurcation (BCB) takes place. In essence, this bifurcation results in a locally unique stable fixed point transforming into an unstable fixed point, accompanied by the emergence of a locally unique attracting period-two orbit. See Figure 2.6.

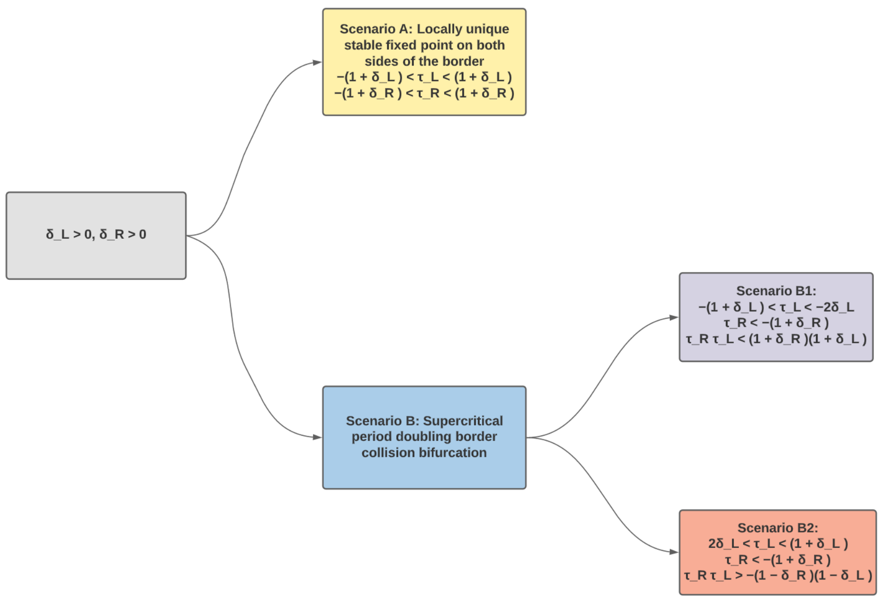

Figure 2.6: Fixed point Scenarios in the case of positive determinant on both sides of the border.

Scenario C: As the parameter μ increases through zero, a locally unique stable fixed point undergoes a transformation, resulting in the emergence of another locally unique stable fixed point.

Scenario D: When the parameter μ is negative, a locally unique stable fixed point positioned to the left of the border undergoes a transition upon crossing the border. It transforms into an unstable state, while simultaneously giving rise to a locally unique two-period orbit as μ is incrementally increased through zero. This particular scenario corresponds to a condition known as supercritical period doubling border collision, which exhibits the absence of extraneous periodic orbits. See Figure 2.7.

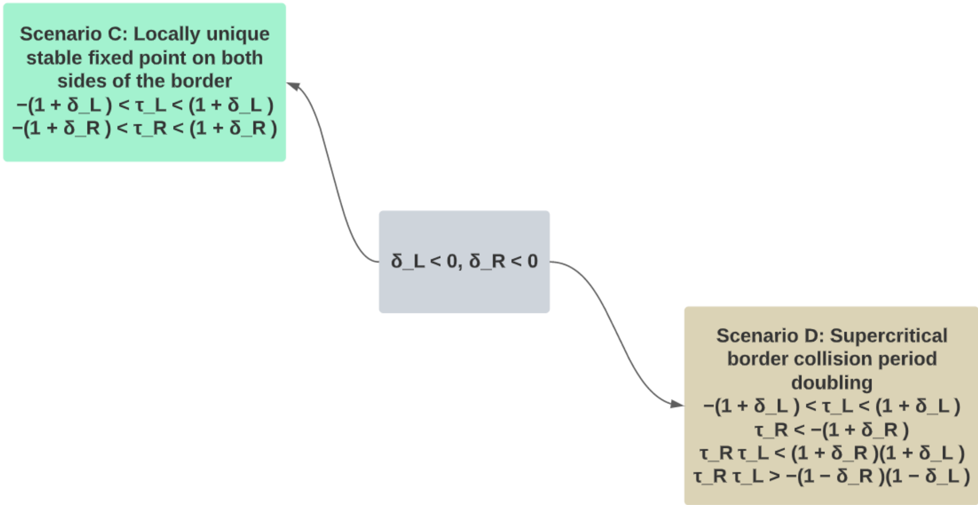

Figure 2.7: Fixed point Scenarios in the case of negative determinant on both sides of the border.

Scenario E: A locally unique fixed point leads to a locally unique fixed point for a sufficient condition concerning the positive determinant to the right side of the border. This case is divided into two sub cases **E1** and **E2**. See Figure 2.8.

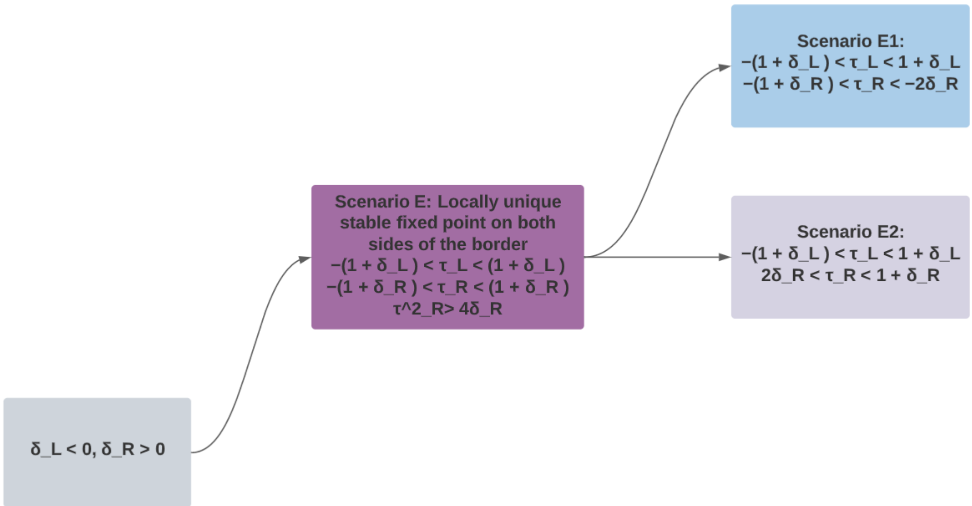

Figure 2.8: Fixed point Scenarios in the case of negative determinant and positive determinant to the left and to the right, respectively.

Scenario F: A locally unique fixed point leads to a locally unique fixed point for a sufficient condition concerning the positive determinant to the left side of the border. Also, this case is divided into two secondary cases **F1** and **F2**. See Figure 2.9

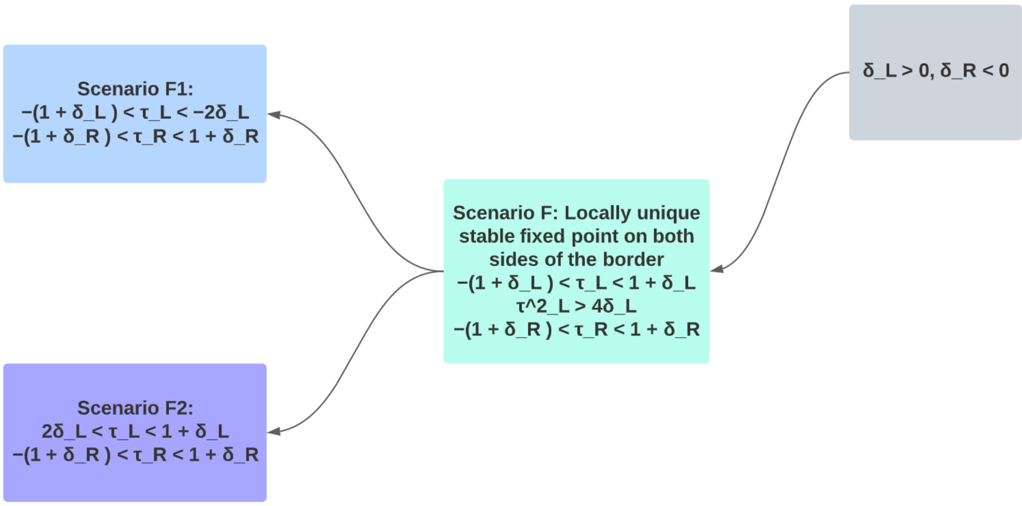

Figure 2.9: Fixed point Scenarios in the case of positive determinant and negative determinant to the left and to the right, respectively.

Remark 2 *To date, no established conditions have been identified for the occurrence of supercritical border collision period doubling without extraneous periodic orbits (EPOs) when the determinants on both sides of the border exhibit opposite signs [2].*

2.4.4 Regions leading to robust chaos

Robust chaos, as we have defined in **Chapter 1**, refers to the persistence of chaotic behavior in a system even when subjected to small perturbations or variations in its parameters. It implies that the chaotic dynamics are persists and not easily destroyed by external influences.

The presence of robust chaos in the border collision normal form implies that even slight changes near the border can lead to chaotic responses, making the system highly sensitive to perturbations. Consequently, it becomes crucial to understand and analyze the dynamics associated with these bifurcations to ensure system stability and avoid undesirable and potentially dangerous outcomes. And that's why the regions for robust chaos can be refered as **undesirable and dangerous bifurcations** (See [55]).

Definition 2.4.1 *The dangerous bifurcations considered begin with a system operating at a stable fixed point on one side of the border, say the left side.*

The possible dangerous bifurcations for a border collision are shown in Figure 2.10:

In the first scenario, there is a convergence between a point and an unstable point. As the parameter μ goes beyond zero these points eventually disappear. This occurrence is similar to saddle node bifurcations often seen in maps. When regional attractors become unstable the systems trajectory changes noticeably, for values of μ.

