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Dedication

First and foremost, I thank "Allah ›› for enabling me to complete this thesis.

I dedicate this work to:

All members of my family who encouraged me to complete my studies career.

To all my respected teachers who taught me.

To all my colleagues who studied with me.

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Abstract

In this thesis, we will study some evolution problems that represent some physical phenomena (Piezoelectric beam, Kirchhoff beam) with some types of delay (for example, distributed delay, neutral delay) acting on linear or nonlinear internal feedbacks. We will prove the well-posedness (existence and uniqueness) of solutions to these systems by semigroup theory or by Faedo–Galerkin method. With regard to the asymptotic behavior of the solutions, we will get the exponential decay of solutions, which represents the rapid decrease of energy, by constructing a Lyapunov functional using the multiplication method. Or we get the blow-up of solutions by using Georgiev and Todorova's method.

Keywords: Piezoelectric beam; Kirchhoff beam; Semigroup theory; Faedo–Galerkin method; Time delay; Lyapunov functional; Exponential decay of solutions; Blow-up of solutions.

Résumé

Dans cette thèse, nous étudierons des problèmes d'évolution qui représentent certains phénomènes physiques (poutre piézoélectrique, poutre de Kirchhoff) avec certains types de retard (par exemple, retard distribué, retard neutre) agissant sur des rétroactions internes linéaires ou non linéaires. Nous démontrerons l'existence et l'unicité des solutions de ces systèmes par la théorie des semi-groupes ou par la méthode Faedo–Galerkin. En ce qui concerne le comportement asymptotique des solutions, nous obtiendrons la décroissance exponentielle des solutions, qui représente la décroissance rapide de l'énergie, en construisant une fonctionnelle de Lyapunov en utilisant la méthode de multiplication. Ou nous obtenons une explosion des solutions en utilisant la méthode de Georgiev et Todorova.

Mots-clés: Poutre piézoélectrique; Poutre de Kirchhoff; Théorie des semi-groupes; Méthode de Faedo-Galerkin; Temps de retard; Fonctionnelle de Lyapunov; Décroissance exponentielle des solutions; Explosion des solutions.

المـلـخــص

في هذه الأطروحة، سوف نقوم بدراسة بعض المشاكل التطورية التي تمثل بعض الظواهر الفيزيائية (العارضة الكهرضغطية، عارضة كيرشوف Kirchhoff) مع بعض أنواع التأخير الزمني (على سبيل المثال، التأخير الموزع أو التأخير المحايد) تعمل على ردود الفعل الداخلية الخطية أو غير الخطية. فيما يخص حسن وضع الحلول لهذه الأنظمة (وجودها وتفردها) نستخدم نظرية شبه المجموعة أو طريقة فايدو-قاليركين Faedo-Galerkin. و فيما يتعلق بالسلوك التقاربي للحلول، فسنتحصل على الاضمحلال الأسي للحلول، الذي يمثل الانخفاض السريع للطاقة، من خلال بناء دالة ليابونوف Lyapunov وذلك باستخدام طريقة الضـــرب. أو نحصل على تفجير الحلول باستخدام طريقة جورجييف وتودوروفا Georgiev and Todorova.

الكلمات المفتاحية: العارضة الكــهرضغــطيــة؛ عارضة كيـرشـــــوف Kirchhoff؛ نظــــرية شبــــه المجــــموعة؛ طريقـــة فايـــدو قالـــيركين Faedo-Galerkin؛ تأخيــر الوقـــت؛ وظيفة ليــابونــوف Lyapunov؛ التناقص الأســـي للحلول؛ إنفجار الحـلول.

Introduction

*I*n physical phenomena and systems, time delay refers to the time interval between the occurrence of a specific event and the appearance of its effect or a change in the system. In general, time delay reflects the time required for information transmission or change from one point to another in a physical system and affects response speed and event timing within the system. For example, continuous combustion systems, including domestic and industrial burners, steam and gas turbines and waste incinerators, are widely used in power generation and heating. There are two major dynamics in a combustion system: flame dynamics and acoustic wave dynamics. They are coupled to form a loop, as shown in the next figure. Due to wave propagation, there is a delay in the wave dynamics. Delays also appear in the measurement and actuator units of the system

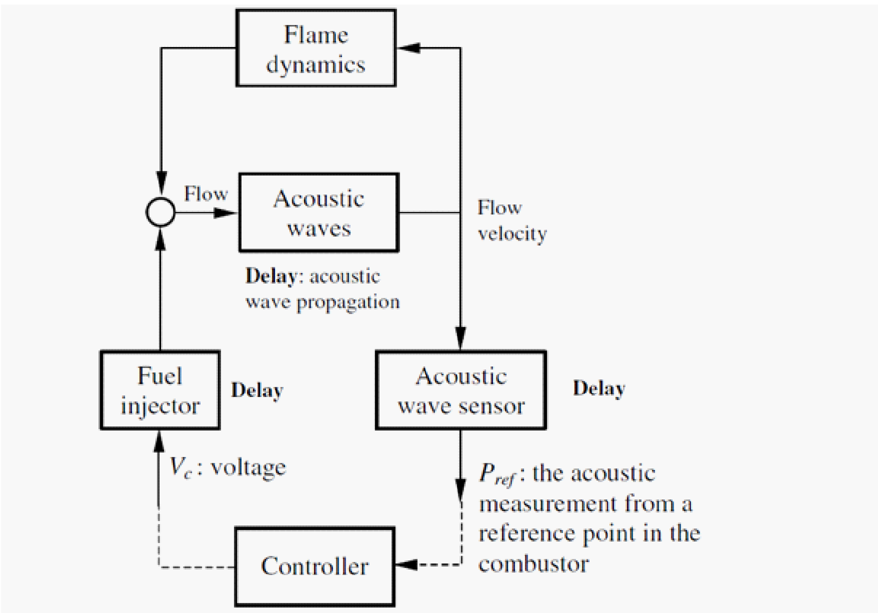

Figure 1 : Dynamics in a combustion system

For more examples, we direct the reader to reference [103].

In the context of mathematical problems, the term "delay" typically refers to a phenomenon known as delay differential equations (DDEs) or functional differential equations (FDEs). It is employed in mathematical models that assume a specified behavior or phenomenon depends on both the current and past states of a system [40]. For this reason, functional differential equations are more applicable than ordinary differential equations (ODE). The simplest type of functional differential equations is of the form:

$$x'(t) = g(t, x(t), x(t-r)).$$

Delay differential equations arise in various fields of science and engineering, such as biology, physics, economics, and control theory [1, 2, 3, 15, 82, 103]. Solving delay differential equations can be challenging due to the need to consider past values of the variables. Numerical methods, such as the method of steps or various approximations, are often employed to approximate the solutions to these equations. Additionally, stability analysis and the existence of solutions are important aspects when dealing with delay differential equations. Datko in [20] proved that uniform asymptotic stability is not necessarily preserved under small perturbations of the delay for infinite dimensional problems with finite lags. Also, Datko et al. [21] considered the equation

$$u_{tt} - u_{xx} + 2au_t + a^2 u = 0, \ 0 < x < 1, \quad t > 0, \tag{1}$$

with time delays in boundary feedback given by

$$\begin{cases} u(0,t) = 0, \\ u_x(1,t) = -ku_t(1, t-\varepsilon), & t > 0. \end{cases} \tag{2}$$

By using some lemmas, an example was given that showed this time delay can destabilize the system (1)-(2) which, in the absence of delays, is uniformly asymptotically stable. Xu et al. in [97] were interested in studying the following wave system

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, & \text{in } (0,1) \times (0,\infty), \\ w(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = -k\mu w_t(1,t) - k(1-\mu) w_t(1, t-\tau), & t \geq 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \\ w_t(1, t-\tau) = f(t-\tau) & t \in (0,\tau). \end{cases} \tag{3}$$

The following cases are proven:

- System (3) is exponentially stable if $\mu > \frac{1}{2}$.
- System (3) becomes unstable when $\mu < \frac{1}{2}$.
- If $\mu = \frac{1}{2}$ and $\tau \in (0,1)$ is rational, so the system is unstable.

- If $\mu = \frac{1}{2}$ and $\tau \in (0, 1)$ is irrational, so the system (3) is asymptotically stable.

In [1] Agrawal et al. present a stability analysis of a single-degree-of-freedom system with time-delayed feedback. And proved by numerical simulation that when the time delay is close to its maximum allowable limit, significant control degradation occurs or may lead to instability. A compensation technique was also introduced by modeling time delay as transportation lag, which ensures the stability of their controlled system. Nicaise and Pignotti [66] studied a wave equation problem with a delay term in the boundary or internal feedbacks. In the case $\mu_2 < \mu_1$, established the exponential stability of the solution. If $\mu_2 \geq \mu_1$, the authors showed the existence of an explicit sequence of arbitrarily small delays that lead to the destabilization of the system. Also, Nicaise and Pignotti in [67] considered the wave equation with the boundary or internal distributed delay by introducing appropriate energy, and by proving some observability inequalities, proved the exponential stability of the solution. For the internal distributed delay, and in the case where some assumptions are not verified, showed that this time delay leads to instability. We direct the reader to the following references [19, 34, 69, 78, 86, 93] for more results related to the instability of some systems due to the time delay.

▸ Stability of some systems with some types of delays

Because delay is the source of instability. In [16], the stability of solutions for a one-dimensional model of a Rao-Nakra sandwich beam with Kelvin-Voigt damping and a time delay was studied by Cabanillas et al. The well-posedness of the problem is established by applying the Lumer-Phillips theorem. The exponential stability is then proven by utilizing the Gearhart-Huang-Prüss' theorem. Feng and Raposo et al. in their search [28], considered in $(0, \Gamma) \times (0, \infty)$ the Rao-Nakra sandwich beam equation with time-varying weight and time-varying delay

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k\left(-u + \nu + \alpha w_x\right) + \mu_1(t)\, u_t + \mu_2(t)\, u_t\left(t - \tau_1(t)\right) = 0, \\ \rho_3 h_3 \nu_{tt} - E_3 h_3 \nu_{xx} + k\left(-u + \nu + \alpha w_x\right) + \tilde{\mu}_1(t)\, \nu_t + \tilde{\mu}_2(t)\, \nu_t\left(t - \tau_2(t)\right) = 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k\left(-u + \nu + \alpha w_x\right)_x + \hat{\mu}_1(t)\, w_t + \hat{\mu}_2(t)\, w_t\left(t - \tau_3(t)\right) = 0. \end{cases}$$

By utilizing the semigroup of the linear operator and employing the Kato variable norm technique, they demonstrated that the system is globally well-posed. When the coefficients of delay are small, they establish an exponential decay of the system by using the multiplier approach (the first method to prove stability). In the last, they showed the inequality of internal observability and the equivalence between stabilization and observability (the second method to prove stability). Feng and Almeida Junior et al. [30] were interested in the asymptotic behavior of the following Bresse-Timoshenko type system with time-dependent delay terms acting on angular rotation

$$\begin{cases} \rho_1 y_{tt} - \kappa\left(y_x + \psi\right)_x = 0, \text{ in }]0, \Gamma[\times]0, \infty[, \\ -\rho_2 y_{ttx} - b\psi_{xx} + \kappa\left(y_x + \psi\right) + \mu_1 \psi_t + \mu_2 \psi_t\left(t - \tau(t)\right) = 0. \end{cases}$$

Through the introduction of a suitable Lyapunov functional and irrespective of any relationship between wave propagation velocities, exponential stability was proven under some assumptions. Finally, this problem was studied again in the case of time-dependent delay and viscous damping acting on vertical displacement, yielding the same results. Feng and Li [26] considered the following nonlinear viscoelastic Kirchhoff plate equation with a time delay term in the internal feedback

$$\begin{cases} u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) - \sigma(t) \int_0^t g(t-s) \Delta^2 u(s)\, ds + \mu_1 |u_t|^{m-1} u_t \\ +\mu_2 |u_t(t-\tau)|^{m-1} u_t(t-\tau) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}^+. \end{cases}$$

By using the energy perturbation method and under suitable assumptions, the general decay of the solution for this problem was established. In the presence of delay feedback, Komornik and Pignotti in [47] considered the Korteweg-de Vries-Burgers (KdV-Burgers) equation

$$\begin{cases} u_t + u_{xxx} - u_{xx} + \lambda_0 u + \lambda u(t-\tau) + \mathrm{u} u_x = 0, & in\ \mathbb{R} \times (0,\infty), \\ u(x,s) = u_0(x,s), & in\ \mathbb{R} \times [-\tau, 0], \end{cases}$$

together with its linear version, i.e. without the term $\mathrm{u}u_x$. The well-posedness of the models and exponential decay estimates were proven under appropriate conditions for the damping coefficients. Their arguments relied on a Lyapunov functional approach and semigroup theory. Mpungu and Apalara [57] investigated a system of laminated beams that incorporates an internal constant delay

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu w_t(t-\tau) = 0, \\ I_\rho (3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3D s_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,\infty)$. The dissipation through structural damping at the interface was proven to be sufficiently strong to achieve exponential stabilization of the system under suitable assumptions on coefficients of wave propagation speed and delay feedback. In [9] Almeida Júnior et al. considered a truncated version of the Bresse-Timoshenko equations with delay and viscous damping acting on displacement

$$\begin{cases} \rho_1 y_{tt} - k(y_x + \psi)_x + \mu_1 y_t + \mu_2 y_t(x, t-\tau) = 0, \quad \text{in }]0,\Gamma[\times]0,\infty[, \\ -\rho_2 y_{ttx} - b\psi_{xx} + k(y_x + \psi) = 0, \quad \text{in }]0,\Gamma[\times]0,\infty[, \end{cases}$$

with the homogeneous boundary conditions of Dirichlet and the following initial conditions

$$y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ \psi(x,0) = \psi_0(x), \ x \in]0,\Gamma[,$$

the same equations with delay and viscous damping acting on angular rotation, equipped with the same previous conditions, were considered. Under certain assumptions and using the Lyapunov functional technique, the exponential decay is obtained in both cases, regardless of any relationship between the system's coefficients. Ouchenane and Zennir in [72] considered a one-dimensional porous-elastic system that incorporates both memory and distributed delay terms in the second equation with second sound. Although the delay is a source of instability, a general decay result was demonstrated under some conditions on the relaxation function. In [48] Khochemane et al. considered a one-dimensional porous-elastic system with distributed delay acting in the second equation. Under some assumptions, the existence and uniqueness of this system were proven by using semi-group theory (Hille-Yosida theorem). Also, exponential stability is obtained by using the energy method. Douib et al. [22] by introducing a suitable Lyaponov functional, proved exponential stability for a flexible structure with distributed delay and fourier's type heat conduction. The Bresse system with delay terms in the internal feedbacks acting in the first, third equations and a distributed delay term in the second equation was studied by Bouzettouta et al. [10] through some theories of semigroup, proved the global existence of solution. Furthermore, the stability of solutions was studied using the multiplier method. Fares Yazid et al. in their paper [102], studied a one-dimensional linear thermoelastic (Cattaneo's law) system of Timoshenko type with distributed delay term. Through an appropriate assumption between the weight of the damping and the weight of the delay and using the energy method, exponential stability was proven without the usual assumption on the wave speeds.

Among the types of functional differential equations (FDEs), we find "**Neutral Delay Differential Equations" (NDDEs)**, where this type of equation relies on both past and present values of the function, it also incorporates derivatives with delays [35, 40, 41, 42, 55]. We provide the reader with some illustrative examples

$$[u(t) - au(t-\tau)]' = f(t, u, u(t-s)),$$

$$u'(t,x) = \Delta u + f(t, u, u'(t-\tau, x)).$$

In fact, neutral delays are commonly applied in the study of vibrating masses attached to an elastic bar and also in some variation problems, heat exchanges, electrodynamics, biological sciences, population ecology, etc [40, 95]. While minor delays can lead to instability in some systems, 'large' neutral delays can stabilize certain systems. In fact, intentional incorporation of neutral delays into a system is done at times to enhance its performance, structure, or stability [89]. In [88], Tatar considered in $[0,1] \times [0,\infty)$ the damped wave equation problem with the inclusion of a neutral delay

$$\begin{cases} u_{tt} - u_{xx} + u_t + \int_0^t h(t-s) u_{tt}(s) ds = 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x). \\ u(t, 0) = 0, \ u(t, 1) = 0. \end{cases} \tag{4}$$

An exponential stability result of (4) was shown in some cases on the kernel h. Mpungu and Apalara, in their research [58] proved the exponential stability of a laminated beam when a neutral delay is present. In $[0,1]\times[0,\infty)$, the thermoelastic laminated system subjected to a neutral delay was investigated by Seghour et al. [84]

$$\begin{cases} \rho w_{tt} + G\left(\psi - w_x\right)_x + Aw_t = 0, \\ I_\rho \left(3s_{tt} - \psi_{tt}\right) - G\left(\psi - w_x\right) - \left(3s - \psi\right)_{xx} + \mu\theta_x = 0, \\ 3I_\rho \left[s_t + \int_0^t h\left(t-s\right) s_t\left(r\right) dr\right]' + 3G\left(\psi - w_x\right) + 4\gamma s - 3s_{xx} = 0, \\ \theta_t - \kappa\theta_{xx} + \mu\left(3s - \psi\right)_{tx} = 0, \end{cases}$$

with the boundary and initial conditions

$$\begin{cases} \psi = s = \theta_x = w_x = 0, \text{ in the case } x = 0, \\ \theta = w = s_x = \psi_x = 0, \text{ in the case } x = 1, \\ \left(w, \psi, s, \theta\right)\left(x, 0\right) = \left(w_0, \psi_0, s_0, \theta_0\right), \ x \in [0,1], \\ \left(w_t, \psi_t, s_t\right)\left(x, 0\right) = \left(w_1, \psi_1, s_1\right). \end{cases}$$

By employing the energy method with certain conditions on the kernel h and system parameters, both exponential and polynomial stability were demonstrated. For further results concerning neutral delay problems with the occurrence of delays in the second derivative (see [23, 51, 62, 87] and the references therein).

▶ Blow-up of solution of some systems with some types of delays

A nonlinear wave equation with delay was examined by Kafini and Messaoudi in [44] and demonstrated that the solution of this problem blows up in a finite time under appropriate conditions for the initial data, the nonlinear source term, the weights of delay, and the damping term. Also, Kafini and Messaoudi in [46] examined the following delayed wave equation with a logarithmic nonlinear source term

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t\left(t-\tau\right) = u\left|u\right|^{p-2} \ln\left|u\right|^k, \ x \in \Omega \text{ and } t > 0,$$

under the conditions

$$\begin{cases} u\left(x,t\right) = 0, \qquad x \in \partial\Omega, \\ u_t\left(x, t-\tau\right) = f_0\left(x, t-\tau\right), \qquad in \ \left(0,\tau\right), \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \ in \ \Omega. \end{cases}$$

The local existence result has been proven using semigroup theory. Furthermore, the blow-up of solutions in finite time with nonpositive initial energy is demonstrated. Yüksekkaya et al. in their work [98] focused on the investigation of the higher-order Kirchhoff-type equation with a delay term in a bounded domain. Firstly, the global existence of the solution was established. Next, the decay of solutions was discussed using Nakao's technique, considering both polynomial and exponential decays. Additionally, they established the blow-up result for negative initial energy under suitable conditions.

Yüksekkaya and Pişkin in [100] considered the following nonlinear viscoelastic plate equation with a distributed delay

$$u_{tt}+\Delta^2 u-\int_0^t g(t-s)\,\Delta^2 u(s)\,ds+\mu_1 u_t+\int_{\tau_1}^{\tau_2} |\mu_2(q)|\,u_t(t-q)\,dq=b\,|u|^{p-2}\,u,\ x\in\Omega \text{ and } t>0,$$

with the conditions

$$\begin{cases} u=\frac{\partial u}{\partial \nu}=0, & x\in\partial\Omega, \\ u_t(x,-t)=f_0(x,t), & (x,t)\in\Omega\times(0,\tau_2), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ in\ \Omega. \end{cases}$$

A blow-up of solutions was successfully obtained under some conditions. Fatima Zohra Mahdi et al. [104] the focus of their paper is to investigate the initial boundary value problem for a system of viscoelastic wave equations of Kirchhoff type with a delay term in a bounded domain. Under some suitable assumptions, the energy decay rate is proved by Nakao's technique. In addition, the blow up of solutions is obtained in different states on the initial energy.

▶ The concept of stability in dynamic systems

Dynamic systems, also known as dynamical systems, refer to mathematical models used to describe and analyze the behavior of systems that evolve or change over time. These systems can be found in various fields, including physics, engineering, biology, economics, and social sciences [74].

In the dynamic systems, understanding the behavior and properties of various states is of utmost importance. Some concepts that play a fundamental role in analyzing system dynamics are equilibrium, stability, asymptotic stability, instability, and system explosions. Stable equilibrium points (also known as rest points or stationary points) are characterized by a system that returns to its original state after experiencing small disturbances. In other words, if the system is slightly displaced from equilibrium, it will eventually return to the same equilibrium state.

Example 0.1 A typical example that illustrates this situation is the pendulum. Pendulums with a rigid rod have two equilibrium points. One equilibrium point occurs when the rod is in a vertical position, with the mass hanging downward, the other is when the mass is up.

Stability is the property of a system to maintain or return to equilibrium after experiencing a disturbance. A stable system resists change and exhibits a tendency to restore its original state. Stability is often evaluated by examining the system's response to small perturbations or deviations from the equilibrium point. If the system's response damps out over time, it is considered stable.

Example 0.2 As an illustrative example, considering the first equilibrium point of the pendulum (when the rod is vertical and the mass is hanging downward) and assuming there is no friction, a small push from this resting position will lead to sustained oscillations with a bounded amplitude around the equilibrium. This implies that the first equilibrium point is stable [43].

Asymptotic stability goes one step further than stability. It implies that a disturbed system not only returns to equilibrium but also the solutions or trajectories starting nearby to it converge (as time approaches infinity) back to it. In other words, the system's response converges towards the equilibrium point, resulting in a progressively diminishing deviation. Asymptotic stability is a desirable property as it guarantees long-term stability and resilience to disturbances.

Example 0.3 Taking the pendulum example once more, if we introduce friction into the problem, will result in damped oscillations around the equilibrium point. Ultimately, the pendulum will cease its oscillations and revert back to its resting position.

On the other hand, instability refers to a system's inability to return to equilibrium after a disturbance. Instead of converging towards a steady state, an unstable system exhibits an ever-increasing deviation from the original state. Small perturbations can trigger significant changes, leading to unpredictable behavior and often resulting in chaotic or explosive dynamics.

Example 0.4 The second equilibrium point of the pendulum with friction, i.e., where the mass is positioned upwards, is considered unstable. If it is slightly disturbed from its equilibrium position, it does not return to that position.

System explosions occur when a system becomes highly unstable, leading to an exponential growth of its variables or components. System explosions are undesirable and often indicate a breakdown in the system's structure or control mechanisms.

Understanding these concepts and their implications is crucial for engineers, scientists, and analysts working with dynamic systems. By assessing equilibrium, stability, asymptotic stability, instability, and the risk of system explosions, experts can design robust and reliable systems, predict their behavior, and identify potential vulnerabilities or failure modes.

▶ Description and objective of the thesis

The main goal of this thesis is to study the well-posedness and asymptotic behavior (exponential decay and blow-up result) of solutions for some evolution problems with different types of boundary conditions and delay terms. This work consists of five chapters:

- **In Chapter 1**, we focused on some mathematical principal concepts, some theorems and lemmas on distributions, Lebesgue and Sobolev spaces, which we need in the proofs of our next results.
- **In Chapter 2**, we study a one-dimensional system of piezoelectric beams with a distributed delay term. The existence of solutions has been obtained by using semigroup theory. Also, by constructing a suitable Lyapunov functional, the exponential stability result of the solution has been established independent of any conditions on the wave speeds $\left(\sqrt{\frac{\alpha}{\rho}},\ \sqrt{\frac{\beta}{\mu}}\right)$ or any system parameters.
- **In Chapter 3**, we focus on a one-dimensional system of piezoelectric beams with distributed delay acting in the mechanical equation, where magnetic and thermal effects governed by Maxwell's equations and Fourier's law are taken into account. Using the same methods and assumptions that we used in chapter 2, we prove exponential stability. Finally, the results are compared to those of the electrostatic case (the magnetic effects are negligible).
- **In Chapter 4**, we will prove the global existence, uniqueness and exponential energy decay of a one-dimensional system of fully dynamic and electrostatic or quasi-static piezoelectric system with distributed delay of neutral type acting on mechanical equation without adding any damping term. Under some assumptions and by using the classical Faedo-Galerkin approximations along with some a priori estimates, we first prove the global existence and uniqueness of the system. Next, using the energy method and constructing a Lyapunov functional we establish that this system is exponentially stable. Our results are associated with specific assumptions only concerning the kernel h. In the end, we get the same results in the case quasi-static or electrostatic system.
- **In Chapter 5,** we consider some problem of Kirchhoff type with variable exponents and time delay. Under suitable hypotheses, the blow-up of solutions is proved by using Georgiev and Todorova's method.

▶ Methodology

In this thesis, we utilize the theory of semigroup to prove the existence and uniqueness (well-posedness) of solutions related to our systems. Particularly, the Hille-Yosida theorem is a fundamental and powerful tool to find the existence, uniqueness and regularity of the solutions of a stationary differential equation

$$\begin{cases} U'(t) = AU(t), \ t \in (0, \infty), \\ U(0) = U_0, \end{cases}$$

where $A : D(A) \subset H \longrightarrow H$ and H generally called the energy space. Or we will adopt Faedo-Galerkin method to show the existence of solutions.

For the stability results, we employ the multiplier technique to construct the Lyapunov function L that is equivalent to the energy E of the solution. This implies the existence of two positive constants, c_1 and c_2 such that

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \ \forall t \geq 0. \tag{5}$$

For exponential stability, it is sufficient to establish that

$$L'(t) \leq -cE(t), \ \forall t \in [0, +\infty[. \tag{6}$$

Where $c > 0$. By introducing the integral on (6) over the interval $(0, t)$ and utilizing the equivalence between the Lyapunov function and energy, as indicated in the inequality (5), we reach the desired result of exponential stability (exponential decay of solutions or exponential energy decay). In fact, the main difficulty lies in determining the appropriate Lyapunov function that guarantees us a stability result.

Remark 0.1 There are other types of stability, that we mention some of them

- Strong stabilization this means $E(t) \underset{t\to\infty}{\longrightarrow} 0$.
- Polynomial stabilization. For example $E(t) \leq ct^{-m}, \quad c, m > 0, \ \forall t > 0$.
- Logarithmic stabilization. For example $E(t) \leq c(\log(t))^{-m}, \quad c, m > 0, \ \forall t > 0$.

For the blow-up result, we employ the Georgiev-Todorova method [37], which is based on searching for $0 < \alpha < 1$ and $\varepsilon > 0$ in such a way that the functional

$$L(t) = [-E(t)]^{1-\alpha} + \varepsilon \int_\Omega u_t u dx,$$

verifies an inequality of the form

$$L'(t) \geq \lambda L^q(t), \ t \geq 0, \ q > 1.$$

This inequality will indeed lead to an explosion in finite time.

▶ Publications

1. Loucif, S., Guefaifia, R., Zitouni, S., Khochemane, H. E.: Global well-posedness and exponential decay of fully dynamic and electrostatic or quasi-static piezoelectric beams subject to a neutral delay. Zeitschrift für angewandte Mathematik und Physik 74(3), 83 (2023)

2. Loucif, S., Guefaifia, R., Zitouni, S., Ardjouni, A.: Well-posedness and exponential decay for piezoelectric beams with distributed delay term. Memoirs on Differential Equations and Mathematical Physics 1-20 (2023)

3. Loucif, S., Guefaifia, R., Zitouni, S., Khochemane, H. E.: Global well-posedness and exponential decay of shear beam model subject to a neutral delay. Eurasian Journal Of Mathematical And Computer Applications 11(2), 67–81 (2023)

4. Zireg, B., Khochemane, H. E., Loucif, S., Zitouni, S.: Theoretical analysis and numerical simulation of a heat Bresse-Timoshenko model. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 30(6), 415–439 (2023)

CHAPTER 1

Preliminaries

*T*he primary aim of this chapter is to introduce several fundamental mathematical concepts, some theorems, definitions and lemmas on distributions, Lebesgue and Sobolev spaces that we may need in the next chapters. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$.

1.1 Spaces of test functions and distributions

Definition 1.1 Let $\Omega \subset \mathbb{R}^n$, if u is a function defined on Ω, we define the support of u to be the set

$$supp(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

Definition 1.2 Let Ω be a domain in $\mathbb{R}^n$. For any nonnegative integer m, let $C^m(\Omega)$ represent the vector space consisting of all functions ϕ along with all their partial derivatives $D^\alpha \phi$ of orders $|\alpha| \leq m$, are continuous on Ω. So that $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and

$$\begin{cases} |\alpha| = \alpha_1 + ... + \alpha_n, \\ D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}. \end{cases}$$

Definition 1.3 We denote by $D(\mathbb{R}^n)$, or simply D, the set of infinitely differentiable functions with bounded support

$$D = \{\varphi \in C^\infty : supp\ \varphi \text{ is bounded}\}.$$

This set is called the base space, and its elements are called base functions (or test functions).

Note that D is an infinite-dimensional vector space.

Definition 1.4 We say that a sequence of functions $(\varphi_k) \in D$ converges in D to a function $\varphi \in D$ if:

(i) All the supports of φ_k are contained within the same compact set K.

(ii) For every $j \in \mathbb{N}$, $0 \leq j \leq m$, the sequence of derivatives $(\varphi_k^{(j)})$ converges uniformly to $\varphi^{(j)}$ on K.

Definition 1.5 A distribution T is defined as a linear continuous functional on D.

(i) A linear functional means that for any $\varphi_1, \varphi_2 \in D$ and $\alpha, \beta \in \mathbb{C}$, we have:

$$\langle T, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha \langle T, \varphi_1 \rangle + \beta \langle T, \varphi_2 \rangle .$$

Instead of linear functional, we also use the term linear form.

(ii) Continuity means that if the sequence (φ_k) converges in D to φ, then $\langle T, \varphi_k \rangle$ converges in the usual sense to $\langle T, \varphi \rangle$.

We also say that a linear functional on D defines a distribution if, for any sequence $(\varphi_k) \in D$ that converges in D to zero, the sequence $\langle T, \varphi_k \rangle$ converges in the usual sense to zero.

Proposition 1.1 *A linear functional on D is a distribution if and only if, for every compact K and for every function $\varphi \in D$ with supp $\varphi \subset K$, there exists a constant $C > 0$ and an integer m such that:*

$$|\langle T, \varphi \rangle| \leq C \sum_{j=0}^{m} \sup_{x \in K} \left| \varphi^{(j)}(x) \right| .$$

Definition 1.6 The derivative T' of a distribution T is defined as the functional determined by the relation

$$\langle T', \varphi \rangle = - \langle T, \varphi' \rangle , \qquad \forall \varphi \in D$$

Example 1.1 The derivative of the Heaviside function, defined by

$$H(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 \text{ if } x > 0, \end{cases}$$

determines a distribution denoted H. The derivative of $H(x)$ does not exist at the point $x = 0$. But in the sense of distributions, we have for $\varphi \in D$

$$\langle H', \varphi \rangle = - \langle H, \varphi' \rangle = - \int_0^{\infty} \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi \rangle ,$$

because $\varphi(+\infty) = 0$. Therefore, $H' = \delta$.

1.2 Lebesgue and Sobolev Spaces

In this part, we introduce Lebesgue and Sobolev spaces of integer order and establish some of their most important properties.

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.7 (The space $L^p(\Omega)$ [4]) Let Ω be a domain in $\mathbb{R}^n$ and let p be a positive number. We denote by $L^p(\Omega)$ the class of all measurable functions u defined on Ω for which

$$\int_{\Omega} |u(x)|^p \, dx < \infty. \tag{1.1}$$

Definition 1.8 (The Lp norm [4]) the functional $\|.\|_p$ defined by

$$\|u\|_p = \left(\int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}},$$

is a norm on $L^p(\Omega)$ provided $1 \leq p < \infty$. (It is not a norm for values of p in the range $0 < p < 1$).

