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Blow up Phenomena For Some Nonlinear Pseudo-Parabolic Equations

Presented by :
Barhoum Aya

Exammination Committee :

Mr. Mecheri Hacene	MCA	University Larbi Tebessi	President
Mrs. Gasri Ahlem	MCA	University Larbi Tebessi	Examiner
Mr. Toualbia Abdelatif	MCB	University Larbi Tebessi	Supervisor

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

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Dedication

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⋆ Barhoum Aya⋆

Abstract

The main objective of this thesis is an analytical study of the blow-up of solutions for certain pseudo-parabolic equations. In the first study, we focused on equations with source and damping terms with fixed exponents. To prove the blow-up, we used the method of differential inequalities. In the second study, we highlighted pseudo-parabolic equations with variable exponents. By also using differential inequalities, we obtained the blow-up of the solution. Finally, we study the blow-up of solutions for pseudo-parabolic equations with variable exponents in the presence of a matrix with variable coefficients.

Keywords: Blow–up , pseudo-parabolic equation , Lower bound , upper bound

الملخص

الهدف الرئيسي من هذه المذكرة هو دراسة تحليلية لانفجار الحل لبعض المعادلات شبه التكافؤية. في الدراسة الاولى ركزنا على المعادلات ذات منبع و كبح و ذات أسس ثابتة. لإثبات الانفجار نستعمل طريقة المراجعات التفاضلية. في الدراسة الثانية سلطنا الضوء على المعادلات شبه تكافؤية ذات أسس متغيرة. أيضا باستخدام المتراجحات التفاضلية تحصلنا على انفجار الحل. اخيرا ندرس انفجار الحل للمعادلات شبه تكافؤية ذات أسس متغيرة و بوجود مصفوفة ذات معاملات متغيرة

الكلمات الرئيسية: الإنفجار، معادلة شبه تكافؤية ، الحد الأدنى، الحد الأقصى.

Résumé

L'objectif principal de cette mémoire est une étude analytique de l'explosion des solutions pour certaines équations pseudo-paraboliques. Dans la première étude, nous nous sommes concentrés sur les équations avec des termes de source et d'amortissement avec des exposants fixes. Pour prouver l'explosion, nous avons utilisé la méthode des inégalités différentielles. Dans la deuxième étude, nous avons mis en lumière les équations pseudo-paraboliques avec des exposants variables. En utilisant également les inégalités différentielles, nous avons obtenu l'explosion de la solution. Enfin, nous étudions l'explosion des solutions pour les équations pseudo-paraboliques avec des exposants variables en présence d'une matrice avec des coefficients variables.

Mots-clés: Explosion, équation pseudo-parabolique, borne inférieure, borne supérieure

Contents

General Introduction

In all this work

* $T > 0$

* Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$, with smooth boundary $\partial\Omega$.

Consider the initial boundary value problem

$$\upsilon(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1}$$

$$\upsilon(x,0) = \upsilon_0(x), \qquad x \in \Omega, \tag{2}$$

for a linear operator differential equations

$$(1 + L_0(x))\,\upsilon_t + L_1(x)\upsilon = f(t,x,\upsilon), \qquad \text{in } \Omega \times (0,\infty) \tag{3}$$

where $L_0(x)$ and $L_1(x)$ are second-order partial differential operators.

Examining the possibility that some evolution problems' solutions blow up in finite time is why we are interested in this work. We are then in the presence of a local time, but not globally.

A variety of nonlinear evolution equations exhibit the blow-up phenomenon. It happens for hyperbolic equations,Schrödinger equations, parabolic equations as well as pseudo-parabolic equations. In this work, we shall deal only with pseudo-parabolic equations.
For the first initial boundary value problem, operators $L_0(x), L_1(x)$ have the form $L_0(x) = L_1(x) = -\Delta$ (Δ is the Laplacian in x) and $f(t,x,\upsilon) = \upsilon^p$. For the second initial boundary value problem $L_0(x) = -\Delta$, $L_1(x) = \Delta_{r(x)}\Delta_{r(x)}$ ($\Delta_{r(x)} = \mathrm{div}(|\nabla\upsilon|^{r(x)-2}\nabla\upsilon)$ is th $r(x)$-Laplacian in x) and $f(t,x,\upsilon) = |\upsilon|^{p(x)-2}\upsilon$. For the third problem $L_0(x) = -\Delta$, $L_1(t,x) = \mathrm{div}(A(x,t)\,|\nabla\upsilon|^{r(x)-2}\nabla\upsilon)$ and $f(t,x,\upsilon) = |\upsilon|^{p(x)-2}\upsilon$.

Historiography.

Many problems of thermodynamics, hydrodynamics, and filtration theory lead to equations of type 3. Let us consider some examples.

1. C. G. Rossby [39] considered one of the earliest equations of type (3) in 1939. It is in the form

$$\Delta D_t \upsilon + \beta D_{x_2} u = 0, \qquad n = 2 \tag{4}$$

It first appeared in research on how certain kinds of ocean waves move. In the literature, it is now referred to as the Rossby wave equation, where Δ represents the Laplacian in x.

2. S. L. Sobolev's equation [44] considered in the study of small oscillations of a rotating ideal fluid is

$$\Delta D_t^2 \upsilon + \omega^2 D_{x_3}^3 u = f(t,x), \qquad n = 3 \tag{5}$$

($\frac{\omega}{2}$ is the angular velocity). S. L. Sobolev developed some new mathematical physics problems in addition to studying the Cauchy problem and the first and second boundary value problems for this equation. This was the first comprehensive analysis of equations that were not solved for the maximum derivative in terms of time. This is why now (5) is called the Sobolev equation.

3. In 1960, G. I. Barenblatt, J. P. Zheltov and I. N. Kochina [5] examined one of the first equations of type (3). It has the form

$$(\eta\Delta - 1)\, D_t \upsilon + \beta \Delta \upsilon = f(t,x), \qquad n = 3 \tag{6}$$

It explains why uniform liquids seep through fissure rocks(Δ is the Laplacian in x).
Moreover, the equation (6), for $n = 1$ appeared in other physical papers unrelated to seepage problems (see, for example, [11], [12]).

4. For the problem of non-stationary processes in semiconductors in the presence of sources, the following equation was found

$$D_t \upsilon - \Delta D_t \upsilon - \Delta \upsilon = f(\upsilon), \tag{7}$$

the term $\Delta \upsilon_t - \upsilon_t$ represents the rate at which the free electron density changes, while $\Delta \upsilon$ represents the linear dissipation of the free charge current. The source term $f(\upsilon)$, which can be expressed as either $f(\upsilon) = \upsilon^{p-1}$ or $f(\upsilon) = |\upsilon|^{p-2}\, \upsilon$, represents a source of free electron current (see [24])

5. Studying the aggregation of populations leads to the equation

$$\upsilon_t - \mu \Delta \upsilon_t - \beta \Delta \upsilon = |\upsilon|^p - \frac{1}{\Omega} \int_\Omega |\upsilon|^p \, dx, \tag{8}$$

the function $\upsilon(x,t)$ is utilized to denote the density of the species at a particular position x and time t. The rate of reproduction is defined as the reaction term $|\upsilon|^p - \frac{1}{\Omega}\int_\Omega |\upsilon|^p \, dx$. The nonlocal term $\int_\Omega |\upsilon|^p \, dx$ can be used to express how, as a result of spatial inhomogeneity, the evolution of a species at a given point in space depends not only on the density close by but also on the mean value of all species present (see [21][10][34]). Nonlocal reaction terms can also be used to characterize the behaviors of cancer cells in response to therapy or the Darwinian evolution of a structured population density (see [29][28]).