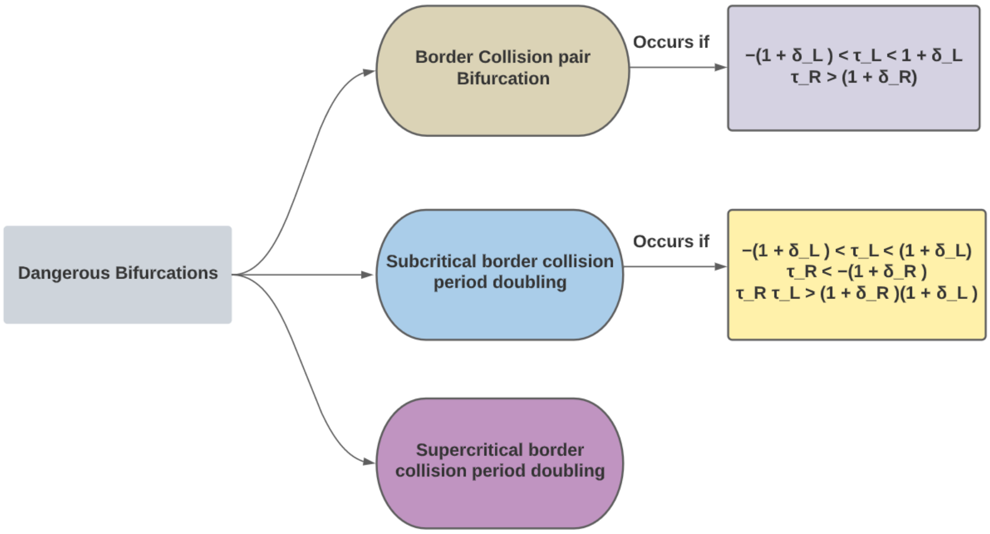

Figure 2.10: Different cases for the dangerous bifurcations occurrence.

In the second case, an undesirable bifurcation arises, whereby an unstable fixed point emerges to the right of the border as μ is increased through zero, replacing a stable fixed point and an unstable period-2 orbit to the left of the border.

The third case, though not deemed dangerous, can be undesirable in some applications, such as the cardiac conduction model presented by [Sun et al. (1996)] as a two-dimensional PWS map, where the atrial His interval A denotes the interval between the excitation of cardiac impulse in the lower inter-atrial septum and the Bundle of His. An example for this case was detailed in [55].

Notably, Banergee et al. (1998-1999) [[11], [7]] demonstrated that robust chaos manifests within specific regions of the map (2.7). The following conditions were identified as crucial for the occurrence of robust chaos:

Firstly,

$$\begin{cases} \tau_L > 1+\delta_L, \\ 0<\delta_L<1, \end{cases} \text{and} \quad \begin{cases} \tau_R < -(1+\delta_R), \\ 0<\delta_R<1, \end{cases} \tag{2.14}$$

while the parameter range for boundary crisis is given by:

$$\delta_L\tau_L\lambda_{1L} - \delta_L\lambda_{1L}\lambda_{2L} + \delta_R\lambda_{2L} - \delta_L\tau_R + \delta_L\tau_L - \delta_L - \lambda_{1L}\delta_L > 0. \tag{2.15}$$

This inequality determines the condition for stability of the chaotic attractor. As τ_L is decreased below $(1+\delta_L)$, the robust chaotic orbit remains persistently intact.

Secondly,

$$\begin{cases} \tau_L > 1+\delta_L, \\ \delta_L<0, \end{cases} \text{and} \quad \begin{cases} \tau_R < -(1+\delta_R), \\ -1<\delta_R<0, \end{cases} \tag{2.16}$$

and

$$\frac{\lambda_{1L}-1}{\tau_L-1-\delta_L} > \frac{\lambda_{2R}-1}{\tau_R-1-\delta_R}. \tag{2.17}$$

The determination of the chaotic attractor's stability is also contingent upon (2.15). Nevertheless, in the event that the condition (2.17) is not met, the criterion for the existence of the chaotic attractor undergoes a modification, leading to the following condition:

$$\frac{\lambda_{2R} - 1}{\tau_R - 1 - \delta_L} < \frac{\tau_L - \delta_L - \lambda_{2L}}{(\tau_L - 1 - \delta_L)(\lambda_{2L} - \tau_R)}. \tag{2.18}$$

Finaly, in certain scenarios, the determination of the remaining ranges for τ_k and δ_k, $k = L, R$, can be achieved by employing a similar logical approach as in the aforementioned two cases. However, it is worth noting that in some instances, there is no analytical criterion available for identifying a boundary crisis. Hence, numerical methods become necessary to ascertain the occurrence of such a crisis.

Chapter 3

Classification of strange attractors

Chaos theory simply suggests that what appears to most people as chaos is not really chaotic, but a series of different types of orders with which the human mind has not yet become familiar.

– Frederick Lenz

3.1 Introduction

Chaotic dynamics are characterized by the existence of "attractors", that represent the asymptotic limits of solutions starting from any initial condition located within a basin of attraction, which is a domain of non-zero volume. It has been notices in the current literature that numerous dynamical systems are chaotic and have strange attractors.

Strange attractors as we said in the first chapter are a special type of chaotic attractors that have fractal structure and exhibit a certain level of unpredictability in their behavior. They arise in many nonlinear dynamical systems, including weather patterns, fluid flows, and biological systems.

The classification of strange attractors involves identifying and categorizing them based on their geometric properties, dynamical behavior, and other features. This can be done using various methods, including visual inspection, topological analysis, and mathematical modeling. One of the key features used to classify strange attractors is their fractal dimension, which characterizes the self-similarity and complexity of their structure. Other features that can be used to classify strange attractors include their Lyapunov exponents, which measure the rate of divergence of nearby trajectories, and their periodic orbits, which are recurrent trajectories that repeat after a certain number of iterations.

It is a challenging task due to the complexity and diversity of their behaviors. Different classification schemes have been proposed based on various properties and characteristics of the strange attractors. According to Anishchenko & Strelkova, Plykin and Zeraoulia [4], [42], [54], [53], they are defined as ideal concepts occurring in chaotic systems satisfying a number of rigorous mathematical properties. They are classified into three principal classes which are: hyperbolic attractors, Lorenz-type attractors and quasi-attractors.

In this chapter we are going to explore this specific classification in more details. In Sec 3.2 we are going to talk about Hyperbolic attractors type with some examples. In Sec 3.3 we'll move towards the second type which is the Lorenz-type attractors. And in Sec 3.4, we'll take the concept of Quasi-attractors. We will provide a structural proof of the existence of a specific type which is a mixture between quasi and hyperbolic attractors called *Quasi-hyperbolic attractors* in a new similar form to Lozi map generalization.

3.2 Hyperbolic attractors

Hyperbolic attractors are a well-studied type of attractors that often arise in dynamical systems exhibiting chaotic behaviors. They are characterized by the presence of hyperbolic fixed points, which are points in phase space where the Jacobian matrix has at least one eigenvalue with positive real part and one eigenvalue with negative real part. The stable and unstable manifolds associated with these hyperbolic fixed points have distinct directions, causing nearby trajectories in phase space to rapidly diverge from each other. This exponential divergence of trajectories is a hallmark of chaotic behavior and it is often associated with some complex phenomena such as turbulence and strange attractors.

The word "True Chaos" is commonly associated with hyperbolic chaos, which derives its name from its homogeneous and topologically stable structure.

Definition 3.2.1 *Let's consider $F : \Omega \in \mathbb{R}^n \longrightarrow \mathbb{R}^n$, a $\mathcal{C}^r$ real function defining a discrete map while Ω is a manifold. Then,*

1. *If for every neighborhood U of such a point X, there exist $k \geq 1$ such that $F^k(U) \cap U \neq 0$ then, X is called a non-wandering point for the map F.*
2. *A non-wandering set of F is the set of all non-wandering points.*
3. *An F-invariant subset Λ of $\mathbb{R}^n$ satisfies $F(\Lambda) \subset \Lambda$.*
4. *Given a diffeomorphism F defined on a compact smooth manifold $\Omega \subset \mathbb{R}^n$, an F-invariant subset Λ of $\mathbb{R}^n$ is considered hyperbolic if there exist constants $0 < \lambda_1 < 1$ and $c > 0$ satisfying the following condition:*
   (a) *$T_\Lambda \Omega = \Sigma^s \bigoplus \Sigma^u$, where $\bigoplus$ means the algebraic direct sum and $T_\Lambda \Omega$ is the tangent space of Ω.*
   (b) *$DF(X)\Sigma^s_X = \Sigma^s_{F(X)}$, and $DF(X)\Sigma^u_X = \Sigma^u_{F(X)}$ For each $X \in \Lambda$.*
   (c) *$\|DF^k V\| \leq c\lambda_1^k \|v\|$, For each $v \in \Sigma^s$ and $k > 0$.*
   (d) *$\|DF^{-k} V\| \leq c\lambda_1^k \|v\|$, For each $v \in \Sigma^u$ and $k > 0$.*

where,

- *Σ^s, Σ^u are the DF-invariant sub-manifolds (respectively, the stable and unstable sub-manifolds of the map F),*
- *Σ^s_X, Σ^u_X are the $DF(X)$-invariant sub-manifolds.*

Definition 3.2.2 *[54] A hyperbolic set Λ is locally maximal (or isolated) if there exists an open set U such that $\Lambda = \bigcap_{n\in\mathbb{Z}} F^n(U)$.*

The Smale horseshoe and the Plykin attractors are examples of locally maximal sets.

The Smale horseshoe was first introduced by the mathematician Stephen Smale in the 1960s. It is a two-dimensional structure that is formed by stretching and folding a square region of the plane, in a way that preserves the topology of the region.