Definition 1.9 (The space $L^{\infty}(\Omega)$[4]) $L^{\infty}(\Omega)$ denotes the measurable real valued functions that are essentially bounded (bounded except on a set of measure zero). For $u \in L^{\infty}(\Omega)$, we define the norms

$$\|u\|_{\infty} = \underset{x \in \Omega}{esssup} |u(x)| = \inf \{M : \mu \{x : u(x) > M\} = 0\},$$

is a norm on $L^{\infty}(\Omega)$.

Theorem 1.1 ([4]) *$L^p(\Omega)$ is a Banach space if $1 \leq p \leq \infty$.*

Corollary 1.1 ([4]) *$L^2(\Omega)$ is a Hilbert space with respect to the inner product given by:*

$$\langle u, \nu \rangle = \int_{\Omega} u(x) \overline{\nu(x)} dx.$$

Theorem 1.2 ([4]) *$L^p(\Omega)$ is separable if $1 \leq p < \infty$.*

Theorem 1.3 (Density theorem [25]) *$D(\Omega) = C_0^{\infty}(\Omega)$ is dense in $L^p(\Omega)$ if $1 \leq p < \infty$.*

Theorem 1.4 ([4]) *$L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$.*

Theorem 1.5 (The Dominated Convergence theorem [4]) *Let $A \subset \mathbb{R}^n$ be measurable, and let $\{f_j\}$ be a sequence of measurable functions converging to a limit pointwise on A. If there exists a function $g \in L^1(A)$ such that*

$$|f_j(x)| \leq g(x),$$

for every j *and all* $x \in A$*, then*

$$\lim_{j\to\infty}\int_A f_j(x)\,dx = \int_A \lim_{j\to\infty} f_j(x)\,dx.$$

Theorem 1.6 (Fubini's theorem [4]) *Let* f *be a measurable function on* $\mathbb{R}^{m+n}$ *and suppose that at least one of the integrals*

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{n+m}} |f(x,y)|\,dxdy, \\ I_2 &= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} |f(x,y)|\,dx\right)dy, \\ I_3 &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} |f(x,y)|\,dy\right)dx, \end{aligned}$$

exists and is finite. For I_2*, we mean by this that there is an integrable function* g *on* $\mathbb{R}^n$ *such that* $g(y)$ *is equal to the inner integral for almost all* y*, and similarly for* I_3*. Then*

(a) $f(.,y) \in L^1(\mathbb{R}^n)$ for almost all $y \in \mathbb{R}^m$.

(b) $f(x,.) \in L^1(\mathbb{R}^m)$ for almost all $x \in \mathbb{R}^n$.

(c) $\int_{\mathbb{R}^m} f(.,y)dy \in L^1(\mathbb{R}^n)$.

(d) $\int_{\mathbb{R}^n} f(x,.)dx \in L^1(\mathbb{R}^m)$.

(e) $I_1 = I_2 = I_3$.

1.2.2 The $L^p(0,T;X)$ spaces

Definition 1.10 Let $-\infty \leq a < b \leq +\infty$ and X be a Banach space with the norm denoted by $\|.\|_X$. We define the spaces $L^p(a,b;X)$, $1 \leq p < \infty$ and $L^\infty(a,b;X)$ respectively, as follows

$$L^p(a,b;X) = \left\{ u : (a,b) \to X \text{ measurable, where } \int_a^b \|u(.)\|_X^P\,dt < +\infty \right\},$$

and

$$L^\infty(a,b;X) = \left\{ u : (a,b) \to X \text{ measurable, where } ess \sup_{t\in(a,b)} \|u(.)\|_X < +\infty \right\}.$$

The space $L^p(a,b;X)$ is a Banach space with respect to the norm

$$\|u\|_{L^p(a,b;X)} = \begin{cases} \left(\int_a^b \|u(t)\|_X^P\,dt\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ ess \sup_{t\in(a,b)} \|u(t)\|_X, & \text{if } p = \infty. \end{cases}$$

Naturally, we have

$$L^p(a,b;L^p(\Omega)) = L^p((a,b)\times\Omega), \quad 1 \leq p \leq \infty.$$

1.2.3 The $W^{m,p}(\Omega)$ spaces

Definition 1.11 (Sobolev spaces [4]) For any positive integer m and $1 \leq p \leq \infty$ we consider the vector space

$$W^{m,p}(\Omega) \equiv \left\{ u \in L^P(\Omega) : D^\alpha u \in L^P(\Omega) \text{ for } 0 \leq |\alpha| \leq m \right\},$$

where $D^\alpha u$ represents the distributional or weak partial derivative. Equipped with the norm

$$||u||_{m,p} = \begin{cases} \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} & if\ 1 \leq p < \infty, \\ \max\limits_{0\leq|\alpha|\leq m} \|D^\alpha u\|_\infty & if\ p = \infty, \end{cases} \tag{1.2}$$

called Sobolev space over Ω.

Theorem 1.7 ([4]) *$W^{m,p}(\Omega)$ equipped with the norm (1.2) is a Banach space.*

Lemma 1.1 ([4]) *Let $u \in L^1_{loc}(\Omega)$ satisfy $\int_\Omega u(x)\phi(x)\,dx = 0$ for every ϕ in $D(\Omega)$. Then $u(x) = 0$ a.e. in Ω.*

Definition 1.12 (Compact sets [4]) A subset A of a normed space X is considered compact if every sequence of points in A contains a subsequence converging in X to an element belonging to A. This definition is equivalent to the compactness definition in a general topological space.

Remark 1.1 Compact sets are both closed and bounded. However, closed and bounded sets may not necessarily be compact unless X is finite dimensional.

Definition 1.13 (precompact sets [4]) A set A in space X is defined as precompact if its closure, denoted by $\bar{A}$ is a compact set in the norm topology of X.

Definition 1.14 A set A is termed weakly sequentially compact if each sequence in A has a subsequence that weakly converges in X to a point belonging to the set A. The reflexivity of a Banach space can be characterized in terms of this property.

Definition 1.15 (Imbeddings) We say the normed space X is imbedded in the normed space Y, and we write $X \hookrightarrow Y$ to designate this imbedding, if these conditions are satisfied:

(i) X is a vector subspace of Y.

(ii) The operator $I : X \to Y$ defined by $Ix = x$ for all $x \in X$ is continuous.

Since I is linear, (ii) is equivalent to the following relationship

$$\exists M' > 0 : \|Ix\|_Y \leq M' \|x\|_X \qquad \forall x \in X.$$

We say that X is compactly imbedded in Y if the imbedding operator I is compact.

Definition 1.16 (Closed operator [49]) Let X and Y are Banach spaces and $D(T) \subset X$ a subspace. A linear operator $T : D(T) \to Y$ is called closed if, for each sequence $\{x_n\}_n \subset D(T)$ check

$$\begin{cases} x_n \xrightarrow{X} x, \\ Tx_n \xrightarrow{Y} y, \end{cases}$$

this imply $x \in D(T)$ and $Tx = y$.

1.3 Some important inequalities

Theorem 1.8 (Hölder's inequality [4]) *Let $1 < p < \infty$ and let p' denote the conjugate exponent defined by*

$$p' = \frac{p}{p-1} \quad \text{that is} \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

which also satisfies $1 < p' < \infty$. If $u \in L^p(\Omega)$ and $\nu \in L^{p'}(\Omega)$, then $u\nu \in L^1(\Omega)$, and

$$\int_\Omega |u(x)\,\nu(x)|\,dx \leq \|u\|_p \|\nu\|_{p'}.$$

Remark 1.2 Holder's inequality for $L^2(\Omega)$ is just the well-known Cauchy-Schwarz inequality

$$|\langle u, \nu \rangle| \leq \|u\|_2 \|\nu\|_2.$$

Theorem 1.9 (Poincaré's inequality[11]) *Assuming I is a bounded interval, then there exists a constant C (which depends on the finite length of I) such that*

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^p(I)} \qquad \forall u \in W_0^{1,p}(I). \tag{1.3}$$

Remark 1.3 Through (1.3) we conclude that on $W_0^{1,p}$, the quantity $\left\|u'\right\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}(I)$ norm.

Theorem 1.10 (Young's inequality [25]) *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b > 0$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 1.11 (Young's inequality with ε [25]) *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b > 0$. Then for any $\varepsilon > 0$,*

$$ab \leq \varepsilon a^p + C(\varepsilon)\, b^q,$$

where

$$C(\varepsilon) = \frac{1}{q\,(\varepsilon p)^{\frac{q}{p}}}.$$

For $p, q = 2$, the inequality takes the form

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon},$$

Proof. Write

$$ab = \left((\varepsilon p)^{\frac{1}{p}} a\right)\left(\frac{b}{(\varepsilon p)^{\frac{1}{p}}}\right),$$

and apply Young's inequality. ■

Lemma 1.2 (Gronwall inequality [105]) *Let $d_1, d_2 > 0$ and f is a nonnegative integrable function. If*

$$f(t) \leq d_1 + d_2 \int_0^t f(s)\, ds,$$

then

$$f(t) \leq d_1 e^{d_2 t} \text{ for } 0 \leq t \leq T.$$

1.4 Some results on the existence and uniqueness

In this section, our focus will be on providing basic definitions and presenting important results related to the existence and uniqueness of solutions.

Definition 1.17 Let H be a Hilbert space. A bilinear form $\tilde{a} : H \times H \to R$ is said to be

(i) continuous if

$$\exists c > 0 : |\tilde{a}(u, \nu)| \leq c \|u\|_H \|\nu\|_H \quad \forall u, \nu \in H,$$

(ii) coercive if

$$\exists \alpha > 0 : \tilde{a}(\nu, \nu) \geq \alpha \|\nu\|_H^2 \quad \forall \nu \in H.$$

Lemma 1.3 (Lax-Milgram lemma [11]) *Consider a bilinear form $\tilde{a}(\cdot,\cdot)$ defined on a Hilbert space H, which is equipped with the norm $\|\cdot\|_H$, and the following properties are satisfied*

i) $\tilde{a}(\cdot,\cdot)$ is continuous and coercive.

ii) The mapping $\tilde{L} : H \to R$ is linear continuous, i.e $\exists \gamma_2 > 0$ such that $\left|\tilde{L}(\nu)\right| \leq \gamma_2 \|\nu\|_H \quad \forall \nu \in H$.

Then there exists a unique element $\tilde{u} \in H$ such that

$$\tilde{a}(\tilde{u}, \nu) = \tilde{L}(\nu) \quad \forall \nu \in H.$$

1.4.1 Some theory of semi-groupe

Definition 1.18 **([73])** Let X be a real or complex Banach space, and X^* be its dul. We denote the value $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}.$$

Definition 1.19 **([73])** A linear operator $A : D(A) \subseteq X \to X$ is dissipative if, for every $x \in D(A)$ there exists $x^* \in F(x)$ such that

$$\mathrm{Re}\langle Ax, x^*\rangle \leq 0.$$

Remark 1.4 **([92])** In the case in which $X = H$ is a real Hilbert space with an inner product $\langle .,.\rangle_H$, a linear operator $A : D(A) \subseteq H \to H$ is dissipative if

$$\langle Ax, x\rangle \leq 0 \qquad \forall x \in D(A).$$

Definition 1.20 **([11])** A linear operator $A : D(A) \subseteq X \to X$ is said to be monotone if the operator $(-A)$ is dissipative, this property is expressed by

$$\mathrm{Re}\langle Ax, x^*\rangle \geq 0 \qquad \forall x \in D(A).$$

Remark 1.5 According to some authors, A is accretive or $-A$ is dissipative is the same thing.

Definition 1.21 **([92])** A collection $\{S(t)\}_{t\geq 0}$ of bounded linear operators in a Banach space X into X is a semigroup of linear operators on X, or simply semigroup if:

(i) $S(0) = I$

(ii) $S(t+s) = S(t)S(s)$ for each $t, s \geq 0$.

If, in addition, it fulfills the condition of continuity at $t = 0$,

$$\lim_{t\to 0} S(t) = I,$$

the semigroup is termed uniformly continuous.

Definition 1.22 **([92])** The infinitesimal generator of the semigroup of linear operators $\{S(t)\}_{t\geq 0}$ is the operator $A : D(A) \subseteq X \to X$, which is defined by

$$D(A) = \left\{x \in X : \exists \lim_{t\to 0} \frac{S(t)x - x}{t}\right\},$$

and

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}.$$

Remark 1.6 **([92])** If $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a semigroup of linear operators, then $D(A)$ is a vector subspace of X and A is a linear operator that may be unbounded.

Theorem 1.12 **([92])** *A linear operator $A : D(A) \subseteq X \to X$ is the generator of a uniformly continuous semigroup if and only if the domain $D(A)$ equals the space X and A is bounded.*

Definition 1.23 **([92])** A semigroup of linear operators $\{S(t)\}_{t\geq 0}$ is called a C_0-semigroup, or semigroup of class C_0, if

$$\lim_{t\to 0} S(t)x = x \qquad \forall x \in X.$$

Definition 1.24 A C_0-semigroup $\{S(t)\}_{t\geq 0}$ is termed a C_0-semigroup of contractions, or of nonexpansive operators, if

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1.$$

Where $\mathcal{L}(X)$ represents the set of all linear bounded operators from X to X.

Theorem 1.13 **(Hille-Yosida [92])** *The linear operator $A : D(A) \subseteq X \to X$ is considered as the infinitesimal generator of a C_0-semigroup of contractions if and only if*

(i) *A is densely defined and closed.*

(ii) $(0,\infty) \subseteq \rho(A)$ *and for each* $\lambda > 0$

$$\|R(\lambda; A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

where $\rho(A)$ denotes the resolvent set of the operator A and $R(\lambda; A) = (\lambda I - A)^{-1}$.

Theorem 1.14 **(Lumer-Phillips [92])** *Let $A : D(A) \subseteq X \to X$ ($D(A)$ dense subspace). A generates a C_0-semigroup of contractions on X if and only if*

(i) *A is dissipative.*

(ii) *There exists $\lambda > 0$ such $\lambda I - A$ is surjective.*

Moreover, if A generates a C_0-semigroup of contractions, then $\lambda I - A$ is surjective for any $\lambda > 0$.

Theorem 1.15 **(Hille–Yosida [11])** *Let A be a maximal monotone operator. Then, given any $\nu_0 \in D(A)$ there exists a unique function*

$$\nu \in C^1([0,+\infty); H) \cap C([0,+\infty);\ D(A))$$

satisfying

$$\begin{cases} \frac{d\nu}{dt} + A\nu = 0 \ \text{on} \ [0,+\infty), \\ \nu(0) = \nu_0. \end{cases}$$

Moreover $\|\nu(t)\| \leq \|\nu_0\|$ *and* $\left\|\frac{d\nu}{dt}(t)\right\| = \|A\nu(t)\| \leq \|A\nu_0\| \ \forall t \geq 0.$

1.4.2 Compactness Method

The method is based on three steps:

1) To apply the Faedo-Galerkin method, we select a set of suitable basis functions from an appropriate Sobolev space. We then solve the approximate problems within a finite-dimensional space spanned by these finite base functions. This approach often leads to an initial value problem for nonlinear ordinary differential equations. According to the well-known local existence theorem for ordinary differential equations, the local existence of a solution to the approximate problem can be guaranteed.

2) Obtain the compactness estimates for the solution of the approximate problem. It also turns out that the solution to the approximate problem globally exists.

3) By utilizing the obtained compactness estimates, it becomes possible to select a subsequence from the solutions of the approximate problem obtained in the second step. This subsequence is chosen in such a way that it converges to a solution of the original problem.

   For more explanation about this method, see [105] and references therein.

CHAPTER 2

Existence, uniqueness and exponential energy decay of piezoelectric system with magnetic effects and distributed delay time

2.1 Introduction

Piezoelectric materials such as barium titanate, quartz and rochelle salt exhibit the property of transforming mechanical energy into electromagnetic energy (see [94]). The direct piezoelectric effect was initially demonstrated by the brothers Pierre and Jacques Curie, in 1880 [91], where single crystal quartz was the first material used in early experiments with piezoelectricity. These same materials, when subjected to an electric field, exhibit a phenomenon known as the reverse piezoelectric effect, which was discovered by Gabriel Lippmann in 1881 [91, 94]

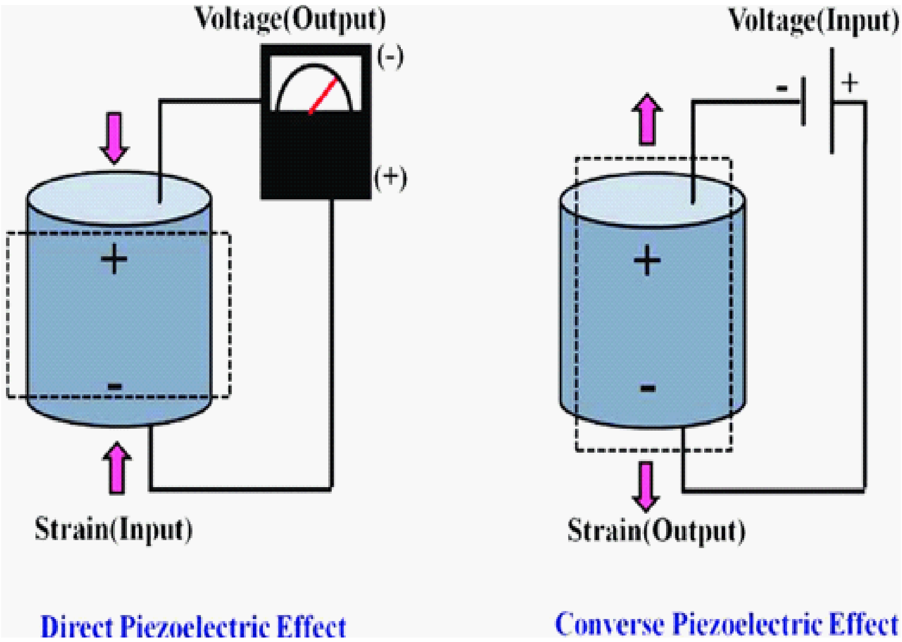

Figure 2 :Direct and converse piezoelectric effect

Various sectors benefit from the utilization of these piezoelectric materials in various industries, including manufacturing, the medical device industry, telecommunications, and information technology. Furthermore, piezoelectric beam refers to an elastic beam that is coated with a piezoelectric material on both its upper and lower surfaces, while the edges are insulated to prevent fringing effects. The beam is also connected to an external electric circuit [59]. To operate piezoelectric materials electrically, there are three fundamental methods: voltage, current, or charge. For more detailed information on these methods, the reader is referred to the references [31, 38]. When modeling piezoelectric systems, it is essential to consider three primary effects and their interrelationships: mechanical, electrical, and magnetic effects. The mechanical effects are commonly represented using small displacement assumptions, such as Kirchhoff, Euler-Bernoulli, or Mindlin-Timoshenko theories. References such as [14, 83] provide further details on these modeling approaches. On the other hand, the incorporation of electrical and magnetic effects in piezoelectric systems can be achieved through three main approaches: electrostatic, quasi-static, and fully dynamic methods. These approaches are discussed in detail in references such as [54, 90] and the related literature. It is important to note that magnetic effects are not considered in the case of electrostatic and quasi-static approaches. In [59], Morris and Özer employed a variational approach to derive the differential equations and boundary conditions that describe a single piezoelectric beam with magnetic effects. By utilizing the Lagrangian and Hamilton's principle, setting the variation of admissible displacements $\{\nu, w, \varphi\}$ of Γ to zero and assuming that the beam is clamped at $x = 0$ and left free at $x = \Gamma$, two distinct sets of equations are obtained. These equations correspond to stretching and bending, respectively, with associated boundary conditions. They ignored the bending equation in favor of studying the stretching equations because the bending equation is completely decoupled from the stretching equations given as follows

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} = 0, \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, \end{cases} \tag{2.1}$$

with the boundary and initial conditions

$$\begin{cases} \nu(0) = \varphi(0) = \alpha\nu_x(\Gamma) - \gamma\beta\varphi_x(\Gamma) = 0,\ \beta\varphi_x(\Gamma) - \gamma\beta\nu_x(\Gamma) = -\frac{V(t)}{h}, \\ (\nu, \varphi, \nu_t, \varphi_t)(0) = (\nu^0, \varphi^0, \nu^1, \varphi^1), \end{cases} \tag{2.2}$$

where $\alpha = \alpha_1 + \gamma^2\beta$ and the parameters $\Gamma,\ \rho,\ \alpha,\ \gamma,\ \mu$ and β represent respectively the length of the beam, the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability and water resistance coefficient. Finally, by using only an electrical feedback controller $V(t) = k\varphi_t(\Gamma)$, they demonstrate that the closed-loop system is strongly stable in the energy space. In [77] in the case $V(t) = 0$, exponential stability has been demonstrated for piezoelectric beams with magnetic effects by incorporating damping $\delta\nu_t$ into the first equation by Ramos et al., and employing the finite difference method, they computed a numerical energy associated with their system. The numerical simulations involved using specific values of $\Gamma,\ \rho,\ \mu,\ \gamma,\ \beta$ and δ. In a recent study [7], Akil and Soufyane et al. investigated a one-dimensional piezoelectric system with partial viscous dampings and established the existence and uniqueness of a solution under Lorenz gauge conditions. The strong stability was obtained by applying the general criteria of Arendt-Batty. Finally, exponential stability is proven to be obtainable by controlling the stretching of the center-line of the beam in the x-direction. In [8] Afilal et al. considered a one-dimensional dissipative system of piezoelectric beams with a magnetic effect and localized damping. The authors proved that the semigroup $S(t) = e^{At}$ associated with their system is exponentially stable. A Multi-dimensional nonlinear piezoelectric beam with viscoelastic infinite memory has been studied by [106] et al. by using semigroup theories and the Banach fixed-point theorem, the well-posedness of this nonlinear coupled system was demonstrated. Also, the exponential decay is established by the energy estimation method. We refer the reader to [56, 60, 61, 70, 79, 101] and the references therein for more results related to piezoelectric systems (in the absence of delay terms). Ramos et al. [76], demonstrated the exponential stability of a system of piezoelectric beams with delayed

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \xi_1\nu_t + \xi_2\nu_t(x, t-\tau) = 0, & \text{in }]0, \Gamma[\times]0, +\infty[, \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, & \text{in }]0, \Gamma[\times]0, +\infty[, \end{cases}$$

with the boundary and initial conditions

$$\begin{cases} \nu(0, t) = \alpha\nu_x(\Gamma, t) - \gamma\beta\varphi_x(\Gamma, t) = 0, & t \geq 0, \\ \varphi(0, t) = \varphi_x(\Gamma, t) - \gamma\nu_x(\Gamma, t) = 0, & t \geq 0, \\ \nu_t(x, t-\tau) = f_0(x, t-\tau), & (x, t) \in]0, \Gamma[\times]0, \tau[\\ \nu(x, 0) = \nu_0(x), \nu_t(x, 0) = \nu_1(x), & x \in]0, \Gamma[, \\ \varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x), & x \in (0, \Gamma), \end{cases} \tag{2.3}$$

where $\xi_2\nu_t(x, t-\tau)$ is the time of delay on vertical displacement, $\tau > 0$ is the respective retardation time. The authors proved this stability under the condition $\xi_1 > \xi_2$. Recently, Kong et al. [50], employed the Kato variable norm technique to demonstrate that the system of magnetic effected piezoelectric beams with time-dependent weights and time-varying delay is well-posed. Furthermore, the application of the multiplier technique allowed them to obtain exponential stability. Finally, the equivalence between stabilization and observability was proven by imposing certain conditions on the time-varying delay term and time-dependent weights. In [85], Soufyane et al. extended the previously mentioned recent finding in [50]. Their investigation focused on stability, using several lemmas. Feng and Özer in [29] considered the following fully dynamic and electrostatic or quasi-static models with clamped-free boundary

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + c_1\nu_t + a_1\nu_t(t-\tau) = 0, & (x,t) \in (0,\Gamma) \times \mathbb{R}^+, \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} + c_2\varphi_t + a_2\varphi_t(t-\tau) = 0, & \\ \nu(0,t) = \varphi(0,t) = 0, & \\ (\alpha\nu_x - \gamma\beta\varphi_x)(\Gamma,t) = -b_1\nu_t(\Gamma,t) - a_1\nu_t(\Gamma,t-\tau), & \\ (\varphi_x - \gamma\nu_x)(\Gamma,t) = -b_2\varphi_t(\Gamma,t) - a_2\varphi_t(\Gamma,t-\tau), & t \in \mathbb{R}^+, \\ (\nu,\nu_t,\varphi,\varphi_t)(x,0) = (\nu_0,\nu_1,\varphi_0,\varphi_1)(x), & x \in (0,\Gamma), \\ (\nu_t,\varphi_t)(\Gamma,t-\tau) = (f_0,g_0)(\Gamma,t-\tau), & t \in (0,\tau), \end{cases}$$

$$\begin{cases} \rho\nu_{tt} - \alpha_1\nu_{xx} + c_1\nu_t + a_1\nu_t(t-\tau) = 0, & (x,t) \in (0,\Gamma) \times \mathbb{R}^+, \\ \nu(0,t) = 0, & \\ \alpha_1\nu_x(\Gamma,t) = -b_1\nu_t(\Gamma,t) - a_1\nu_t(\Gamma,t-\tau), & t \in \mathbb{R}^+ \\ (\nu,\nu_t)(x,0) = (\nu_0,\nu_1)(x), & x \in (0,\Gamma) \\ \nu_t(x,t-\tau) = f_0(x,t-\tau), & t \in (0,\tau). \end{cases}$$

Their study is noteworthy as it focused on investigating boundary feedback controllers and their interactions with both internally and boundary distributed delay feedback controllers (i.e. $b_1, b_2 \neq 0$ and $c_1 = c_2 = 0$). The well-posedness of these models was determined using semigroup theory. In each model, the exponential stability has been proven through the Lyapunov theory by satisfying some conditions.

2.2 Problem statement

In the present chapter, we are concerned one dimensional piezoelectric beams with distributed delay terms, which has the form

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \mu_1\nu_t + \int_{\tau_1}^{\tau_2}\zeta(\hbar)\,\nu_t(x,t-\hbar)\,d\hbar = 0, & \text{in } (0,\Gamma) \times (0,\infty), \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, & \\ \nu(0,t) = \nu_x(\Gamma,t) = \varphi(0,t) = \varphi_x(\Gamma,t) = 0, & t \in (0,\infty), \\ \nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x), & x \in (0,\Gamma), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), & \\ \nu_t(x,-t) = f_0(x,t), & (x,t) \in (0,\Gamma) \times (0,\tau_2), \end{cases} \tag{2.4}$$

where τ_1, τ_2, μ_1 are positive numbers and $\zeta : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying the following assumption

$$\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\,d\hbar \leq \mu_1. \tag{2.5}$$

2.3 Existence, uniqueness

In this section, we will show the existence and uniqueness of solutions for (2.4) through the application of semigroup theory.

Following the method used in [67], we introduce the new variable.

$$Y\left(x,\rho,t,\hbar\right)=\nu_t\left(x,t-\rho\hbar\right), \qquad x\in\left(0,\Gamma\right),\ \rho\in\left(0,1\right),\ \hbar\in\left(\tau_1,\tau_2\right),\ t\geq 0.$$

Therefore, we achieve

$$\hbar Y_t\left(x,\rho,t,\hbar\right)+Y_\rho\left(x,\rho,t,\hbar\right)=0.$$

The problem (2.4), take the form

$$\begin{cases}\rho\nu_{tt}-\alpha\nu_{xx}+\gamma\beta\varphi_{xx}+\mu_1\nu_t+\int_{\tau_1}^{\tau_2}\zeta\left(\hbar\right)Y\left(x,1,t,\hbar\right)d\hbar=0,\\ \mu\varphi_{tt}-\beta\varphi_{xx}+\gamma\beta\nu_{xx}=0,\\ \hbar Y_t\left(x,\rho,t,\hbar\right)+Y_\rho\left(x,\rho,t,\hbar\right)=0,\end{cases} \tag{2.6}$$

with the following conditions

$$\begin{cases}\nu\left(0,t\right)=\alpha\nu_x\left(\Gamma,t\right)-\gamma\beta\varphi_x\left(\Gamma,t\right)=0, \quad t\geq 0,\\ \varphi\left(0,t\right)=\varphi_x\left(\Gamma,t\right)-\gamma\nu_x\left(\Gamma,t\right)=0,\\ \nu\left(x,0\right)=\nu_0\left(x\right),\ \nu_t\left(x,0\right)=\nu_1\left(x\right),\ \forall x\in\left(0,\Gamma\right),\\ \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right),\\ Y\left(x,\rho,0,\hbar\right)=f_0\left(x,\rho,\hbar\right),\ x\in\left(0,\Gamma\right),\ \rho\in\left(0,1\right),\ \hbar\in\left(0,\tau_2\right).\end{cases}$$

By using the following notations

$$\nu_t=u,\ \varphi_t=q \qquad \text{and} \qquad U=\left(\nu,u,\varphi,q,Y\right)^T,$$

$$\partial_t U=\left(\nu_t,u_t,\varphi_t,q_t,Y_t\right)^T,$$

therefore, the problem (2.6) can be represented as follows

$$\begin{cases}\partial_t U=AU,\\ U\left(0\right)=U_0=\left(\nu_0,\nu_1,\varphi_0,\varphi_1,f_0\right),\end{cases} \tag{2.7}$$

where the operator $A:D(A)\subset H\to H$ is defined by

$$AU:=\begin{pmatrix}\nu_t\\ \frac{\alpha}{\rho}\nu_{xx}-\frac{\mu_1}{\rho}\nu_t-\frac{\gamma\beta}{\rho}\varphi_{xx}-\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta\left(\hbar\right)Y\left(x,1,t,\hbar\right)d\hbar\\ \varphi_t\\ -\frac{\gamma\beta}{\mu}\nu_{xx}+\frac{\beta}{\mu}\varphi_{xx}\\ -\frac{1}{\hbar}Y_\rho\end{pmatrix}, \tag{2.8}$$

we consider the following spaces

$$\hat{H}^1(0,\Gamma) = \left\{\nu \in H^1(0,\Gamma) : \nu(0) = 0\right\},$$

$$\hat{H}^2(0,\Gamma) = H^2(0,\Gamma) \cap \hat{H}^1(0,\Gamma),$$

and we define the previous Hilbert space H as

$$H := \hat{H}^1(0,\Gamma) \times L^2(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times L^2(0,\Gamma) \times L^2((0,\Gamma) \times (0,1) \times (\tau_1,\tau_2)).$$

We define the inner product on H as follows

$$\begin{aligned}\langle U,\tilde{U}\rangle_H &= \rho\int_0^\Gamma \nu_t\tilde{\nu}_t dx + \mu\int_0^\Gamma \varphi_t\tilde{\varphi}_t dx + \alpha_1\int_0^\Gamma \nu_x\tilde{\nu}_x dx + \beta\int_0^\Gamma (\gamma\nu_x - \varphi_x)(\gamma\tilde{\nu}_x - \tilde{\varphi}_x)\,dx \\ &+ \int_0^\Gamma\int_{\tau_1}^{\tau_2} \hbar\,|\zeta(\hbar)| \int_0^1 Y(x,\rho,t,\hbar)\,\tilde{Y}(x,\rho,t,\hbar)\,d\rho d\hbar dx, \\ &= \rho\int_0^\Gamma \nu_t\tilde{\nu}_t dx + \mu\int_0^\Gamma \varphi_t\tilde{\varphi}_t dx - \gamma\beta\int_0^\Gamma \nu_x\tilde{\varphi}_x dx - \gamma\beta\int_0^\Gamma \tilde{\nu}_x\varphi_x dx + \alpha\int_0^\Gamma \nu_x\tilde{\nu}_x dx \\ &+ \beta\int_0^\Gamma \varphi_x\tilde{\varphi}_x dx + \int_0^\Gamma\int_{\tau_1}^{\tau_2} \hbar\,|\zeta(\hbar)| \int_0^1 Y(x,\rho,t,\hbar)\,\tilde{Y}(x,\rho,t,\hbar)\,d\rho d\hbar dx. \end{aligned} \tag{2.9}$$

Now, we defined the previous domain of operator A as follows

$$\begin{aligned} D(A) := \Big\{ &(\nu,\nu_t,\varphi,\varphi_t,Y) \in \hat{H}^2(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times \hat{H}^2(0,\Gamma) \times \hat{H}^1(0,\Gamma) \\ &\times L^2((0,\Gamma) \times (0,1) \times (\tau_1,\tau_2)) : \ \nu_x(\Gamma) = \varphi_x(\Gamma) = 0\Big\}. \end{aligned} \tag{2.10}$$

$D(A)$ is clearly dense in H.