The appearance of equations of type (3) - which are also known as Sobolev-type equations or Sobolev-Galpern-type equations, was first introduced by S. Sobolev[32]- in many physical applications simulated the interest of mathematicians in them. Sobolev-type equations have taken several different paths since the 1950s. In particular, the qualitative behavior of solutions to certain initial boundary value problems has been examined in conjunction with spectral problems. A general theory of boundary value problems for those equations was constructed, and it was the subject of numerous papers.

H. Di's, X. Zhu's, M.I. Vishik's, G. I. Eskin's, S. A. Galpern's, Y. Zheng's, J. Zhou's, Pavlov's, R. Z. Xu's, B. K. Romanko's, R. E. Showalter's, A. G. kostyuchenko's and other works were devoted to the construction of a general theory of boundary value problems for Sobolev-type equations (see, for example, the bibliography in [46][14]).

Object, method, and aim.

The object of this work is to answer the questions usually posed in the study of the blow-up phenomenon, which includes which solutions blow up and where and how they do. We use a differential inequality technique. The essence of the method is to show that $G = \|v\|_{H_0^1(\Omega)}$ satisfies a differential inequality which leads to blow up in finite time. This method is used in([14][25][31] [33][38]...). The first goal considers the study of the blow-up of solution for the nonlinear pseudo-parabolic equation with damping and source terms. The second goal is also centered on the study of these problems for the nonlinear pseudo-parabolic equation with damping and source terms of variable-exponent types. The third one is to prove that the solution of the pseudo-parabolic equation with damping and source of variable-exponent type with the presence of a matrix, blows up in finite time.

Contents.

The monograph contains four chapters, except Introduction and References. **The first chapter** is auxiliary and contains useful later facts about functional analysis, variable exponent space, and function theory. **In the second chapter**, we consider the following pseudo-parabolic equation with source and damping terms

$$\begin{cases} v_t - \Delta v - \Delta v_t = v^p, & \text{in } \Omega \times (0,\infty), \\ \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $p > 1$. First, we present the theorem of the existence of a solution. Next, we show that the energy is decreasing and we use some assumptions for initial data to prove a blow-up result. **In the third chapter**, we consider the following nonlinear pseudo-parabolic equation with damping and source terms of variable-exponent types

$$\begin{cases} v_t - \operatorname{div}(|\nabla v|^{r(x)-2}\nabla v) - \Delta v_t = |v|^{p(x)-2}v, & \text{in } \Omega \times (0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $r(.)$ and $p(.)$ are measurable functions. First, we present the result concerning the existence of the local solution of this system. Next, we show that the energy is decreasing, and by some assumptions for the variable exponents $r(.), s(.)$, and the initial data, we obtain the blow-up results. **In the fourth chapter**, we consider the following nonlinear pseudo-parabolic equation with damping, source, and with presence of a matrix with variable entries in the divergence operator

$$\begin{cases} v_t - \Delta v_t - \operatorname{div}(A(x,t)\,|\nabla v|^{r(x)-2}\nabla v) = |v|^{s(x)-2}v, & \text{in } \Omega \times (0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $A(x,t) = (a_{ij}(x,t))_{i,j}$ is a matrix that satisfies some conditions to be specified later. We assume that the conditions on $p(x)$ and $r(x)$ given in Chapter 3, hold. First, we present the result concerning the existence of the local solution of this system, then we show the usual energy is decreasing. After that, we use some assumptions for the variable exponents $r(.), s(.)$, the initial data, and the matrix $A(., t)$ to prove that the solution becomes unbounded at a finite time T, and find an upper bound for this time with a negative initial energy

Chapter 1

Auxiliary material

Elements of Functional Analysis - Variable Exponent Spaces - Important Lemmas - Notion of blow-up.

In this chapter, We specify some of the symbols we will constantly use throughout the memory and recall some basic notions about differential operators, elements of Functional Analysis, Variable Exponent Spaces, and the notion of blow-up. For more information see [41] [15] [8].

1.1 Elements of Functional Analysis

1.1.1 Basic Notations and Facts

* The gradient of a function u is defined by:

$$\operatorname{grad} u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right), \text{ then } |\nabla u|^2 = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2. \tag{1.1}$$

* The divergence of a function u is defined by:

$$\operatorname{div} u = \nabla \cdot u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + ... + \frac{\partial u}{\partial x_n} = \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}. \tag{1.2}$$

* The Laplacian of u

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}. \tag{1.3}$$

* $\Delta_{r(x)}$ is the so-called $r(x)$-Laplace operator defined as

$$\Delta_{r(x)} u = \operatorname{div}(|\nabla u|^{r(x)-2}\,\nabla u), \tag{1.4}$$

* If u has continuous partials up to the order k (included) in the domain Ω, we say that u is of class $C^k(\Omega), k \geq 1$. The class of functions that are continuously differentiable with any order in Ω is represented by $C^\infty(\Omega)$.
* $C_0^k(\Omega), 0 \leq k \leq \infty$ denotes the vector subspace of the compactly supported $C^k(\Omega)$ functions in Ω.
* The space $C_0^\infty(\Omega)$which we will also note $D(\Omega)$, is called the test function space on Ω.
* An operator A is called linear, if $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$ for any $x, y \in D(A)$ and for any $\lambda, \mu \in \mathbb{R}$ $(\mathbb{C})$

Lemma 1.1 *Let u and v two functions of $C^1(\Omega)$, for all : $1 \leq i \leq n$ we have*

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int\limits_{\Omega} \frac{\partial v}{\partial x_i} u dx + \int\limits_{\partial\Omega} v u \eta_i ds \tag{1.5}$$

Where $\eta_i(x) = \cos(\eta, x_i)$ cosinus direction of the angle between external regulator on $\partial\Omega$ in x point and axis of x_i

Corollary 1.1 *(**Green's Formula**). Let Ω be a bounded open of class C^1. Then for all functions $u \in C^2\left(\bar{\Omega}\right)$ and $v \in C^1\left(\bar{\Omega}\right)$ we have*

$$\int\limits_{\Omega} \Delta u(x) v(x) dx = \int\limits_{\Gamma} \frac{\partial u}{\partial \eta}(x) v(x) d\Gamma - \int\limits_{\Omega} \nabla u(x).\nabla v(x) dx, \tag{1.6}$$

Where $\frac{\partial u}{\partial \eta} = \nabla u(x) \cdot \eta(x)$ (normal derivative of u).