The dynamics of the Smale horseshoe are characterized by a set of non-linear equations that describe the evolution of points in the plane. These equations are similar to those used to describe the Lorenz attractor, but the horseshoe has a different geometry and different qualitative properties. They are given as follow while ΔT represents the time step size parameter:

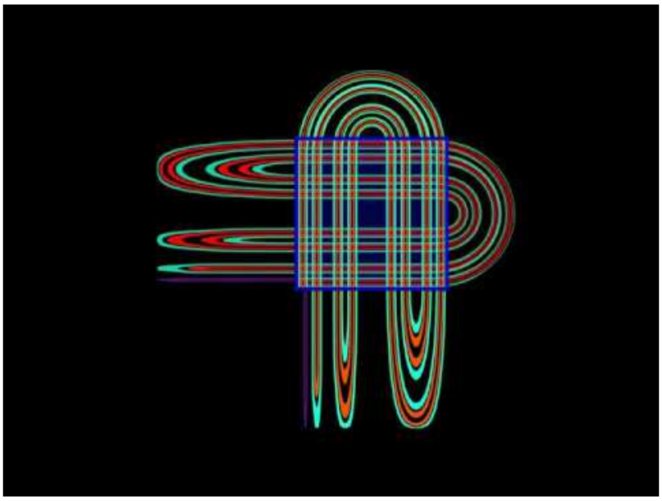

Figure 3.1: The Smale horseshoe map iterations.

$$\begin{cases} X_{n+1} = X_n + \Delta T \cdot (Y_n + a \cdot X_n) \\ Y_{n+1} = Y_n + \Delta T \cdot Z_n \\ Z_{n+1} = Z_n + \Delta T \cdot (-X_n) \end{cases}$$

The Plykin attractors were first introduced by the mathematician Andrei Plykin in the 1980s as a modification of the Smale horseshoe attractor. The Plykin attractor is a two-dimensional structure that is formed by stretching and folding a rectangular region of the plane, in a way that preserves the topology of the region.

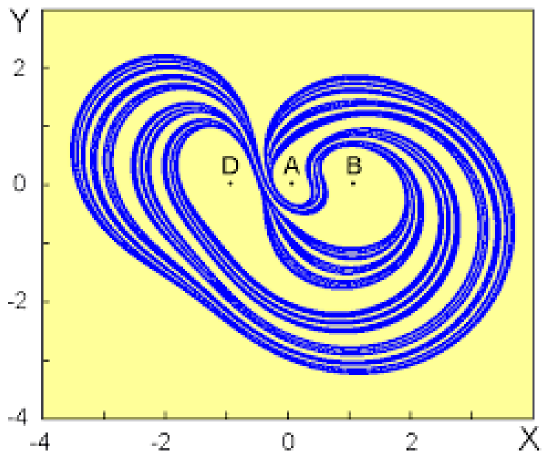

Figure 3.2: The Plykin attractor.

Like the Smale horseshoe, it exhibits sensitive dependence on initial conditions and has a fractal structure. One of the distinguishing features of the Plykin attractor is that it has a pair of stable periodic orbits that are contained within the attractor itself. These periodic orbits are connected by a chain of unstable periodic orbits that wind around the attractor in a spiral-like pattern. This spiral structure is what gives the Plykin attractor its distinctive appearance.

The dynamics of the Plykin attractor are described by a set of non-linear equations that are similar to those used to describe the Smale horseshoe. These equations are based on a discrete-time map that describes the evolution of points in the plane over a sequence of discrete steps and they are as follow:

$$\begin{cases} X_{n+1} = X_n + \Delta T \cdot (-Y_n - Z_n) \\ Y_{n+1} = Y_n + \Delta T \cdot (X_n + a \cdot Y_n) \\ Z_{n+1} = Y_n + \Delta T \cdot (a + Z_n \cdot (X_n - 10.0)) \end{cases}$$

where ΔT represents the time step size parameter.

If Λ represents a hyperbolic set with a nonempty interior for a map F, the map F is classified as Anosov provided it possesses the properties of transitivity, local maximality, and Ω being a surface. This class of attractors exhibits several noteworthy characteristics, including the shadowing property, structural stability, Markov partitions, and the presence of SRB measures. In the scenario where $\Lambda = \Omega$, the diffeomorphism F is referred to as an Anosov diffeomorphism or a uniformly hyperbolic map. An exemplary instance of hyperbolic maps is the Arnold cat map (See [5]) which was named after mathematician Vladimir Arnold, and it is defined as follows:

$$\begin{cases} X_{n+1} = X_n + Y_n, mod\ 1 \\ Y_{n+1} = X_n + 2Y_n, mod\ 1 \end{cases}$$

It operates on a two-dimensional grid, typically represented as a square lattice, and involves a periodic boundary condition. In these equations, $\mod 1$ indicates that the result is taken modulo 1, meaning it wraps around to the interval $[0, 1)$ to satisfy the periodic boundary condition.

The Arnold cat map demonstrates interesting mixing and scrambling properties. It is often used as a simple example to study chaotic dynamics and and is widely explored in the field of chaos-based image encryption as it is mainly used for the confusion of pixels.

3.3 Lorenz-type attractors

Often referred to as *pseudo-hyperbolic* attractors, represent a class of attractors that typically lack structural stability. While their homoclinic and heteroclinic orbits exhibit structural stability, these attractors do not display stable periodic orbits when subjected to small variations in parameters. Among the extensively studied instances within this category is the Lorenz system (See [33]), which is defined as follows:

$$\begin{cases} \dot{X} = \sigma(X - Y) \\ \dot{Y} = rX - Y - XZ \\ \dot{Z} = -bZ + XY. \end{cases}$$

It was first described by the meteorologist Edward Lorenz in 1963. Lorenz discovered a simple set of differential equations that could produce a complex, chaotic behavior and it has since become known as the Lorenz attractor. The Lorenz attractor is a three-dimensional structure that resembles a butterfly or a figure-eight. It is formed by the motion of a point in three-dimensional space as it follows the solutions of the Lorenz equations. The equations themselves are non-linear and describe the behavior of a system that is sensitive to its initial conditions.

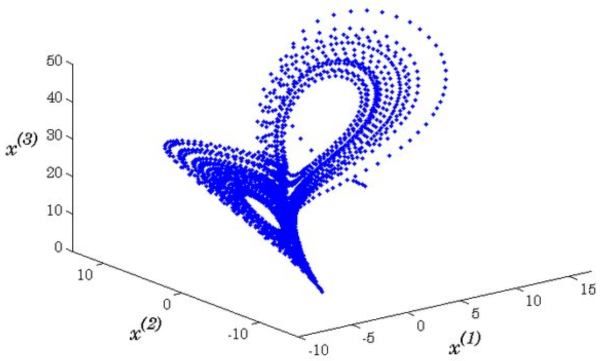

Figure 3.3: The Lorenz attractor.

Lorenz-type attractors were considered as examples of truly strange attractors and the literature contains a finite number of these well-documented attractors.

The characteristics of Lorenz-like attractors can be summarized as follows: Let Λ be an attractor of a Lorenz-like system in a compact, boundary less 3-manifold M. Let $C^1(M)$ denote the set of C^1 vector fields on M, equipped with the C^1 topology. These systems possess the following properties:

1. An invariant foliation exists, with its leaves forward-contracted by the flow.
2. A positive Lyapunov exponent exists at every orbit.
3. The systems exhibit expansiveness and thus, sensitivity with respect to initial data.
4. If the flow is C^2, they have zero volume.
5. A unique physical measure is present, whose support encompasses the entire attractor, and is an equilibrium state concerning the center unstable Jacobian.

3.4 Quasi-attractors

The focus of our study in this thesis is concerned with this particular category of attractors. It is noteworthy to mention that the majority of observed chaotic attractors in [2], [32], [43], [46], [37], are classified as *quasi-attractors*. This designation implies that they do not qualify as either *robust hyperbolic attractors*, as described in the first defined case, or *pseudo-hyperbolic attractors*, as described in the second defined case.

The intricate nature of the embedded basin structures stemming from quasi-attractors is not limited by the complexity exhibited through the selection of initial conditions and a set of bifurcation parameters of non-zero measure. The principal source of this complexity arises from the homoclinic tangency observed between the stable and unstable manifolds of saddle points in the Poincaré section. These attractors possess the following properties:

1. They comprise distinct loops formed by saddle-foci or homoclinic orbits of saddle cycles at the moment of tangency between their stable and unstable manifolds. These loops enclose non-robust singular trajectories, which are considered critical due to their association with dangerous border collision bifurcations. Such bifurcations occur when the basins of attraction of stable fixed points diminish to a set of zero measure as the parameter approaches the bifurcation value from either side.

2. In the vicinity of their trajectories, a map of Smale's horseshoe-type emerges. This map encompasses both non-trivial hyperbolic subsets of trajectories and a denumerable subset of stable periodic orbits, as indicated by [48] and Newhouse's theorem [38].

3. The quasi-attractor serves as the unified limit set encompassing the entire attracting set of trajectories, including a subset of both chaotic and stable periodic trajectories characterized by long periods, narrow basins of attraction, and regions of stability. This characteristic arises from the fact that this attractor contains a collection of basins of attraction associated with different periodic orbits.

4. The basins of attraction for stable cycles exhibit a remarkably narrow width.

5. Lastly, it is noteworthy that certain orbits may not manifest themselves explicitly in numerical simulations, except within sufficiently large stability windows where they become distinctly visible.

3.4.1 Defining a new 2-D piecewise smooth map

It is established that the Lozi map [54] serves as the piecewise-affine adaptation of the Hénon map, which has garnered considerable attention. The Lozi map has assumed a prominent position in significant research endeavors due to its favorable structure for analysis and the emergence of numerous novel chaotic phenomena within it. Moreover, recent studies have extensively explored the examination of a smooth variant of the piecewise linear Lozi map [22] and [19].