Theorem 2.1 *Let $U_0 \in H$, then problem (2.7) admits a unique solution $U \in C(\mathbb{R}^+, H)$. Moreover, if $U_0 \in D(A)$ then $U \in C(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$.*

Proof. Our initial step is to show that the operator A is dissipative.

Let $U = (\nu,\nu_t,\varphi,\varphi_t,Y)^T \in D(A)$, by employing the inner product defined earlier, we get

$$\langle AU,U\rangle_H = \left\langle \begin{pmatrix} \nu_t \\ \frac{\alpha}{\rho}\nu_{xx} - \frac{\mu_1}{\rho}\nu_t - \frac{\gamma\beta}{\rho}\varphi_{xx} - \frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta(\hbar)\,Y(x,1,t,\hbar)\,d\hbar \\ \varphi_t \\ -\frac{\gamma\beta}{\mu}\nu_{xx} + \frac{\beta}{\mu}\varphi_{xx} \\ -\frac{1}{\hbar}Y_\rho \end{pmatrix}, \begin{pmatrix} \nu \\ \nu_t \\ \varphi \\ \varphi_t \\ Y \end{pmatrix} \right\rangle_H. \tag{2.11}$$

After integrating by parts and taking the boundary conditions into account, we obtain

$$\begin{aligned}\langle AU,U\rangle_H &= \rho\int_0^\Gamma\left(\frac{\alpha}{\rho}\nu_{xx}-\frac{\mu_1}{\rho}\nu_t-\frac{\gamma\beta}{\rho}\varphi_{xx}-\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar\right)\nu_t dx\\
&+\mu\int_0^\Gamma\left(-\frac{\gamma\beta}{\mu}\nu_{xx}+\frac{\beta}{\mu}\varphi_{xx}\right)\varphi_t dx-\gamma\beta\int_0^\Gamma\nu_{tx}\varphi_x dx-\gamma\beta\int_0^\Gamma\varphi_{tx}\nu_x dx\\
&+\alpha\int_0^\Gamma\nu_{tx}\nu_x dx+\beta\int_0^\Gamma\varphi_{tx}\varphi_x dx\\
&-\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\int_0^1 Y_\rho(x,\rho,t,\hbar)Y(x,\rho,t,\hbar)\,d\rho d\hbar dx\\
&=-\mu_1\int_0^\Gamma\nu_t^2 dx-\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx\\
&-\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\int_0^1 Y_\rho(x,\rho,t,\hbar)Y(x,\rho,t,\hbar)\,d\rho d\hbar dx.\end{aligned}\tag{2.12}$$

Additionally, by integrating with respect to ρ, we find

$$\begin{aligned}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\int_0^1 Y_\rho(x,\rho,t,\hbar)Y(x,\rho,t,\hbar)\,d\rho d\hbar dx &= \frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx\\
&-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\int_0^\Gamma\nu_t^2 dx,\end{aligned}\tag{2.13}$$

applying Young's inequality, we find

$$\begin{aligned}&-\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx\\
&\le\frac{1}{2}\int_0^\Gamma\nu_t^2 dx\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar+\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx,\end{aligned}\tag{2.14}$$

by (2.13)-(2.14) and condition (2.5) we obtain

$$\langle AU,U\rangle_H\le-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\right)\int_0^\Gamma\nu_t^2 dx.$$

Hence, we get that A is a dissipative operator.

We will now prove the surjectivity of the operator $(I-A)$.

Given $M=(g_1,g_2,g_3,g_4,g_5)^T\in H$, we demonstrate that there exists a unique $U=(\nu,u,\varphi,q,Y)^T\in D(A)$ so that

$$(I-A)U=M,\tag{2.15}$$

i.e

$$\begin{pmatrix}\nu\\ u\\ \varphi\\ q\\ Y\end{pmatrix}-\begin{pmatrix}u\\ \frac{\alpha}{\rho}\nu_{xx}-\frac{\mu_1}{\rho}u-\frac{\gamma\beta}{\rho}\varphi_{xx}-\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar\\ q\\ -\frac{\gamma\beta}{\mu}\nu_{xx}+\frac{\beta}{\mu}\varphi_{xx}\\ -\frac{1}{\hbar}Y_\rho\end{pmatrix}=\begin{pmatrix}g_1\\ g_2\\ g_3\\ g_4\\ g_5\end{pmatrix},\tag{2.16}$$

then, by (2.16), we get

$$\begin{cases} \nu - u = g_1, \\ \rho u - \alpha \nu_{xx} + \mu_1 u + \gamma\beta\varphi_{xx} + \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar = \rho g_2, \\ \varphi - q = g_3, \\ \mu q + \gamma\beta\nu_{xx} - \beta\varphi_{xx} = \mu g_4, \\ Y + \frac{1}{\hbar} Y_\rho = g_5. \end{cases} \tag{2.17}$$

Also, by using (2.17), we have

$$\begin{cases} u = \nu - g_1, \\ q = \varphi - g_3, \end{cases} \tag{2.18}$$

as

$$Y(x,0,t,\hbar) = u(x,t) = \nu_t(x,t), \qquad \text{for } x \in (0,\Gamma),\ \hbar \in (\tau_1,\tau_2), t \geq 0,$$

and by $(2.17)_5$ we get

$$Y(x,\rho,t,\hbar) + \frac{1}{\hbar} Y_\rho(x,\rho,t,\hbar) = g_5(x,\rho,\hbar), \tag{2.19}$$

that implies

$$Y(x,\rho,t,\hbar) = \hbar e^{-\hbar\rho} \int_0^\rho g_5(x,\tau,\hbar) e^{\hbar\tau} d\tau + u e^{-\hbar\rho}, \tag{2.20}$$

in particular

$$Y(x,1,t,\hbar) = \hbar e^{-\hbar} \int_0^1 g_5(x,\tau,\hbar) e^{\hbar\tau} d\tau + u e^{-\hbar}. \tag{2.21}$$

Now by using (2.18)-(2.21) in the other equations for (2.17), we obtain

$$\begin{aligned} &\rho(\nu - g_1) - \alpha\nu_{xx} + \mu_1(\nu - g_1) + \gamma\beta\varphi_{xx} + \int_{\tau_1}^{\tau_2} \zeta(\hbar) \hbar e^{-\hbar} \int_0^1 g_5(x,\tau,\hbar) e^{\hbar\tau} d\tau d\hbar \\ &+ (\nu - g_1) \int_{\tau_1}^{\tau_2} \zeta(\hbar) e^{-\hbar} d\hbar \ = \ \rho g_2, \\ &\mu(\varphi - g_3) + \gamma\beta\nu_{xx} - \beta\varphi_{xx} \ = \ \mu g_4, \end{aligned} \tag{2.22}$$

then we get

$$\begin{cases} -\alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \varpi_1 \nu = Q_1 \in L^2(0,\Gamma), \\ \\ \gamma\beta\nu_{xx} - \beta\varphi_{xx} + \mu\varphi = Q_2 \in L^2(0,\Gamma), \end{cases} \tag{2.23}$$

where

$$\begin{aligned} \varpi_1 &= (\mu_1 + \rho) + \int_{\tau_1}^{\tau_2} \zeta(\hbar) e^{-\hbar} d\hbar, \\ Q_1 &= \varpi_1 g_1 + \rho g_2 - \int_{\tau_1}^{\tau_2} \zeta(\hbar) \hbar e^{-\hbar} \int_0^1 g_5(x,\tau,\hbar) e^{\hbar\tau} d\tau d\hbar, \\ Q_2 &= \mu(g_4 + g_3). \end{aligned} \tag{2.24}$$

Multiplying $(2.23)_1$, $(2.23)_2$ respectively by $\tilde{\nu}$, $\tilde{\varphi} \in \hat{H}^1(0,\Gamma)$, and integrating by parts together with the boundary conditions, we have

$$\begin{cases} \alpha \int_0^\Gamma \nu_x \tilde{\nu}_x dx - \gamma\beta \int_0^\Gamma \varphi_x \tilde{\nu}_x dx + \varpi_1 \int_0^\Gamma \nu \tilde{\nu} dx = \int_0^\Gamma Q_1 \tilde{\nu} dx, \\ -\gamma\beta \int_0^\Gamma \nu_x \tilde{\varphi}_x dx + \beta \int_0^\Gamma \varphi_x \tilde{\varphi}_x dx + \mu \int_0^\Gamma \varphi \tilde{\varphi} dx = \int_0^\Gamma Q_2 \tilde{\varphi} dx, \end{cases} \tag{2.25}$$

consequently, problem (2.25) is equivalent to the problem

$$a((\nu,\varphi),(\tilde{\nu},\tilde{\varphi})) = b(\tilde{\nu},\tilde{\varphi}). \tag{2.26}$$

Where
$a : \left(\hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma)\right)^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} a((\nu,\varphi),(\tilde{\nu},\tilde{\varphi})) = {} & \alpha \int_0^\Gamma \nu_x \tilde{\nu}_x dx + \beta \int_0^\Gamma \varphi_x \tilde{\varphi}_x dx - \gamma\beta \int_0^\Gamma \varphi_x \tilde{\nu}_x dx - \gamma\beta \int_0^\Gamma \nu_x \tilde{\varphi}_x dx \\ & + \varpi_1 \int_0^\Gamma \nu \tilde{\nu} dx + \mu \int_0^\Gamma \varphi \tilde{\varphi} dx, \end{aligned} \tag{2.27}$$

$b : \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma) \longrightarrow \mathbb{R}$ is the linear form given by

$$b(\tilde{\nu},\tilde{\varphi}) = \int_0^\Gamma Q_1 \tilde{\nu} dx + \int_0^\Gamma Q_2 \tilde{\varphi} dx. \tag{2.28}$$

Now, for $\tilde{H} := \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma)$ equipped by this norm

$$\|(\nu,\varphi)\|_{\tilde{H}} = \left(\left\| \left(\nu_x - \frac{\gamma\beta}{\alpha} \varphi_x \right) \right\|_2^2 + \|\nu\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 \right)^{\frac{1}{2}}. \tag{2.29}$$

Proving the continuity of both the bilinear form a and the linear form b is simple. Moreover, we have

$$\begin{aligned} a((\nu,\varphi),(\nu,\varphi)) = {} & \alpha \int_0^\Gamma \left(\nu_x - \frac{\gamma\beta}{\alpha} \varphi_x \right)^2 dx + \left(\beta - \frac{(\gamma\beta)^2}{\alpha} \right) \int_0^\Gamma \varphi_x^2 dx + \varpi_1 \int_0^\Gamma \nu^2 dx \\ & + \mu \int_0^\Gamma \varphi^2 dx \geq \hat{m} \|(\nu,\varphi)\|_{\tilde{H}}^2, \end{aligned} \tag{2.30}$$

where

$$\hat{m} = \min\left(\alpha, \left(\beta - \frac{(\gamma\beta)^2}{\alpha} \right), \varpi_1, \mu \right). \tag{2.31}$$

For all $\varpi_1 \geq 0$ thus a is coercive, by using the Lax-Milgram theorem, we can conclude that the system (2.26) has a unique solution

$$(\nu,\varphi) \in \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma).$$

Substituting $\nu,\ \varphi$ in (2.18), we obtain

$$(u,q)\in \hat{H}^1(0,\Gamma)\times \hat{H}^1(0,\Gamma),$$

also by substituting u in (2.20) and $(2.17)_5$ we get

$$Y,Y_\rho\in L^2((0,\Gamma)\times(0,1)\times(\tau_1,\tau_2)),$$

and by (2.23) we get

$$\nu_{xx}=\frac{\varpi_1}{\alpha_1}\nu+\frac{\gamma\mu}{\alpha_1}\varphi-\frac{1}{\alpha_1}Q_1-\frac{\gamma}{\alpha_1}Q_2\in L^2(0,\Gamma)\Longrightarrow \nu\in H^2(0,\Gamma)\Longrightarrow \varphi\in H^2(0,\Gamma), \tag{2.32}$$

also $(2.25)_1$ implies

$$-\alpha\nu_{xx}+\gamma\beta\varphi_{xx}+\varpi_1\nu=Q_1,\ \text{in the distribution sense.} \tag{2.33}$$

Multiplying (2.33) by $\tilde{\nu}\in \hat{H}^1(0,\Gamma)$ and using integration by parts, we get by using $(2.25)_1$ again

$$-\alpha\nu_x(\Gamma)\tilde{\nu}(\Gamma)+\gamma\beta\varphi_x(\Gamma)\tilde{\nu}(\Gamma)=0 \qquad \forall\tilde{\nu}\in \hat{H}^1(0,\Gamma),$$

we choose

$$\tilde{\nu}(x)=\frac{x}{\Gamma}, \tag{2.34}$$

then we obtain

$$\gamma\beta\varphi_x(\Gamma)=\alpha\nu_x(\Gamma), \tag{2.35}$$

also $(2.25)_2$, implies

$$\gamma\beta\nu_{xx}-\beta\varphi_{xx}+\mu\varphi=Q_2,\ \text{in the distribution sense.} \tag{2.36}$$

Multiplying (2.36) by $\tilde{\varphi}\in \hat{H}^1(0,\Gamma)$ and using integration by parts we get by using $(2.25)_2$ again

$$\gamma\beta\nu_x(\Gamma)\tilde{\varphi}(\Gamma)-\beta\varphi_x(\Gamma)\tilde{\varphi}(\Gamma)=0, \qquad \forall\tilde{\varphi}\in \hat{H}^1(0,\Gamma),$$

we choose

$$\tilde{\varphi}(x)=\frac{x}{\Gamma},$$

then we obtain

$$\gamma\beta\nu_x(\Gamma)-\beta\varphi_x(\Gamma)=0, \tag{2.37}$$

using (2.35) in (2.37), then we get

$$\nu_x(\Gamma)=\varphi_x(\Gamma)=0, \tag{2.38}$$

then, by (2.32) and (2.38) we obtain

$$\nu,\varphi\in \hat{H}^2(0,\Gamma):\varphi_x(\Gamma)=\nu_x(\Gamma)=0,$$

then the operator $(I-A)$ is surjective.

Therefore, A is a maximal dissipative operator, and by applying the Hille-Yosida theorem, we get the desired result. ■

2.4 Exponential stability

In this section, we present and demonstrate the technical lemmas necessary for establishing the proof of our stability result.

Lemma 2.1 *Let (ν, φ, Y) represent a solution to (2.6), then the expression of energy $E(t)$ is defined as follows*

$$E(t) = \frac{1}{2}\int_0^\Gamma \left(\rho\nu_t^2 + \mu\varphi_t^2 + \alpha_1\nu_x^2 + \beta(\gamma\nu_x - \varphi_x)^2 + \int_0^1\int_{\tau_1}^{\tau_2} \hbar\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho\right) dx, \tag{2.39}$$

and satisfies

$$\frac{d}{dt}E(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \left|\zeta(\hbar)\right| d\hbar\right)\int_0^\Gamma \nu_t^2 dx. \tag{2.40}$$

Proof. Multiplying $(2.6)_1$, $(2.6)_2$ by ν_t, φ_t respectively, and integrating over the interval $(0,\Gamma)$, we get

$$\begin{aligned}
&\rho\frac{d}{2dt}\int_0^\Gamma \nu_t^2 dx + \mu\frac{d}{2dt}\int_0^\Gamma \varphi_t^2 dx + \alpha_1\frac{d}{2dt}\int_0^\Gamma \nu_x^2 dx \\
&+\gamma\beta\int_0^\Gamma (\gamma\nu_x - \varphi_x)\nu_{xt} dx - \beta\int_0^\Gamma (\gamma\nu_x - \varphi_x)\varphi_{xt} dx \\
&+\mu_1\int_0^\Gamma \nu_t^2 dx + \int_0^\Gamma \nu_t \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx = 0,
\end{aligned} \tag{2.41}$$

by (2.41), we find

$$\begin{aligned}
&\rho\frac{d}{2dt}\int_0^\Gamma \nu_t^2 dx + \mu\frac{d}{2dt}\int_0^\Gamma \varphi_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^\Gamma \beta(\gamma\nu_x - \varphi_x)^2 dx \\
&+\alpha_1\frac{d}{2dt}\int_0^\Gamma \nu_x^2 dx + \mu_1\int_0^\Gamma \nu_t^2 dx + \int_0^\Gamma \nu_t \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx = 0.
\end{aligned} \tag{2.42}$$

After multiplying $(2.6)_3$ by $\left|\zeta(\hbar)\right| Y(x,\rho,t,\hbar)$ and integration over $(0,\Gamma)\times(0,1)\times(\tau_1,\tau_2)$ with respect to x, ρ and $\hbar$, we obtain

$$\begin{aligned}
&\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\left|\zeta(\hbar)\right| Y(x,\rho,t,\hbar) Y_t(x,\rho,t,\hbar)\, d\hbar d\rho dx \\
&+\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \left|\zeta(\hbar)\right| Y(x,\rho,t,\hbar) Y_\rho(x,\rho,t,\hbar)\, d\hbar d\rho dx = 0,
\end{aligned} \tag{2.43}$$

then we have

$$\begin{aligned}
&\frac{d}{2dt}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx \\
&+\frac{1}{2}\int_0^\Gamma\int_0^1 \frac{d}{d\rho}\int_{\tau_1}^{\tau_2} \left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx = 0,
\end{aligned} \tag{2.44}$$

as

$$\frac{1}{2}\int_0^\Gamma\int_0^1\frac{d}{d\rho}\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,\rho,t,\hbar)\,d\hbar d\rho dx=\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx$$
$$-\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\nu_t^2 d\hbar dx, \tag{2.45}$$

then we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\Gamma\left(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta(\gamma\nu_x-\varphi_x)^2+\int_0^1\int_{\tau_1}^{\tau_2}\hbar|\zeta(\hbar)|Y^2(x,\rho,t,\hbar)\,d\hbar d\rho\right)dx$$
$$+\mu_1\int_0^\Gamma\nu_t^2dx+\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx$$
$$+\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx-\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\nu_t^2d\hbar dx=0, \tag{2.46}$$

Since $\alpha_1=\alpha-\gamma^2\beta>0$, we get

$$\frac{d}{dt}E(t)=-\mu_1\int_0^\Gamma\nu_t^2dx-\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx$$
$$-\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\int_0^\Gamma\nu_t^2dx, \tag{2.47}$$

using Young's inequality, we obtain

$$-\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx\le\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\nu_t|\,|\zeta(\hbar)|^{\frac{1}{2}}\,|\zeta(\hbar)|^{\frac{1}{2}}\,|Y(x,1,t,\hbar)|\,d\hbar dx$$
$$\le\frac{1}{2}\int_0^\Gamma\nu_t^2dx\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar$$
$$+\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|Y^2(x,1,t,\hbar)\,d\hbar dx, \tag{2.48}$$

then we have by using (2.47)-(2.48)

$$\frac{d}{dt}E(t)\le-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\right)\int_0^\Gamma\nu_t^2dx,$$

also, by using (2.5), we obtain

$$\frac{d}{dt}E(t)\le 0.$$

■

Lemma 2.2 *Let (ν,φ,Y) represent a solution to (2.6), then the first functional*

$$I_1(t)=\rho\int_0^\Gamma\nu_t\nu dx+\gamma\mu\int_0^\Gamma\varphi_t\nu dx+\frac{\mu_1}{2}\int_0^\Gamma\nu^2dx,\forall t\ge 0,$$

satisfies for some positive constant ε_1

$$I_1'(t) \leq -\frac{\alpha_1}{2}\int_0^\Gamma \nu_x^2 dx + \left(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\right)\int_0^\Gamma \nu_t^2 dx + \varepsilon_1 \int_0^\Gamma \varphi_t^2 dx$$
$$+ \frac{c_0\mu_1}{2\alpha_1}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx. \tag{2.49}$$

Proof. By integrating equation $(2.6)_1$ multiplied by ν over the interval $(0,\Gamma)$ with respect to x, we arrive at the following expression

$$\frac{d}{dt}\rho\int_0^\Gamma \nu_t\nu dx - \rho\int_0^\Gamma \nu_t^2 dx + \alpha_1\int_0^\Gamma \nu_x^2 dx + \gamma\int_0^\Gamma (\beta\varphi_{xx} - \gamma\beta\nu_{xx})\,\nu dx$$
$$+ \frac{d}{dt}\frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx + \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar = 0, \tag{2.50}$$

also by using the equation $(2.6)_2$, we get

$$\frac{d}{dt}\rho\int_0^\Gamma \nu_t\nu dx - \rho\int_0^\Gamma \nu_t^2 dx + \alpha_1\int_0^\Gamma \nu_x^2 dx + \gamma\mu\int_0^\Gamma \varphi_{tt}\nu dx$$
$$+ \frac{d}{dt}\frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx + \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar dx = 0, \tag{2.51}$$

(2.51), satisfies the equation

$$\frac{d}{dt}\left(\rho\int_0^\Gamma \nu_t\nu dx + \gamma\mu\int_0^\Gamma \varphi_t\nu dx + \frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx\right) =$$
$$\rho\int_0^\Gamma \nu_t^2 dx - \alpha_1\int_0^\Gamma \nu_x^2 dx + \gamma\mu\int_0^\Gamma \varphi_t\nu_t dx - \int_0^\Gamma \nu\int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar dx. \tag{2.52}$$

When applying Cauchy-Schwarz, Young's, and Poincaré's inequalities, the following inequality holds for any $\varepsilon_1 > 0$

$$\gamma\mu\int_0^\Gamma \varphi_t\nu_t dx \leq \varepsilon_1\int_0^\Gamma \varphi_t^2 dx + \frac{(\gamma\mu)^2}{4\varepsilon_1}\int_0^\Gamma \nu_t^2 dx, \tag{2.53}$$

$$-\int_0^\Gamma \nu\int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar dx$$
$$\leq \frac{\alpha_1}{2}\int_0^\Gamma \nu_x^2 dx + \frac{c_0\mu_1}{2\alpha_1}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx, \tag{2.54}$$

by using (2.53)-(2.54) in (2.52) we get (2.49). ■

Lemma 2.3 *Let* (ν,φ,Y) *represent a solution to* (2.6)*, then the functional*

$$I_2(t) = \mu\int_0^\Gamma \varphi_t\varphi dx + \rho\int_0^\Gamma \nu_t\nu dx,$$

its derivative satisfies

$$I_2'(t) \le -\beta \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx - \frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx + \left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\int_0^\Gamma \nu_t^2 dx + \mu\int_0^\Gamma \varphi_t^2 dx$$
$$+ \frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx. \tag{2.55}$$

Proof. By differentiating $I_2(t)$ and using $(2.6)_1$, $(2.6)_2$, we have

$$\begin{aligned} I_2'(t) &= \mu\int_0^\Gamma \varphi_t^2 dx + \mu\int_0^\Gamma \varphi_{tt}\varphi dx + \rho\int_0^\Gamma \nu_t^2 dx + \rho\int_0^\Gamma \nu_{tt}\nu dx \\ &= \mu\int_0^\Gamma \varphi_t^2 dx - \beta\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx + \rho\int_0^\Gamma \nu_t^2 dx \\ &- \alpha_1\int_0^\Gamma \nu_x^2 dx - \mu_1\int_0^\Gamma \nu_t\nu dx - \int_0^\Gamma \nu\int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx, \end{aligned} \tag{2.56}$$

employing Cauchy-Schwarz, Young's, and Poincaré's inequalities, we get

$$-\mu_1\int_0^\Gamma \nu_t\nu dx \le \frac{\alpha_1}{2}\int_0^\Gamma \nu_x^2 dx + \frac{c_0\mu_1^2}{2\alpha_1}\int_0^\Gamma \nu_t^2 dx, \tag{2.57}$$

and

$$\begin{aligned} &\int_0^\Gamma \nu\int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx \\ &\le \frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx + \frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \tag{2.58}$$

By using (2.57)-(2.58) in (2.56) we get (2.55). ∎

Lemma 2.4 *Let (ν,φ,Y) satisfy system (2.6) then the functional*

$$I_3(t) = \rho\int_0^\Gamma \nu_t(\gamma\nu - \varphi)\, dx + \gamma\mu\int_0^\Gamma \varphi_t(\gamma\nu - \varphi)\, dx,$$

satisfies for any ε_2, ε_3, $\varepsilon_4 > 0$

$$\begin{aligned} I_3'(t) &\le -\frac{\gamma\mu}{2}\int_0^\Gamma \varphi_t^2 dx + (\varepsilon_2 + \varepsilon_3 c_0 + \varepsilon_4 c_0)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ &+ \left(\frac{\mu_1^2}{4\varepsilon_3} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\right)\int_0^\Gamma \nu_t^2 dx \\ &+ \frac{\alpha_1^2}{4\varepsilon_2}\int_0^\Gamma \nu_x^2 dx + \frac{\mu_1}{4\varepsilon_4}\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \tag{2.59}$$

Proof. *By differentiating $I_3(t)$ and using (2.6)$_1$, (2.6)$_2$ then we have*

$$
\begin{aligned}
I_3'(t) &= -\gamma\mu\int_0^\Gamma \varphi_t^2 dx - \alpha_1\int_0^\Gamma \nu_x(\gamma\nu_x-\varphi_x)\,dx - \mu_1\int_0^\Gamma \nu_t(\gamma\nu-\varphi)\,dx \\
&- \int_0^\Gamma (\gamma\nu-\varphi)\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx + \rho\gamma\int_0^\Gamma \nu_t^2 dx \\
&+ \underbrace{(\gamma^2\mu-\rho)}_{\varkappa}\int_0^\Gamma \nu_t\varphi_t dx,
\end{aligned} \tag{2.60}
$$

by using Cauchy-Schwarz, Young's, and Poincaré's inequalities, we obtain

$$
-\alpha_1\int_0^\Gamma \nu_x(\gamma\nu_x-\varphi_x)\,dx \le \varepsilon_2\int_0^\Gamma (\gamma\nu_x-\varphi_x)^2 dx + \frac{\alpha_1^2}{4\varepsilon_2}\int_0^\Gamma \nu_x^2 dx, \forall\varepsilon_2>0, \tag{2.61}
$$

and

$$
-\mu_1\int_0^\Gamma \nu_t(\gamma\nu-\varphi)\,dx \le \varepsilon_3 c_0\int_0^\Gamma (\gamma\nu_x-\varphi_x)^2 dx + \frac{\mu_1^2}{4\varepsilon_3}\int_0^\Gamma \nu_t^2 dx, \forall\varepsilon_3>0, \tag{2.62}
$$

$$
\begin{aligned}
&-\int_0^\Gamma (\gamma\nu-\varphi)\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx \le \varepsilon_4 c_0\int_0^\Gamma (\gamma\nu_x-\varphi_x)^2 dx \\
&+\frac{\mu_1}{4\varepsilon_4}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx, \forall\varepsilon_4>0,
\end{aligned} \tag{2.63}
$$

and

$$
\varkappa\int_0^\Gamma \nu_t\varphi_t dx \le \frac{\gamma\mu}{2}\int_0^\Gamma \varphi_t^2 dx + \frac{\varkappa^2}{2\gamma\mu}\int_0^\Gamma \nu_t^2 dx. \tag{2.64}
$$

By using (2.61)-(2.62)-(2.63)-(2.64) in (2.60) we get (2.59). ■

Lemma 2.5 *Let (ν,φ,Y) satisfy system (2.6), then the functional*

$$
I_4(t) := \int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar e^{-\hbar\rho}\,|\zeta(\hbar)|\,Y^2(x,\rho,t,\hbar)\,d\hbar d\rho dx,
$$

satisfies

$$
\begin{aligned}
I_4'(t) &\le -e^{-\tau_2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx + \mu_1\int_0^\Gamma \nu_t^2 dx \\
&- e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar\,|\zeta(\hbar)|\,Y^2(x,\rho,t,\hbar)\,d\hbar d\rho dx.
\end{aligned}
$$

Proof. By differentiating $I_4(t)$ and using $(2.6)_3$, then we have

$$\begin{aligned} I_4'(t) &= -2\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} e^{-\hbar\rho}\left|\zeta(\hbar)\right| Y(x,\rho,t,\hbar)\, Y_\rho(x,\rho,t,\hbar)\, d\hbar d\rho dx \\ &= -\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\left|\zeta(\hbar)\right|\frac{d}{d\rho}\left(e^{-\hbar\rho}Y^2(x,\rho,t,\hbar)\right) d\hbar d\rho dx \\ &\quad -\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar e^{-\hbar\rho}\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx \\ &= -\int_0^\Gamma\int_{\tau_1}^{\tau_2} e^{-\hbar}\left|\zeta(\hbar)\right| Y^2(x,1,t,\hbar)\, d\hbar dx + \int_{\tau_1}^{\tau_2}\left|\zeta(\hbar)\right| d\hbar\int_0^\Gamma \nu_t^2 dx \\ &\quad -\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar e^{-\hbar\rho}\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx, \end{aligned}$$

by using the following relation $e^{-\hbar} \leq e^{-\hbar\rho} \leq 1,\ \forall\ 0 \leq \rho \leq 1$, we get

$$\begin{aligned} I_4'(t) &\leq -\int_0^\Gamma\int_{\tau_1}^{\tau_2} e^{-\hbar}\left|\zeta(\hbar)\right| Y^2(x,1,t,\hbar)\, d\hbar dx + \int_{\tau_1}^{\tau_2}\left|\zeta(\hbar)\right| d\hbar\int_0^\Gamma \nu_t^2 dx \\ &\quad -\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar e^{-\hbar}\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx. \end{aligned}$$

Since $\left(-e^{-\hbar}\right)' = e^{-\hbar} \geq 0$, we conclude that $-e^{-\hbar} \leq -e^{-\tau_2}$, $\forall \hbar \in (\tau_1, \tau_2)$, then we get

$$\begin{aligned} I_4'(t) &\leq -e^{-\tau_2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}\left|\zeta(\hbar)\right| Y^2(x,1,t,\hbar)\, d\hbar dx + \mu_1\int_0^\Gamma \nu_t^2 dx \\ &\quad - e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar\left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx. \end{aligned}$$

■

Now, for a large enough N, the Lyapunov functional is defined as follows

$$L(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t),$$

where $N_1,\ N_2,\ N_3$ and N_4 are positive constants, to be chosen later.