1.1.2 Norms and Banach Spaces

Definition 1.1 *: Over the scalar field $\mathbb{R}$ or $\mathbb{C}$, let X a linear space . A real function that satisfies the following properties for every $x, y \in X$ and every scalar λ is called a norm in X*

1. $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$ (positivity)
2. $\|\lambda x\| = |\lambda| \, \|x\|$ (homogeneity)
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality)

Definition 1.2 *: A linear space X with a norm is called a normed space.With a norm determined by the distance between two vectors, which is given by*

$$d(x, y) = \|x - y\|$$

which makes X a metric space and allows us to define a topology in X and a notion of convergence in a very simple way.

Notation 1.1 *: A normed space in which every Cauchy sequence converges is called* ***complete*** *and deserves a special name.*

Notation 1.2 *: Banach space is the name given to a complete, normed linear space.*

Definition 1.3 *: X, Y linear spaces, endowed with the norms X and Y, respectively, and let $F : X \to Y$. We say that F is continuous at $x \in X$ if*

$$\|F(y) - F(x)\|_Y \to 0 \quad \textit{when} \ \|y - x\|_X$$

or, equivalently, if, for every sequence $\{x_m\} \subset X$,

$$\|x_m - x\|_X \to 0 \quad \textit{implies} \ \|F(x_m) - F(x)\|_Y \to 0$$

** If F is continuous at every x in X, then it is continuous in X. Specifically:*

Proposition 1.1 *[41]. Every norm in a linear space X is continuous in X.*

Notation 1.3 *: A few illustrations are necessary.*

Spaces of continuous functions: Let $X = C(A)$ represent the set of continuous functions (real or complex) on A, where A is a compact subset of $\mathbb{R}^n$ that has the norm

$$\|f\|_{C(A)} = \max_A |f| \tag{1.7}$$

A sequence $\{f_m\}$ converges to f in $C(A)$ if

$$\max_A |f_m - f| \to 0$$

in other words, if f_m converges to f in A uniformly. $C(A)$ is a **Banach space** because a uniform limit of continuous functions is continuous.

Summable and bounded functions: Let p a natural number and Ω an open set in $\mathbb{R}^n$. The set of functions f such that $|f|^p$ is Lebesgue integrable in Ω is denoted by $X = L^p(\Omega)$.

Equipped with the norm

$$\|\zeta\|_{L^p(\Omega)} = \left(\int_\Omega |\zeta|^p\right)^{\frac{1}{p}} \tag{1.8}$$

$L^p(\Omega)$ becomes a **Banach space** when equipped with the norm

Definition 1.4 *Let $X = L^\infty(\Omega)$ the set of essentially bounded functions in Ω. Remember that if there exists M such that*

$$|f(x)| \leq M \qquad a.e. on\ \Omega \tag{1.9}$$

then $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is effectively bounded.

The essential supremum of f is the infimum of all numbers M having the property (1.9), and it is represented by

$$\|f\|_{L^\infty(\Omega)} = ess \sup_\Omega |f(x)| \tag{1.10}$$

$\|f\|_{L^\infty(\Omega)}$ is a norm in $L^\infty(\Omega)$, and $L^\infty(\Omega)$ becomes a **Banach space**

Lemma 1.2 ***(Young's Inequality).*** *Let $p, r \in]1, \infty[, s \geq 1$ such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$. Then, for all $a, b \geq 0$, we have*

$$\frac{(ab)^s}{s} \leq \frac{a^p}{p} + \frac{b^r}{r} \tag{1.11}$$

By taking $s = 1$. It follows that for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + c(\varepsilon) b^r, \text{ where } c(\varepsilon) = 1/r(\varepsilon p)^{\frac{r}{p}}.$$

For $p = s = 2$, it comes

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 1.3 ***(Hölder's Inequality).*** *Let $p, r \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{r} = 1$. If $f \in L^p(\Omega)$ and $g \in L^r(\Omega)$, then $fg \in L^1(\Omega)$ with*

$$\|fg\|_1 \leq \|f\|_p \|g\|_r \tag{1.12}$$

*By taking $p = q = 2$, we obtain the **Cauchy-Schwarz inequality***

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 \tag{1.13}$$

1.1.3 Hilbert Spaces

Definition 1.5 : *Let X a linear space over $\mathbb{R}$. An inner or scalar product in X is a function*

$$(.,.) : X \times X \to \mathbb{R}$$

with the following three properties. For every $x, y, z \in X$ and scalars $\lambda, \mu \in \mathbb{R}$:

1. $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$
2. $(x, y) = (y, x)$
3. $(\mu x + \lambda y, z) = \mu(x, z) + \lambda(y, z)$

A linear space endowed with an inner product is called an inner product space.

An inner product induces a norm, given by:

$$\|x\| = \sqrt{(x, x)} \tag{1.14}$$

Definition 1.6 *Let H an inner product space. We say that H is a **Hilbert space** if it is complete with respect to the norm (1.14), induced by the inner product.*

Example 1.1 *$\mathbb{R}^n$ is a Hilbert space with respect to the usual inner product*

$$(X, Y)_{\mathbb{R}^n} = X.Y = \sum_{i=1}^{n} x_i y_i, \quad X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$$

The induced norm is

$$\|X\| = \sqrt{X.X} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \tag{1.15}$$

Example 1.2 *$L^2(\Omega)$ is a Hilbert space (perhaps the most important one) with respect to the inner product*

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} uv dx \tag{1.16}$$

If Ω is fixed, we will simply use the notations (u, v) instead of $(u, v)_{L^2(\Omega)}$ and $\|u\|$ instead of $\|u\|_{L^2(\Omega)}$.

Definition 1.7 *(**Weak Derivative**) Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that and $u \in L^1_{loc}(\Omega)$. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ be a multi-indice. If there exists $g \in L^1_{loc}(\Omega)$ such that*

$$\int_{\Omega} u \frac{\partial^{\alpha_1+\alpha_2,...+\alpha_n}\psi}{\partial^{\alpha_1}x_1...\partial^{\alpha_n}x_n} dx = (-1)^{\alpha_1+\alpha_2,...+\alpha_n} \int_{\Omega} \psi g dx, \quad \forall \psi \in C_0^{\infty}(\Omega)$$

then g is called a weak partial derivative of u of order α.
The function g is denoted by $D^{|\alpha|}u$ or by $\frac{\partial^{\alpha_1+\alpha_2,...+\alpha_n}u}{\partial^{\alpha_1}x_1...\partial^{\alpha_n}x_n}$

Definition 1.8 (***Sobolev Spaces***) *Let $m, p \in \mathbb{N}$. We define the constant exponent Sobolev space $W^{m,p}(\Omega)$ as follows:*

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \text{ such that } D^{|\alpha|}u \in L^p(\Omega) \text{ with } |\alpha| \leq k \right\}.$$

where $|\alpha| = \alpha_1 + \alpha_2, ... + \alpha_n$ equipped with the following norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0<|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)} \tag{1.17}$$

Clearly

$$W^{0,p}(\Omega) = L^p(\Omega)$$

and

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \text{ such that } \nabla u \text{ exists and } \nabla u \in L^p(\Omega)\} \tag{1.18}$$

equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \tag{1.19}$$

we denote $H^1(\Omega) = W^{1,2}(\Omega)$.