In the subsequent discourse, we aim to introduce and scrutinize a novel finding pertaining to the generalization of the Lozi map, referred to as the border collision normal form $(BCNF_{RC})$ [7], and which is defined in the second chapter in equation (2.7). Our objective is to establish that a quasi-hyperbolic attractor emerges for a specific parameter set, exhibiting a form akin to the $(BCNF_{RC})$. Through our analysis, we will provide some evidences that support for this claim.

Let's first consider the new mapping system which has some resemblance with system (2.7), and we will show later why we would better take this system rather than the $(BCNF_{RC})$ system.

Let F be the map defined by:

$$F(X,Y) = \begin{pmatrix} G(X,Y) \\ H(X,Y) \end{pmatrix} = \begin{cases} \begin{pmatrix} \alpha & \tau_L \\ 0 & -\delta_L \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, & X < 0 \\ \begin{pmatrix} \alpha & \tau_R \\ 0 & -\delta_R \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, & X \geq 0 \end{cases} \tag{3.1}$$

where,

$$G(X,Y) = \begin{cases} \tau_L Y + \alpha X + \mu, & X < 0, \\ \tau_R Y + \alpha X + \mu, & X \geq 0, \end{cases}$$

and,

$$H(X,Y) = \begin{cases} -\delta_L Y, & X < 0, \\ -\delta_R Y, & X \geq 0. \end{cases}$$

The system has a single fixed point which is :

$$Y^* = \left(\frac{\mu}{1-\alpha}, 0\right).$$

Its stability is determined by the eigenvalues $\lambda_1 = \alpha$ and $\lambda_2 = -\delta_k$, $k = L, R$.

3.4.2 Hyperbolicity

We will introduce certain assumptions regarding the parameters α and δ_k in order to establish the sufficient conditions for the hyperbolicity theorem presented in [[2], Chap 4]:

Theorem 3.4.1 (Sufficient conditions for hyperbolicity) *Consider a C^1-map $F : U \longrightarrow \mathbb{R}^{m+n}$, where W is an open convex subset of $\mathbb{R}^{m+n}$, such that $F(X,Y) = (\bar{X}, \bar{Y})$, $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$,*

$$\bar{X} = G(X,Y), \quad \bar{Y} = H(X,Y).$$

if,

$$\|G_X\| < 1, \tag{3.2}$$
$$\|H_Y^{-1}\| < 1, \tag{3.3}$$
$$1 - \|G_X\|\|H_Y^{-1}\| > 2\sqrt{\|G_Y \cdot H_Y^{-1}\|\|H_X\|\|H_Y^{-1}\|}, \tag{3.4}$$
$$(1 - \|G_X\|)\left(1 - \|H_Y^{-1}\|\right) > \|G_Y \cdot H_Y^{-1}\| \cdot \|H_X\|, \tag{3.5}$$

where $\|\cdot\| = \sup_{(X,Y)\in U} |\cdot|$, *and subscripts means differentiation with respect to the corresponding coordinates, then:*

1. *Any compact invariant set* Λ *in* W *is hyperbolic,*
2. *The stable subspace* $E^s_{p_0}$, $p_0 = (X_0, Y_0) \in \Lambda$, *can be represented as the graph of an affine map*
$$Y = L_{p_0}(X - X_0) + Y_0 \text{ and } \|L_{p_0}\| \leq L.$$

The unstable subspace $E^u_{p_0}$ *can be represented as the graph of an affine map*

$$X = M_{p_0}(Y - Y_0) + X_0,$$

and $\|M_{p_0}\| \leq M$, *where* L *and* M *are constants and*

$$LM < 1.$$

Proof 2 *By applying the aforementioned theorem 3.4.1 to the new mapping system (3.1), the following conditions are obtained:*

$$\|\frac{\partial G}{\partial X}\| = \alpha, \ \|\frac{\partial G}{\partial Y}\| = \tau_k, \ \|\frac{\partial H}{\partial X}\| = 0, \ \|(\frac{\partial H}{\partial Y})^{-1}\| \leq \frac{1}{\delta_k}. \tag{3.6}$$

The inequality (3.2) holds if $\alpha < 1$, *and the inequality (3.3) holds if* $\delta_k > 1$.

Since $|H_X| = 0$, *the inequality (3.4) is equivalent to*

$$1 - \|G_X\|\|H_Y^{-1}\| > 0,$$

By using the inequalities (3.2) and (3.3), we get

$$1 - \frac{\alpha}{\delta_k} > 0,$$

and subsequently,

$$\delta_k - \alpha > 0.$$

This implies that the inequality (3.4) holds if $\delta_k > \alpha$.

Given that $\|H_X\| = 0$, *the inequality (3.5) can be equivalently expressed as:*

$$(1 - \|G_X\|)\left(1 - \|H_Y^{-1}\|\right) > 0.$$

By using the inequalities (3.2) and (3.3), we obtain:

$$(1 - \alpha)\left(1 - \frac{1}{\delta_k}\right) > 0.$$

Consequently, we have:

$$1 - \alpha > \frac{1 - \alpha}{\delta_k}.$$

Since $\alpha < 1$, *we know that* $1 - \alpha > 0$. *By substituting this into the previous inequality, we deduce that the inequality (3.5) holds if* $\delta_k > 1$. *Therefore, considering all of these results, we can conclude that every orbit of the new normal form (3.1) is hyperbolic if:*

$$\delta_k > 1 > \alpha, \quad \textit{with} \quad 0 < \alpha < 1.$$

As $0 < \alpha < 1$, *we can establish a more suitable condition, namely:*

$$1 < \delta_k < \frac{1}{\alpha}, \quad \textit{with} \quad 0 < \alpha < 1. \tag{3.7}$$

With this condition, we deduce that our fixed point is a saddle on both sides, where $\det(J) < 0$ *and tr*$(J) < 0$.

The eigenvectors are $V_1 = (1, 0)$ *and* $V_2 = \left(1, \frac{-(\alpha+\delta_k)}{\tau_k}\right)$.

Remark 3 *Applying Theorem 3.4.1 to* $BCNF_{RC}$ *yields:*

$$\|\frac{\partial G}{\partial X}\| = \tau_k, \ \|\frac{\partial G}{\partial Y}\| = 1, \ \|\frac{\partial H}{\partial X}\| = -\delta_k, \ \|\frac{\partial H}{\partial Y}\| = 0.$$

Thus, it can be observed that the inequality (3.3) in Theorem 3.4.1 cannot be satisfied in this case. Therefore, it can be concluded that Theorem 3.4.1 cannot be applied to the normal form $BCNF_{RC}$. *This suggests that the conditions of the theorem are not applicable, and it is possible that the normal form with such a structure does not exhibit a quasi-hyperbolic regime.*

3.4.3 Quasi-hyperbolic regime and absorbing region

In this section, our objective is to identify a trapping region and establish the essential conditions for its existence. To accomplish this, we will rely on the following theorem presented in [[2], Chap 4].

Theorem 3.4.2 ***(Quasi-hyperbolic attractors)****. Let F be a piecewise smooth map of $\mathbb{R}^2$ and L be the union of points of discontinuity of F and DF. Let $D = \bigcup_{i=0}^{\infty} F^{-i}L$. if $X_0 \notin D$ then $F^n(X_0) \notin L$, $n \in \mathbb{Z}_+$. Assume that R is an absorbing region, i.e., $F(X) \in IntR$ if $X \in R$ and $X \notin D$. Then an attractor A is the set of all limit points of all orbits $F^i(X_0)$, $F \in \mathbb{Z}_+$, $X_0 \notin D$, i.e., A = closure($(\bar{R}\ D) \cap F^2(\bar{R}\ D) \cap ...$). We call it quasi-hyperbolic (or generalized hyperbolic, following Ya. Pesin) if For $X_0 \notin D$ every orbit $X_0, F(X_0), F^2(X_0), ...$ is hyperbolic.*

Theorem 3.4.3 *[28] There exists a quasi-hyperbolic regime in the 2-D piecewise linear smooth map F given in equation (3.1).*

Proof 3 *Let us define Ω as the following domain*

$$\Omega(A, B, C, D) = \{X, Y \in \mathbb{R},\ X_3 \leq X \leq X_1,\ Y_3 \leq Y \leq Y_1\}.$$

Let us consider the points $A(X_1, Y_1)$, $B(X_2, Y_2)$, $C(X_3, Y_3)$, and $D(X_4, Y_4)$. We will determine under what conditions those points, their images by the formula (3.1), and the images of the connecting lines between the points are in our domain Ω, by taking $X_1 > 0$, $X_2 > 0$, $X_3 < 0$ and $X_4 < 0$.

As a first step, the points $A(X_1, Y_1)$, $B(X_2, Y_2)$, $C(X_3, Y_3)$ and $D(X_4, Y_4)$ are in Ω if

$$X_3 \leq X_2 \leq X_1,\ Y_3 \leq Y_2 \leq Y_1,$$
$$X_3 \leq X_4 \leq X_1,\ Y_3 \leq Y_4 \leq Y_1.$$

Subsequently, we will find the conditions under which the images of points A, B, C, and D, as given by the formula (3.1), are located within the domain Ω.