Theorem 2.2 *Let (ν, φ, Y) satisfy system (2.6), then there exist two positive constants $c_1, c_2 > 0$ that satisfy*

$$c_1 E(t) \leq L(t) \leq c_2 E(t),\ \forall t \geq 0. \tag{2.65}$$

Proof. *Let*

$$\Im(t) = L(t) - NE(t) = N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t), \tag{2.66}$$

$$\begin{aligned}
|\Im(t)| &= |L(t) - NE(t)| \\
&\leq N_1 \left(\rho \int_0^\Gamma |\nu_t \nu| \, dx + \gamma\mu \int_0^\Gamma |\varphi_t \nu| \, dx + \frac{\mu_1}{2} \int_0^\Gamma \nu^2 dx \right) \\
&+ N_2 \left(\mu \int_0^\Gamma |\varphi_t \varphi| \, dx + \rho \int_0^\Gamma |\nu_t \nu| \, dx \right) \\
&+ N_3 \left(\rho \int_0^\Gamma |\nu_t (\gamma\nu - \varphi)| \, dx + \gamma\mu \int_0^\Gamma |\varphi_t (\gamma\nu - \varphi)| \, dx \right) \\
&+ N_4 \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar\rho} |\zeta(\hbar)| \, Y^2(x, \rho, t, \hbar) \, d\hbar d\rho dx, \qquad (2.67)
\end{aligned}$$

using Poincaré's and Young's inequalities in (2.67), we find for any $\varepsilon > 0$

$$\begin{aligned}
|\Im(t)| &\leq \underbrace{\left(\frac{N_1 \rho^2}{4\varepsilon} + N_2 \rho^2 \varepsilon + \frac{N_3 \rho^2}{4\varepsilon} \right)}_{\theta_1} \int_0^\Gamma \nu_t^2 dx \\
&+ \underbrace{\left(N_1 \frac{(\gamma\mu)^2}{4\varepsilon} + N_2 \frac{\mu^2}{4\varepsilon} + N_3 \frac{(\gamma\mu)^2}{4\varepsilon} \right)}_{\theta_2} \int_0^\Gamma \varphi_t^2 dx \\
&+ \underbrace{\left(N_1 \left(2\varepsilon c_0 + \frac{c_0 \mu_1}{2} \right) + N_2 \left(2\varepsilon \gamma^2 c_0 + \frac{c_0}{4\varepsilon} \right) \right)}_{\theta_3} \int_0^\Gamma \nu_x^2 dx \\
&+ \underbrace{(2N_2 \varepsilon c_0 + 2N_3 \varepsilon c_0)}_{\theta_4} \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 \, dx \\
&+ N_4 \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar \, |\zeta(\hbar)| \, Y^2(x, \rho, t, \hbar) \, d\hbar d\rho dx,
\end{aligned}$$

then

$$|\Im(t)| \leq CE(t),$$

where

$$C = \max\left(\frac{2}{\rho}\theta_1, \frac{2}{\mu}\theta_2, \frac{2}{\alpha_1}\theta_3, \frac{2}{\beta}\theta_4, 2N_4 \right),$$

then we obtain

$$\underbrace{(-C + N)}_{c_1} E(t) \leq L(t) \leq \underbrace{(C + N)}_{c_2} E(t).$$

■

Theorem 2.3 *Let* (ν, φ, Y) *satisfies system (2.6), then there exist two positive constants* k *and* λ*, such that*

$$E(t) \leq ke^{-\lambda t}, \qquad \forall t \geq 0. \qquad (2.68)$$

Proof. Using the previous lemmas, we get

$$L'(t) = NE'(t) + N_1 I_1'(t) + N_2 I_2'(t) + N_3 I_3'(t) + N_4 I_4'(t).$$

This leads to

$$\begin{aligned} L'(t) \leq & -\left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - N_1\left(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right. \\ & \left. -N_3\left(\frac{\mu_1^2}{4\varepsilon_3} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right)\int_0^\Gamma \nu_t^2 dx \\ & - \left(\frac{\gamma\mu N_3}{2} - N_1\varepsilon_1 - N_2\mu\right)\int_0^\Gamma \varphi_t^2 dx - \left(\frac{N_1\alpha_1}{2} + \frac{N_2\alpha_1}{4} - N_3\frac{\alpha_1^2}{4\varepsilon_2}\right)\int_0^\Gamma \nu_x^2 dx \\ & - \left(N_2\beta - (N_3\varepsilon_2 + N_3\varepsilon_3 c_0 + N_3\varepsilon_4 c_0)\right)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ & - N_4 e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\,|\zeta(\hbar)|\, Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx \\ & - \left(N_4 e^{-\tau_2} - \frac{N_3\mu_1}{4\varepsilon_4} - N_2\frac{c_0\mu_1}{\alpha_1} - N_1\frac{c_0\mu_1}{2\alpha_1}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx, \end{aligned}$$

we choose the following values

$$\varepsilon_1 = \frac{1}{N_1},\ \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1}{N_3},$$

we get

$$\begin{aligned} L'(t) \leq & -\left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - N_1\left(\rho + \frac{N_1(\gamma\mu)^2}{4}\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right. \\ & \left. -N_3\left(\frac{N_3\mu_1^2}{4} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right)\int_0^\Gamma \nu_t^2 dx \\ & - \left(\frac{\gamma\mu N_3}{2} - 1 - N_2\mu\right)\int_0^\Gamma \varphi_t^2 dx - \left(\frac{N_1\alpha_1}{2} - \frac{\alpha_1^2}{4}N_3^2\right)\int_0^\Gamma \nu_x^2 dx \\ & - \left(N_2\beta - (1 + 2c_0)\right)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ & - N_4 e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\,|\zeta(\hbar)|\, Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx \\ & - \left(N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\frac{c_0\mu_1}{\alpha_1} - N_1\frac{c_0\mu_1}{2\alpha_1}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \quad (2.69)$$

First, in (2.69), we choose N_2 until it becomes

$$N_2\beta - (1 + 2c_0) > 0.$$

We also choose N_3 until it becomes

$$\frac{\gamma\mu N_3}{2} - 1 - N_2\mu > 0.$$

Now, we choose N_1 large enough so that

$$\frac{N_1\alpha_1}{2} - \frac{\alpha_1^2}{4}N_3^2 > 0.$$

We also choose N_4 large enough so that

$$N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\frac{c_0\mu_1}{\alpha_1} - N_1\frac{c_0\mu_1}{2\alpha_1} > 0.$$

Finally, we choose a very large N so that

$$\begin{aligned}&\left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\right) - N_1\left(\rho + \frac{(\gamma\mu)^2}{4}N_1\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right.\\ &\left. -N_3\left(\frac{N_3\mu_1^2}{4} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right) \ > \ 0.\end{aligned}$$

As

$$-\left(N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\frac{c_0\mu_1}{\alpha_1} - N_1\frac{c_0\mu_1}{2\alpha_1}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx \le 0,$$

then we get

$$L'(t) \le -mE(t),$$

by (2.65) we get

$$L'(t) \le -\frac{m}{c_2}L(t),$$

this implies

$$L(t) \le L(0)\,e^{-\frac{m}{c_2}t},$$

using (2.65) again, we obtain (2.68). ■

CHAPTER 3

Existence, uniqueness and exponential energy decay of piezoelectric system with thermal effect and distributed delay time in the presence or absence of magnetic effects

3.1 Introduction

Fourier's law, also known as the law of heat conduction, is a fundamental principle in the field of thermal conduction. It states that the rate of heat transfer through a material is proportional to the negative gradient in temperature and to the area, at right angles to that gradient, through which the heat flows. This law can be stated in two equivalent forms: the integral form, which looks at the amount of energy flowing into or out of a body as a whole, and the differential form, which looks at the flow rates or fluxes of energy locally. The differential form of Fourier's law is given by the equation $q = -k\nabla\theta$, where q is the heat flux, k is the thermal conductivity of the material, and $\nabla\theta$ is the temperature gradient. The integral form of Fourier's law is given by the equation $Q = -kA(d\theta/dx)$, where Q is the amount of heat transferred per unit time, A is an oriented surface area element, and $d\theta/dx$ is the temperature gradient. To solve Fourier's law, the relationship of geometry, temperature difference, and thermal conductivity of the material is derived. Joseph Fourier first introduced this law in 1822 and concluded that "the heat flux resulting from thermal conduction is proportional to the magnitude of the temperature gradient and opposite to it in sign [68]. Thermoelastic damping is a source of intrinsic material damping due to the thermoelasticity present in almost all materials. As the name thermoelastic suggests, it describes the coupling between the elastic field in the structure caused by deformation and the temperature field. The earliest study of thermoelastic damping can be found in Zener's classical work, [107] in 1937/1938, which studied thermoelastic damping in beams undergoing flexural vibrations. Messaoudi et al.[65] studied piezoelectric beams with thermal and magnetic effects in the presence of a nonlinear damping term acting on the mechanical equation. A general decay result of the solution was shown, from which the exponential and polynomial decay are only special cases. Keddi and al. [53] by using semigroup theory, studied the well-posedness of a linear thermoelastic Timoshenko system free of a second spectrum where the heat conduction is given by Cattaneo's law. The asymptotic stability of this system was also proven. Finally, they further clarified their theoretical results through some numerical tests. Afilal et al. [8] considered the thermoelastic Timoshenko system with past history, where the thermal effects are given by Cattaneo and Fourier laws. By using the energy method in Fourier space to build appropriate Lyapunov functionals, it was obtained that both systems, have the same rate of decay $(1+t)^{-\frac{1}{4}}$. Rivera et al. [81] in their paper, by using semigroup theory, demonstrated the lack of exponential stability (the wave speeds are different) of linear Timoshenko systems coupled with heat conduction given by Fourier law.

3.2 Problem statement

Based on the following points:

- Since the model of piezoelectric beams with magnetic effects is proven to not be exactly observable/exponentially stabilizable in the energy space for all choices of material parameters. Additionally, achieving strong stability is not possible for many material parameter values [71].
- Since the time delay can destabilize the systems.
- Since many authors have proven the lack of exponential stability for some systems coupled with heat equation governed by Fourier's law.

In the present chapter, we consider the following fully dynamic piezoelectric beams with thermal effects

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \delta\theta_x + \mu_1\nu_t \\ + \int_{\tau_1}^{\tau_2} \zeta(\hbar)\,\nu_t(x,t-\hbar)\,d\hbar = 0, & (x,t) \in (0,\Gamma)\times(0,\infty), \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, \\ c\theta_t - \kappa\theta_{xx} + \delta\nu_{tx} = 0, \end{cases} \tag{3.1}$$

$$\begin{cases} \nu(0,t) = \alpha\nu_x(\Gamma,t) - \gamma\beta\varphi_x(\Gamma,t) = 0, & t \geq 0, \\ \varphi(0,t) = \varphi_x(\Gamma,t) - \gamma\nu_x(\Gamma,t) = 0, \\ \theta(0,t) = \theta(\Gamma,t) = 0, \\ \nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x),\ \varphi(x,0) = \varphi_0(x), & x \in (0,\Gamma) \\ \varphi_t(x,0) = \varphi_1(x), \theta(x,0) = \theta_0(x), \\ \nu_t(x,-t) = f_0(x,-t), & t \in (0,\tau_2), \end{cases} \tag{3.2}$$

where c, κ and δ are positive physical constants (see [32, 59]).

The important question we ask here is whether the linear damping is strong enough to achieve exponential stability (rapid decrease in the energy) in the presence of magnetic and thermal effects with distributed delay.

3.3 Existence, uniqueness

In this section, we will establish the existence and uniqueness of solutions for system (3.1)- (3.2) by employing semigroup theory. As stated in the work [67], we introduce the new variable

$$Y(x,\rho,t,\hbar)=\nu_t(x,t-\rho\hbar), \qquad x\in(0,\Gamma),\ \rho\in(0,1),\ \hbar\in(\tau_1,\tau_2),\ t>0.$$

Then, we find the new equivalent problem

$$\begin{cases} \rho\nu_{tt}-\alpha\nu_{xx}+\gamma\beta\varphi_{xx}+\delta\theta_x+\mu_1\nu_t & (x,t)\in(0,\Gamma)\times(0,\infty), \\ +\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar=0, & \\ \mu\varphi_{tt}-\beta\varphi_{xx}+\gamma\beta\nu_{xx}=0, & \\ c\theta_t-\kappa\theta_{xx}+\delta\nu_{tx}=0, & \\ \hbar Y_t(x,\rho,t,\hbar)+Y_\rho(x,\rho,t,\hbar)=0, & (\rho,\hbar)\in(0,1)\times(\tau_1,\tau_2), \end{cases} \tag{3.3}$$

with the following initial and boundary conditions:

$$\begin{cases} \nu(0,t)=\alpha\nu_x(\Gamma,t)-\gamma\beta\varphi_x(\Gamma,t)=0, & t\geq 0, \\ \varphi(0,t)=\varphi_x(\Gamma,t)-\gamma\nu_x(\Gamma,t)=0, & \\ \theta(0,t)=\theta(\Gamma,t)=0, & \\ \nu(x,0)=\nu_0(x),\nu_t(x,0)=\nu_1(x),\ \varphi(x,0)=\varphi_0(x), & x\in(0,\Gamma) \\ \varphi_t(x,0)=\varphi_1(x),\theta(x,0)=\theta_0(x), & \\ Y(x,\rho,0,\hbar)=f_0(x,\rho,\hbar). & (\rho,\hbar)\in(0,1)\times(0,\tau_2). \end{cases} \tag{3.4}$$

By using the following notations

$$\nu_t=u,\ \varphi_t=q, \qquad \text{and} \qquad V=(\nu,u,\varphi,q,\theta,Y)^T,$$
$$\partial_t V=(\nu_t,u_t,\varphi_t,q_t,\theta_t,Y_t)^T,$$

therefore, the problem (3.3)-(3.4) can be reformulated as

$$\begin{cases} \partial_t V=BV, \\ V(0)=V_0=(\nu_0,\nu_1,\varphi_0,\varphi_1,\theta_0,f_0), \end{cases} \tag{3.5}$$

where the operator $B:D(B)\subset H_1\to H_1$ is defined by

$$BV:=\begin{pmatrix} \nu_t \\ \frac{\alpha}{\rho}\nu_{xx}-\frac{\gamma\beta}{\rho}\varphi_{xx}-\frac{\delta}{\rho}\theta_x-\frac{\mu_1}{\rho}\nu_t-\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar \\ \varphi_t \\ \frac{\beta}{\mu}\varphi_{xx}-\frac{\gamma\beta}{\mu}\nu_{xx} \\ \frac{\kappa}{c}\theta_{xx}-\frac{\delta}{c}\nu_{tx} \\ -\frac{1}{\hbar}Y_\rho \end{pmatrix}. \tag{3.6}$$

We consider the following spaces

$$\hat{H}^1(0,\Gamma) = \left\{\nu \in H^1(0,\Gamma) : \nu(0) = 0\right\},$$
$$\tilde{H}^2(0,\Gamma) = \left\{\nu \in H^2(0,\Gamma) : \nu_x(\Gamma) = 0\right\}.$$

Furthermore, we define the aforementioned Hilbert space H_1 as follows:

$$H_1 := \hat{H}^1(0,\Gamma) \times L^2(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times L^2(0,\Gamma) \times L^2(0,\Gamma) \times L^2((0,\Gamma)\times(0,1)\times(\tau_1,\tau_2)).$$

The inner product on H_1 is defined as follows:

$$\begin{aligned}\left\langle V,\tilde{V}\right\rangle &= \rho\int_0^\Gamma \nu_t\tilde{\nu}_t dx + \mu\int_0^\Gamma \varphi_t\tilde{\varphi}_t dx - \gamma\beta\int_0^\Gamma \nu_x\tilde{\varphi}_x dx - \gamma\beta\int_0^\Gamma \tilde{\nu}_x\varphi_x dx \\ &+\alpha\int_0^\Gamma \nu_x\tilde{\nu}_x dx + \beta\int_0^\Gamma \varphi_x\tilde{\varphi}_x dx + c\int_0^\Gamma \theta\tilde{\theta} dx \\ &+\int_0^\Gamma\int_{\tau_1}^{\tau_2} \hbar\left|\zeta(\hbar)\right|\int_0^1 Y(x,\rho,t,\hbar)\tilde{Y}(x,\rho,t,\hbar)\, d\rho d\hbar dx. \end{aligned} \tag{3.7}$$

Now, we defined the previous domain of operator B as

$$\begin{aligned} D(B) := \Big\{(\nu,\nu_t,\varphi,\varphi_t,\theta,Y) \;\in\; &\tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)\times\hat{H}^1(0,\Gamma)\times\tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma) \\ &\times\hat{H}^1(0,\Gamma)\times H^2(0,\Gamma)\cap H_0^1(0,\Gamma)\times L^2((0,\Gamma)\times(0,1)\times(\tau_1,\tau_2))\Big\}. \end{aligned} \tag{3.8}$$

Clearly, $D(B)$ is dense in H_1.

Theorem 3.1 *Let $V_0 \in D(B)$. Then, the problem mentioned (3.3)-(3.4) has a unique solution $V \in C(\mathbb{R}^+, D(B)) \cap C^1(\mathbb{R}^+, H_1)$.*

Proof. Firstly, we establish the dissipativity of the operator B.

Let $V = (\nu,\nu_t,\varphi,\varphi_t,\theta,Y)^T \in D(B)$. By utilizing the previous inner product, we get:

$$\langle BV,V\rangle_{H_1} = \left\langle \begin{pmatrix} \nu_t \\ \frac{\alpha}{\rho}\nu_{xx} - \frac{\gamma\beta}{\rho}\varphi_{xx} - \frac{\delta}{\rho}\theta_x - \frac{\mu_1}{\rho}\nu_t - \frac{1}{\rho}\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar \\ \varphi_t \\ \frac{\beta}{\mu}\varphi_{xx} - \frac{\gamma\beta}{\mu}\nu_{xx} \\ \frac{\kappa}{c}\theta_{xx} - \frac{\delta}{c}\nu_{tx} \\ -\frac{1}{\hbar}Y_\rho \end{pmatrix}, \begin{pmatrix}\nu \\ \nu_t \\ \varphi \\ \varphi_t \\ \theta \\ Y\end{pmatrix}\right\rangle_{H_1}. \tag{3.9}$$

By integrating by parts and taking into account the boundary conditions, we obtain:

$$\begin{aligned}\langle BV,V\rangle_{H_1} &= -\mu_1\int_0^\Gamma \nu_t^2 dx - \kappa\int_0^\Gamma \theta_x^2 dx - \int_0^\Gamma \nu_t\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx \\ &- \int_0^\Gamma\int_{\tau_1}^{\tau_2}\left|\zeta(\hbar)\right|\int_0^1 Y_\rho(x,\rho,t,\hbar)Y(x,\rho,t,\hbar)\,d\rho d\hbar dx, \end{aligned} \tag{3.10}$$

also, by integrating with respect to ρ, we find

$$-\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| \int_0^1 Y_\rho(x,\rho,t,\hbar) Y(x,\rho,t,\hbar)\, d\rho d\hbar dx = -\frac{1}{2}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar \int_0^\Gamma \nu_t^2 dx, \quad (3.11)$$

by applying Young's and Cauchy-Schwarz inequalities, we get

$$-\int_0^\Gamma \nu_t \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx \leq \frac{1}{2}\int_0^\Gamma \nu_t^2 dx \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar + \frac{1}{2}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx, \quad (3.12)$$

by (3.11), (3.12), we obtain

$$\langle BV, V\rangle_{H_1} \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) \int_0^\Gamma \nu_t^2 dx - \kappa \int_0^\Gamma \theta_x^2 dx.$$

Consequently, throught the condition (2.5), we conclude that B is a dissipative operator. Next, we will proceed to prove that the operator $(I-B)$ is surjective. Given $M_1 = (g_1, g_2, g_3, g_4, g_5, g_6)^T \in H_1$, we show that there exists a unique $V = (\nu, u, \varphi, q, \theta, Y)^T \in D(B)$ such that

$$(I-B)V = M_1, \quad (3.13)$$

this implies

$$\begin{cases} \nu - u = g_1, \\ \rho u - \alpha \nu_{xx} + \gamma\beta\varphi_{xx} + \mu_1 u + \delta\theta_x + \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar = \rho g_2, \\ \varphi - q = g_3, \\ \mu q - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = \mu g_4, \\ c\theta - \kappa\theta_{xx} + \delta u_x = c g_5, \\ Y + \frac{1}{\hbar} Y_\rho = g_6. \end{cases} \quad (3.14)$$

Using $(3.14)_1$ and $(3.14)_3$, we have

$$\begin{cases} u = \nu - g_1, \\ q = \varphi - g_3. \end{cases} \quad (3.15)$$

Because

$$Y(x,0,t,\hbar) = \nu_t(x,t) = u(x,t), \qquad \text{for } x \in (0,\Gamma),\ \hbar \in (\tau_1,\tau_2), t \geq 0,$$

and according to equation $(3.14)_6$, we obtain

$$Y(x,\rho,t,\hbar) = \hbar e^{-\hbar\rho} \int_0^\rho g_6(x,\tau,\hbar) e^{\hbar\tau} d\tau + u e^{-\hbar\rho}, \quad (3.16)$$

in particular

$$Y(x,1,t,\hbar) = \hbar e^{-\hbar} \int_0^1 g_6(x,\tau,\hbar) e^{\hbar\tau} d\tau + u e^{-\hbar}. \tag{3.17}$$

Now, by using (3.15)-(3.17) in the remaining equations for (3.14), we get

$$\begin{cases} -\alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \delta\theta_x + \varpi_1\nu = Q_1 \in L^2(0,\Gamma), \\ \gamma\beta\nu_{xx} - \beta\varphi_{xx} + \mu\varphi = Q_2 \in L^2(0,\Gamma), \\ c\theta - \kappa\theta_{xx} + \delta\nu_x = Q_3 \in L^2(0,\Gamma). \end{cases} \tag{3.18}$$

Where

$$\begin{aligned} \varpi_1 &= (\mu_1 + \rho) + \int_{\tau_1}^{\tau_2} \zeta(\hbar) e^{-\hbar} d\hbar, \\ Q_1 &= \varpi_1 g_1 + \rho g_2 - \int_{\tau_1}^{\tau_2} \zeta(\hbar) \hbar e^{-\hbar} \int_0^1 g_6(x,\tau,\hbar) e^{\hbar\tau} d\tau d\hbar, \\ Q_2 &= \mu(g_4 + g_3), \\ Q_3 &= c g_5 + \delta g_{1x}. \end{aligned} \tag{3.19}$$

Multiplying $(3.18)_1$, $(3.18)_2$, $(3.18)_3$ respectively by $\tilde{\nu}$, $\tilde{\varphi} \in \hat{H}^1(0,\Gamma)$ and $\tilde{\theta} \in H_0^1(0,\Gamma)$, and using integration by parts while considering the boundary conditions, we find

$$\begin{cases} \alpha \int_0^\Gamma \nu_x\tilde{\nu}_x dx - \gamma\beta \int_0^\Gamma \varphi_x\tilde{\nu}_x dx + \delta \int_0^\Gamma \theta_x\tilde{\nu} dx + \varpi_1 \int_0^\Gamma \nu\tilde{\nu} dx = \int_0^\Gamma Q_1\tilde{\nu} dx, \\ -\gamma\beta \int_0^\Gamma \nu_x\tilde{\varphi}_x dx + \beta \int_0^\Gamma \varphi_x\tilde{\varphi}_x dx + \mu \int_0^\Gamma \varphi\tilde{\varphi} dx = \int_0^\Gamma Q_2\tilde{\varphi} dx, \\ c \int_0^\Gamma \theta\tilde{\theta} dx + \kappa \int_0^\Gamma \theta_x\tilde{\theta}_x dx - \delta \int_0^\Gamma \nu\tilde{\theta}_x dx = \int_0^\Gamma Q_3\tilde{\theta} dx. \end{cases} \tag{3.20}$$

Consequently, problem (3.20) is equivalent to the following variational problem

$$a_1\left((\nu,\varphi,\theta), \left(\tilde{\nu},\tilde{\varphi},\tilde{\theta}\right)\right) = b_1\left(\tilde{\nu},\tilde{\varphi},\tilde{\theta}\right). \tag{3.21}$$

Where $a_1 : \left[\hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times H_0^1(0,\Gamma)\right]^2 \to \mathbb{R}$ is the bilinear form defined as follows

$$\begin{aligned} a_1\left((\nu,\varphi,\theta), \left(\tilde{\nu},\tilde{\varphi},\tilde{\theta}\right)\right) = &\ \alpha \int_0^\Gamma \nu_x\tilde{\nu}_x dx + \beta \int_0^\Gamma \varphi_x\tilde{\varphi}_x dx - \gamma\beta \int_0^\Gamma \varphi_x\tilde{\nu}_x dx - \gamma\beta \int_0^\Gamma \nu_x\tilde{\varphi}_x dx \\ &+ \varpi_1 \int_0^\Gamma \nu\tilde{\nu} dx + \mu \int_0^\Gamma \varphi\tilde{\varphi} dx + \delta \int_0^\Gamma \theta_x\tilde{\nu} dx + c \int_0^\Gamma \theta\tilde{\theta} dx \\ &+ \kappa \int_0^\Gamma \theta_x\tilde{\theta}_x dx - \delta \int_0^\Gamma \nu\tilde{\theta}_x dx, \end{aligned} \tag{3.22}$$

$b_1 : \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times H_0^1(0,\Gamma) \to \mathbb{R}$ is the linear form given by

$$b_1\left(\tilde{\nu}, \tilde{\varphi}, \tilde{\theta}\right) = \int_0^\Gamma Q_1 \tilde{\nu} dx + \int_0^\Gamma Q_2 \tilde{\varphi} dx + \int_0^\Gamma Q_3 \tilde{\theta} dx. \tag{3.23}$$

Now, for $\tilde{H}_1 := \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times H_0^1(0,\Gamma)$ equipped with this norm

$$\|(\nu, \varphi, \theta)\|_{\tilde{H}_1} = \left(\left\| \left(\nu_x - \frac{\gamma\beta}{\alpha} \varphi_x \right) \right\|_2^2 + \|\nu\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 \right)^{\frac{1}{2}}. \tag{3.24}$$

The continuity of the bilinear form a_1 and the linear form b_1 can be easily established. Additionally, we have

$$\begin{aligned} a_1\left((\nu, \varphi, \theta), (\nu, \varphi, \theta)\right) &= \alpha \int_0^\Gamma \left(\nu_x - \frac{\gamma\beta}{\alpha} \varphi_x \right)^2 dx + \left(\beta - \frac{(\gamma\beta)^2}{\alpha} \right) \int_0^\Gamma \varphi_x^2 dx + \varpi_1 \int_0^\Gamma \nu^2 dx \\ &+ \mu \int_0^\Gamma \varphi^2 dx + c \int_0^\Gamma \theta^2 dx + \kappa \int_0^\Gamma \theta_x^2 dx \geq \hat{m} \|(\nu, \varphi, \theta)\|_{\tilde{H}_1}^2, \end{aligned} \tag{3.25}$$

where

$$\hat{m} = \min\left(\alpha, \left(\beta - \frac{(\gamma\beta)^2}{\alpha} \right), \varpi_1, \mu, c, \kappa \right). \tag{3.26}$$

For all $\varpi_1 \geq 0$ the bilinear form a_1 is coercive. Therefore, by applying the Lax-Milgram theorem, it follows that the system (3.21) possesses a unique solution

$$(\nu, \varphi, \theta) \in \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma) \times H_0^1(0,\Gamma).$$

Therefore, through (3.15), we find

$$(u, q) \in \hat{H}^1(0,\Gamma) \times \hat{H}^1(0,\Gamma),$$

also, by substituting u in (3.16) and $(3.14)_6$, we obtain

$$Y, Y_\rho \in L^2\left((0,\Gamma) \times (0,1) \times (\tau_1, \tau_2)\right).$$

We consider the following cases $(\tilde{\nu}, 0, 0)$, $(0, \tilde{\varphi}, 0)$, $\left(0, 0, \tilde{\theta}\right)$ and we apply the derivative in the distribution sense, we find that the unique solution (ν, φ, θ) satisfies (3.18).

Now using $(3.18)_1$ and $(3.18)_2$, we get

$$\nu_{xx} = \frac{\varpi_1}{\alpha_1}\nu + \frac{\gamma\mu}{\alpha_1}\varphi + \frac{\delta}{\alpha_1}\theta_x - \frac{1}{\alpha_1}Q_1 - \frac{\gamma}{\alpha_1}Q_2 \in L^2(0,\Gamma) \Longrightarrow \nu \in H^2(0,\Gamma) \Longrightarrow \varphi \in H^2(0,\Gamma). \tag{3.27}$$

Multiplying $(3.18)_1$ by the function $\tilde{\nu} \in \hat{H}^1(0,\Gamma)$ and applying integration by parts, we obtain by using $(3.20)_1$

$$-\alpha\nu_x(\Gamma)\tilde{\nu}(\Gamma) + \gamma\beta\varphi_x(\Gamma)\tilde{\nu}(\Gamma) = 0 \qquad \forall \tilde{\nu} \in \hat{H}^1(0,\Gamma),$$

we select

$$\tilde{\nu}(x)=\frac{x}{\Gamma}, \tag{3.28}$$

then we get

$$\gamma\beta\varphi_x(\Gamma)=\alpha\nu_x(\Gamma). \tag{3.29}$$

Multiplying $(3.18)_2$ by $\tilde{\varphi}\in\hat{H}^1(0,\Gamma)$ and using integration by parts, we find by using $(3.20)_2$

$$\gamma\beta\nu_x(\Gamma)\tilde{\varphi}(\Gamma)-\beta\varphi_x(\Gamma)\tilde{\varphi}(\Gamma)=0, \qquad \forall\tilde{\varphi}\in\hat{H}^1(0,\Gamma),$$

we choose

$$\tilde{\varphi}(x)=\frac{x}{\Gamma},$$

then we get

$$\gamma\beta\nu_x(\Gamma)-\beta\varphi_x(\Gamma)=0, \tag{3.30}$$

by utilizing equation (3.29) in equation (3.30), we obtain

$$\nu_x(\Gamma)=\varphi_x(\Gamma)=0. \tag{3.31}$$

Through the results we obtained in (3.27) and (3.31), we have

$$\nu,\varphi\in\hat{H}^2(0,\Gamma),$$

and by $(3.18)_3$, we obtain

$$\theta_{xx}=-\frac{1}{\kappa}(Q_3-c\theta-\delta\nu_x)\in L^2(0,\Gamma).$$

Consequentially, the operator $(I-B)$ is surjective.

Hence, B is a maximal dissipative operator, then we can utilize the Hille-Yosida theorem and get the well-posedness result of a solution for the problem (3.5). ■

3.4 Exponential stability

In this section, we will state and provide the proofs of the necessary technical lemmas that are required for establishing the proof of our stability result.