Definition 1.9 *The space $W^{1,2}(\Omega)$ equipped with the norm*

$$\|u\|_{H^1}^2 = \|u\|^2 + \|\nabla u\|^2 \tag{1.20}$$

and the inner product

$$(u, v)_{H^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2} \tag{1.21}$$

is a Hilbert space.

* Let $\Omega \subseteq \mathbb{R}^n$. We introduce an important subspace of $H^1(\Omega)$

Definition 1.10 *We denote by $H_0^1(\Omega)$ the closure of $D(\Omega)$ in $H^1(\Omega)$.*

* The following Poincaré inequality represents a significant property of $H_0^1(\Omega)$, which is especially helpful when solving boundary value problems.

Theorem 1.1 *(**Poincaré's inequality**)[41]. Assume that the domain $\Omega \subseteq \mathbb{R}^n$ is bounded. There exists a positive constant C_p (Poincaré's constant) such that, for every $u \in H_0^1(\Omega)$*

$$\|u\| \leq C_p \|\nabla u\| \tag{1.22}$$

1.2 Variable Exponent Spaces

1.2.1 Lebesgue Spaces With Variable Exponents

Definition 1.11 *The Lebesgue space $L^{p(.)}(\Omega)$ is defined by*

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable in } \Omega \ : \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty \text{ for some } \lambda > 0 \right\}$$

where p is a variable-exponent
$L^{p(.)}(\Omega)$ is endowed with the following Luxembourg-type norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

Lemma 1.4 *If $p(.) = p$, where p is constant. Then*

$$\|u\|_{p(.)} = \left(\int_\Omega |u(x)|^p\, dx \right)^{\frac{1}{p}}$$

if $p = 2$

$$\|u\|_2 = \left(\int_\Omega |u(x)|^2\, dx \right)^{\frac{1}{2}}$$

In order to obtain the Poincaré inequality in the variable case, we now introduce the most crucial condition on the variable exponent, known as the log-Hölder continuity condition:

Definition 1.12 *We say that a function $r : \Omega \to \mathbb{R}$ is a log-hölder continuous on Ω, if there exists constant $a > 0$ such that for all $0 < \delta < 1$, we have*

$$|r(x) - r(y)| = \frac{-a}{\log|x-y|} \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta. \tag{1.23}$$

Theorem 1.2 *[15] If $r : \Omega \to [1, \infty[$ is a measurable functions, then $L^{r(.)}(\Omega)$ is a Banach space.*

* These are the Young's and Hölder's inequalities, just as they are in the case of constant exponent.

Lemma 1.5 ***(Young's Inequality)**[15] Let $p, r, s \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad \text{for a.e } x \in \Omega.$$

Then, for all $a, b \geq 0$, *we have*

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{p(.)}}{p(.)} + \frac{(b)^{r(.)}}{r(.)}$$

By taking $s = 1$ *and* $1 < p, r < \infty$, *it follows that, for any* $\varepsilon > 0$, *we have*

$$ab \leq \varepsilon a^p + c(\varepsilon)b^r, \text{ where } c(\varepsilon) = 1/r(\varepsilon p)^{\frac{r}{p}}.$$

For $p = s = 2$, *it comes*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 1.6 *(**Hölder's Inequality**)[15] Let* $p, r, s \geq 1$ *be measurable functions defined on* Ω *satisfying*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad \text{for a.e } x \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ *and* $g \in L^{r(.)}(\Omega)$ *then* $fg \in L^{s(.)}(\Omega)$ *and*

$$\|fg\|_{s(.)} \leq \|f\|_{p(.)} \|g\|_{r(.)} . \tag{1.24}$$

Case $p = q = 2$ *yields the* ***Cauchy-Schwarz inequality***.

1.2.2 Sobolev Spaces With Variable Exponents

The Sobolev space is a vector space of functions with weak derivatives. One motivation for studying these spaces is that solutions of partial differential equations belong naturally to Sobolev spaces. In this section, we define the variable exponent Sobolev spaces and cite some important properties and results related to this class of spaces.

Definition 1.13 *Let* $m \in \mathbb{N}$. *We define the variable exponent Sobolev space* $W^{m,p}(\Omega)$ *as follows:*

$$W^{m,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } D^{|\alpha|}u \in L^{p(.)}(\Omega) \text{ with } |\alpha| \leq k\right\}.$$

where $|\alpha| = \alpha_1 + \alpha_2, ... + \alpha_n$ *equipped with the following norm*

$$\|u\|_{W^{m,p(.)}(\Omega)} = \|u\|_{L^{p(.)}(\Omega)} + \sum_{0<|\alpha|\leq m} \|D^\alpha u\|_{L^{p(.)}(\Omega)} \tag{1.25}$$

Theorem 1.3 *The space* $W^{m,p(.)}(\Omega)$ *is a Banach space, which is seperable if* p *is bounded and reflexive if* $1 < p^- \leq p^+ < \infty$

Remark 1.1 *If* $p(.) = 2$ *and* $m = 1$ *then we set* $H_0^1(\Omega) = W_0^{1,2}(\Omega)$

The version of the Poincaré inequality, in the variable exponent case, is presented in the following theorem.

1.3 Important Lemmas

The version of the Poincaré inequality, in the variable exponent case, is presented in the following theorem.

Theorem 1.4 ***(Poincaré's inequality)***[15]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If p satisfies the log-Hölder inequality (1.23)on Ω, then*

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)} \text{ , for all } u \in W_0^{1,p(.)}(\Omega) \tag{1.26}$$

where C is a positive constant deponding on Ω and $p(.)$. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)} \tag{1.27}$$

Lemma 1.7 ***(Embedding Proprety)*** [15]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Assume that $p, q \in C(\overline{\Omega})$ such that*

$$1 < p^- \leq p(x) \leq p^+ < \infty \text{ and } 1 < q^- \leq q(x) \leq q^+ < \infty \text{ for all } x \in \overline{\Omega}$$

and $p(x) < q^*(x)$ in $\overline{\Omega}$ with $q^* = \begin{cases} \frac{nq(x)}{n-q(x)}, & \text{if } n > q^+, \\ \infty, & \text{if } n \leq q^+, \end{cases}$
then we have continuous and compact embedding $W_0^{1,q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$.