Firstly,

$$F(A) = \begin{cases} \tau_R Y_1 + \alpha X_1 + \mu, \\ -\delta_R Y_1, \end{cases}$$

The condition for the image $F(A)$ to belong to Ω is as follows:

$$X_3 \leq \tau_R Y_1 + \alpha X_1 + \mu \leq X_1,$$

and

$$Y_3 \leq -\delta_R Y_1 \leq Y_1,$$

which means that

$$\frac{\tau_R Y_1 + \mu}{1 - \alpha} \leq X_1, \tag{3.8}$$

and

$$Y_1 > 0. \tag{3.9}$$

Secondly,

$$F(B) = \begin{cases} \tau_R Y_2 + \alpha X_2 + \mu, \\ -\delta_R Y_2, \end{cases}$$

$F(B)$ *belongs to* Ω *if*

$$X_3 \leq \tau_R Y_2 + \alpha X_2 + \mu \leq X_1,$$

and

$$Y_3 \leq -\delta_R Y_2 \leq Y_1,$$

which means that

$$\frac{X_3 - \tau_R Y_2 - \mu}{\alpha} \leq X_2 \leq \frac{X_1}{\alpha}, \tag{3.10}$$

and

$$\frac{-Y_1}{\delta_R} < Y_2 < \frac{-Y_3}{\delta_R}. \tag{3.11}$$

Thirdly,

$$F(C) = \begin{cases} \tau_L Y_3 + \alpha X_3 + \mu, \\ -\delta_L Y_3, \end{cases}$$

the image $F(C)$ *belongs to* Ω *if*

$$X_3 \leq \tau_L Y_3 + \alpha X_3 + \mu \leq X_1,$$

and

$$Y_3 \leq -\delta_L Y_3 \leq Y_1,$$

which means that

$$X_3 < max\{\frac{\mu}{1-\alpha}, \frac{X_1 - \tau_L Y_3 - \mu}{\alpha}\}, \tag{3.12}$$

and

$$Y_3 < 0. \tag{3.13}$$

Finally,

$$F(D) = \begin{cases} \tau_L Y_4 + \alpha X_4 + \mu, \\ -\delta_L Y_4, \end{cases}$$

the image $F(D)$ belongs to Ω if

$$X_3 \leq \tau_L Y_4 + \alpha X_4 + \mu \leq X_1,$$

and

$$Y_3 \leq -\delta_L Y_4 \leq Y_1,$$

which means that

$$\frac{X_3 - \tau_L Y_4 - \mu}{\alpha} \leq X_4 < \frac{X_1 - \tau_L Y_4 - \mu}{\alpha}, \tag{3.14}$$

and

$$\frac{-Y_1}{\delta_R} < Y_4 < \frac{-Y_3}{\delta_R}. \tag{3.15}$$

Now, let us determine the essential conditions under which the images of the line segments (AB), (BC), (AD), and (DC) lies within the domain Ω.

To begin with, the line segment [AB] can be represented by the equation in the form $Y = aX + b$, given by:

$$Y = (\frac{Y_2 - Y_1}{X_2 - X_1})X + \frac{X_2 Y_1 - X_1 Y_2}{X_2 - X_1}.$$

The image of the line segment (AB) under the mapping (3.1), denoted as $F(AB)$, is defined by:

$$F(AB) = \begin{cases} \tau_k\big((\frac{Y_2-Y_1}{X_2-X_1})X + \frac{X_2Y_1-X_1Y_2}{X_2-X_1}\big) + \alpha X + \mu, \\ -\delta_k\big((\frac{Y_2-Y_1}{X_2-X_1})X + \frac{X_2Y_1-X_1Y_2}{X_2-X_1}\big), \end{cases}$$

So, $F(AB)$ is in Ω if

$$X_3 \leq \tau_k\big((\frac{Y_2 - Y_1}{X_2 - X_1})X + \frac{X_2 Y_1 - X_1 Y_2}{X_2 - X_1}\big) + \alpha X + \mu \leq X_1,$$

and

$$Y_3 \leq -\delta_k\big((\frac{Y_2-Y_1}{X_2-X_1})X+\frac{X_2Y_1-X_1Y_2}{X_2-X_1}\big) \leq Y_1.$$

Starting with the first inequality

$$X_3-\mu-\tau_k(\frac{X_2Y_1-X_1Y_2}{X_2-X_1}) \leq \big(\tau_k(\frac{Y_2-Y_1}{X_2-X_1})+\alpha\big)X \leq X_1-\mu-\tau_k(\frac{X_2Y_1-X_1Y_2}{X_2-X_1}),$$

which means

$$\frac{(X_3-\mu)(X_2-X_1)-\tau_k(X_2Y_1-X_1Y_2)}{\tau_k(Y_2-Y_1)+\alpha(X_2-X_1)} \leq X \leq \frac{(X_1-\mu)(X_2-X_1)-\tau_k(X_2Y_1-X_1Y_2)}{\tau_k(Y_2-Y_1)+\alpha(X_2-X_1)}. \tag{3.16}$$

Passing to the second inequality, we have

$$\frac{-Y_1}{\delta_k}-(\frac{X_2Y_1-X_1Y_2}{X_2-X_1}) \leq (\frac{Y_2-Y_1}{X_2-X_1})X \leq \frac{-Y_3}{\delta_k}-(\frac{X_2Y_1-X_1Y_2}{X_2-X_1}),$$

which means

$$\frac{-Y_1}{\delta_k}(\frac{X_2-X_1}{Y_2-Y_1})-(\frac{X_2Y_1-X_1Y_2}{Y_2-Y_1}) \leq X \leq \frac{-Y_3}{\delta_k}(\frac{X_2-X_1}{Y_2-Y_1})-(\frac{X_2Y_1-X_1Y_2}{Y_2-Y_1}). \tag{3.17}$$

Secondly, the line segment [BC] can be described by the equation in the form $Y=aX+b$*, which is given by:*

$$Y=(\frac{Y_3-Y_2}{X_3-X_2})X+\frac{X_3Y_2-X_2Y_3}{X_3-X_2}.$$

Then, the image F(BC) of (BC) by (3.1) can be defined as follows:

$$F(BC)=\begin{cases} \tau_k\big((\frac{Y_3-Y_2}{X_3-X_2})X+\frac{X_3Y_2-X_2Y_3}{X_3-X_2}\big)+\alpha X+\mu, \\ -\delta_k\big((\frac{Y_3-Y_2}{X_3-X_2})X+\frac{X_3Y_2-X_2Y_3}{X_3-X_2}\big). \end{cases}$$

$F(BC)$ *is in* Ω *if*

$$X_3 \leq \tau_k\big((\frac{Y_3-Y_2}{X_3-X_2})X+\frac{X_3Y_2-X_2Y_3}{X_3-X_2}\big)+\alpha X+\mu \leq X_1,$$

and

$$Y_3 \leq -\delta_k\big((\frac{Y_3-Y_2}{X_3-X_2})X+\frac{X_3Y_2-X_2Y_3}{X_3-X_2}\big) \leq Y_1.$$

which means

$$\frac{(X_3-\mu)(X_3-X_2)-\tau_k(X_3Y_2-X_2Y_3)}{\tau_k(Y_3-Y_2)+\alpha(X_3-X_2)} \leq X \leq \frac{(X_1-\mu)(X_3-X_2)-\tau_k(X_3Y_2-X_2Y_3)}{\tau_k(Y_3-Y_2)+\alpha(X_3-X_2)}. \tag{3.18}$$

and

$$\frac{-Y_1}{\delta_k}(\frac{X_3-X_2}{Y_3-Y_2})-(\frac{X_3Y_2-X_2Y_3}{Y_3-Y_2}) \leq X \leq \frac{-Y_3}{\delta_k}(\frac{X_3-X_2}{Y_3-Y_2})-(\frac{X_3Y_2-X_2Y_3}{Y_3-Y_2}). \tag{3.19}$$

Thirdly, the line segment [DC] can be described by the equation in the form $Y = aX + b$, which is given by:

$$Y = ((\frac{Y_3-Y_4}{X_3-X_4})X + \frac{X_3Y_4-X_4Y_3}{X_3-X_4}).$$

Then, the image $F(DC)$ of (DC) by (3.1) can be defined as follows:

$$F(DC) = \begin{cases} \tau_k\big((\frac{Y_3-Y_4}{X_3-X_4})X + \frac{X_3Y_4-X_4Y_3}{X_3-X_4}\big) + \alpha X + \mu, \\ -\delta_k\big((\frac{Y_3-Y_4}{X_3-X_4})X + \frac{X_3Y_4-X_4Y_3}{X_3-X_4}\big), \end{cases}$$

So, $F(DC)$ is in Ω if

$$\frac{(X_3-\mu)(X_3-X_4)-\tau_k(X_3Y_4-X_4Y_3)}{\tau_k(Y_3-Y_4)+\alpha(X_3-X_4)} \leq X \leq \frac{(X_1-\mu)(X_3-X_4)-\tau_k(X_3Y_4-X_4Y_3)}{\tau_k(Y_3-Y_4)+\alpha(X_3-X_4)}, \tag{3.20}$$

and

$$\frac{-Y_1}{\delta_k}(\frac{X_3-X_4}{Y_3-Y_4})-(\frac{X_3Y_4-X_4Y_3}{Y_3-Y_4}) \leq X \leq \frac{-Y_3}{\delta_k}(\frac{X_3-X_2}{Y_3-Y_2})-(\frac{X_3Y_2-X_2Y_3}{Y_3-Y_2}). \tag{3.21}$$

Finally, the line segment [AD] can be described by the equation in the form $Y = aX + b$, which is given by:

$$Y = ((\frac{Y_4-Y_1}{X_4-X_1})X + \frac{X_4Y_1-X_1Y_4}{X_4-X_1}).$$

Then, the image $F(AD)$ of (AD) by (3.1) can be defined as follows:

$$F(AD) = \begin{cases} \tau_k\big((\frac{Y_4-Y_1}{X_4-X_1})X + \frac{X_4Y_1-X_1Y_4}{X_4-X_1}\big) + \alpha X + \mu, \\ -\delta_k\big((\frac{Y_4-Y_1}{X_4-X_1})X + \frac{X_4Y_1-X_1Y_4}{X_4-X_1}\big), \end{cases}$$