Lemma 3.1 *Let (ν,φ,θ,Y) be a solution of (3.3)-(3.4), in that case, the expression of energy $E(t)$ defined as follows*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^\Gamma \Big(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta(\gamma\nu_x-\varphi_x)^2+c\theta^2 \\ &\quad +\int_0^1\int_{\tau_1}^{\tau_2}\hbar|\zeta(\hbar)|Y^2(x,\rho,t,\hbar)\,d\hbar d\rho\Big)dx, \end{aligned} \tag{3.32}$$

and satisfies

$$\frac{d}{dt}E(t)\le-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,d\hbar\right)\int_0^\Gamma\nu_t^2dx-\kappa\int_0^\Gamma\theta_x^2dx. \tag{3.33}$$

Proof. Multiplying the first equation of (3.3) by ν_t, the second equation by φ_t and the third by θ, integrating over the interval $(0,\Gamma)$, with respect to x, we get

$$\begin{aligned}
&\rho\frac{d}{2dt}\int_0^\Gamma \nu_t^2dx+\mu\frac{d}{2dt}\int_0^\Gamma \varphi_t^2dx+\frac{1}{2}\frac{d}{dt}\int_0^\Gamma \beta\left(\gamma\nu_x-\varphi_x\right)^2dx\\
&+\alpha_1\frac{d}{2dt}\int_0^\Gamma \nu_x^2dx+c\frac{d}{2dt}\int_0^\Gamma \theta^2dx+\kappa\int_0^\Gamma \theta_x^2dx+\mu_1\int_0^\Gamma \nu_t^2dx\\
&+\int_0^\Gamma \nu_t\int_{\tau_1}^{\tau_2}\zeta\left(\hbar\right)Y\left(x,1,t,\hbar\right)d\hbar dx\;=\;0.
\end{aligned}\tag{3.34}$$

Next, multiplying equation $(3.3)_4$ by $|\zeta(\hbar)|\,Y(x,\rho,t,\hbar)$ and integrating over $(0,\Gamma)\times(0,1)\times(\tau_1,\tau_2)$ with respect to x, ρ and $\hbar$, we find

$$\begin{aligned}
&\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar\left|\zeta\left(\hbar\right)\right|Y\left(x,\rho,t,\hbar\right)Y_t\left(x,\rho,t,\hbar\right)d\hbar d\rho dx\\
&+\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|Y\left(x,\rho,t,\hbar\right)Y_\rho\left(x,\rho,t,\hbar\right)d\hbar d\rho dx\;=\;0,
\end{aligned}\tag{3.35}$$

then we obtain

$$\begin{aligned}
&\frac{d}{2dt}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2}\hbar\left|\zeta\left(\hbar\right)\right|Y^2\left(x,\rho,t,\hbar\right)d\hbar d\rho dx\\
&+\frac{1}{2}\int_0^\Gamma\int_0^1\frac{d}{d\rho}\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|Y^2\left(x,\rho,t,\hbar\right)d\hbar d\rho dx\;=\;0,
\end{aligned}\tag{3.36}$$

because

$$\begin{aligned}
\frac{1}{2}\int_0^\Gamma\int_0^1\frac{d}{d\rho}\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|Y^2\left(x,\rho,t,\hbar\right)d\hbar d\rho dx&=\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|Y^2\left(x,1,t,\hbar\right)d\hbar dx\\
&-\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|\nu_t^2d\hbar dx,
\end{aligned}\tag{3.37}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\Gamma\left(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta\left(\gamma\nu_x-\varphi_x\right)^2+c\theta^2\right.\\
&\left.+\int_0^1\int_{\tau_1}^{\tau_2}\hbar\left|\zeta\left(\hbar\right)\right|Y^2\left(x,\rho,t,\hbar\right)d\hbar d\rho\right)dx\;=\;-\kappa\int_0^\Gamma\theta_x^2dx-\mu_1\int_0^\Gamma\nu_t^2dx\\
&-\int_0^\Gamma\nu_t\int_{\tau_1}^{\tau_2}\zeta\left(\hbar\right)Y\left(x,1,t,\hbar\right)d\hbar dx-\frac{1}{2}\int_0^\Gamma\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|Y^2\left(x,1,t,\hbar\right)d\hbar dx\\
&+\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\zeta\left(\hbar\right)\right|d\hbar\int_0^\Gamma\nu_t^2dx.
\end{aligned}\tag{3.38}$$

3.4. Exponential stability

By applying Young's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}-\int_0^\Gamma \nu_t \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx &\le \int_0^\Gamma \int_{\tau_1}^{\tau_2} |\nu_t| |\zeta(\hbar)|^{\frac{1}{2}} |\zeta(\hbar)|^{\frac{1}{2}} |Y(x,1,t,\hbar)|\, d\hbar dx \\ &\le \frac{1}{2}\int_0^\Gamma \nu_t^2 dx \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar \\ &+ \frac{1}{2}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx, \end{aligned} \tag{3.39}$$

by employing the inequality (3.39) in (3.38), we get

$$\frac{d}{dt}E(t) \le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) \int_0^\Gamma \nu_t^2 dx - \kappa \int_0^\Gamma \theta_x^2 dx,$$

also, by using (2.5), we obtain

$$\frac{d}{dt}E(t) \le 0.$$

■

Lemma 3.2 *Let* $(\nu, \varphi, \theta, Y)$ *satisfies (3.3)-(3.4) then the functional*

$$I_1(t) = \rho \int_0^\Gamma \nu_t \nu dx + \gamma\mu \int_0^\Gamma \varphi_t \nu dx + \frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx, \forall t \ge 0,$$

satisfies for any positive constant ε_1

$$\begin{aligned} I_1'(t) \le& -\frac{\alpha_1}{2}\int_0^\Gamma \nu_x^2 dx + \left(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\right)\int_0^\Gamma \nu_t^2 dx + \varepsilon_1 \int_0^\Gamma \varphi_t^2 dx \\ &+ \frac{\delta^2 c_0}{\alpha_1}\int_0^\Gamma \theta_x^2 dx + \frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \tag{3.40}$$

Proof. By multiplying equation $(3.3)_1$ by ν and integrating with respect to x in $(0,\Gamma)$, we get

$$\begin{aligned} &\frac{d}{dt}\rho \int_0^\Gamma \nu_t \nu dx - \rho \int_0^\Gamma \nu_t^2 dx + \alpha_1 \int_0^\Gamma \nu_x^2 dx + \gamma \int_0^\Gamma (\beta\varphi_{xx} - \gamma\beta\nu_{xx})\, \nu dx \\ &+ \delta \int_0^\Gamma \theta_x \nu dx + \frac{d}{dt}\frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx + \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx = 0. \end{aligned} \tag{3.41}$$

Furthermore, by employing equation $(3.3)_2$, we obtain

$$\begin{aligned} &\frac{d}{dt}\rho \int_0^\Gamma \nu_t \nu dx - \rho \int_0^\Gamma \nu_t^2 dx + \alpha_1 \int_0^\Gamma \nu_x^2 dx + \gamma\mu \int_0^\Gamma \varphi_{tt} \nu dx \\ &+ \delta \int_0^\Gamma \theta_x \nu dx + \frac{d}{dt}\frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx + \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx = 0, \end{aligned} \tag{3.42}$$

(3.42), can be written as follows

$$\frac{d}{dt}\left(\rho\int_0^\Gamma \nu_t\nu dx+\gamma\mu\int_0^\Gamma \varphi_t\nu dx+\frac{\mu_1}{2}\int_0^\Gamma \nu^2 dx\right)=$$
$$\rho\int_0^\Gamma \nu_t^2 dx-\alpha_1\int_0^\Gamma \nu_x^2 dx+\gamma\mu\int_0^\Gamma \varphi_t\nu_t dx-\delta\int_0^\Gamma \theta_x\nu dx-\int_0^\Gamma \nu\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx. \tag{3.43}$$

By utilizing Young's, Poincaré's and Cauchy-Schwarz inequalities, we obtain the following results for any $\varepsilon_1>0$

$$\gamma\mu\int_0^\Gamma \varphi_t\nu_t dx\le\varepsilon_1\int_0^\Gamma \varphi_t^2 dx+\frac{(\gamma\mu)^2}{4\varepsilon_1}\int_0^\Gamma \nu_t^2 dx, \tag{3.44}$$

$$-\int_0^\Gamma \nu\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx$$
$$\le\frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx+\frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx, \tag{3.45}$$

$$-\delta\int_0^\Gamma \theta_x\nu dx\le\frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx+\frac{\delta^2c_0}{\alpha_1}\int_0^\Gamma \theta_x^2 dx, \tag{3.46}$$

by using (3.44), (3.45) and (3.46) in (3.43), we get (3.40). ■

Lemma 3.3 *Let (ν,φ,θ,Y) be the solution of system (3.3)-(3.4) then the functional*

$$I_2(t)=\mu\int_0^\Gamma \varphi_t\varphi dx+\rho\int_0^\Gamma \nu_t\nu dx,$$

satisfies

$$I_2'(t)\le-\beta\int_0^\Gamma(\gamma\nu_x-\varphi_x)^2 dx+\mu\int_0^\Gamma \varphi_t^2 dx+\left(\rho+\frac{c_0\mu_1^2}{2\alpha_1}\right)\int_0^\Gamma \nu_t^2 dx$$
$$+\frac{\delta^2c_0}{\alpha_1}\int_0^\Gamma \theta_x^2 dx+\frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx. \tag{3.47}$$

Proof. By multiplying the first equation of (3.3) by ν, we obtain

$$\rho\frac{d}{dt}\int_0^\Gamma \nu_t\nu dx-\rho\int_0^\Gamma \nu_t^2 dx+\alpha_1\int_0^\Gamma \nu_x^2 dx-\gamma\beta\int_0^\Gamma \varphi_x\nu_x dx+\delta\int_0^\Gamma \theta_x\nu dx$$
$$+\mu_1\int_0^\Gamma \nu_t\nu dx+\int_0^\Gamma \nu\int_{\tau_1}^{\tau_2}\zeta(\hbar)Y(x,1,t,\hbar)\,d\hbar dx+\gamma^2\beta\int_0^\Gamma \nu_x^2 dx=0. \tag{3.48}$$

Furthermore, by multiplying equation $(3.3)_2$ by φ, we obtain

$$\mu\frac{d}{dt}\int_0^\Gamma \varphi_t\varphi dx-\mu\int_0^\Gamma \varphi_t^2 dx+\beta\int_0^\Gamma \varphi_x^2 dx-\gamma\beta\int_0^\Gamma \nu_x\varphi_x dx=0, \tag{3.49}$$

adding (3.48) to (3.49) gives us

$$\left\{ \begin{array}{c} I_2'(t) = \mu \int_0^\Gamma \varphi_t^2 dx - \beta \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx + \rho \int_0^\Gamma \nu_t^2 dx \\ -\alpha_1 \int_0^\Gamma \nu_x^2 dx - \delta \int_0^\Gamma \theta_x \nu dx - \mu_1 \int_0^\Gamma \nu_t \nu dx \\ - \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar) d\hbar dx. \end{array} \right. \tag{3.50}$$

By using Young's, Poincaré's and Cauchy-Schwarz inequalities, we obtain

$$-\mu_1 \int_0^\Gamma \nu_t \nu dx \leq \frac{\alpha_1}{2} \int_0^\Gamma \nu_x^2 dx + \frac{c_0 \mu_1^2}{2\alpha_1} \int_0^\Gamma \nu_t^2 dx, \tag{3.51}$$

$$-\delta \int_0^\Gamma \theta_x \nu dx \leq \frac{\alpha_1}{4} \int_0^\Gamma \nu_x^2 dx + \frac{\delta^2 c_0}{\alpha_1} \int_0^\Gamma \theta_x^2 dx, \tag{3.52}$$

and

$$\begin{aligned} & - \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar) d\hbar dx \\ & \leq \frac{\alpha_1}{4} \int_0^\Gamma \nu_x^2 dx + \frac{c_0 \mu_1}{\alpha_1} \int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar) d\hbar dx. \end{aligned} \tag{3.53}$$

By utilizing (3.51), (3.52) and (3.53) in (3.50), we get (3.47). ∎

Lemma 3.4 *Let $(\nu, \varphi, \theta, Y)$ satisfies (3.3)-(3.4) then the functional*

$$I_3(t) = \rho \int_0^\Gamma \nu_t (\gamma\nu - \varphi) dx + \gamma\mu \int_0^\Gamma \varphi_t (\gamma\nu - \varphi) dx,$$

satisfies for any $\varepsilon_2, \ \varepsilon_3, \ \varepsilon_4, \ \varepsilon_5 > 0$

$$\begin{aligned} \frac{d}{dt} I_3(t) \leq & -\frac{\gamma\mu}{2} \int_0^\Gamma \varphi_t^2 dx + (\varepsilon_2 + \varepsilon_3 c_0 + \varepsilon_4 c_0 + \varepsilon_5 c_0) \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ & + \left(\rho\gamma + \frac{\mu_1^2}{4\varepsilon_3} + \frac{\varkappa^2}{2\gamma\mu}\right) \int_0^\Gamma \nu_t^2 dx + \frac{\alpha_1^2}{4\varepsilon_2} \int_0^\Gamma \nu_x^2 dx \\ & + \frac{\delta^2}{4\varepsilon_5} \int_0^\Gamma \theta_x^2 dx + \frac{\mu_1}{4\varepsilon_4} \int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar) d\hbar dx. \end{aligned} \tag{3.54}$$

Proof. *By multiplying the first equation of (3.3) by $\gamma\nu$, the second by $-\gamma\varphi$, we obtain*

$$\begin{aligned} \gamma\rho \frac{d}{dt} \int_0^\Gamma \nu_t \nu dx = & \ \gamma\rho \int_0^\Gamma \nu_t^2 dx - \alpha\gamma \int_0^\Gamma \nu_x^2 dx + \gamma^2 \beta \int_0^\Gamma \varphi_x \nu_x dx \\ & -\gamma\delta \int_0^\Gamma \theta_x \nu dx - \gamma\mu_1 \int_0^\Gamma \nu_t \nu dx \\ & - \int_0^\Gamma \gamma\nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar) d\hbar dx, \end{aligned} \tag{3.55}$$

$$-\gamma\mu\frac{d}{dt}\int_0^\Gamma \varphi_t\varphi dx = -\gamma\mu\int_0^\Gamma \varphi_t^2 dx + \gamma\beta\int_0^\Gamma \varphi_x^2 dx - \gamma^2\beta\int_0^\Gamma \nu_x\varphi_x dx, \tag{3.56}$$

again, by multiplying the first equation of (3.3) by $-\varphi$, the second by $\gamma^2\nu$, we get

$$\begin{aligned} -\rho\frac{d}{dt}\int_0^\Gamma \nu_t\varphi dx &= -\rho\int_0^\Gamma \nu_t\varphi_t dx + \alpha\int_0^\Gamma \nu_x\varphi_x dx - \gamma\beta\int_0^\Gamma \varphi_x^2 dx \\ &+\delta\int_0^\Gamma \theta_x\varphi dx + \mu_1\int_0^\Gamma \nu_t\varphi dx \\ &+\int_0^\Gamma \varphi\int_{\tau_1}^{\tau_2}\zeta(\hbar)\,Y(x,1,t,\hbar)\,d\hbar dx, \end{aligned} \tag{3.57}$$

$$\mu\gamma^2\frac{d}{dt}\int_0^\Gamma \varphi_t\nu dx = \mu\gamma^2\int_0^\Gamma \varphi_t\nu_t dx - \gamma^2\beta\int_0^\Gamma \varphi_x\nu_x dx + \gamma^3\beta\int_0^\Gamma \nu_x^2 dx. \tag{3.58}$$

By summing (3.55), (3.56), (3.57) and (3.58) together, we get

$$\begin{aligned} \frac{d}{dt}I_3(t) = &-\gamma\mu\int_0^\Gamma \varphi_t^2 dx - \alpha_1\int_0^\Gamma \nu_x(\gamma\nu_x-\varphi_x)\,dx - \mu_1\int_0^\Gamma \nu_t(\gamma\nu-\varphi)\,dx \\ &+\rho\gamma\int_0^\Gamma \nu_t^2 dx + \underbrace{\left(\gamma^2\mu-\rho\right)}_{\varkappa}\int_0^\Gamma \nu_t\varphi_t dx - \delta\int_0^\Gamma \theta_x(\gamma\nu-\varphi)\,dx \\ &-\int_0^\Gamma(\gamma\nu-\varphi)\int_{\tau_1}^{\tau_2}\zeta(\hbar)\,Y(x,1,t,\hbar)\,d\hbar dx, \end{aligned} \tag{3.59}$$

by utilizing Young's, Poincaré's, and Cauchy-Schwarz inequalities, we get

$$-\alpha_1\int_0^\Gamma \nu_x(\gamma\nu_x-\varphi_x)\,dx \le \varepsilon_2\int_0^\Gamma(\gamma\nu_x-\varphi_x)^2 dx + \frac{\alpha_1^2}{4\varepsilon_2}\int_0^\Gamma \nu_x^2 dx, \forall\varepsilon_2>0, \tag{3.60}$$

and

$$-\mu_1\int_0^\Gamma \nu_t(\gamma\nu-\varphi)\,dx \le \varepsilon_3 c_0\int_0^\Gamma(\gamma\nu_x-\varphi_x)^2 dx + \frac{\mu_1^2}{4\varepsilon_3}\int_0^\Gamma \nu_t^2 dx, \forall\varepsilon_3>0, \tag{3.61}$$

$$\begin{aligned} -\int_0^\Gamma(\gamma\nu-\varphi)\int_{\tau_1}^{\tau_2}\zeta(\hbar)\,Y(x,1,t,\hbar)\,d\hbar dx \le \varepsilon_4 c_0\int_0^\Gamma(\gamma\nu_x-\varphi_x)^2 dx \\ +\frac{\mu_1}{4\varepsilon_4}\int_0^\Gamma\int_{\tau_1}^{\tau_2}|\zeta(\hbar)|\,Y^2(x,1,t,\hbar)\,d\hbar dx, \forall\varepsilon_4>0, \end{aligned} \tag{3.62}$$

$$\varkappa\int_0^\Gamma \nu_t\varphi_t dx \le \frac{\gamma\mu}{2}\int_0^\Gamma \varphi_t^2 dx + \frac{\varkappa^2}{2\gamma\mu}\int_0^\Gamma \nu_t^2 dx. \tag{3.63}$$

$$-\delta\int_0^\Gamma \theta_x(\gamma\nu-\varphi)\,dx \le \varepsilon_5 c_0\int_0^\Gamma(\gamma\nu_x-\varphi_x)^2 dx + \frac{\delta^2}{4\varepsilon_5}\int_0^\Gamma \theta_x^2 dx. \tag{3.64}$$

Using the inequalities (3.60) to (3.64) in the relationship (3.59), we find (3.54). ■

3.4. Exponential stability

Lemma 3.5 *Let $(\nu, \varphi, \theta, Y)$ be a solution of system (3.3)-(3.4) then the functional*

$$I_4(t) := \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar\rho} |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) d\hbar d\rho dx, \tag{3.65}$$

satisfies

$$\begin{aligned} I_4'(t) &\leq -e^{-\tau_2} \int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x, 1, t, \hbar) d\hbar dx + \mu_1 \int_0^\Gamma \nu_t^2 dx \\ &- e^{-\tau_2} \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) d\hbar d\rho dx. \end{aligned} \tag{3.66}$$

Proof. By multiplying the fourth equation of (3.3) by $e^{-\hbar\rho} |\zeta(\hbar)| Y(x, \rho, t, \hbar)$, we have

$$\hbar e^{-\hbar\rho} |\zeta(\hbar)| Y(x, \rho, t, \hbar) Y_t(x, \rho, t, \hbar) + e^{-\hbar\rho} |\zeta(\hbar)| Y(x, \rho, t, \hbar) Y_\rho(x, \rho, t, \hbar) = 0, \tag{3.67}$$

by integrating with respect to x, ρ and $\hbar$ over $(0, \Gamma) \times (0, 1) \times (\tau_1, \tau_2)$ in (3.67), we get

$$\begin{aligned} &\frac{d}{dt} \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar\rho} |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) d\hbar d\rho dx \\ &+2 \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\hbar\rho} |\zeta(\hbar)| Y(x, \rho, t, \hbar) Y_\rho(x, \rho, t, \hbar) d\hbar d\rho dx = 0, \end{aligned} \tag{3.68}$$

so we find

$$\begin{aligned} \frac{d}{dt} I_4(t) &= -\int_0^\Gamma \int_{\tau_1}^{\tau_2} e^{-\hbar} |\zeta(\hbar)| Y^2(x, 1, t, \hbar) d\hbar dx + \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| d\hbar \int_0^\Gamma \nu_t^2 dx \\ &- \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar\rho} |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) d\hbar d\rho dx, \end{aligned} \tag{3.69}$$

by using the following relation $\quad e^{-\hbar} \leq e^{-\hbar\rho} \leq 1, \ \forall \ 0 \leq \rho \leq 1$, we get

$$\begin{aligned} \frac{d}{dt} I_4(t) \leq &-\int_0^\Gamma \int_{\tau_1}^{\tau_2} e^{-\hbar} |\zeta(\hbar)| Y^2(x, 1, t, \hbar) d\hbar dx + \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| d\hbar \int_0^\Gamma \nu_t^2 dx \\ &- \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar} |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) d\hbar d\rho dx. \end{aligned} \tag{3.70}$$

Since $\left(-e^{-\hbar}\right)' = e^{-\hbar} \geq 0$, we conclude that $-e^{-\hbar} \leq -e^{-\tau_2}$, $\forall \hbar \in (\tau_1, \tau_2)$, then we obtain directly (3.66). ■

Now, we define the Lyapunov functional as follows

$$L(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t),$$

where $N, \ N_1, \ N_2, \ N_3, \ N_4$ are positive constants to be determined later.

3.4. Exponential stability

Theorem 3.2 *Let $(\nu, \varphi, \theta, Y)$ be the solution of system (3.3)-(3.4). Then there exist two positive constants $c_1, c_2 > 0$ satisfies*

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \ \forall t \geq 0. \tag{3.71}$$

Proof. *Let*

$$\Im(t) = L(t) - NE(t) = \sum_{i=1}^{i=4} N_i I_i(t), \tag{3.72}$$

then

$$\begin{aligned} |\Im(t)| &= |L(t) - NE(t)| \\ &\leq N_1 \left(\rho \int_0^\Gamma |\nu_t \nu| \, dx + \gamma\mu \int_0^\Gamma |\varphi_t \nu| \, dx + \frac{\mu_1}{2} \int_0^\Gamma \nu^2 dx \right) \\ &+ N_2 \left(\mu \int_0^\Gamma |\varphi_t \varphi| \, dx + \rho \int_0^\Gamma |\nu_t \nu| \, dx \right) \\ &+ N_3 \left(\rho \int_0^\Gamma |\nu_t (\gamma\nu - \varphi)| \, dx + \gamma\mu \int_0^\Gamma |\varphi_t (\gamma\nu - \varphi)| \, dx \right) \\ &+ N_4 \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar e^{-\hbar\rho} |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) \, d\hbar d\rho dx. \end{aligned} \tag{3.73}$$

By utilizing Young's and Poincaré's inequalities in (3.73), we obtain the following inequality for any $\varepsilon > 0$:

$$\begin{aligned} |\Im(t)| \leq& \underbrace{\left(\frac{N_1 \rho^2}{4\varepsilon} + N_2 \rho^2 \varepsilon + \frac{N_3 \rho^2}{4\varepsilon} \right)}_{\theta_1} \int_0^\Gamma \nu_t^2 dx \\ &+ \underbrace{\left(N_1 \frac{(\gamma\mu)^2}{4\varepsilon} + N_2 \frac{\mu^2}{4\varepsilon} + N_3 \frac{(\gamma\mu)^2}{4\varepsilon} \right)}_{\theta_2} \int_0^\Gamma \varphi_t^2 dx \\ &+ \underbrace{\left(N_1 \left(2\varepsilon c_0 + \frac{c_0 \mu_1}{2} \right) + N_2 \left(2\varepsilon\gamma^2 c_0 + \frac{c_0}{4\varepsilon} \right) \right)}_{\theta_3} \int_0^\Gamma \nu_x^2 dx \\ &+ \underbrace{(2N_2 \varepsilon c_0 + 2N_3 \varepsilon c_0)}_{\theta_4} \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ &+ N_4 \int_0^\Gamma \int_0^1 \int_{\tau_1}^{\tau_2} \hbar |\zeta(\hbar)| Y^2(x, \rho, t, \hbar) \, d\hbar d\rho dx, \end{aligned}$$

for each constant value of ε, there is a positive constant C such that

$$|\Im(t)| \leq CE(t),$$

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where

$$C = \max\left(\frac{2}{\rho}\theta_1, \frac{2}{\mu}\theta_2, \frac{2}{\alpha_1}\theta_3, \frac{2}{\beta}\theta_4, 2N_4\right),$$

then we obtain

$$\underbrace{(-C+N)}_{c_1}E(t) \leq L(t) \leq \underbrace{(C+N)}_{c_2}E(t).$$

■

Theorem 3.3 *Let $(\nu, \varphi, \theta, Y)$ be a solution of system (3.3)-(3.4). Then there exist two positive constants k and λ, such that the following inequality is satisfied*

$$E(t) \leq ke^{-\lambda t}, \qquad \forall t \geq 0. \tag{3.74}$$

Proof. By utilizing the previous lemmas, we get the following result

$$\begin{aligned}
L'(t) \leq & -\left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - N_1\left(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right. \\
& \left. -N_3\left(\rho\gamma + \frac{\mu_1^2}{4\varepsilon_3} + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right)\int_0^\Gamma \nu_t^2 dx \\
& -\left(N_3\frac{\gamma\mu}{2} - N_1\varepsilon_1 - N_2\mu\right)\int_0^\Gamma \varphi_t^2 dx - \left(N_1\frac{\alpha_1}{2} - N_3\frac{\alpha_1^2}{4\varepsilon_2}\right)\int_0^\Gamma \nu_x^2 dx \\
& -\left(N_2\beta - N_3\left(\varepsilon_2 + \varepsilon_3 c_0 + \varepsilon_4 c_0 + \varepsilon_5 c_0\right)\right)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2\, dx \\
& -\left(N\kappa - N_1\frac{\delta^2 c_0}{\alpha_1} - N_2\frac{\delta^2 c_0}{\alpha_1} - N_3\frac{\delta^2}{4\varepsilon_5}\right)\int_0^\Gamma \theta_x^2 dx \\
& -\left(N_4 e^{-\tau_2} - N_1\frac{c_0\mu_1}{\alpha_1} - N_2\frac{c_0\mu_1}{\alpha_1} - N_3\frac{\mu_1}{4\varepsilon_4}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx \\
& - N_4 e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\, |\zeta(\hbar)|\, Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx.
\end{aligned}$$

We select the following values as follows

$$\varepsilon_1 = \frac{1}{N_1},\ \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \frac{1}{N_3},$$

we get

$$
\begin{aligned}
L'(t) \leq & -\left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - N_1\left(\rho + \frac{N_1(\gamma\mu)^2}{4}\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right. \\
& \left. -N_3\left(\rho\gamma + \frac{N_3\mu_1^2}{4} + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right)\int_0^\Gamma \nu_t^2 dx \\
& -\left(N_3\frac{\gamma\mu}{2} - 1 - N_2\mu\right)\int_0^\Gamma \varphi_t^2 dx - \left(N_1\frac{\alpha_1}{2} - N_3^2\frac{\alpha_1^2}{4}\right)\int_0^\Gamma \nu_x^2 dx \\
& -\left(N_2\beta - (1+3c_0)\right)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2\, dx \\
& -\left(N\kappa - N_1\frac{\delta^2 c_0}{\alpha_1} - N_2\frac{\delta^2 c_0}{\alpha_1} - N_3^2\frac{\delta^2}{4}\right)\int_0^\Gamma \theta_x^2 dx \\
& -\left(N_4 e^{-\tau_2} - N_1\frac{c_0\mu_1}{\alpha_1} - N_2\frac{c_0\mu_1}{\alpha_1} - N_3^2\frac{\mu_1}{4}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx \\
& - N_4 e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\, |\zeta(\hbar)|\, Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx.
\end{aligned} \tag{3.75}
$$

We select N_2 as the first option in (3.75) until it becomes

$$N_2\beta - (1+3c_0) > 0.$$

We also choose N_3 until it becomes

$$\frac{\gamma\mu N_3}{2} - N_2\mu > 1.$$

Now, we select N_1 to be sufficiently large such that

$$N_1\frac{\alpha_1}{2} - N_3^2\frac{\alpha_1^2}{4} > 0.$$

Additionally, we choose N_4 large enough so that

$$N_4 e^{-\tau_2} - N_1\frac{c_0\mu_1}{\alpha_1} - N_2\frac{c_0\mu_1}{\alpha_1} - N_3^2\frac{\mu_1}{4} > 0.$$

Lastly, we choose an exceptionally large value for N in order to ensure that

$$
\begin{aligned}
& \left(N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - N_1\left(\rho + \frac{N_1(\gamma\mu)^2}{4}\right) - N_2\left(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\right)\right. \\
& \left. -N_3\left(\rho\gamma + \frac{N_3\mu_1^2}{4} + \frac{\varkappa^2}{2\gamma\mu}\right) - N_4\mu_1\right) > 0.
\end{aligned}
$$

And

$$\left(N\kappa - N_1\frac{\delta^2 c_0}{\alpha_1} - N_2\frac{\delta^2 c_0}{\alpha_1} - N_3^2\frac{\delta^2}{4}\right) > 0.$$

3.4. Exponential stability

Since

$$-\left(N_4 e^{-\tau_2} - \frac{N_3^2 \mu_1}{4} - N_2 \frac{c_0 \mu_1}{\alpha_1} - N_1 \frac{c_0 \mu_1}{2\alpha_1}\right) \int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)| Y^2(x,1,t,\hbar)\, d\hbar dx \le 0.$$

By employing Poincaré's inequality, we obtain

$$L'(t) \le -mE(t).$$

By (3.71), we find

$$L'(t) \le -\frac{m}{c_2} L(t). \tag{3.76}$$

By integrating (3.76) over the interval $(0,t)$, we get

$$L(t) \le L(0)\, e^{-\frac{m}{c_2} t}.$$

When we use (3.71) once more, we get (3.74). ■

3.5 Exponential energy decay when the magnetic effects are neglected

By neglecting the magnetic effects, we can achieve the electrostatic and quasi-static cases. For a beam of a length Γ and thickness h, we consider the system of stretching motion subjected to a distributed delay term coupled with the parabolic equation governed by Fourier's law

$$\begin{cases} \rho\nu_{tt} - \alpha_1 \nu_{xx} + \delta\theta_x + \mu_1 \nu_t & \text{in } (0,\Gamma)\times(0,\infty), \\ + \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, \nu_t(x, t-\hbar)\, d\hbar = 0, & \\ c\theta_t - \kappa\theta_{xx} + \delta\nu_{tx} = 0, & \\ \nu(0,t) = \nu_x(\Gamma,t) = \theta(0,t) = \theta(\Gamma,t) = 0, & t \ge 0, \\ (\nu, \nu_t, \theta)(x,0) = (\nu_0, \nu_1, \theta_0)(x) & x \in (0,\Gamma) \\ \nu_t(x,-t) = f_0(x,-t), & t \in (0,\tau_2). \end{cases} \tag{3.77}$$

As in [67], we introduce the new variable

$$Y(x,\rho,t,\hbar) = \nu_t(x, t-\rho\hbar), \qquad x \in (0,\Gamma),\ \rho \in (0,1),\ \hbar \in (\tau_1,\tau_2),\ t \ge 0,$$

then we get

$$\hbar Y_t(x,\rho,t,\hbar) + Y_\rho(x,\rho,t,\hbar) = 0, \qquad x \in (0,\Gamma),\ \rho \in (0,1),\ \hbar \in (\tau_1,\tau_2),\ t \ge 0.$$

Consequently, the problem (3.77) rewritten as follows

$$\begin{cases} \rho\nu_{tt} - \alpha_1 \nu_{xx} + \delta\theta_x + \mu_1 \nu_t & (x,t) \in (0,\Gamma)\times(0,\infty), \\ + \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar = 0, & \\ c\theta_t - \kappa\theta_{xx} + \delta\nu_{tx} = 0, & \\ \hbar Y_t(x,\rho,t,\hbar) + Y_\rho(x,\rho,t,\hbar) = 0, & (\rho,\hbar) \in (0,1)\times(\tau_1,\tau_2), \end{cases} \tag{3.78}$$

with the given initial and boundary conditions

$$\begin{cases} \nu(0,t) = \nu_x(\Gamma,t) = \theta(0,t) = \theta(\Gamma,t) = 0, & t \geq 0, \\ (\nu, \nu_t, \theta)(x,0) = (\nu_0, \nu_1, \theta_0)(x) & x \in (0,\Gamma) \\ Y(x,\rho,0,\hbar) = f_0(x,\rho,\hbar). & (\rho,\hbar) \in (0,1) \times (0,\tau_2). \end{cases} \tag{3.79}$$

Moreover, the energy associated with the system is expressed as follows

$$\tilde{E}(t) = \frac{1}{2} \int_0^\Gamma \left(\rho\nu_t^2 + \alpha_1\nu_x^2 + c\theta^2 + \int_0^1 \int_{\tau_1}^{\tau_2} \hbar \left|\zeta(\hbar)\right| Y^2(x,\rho,t,\hbar)\, d\hbar d\rho \right) dx, \tag{3.80}$$

and satisfies

$$\frac{d}{dt}\tilde{E}(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \left|\zeta(\hbar)\right| d\hbar \right) \int_0^\Gamma \nu_t^2 dx - \kappa \int_0^\Gamma \theta_x^2 dx \quad \forall t \geq 0. \tag{3.81}$$

We now demonstrate that the system (3.78)-(3.79) is exponentially stable.