Corollary 1.2 *If $q \in C(\overline{\Omega})$ such that $q \geq 2$ and $q(x) < 2^*$ in $\overline{\Omega}$ with*

$$2^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2, \\ \infty, & \text{if } n \leq 2, \end{cases}$$

then we have continuous and compact embedding $H_0^1(\Omega) \hookrightarrow L^{q(.)}(\Omega)$. So, there exists $C > 0$ such that

$$\|u\|_{L^{q(.)}(\Omega)} \leq C \|u\|_{H_0^1(\Omega)} . \tag{1.28}$$

1.4 Notion of blow-up

1.4.1 Elementary example. Blow-up in ODE.

The ODE problem

$$\begin{cases} \frac{du}{dt} = u^2 & \text{for } t > 0 \\ \\ u(0) = \alpha > 0 \end{cases} \tag{1.29}$$

is the simplest example in which the phenomenon of blow-up appears. Only the finite interval $[0, T[$, where $T = 1/\alpha$, defines the unique solution :

$$u(t) = \frac{1}{T - t} \tag{1.30}$$

and satisfies $\lim_{t \to T} u(t) = \infty$. Inspired by this example, blow-up is defined as a phenomenon for which there is no globally defined solution because it tends to infinity in a finite amount of time.

1.4.2 Blow-up in PDE.

When a problem involves multiple variables, or partial derivatives, the study of blow-up becomes much more complex and fascinating from a mathematical perspective. The usual case is a PDE where the solution depends on a spatial variable $x \in \mathbb{R}^n$, $n \geq 1$ and a time variable $u = u(x, t)$. The so-called pseudo-parabolic equations are a special class of these evolution equations that first appear in the 19th century and are primarily used to model biological and physical processes. We emphasize the use of mechanics, technology, biology, and ecology. Thus we have equations in divergence form

$$\frac{\partial v}{\partial t} = \operatorname{div} A(v, \nabla v, \nabla v_t, x, t) + B(v, \nabla v, x, t), \tag{1.31}$$

the prototype being the semilinear equation

$$\frac{\partial v}{\partial t} = \Delta v + \Delta v_t + f(v) \tag{1.32}$$

We complement our equation with an initial datum

$$v(x, 0) = v_0(x)$$

and also with some boundary condition, usually $v = 0$ at $\partial\Omega$, if Ω is not all of $\mathbb{R}^n$.
A local theory must be established first in the study of blow up; **Theorem** (2.1), **Theorem** (3.1) and **Theorem** (4.1) show that the solution exists and is unique for a small time interval $0 < t < t_0$.
When u is bounded for every $0 < t < T$ but tends to infinity at some point(s), that is the simplest scenario in which T can be finite,

$$u(., t) \in L^\infty(\Omega) \;\; \forall 0 \leq t < T, \qquad \limsup_{t \to T} \|u(., t)\|_\infty = \infty$$

Then we say that u blows up at T, which is the blow-up time.
The works of Kaplan ([23]), Fujita ([18] [19]), Friedman [20]) and others marked the beginning of the mathematical theory of blow-up in the 1960s (of the previous century). The books [6]) and ([42]) are the best sources to start with.

Chapter 2

Blowing-up solution to a pseudo-parabolic equation with source and damping terms

- Introduction - Main tools in the study of blow-up - Blow-up result

2.1 Introduction

In this chapter, we consider the following pseudo-parabolic equation

$$\begin{cases} v_t - \Delta v - \Delta v_t = v^p, & \text{in } \Omega \times (0, \infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0, \infty), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary, $v_0(x) \in H_0^1(\Omega), T \in [0, \infty[$.
Problem 2.1 describes a variety of significant physical and biological phenomena, such as the analysis of nonstationary processes as discussed in [24], and the aggregation of populations as explored in [49].Moreover, equation 2.1 can be regarded as a Sobolev type equation as demonstrated in [44].
For problem 2.1, many results have been obtained, such as the existence and uniqueness in [43], the maximum in [7], asymptotic behavior discussed in ([30], [50]), blow-up phenomena in ([50], [51]), and homogenization explored in ([37]). Especially, in ([50]), the authors proved that there are solutions that blow up in finite time T in $H_0^1(\Omega)$-norm.

In Section 2, first we present the theorem of existence of solution. Next, we show that the energy is decreasing. **In Section 3**, we use some assumptions for initial data to prove a blow-up result. By means of a differential inequality technique, we obtain an upper bound for blow-up time. Also, a lower bound for blow-up time is determined under some other conditions.

Most of results in the chapter were obtained by Peng Luo ([31]) (**2015**). Similar result was obtained by Xu and Su ([50]) (**2013**) before.

2.2 Main tools in the study of blow-up.

We devote this section to enumerating the main tools and techniques used in the study of blow-up for the problem 2.1. We first start with the following existence and uniqueness of local solution, which can be obtained by using Faedo-Galerkin methods as in ([50]).

Let us introduce the definition of a weak solution for our problem.

Definition 2.1 *(**Weak solution**) Let $v_0 \in H_0^1(\Omega)$ be given. Any functions v such that*

$$v \in L^{\infty}([0,T_0], H_0^1(\Omega)), \qquad v_t \in L^2([0,T_0], H_0^1(\Omega))$$

is called a weak solution of (2.1) on [0;T), if

$$(v_t, w) + (\nabla v_t, \nabla w) + (\nabla v, \nabla w) = (v^p, w),$$

for a.e. $t \in [0, T_0]$ and all test function $w \in H_0^1(\Omega)$.

The local existence of solutions to 2.1 is assured by the

Theorem 2.1 *Under the condition $p > 1$, and for $v_0 \in H_0^1(\Omega)$, the problem 2.1 has a unique local weak solution v on $[0, T)$ in the sense of Definition 2.1. Moreover, v can be extended to the whole of $[0, \infty)$ or there is $T < \infty$ such that $\lim_{t\to T} \|v\|_{H_0^1(\Omega)} = \infty$*

In order to state and prove our result, we introduce the following functionals,

$$J(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 \, dx - \frac{1}{p+1}\int_{\Omega} v^{p+1} dx, \tag{2.2}$$

sometimes called **energy**, and

$$I(v) = \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} v^{p+1} dx. \tag{2.3}$$

The decay of the energy of the system 2.1 is given in the following lemma:

Lemma 2.1 ***The energy functional*** *J is a decreasing function.*

Proof. By multiplying υ_t on both sides of first equation in 2.1 and performing integration, we obtain:

$$\int_\Omega \upsilon_t \upsilon_t dx - \int_\Omega \Delta \upsilon_t \upsilon_t dx - \int_\Omega \Delta \upsilon \upsilon_t dx = \int_\Omega \upsilon^p \upsilon_t dx$$

Then,we use the generalized Green formula and the boundary conditions, to find

$$\int_\Omega \left(|\upsilon_t|^2 x + |\nabla \upsilon_t|^2 \right) dx + \int_\Omega \nabla \upsilon \nabla \upsilon_t dx = \frac{1}{p+1} \frac{d}{dt} \int_\Omega \upsilon^{p+1} dx$$

This implies that

$$\int_\Omega \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2 \right) dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \upsilon|^2 dx = \frac{1}{p+1} \frac{d}{dt} \int_\Omega \upsilon^{p+1} dx.$$

So

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} |\nabla \upsilon|^2 dx - \frac{1}{p+1} \upsilon^{p+1} \right) dx = - \int_\Omega \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2 \right) dx$$

Then, we find

$$\frac{d}{dt} J(\upsilon) \leq 0$$

∎

2.3 Blow-up result

In this section, we study the blow up time for a solution υ to problem 2.1 that blows up at a certain time $T > 0$. By means of a differential inequality technique, we obtain an upper bound for blow-up time if constant exponents p and the initial data satisfy some conditions. Also, a lower bound for blow-up time is determined under some other conditions.