So, $F(AD)$ is in Ω if

$$\frac{(X_3-\mu)(X_4-X_1)-\tau_k(X_4Y_1-X_1Y_4)}{\tau_k(Y_4-Y_1)+\alpha(X_4-X_1)} \le X \le \frac{(X_1-\mu)(X_4-X_1)-\tau_k(X_4Y_1-X_1Y_4)}{\tau_k(Y_4-Y_1)+\alpha(X_4-X_1)}, \tag{3.22}$$

and

$$\frac{-Y_1}{\delta_k}\left(\frac{X_4-X_1}{Y_4-Y_1}\right)-\left(\frac{X_4Y_1-X_1Y_4}{Y_4-Y_1}\right) \le X \le \frac{-Y_3}{\delta_k}\left(\frac{X_4-X_1}{Y_4-Y_1}\right)-\left(\frac{X_4Y_1-X_1Y_4}{Y_4-Y_1}\right). \tag{3.23}$$

As a final step, for any point $(X,Y)\in\Omega$, the image $F(X,Y)$ of this point under the formula (3.1) will lie within Ω if:

$$X_3 \le \tau_k Y + \alpha X + \mu \le X_1,$$

and

$$Y_3 \le -\delta_k Y \le Y_1,$$

which means

$$\frac{X_3-\tau_k Y-\mu}{\alpha} \le X \le \frac{X_1-\tau_k Y-\mu}{\alpha}, \tag{3.24}$$

and by applying the condition (3.7) to the second inequality

$$-Y_1 < Y < -\alpha Y_3. \tag{3.25}$$

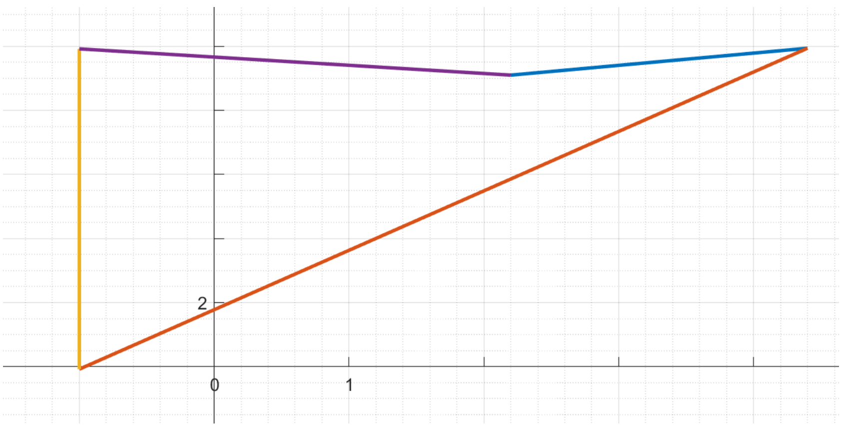

Figure 3.4: The absorbing region for α = 0.5, δ_k = 1.2, τ_L = -1 and τ_R = 1.

By taking into account all previous results in conjunction with the hyperbolicity condition derived in equation (3.7), it can be observed that Ω exhibits an absorbing region for the map *described in equation (3.1). A numerical approximation of this absorbing region is depicted in Figure 3.4, providing evidence that the set of parameters for the map (3.1) satisfying the aforementioned conditions is not empty. It is worth noting that in this particular case, all the conditions specified in Theorem 3.4.1 are met. Specifically, the values $\alpha = 0.5$ and $\delta = 1.2$ fulfill the requirement stated in equation (3.7).*

3.4.4 Generalizing to n-D mappings

Consider the n-D map denoted as $F : D \longrightarrow D$, where D is a subset of $\mathbb{R}^n$. This map, introduced in [51], is defined as follows:

$$Z_{k+1} = F(Z_k) = A_i Z_k + b_i, \text{ if } Z_k \in D_i, \text{i = 1, 2, ..., m}. \tag{3.26}$$

where $A_i = (a^i_{jl})_{1\leq j,l\leq n}$ and $b_i = (b^i_j)_{1\leq j\leq n}$, are respectively $n \times n$ and $n \times 1$ real matrices, For all $i = 1, 2, ..., m$, $Z_k = (Z^j_k)_{1\leq j\leq n} \in \mathbb{R}^n$ is the state variable, and m is the number of disjoint domains on which D is partitioned.

Let us consider that : $Z_{k+1} = (X_{k+1}, Y_{k+1})$ while:

$$X_{k+1} = (Z^1_{k+1}, Z^2_{k+1}, ..., Z^{n-1}_{k+1}), \text{ and } Y_{k+1} = Z^n_{k+1}.$$

So, we can write the system (3.26) as follow:

$$Z_{k+1} = \begin{cases} B'_i X_k + C'_i Y_k + b'_i \\ B''_i X_k + C''_i Y_k + b''_i \end{cases} \tag{3.27}$$

where,
$B'_i = (a^i_{jl})_{1\leq j,l\leq n-1}$, $C'_i = (a^i_{jn})_{1\leq j\leq n-1}$, $b'_i = (b^i_j)_{1\leq j\leq n-1}$,
$B''_i = (a^i_{nl})_{1\leq l\leq n-1}$, $C''_i = (a^i_{nn})$ and $b''_i = b^n_i$.

By applying the Theorem 3.4.1 [see [2], [28]] on the last system and by considering $X_{k+1} = F(X_k, Y_k)$ and $Y_{k+1} = G(X_k, Y_k)$ we get:

$$\frac{\partial F}{\partial X} = B'_i, \ \frac{\partial F}{\partial Y} = C'_i, \ \frac{\partial G}{\partial X} = B''_i, \ \frac{\partial G}{\partial Y} = C''_i.$$

which means that:

The inequality (3) holds if, $\max(\sum_{j=1}^{n-1} |a^i_{jl}|) < 1$, For all $i = 1, 2, ..., m$,

The inequality (4) holds if, $|a^i_{nn}| > 1$,

The inequality (5) holds if, $S_l > 2\sqrt{S_k S_m} - |a^i_{nn}|$ where

$S_l = \|F_X\| = \max(\sum_{j=1}^{n-1} |a^i_{jl}|)$, $S_k = \|F_Y\| = \sum_{j=1}^{n-1} |a^i_{jn}|$ and $S_m = \|G_X\| = \sum_{l=1}^{n-1} |a^i_{nl}|$,

And, the inequality (6) holds if, $(1 - S_l)(1 - \frac{1}{|a^i_{nn}|}) > \frac{1}{|a^i_{nn}|} S_k S_m$.

Due to the shape of the vector field F of the map (3.26) the plane can be divided into m regions denoted by $(D_i)_{1\leq i\leq m}$, which means that Z_{k+1} can be written as follow:

$$Z_{k+1} = \begin{cases} A_1Z_k + b_1, & Z_k \in D_1 \\ A_2Z_k + b_2, & Z_k \in D_2 \\ A_3Z_k + b_3, & Z_k \in D_3 \\ & . \\ & . \\ & . \\ A_mZ_k + b_m, & Z_k \in D_m \end{cases} \tag{3.28}$$

We are going to change the notations just for the understanding and reading-ability's sake.

Let us consider the following system such that $D_1 \cup D_2 = \mathbb{R}^n$.

$$F(Z) = \begin{cases} A_1Z + b, & Z_1 \in D_1 \\ A_2Z + c, & Z_1 \in D_2 \end{cases} \tag{3.29}$$

where, $Z = (Z_1, Z_2, ..., Z_n)^t$, $b = (b_1, b_2, ..., b_n)^t$ and $c = (c_1, c_2, ..., c_n)^t$.

Let's suppose that when $Z_1 \geq 0$ then $Z_1 \in D_1$ and $Z_1 < 0$ then $Z_1 \in D_0$. So, A is in D_1 implies that $Z_1 \geq 0$ and A is in D_2 implies that $Z_1 < 0$, which means that the system (3.29) is equivalent to the following system

$$F(Z) = \begin{cases} A_1Z + b, & Z_1 \geq 0 \\ A_2Z + c, & Z_1 < 0 \end{cases} \tag{3.30}$$

Let Ω be the domain relies the points $A(X_1, X_2, ..., X_n), B(Y_1, Y_2, ..., Y_n), C(t_1, t_2, ..., t_n)$ and $D(s_1, s_2, ..., s_n)$, and suppose that the points A and B are in D_1 while the points C and D are in D_2.

We are going to prove that the images of the four points and the images of the connecting lines between the points by the formula (3.29) does not go outside the domain Ω. The idea is to get a regressive relationship between such image of a point and the other next point.