Lemma 3.6 *Let (ν, θ, Y) be a solution of the system (3.78)-(3.79). Then the functional defined as follows*

$$\tilde{K}_1(t) = \rho \int_0^\Gamma \nu_t \nu dx + \frac{\mu_1}{2} \int_0^\Gamma \nu^2 dx \quad \forall t \geq 0, \tag{3.82}$$

satisfies

$$\begin{aligned} \tilde{K}_1'(t) \;\leq\; & -\frac{\alpha_1}{2} \int_0^\Gamma \nu_x^2 dx + \rho \int_0^\Gamma \nu_t^2 dx + \frac{\delta^2 c_0}{\alpha_1} \int_0^\Gamma \theta_x^2 dx \\ & + \frac{c_0\mu_1}{\alpha_1} \int_0^\Gamma \int_{\tau_1}^{\tau_2} \left|\zeta(\hbar)\right| Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \tag{3.83}$$

Proof. By multiplying equation $(3.78)_1$ by ν and integrating with respect to x in $(0,\Gamma)$, we get the following expression

$$\begin{aligned} & \rho\frac{d}{dt} \int_0^\Gamma \nu_t \nu dx - \rho \int_0^\Gamma \nu_t^2 dx + \alpha_1 \int_0^\Gamma \nu_x^2 dx + \delta \int_0^\Gamma \theta_x \nu dx + \mu_1 \frac{d}{2dt} \int_0^\Gamma \nu^2 dx \\ & + \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar dx \;=\; 0, \end{aligned} \tag{3.84}$$

(3.84), satisfies the equation

$$\begin{aligned} & \frac{d}{dt}\left(\rho \int_0^\Gamma \nu_t \nu dx + \frac{\mu_1}{2} \int_0^\Gamma \nu^2 dx \right) \;=\; \\ & \rho \int_0^\Gamma \nu_t^2 dx - \alpha_1 \int_0^\Gamma \nu_x^2 dx - \delta \int_0^\Gamma \theta_x \nu dx - \int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar)\, Y(x,1,t,\hbar)\, d\hbar dx, \end{aligned} \tag{3.85}$$

by utilizing Young's, Poincaré's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} &-\int_0^\Gamma \nu \int_{\tau_1}^{\tau_2} \zeta(\hbar) Y(x,1,t,\hbar)\, d\hbar dx \\ &\le \frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx + \frac{c_0\mu_1}{\alpha_1}\int_0^\Gamma \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx, \end{aligned} \tag{3.86}$$

$$-\delta\int_0^\Gamma \theta_x \nu dx \le \frac{\alpha_1}{4}\int_0^\Gamma \nu_x^2 dx + \frac{\delta^2 c_0}{\alpha_1}\int_0^\Gamma \theta_x^2 dx. \tag{3.87}$$

Using the inequalities (3.86) and (3.87) in (3.85), we find (3.83). ■
We define the Lyapunov functional as follows

$$\tilde{\mathcal{L}}(t) = \tilde{N}\tilde{E}(t) + \tilde{K}_1(t) + \tilde{N}_4 I_4(t). \tag{3.88}$$

As stated in the theorem (3.2), it is evident that there exist two positive constants $\tilde{c}_1$ and $\tilde{c}_2 > 0$ that satisfy

$$\tilde{c}_1\tilde{E}(t) \le \tilde{\mathcal{L}}(t) \le \tilde{c}_2\tilde{E}(t), \ \forall t \ge 0. \tag{3.89}$$

Theorem 3.4 *Let (ν, θ, Y) solution of system (3.78)-(3.79), then there are two positive constants $\tilde{k}$ and $\tilde{\lambda}$, such that*

$$\tilde{E}(t) \le \tilde{k}e^{-\tilde{\lambda}t}, \qquad \forall t \ge 0. \tag{3.90}$$

Proof. Differentiating $\tilde{\mathcal{L}}(t)$ and exploiting (3.66)-(3.81)-(3.83), we get

$$\begin{aligned} \tilde{\mathcal{L}}'(t) \le\ & -\left(\tilde{N}\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - \tilde{N}_4\mu_1 - \rho\right)\int_0^\Gamma \nu_t^2 dx - \frac{\alpha_1}{2}\int_0^\Gamma \nu_x^2 dx \\ & - \left(\tilde{N}\kappa - \frac{\delta^2 c_0}{\alpha_1}\right)\int_0^\Gamma \theta_x^2 dx - \tilde{N}_4 e^{-\tau_2}\int_0^\Gamma\int_0^1\int_{\tau_1}^{\tau_2} \hbar\, |\zeta(\hbar)|\, Y^2(x,\rho,t,\hbar)\, d\hbar d\rho dx \\ & - \left(\tilde{N}_4 e^{-\tau_2} - \frac{c_0\mu_1}{\alpha_1}\right)\int_0^\Gamma\int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, Y^2(x,1,t,\hbar)\, d\hbar dx. \end{aligned} \tag{3.91}$$

We select $\tilde{N}_4$ so that

$$\left(\tilde{N}_4 e^{-\tau_2} - \frac{c_0\mu_1}{\alpha_1}\right) > 0.$$

Furthermore, we choose $\tilde{N}$ big enough so that

$$\left(\tilde{N}\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\zeta(\hbar)|\, d\hbar\right) - \tilde{N}_4\mu_1 - \rho\right) > 0.$$

And

$$\left(\tilde{N}\kappa - \frac{\delta^2 c_0}{\alpha_1}\right) > 0.$$

Using the equivalence between energy and the Lyapunov functional, we have completed the proof. ■

3.5. Exponential energy decay when the magnetic effects are neglected

CHAPTER 4

Global well-posedness and asymptotic stability of piezoelectric system with neutral delay time in the presence or absence of magnetic effects

4.1 Presentation of the problem

*I*n the investigations of piezoelectric beams, it is consistently observed that in studies of these beams with different types of boundary and distributed delays, there is a relationship between the coefficient of the delay term and the coefficient of the damping term. The question posed here is whether certain types of delays can lead to the stability of piezoelectric beams without any conditions between delay and damping coefficients or when the damping is disregarded.

In the present chapter, we consider the following initial boundary value problem for fully dynamic piezoelectric beams (the magnetic effects are not negligible) subject to a neutral delay. The system is written as

$$\begin{cases} \rho\left(\nu_t + \int_0^t h\left(t-s\right)\nu_t\left(s\right)ds\right)' - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} = 0, & \text{in } (0,\Gamma)\times(0,\infty), \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, & \\ \nu\left(0,t\right) = \alpha\nu_x\left(\Gamma,t\right) - \gamma\beta\varphi_x\left(\Gamma,t\right) = 0, & t \geq 0, \\ \varphi\left(0,t\right) = \varphi_x\left(\Gamma,t\right) - \gamma\nu_x\left(\Gamma,t\right) = 0, & \\ \nu\left(x,0\right) = \nu_0\left(x\right), \nu_t\left(x,0\right) = \nu_1\left(x\right), & x \in (0,\Gamma), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right), & \end{cases} \tag{4.1}$$

the initial data ν_0, φ_0, ν_1, φ_1 belong to an appropriate functional space, and the neutral delay is defined by the convolution term involving the kernel h.

4.2 Preliminaries

In this section, we introduce our assumptions regarding the kernel h and present some results that are essential for the subsequent sections.
The assumptions concerning the kernel h are as follows:

(H_1) The kernel h is a nonnegative function that is continuously differentiable and meets the following conditions:

$$\forall t \geq 0 \qquad h'(t) \leq 0, \qquad \bar{h} = \int_0^\infty h(s)\,ds < \infty,$$

(H_2) $\exp(\varsigma .)\, h(.) \in L^1(\mathbb{R}_+)$ for any $\varsigma > 0$.

Lemma 4.1 ([84]) *For any function $\nu \in C^1(\mathbb{R}_+; L^2(0,\Gamma))$ and any $h \in C^1[0,\infty)$, we have*

$$\begin{aligned} &\int_0^\Gamma \nu(t)\left(\int_0^t h(t-s)\,\nu_t(s)\,ds\right)dx \\ =\; & -\frac{1}{2}(h'\Box\nu)(t) + \frac{1}{2}\frac{d}{dt}\int_0^t h(t-s)\,\|\nu(s)\|^2\,ds \\ & + \frac{h(t)}{2}\|\nu\|^2 - h(t)\int_0^\Gamma \nu(0)\,\nu(t)\,dx, \;\; \forall t \geq 0. \end{aligned} \tag{4.2}$$

Where

$$(h\Box\nu)(t) = \int_0^t h(t-s)\parallel \nu(t) - \nu(s)\parallel^2 ds, \;\; \forall t \geq 0, \tag{4.3}$$

and $\|\|$ represents the norm in $L^2(0,\Gamma)$.

Theorem 4.1 (Aubin-Lions-Simon [12] (Page 102)) *Let $B_0 \subset B_1 \subset B_2$ represent three Banach spaces. We assume that*

1) *The embedding of B_1 into B_2 is continuous.*

2) *The embedding of B_0 into B_1 is compact.*

Let p and r be real numbers such that $1 \leq p, r \leq +\infty$. For any $T > 0$, we define the space $E_{p,r}$ as:

$$E_{p,r} = \left\{ \nu \in L^p(]0,T[\,, B_0), \qquad \frac{d\nu}{dt} \in L^r(]0,T[\,, B_2) \right\}.$$

We have the following properties:

1) *The compactness of the embedding of $E_{p,r}$ in $L^p(]0,T[, B_1)$ is guaranteed when p is finite.*

2) *In the case where $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0([0,T], B_1)$ remains compact.*

Remark 2.1 Note that because

$$\rho\left(\int_0^t h(t-s)\,\nu_t(s)\,ds\right)' = \rho h(t)\,\nu_t(0) + \rho\int_0^t h(s)\,\nu_{tt}(t-s)\,ds.$$

Then our problem (4.1) can be written as follows

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \rho h(t)\,\nu_t(0) + \rho\int_0^t h(s)\,\nu_{tt}(t-s)\,ds = 0, & \text{in } (0,\Gamma)\times(0,\infty), \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, & \\ \nu(0,t) = \alpha\nu_x(\Gamma,t) - \gamma\beta\varphi_x(\Gamma,t) = 0, & t\geq 0, \\ \varphi(0,t) = \varphi_x(\Gamma,t) - \gamma\nu_x(\Gamma,t) = 0, & \\ \nu(x,0) = \nu_0(x),\, \nu_t(x,0) = \nu_1(x), & x\in(0,\Gamma), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & \end{cases}$$

4.3 Global well-posedness

In this section, our aim is to establish the global well posedness of a solution for the system mentioned in reference (4.1). To achieve this, we employ the classical Faedo-Galerkin method. For more details about this method, we refer the reader to see [13, 17, 18, 39, 75]

Theorem 4.2 *Let* $(\nu_0,\nu_1,\varphi_0,\varphi_1)\in H = \left[\hat{H}^1(0,\Gamma)\times L^2(0,\Gamma)\right]^2$. *Then the system (4.1) possesses a unique global strong solution and satisfies*

$$\nu,\varphi\in C\left(\mathbb{R}_+,\ \tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)\right)\cap C^2\left(\mathbb{R}_+,\ \hat{H}^1(0,\Gamma)\right). \tag{4.4}$$

Proof. To prove this theorem, we will utilize the following four main steps

- **Step 1: Approximate Problem**

Let $\{\delta_j\}_{j\geq 1}$ be an orthogonal basis of $\tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)$ and $L^2(0,\Gamma)$. For any $n\geq 1$, let

$$M_n = span\{\delta_1,\delta_2,...,\delta_n\},$$

if the initial data $(\nu_0,\nu_1,\varphi_0,\varphi_1)\in H$, we aim to find functions $y_j^n,\ R_j^n$ in the space $C^2[0,T]$, such that the following approximations hold:

$$\begin{cases} \nu_n(x,t) = \sum_{j=1}^{j=n} y_j^n(t)\,\delta_j(x), \\ \varphi_n(x,t) = \sum_{j=1}^{j=n} R_j^n(t)\,\delta_j(x), \end{cases}$$

and satisfies the following approximate system

$$\begin{cases} \rho\nu_{ntt} + \rho\left(\int_0^t h(t-s)\,\nu_{nt}(s)\,ds\right)' - \alpha\nu_{nxx} + \gamma\beta\varphi_{nxx} = 0, \\ \mu\varphi_{ntt} - \beta\varphi_{nxx} + \gamma\beta\nu_{nxx} = 0, \end{cases} \tag{4.5}$$

with the initial conditions:

$$\begin{cases} (\nu_n, \nu_{nt})(x,0) = (\nu_0^n, \nu_1^n)(x), \\ (\varphi_n, \varphi_{nt})(x,0) = (\varphi_0^n, \varphi_1^n)(x), \end{cases} \tag{4.6}$$

which satisfies

$$\begin{cases} \nu_0^n \text{ converges strongly to } \nu_0 \text{ in } \hat{H}^1(0,\Gamma), \\ \nu_1^n \text{ converges strongly to } \nu_1 \text{in } L^2(0,\Gamma), \\ \varphi_0^n \text{ converges strongly to } \varphi_0 \text{ in } \hat{H}^1(0,\Gamma), \\ \varphi_1^n \text{ converges strongly to } \varphi_1 \text{ in } L^2(0,\Gamma). \end{cases} \tag{4.7}$$

By using (4.5), we get

$$\begin{cases} \rho \langle \nu_{ntt}, \delta_k \rangle_{L^2(0,\Gamma)} + \rho \left\langle \left(\int_0^t h(t-s)\nu_{nt}(s)\,ds \right)', \delta_k \right\rangle_{L^2(0,\Gamma)} \\ -\alpha \langle \nu_{nxx}, \delta_k \rangle_{L^2(0,\Gamma)} + \gamma\beta \langle \varphi_{nxx}, \delta_k \rangle_{L^2(0,\Gamma)} = 0, \\ \mu \langle \varphi_{ntt}, \delta_k \rangle_{L^2(0,\Gamma)} - \beta \langle \varphi_{nxx}, \delta_k \rangle_{L^2(0,\Gamma)} + \gamma\beta \langle \nu_{nxx}, \delta_k \rangle_{L^2(0,\Gamma)} = 0, \end{cases} \tag{4.8}$$

using the Caratheodory theorem for standard ordinary differential equations theory, system (4.8) has a solutions $\left(y_j^n,\ R_j^n\right)_{j=1,n} \in \left(C^2[0,t_n)\right)^2$. The first estimate below will guarantee that $t_n = T$, for any given $T > 0$.

- **Step 2: A priori estimate**

In this section, we will get two a priori estimates that are necessary to extend these solutions, as well as in the later part (Passage to limit).

A priori estimate I

For any $n \geq 1$, following integration by parts with respect to x on the interval $(0,\Gamma)$ in equation (4.8), we obtain:

$$\begin{cases} \rho \int_0^\Gamma \nu_{ntt}\delta_k dx + \rho \int_0^\Gamma \delta_k \left(\int_0^t h(t-s)\nu_{nt}(s)\,ds \right)' dx + \alpha \int_0^\Gamma \nu_{nx}\delta_{kx} dx \\ -\gamma\beta \int_0^\Gamma \varphi_{nx}\delta_{kx} dx = 0, \\ \mu \int_0^\Gamma \varphi_{ntt}\delta_k dx + \beta \int_0^\Gamma \varphi_{nx}\delta_{k_x} dx - \gamma\beta \int_0^\Gamma \nu_{nx}\delta_{kx} dx = 0, \qquad \forall k = 1,..,n. \end{cases} \tag{4.9}$$

Multiplying $(4.9)_1$ by $(y_k^n)'$, $(4.9)_2$ by $(R_k^n)'$ and using integration by parts, we get

$$\begin{cases} \rho \int_0^\Gamma \nu_{ntt}\nu_{nt} dx + \rho \int_0^\Gamma \nu_{nt} \left(\int_0^t h(t-s)\nu_{nt}(s)\,ds \right)' dx + \alpha \int_0^\Gamma \nu_{nx}\nu_{nxt} dx \\ -\gamma\beta \int_0^\Gamma \varphi_{nx}\nu_{nxt} dx = 0, \\ \mu \int_0^\Gamma \varphi_{ntt}\varphi_{nt} dx + \beta \int_0^\Gamma \varphi_{nx}\varphi_{nxt} dx - \gamma\beta \int_0^\Gamma \nu_{nx}\varphi_{nxt} dx = 0, \end{cases} \tag{4.10}$$

as

$$\begin{aligned} \left(\int_0^t h(t-s)\nu_{nt}(s)\,ds \right)' &= h(t)\nu_{nt}(0) + \int_0^t h(s)\nu_{ntt}(t-s)\,ds \\ &= h(t)\nu_{nt}(0) + \int_0^t h(t-s)\nu_{ntt}(s)\,ds. \end{aligned} \tag{4.11}$$

By utilizing Lemma (4.1), we obtain:

$$\rho\int_0^\Gamma \nu_{nt}\int_0^t h(t-s)\,\nu_{ntt}(s)\,dsdx = -\frac{\rho}{2}\left(h'\Box\nu_{nt}\right)(t)+\frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\left\|\nu_{nt}(s)\right\|^2 ds + \frac{\rho h(t)}{2}\left\|\nu_{nt}\right\|^2-\rho h(t)\int_0^t \nu_{nt}(t)\,\nu_{nt}(0)\,dx. \quad (4.12)$$

By employing (4.11), (4.12) in the system (4.10), we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\Gamma \nu_{nt}^2dx-\frac{\rho}{2}\left(h'\Box\nu_{nt}\right)(t)+\frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\left\|\nu_{nt}(s)\right\|^2 ds + \frac{\rho h(t)}{2}\left\|\nu_{nt}\right\|^2+\frac{\alpha}{2}\frac{d}{dt}\int_0^\Gamma \nu_{nx}^2dx-\gamma\beta\frac{d}{dt}\int_0^\Gamma \varphi_{nx}\nu_{nx}dx + \frac{\mu}{2}\frac{d}{dt}\int_0^\Gamma \varphi_{nt}^2dx+\frac{\beta}{2}\frac{d}{dt}\int_0^\Gamma \varphi_{nx}^2dx = 0, \quad (4.13)$$

it follows that,

$$\frac{1}{2}\frac{d}{dt}\int_0^\Gamma\left(\rho\nu_{nt}^2+\mu\varphi_{nt}^2+\alpha_1\nu_{nx}^2+\beta\left(\gamma\nu_{nx}-\varphi_{nx}\right)^2\right)dx+\frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\left\|\nu_{nt}(s)\right\|^2 ds - \frac{\rho}{2}\left(h'\Box\nu_{nt}\right)(t) + \frac{\rho h(t)}{2}\left\|\nu_{nt}\right\|^2 = 0, \quad (4.14)$$

from (4.14), we deduce that

$$\frac{d}{dt}E^n(t) = \frac{\rho}{2}\left(h'\Box\nu_{nt}\right)(t)-\frac{\rho h(t)}{2}\left\|\nu_{nt}\right\|^2,$$

where

$$E^n(t)=\frac{1}{2}\int_0^\Gamma\left(\rho\nu_{nt}^2+\mu\varphi_{nt}^2+\alpha_1\nu_{nx}^2+\beta\left(\gamma\nu_{nx}-\varphi_{nx}\right)^2\right)dx+\frac{\rho}{2}\int_0^t h(t-s)\left\|\nu_{nt}(s)\right\|^2 ds.$$

By employing (4.7) and the assumptions of (H_1) associated with the kernel h, it is possible to find a positive constant $\tilde{C}$ that does not depend on n such that

$$E^n(t)\leq E^n(0)\leq\tilde{C}. \quad (4.15)$$

From the relationship (4.15), we obtain

$$\int_0^\Gamma \nu_{nt}^2dx+\int_0^\Gamma \varphi_{nt}^2dx+\int_0^\Gamma \nu_{nx}^2dx+\int_0^\Gamma\left(\gamma\nu_{nx}-\varphi_{nx}\right)^2dx + \int_0^t h(t-s)\left\|\nu_{nt}(s)\right\|^2 ds \leq C, \quad (4.16)$$

through (4.16), we can deduce that $t_n=T$ for every $T>0$.

A priori estimate II

Because

1.

$$\nu_n(x,t) = \sum_{j=1}^{j=n} \delta_j(x)\, y_j^n(t),$$
$$\varphi_n(x,t) = \sum_{j=1}^{j=n} \delta_j(x)\, R_j^n(t),$$

2. $\left(y_j^n,\ R_j^n\right)_{j=1,..,n} \in \left(C^2[0,T]\right)^2$

3. $(\delta_j)_{j\geq 1} \subset H^1(0,\Gamma) \hookrightarrow C(0,\Gamma)$, (continuous embedding) .

So, we can deduce that

$$\nu_n,\ \varphi_n \in C^2\left(0,T;\ \tilde{H}^2(0,\Gamma) \cap \hat{H}^1(0,\Gamma)\right), \tag{4.17}$$

where $X = C^2\left(0,T;\ \tilde{H}^2(0,\Gamma) \cap \hat{H}^1(0,\Gamma)\right)$, is a Banach space equipped with the norm

$$\|\nu_n\|_X = \sup_{t\in[0,T]} \|\nu_n(.,t)\|_{H^2(0,\Gamma)} + \sup_{t\in[0,T]} \|\nu_{nt}(.,t)\|_{H^2(0,\Gamma)} + \sup_{t\in[0,T]} \|\nu_{ntt}(.,t)\|_{H^2(0,\Gamma)}, \tag{4.18}$$

using (4.17)-(4.18), we get

$$\int_0^\Gamma \left(\nu_n^2 + \nu_{nxx}^2 + \varphi_n^2 + \varphi_{nx}^2 + \varphi_{nxx}^2\right) dx < \infty, \quad \forall t \in [0,T]. \tag{4.19}$$

- **Step 3: Passage to limit**

By utilizing equations (4.16)-(4.19), we conclude that

$$\begin{aligned} &(\nu_n)_{n\in\mathbb{N}^*} && \text{is a bounded sequence in the space } L^\infty\left(0,T;\ \tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)\right),\\ &(\nu_{nt})_{n\in\mathbb{N}^*} && \text{is a bounded sequence in the space } L^\infty\left(0,T;\ L^2(0,\Gamma)\right), \end{aligned} \tag{4.20}$$

$$\begin{aligned} &(\varphi_n)_{n\in\mathbb{N}^*} && \text{is a bounded sequence in the space } L^\infty\left(0,T;\ \tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)\right),\\ &(\varphi_{nt})_{n\in\mathbb{N}^*} && \text{is a bounded sequence in the space } L^\infty\left(0,T;\ L^2(0,\Gamma)\right). \end{aligned} \tag{4.21}$$

By employing Aubin–Lions–Simon theorem (4.1), as

1. The space $\hat{H}^1(0,\Gamma)$ is continuously embedded in $L^2(0,\Gamma)$.

2. The embedding of $\tilde{H}^2(0,\Gamma)\cap\hat{H}^1(0,\Gamma)$ into $\hat{H}^1(0,\Gamma)$ is compact.

Consequently, we can infer that the embedding $E_{\infty,\infty}$ in $C(0,T;\ \hat{H}^1(0,\Gamma))$ is compact. Where

$$E_{\infty,\infty} = \Big\{\nu_n/\ \nu_n \in L^\infty\left(0,T;\ \tilde{H}^2(0,\Gamma) \cap \hat{H}^1(0,\Gamma)\right),\ \nu_{nt} = \frac{d\nu_n}{dt} \in L^\infty\left(0,T;\ L^2(0,\Gamma)\right),\ n \geq 1\Big\}.$$

Note that by referencing (4.20), we get $(\nu_n)_{n\geq 1}$ bounded in $E_{\infty,\infty}$. Consequently, there exists a subsequence $(\nu_m)_{m\geq 1}$ of $(\nu_n)_{n\geq 1}$ such that

$$\nu_m \overset{m\to\infty}{\rightarrow} \nu \text{ strongly in } W = C(0,T;\ \hat{H}^1(0,\Gamma)), \tag{4.22}$$

since

$$\nu_{mxx}(x,t) = \lambda \nu_m(x,t), \tag{4.23}$$

using (4.22) and closed operator definition, we deduce the following result:

$$\nu_m \overset{m\to\infty}{\rightarrow} \nu \text{ strongly in } C(0,T;\ \tilde{H}^2(0,\Gamma) \cap \hat{H}^1(0,\Gamma)).$$

by using (4.17), (4.22), in addition to applying the dominated convergence theorem and closed operator definition, we can derive the following result:

$$\|\nu_{mt} - \nu_t\|_W = \left\|\frac{d}{dt}\nu_m - \nu_t\right\|_W \overset{m\to\infty}{\rightarrow} 0,$$

this implies

$$\nu_{mt} \overset{m\to\infty}{\rightarrow} \nu_t \text{ strongly in } C(0,T;\ \hat{H}^1(0,\Gamma)), \tag{4.24}$$

(4.22), (4.24) implies that

$$\nu_m \overset{m\to\infty}{\rightarrow} \nu \text{ strongly in } C^1(0,T;\ \hat{H}^1(0,\Gamma)) \quad \forall T > 0. \tag{4.25}$$

Once again, using (4.17), (4.24), the dominated convergence theorem and closed operator definition, we get

$$\|\nu_{mtt} - \nu_{tt}\|_W = \left\|\frac{d^2}{dt^2}\nu_m - \nu_{tt}\right\|_W \overset{m\to\infty}{\rightarrow} 0,$$

this means that

$$\nu_{mtt} \overset{m\to\infty}{\rightarrow} \nu_{tt} \text{ strongly in } C(0,T;\ \hat{H}^1(0,\Gamma)), \tag{4.26}$$

through (4.25), (4.26), we obtain the following result:

$$\nu_m \overset{m\to\infty}{\rightarrow} \nu \text{ strongly in } C^2(0,T;\ \hat{H}^1(0,\Gamma)).$$

We apply the same proof technique to $(\varphi_n)_{n\geq 1}$. By passing to the limit in (4.6)-(4.9), we get that the problem (4.1) accepts a strong solution that satisfies (4.4).

4.3. Global well-posedness

- **Step 4: Uniqueness of solution**

Assume that $(\tilde{\nu}, \tilde{\varphi})$ and (ν, φ) are two global solutions of (4.1), then the pair $(\vartheta, \omega) = (\nu - \tilde{\nu},\ \varphi - \tilde{\varphi})$ satisfies

$$\begin{cases} \rho\left(\vartheta_t + \int_0^t h(t-s)\,\vartheta_t(s)\,ds\right)' - \alpha\vartheta_{xx} + \gamma\beta\omega_{xx} = 0, & \text{in } (0,\Gamma)\times(0,\infty), \\ \mu\omega_{tt} - \beta\omega_{xx} + \gamma\beta\vartheta_{xx} = 0, & \\ \vartheta(0,t) = \alpha\vartheta_x(\Gamma,t) - \gamma\beta\omega_x(\Gamma,t) = 0, & t \geq 0, \\ \omega(0,t) = \omega_x(\Gamma,t) - \gamma\vartheta_x(\Gamma,t) = 0, & \\ (\vartheta, \vartheta_t)(x,0) = 0, & x \in (0,\Gamma), \\ (\omega, \omega_t)(x,0) = 0, & \end{cases} \tag{4.27}$$

multiplying $(4.27)_1$ by ϑ_t, $(4.27)_2$ by ω_t and integrating over the interval $(0,\Gamma)$, we get as the steps witch used in a priori estimate I

$$\begin{aligned} &\frac{d}{2dt}\int_0^\Gamma \left(\rho\vartheta_t^2 + \mu\omega_t^2 + \alpha_1\vartheta_x^2 + \beta(\gamma\vartheta_x - \omega_x)^2\right)dx + \frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\left\|\vartheta_t(s)\right\|^2 ds \\ &- \frac{\rho}{2}\left(h'\Box\vartheta_t\right)(t) + \frac{\rho h(t)}{2}\left\|\vartheta_t\right\|^2 \ = 0, \end{aligned} \tag{4.28}$$

by using the assumptions of (H_1) associated with the kernel h, we obtain

$$\frac{d}{dt}\bar{E}(t) = \frac{\rho}{2}\left(h'\Box\vartheta_t\right)(t) - \frac{\rho}{2}h(t)\left\|\vartheta_t\right\|^2 \leq 0 \qquad \forall t \geq 0,$$

this implies that

$$\bar{E}(t) \leq 0,$$

where

$$\bar{E}(t) = \frac{1}{2}\int_0^\Gamma \rho\vartheta_t^2 + \mu\omega_t^2 + \alpha_1\vartheta_x^2 + \beta(\gamma\vartheta_x - \omega_x)^2\,dx + \frac{\rho}{2}\int_0^t h(t-s)\left\|\vartheta_t^2\right\|ds. \tag{4.29}$$

We get directly

$$(\vartheta, \omega) = (0,0).$$

Then, there exists only one global strong solution to the problem (4.1). ■

4.4 Exponential stability

In this section, we will present and demonstrate the technical lemmas required to demonstrate our stability theorem.

Lemma 4.2 *Let (ν, φ) satisfies (4.1), then the expression of energy $E(t)$ is given by the following*

$$E(t) = \frac{1}{2}\int_0^\Gamma \left(\rho\nu_t^2 + \mu\varphi_t^2 + \alpha_1\nu_x^2 + \beta(\gamma\nu_x - \varphi_x)^2 + \rho\int_0^t h(t-s)(\nu_t(s))^2 ds\right) dx, \qquad (4.30)$$

and satisfies

$$\frac{d}{dt}E(t) = \frac{\rho}{2}(h'\Box\nu_t)(t) - \frac{\rho h(t)}{2}\|\nu_t\|^2. \qquad (4.31)$$

Proof. When we multiply equation $(4.1)_1$ by ν_t and equation $(4.1)_2$ by φ_t, and integrate both equations over the interval $(0, \Gamma)$ with respect to x, we get the following result:

$$\begin{aligned} &\rho\int_0^\Gamma \nu_{tt}\nu_t dx + \rho\int_0^\Gamma \nu_t\left(\int_0^t h(t-s)\nu_t(s)\,ds\right)' dx - \alpha\int_0^\Gamma \nu_{xx}\nu_t dx \\ &+ \gamma\beta\int_0^\Gamma \varphi_{xx}\nu_t dx = 0, \\ &\mu\int_0^\Gamma \varphi_{tt}\varphi_t dx - \beta\int_0^\Gamma \varphi_t\varphi_{xx}dx + \gamma\beta\int_0^\Gamma \varphi_t\nu_{xx}dx = 0, \end{aligned} \qquad (4.32)$$

by (4.32), we obtain

$$\begin{aligned} &\rho\frac{d}{2dt}\int_0^\Gamma \nu_t^2 dx + \rho\int_0^\Gamma \nu_t\left(\int_0^t h(t-s)\nu_t(s)\,ds\right)' dx + \alpha\int_0^\Gamma \nu_x\nu_{tx}dx - \gamma\beta\int_0^\Gamma \varphi_x\nu_{tx}dx \\ &+ \mu\frac{d}{2dt}\int_0^\Gamma \varphi_t^2 dx + \beta\int_0^\Gamma \varphi_{tx}\varphi_x dx - \gamma\beta\int_0^\Gamma \varphi_{tx}\nu_x dx = 0, \end{aligned} \qquad (4.33)$$

as

$$\left(\int_0^t h(t-s)\nu_t(s)\,ds\right)' = \left(\int_0^t h(s)\nu_t(t-s)\,ds\right)' = h(t)\nu_t(0) + \int_0^t h(t-s)\nu_{tt}(s)\,ds.$$

Then

$$\rho\int_0^\Gamma \nu_t\left(\int_0^t h(t-s)\nu_t(s)\,ds\right)' dx = \rho h(t)\int_0^\Gamma \nu_t\nu_1 dx + \rho\int_0^\Gamma \nu_t\int_0^t h(t-s)\nu_{tt}(s)\,dsdx, \qquad (4.34)$$

using Lemma (4.1), we get

$$\begin{aligned} \rho\int_0^\Gamma \nu_t\int_0^t h(t-s)\nu_{tt}(s)\,dsdx = &-\frac{\rho}{2}(h'\Box\nu_t)(t) + \frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\int_0^\Gamma (\nu_t(s))^2 dxds \\ &+ \frac{\rho h(t)}{2}\|\nu_t\|^2 - \rho h(t)\int_0^t \nu_t\nu_t(0)\,dx, \end{aligned} \qquad (4.35)$$

then

$$\begin{aligned} \rho\int_0^\Gamma \nu_t\left(\int_0^t h(t-s)\nu_t(s)\,ds\right)' dx = &\frac{\rho}{2}\frac{d}{dt}\int_0^t h(t-s)\int_0^\Gamma (\nu_t(s))^2 dxds \\ &-\frac{\rho}{2}(h'\Box\nu_t)(t) + \frac{\rho h(t)}{2}\|\nu_t\|^2, \end{aligned} \qquad (4.36)$$

then by using (4.36) in (4.33), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\Gamma \left(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta\left(\gamma\nu_x-\varphi_x\right)^2\right)dx+\frac{\rho}{2}\frac{d}{dt}\int_0^t\int_0^\Gamma h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2dxds$$
$$-\frac{\rho}{2}\left(h'\square\nu_t\right)(t)+\frac{\rho h\left(t\right)}{2}\left\|\nu_t\right\|^2=0. \tag{4.37}$$

Fubini's theorem allows us to conclude the following:

$$\int_0^\Gamma\int_0^t h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2dsdx=\int_0^t\int_0^\Gamma h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2dxds\quad\forall t\geq 0, \tag{4.38}$$

then, (4.37) takes the following form

$$\frac{1}{2}\frac{d}{dt}\int_0^\Gamma\left(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta\left(\gamma\nu_x-\varphi_x\right)^2+\rho\int_0^t h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2ds\right)dx$$
$$-\frac{\rho}{2}\left(h'\square\nu_t\right)(t)+\frac{\rho h\left(t\right)}{2}\left\|\nu_t\right\|^2=0. \tag{4.39}$$