2.3.1 Upper bound for blow-up time

The first main result of this chapter is given in the following theorem.

Theorem 2.2 *[31] For any $p > 1$, if $\upsilon_o \in H_0^1(\Omega) \cap L^{p+1}(\Omega), J(\upsilon_0) < 0$, $\upsilon(x,t)$ is a solution of problem 2.1, then $\upsilon(x,t)$ blows up in finite time T in $H_0^1(\Omega)$-norm. Moreover, an upper bound for blow-up time T is given by*

$$T_{\max} \leq \frac{\|\upsilon_0\|^2_{H_0^1(\Omega)}}{(1-p^2)J(\upsilon_0)} \tag{2.4}$$

Proof. Let us define the auxiliary function

$$\varphi(t) = \|\upsilon(.,t)\|^2_{H_0^1(\Omega)} = \int_\Omega \upsilon^2(x,t)dx + \int_\Omega |\nabla \upsilon(x,t)|^2 \, dx \tag{2.5}$$

and

$$\Psi(t) = -2(p+1)J(\upsilon) = -(p+1)\int\limits_\Omega |\nabla \upsilon|^2 \, dx + 2\int\limits_\Omega \upsilon^{p+1} dx. \tag{2.6}$$

Multiplying υ on two sides of equation 2.1, and integrating by part, we have

$$\int_\Omega \left(\upsilon\upsilon_t dx + \nabla\upsilon\nabla\upsilon_t\right) dx = -\int_\Omega |\nabla \upsilon|^2 \, dx + \int\limits_\Omega \upsilon^{p+1} dx. \tag{2.7}$$

By differentiate $\varphi(t)$ with respect to t, we obtain

$$\varphi'(t) = 2\int_\Omega \left(\upsilon\upsilon_t dx + \nabla\upsilon\nabla\upsilon_t\right) dx. \tag{2.8}$$

Combining 2.6, 2.7 and 2.8, we get

$$\varphi'(t) = -2\int\limits_\Omega |\nabla \upsilon|^2 \, dx + 2\int\limits_\Omega \upsilon^{p+1} dx \geq \Psi(t). \tag{2.9}$$

Now differentiate $\Psi(t)$ with respect to t to obtain

$$\begin{aligned} \Psi'(t) &= -2(p+1)\int\limits_\Omega \nabla\upsilon.\nabla\upsilon_t dx + 2(p+1)\int\limits_\Omega \upsilon^p \upsilon_t dx \\ &= 2(p+1)\left(-\int\limits_\Omega \nabla\upsilon.\nabla\upsilon_t dx + \int\limits_\Omega \upsilon^p \upsilon_t dx\right) \end{aligned} \tag{2.10}$$

Multiplying υ_t on two sides of equation 2.1 and integrating by part, we have

$$\int_{\Omega}\left(v_t^2 dx+|\nabla v_t|^2\right)dx = -\int_{\Omega}\nabla v.\nabla v_t dx+\int_{\Omega}v^p v_t dx \tag{2.11}$$

We then substitute for $\left(-\int\limits_{\Omega}\nabla v.\nabla v_t dx+\int\limits_{\Omega}v^p v_t dx\right)$ from 2.11, hence 2.10 becomes

$$\Psi'(t)=2(p+1)\left(\int\limits_{\Omega}v_t^2 dx+\int\limits_{\Omega}|\nabla v_t|^2 dx\right). \tag{2.12}$$

By using Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned}\varphi(t)\Psi'(t) &= 2(p+1)\left(\int\limits_{\Omega}v^2 dx+\int\limits_{\Omega}|\nabla v|^2 dx\right)\left(\int\limits_{\Omega}v_t^2 dx+\int\limits_{\Omega}|\nabla v_t|^2 dx\right)\\ &\geq 2(p+1)\left(\int\limits_{\Omega}v v_t dx+\int\limits_{\Omega}\nabla v\nabla v_t dx\right)^2\\ &= \frac{(p+1)}{2}\left[\varphi'(t)\right]^2\end{aligned}$$

By Lemma2.1 and the fact that $J(u_0)<0$, it follows that $\Psi(t)>0$ for all $t\geq 0$. Hence, by 2.9 we obtain:

$$\varphi(t)\Psi'(t) \geq \frac{(p+1)}{2}\varphi'(t)\Psi(t)$$

This can be expressed as:

$$\frac{\Psi'(t)}{\Psi(t)} \geq \frac{(p+1)}{2}\frac{\varphi'(t)}{\varphi(t)} \tag{2.13}$$

By integrating 2.13 from 0 to t, we obtain:

$$\int\limits_0^t\frac{d\Psi(\xi)}{\Psi(\xi)} \geq \int\limits_0^t\frac{(p+1)}{2}\frac{d\varphi(\xi)}{\varphi(\xi)}$$

Then

$$[\ln \Psi(\xi)]_0^t \quad \geq \quad \left[\frac{(p+1)}{2} \ln \varphi(x)\right]_0^t$$

so

$$\ln \Psi(t) - \ln \Psi(0) \quad \geq \quad \ln(\varphi(t))^{\frac{(p+1)}{2}} - \ln(\varphi(0))^{\frac{(p+1)}{2}}$$

$$\frac{\Psi(t)}{(\varphi(t))^{\frac{(p+1)}{2}}} \quad \geq \quad \frac{\Psi(0)}{(\varphi(0))^{\frac{(p+1)}{2}}}$$

using 2.9, we obtain

$$\frac{\varphi'(t)}{(\varphi(t))^{\frac{(p+1)}{2}}} \quad \geq \quad \frac{\Psi(0)}{(\varphi(0))^{\frac{(p+1)}{2}}} \tag{2.14}$$

Integrating inequality 2.14 from 0 to t, we see

$$\left[-\frac{2}{p+1}\frac{1}{(\varphi(t))^{\frac{P-1}{2}}}\right]_0^t \quad \geq \quad \frac{\Psi(0)}{(\varphi(0))^{\frac{(p+1)}{2}}}t$$

then

$$\frac{1}{(\varphi(t))^{\frac{P-1}{2}}} \quad \leq \quad \frac{1}{(\varphi(0))^{\frac{P-1}{2}}} - \frac{p-1}{2}\frac{\Psi(0)}{(\varphi(0))^{\frac{(p+1)}{2}}}t$$

so

$$\varphi(t) \quad \geq \quad \frac{1}{\left(\frac{1}{(\varphi(0))^{\frac{P-1}{2}}} - \frac{p-1}{2}\frac{\Psi(0)}{(\varphi(0))^{\frac{(p+1)}{2}}}t\right)^{\frac{2}{p-1}}} \tag{2.15}$$

Clearly, 2.15 cannot hold for all time, this means $\upsilon(x.t)$ blows up in finite time T in $H_0^1(\Omega)$-norm. In fact, let $t \to T$, 2.6 and 2.15 yield:

$$T \leq \frac{\|\upsilon_0\|^2_{H_0^1(\Omega)}}{(1-p^2)\, J(\upsilon_0)}$$

■

2.3.2 Lower bound for blow-up time

The second main result of this chapter is given in the following theorem.