Let us Consider that $F(A) = B$, so this is means that $A_1X + b = B$, i.e.,

$$\begin{pmatrix} a^1_{11} & a^1_{12} & \dots & a^1_{1n} \\ a^1_{21} & a^1_{22} & \dots & a^1_{2n} \\ & & & . \\ & & & . \\ & & & . \\ a^1_{n1} & a^1_{n2} & \dots & a^1_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{pmatrix} \tag{3.31}$$

i.e.,

$$\begin{pmatrix} a_{11}^1X_1+a_{12}^1X_2+...+a_{1n}^1X_n+b_1 \\ a_{21}^1X_1+a_{22}^1X_2+...+a_{2n}^1X_n+b_2 \\ . \\ . \\ . \\ a_{n1}^1X_1+a_{n2}^1X_2+...+a_{nn}^1X_n+b_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{pmatrix} \tag{3.32}$$

which is equivalent to the system

$$\begin{cases} a_{11}^1X_1+a_{12}^1X_2+...+a_{1n}^1X_n+b_1=Y_1 \\ a_{21}^1X_1+a_{22}^1X_2+...+a_{2n}^1X_n+b_2=Y_2 \\ . \\ . \\ . \\ a_{n1}^1X_1+a_{n2}^1X_2+...+a_{nn}^1X_n+b_n=Y_n \end{cases}$$

giving a simple form of the last system

$$\begin{cases} \sum_{i=1}^n a_{1i}^1X_i+b_1=Y_1 \\ \sum_{i=1}^n a_{2i}^1X_i+b_2=Y_2 \\ . \\ . \\ . \\ \sum_{i=1}^n a_{ni}^1X_i+b_n=Y_n \end{cases} \tag{3.33}$$

Secondly, $F(B)=C$ means that $A_1Y+b=C$, i.e.,

$$\begin{pmatrix} a_{11}^1 & a_{12}^1 & ... & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & ... & a_{2n}^1 \\ & & & . \\ & & & . \\ & & & . \\ a_{n1}^1 & a_{n2}^1 & ... & a_{nn}^1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ . \\ . \\ . \\ t_n \end{pmatrix} \tag{3.34}$$

which means

$$\begin{cases} a_{11}^1Y_1+a_{12}^1Y_2+...+a_{1n}^1Y_n+b_1=t_1 \\ a_{21}^1Y_1+a_{22}^1Y_2+...+a_{2n}^1Y_n+b_2=t_2 \\ . \\ . \\ . \\ a_{n1}^1Y_1+a_{n2}^1Y_2+...+a_{nn}^1Y_n+b_n=t_n \end{cases}$$

which is equivalent to the system:

$$\begin{cases} \sum_{i=1}^{n} a_{1i}^{1} Y_i + b_1 = t_1 \\ \sum_{i=1}^{n} a_{2i}^{1} Y_i + b_2 = t_2 \\ . \\ . \\ . \\ \sum_{i=1}^{n} a_{ni}^{1} Y_i + b_n = t_n \end{cases} \tag{3.35}$$

using the system (3.33), we get

$$\begin{cases} \sum_{i=1}^{n} a_{1i}^{1} (\sum_{j,l=1}^{n} a_{jl}^{1} X_l + b_j) + b_1 = t_1 \\ \sum_{i=1}^{n} a_{2i}^{1} (\sum_{j,l=1}^{n} a_{jl}^{1} X_l + b_j) + b_2 = t_2 \\ . \\ . \\ . \\ \sum_{i=1}^{n} a_{ni}^{1} (\sum_{j,l=1}^{n} a_{jl}^{1} X_l + b_j) + b_n = t_n \end{cases} \tag{3.36}$$

Thirdly, $F(C) = D$ means that $A_2 T + c = D$, i.e.,

$$\begin{pmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ & & & . \\ & & & . \\ & & & . \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ . \\ . \\ . \\ t_n \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ . \\ . \\ . \\ s_n \end{pmatrix} \tag{3.37}$$

which means

$$\begin{cases} a_{11}^2 t_1 + a_{12}^2 t_2 + \dots + a_{1n}^2 t_n + c_1 = s_1 \\ a_{21}^2 t_1 + a_{22}^2 t_2 + \dots + a_{2n}^2 t_n + c_2 = s_2 \\ . \\ . \\ . \\ a_{n1}^2 t_1 + a_{n2}^2 t_2 + \dots + a_{nn}^2 t_n + c_n = s_n \end{cases}$$

which is equivalent to the system:

$$\begin{cases} \sum_{i=1}^{n} a_{1i}^{2} t_i + c_1 = s_1 \\ \sum_{i=1}^{n} a_{2i}^{2} t_i + c_2 = s_2 \\ . \\ . \\ . \\ \sum_{i=1}^{n} a_{ni}^{2} t_i + c_n = s_n \end{cases} \tag{3.38}$$

Finally, $F(D) = A$ means that $A_2S + c = A$, i.e.,

$$\begin{pmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ & & & . \\ & & & . \\ & & & . \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ . \\ . \\ . \\ s_n \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} \tag{3.39}$$

which means

$$\begin{cases} a_{11}^2 s_1 + a_{12}^2 s_2 + \dots + a_{1n}^2 s_n + c_1 = X_1 \\ a_{21}^2 s_1 + a_{22}^2 s_2 + \dots + a_{2n}^2 s_n + c_2 = X_2 \\ . \\ . \\ . \\ a_{n1}^2 s_1 + a_{n2}^2 s_2 + \dots + a_{nn}^2 s_n + c_n = X_n \end{cases}$$

which is equivalent to the system:

$$\begin{cases} \sum_{i=1}^{n} a_{1i}^2 s_i + c_1 = X_1 \\ \sum_{i=1}^{n} a_{2i}^2 s_i + c_2 = X_2 \\ . \\ . \\ . \\ \sum_{i=1}^{n} a_{ni}^2 s_i + c_n = X_n \end{cases} \tag{3.40}$$

using the system (3.38), we get

$$\begin{cases} \sum_{i=1}^{n} a_{1i}^2 (\sum_{j,l=1}^{n} a_{jl}^2 t_l + c_j) + c_1 = X_1 \\ \sum_{i=1}^{n} a_{2i}^2 (\sum_{j,l=1}^{n} a_{jl}^2 t_l + c_j) + c_2 = X_2 \\ . \\ . \\ . \\ \sum_{i=1}^{n} a_{ni}^2 (\sum_{j,l=1}^{n} a_{jl}^2 t_l + c_j) + c_n = X_n \end{cases} \tag{3.41}$$

Passing now to prove that the images of the lines (AB), (BC), (CD) and (DA) by our formula are in Ω. We will go through the same idea of the first part of the proof.

Let $M(Z_1, Z_2, \dots, Z_n)$ be a point from the line (AB), so the vectorial equation of the line segment is defined by $\overrightarrow{AM} = \beta \cdot \overrightarrow{AB}$,and we have

$$\begin{pmatrix} Z_1 - X_1 \\ Z_2 - X_2 \\ . \\ . \\ . \\ Z_n - X_n \end{pmatrix} = \beta \begin{pmatrix} Y_1 - X_1 \\ Y_2 - X_2 \\ . \\ . \\ . \\ Y_n - X_n \end{pmatrix}$$

i.e.,

$$\begin{cases} Z_1 = X_1 + \beta(Y_1 - X_1) \\ Z_2 = X_2 + \beta(Y_2 - X_2) \\ . \\ . \\ . \\ Z_n = X_n + \beta(Y_n - X_n) \end{cases}$$

which is equivalent to

$$\begin{cases} Z_1 = (1-\beta)X_1 + \beta Y_1 \\ Z_2 = (1-\beta)X_2 + \beta Y_2 \\ . \\ . \\ . \\ Z_n = (1-\beta)X_n + \beta Y_n \end{cases} \tag{3.42}$$

Then $F(\overrightarrow{AB}) = \overrightarrow{BC}$ means

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ & & & . \\ & & & . \\ & & & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \\ . \\ . \\ . \\ b'_n \end{pmatrix} = \begin{pmatrix} t_1 - Y_1 \\ t_2 - Y_2 \\ . \\ . \\ . \\ t_n - Y_n \end{pmatrix} \tag{3.43}$$

i.e.,

$$\begin{cases} a_{11}[(1-\beta)X_1 + \beta Y_1] + Y_1 + a_{12}[(1-\beta)X_2 + \beta Y_2] + ... + a_{1n}[(1-\beta)X_n + \beta Y_n] + b'_1 = t_1 \\ a_{21}[(1-\beta)X_1 + \beta Y_1] + a_{12}[(1-\beta)X_2 + \beta Y_2] + Y_2 + ... + a_{1n}[(1-\beta)X_n + \beta Y_n] + b'_2 = t_2 \\ . \\ . \\ . \\ a_{11}[(1-\beta)X_1 + \beta Y_1] + a_{12}[(1-\beta)X_2 + \beta Y_2] + ... + a_{1n}[(1-\beta)X_n + \beta Y_n] + Y_n + b'_n = t_n \end{cases} \tag{3.44}$$

which is equivalent to the system:

$$\begin{cases} a_{11}(1-\beta)X_1 + (1 + a_{11}\beta)Y_1 + \sum_{j=2}^{n} a_{1j}(X_j + \beta(Y_j - X_j)) + b'_1 = t_1 \\ a_{22}(1-\beta)X_2 + (1 + a_{22}\beta)Y_2 + \sum_{j=1, j\neq 2}^{n} a_{2j}(X_j + \beta(Y_j - X_j)) + b'_2 = t_2 \\ . \\ . \\ . \\ a_{nn}(1-\beta)X_n + (1 + a_{nn}\beta)Y_n + \sum_{j=1}^{n_1} a_{nj}(X_j + \beta(Y_j - X_j)) + b'_n = t_n \end{cases}$$

From the last system we can get the following recursive relationship

$$a_{ii}(1-\beta)X_i + (1+a_{ii}\beta)Y_i + \sum_{i,j=1/i\neq j}^{n_1} a_{ij}(X_j + \beta(Y_j - X_j)) + b'_i = t_i \tag{3.45}$$