Utilizing assumptions (H_1), we arrive at

$$E\left(t\right)=\frac{1}{2}\int_0^\Gamma\left(\rho\nu_t^2+\mu\varphi_t^2+\alpha_1\nu_x^2+\beta\left(\gamma\nu_x-\varphi_x\right)^2+\rho\int_0^t h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2ds\right)dx,$$

and

$$\frac{d}{dt}E\left(t\right)=\frac{\rho}{2}\left(h'\square\nu_t\right)(t)-\frac{\rho h\left(t\right)}{2}\left\|\nu_t\right\|^2\leq-\frac{\rho h\left(t\right)}{2}\left\|\nu_t\right\|^2\leq 0.$$

■

Lemma 4.3 *Let (ν,φ) be a solution of the system described in (4.1). Then the functional M_1 is defined as follows*

$$M_1\left(t\right)=\rho\int_0^\Gamma\nu\left(\nu_t+\int_0^t h\left(t-s\right)\nu_t\left(s\right)ds\right)dx+\gamma\mu\int_0^\Gamma\varphi_t\nu dx,\ \forall t\geq 0, \tag{4.40}$$

satisfies the following identity

$$M_1'\left(t\right)\leq-\alpha_1\int_0^\Gamma\nu_x^2dx+\left(\rho+\frac{\left(\gamma\mu\right)^2}{4\varepsilon_0}+\frac{\rho^2}{4\varepsilon_1}\right)\int_0^\Gamma\nu_t^2dx+\varepsilon_0\int_0^\Gamma\varphi_t^2dx$$
$$+\varepsilon_1\bar{h}\int_0^\Gamma\int_0^t h\left(t-s\right)\left(\nu_t\left(s\right)\right)^2dsdx,\qquad\forall\ \varepsilon_0,\varepsilon_1>0. \tag{4.41}$$

Proof. By differentiating $M_1\left(t\right)$, and employing equations $(4.1)_1$, $(4.1)_2$, then we have, after integrating by parts taking into account the boundary conditions of (4.1)

$$M_1'\left(t\right)=\rho\int_0^\Gamma\nu_t^2dx+\rho\int_0^\Gamma\nu_t\int_0^t h\left(t-s\right)\nu_t\left(s\right)dsdx-\alpha_1\int_0^\Gamma\nu_x^2dx+\gamma\mu\int_0^\Gamma\varphi_t\nu_tdx, \tag{4.42}$$

through the application of Cauchy-Schwarz and Young's inequalities, we get

$$\gamma\mu \int_0^\Gamma \varphi_t \nu_t dx \leq \varepsilon_0 \int_0^\Gamma \varphi_t^2 dx + \frac{(\gamma\mu)^2}{4\varepsilon_0} \int_0^\Gamma \nu_t^2 dx \qquad \forall \varepsilon_0 > 0, \tag{4.43}$$

$$\begin{aligned} \rho \int_0^\Gamma \nu_t \int_0^t h(t-s)\, \nu_t(s)\, ds dx &\leq \varepsilon_1 \bar{h} \int_0^\Gamma \int_0^t h(t-s)\,(\nu_t(s))^2\, ds dx \\ &+ \frac{\rho^2}{4\varepsilon_1} \int_0^\Gamma \nu_t^2 dx \quad \forall \varepsilon_1 > 0, \end{aligned} \tag{4.44}$$

by using (4.43)-(4.44) in (4.42), we get (4.41). ■

Lemma 4.4 *Let (ν, φ) be a solution of the system described in (4.1). Then the functional M_2 is defined as follows*

$$M_2(t) = \mu \int_0^\Gamma \varphi_t \varphi dx + \rho \int_0^\Gamma \left(\nu_t + \int_0^t h(t-s)\, \nu_t(s)\, ds \right) \nu dx, \tag{4.45}$$

satisfies

$$\begin{aligned} M_2'(t) \leq &-\beta \int_0^\Gamma (\gamma \nu_x - \varphi_x)^2 dx + \mu \int_0^\Gamma \varphi_t^2 dx + \left(\rho + \frac{\rho^2}{2} \right) \int_0^\Gamma \nu_t^2 dx \\ &+ \frac{\bar{h}}{2} \int_0^\Gamma \int_0^t h(t-s)\,(\nu_t(s))^2\, ds dx. \end{aligned} \tag{4.46}$$

Proof. By differentiating $M_2(t)$ and utilizing equations $(4.1)_1$ and $(4.1)_2$, we obtain

$$\begin{aligned} M_2'(t) = &\mu \int_0^\Gamma \varphi_t^2 dx + \beta \int_0^\Gamma \varphi \varphi_{xx} dx - \gamma\beta \int_0^\Gamma \nu_{xx} \varphi dx + \rho \int_0^\Gamma \nu_t^2 dx \\ &+ \rho \int_0^\Gamma \nu_t \int_0^t h(t-s)\, \nu_t(s)\, ds dx + \alpha \int_0^\Gamma \nu_{xx} \nu dx \\ &- \gamma\beta \int_0^\Gamma \varphi_{xx} \nu dx, \end{aligned} \tag{4.47}$$

by using (4.47), we get

$$\begin{aligned} M_2'(t) = &\mu \int_0^\Gamma \varphi_t^2 dx - \beta \int_0^\Gamma (\gamma \nu_x - \varphi_x)^2 dx - \alpha_1 \int_0^\Gamma \nu_x^2 dx + \rho \int_0^\Gamma \nu_t^2 dx \\ &+ \rho \int_0^\Gamma \nu_t \int_0^t h(t-s)\, \nu_t(s)\, ds dx. \end{aligned} \tag{4.48}$$

Utilizing Cauchy-Schwarz and Young's inequalities, we can deduce the following:

$$\rho \int_0^\Gamma \nu_t \int_0^t h(t-s)\, \nu_t(s)\, ds dx \leq \frac{\bar{h}}{2} \int_0^\Gamma \int_0^t h(t-s)\,(\nu_t(s))^2\, ds dx + \frac{\rho^2}{2} \int_0^\Gamma \nu_t^2 dx, \tag{4.49}$$

using (4.49) in (4.48), we get (4.46). ■

4.4. Exponential stability

Lemma 4.5 *Let (ν, φ) be a solution of the system referred to in (4.1). Then the functional M_3 is defined as follows*

$$M_3(t) = \rho \int_0^\Gamma \left(\nu_t + \int_0^t h(t-s)\nu_t(s)\,ds \right)(\gamma\nu - \varphi)\,dx + \gamma\mu \int_0^\Gamma \varphi_t(\gamma\nu - \varphi)\,dx, \tag{4.50}$$

satisfies, for any $\varepsilon_4 > 0$

$$\begin{aligned} M_3'(t) \leq & -\frac{\gamma\mu}{2}\int_0^\Gamma \varphi_t^2 dx + \left(\rho\gamma + \frac{(\rho\gamma)^2}{2} + \frac{\sigma^2}{\gamma\mu}\right)\int_0^\Gamma \nu_t^2 dx + \varepsilon_4 \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\ & + \left(\frac{\bar{h}}{2} + \frac{\rho^2\bar{h}}{\gamma\mu}\right)\int_0^\Gamma \int_0^t h(t-s)(\nu_t(s))^2 dsdx + \frac{\alpha_1^2}{4\varepsilon_4}\int_0^\Gamma \nu_x^2 dx, \end{aligned} \tag{4.51}$$

where $\sigma = (\gamma^2\mu - \rho)$.

Proof. *By differentiating $M_3(t)$, and using $(4.1)_1$, $(4.1)_2$, then we get*

$$\begin{aligned} M_3'(t) = & -\gamma\mu \int_0^\Gamma \varphi_t^2 dx + \rho\gamma \int_0^\Gamma \nu_t^2 dx + \rho\gamma \int_0^\Gamma \nu_t \int_0^t h(t-s)\nu_t(s)\,dsdx \\ & - \rho \int_0^\Gamma \varphi_t \int_0^t h(t-s)\nu_t(s)\,dsdx - \alpha_1 \int_0^\Gamma \nu_x(\gamma\nu_x - \varphi_x)\,dx \\ & + \underbrace{(\gamma^2\mu - \rho)}_{\sigma} \int_0^\Gamma \varphi_t \nu_t dx, \end{aligned} \tag{4.52}$$

by applying Cauchy-Schwarz and Young's inequalities, we can infer that

$$\sigma \int_0^\Gamma \varphi_t \nu_t dx \leq \frac{\gamma\mu}{4}\int_0^\Gamma \varphi_t^2 dx + \frac{\sigma^2}{\gamma\mu}\int_0^\Gamma \nu_t^2 dx, \tag{4.53}$$

and

$$-\rho \int_0^\Gamma \varphi_t \int_0^t h(t-s)\nu_t(s)\,dsdx \leq \frac{\gamma\mu}{4}\int_0^\Gamma \varphi_t^2 dx + \frac{\rho^2\bar{h}}{\gamma\mu}\int_0^\Gamma \int_0^t h(t-s)(\nu_t(s))^2 dsdx, \tag{4.54}$$

$$\begin{aligned} \rho\gamma \int_0^\Gamma \nu_t \int_0^t h(t-s)\nu_t(s)\,dsdx \leq & \frac{\bar{h}}{2}\int_0^\Gamma \int_0^t h(t-s)(\nu_t(s))^2 dsdx \\ & + \frac{(\rho\gamma)^2}{2}\int_0^\Gamma \nu_t^2 dx, \end{aligned} \tag{4.55}$$

$$-\alpha_1 \int_0^\Gamma \nu_x(\gamma\nu_x - \varphi_x)\,dx \leq \varepsilon_4 \int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx + \frac{\alpha_1^2}{4\varepsilon_4}\int_0^\Gamma \nu_x^2 dx \qquad \forall \varepsilon_4 > 0, \tag{4.56}$$

using (4.53)-(4.54)-(4.55)-(4.56) in (4.52), we get (4.51). ■

4.4. Exponential stability

Lemma 4.6 *Let (ν, φ) be a solution of the system mentioned in (4.1). Then the functional M_4 is given by the following relation:*

$$M_4(t) := e^{-\varsigma t} \int_0^\Gamma \left(\int_0^t e^{\varsigma s} \tilde{H}_1(t-s) (\nu_t(s))^2 ds \right) dx, \tag{4.57}$$

its derivative is

$$M_4'(t) = -\varsigma M_4(t) + \tilde{H}_1(0) \int_0^\Gamma \nu_t^2(s) dx - \int_0^\Gamma \int_0^t h(t-s)(\nu_t(s))^2 ds dx, \quad \forall \varsigma > 0, \tag{4.58}$$

where $\tilde{H}_1(t) = \int_t^\infty e^{\varsigma s} h(s) ds$.

Proof. By differentiating $M_4(t)$, we directly get (4.58). ■

Now, for a sufficiently large N, we define the Lyapunov functional as follows:

$$\mathcal{L}(t) = NE(t) + N_1 M_1(t) + M_2(t) + N_2 M_3(t) + N_3 M_4(t), \tag{4.59}$$

where N, N_1, N_2, and N_3 are positive constants, which will be determined later.

Theorem 4.3 *Let (ν, φ) be a solution of the system (4.1). Then there exist two positive constants c_1 and $c_2 > 0$ which fulfill the following*

$$c_1(E(t) + M_4(t)) \le \mathcal{L}(t) \le c_2(E(t) + M_4(t)), \ \forall t \ge 0. \tag{4.60}$$

Proof. *Let*

$$\Im(t) = \mathcal{L}(t) - NE(t) - N_3 M_4(t) = N_1 M_1(t) + M_2(t) + N_2 M_3(t), \tag{4.61}$$

then

$$\begin{aligned} |\Im(t)| \le{}& N_1 \left(\rho \int_0^\Gamma |\nu \nu_t| dx + \rho \int_0^\Gamma \left| \nu \int_0^t h(t-s) \nu_t(s) ds \right| dx + \gamma\mu \int_0^\Gamma |\varphi_t \nu| dx \right) \\ &+ \left(\mu \int_0^\Gamma |\varphi_t \varphi| dx + \rho \int_0^\Gamma |\nu_t \nu| dx + \rho \int_0^\Gamma \left| \nu \int_0^t h(t-s) \nu_t(s) ds \right| dx \right) \\ &+ N_2 \left(\rho \int_0^\Gamma |\nu_t(\gamma\nu - \varphi)| dx + \rho \int_0^\Gamma \left| \int_0^t h(t-s) \nu_t(s) ds (\gamma\nu - \varphi) \right| dx \right. \\ &\left. + \gamma\mu \int_0^\Gamma |\varphi_t(\gamma\nu - \varphi)| dx \right). \end{aligned} \tag{4.62}$$

By employing Young's, Cauchy-Schwarz, and Poincaré's inequalities, we get $\forall\varepsilon > 0$*:*

$$\begin{aligned}
|\Im(t)| &\leq \left(N_1\frac{\rho^2}{4\varepsilon} + \frac{\rho^2}{4\varepsilon} + N_2\frac{\rho^2}{4\varepsilon}\right)\int_0^\Gamma \nu_t^2 dx \\
&+ \left(N_1\frac{(\gamma\mu)^2}{4\varepsilon} + \frac{\mu^2}{4\varepsilon} + N_2\frac{(\gamma\mu)^2}{4\varepsilon}\right)\int_0^\Gamma \varphi_t^2 dx \\
&+ \left(3N_1\varepsilon c_0 + 2\gamma^2\varepsilon c_0 + 2\varepsilon c_0\right)\int_0^\Gamma \nu_x^2 dx \\
&+ \left(2\varepsilon c_0 + 3N_2\varepsilon c_0\right)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\
&+ \left(N_1\frac{\rho^2\bar{h}}{4\varepsilon} + \frac{\rho^2\bar{h}}{4\varepsilon} + N_2\frac{\rho^2\bar{h}}{4\varepsilon}\right)\int_0^\Gamma\int_0^t h(t-s)(\nu_t(s))^2 dsdx,
\end{aligned}$$

Here, c_0 *represents the constant in Poincaré's inequality. Then, we get*

$$|\Im(t)| \leq mE(t), \tag{4.63}$$

therefore

$$(N-m)E(t) + N_3M_4(t) \leq \mathcal{L}(t) \leq (N+m)E(t) + N_3M_4(t), \tag{4.64}$$

then we get (4.60), with

$$c_1 = \min((N-m), N_3) \qquad c_2 = \max((N+m), N_3).$$

■

Theorem 4.4 *Let* (ν,φ) *be a solution of the system (4.1), then there exists a positive constant* $\gamma_1 > 0$ *such that*

$$\mathcal{L}'(t) \leq -\gamma_1(E(t) + M_4(t)). \tag{4.65}$$

Proof. differentiating $\mathcal{L}(t)$ and utilizing (4.31)-(4.41)-(4.46)-(4.51)-(4.58), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq &-\left(N\frac{\rho h(t)}{2} - N_1\left(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_0} + \frac{\rho^2}{4\varepsilon_1}\right) - \left(\rho + \frac{\rho^2}{2}\right) - N_3\tilde{H}_1(0)\right. \\
&\left.-N_2\left(\rho\gamma + \frac{(\rho\gamma)^2}{2} + \frac{\sigma^2}{\gamma\mu}\right)\right)\int_0^\Gamma \nu_t^2 dx - \left(N_2\frac{\gamma\mu}{2} - \mu - N_1\varepsilon_0\right)\int_0^\Gamma \varphi_t^2 dx \\
&-\left(\alpha_1N_1 - N_2\frac{\alpha_1^2}{4\varepsilon_4}\right)\int_0^\Gamma \nu_x^2 dx - (\beta - N_2\varepsilon_4)\int_0^\Gamma (\gamma\nu_x - \varphi_x)^2 dx \\
&-\left(N_3 - \frac{\bar{h}}{2} - N_2\left(\frac{\bar{h}}{2} + \frac{\rho^2\bar{h}}{\gamma\mu}\right) - N_1\varepsilon_1\bar{h}\right)\int_0^\Gamma\int_0^t h(t-s)(\nu_t(s))^2 dsdx \\
&-\varsigma N_3M_4(t).
\end{aligned} \tag{4.66}$$

First in (4.66), we choose N_2 large enough so that

$$N_2\frac{\gamma\mu}{2}-\mu>0.$$

Since N_2 is fixed, we choose ε_4 small such that

$$\chi_1=\beta-N_2\varepsilon_4>0.$$

Next, we choose N_1 sufficiently large to satisfy the relationship

$$\chi_2=\alpha_1N_1-N_2\frac{\alpha_1^2}{4\varepsilon_4}>0.$$

Once N_1 is fixed, we pick ε_0 small enough so that

$$\chi_3=N_2\frac{\gamma\mu}{2}-\mu-N_1\varepsilon_0>0.$$

Also, we choose N_3 large enough such that

$$N_3-\frac{\bar{h}}{2}-N_2\left(\frac{\bar{h}}{2}+\frac{\rho^2\bar{h}}{\gamma\mu}\right)>0.$$

Now, we choose ε_1 sufficiently small such that until it becomes

$$\chi_4=N_3-\frac{\bar{h}}{2}-N_2\left(\frac{\bar{h}}{2}+\frac{\rho^2\bar{h}}{\gamma\mu}\right)-N_1\varepsilon_1\bar{h}>0.$$

Finally, we choose N large enough (even larger such that the relationship (4.60) remains valid) so that

$$\begin{aligned}&\left(N\frac{\rho h(t)}{2}-N_1\left(\rho+\frac{(\gamma\mu)^2}{4\varepsilon_0}+\frac{\rho^2}{4\varepsilon_1}\right)-\left(\rho+\frac{\rho^2}{2}\right)-N_3\tilde{H}_1(0)\right.\\&\left.-N_2\left(\rho\gamma+\frac{(\rho\gamma)^2}{2}+\frac{\sigma^2}{\gamma\mu}\right)\right) \; > \; \min(\chi_1,\chi_2,\chi_3,\chi_4) \quad \forall t\geq 0.\end{aligned}$$

Then we obtain

$$\mathcal{L}'(t)\leq-\gamma_1(E(t)+M_4(t)).$$

■

Theorem 4.5 *Let (ν,φ) be the solution of system (4.1) with the conditions (H_1) and (H_2) satisfied. Then there exist two positive constants s and η, such that*

$$E(t)\leq se^{-\eta t},\qquad \forall t\geq 0. \tag{4.67}$$

Proof. By utilizing equations (4.60) and (4.65), we obtain:

$$\mathcal{L}'(t)\leq-\frac{\gamma_1}{c_2}\mathcal{L}(t),$$

then, we find

$$c_1(E(t)+M_4(t))\leq e^{-\frac{\gamma_1}{c_2}t}.\mathcal{L}(0).$$

Because $M_4(t)$ is positive. The proof is complete. ■

4.4. Exponential stability

4.5 Exponential energy decay of quasi-static / electrostatic piezoelectric beams (the magnetic effects are negligible) subject to a neutral delay

For a length Γ and thickness h beam, in this case the stretching motion subject to a neutral delay is described as follows:

$$\begin{cases} \rho\left(\nu_t + \int_0^t h(t-s)\,\nu_t(s)\,ds\right)' - \alpha_1\nu_{xx} = 0, & \text{in } (0,\Gamma)\times(0,\infty), \\ \nu(0,t) = \nu_x(\Gamma,t) = 0, & t \geq 0, \\ \nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x), & x \in (0,\Gamma). \end{cases} \tag{4.68}$$

Furthermore, the system's energy is given by the following expression

$$\tilde{E}(t) = \frac{1}{2}\int_0^\Gamma \left(\rho\nu_t^2 + \alpha_1\nu_x^2 + \rho\int_0^t h(t-s)\,\nu_t^2(s)\,ds\right)dx, \tag{4.69}$$

and it satisfies

$$\frac{d}{dt}\tilde{E}(t) = \frac{\rho}{2}\left(h'\Box\nu_t\right)(t) - \frac{\rho h(t)}{2}\|\nu_t\|^2 \quad \forall t \geq 0. \tag{4.70}$$

Now, we will proceed to prove the exponential stability of the system.

Lemma 4.7 *Let ν is a solution to the system referred to in (4.68). Then, the functional*

$$\tilde{M}_1(t) = \rho\int_0^\Gamma \nu\left(\nu_t + \int_0^t h(t-s)\,\nu_t(s)\,ds\right)dx \quad \forall t \geq 0, \tag{4.71}$$

satisfies

$$\begin{aligned} \tilde{M}_1'(t) \;\leq\; & -\alpha_1\int_0^\Gamma \nu_x^2dx + \left(\rho + \frac{\rho^2}{4\tilde{\varepsilon}_1}\right)\int_0^\Gamma \nu_t^2dx \\ & + \tilde{\varepsilon}_1\bar{h}\int_0^\Gamma\int_0^t h(t-s)\left(\nu_t(s)\right)^2 dsdx, \qquad \forall\ \tilde{\varepsilon}_1 > 0. \end{aligned} \tag{4.72}$$

Proof. By differentiating $\tilde{M}_1(t)$ and utilizing (4.68), then we have

$$\tilde{M}_1'(t) = -\alpha_1\int_0^\Gamma \nu_x^2dx + \rho\int_0^\Gamma \nu_t^2dx + \rho\int_0^\Gamma \nu_t\int_0^t h(t-s)\,\nu_t(s)\,dsdx, \tag{4.73}$$

through Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned} \rho\int_0^\Gamma \nu_t\int_0^t h(t-s)\,\nu_t(s)\,dsdx \leq\; & \tilde{\varepsilon}_1\bar{h}\int_0^\Gamma\int_0^t h(t-s)\left(\nu_t(s)\right)^2 dsdx \\ & + \frac{\rho^2}{4\tilde{\varepsilon}_1}\int_0^\Gamma \nu_t^2dx \quad \forall\tilde{\varepsilon}_1 > 0. \end{aligned} \tag{4.74}$$

By employing (4.74) in (4.73), we obtain (4.72). ■
Now, for a $\tilde{N}$ sufficiently large, we define the Lyapunov functional as follows:

$$\tilde{\mathcal{L}}(t) = \tilde{N}\tilde{E}(t) + \tilde{M}_1(t) + M_4(t). \tag{4.75}$$

It is clear, as stated in the theorem (4.3), that there exist two positive constants $\tilde{d}_1,\ \tilde{d}_2 > 0$ satisfying

$$\tilde{d}_1\left(\tilde{E}(t) + M_4(t)\right) \leq \tilde{\mathcal{L}}(t) \leq \tilde{d}_2\left(\tilde{E}(t) + M_4(t)\right),\ \forall t \geq 0. \tag{4.76}$$

Theorem 4.6 *If ν is a solution of the system (4.68), then there exist positive constant $\tilde{\gamma}_1 > 0$ fulfills this inequality*

$$\tilde{\mathcal{L}}'(t) \leq -\tilde{\gamma}_1\left(\tilde{E}(t) + M_4(t)\right). \tag{4.77}$$

Proof. Differentiating $\tilde{\mathcal{L}}(t)$ and exploiting (4.58)-(4.70)-(4.72), we get

$$\begin{aligned}\tilde{\mathcal{L}}'(t) \leq\ & -\left(\frac{\tilde{N}\rho h(t)}{2} - \tilde{H}_1(0) - \left(\rho + \frac{\rho^2}{4\tilde{\varepsilon}_1}\right)\right)\int_0^\Gamma \nu_t^2 dx \\ & -\alpha_1\int_0^\Gamma \nu_x^2 dx - \left(1-\tilde{\varepsilon}_1\bar{h}\right)\int_0^\Gamma\int_0^t k(t-s)\,\nu_t^2(s)\,dsdx \\ & -\varsigma M_4(t),\end{aligned} \tag{4.78}$$

we choose $\tilde{\varepsilon}_1$ small enough so that

$$1 - \tilde{\varepsilon}_1\bar{h} > 0.$$

Also, we choose $\tilde{N}$ big enough so that

$$\left(\frac{\tilde{N}\rho h(t)}{2} - \tilde{H}_1(0) - \left(\rho + \frac{\rho^2}{4\tilde{\varepsilon}_1}\right)\right) > \min\left(\left(1-\tilde{\varepsilon}_1\bar{h}\right), \alpha_1\right) \quad \forall t \geq 0.$$

■

Now, by using (4.76) and (4.77), we obtain

$$\tilde{\mathcal{L}}'(t) \leq -\frac{\tilde{\gamma}_1}{\tilde{c}_2}\tilde{\mathcal{L}}(t),$$

this implies

$$\tilde{c}_1\left(\tilde{E}(t) + M_4(t)\right) \leq e^{-\frac{\tilde{\gamma}_1}{\tilde{c}_2}t}.\tilde{\mathcal{L}}(0), \tag{4.79}$$

also, using the fact that $M_4(t)$ is positive, we obtain

$$\tilde{E}(t) \leq e^{-\frac{\tilde{\gamma}_1}{\tilde{c}_2}t}\frac{\tilde{\mathcal{L}}(0)}{\tilde{c}_1}. \tag{4.80}$$

In other words, the energy we defined in (4.69) which is related to the system (4.68), exponentially decreasing.

Now, we give a simple example of the kernel h that satisfies the following hypotheses: $(H_1) - (H_2)$

4.5. Exponential energy decay of quasi-static / electrostatic piezoelectric beams (the magnetic effects are negligible) subject to a neutral delay

Example 4.1 For any $\varsigma > 0$, let h be the function defined as follows

$$\begin{array}{rcl} h : \mathbb{R}_+ & \rightarrow & \mathbb{R}_+ \\ t & \rightarrow & e^{-2\varsigma t}. \end{array}$$

1. It is clear that the function h is nonnegative and continuously differentiable.
2. $\forall t \geq 0 \quad h'(t) = -2\varsigma e^{-2\varsigma t} \leq 0,\ \bar{h} = \int_0^\infty h(t)\,dt = \int_0^\infty e^{-2\varsigma t}dt = \left[-\frac{1}{2\varsigma}e^{-\varsigma t}\right]_0^\infty < \infty.$
3. $\int_0^\infty e^{\varsigma t} h(t)\,dt = \int_0^\infty e^{-\varsigma t}dt = \left[-\frac{1}{\varsigma}e^{-\varsigma t}\right]_0^\infty < \infty$, this implies that $e^{\varsigma t}h \in L^1(\mathbb{R}_+)$.

Conclusion and open problem: In this chapter, a one-dimensional system of piezoelectric beams has been considered in the presence of a distributed delay of neutral type added to the first equation. Our main goal in this research is to determine the asymptotic behavior (stability or instability) of this system without adding any damping term. Under some appropriate assumptions on the kernel of the neutral delay term, we proved the global well-posedness of the system by using the classical Faedo-Galerkin method. Furthermore, based on the energy method, which depends on constructing a suitable Lyapunov functional, we showed that, despite delays are known to be of a destructive nature in the general case, this system is exponentially stable without any relationship between the system parameters. Finally, we obtained the same results in the electrostatic case.

It is an interesting open problem to study the stability or instability of the following system

$$\begin{cases} \rho\nu_{tt} - \alpha\nu_{xx} + \gamma\beta\varphi_{xx} + \rho\int_0^t h(s)\,\nu_{tt}(t-s)\,ds = 0, & \text{in } (0,\Gamma)\times(0,\infty), \\ \mu\varphi_{tt} - \beta\varphi_{xx} + \gamma\beta\nu_{xx} = 0, & \\ \nu(0,t) = \alpha\nu_x(\Gamma,t) - \gamma\beta\varphi_x(\Gamma,t) = 0, & t \geq 0, \\ \varphi(0,t) = \varphi_x(\Gamma,t) - \gamma\nu_x(\Gamma,t) = 0, & \\ \nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x), & x \in (0,\Gamma), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x). & \end{cases}$$

That is to say, the same problem that we studied, but in the absence of the next term $\rho h(t)\,\nu_t(0)$.

CHAPTER 5

Finite time blow up of solutions for a Kirchhoff beam equation with delay and variable exponent

5.1 Introduction

*L*ebesgue spaces with variable exponents, often referred to as variable exponent Lebesgue spaces are an extension of the classical Lebesgue spaces where the exponent is vary. The problems with variable exponents arise in many branches of the sciences, such as electrorheological fluids, nonlinear elasticity theory, and image processing (see [24, 80]). In 1883, Kirchhoff [52] first proposed the following wave equation problem which represents the nonlinear vibration of an elastic string

$$\rho h\frac{\partial^2 u}{\partial^2 t}-\left(p_0+\frac{Eh}{2\Gamma}\int_0^\Gamma\left(\frac{du}{dx}\right)^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,\ 0<x<\Gamma,\ t\geq 0,$$

where ρ represents the mass density of the string, E is the Young coefficient, h represents the cross-sectional area of the stretched string, p_0 represents the initial axial tension, Γ represents the length of the string, and $u=u(x,t)$ is the transverse displacement in space x and time t. The expansion model of the equation in higher dimensional space is as follows:

$$u_{tt}-M\left(\int_\Omega|\nabla u|^2dx\right)\Delta u=f(x,u),\ x\in\Omega\subseteq\mathbb{R}^n,$$

where, u denotes the vibration displacement of the string, $f(x,u)$ denotes the external force. Woinowsky-Krieger [96], the author first introduced the one-dimensional nonlinear equation of vibration of beams, which is given by

$$u_{tt}(x,t)+\alpha u_{xxxx}(x,t)-\left(\beta+\gamma\int_0^\Gamma u_x^2dx\right)u_{xx}(x,t)=0,$$

where γ, β, α are positive physical constants. Guo, Bao-Zhu and Guo, Wei [36] considered the following Kirchhoff-type nonlinear beam

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) - F\left(\int_0^\Gamma y_x^2(x,t)\,dx\right) y_{xx}(x,t) = 0, \\ y(0,t) = y_x(0,t) = y_{xx}(\Gamma,t) = 0,\ t \geq 0, \\ y_{xxx}(\Gamma,t) - F\left(\int_0^\Gamma y_x^2(x,t)\,dx\right) y_x(\Gamma,t) = u(t),\ t \geq 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), \\ y_{out}(t) = y_t(\Gamma,t), \end{cases}$$

where $y_{out}(t)$ denotes the measured signal of the system at time t. The existence and uniqueness of the classical solution of the problem are justified by Galerkin approximation, and by the energy multiplier method, they proved the exponential stability that is dependent on initial data. In [26] a nonlinear viscoelastic Kirchhoff plate equation that incorporated a time delay term in the internal feedback is considered by Feng and Li. Under the appropriate assumptions, the energy perturbation method was employed to establish the general rates of energy decay for this problem. Feng in [27] studied the following plate equation with a time delay and a memory term in the internal feedback

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(u) = 0,$$

by employing the Faedo-Galerkin approximations that depend on some energy estimates. global existence and uniqueness of the problem were established. Moreover, under suitable assumptions and by using energy method managed to prove the general decay result of the solution. In [99] Yüksekkaya and Pişkin under suitable conditions, established the growth of the solution of the delayed Kirchhoff-type viscoelastic equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t u(t-q)\,\Delta u(q)\,dq + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)|\, u_t(t-q)\,dq = b\,|u|^{p-2}\,u,$$

where $b > 0,\ p > 2$ and $M(s) = 1 + s^\gamma$. Kafini and Messaoudi [46] considered the following delay wave equation, incorporating a logarithmic nonlinear source term

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x,t-\tau) = u|u|^{p-2}\ln|u|^k \quad , \quad x \in \Omega, \quad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{5.1}$$

under the assumption $|\mu_2| \leq \mu_1$, the local well-posedness was established using semigroup theory and proved a finite time blow up result. Also, Kafini and Messaoudi in [45] took care of the study of the nonlinear wave equation

$$u_{tt} - \Delta u + \mu_1 u_t\,|u_t|^{m(x)-2} + \mu_2 u_t(t-\tau)\,|u_t|^{m(x)-2}(t-\tau) = b\,|u|^{p(x)-2}\,u, \tag{5.2}$$

in the absence of the source term, this means $b = 0$, proved a decay result, while in the presence of the source term ($b \neq 0$), they proved a global nonexistence result. Recently, in [33] Jorge Ferreira et al. extended the work of Kafini and Messaoudi [45] in the absence of the source term $b\,|u|^{p(x)-2}\,u$ (this means $b = 0$) to the following Kirchhoff beam problem

$$\begin{aligned} &u_{tt} + \Delta^2 u - M\left(\|\nabla u\|_2^2\right)\Delta u + \mu_1 u_t\,(x,t)\,|u_t|^{m(x)-2}\,(x,t) + \\ &\mu_2 u_t\,(x,t-\tau)\,|u_t|^{m(x)-2}\,(x,t-\tau) \;=\; 0, \end{aligned}$$

they proved the exponential and polynomial stability results based on Komornik's inequality. Based on the paper [33] we consider the next problem

$$\begin{cases} \varpi_{tt} + \Delta^2\varpi - M\left(\|\nabla\varpi\|_2^2\right)\Delta\varpi + \mu_1\varpi_t\,(x,t)\,|\varpi_t|^{m(x)-2}\,(x,t) & \\ +\mu_2\varpi_t\,(x,t-\tau)\,|\varpi_t|^{m(x)-2}\,(x,t-\tau) = b\varpi\,|\varpi|^{p(x)-2}\,, & \text{in } \Omega\times[0,\infty)\,, \\ \varpi\,(x,t) = \Delta\varpi = 0, & \text{in } \partial\Omega\times[0,\infty)\,, \\ \varpi\,(x,0) = \varpi_0\,(x)\,, & \text{in } \Omega, \\ \varpi_t\,(x,0) = \varpi_1\,(x)\,, & \text{in } \Omega, \\ \varpi_t\,(x,t-\tau) = f_0\,(x,t-\tau)\,, & \text{in } \Omega\times(0,\tau)\,. \end{cases} \tag{5.3}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. $\tau > 0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number it satisfies

$$|\mu_2| < \mu_1. \tag{5.4}$$

M is a positive C^1-function given by the relation

$$M\,(s) = 1 + s^{\gamma},$$

for $s \geq 0$, and $\gamma > 0$. The exponents $m(.)$ and $p(.)$ are given continuous functions on $\bar{\Omega}$ and satisfy

$$2 \leq m^- \leq m\,(x) \leq m^+ \leq p^- \leq p\,(x) \leq p^+ \leq \frac{2\,(n-1)}{n-2} \qquad n \geq 3, \tag{5.5}$$

where

$$\begin{aligned} m^- &= ess\inf_{x\in\Omega}\, m\,(x)\,,\; m^+ = ess\sup_{x\in\Omega}\, m\,(x)\,, \\ p^- &= ess\inf_{x\in\Omega}\, p\,(x)\,,\; \text{and } p^+ = ess\sup_{x\in\Omega}\, p\,(x)\,. \end{aligned}$$

5.2 Preliminaries

We define the variable-exponent in Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(.)}\,(\Omega) = \left\{\varpi : \Omega \to \mathbb{R} \qquad \text{measurable in } \Omega : \int_\Omega |\varpi|^{p(.)}\,dx < \infty\right\},$$

and the Luxemburg-type norm by

$$\|\varpi\|_{p(.)} = \inf\left\{\lambda > 0 : \int_{\Omega}\left|\frac{\varpi}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Equipped with this norm, the space $L^{p(\cdot)}(\Omega)$ is a Banach space [5].