Theorem 2.3 *[31] Suppose* $p \in [1, \frac{n+2}{n-2}]$, $\upsilon_0 \in H_0^1(\Omega), J(\upsilon_0) < d, I(u_0) < 0$ *, then the solution* $\upsilon(x,t)$ *of problem 2.1 blows up in finite time* T *in* $H_0^1(\Omega)-$*norm. Moreover,* T*is bounded in the succeeding text by*

$$\frac{\|\upsilon_0\|_{H_0^1(\Omega)}^{-p+1}}{(p-1)C^{p+1}} \tag{2.16}$$

where C *is the Sobolev embedding constants for* $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Proof. Consider $\varphi(t)$ as in 2.5

$$\varphi(t) = \|\upsilon(.,t)\|_{H_0^1(\Omega)}^2 = \int\limits_{\Omega} \upsilon^2(x,t)dx + \int\limits_{\Omega} |\nabla \upsilon(x,t)|^2 \, dx.$$

By multiplying $\upsilon(x,t)$ on both sides of equation 2.1 and performing integration by parts, we obtain:

$$\int\limits_{\Omega} (\upsilon \upsilon_t + \nabla \upsilon \nabla \upsilon_t) \, dx = -\int\limits_{\Omega} |\nabla \upsilon|^2 \, dx + \int\limits_{\Omega} \upsilon^{p+1} dx \tag{2.17}$$

A direct differentiation of $\varphi(t)$ yields:

$$\varphi^{'}(t) = 2 \int_{\Omega} (\upsilon \upsilon_t + \nabla \upsilon \nabla \upsilon_t) \, dx. \tag{2.18}$$

From 2.17 and 2.18, we obtain

$$\varphi'(t) = -2 \int\limits_{\Omega} |\nabla \upsilon|^2 \, dx + 2 \int\limits_{\Omega} \upsilon^{p+1} dx \tag{2.19}$$

By the Sobolev embeddings (See Lemma 1.7), we have

$$\int_{\Omega} |\upsilon|^{p+1} \, dx \leq C^{p+1} \left(\int_{\Omega} |\nabla \upsilon|^2 \, dx \right)^{\frac{p+1}{2}} \tag{2.20}$$

Then 2.20 and 2.19 imply

$$\varphi'(t) \leq 2C^{p+1} \left(\varphi(t)\right)^{\frac{p+1}{2}}$$

If there exists $t_0 \in [0, T)$ such that $\varphi(t_0) = 0$, then we can obtain $\varphi(T) = 0$. ,which contradicts the fact that $u(x,t)$ blows up at T in the H^1-norm. So we see $\varphi(t) > 0$ and the following inequality,

$$\frac{\varphi'(t)}{\left(\varphi(t)\right)^{\frac{p+1}{2}}} \leq 2C^{p+1} \tag{2.21}$$

By integrating inequality 2.21 from 0 to t, we obtain:

$$\varphi(0)^{\frac{1-p}{2}} - \varphi(t)^{\frac{1-p}{2}} \leq (p-1)C^{p+1}t \tag{2.22}$$

so

$$t \geq \frac{\varphi(0)^{\frac{1-p}{2}} - \varphi(t)^{\frac{1-p}{2}}}{(p-1)C^{p+1}}$$

If v blow-up in H_0^1-norm, then we establish a lower bound for $T_{\min}$ by the form:

$$T \geq \frac{\|v_0\|_{H_0^1(\Omega)}^{-p+1}}{(p-1)C^{p+1}}$$

■

Chapter 3

Blowing-up solution to a nonlinear pseudo-parabolic equation with source and damping terms with variable exponents nonlinearities

- Introduction - Main tools in the study of blow-up - Blow-up result

3.1 Introduction

In this chapter, we consider the following pseudo-parabolic equation

$$\begin{cases} v_t - \operatorname{div}(|\nabla v|^{r(x)-2}\nabla v) - \mu\Delta v_t = |v|^{p(x)-2}v, & \text{in } \Omega\times(0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{3.1}$$

Where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The nonlinear term $\operatorname{div}(|\nabla v|^{r(x)-2}\nabla v)$ is the so called $r(x)$-Laplace operator. The term with a variable exponent $|v|^{p(x)-2}v$ plays the role of a source, and the dissipative term Δv_t is a linear strong damping term. The exponents $r(.)$ and $p(.)$ are given continuous functions defined on $\overline{\Omega}$ and satisfy

$$2 < r_- \leq r(x) \leq r_+ < p_- \leq p(x) \leq p_+ < \infty, \tag{3.2}$$

where

$$r_- = ess\ \inf r(x),\quad r_+ = ess\ \sup r(x)$$
$$p_- = ess\ \inf\ p(x),\quad p_+ = ess\ \sup p(x)$$

and the Zhikov–Fan conditions:

$$|r(x) - r(y)| = \frac{-a}{\log|x-y|} \text{ and } |p(x) - p(y)| = \frac{-b}{\log|x-y|} \tag{3.3}$$

for all $x, y \in \Omega$ with $|x - y| < \delta$, where $a, b > 0$ and $0 < \delta < 1$.
Problem (3.1) occurs in the mathematical modeling of various physical phenomena, e.g., the flows of electrorheological fluids, nonlinear viscoelasticity, fluids with temperature-dependent viscosity, processes of filtration through a porous media and image processing, and so on... See [2] [24] [3] [40].