Secondly, the vectorial equation for the line segment (BC) is given by $\overrightarrow{BM} = \beta\overrightarrow{BC}$, which means

$$\begin{cases} Z_1 = (1-\beta)Y_1 + \beta t_1 \\ Z_2 = (1-\beta)Y_2 + \beta t_2 \\ \quad . \\ \quad . \\ \quad . \\ Z_n = (1-\beta)Y_n + \beta t_n \end{cases} \tag{3.46}$$

so, $F(\overrightarrow{BC}) = \overrightarrow{CD}$ means

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & . \\ & & & . \\ & & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \\ . \\ . \\ . \\ b'_n \end{pmatrix} = \begin{pmatrix} s_1 - t_1 \\ s_2 - t_2 \\ . \\ . \\ . \\ s_n - t_n \end{pmatrix} \tag{3.47}$$

i.e.,

$$\begin{cases} a_{11}[(1-\beta)Y_1 + \beta t_1] + t_1 + a_{12}[(1-\beta)Y_2 + \beta t_2] + \dots + a_{1n}[(1-\beta)Y_n + \beta t_n] + b'_1 = s_1 \\ a_{21}[(1-\beta)Y_1 + \beta t_1] + a_{12}[(1-\beta)Y_2 + \beta t_2] + t_2 + \dots + a_{1n}[(1-\beta)Y_n + \beta t_n] + b'_2 = s_2 \\ . \\ . \\ . \\ a_{11}[(1-\beta)Y_1 + \beta t_1] + a_{12}[(1-\beta)Y_2 + \beta t_2] + \dots + a_{1n}[(1-\beta)Y_n + \beta t_n] + t_n + b'_n = s_n \end{cases} \tag{3.48}$$

which is equivalent to the system:

$$\begin{cases} a_{11}(1-\beta)Y_1 + (1+a_{11}\beta)t_1 + \sum_{j=2}^{n} a_{1j}(Y_j + \beta(t_j - Y_j)) + b'_1 = s_1 \\ a_{22}(1-\beta)Y_2 + (1+a_{22}\beta)t_2 + \sum_{j=1, j\neq 2}^{n} a_{2j}(Y_j + \beta(t_j - Y_j)) + b'_2 = s_2 \\ . \\ . \\ . \\ a_{nn}(1-\beta)Y_n + (1+a_{nn}\beta)t_n + \sum_{j=1}^{n_1} a_{nj}(Y_j + \beta(t_j - Y_j)) + b'_n = s_n \end{cases}$$

From the last system we can get the following recursive relationship

$$a_{ii}(1-\beta)Y_i + (1+a_{ii}\beta)t_i + \sum_{i,j=1/i\neq j}^{n_1} a_{ij}(Y_j + \beta(t_j - Y_j)) + b'_i = s_i \tag{3.49}$$

Thirdly, the vectorial equation for the line segment (CD) is given by $\overrightarrow{CM} = \beta\overrightarrow{CD}$, which means

$$\begin{cases} Z_1 = (1-\beta)t_1 + \beta s_1 \\ Z_2 = (1-\beta)t_2 + \beta s_2 \\ \quad . \\ \quad . \\ \quad . \\ Z_n = (1-\beta)t_n + \beta s_n \end{cases} \tag{3.50}$$

So, $F(\overrightarrow{CD}) = \overrightarrow{DA}$ means

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & . \\ & & & . \\ & & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \\ . \\ . \\ . \\ b'_n \end{pmatrix} = \begin{pmatrix} X_1 - s_1 \\ X_2 - s_2 \\ . \\ . \\ . \\ X_n - s_n \end{pmatrix} \tag{3.51}$$

i.e.,

$$\begin{cases} a_{11}[(1-\beta)t_1 + \beta s_1] + s_1 + a_{12}[(1-\beta)t_2 + \beta s_2] + \dots + a_{1n}[(1-\beta)t_n + \beta s_n] + b'_1 = X_1 \\ a_{21}[(1-\beta)t_1 + \beta s_1] + a_{12}[(1-\beta)t_2 + \beta s_2] + s_2 + \dots + a_{1n}[(1-\beta)t_n + \beta s_n] + b'_2 = X_2 \\ \quad . \\ \quad . \\ \quad . \\ a_{11}[(1-\beta)t_1 + \beta s_1] + a_{12}[(1-\beta)t_2 + \beta s_2] + \dots + a_{1n}[(1-\beta)t_n + \beta s_n] + s_n + b'_n = X_n \end{cases} \tag{3.52}$$

which is equivalent to the system:

$$\begin{cases} a_{11}(1-\beta)t_1 + (1 + a_{11}\beta)s_1 + \sum_{j=2}^{n} a_{1j}(t_j + \beta(s_j - t_j)) + b'_1 = X_1 \\ a_{22}(1-\beta)t_2 + (1 + a_{22}\beta)s_2 + \sum_{j=1, j\neq 2}^{n} a_{2j}(t_j + \beta(s_j - t_j)) + b'_2 = X_2 \\ \quad . \\ \quad . \\ \quad . \\ a_{nn}(1-\beta)t_n + (1 + a_{nn}\beta)s_n + \sum_{j=1}^{n_1} a_{nj}(t_j + \beta(s_j - t_j)) + b'_n = X_n \end{cases}$$

From the last system we can get the following recursive relationship

$$a_{ii}(1-\beta)t_i + (1 + a_{ii}\beta)s_i + \sum_{i,j=1/i\neq j}^{n_1} a_{ij}(t_j + \beta(s_j - t_j)) + b'_i = X_i \tag{3.53}$$

Finally, the vectorial equation for the line segment (DA) is given by $\overrightarrow{DM} = \beta\overrightarrow{DA}$, which means

$$\begin{cases} Z_1 = (1-\beta)s_1 + \beta X_1 \\ Z_2 = (1-\beta)s_2 + \beta X_2 \\ \quad . \\ \quad . \\ \quad . \\ Z_n = (1-\beta)s_n + \beta X_n \end{cases} \tag{3.54}$$

So, $F(\overrightarrow{DA}) = \overrightarrow{AB}$ means

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & . \\ & & & . \\ & & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \\ . \\ . \\ . \\ b'_n \end{pmatrix} = \begin{pmatrix} Y_1 - X_1 \\ Y_2 - X_2 \\ . \\ . \\ . \\ Y_n - X_n \end{pmatrix} \tag{3.55}$$

i.e.,

$$\begin{cases} a_{11}[(1-\beta)s_1 + \beta X_1] + X_1 + a_{12}[(1-\beta)s_2 + \beta X_2] + \dots + a_{1n}[(1-\beta)s_n + \beta X_n] + b'_1 = Y_1 \\ a_{21}[(1-\beta)s_1 + \beta X_1] + a_{12}[(1-\beta)s_2 + \beta X_2] + X_2 + \dots + a_{1n}[(1-\beta)s_n + \beta X_n] + b'_2 = Y_2 \\ . \\ . \\ . \\ a_{11}[(1-\beta)s_1 + \beta X_1] + a_{12}[(1-\beta)s_2 + \beta X_2] + \dots + a_{1n}[(1-\beta)s_n + \beta X_n] + X_n + b'_n = Y_n \end{cases} \tag{3.56}$$

which is equivalent to the system:

$$\begin{cases} a_{11}(1-\beta)s_1 + (1+a_{11}\beta)X_1 + \sum_{j=2}^{n} a_{1j}(s_j + \beta(X_j - s_j)) + b'_1 = Y_1 \\ a_{22}(1-\beta)s_2 + (1+a_{22}\beta)X_2 + \sum_{j=1, j\neq 2}^{n} a_{2j}(s_j + \beta(X_j - s_j)) + b'_2 = Y_2 \\ . \\ . \\ . \\ a_{nn}(1-\beta)s_n + (1+a_{nn}\beta)X_n + \sum_{j=1}^{n_1} a_{nj}(s_j + \beta(X_j - s_j)) + b'_n = Y_n \end{cases}$$

From the last system we can get the following recursive relationship

$$a_{ii}(1-\beta)s_i + (1+a_{ii}\beta)X_i + \sum_{i,j=1/i\neq j}^{n_1} a_{ij}(s_j + \beta(X_j - s_j)) + b'_i = Y_i \tag{3.57}$$

By combining all of the previous findings and observations, we can deduce that

Theorem 3.4.4 *It can be asserted that the n-D piecewise linear smooth map F defined in equation (3.27) follows a quasi-hyperbolic regime.*

General Conclusion and Perspectives

Building a structural proof exploring a complex chaotic behavior may not be evident. But the goal of all mathematicians is to establish a brave theory that can give clearance to application field easily. This goal clarifies the vast amount of studies on dynamical systems and their applications.

This thesis is mainly captivating realm of chaotic dynamics, offering a comprehensive exploration of piecewise smooth maps and the classification of strange attractors. Throughout the three chapters, a cohesive narrative unfolds, shedding light on the intricate behaviors and properties exhibited within chaotic systems.

The first chapter presents general concepts on chaotic dynamics. We have discussed the different concepts of the chaos theory. We end the chapter with some existing definitions of chaotic attractors and we give an entry to the robust chaos theory.

The second chapter focuses on the analysis of systems governed by piecewise smooth maps. The chapter explores both one-dimensional and two-dimensional piecewise smooth maps, unraveling their unique dynamics and properties. Additionally, the chapter investigates border collision bifurcation scenarios and their impact on the robustness of chaotic behavior. By studying the behaviors of piecewise smooth maps, this chapter contributes to the understanding of chaos dynamics, paving the way for further exploration.

The third chapter delves into the classification and characterization of various types of strange attractors. we have proved the existence of a quasi-hyperbolic attractor for a particular family of 2D piecewise linear maps. The main analysis is based on finding an absorbing region and then proving that it contains a quasi-hyperbolic attractor for the form (3.1) under some specific conditions.

Over-ally, this thesis advances our knowledge of chaotic dynamics by delving into the complexities of piecewise smooth maps and the classification of strange attractors. Through theoretical analysis, numerical simulations, and rigorous mathematical techniques, the thesis contributes to the broader field of nonlinear dynamics. By revealing the intricate behaviors and properties of chaotic systems, this research opens doors to new perspectives and avenues for future explorations. Ultimately, this thesis deepens our comprehension of the underlying principles that govern complex systems, furthering our understanding of chaotic dynamics and their applications across various disciplines.

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