Now, we proceed to define the variable-exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{\varpi \in L^{p(\cdot)}(\Omega) : |\nabla\varpi| \text{ exists and } |\nabla\varpi| \in L^{p(\cdot)}(\Omega)\right\},$$

the Sobolev space with a variable exponent with respect to the norm

$$\|\varpi\|_{1,p(.)} = \|\varpi\|_{p(.)} + \|\nabla\varpi\|_{p(.)},$$

constitutes a Banach space. The space $W_0^{1,p(.)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. For $\varpi \in W_0^{1,p(.)}(\Omega)$, we can define an equivalent norm

$$\|\varpi\|_{1,p(.)} = \|\nabla\varpi\|_{p(.)}.$$

We also suppose that $p(.)$, $m(.)$ satisfy the log-Holder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{5.6}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 5.1 **([6])** *If $P : \bar{\Omega} \to [1, \infty)$ is continuous, and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2}, \ n \geq 3,$$

then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 5.2 (Poincaré inequality [6]) *Let Ω be a bounded domain of $\mathbb{R}^n$ and suppose $p(.)$ satisfies (5.6). Then*

$$\exists C > 0 : \quad \|\varpi\|_{p(.)} \leq C\|\nabla\varpi\|_{p(.)} \qquad \text{for all } \varpi \in W_0^{1,p(.)}(\Omega),$$

where $C = C(p^-, p^+, |\Omega|) > 0$.

Lemma 5.3 **([24])** *If $p : \Omega \to [1, \infty)$ is a measurable function and $p^+ < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Lemma 5.4 (Hölder's Inequality [24]) *Let $p, q, s \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \qquad \text{for a.e. } y \in \Omega,$$

if $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(.)} \leq 2\|f\|_{p(.)}\|g\|_{q(.)}.$$

Lemma 5.5 (Unit Ball Property [24]) *Let p be a measurable function on Ω. Then*

$$\|f\|_{p(.)} \leq 1 \text{ if and only if } \varrho_{p(.)}(f) \leq 1,$$

where

$$\varrho_{p(.)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx.$$

Lemma 5.6 ([6]) *If $p \geq 1$ is a measurable function on Ω. then*

$$\min\left\{\|\varpi\|_{p(.)}^{p^-}, \|\varpi\|_{p(.)}^{p^+}\right\} \leq \varrho_{p(.)}(\varpi) \leq \max\left\{\|\varpi\|_{p(.)}^{p^-}, \|\varpi\|_{p(.)}^{p^+}\right\},$$

for a.e. $x \in \Omega$ and for any $\varpi \in L^{p(.)}(\Omega)$.

5.3 Existence of solutions

As in [66] we introduce the new variable

$$z(x, \rho, t) = \varpi_t(x, t - \rho\tau) \;\; x \in \Omega, \;\; \rho \in (0,1), \;\; t > 0.$$

Subsequently, the problem (5.3) can be expressed as:

$$\begin{cases} \varpi_{tt} + \Delta^2\varpi - M\left(\|\nabla\varpi\|_2^2\right)\Delta\varpi + \mu_1\varpi_t(x,t)|\varpi_t|^{m(x)-2}(x,t) & \text{in } \Omega \times [0,\infty), \\ +\mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} = b\varpi|\varpi|^{p(x)-2}, & \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Omega, \;\; \rho \in (0,1), \;\; t > 0 \\ \varpi(x,t) = \Delta\varpi = 0, & \text{in } \partial\Omega \times [0,\infty), \\ \varpi(x,0) = \varpi_0(x), & \text{in } \Omega, \\ \varpi_t(x,0) = \varpi_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega \times (0,1). \end{cases} \tag{5.7}$$

Similar to the research conducted by Kafini and Messaoudi [46], we can write the following definition:

Definition 5.1 Fix $T > 0$. We call (ϖ, z) a strong solution of (5.7) if

$$\begin{aligned} \varpi &\in W^{2,\infty}\left((0,T), L^2(\Omega)\right) \cap W^{1,\infty}\left((0,T), H_0^1(\Omega)\right) \cap L^\infty\left((0,T), H^2(\Omega) \cap H_0^1(\Omega)\right), \\ \varpi_t &\in L^{m(.)}(\Omega \times (0,T)), \\ z &\in W^{1,\infty}\left([0,1] \times (0,T), \; L^2(\Omega)\right) \cap L^\infty\left([0,1], \; L^{m(.)}(\Omega \times (0,T))\right), \end{aligned}$$

and (ϖ, z) satisfies (5.7) in the following sense

$$\begin{aligned} &\int_{\Omega} \varpi_{tt}\nu dx + \int_{\Omega} \Delta^2\varpi\nu dx - M\left(\|\nabla\varpi\|_2^2\right)\int_{\Omega} \Delta\varpi\nu dx + \mu_1\int_{\Omega} \varpi_t(x,t)|\varpi_t|^{m(x)-2}(x,t)\nu dx \\ &+\mu_2\int_{\Omega} z(x,1,t)|z(x,1,t)|^{m(x)-2}\nu dx \; = b\int_{\Omega} \varpi|\varpi|^{p(x)-2}\nu dx, \end{aligned}$$

and

$$\tau \int_{\Omega} z_t(x,\rho,t)\, w dx + \int_{\Omega} z_\rho(x,\rho,t)\, w dx = 0,$$

for a.e. $t \in [0,T)$ and for $(\nu, w) \in H_0^2(\Omega) \cap L^2(\Omega)$.

5.4 Global nonexistence

Lemma 5.7 *The energy function E, given by*

$$\begin{aligned} E(t) &= \left(\frac{1}{2}\|\varpi_t\|_2^2 + \frac{1}{2}\|\Delta\varpi\|_2^2 + \frac{1}{2}\|\nabla\varpi\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla\varpi\|_2^{2(\gamma+1)} \right. \\ & \left. -b\int_{\Omega}\frac{1}{p(x)}|\varpi|^{p(x)}dx + \int_{\Omega}\int_0^1 \frac{1}{m(x)}\xi(x)|z(x,\rho,t)|^{m(x)}d\rho dx \right). \end{aligned} \tag{5.8}$$

Its derivative achieves the following

$$E'(t) \le -C_0\left(\int_{\Omega} |\varpi_t|^{m(x)}(x,t)\,dx + \int_{\Omega}|z(x,1,t)|^{m(x)}dx \right), \tag{5.9}$$

where ξ is a continuous function satisfying

$$\tau|\mu_2|(m(x)-1) < \xi(x) < \tau(\mu_1 m(x) - |\mu_2|) \qquad x \in \bar{\Omega}. \tag{5.10}$$

Proof. By multiplying equation $(5.7)_1$ by ϖ_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{2dt}\int_{\Omega}\varpi_t^2 dx + \int_{\Omega}\Delta^2\varpi\varpi_t dx + \left(1+\|\nabla\varpi\|_2^{2\gamma}\right)\frac{d}{2dt}\int_{\Omega}(\nabla\varpi)^2dx + \mu_1\int_{\Omega}|\varpi_t|^{m(x)}(x,t)\,dx \\ &+\mu_2\int_{\Omega}\varpi_t z(x,1,t)|z(x,1,t)|^{m(x)-2}dx \;=\; b\int_{\Omega}\varpi_t\varpi|\varpi|^{p(x)-2}dx. \end{aligned} \tag{5.11}$$

Through (5.11), we find

$$\begin{aligned} &\frac{d}{2dt}\left[\|\varpi_t\|_2^2 + \|\Delta\varpi\|_2^2 + \|\nabla\varpi\|_2^2 + \frac{1}{(\gamma+1)}\|\nabla\varpi\|_2^{2(\gamma+1)}\right] - b\frac{d}{dt}\int_{\Omega}\frac{1}{p(x)}|\varpi|^{p(x)}dx \\ &= \;-\mu_1\int_{\Omega}|\varpi_t|^{m(x)}(x,t)\,dx - \mu_2\int_{\Omega}\varpi_t|z(x,1,t)|^{m(x)-1}dx. \end{aligned} \tag{5.12}$$

Now multiplying $(5.7)_2$ by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$ and integrating over $\Omega\times(0,1)$, we get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega}\int_0^1\frac{1}{m(x)}\xi(x)|z(x,\rho,t)|^{m(x)}d\rho dx &= -\frac{1}{\tau}\int_{\Omega}\frac{\xi(x)}{m(x)}|z(x,1,t)|^{m(x)}dx \\ &+\frac{1}{\tau}\int_{\Omega}\frac{\xi(x)}{m(x)}|\varpi_t|^{m(x)}dx, \end{aligned} \tag{5.13}$$

summing (5.12) and (5.13) side to side we directly get

$$\begin{aligned}
&\frac{d}{2dt}\|\varpi_t\|_2^2+\frac{d}{2dt}\|\Delta\varpi\|_2^2+\frac{d}{2dt}\|\nabla\varpi\|_2^2+\frac{1}{2(\gamma+1)}\frac{d}{dt}\|\nabla\varpi\|_2^{2(\gamma+1)}\\
&-b\frac{d}{dt}\int_\Omega\frac{1}{p(x)}|\varpi|^{p(x)}dx+\frac{d}{dt}\int_\Omega\int_0^1\frac{1}{m(x)}\xi(x)|z(x,\rho,t)|^{m(x)}d\rho dx=\\
&-\mu_1\int_\Omega|\varpi_t|^{m(x)}(x,t)\,dx+\frac{1}{\tau}\int_\Omega\frac{\xi(x)}{m(x)}|\varpi_t|^{m(x)}dx-\frac{1}{\tau}\int_\Omega\frac{\xi(x)}{m(x)}|z(x,1,t)|^{m(x)}dx\\
&-\mu_2\int_\Omega\varpi_t|z(x,1,t)|^{m(x)-1}dx. \qquad (5.14)
\end{aligned}$$

Employing Young's inequality with $p=m(x)$ and $q=\frac{m(x)}{m(x)-1}$, we find

$$-\mu_2\int_\Omega\varpi_t|z(x,1,t)|^{m(x)-1}dx\le|\mu_2|\int_\Omega\frac{|\varpi_t|^{m(x)}}{m(x)}dx+|\mu_2|\int_\Omega\frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}dx. \qquad (5.15)$$

Substituting (5.15) into relation (5.14), we get

$$\begin{aligned}
E'(t)\ \le\ &-\int_\Omega\left[\mu_1-\left(\frac{|\mu_2|}{m(x)}+\frac{\xi(x)}{\tau m(x)}\right)\right]|\varpi_t|^{m(x)}(x,t)\,dx\\
&-\int_\Omega\left(\frac{\xi(x)}{\tau m(x)}-|\mu_2|\frac{m(x)-1}{m(x)}\right)|z(x,1,t)|^{m(x)}dx. \qquad (5.16)
\end{aligned}$$

We define $C_0=\min\left(\min_{x\in\bar\Omega}f_1(x),\min_{x\in\bar\Omega}f_2(x)\right)$, where

$$\begin{aligned}
f_1(x)&=\left[\mu_1-\left(\frac{|\mu_2|}{m(x)}+\frac{\xi(x)}{\tau m(x)}\right)\right],\\
f_2(x)&=\left(\frac{\xi(x)}{\tau m(x)}-|\mu_2|\frac{m(x)-1}{m(x)}\right).
\end{aligned}$$

Using condition (5.10), the proof is finished. ■

Global nonexistence result. We assume that $E(0)<0$. Also, we put

$$H(t)=-E(t), \qquad (5.17)$$

through (5.9), we get

$$H'(t)=-E'(t)\ge 0. \qquad (5.18)$$

This implies that

$$0<H(0)\le H(t)\le b\int_\Omega\frac{|\varpi|^{p(x)}}{p(x)}dx\le\frac{b}{p^-}\varrho(\varpi), \qquad (5.19)$$

where

$$\varrho(\varpi)=\varrho_{p(.)}(\varpi)=\int_\Omega|\varpi|^{p(x)}dx.$$

We state without proof the following technical lemmas and corollaries (see [63] for the proofs).

Lemma 5.8 *Suppose that condition (5.5) holds. Then there exists $C > 1$ depending on Ω only, such that*

$$\varrho^{\frac{s}{p^-}}(\varpi) \leq C\left(\|\nabla \varpi(t)\|_2^2 + \varrho(\varpi)\right),$$

$\forall \varpi \in H_0^1(\Omega)$ *and* $2 \leq s \leq p^-$.

Corollary 5.1 *Let the assumptions of Lemma (5.8) hold. Then*

$$\|\varpi\|_{p^-}^s \leq C\left(|H(t)| + \|\varpi_t\|_2^2 + \|\varpi\|_{p^-}^{p^-} + \int_\Omega \int_0^1 \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} d\rho dx\right),$$

$\forall \varpi \in H_0^1(\Omega)$ *and* $2 \leq s \leq p^-$.

Lemma 5.9 *Assuming the conditions of Lemma (5.8) hold and (ϖ, z) be a strong solution of (5.7). Then*

$$\varrho(\varpi) \geq C\|\varpi(t)\|_{p^-}^{p^-}.$$

Lemma 5.10 *Let the assumptions of Lemma (5.8) hold and (ϖ, z) be a strong solution of (5.7). Then*

$$\int_\Omega |\varpi|^{m(x)} dx \leq C\left(\varrho^{\frac{m^-}{p^-}}(\varpi) + \varrho^{\frac{m^+}{p^-}}(\varpi)\right).$$

Theorem 5.1 *Assuming that (5.5) and (5.6) are satisfied. Assume further $E(0) < 0$. Then the solution of (5.7) blows up in finite time.*

Proof. We define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega \varpi \varpi_t dx, \tag{5.20}$$

for a sufficiently small ε that will be determined later and

$$0 < \alpha \leq \min\left(\frac{p^- - 2}{2p^-}, \frac{p^- - m^+}{p^-(m^+ - 1)}\right). \tag{5.21}$$

Performing a straightforward differentiation of $L(t)$, we obtain

$$\begin{aligned} L'(t) \quad : &= (1-\alpha) H'(t) H^{-\alpha}(t) + \varepsilon \int_\Omega \varpi_t^2 dx + \varepsilon \int_\Omega \varpi \varpi_{tt} dx \\ : &= (1-\alpha) H'(t) H^{-\alpha}(t) + \varepsilon \int_\Omega \varpi_t^2 dx \\ &+ b\varepsilon \int_\Omega |\varpi|^{p(x)} dx - \varepsilon \|\Delta \varpi\|_2^2 - \varepsilon \|\nabla \varpi\|_2^2 - \varepsilon \|\nabla \varpi\|_2^{2(\gamma+1)} \\ &- \varepsilon \int_\Omega \mu_1 \varpi_t \varpi |\varpi_t|^{m(x)-2} dx - \varepsilon \int_\Omega \mu_2 \varpi z(x,1,t) |z(x,1,t)|^{m(x)-2} dx. \end{aligned} \tag{5.22}$$

Utilizing the definition of $H(t)$ and considering $0 < a < 1$, we deduce

$$\begin{aligned}
&\varepsilon p^{-}(1-a) H(t)+\frac{\varepsilon p^{-}(1-a)}{2}\|\varpi_t\|_2^2+\frac{\varepsilon p^{-}(1-a)}{2}\|\Delta \varpi\|_2^2 \\
&+\frac{\varepsilon p^{-}(1-a)}{2}\|\nabla \varpi\|_2^2+\frac{\varepsilon p^{-}(1-a)}{2(\gamma+1)}\|\nabla \varpi\|_2^{2(\gamma+1)} \\
&+\varepsilon p^{-}(1-a) \int_{\Omega} \int_0^1 \frac{1}{m(x)} \xi(x)|z(x, \rho, t)|^{m(x)} d\rho dx \\
&\leq (1-a) \varepsilon b \int_{\Omega}|\varpi|^{p(x)} dx. \qquad (5.23)
\end{aligned}$$

So, we get

$$\begin{aligned}
L'(t) \; : \; &\geq C_0(1-\alpha) H^{-\alpha}(t)\left(\int_{\Omega}|\varpi_t|^{m(x)}(x, t) dx+\int_{\Omega}|z(x, 1, t)|^{m(x)} dx\right) \\
&+\varepsilon\left(\frac{p^{-}(1-a)+2}{2}\right) \int_{\Omega} \varpi_t^2 dx+\varepsilon p^{-}(1-a) H(t) \\
&+\varepsilon\left(\frac{p^{-}(1-a)-2}{2}\right)\|\Delta \varpi\|_2^2+\varepsilon\left(\frac{p^{-}(1-a)-2}{2}\right)\|\nabla \varpi\|_2^2 \\
&+\varepsilon\left(\frac{p^{-}(1-a)-2(\gamma+1)}{2(\gamma+1)}\right)\|\nabla \varpi\|_2^{2(\gamma+1)}+ab\varepsilon \int_{\Omega}|\varpi|^{p(x)} dx \\
&+\varepsilon p^{-}(1-a) \int_{\Omega} \int_0^1 \frac{\xi(x)|z(x, \rho, t)|^{m(x)}}{m(x)} d\rho dx \\
&-\varepsilon \int_{\Omega} \mu_1 \varpi_t \varpi|\varpi_t|^{m(x)-2} dx-\varepsilon \int_{\Omega} \mu_2 \varpi z(x, 1, t)|z(x, 1, t)|^{m(x)-2} dx. \qquad (5.24)
\end{aligned}$$

By employing Young's inequality, we find for $\delta_1 > 0$

$$\begin{aligned}
\varepsilon \mu_1 \int_{\Omega}|\varpi||\varpi_t|^{m(x)-1} dx \;\leq\; &\varepsilon \mu_1 \int_{\Omega} \frac{1}{m(x)} \delta_1^{m(x)}|\varpi|^{m(x)} dx \\
&+\varepsilon \mu_1 \int_{\Omega} \frac{m(x)-1}{m(x)} \delta_1^{-\frac{m(x)}{m(x)-1}}|\varpi_t|^{m(x)} dx. \qquad (5.25)
\end{aligned}$$

And from it, we find

$$\varepsilon \mu_1 \int_{\Omega}|\varpi||\varpi_t|^{m(x)-1} dx \leq \varepsilon \mu_1 \frac{1}{m^{-}} \int_{\Omega} \delta_1^{m(x)}|\varpi|^{m(x)} dx+\varepsilon \mu_1 \frac{m^{+}-1}{m^{+}} \int_{\Omega} \delta_1^{-\frac{m(x)}{m(x)-1}}|\varpi_t|^{m(x)} dx, \qquad (5.26)$$

and

$$\begin{aligned}
\varepsilon \int_{\Omega} \mu_2|\varpi||z(x, 1, t)|^{m(x)-1} dx \;\leq\; &\varepsilon|\mu_2|\left[\frac{1}{m^{-}} \int_{\Omega} \delta_1^{m(x)}|\varpi|^{m(x)} dx+\right. \\
&\left.\frac{m^{+}-1}{m^{+}} \int_{\Omega} \delta_1^{-\frac{m(x)}{m(x)-1}}|z(x, 1, t)|^{m(x)} dx\right]. \qquad (5.27)
\end{aligned}$$

5.4. Global nonexistence

As in [64], estimates (5.26) and (5.27) remain valid even if δ_1 is time-dependent. Hence, by choosing δ_1 such that

$$\delta_1^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),$$

this, implies

$$\delta_1^{m(x)} = k^{1-m(x)} H^{\alpha(m(x)-1)}(t),$$

for a sufficiently large $k \geq 1$ to be specified later, we get

$$\int_\Omega \delta_1^{-\frac{m(x)}{m(x)-1}} |\varpi_t|^{m(x)} dx = kH^{-\alpha}(t) \int_\Omega |\varpi_t|^{m(x)} dx, \tag{5.28}$$

$$\int_\Omega \delta_1^{-\frac{m(x)}{m(x)-1}} |z(x,1,t)|^{m(x)} dx = kH^{-\alpha}(t) \int_\Omega |z(x,1,t)|^{m(x)} dx, \tag{5.29}$$

$$\begin{aligned} \int_\Omega \delta_1^{m(x)} |\varpi|^{m(x)} dx &= \int_\Omega k^{1-m(x)} H^{-\alpha(1-m(x))}(t) |\varpi|^{m(x)} dx \\ &\leq k^{1-m^-} H^{\alpha\left(m^+-1\right)}(t) \int_\Omega |\varpi|^{m(x)} dx. \end{aligned} \tag{5.30}$$

Utilizing (5.19) and Lemma (5.10), we get

$$H^{\alpha\left(m^+-1\right)} \int_\Omega |\varpi|^{m(x)} dx \leq \tilde{C}\left(\varrho^{\frac{m^-}{p^-}+\alpha\left(m^+-1\right)}(\varpi) + \varrho^{\frac{m^+}{p^-}+\alpha\left(m^+-1\right)}(\varpi)\right). \tag{5.31}$$

Therefore, according to Lemma (5.8) yields

$$H^{\alpha\left(m^+-1\right)} \int_\Omega |\varpi|^{m(x)} dx \leq \tilde{K}\left(\|\nabla\varpi(t)\|_2^2 + \varrho(\varpi)\right), \tag{5.32}$$

$$\begin{aligned} \varepsilon\mu_1 \int_\Omega \varpi |\varpi_t|^{m(x)-1}(x,t)\, dx &\leq \varepsilon\mu_1 \left[\frac{\tilde{K}k^{1-m^-}}{m^-}\left(\|\nabla\varpi(t)\|_2^2 + \varrho(\varpi)\right)\right. \\ &\quad \left. + \frac{(m^+-1)}{m^+} kH^{-\alpha}(t) \int_\Omega |\varpi_t|^{m(x)} dx\right], \end{aligned}$$

and

$$\begin{aligned} \varepsilon \int_\Omega \mu_2 \varpi |z(x,1,t)|^{m(x)-1} dx &\leq \varepsilon|\mu_2| \left[\frac{\tilde{K}k^{1-m^-}}{m^-}\left(\|\nabla\varpi(t)\|_2^2 + \varrho(\varpi)\right)\right. \\ &\quad \left. + \frac{(m^+-1)}{m^+} kH^{-\alpha}(t) \int_\Omega |z(x,1,t)|^{m(x)} dx\right], \end{aligned}$$

this implies that by using (5.24)

$$\begin{aligned} L'(t) \quad : \quad & \geq (1-\alpha) H^{-\alpha}(t) \left(C_0 - \varepsilon \frac{(m^+-1) k\mu_1}{m^+(1-\alpha)} \right) \int_\Omega |\varpi_t|^{m(x)}(x,t)\,dx \\ & + (1-\alpha) H^{-\alpha}(t) \left(C_0 - \varepsilon \frac{(m^+-1) k |\mu_2|}{m^+(1-\alpha)} \right) \int_\Omega |z(x,1,t)|^{m(x)}\,dx \\ & + \varepsilon p^-(1-a) H(t) + \varepsilon \frac{p^-(1-a)-2}{2} \|\Delta\varpi\|_2^2 + \varepsilon \frac{p^-(1-a)+2}{2} \|\varpi_t\|_2^2 \\ & + \varepsilon \left(\frac{p^-(1-a)}{2(\gamma+1)} - 1 \right) \|\nabla\varpi\|_2^{2(\gamma+1)} + \varepsilon \left(ab - \mu_1 \frac{\tilde{K} k^{1-m^-}}{m^-} - |\mu_2| \frac{\tilde{K} k^{1-m^-}}{m^-} \right) \varrho(\varpi) \\ & + \varepsilon \left(\frac{p^-(1-a)-2}{2} - \mu_1 \frac{\tilde{K} k^{1-m^-}}{m^-} - |\mu_2| \frac{\tilde{K} k^{1-m^-}}{m^-} \right) \|\nabla\varpi\|_2^2 \\ & + \varepsilon p^-(1-a) \int_\Omega \int_0^1 \frac{1}{m(x)} \xi(x) |z(x,\rho,t)|^{m(x)}\,d\rho dx. \end{aligned} \tag{5.33}$$

At this point, we select a small enough such that

$$\frac{p^-(1-a)-2}{2} > 0.$$

And

$$\frac{p^-(1-a)}{2(\gamma+1)} - 1 > 0.$$

And k sufficiently large so that

$$ab - \mu_1 \frac{\tilde{K}}{m^- k^{m^- -1}} - |\mu_2| \frac{\tilde{K}}{m^- k^{m^- -1}} > 0.$$

and

$$\frac{p^-(1-a)-2}{2} - \mu_1 \frac{\tilde{K}}{m^- k^{m^- -1}} - |\mu_2| \frac{\tilde{K}}{k^{m^- -1} m^-} > 0.$$

After determining the values of a and k, we then select ε to be sufficiently small such that

$$C_0 - \varepsilon \frac{(m^+-1) k\mu_1}{m^+(1-\alpha)} > 0.$$

And

$$C_0 - \varepsilon \frac{(m^+-1) k |\mu_2|}{m^+(1-\alpha)} > 0.$$

And

$$L(0) := H^{1-\alpha}(0) + \varepsilon \int_\Omega \varpi_0(x) \varpi_1(x)\,dx > 0.$$

Thus, (5.33) takes the form

$$\begin{aligned} L'(t) \quad : \quad & \geq \tilde{C}_0 \Big[H(t) + \|\Delta\varpi\|_2^2 + \|\varpi_t\|_2^2 + \|\nabla\varpi\|_2^{2(\gamma+1)} + \varrho(\varpi) + \|\nabla\varpi\|_2^2 \\ & + \int_\Omega \int_0^1 \frac{1}{m(x)} \xi(x) |z(x,\rho,t)|^{m(x)}\,d\rho dx \Big]. \end{aligned} \tag{5.34a}$$

5.4. Global nonexistence

Next, our objective is to demonstrate the existence of constants σ and ξ, both greater than zero, such that

$$L'(t) > \xi L^{\sigma}(t),$$

by using Holder's and Young's inequalities, we have

$$\|\varpi\|_2 = \left(\int_\Omega \varpi^2 dx\right)^{\frac{1}{2}} \leq \left[\left(\int_\Omega \left(\varpi^2\right)^{\frac{p^-}{2}} dx\right)^{\frac{2}{p^-}} \left(\int_\Omega 1dx\right)^{1-\frac{2}{p^-}}\right]^{\frac{1}{2}} \leq c\|\varpi\|_{p^-}. \tag{5.35}$$

And

$$\left|\int_\Omega \varpi\varpi_t dx\right| \leq \|\varpi_t\|_2 \|\varpi\|_2 \leq c\|\varpi_t\|_2 \|\varpi\|_{p^-},$$

then

$$\left|\int_\Omega \varpi\varpi_t dx\right|^{\frac{1}{1-\alpha}} \leq c\|\varpi_t\|_2^{\frac{1}{1-\alpha}} \|\varpi\|_{p^-}^{\frac{1}{1-\alpha}} \leq c\left[\|\varpi_t\|_2^{\frac{\theta}{1-\alpha}} + \|\varpi\|_{p^-}^{\frac{\mu}{1-\alpha}}\right],$$

where

$$\frac{1}{\mu} + \frac{1}{\theta} = 1.$$

We take $\theta = 2(1-\alpha)$, which implies by (5.21)

$$s = \frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p^-,$$

then we obtain

$$\left|\int_\Omega \varpi\varpi_t dx\right|^{\frac{1}{1-\alpha}} \leq c\left[\|\varpi_t\|_2^2 + \|\varpi\|_{p^-}^s\right].$$

Corollary (5.1) gives

$$\left|\int_\Omega \varpi\varpi_t dx\right|^{\frac{1}{1-\alpha}} \leq \tilde{c}\left[H(t) + \|\varpi_t\|_2^2 + \|\varpi\|_{p^-}^{p^-} + \int_\Omega\int_0^1 \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)} d\rho dx\right].$$

Also by lemma (5.9), we obtain

$$\left|\int_\Omega \varpi\varpi_t dx\right|^{\frac{1}{1-\alpha}} \leq \tilde{c}\left[H(t) + \|\varpi_t\|_2^2 + \varrho(\varpi) + \int_\Omega\int_0^1 \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)} d\rho dx\right].$$

Subsequently,

$$
\begin{aligned}
L^{\frac{1}{1-\alpha}}(t) \quad : \quad &= \left[H^{1-\alpha}(t) + \varepsilon \int_{\Omega} \varpi\varpi_t dx\right]^{\frac{1}{1-\alpha}} \\
&\leq 2^{\frac{\alpha}{1-\alpha}} \left[H(t) + \left|\int_{\Omega} \varpi\varpi_t dx\right|^{\frac{1}{1-\alpha}}\right] \\
&\leq \tilde{c}\left[H(t) + \|\varpi_t\|_2^2 + \varrho(\varpi) + \int_{\Omega}\int_0^1 \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} d\rho dx\right] \\
&\leq \tilde{c}\Big[H(t) + \|\Delta\varpi\|_2^2 + \|\varpi_t\|_2^2 + \|\nabla\varpi\|_2^{2(\gamma+1)} + \|\nabla\varpi\|_2^2 + \varrho(\varpi) \\
&\quad + \int_{\Omega}\int_0^1 \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} d\rho dx\Big].
\end{aligned} \tag{5.36}
$$

Based on (5.34a) and (5.36), we find

$$
L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t). \tag{5.37}
$$

By integrating (5.37) over $(0, t)$, we obtain

$$
L^{\frac{\alpha}{\alpha-1}}(t) \geq \frac{1}{L^{-\frac{\alpha}{\alpha-1}}(0) - \xi\frac{\alpha}{1-\alpha}t}.
$$

Hence, the solution blows up in a finite time T^*, such that

$$
T^* = \frac{1-\alpha}{\alpha\xi L^{\frac{\alpha}{\alpha-1}}(0)},
$$

the proof is completed. ■

5.4. Global nonexistence

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