In the case when r, p are constants, there have been many results about the existence and blow-up properties of the solutions,we refer the readers to the bibliography given in [2], [16], [24] and [43]. Obviously, if $\mu = 1$, $r(x) = 2, p(x) = p$, then Eq (3.1) reduces to the following pseudo-parabolic equation

$$\upsilon_t - \Delta\upsilon_t - \Delta\upsilon = |\upsilon|^{p-2}\upsilon, \quad \text{in } \Omega\times(0,T). \tag{3.4}$$

In their work [50], Xu and Su proved that the solutions to the problem (3.4) blow up in a finite time in $H_0^1(\Omega)$-norm. In other studies [31], Luo considered the same problem treated in the work of Xu and Su [50], and he obtained an upper bound and a lower bound of the blowup rate.
In the absence of the damped term ($\mu = 0$), Eq. 3.1 becomes

$$\upsilon_t - \mathrm{div}(|\nabla\upsilon|^{r(x)-2}\nabla\upsilon) = |\upsilon|^{p(x)-2}\upsilon, \tag{3.5}$$

Alaoui, Messaoudi and Khenous [1] proved that any solutions of this equation with nontrivial initial datum blow up in finite time. For the constant exponents case ($r(x) = r$, $p(x) = p$), Eq. 3.5 has been extensively studied and results concerning existence, nonexistence and asymptotic behavior have been established by many authors [[35]–[4]]. For instance, Payne et al. [[35],[36]] obtained the upper and lower bounds on blow up time when blow up does occur by applying the differential inequality techniques.
In Section 2, we present a main tools to study blow up (local existence ,energy). **In Section 3**, the blow up in finite time of solutions to the problem 3.1 is proved. The proof is based on differential inequality techniques. We dedicate first the upper bound for blow up time to problem 3.1 under suitable conditions on $r(\cdot)$, $p(\cdot)$ and the initial data. Also, the lower bound of blow up time is obtained under some other conditions.

The results presented in this chapter were mostly obtained by **Di, Shang and Peng** [14] (**2017**).

3.2 Main tools in the study of blow-up.

We first start with the following existence and uniqueness of local solution for the problem (3.1), which can be obtained by using Faedo-Galerkin methods as in ([2] [17] [27]). Here, the proof is thus omitted. For simplicity, we set $\mu = 1$.

Theorem 3.1 *Let $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)$ be given. Assume that the conditions on $r(x), p(x)$, given in Section 3.1, hold. Then, problem (3.1) has a unique local solution υ on $[0, T_0)$*

$$\upsilon \in L^{\infty}([0, T_0]; W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)), \quad \upsilon_t \in L^2([0, T_0]; W_0^{1,2}(\Omega))$$

for some $T_0 > 0$, satisfying

$$(\upsilon_t, w) + (\nabla \upsilon_t, \nabla w) + (|\nabla \upsilon|^{r(x)-2} \nabla \upsilon, \nabla w) = (|\upsilon|^{p(x)-2} \upsilon, w), \text{ for all } w \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega) \quad (3.6)$$

Moreover, the following alternatives hold

$$i)\ T_0 = +\infty \quad \text{or}$$

$$ii)\ T_0 < +\infty \text{ and } \lim_{t \longrightarrow T} \|\nabla \upsilon\|_2^2 + \|\upsilon\|_2^2 = +\infty$$

The decay of the energy of the system (3.1) is given in the following lemma:

Lemma 3.1 *The energy functional E of the problem (3.1) is a decreasing function. Here*

$$E(t) = \int_{\Omega} \left[\frac{1}{r(x)} |\nabla \upsilon|^{r(x)} - \frac{1}{p(x)} |\upsilon|^{p(x)} \right] dx$$

Proof. Replacing w by υ_t in the Eq. 3.6, we have

$$\int_{\Omega} \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2 \right) dx + \frac{d}{dt} \int_{\Omega} \frac{1}{r(x)} |\nabla \upsilon|^{r(x)} dx = \frac{d}{dt} \int_{\Omega} \frac{1}{p(x)} |\upsilon|^{p(x)} dx \quad (3.7)$$

We then define the energy by:

$$E(t) = \int_{\Omega} \left[\frac{1}{r(x)} |\nabla \upsilon|^{r(x)} - \frac{1}{p(x)} |\upsilon|^{p(x)} \right] dx \quad (3.8)$$

Therefore, from equations 3.7 and 3.8, we obtain:

$$E'(t) = -\int_{\Omega} \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2 \right) dx \leq 0 \quad (3.9)$$

■

3.3 Blow-up result

In this section, we derive an upper and lower bound for the blow-up time of problem (3.1) under certain conditions on the variable exponents $r(.)$, $p(.)$, and initial data. For simplicity, we set $\mu = 1$.

3.3.1 Upper bound for blow-up time

Theorem 3.2 *[14]Assume that (3.2) and (3.3) hold. Let $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)$ such that*

$$\int_\Omega \left[\frac{1}{p(x)} |\upsilon_0|^{p(x)} - \frac{1}{r(x)} |\nabla \upsilon_0|^{r(x)}\right] \geq 0 \tag{3.10}$$

Then, the solution υ of the problem 3.1 blow up in finite time T^*in $H_0^1(\Omega)$-norm. Moreover, an upper bound for blow up time is given by

$$T^* \leq \frac{2(F(0))^{1-\frac{r_-}{2}}}{(r-2)\beta} \tag{3.11}$$

where β is a suitable positive constant given later and $F(0) = \|\upsilon_0\|_{H_0^1}^2$

Proof. We introduce an auxiliary function:

$$F(t) = \int_\Omega \upsilon^2 dx + \int_\Omega |\nabla \upsilon|^2 dx \tag{3.12}$$

By multiplying υ on both sides of problem 3.1 and integrating by parts, we obtain:

$$\int_\Omega \upsilon \upsilon_t dx + \int_\Omega \nabla \upsilon . \nabla \upsilon_t dx = \int_\Omega |\upsilon|^{p(x)} dx - \int_\Omega |\nabla \upsilon|^{r(x)} dx \tag{3.13}$$

By differentiating $F(t)$ with respect to t, we obtain:

$$\begin{aligned} F'(t) &= 2\int_\Omega \upsilon \upsilon_t dx + 2\int_\Omega \nabla \upsilon . \nabla \upsilon_t dx = 2\int_\Omega |\upsilon|^{p(x)} dx - 2\int_\Omega |\nabla \upsilon|^{r(x)} dx \\ &= 2\int_\Omega p(x) \left[\frac{|\upsilon|^{p(x)}}{p(x)} - \frac{|\nabla \upsilon|^{r(x)}}{r(x)}\right] dx + 2\int_\Omega p(x) \left[\frac{1}{r(x)} - \frac{1}{p(x)}\right] |\nabla \upsilon|^{r(x)} dx \end{aligned} \tag{3.14}$$

As $E'(t) \leq 0$ see Lemma 3.1, we have:

$$\begin{aligned} \int_\Omega p(x) \left[\frac{|\upsilon|^{p(x)}}{p(x)} - \frac{|\nabla \upsilon|^{r(x)}}{r(x)}\right] dx &\geq \int_\Omega p(x) \left[\frac{|\upsilon_0|^{p(x)}}{p(x)} - \frac{|\nabla \upsilon_0|^{r(x)}}{r(x)}\right] dx \\ &\geq \int_\Omega p_- \left[\frac{|\upsilon_0|^{p(x)}}{p(x)} - \frac{|\nabla \upsilon_0|^{r(x)}}{r(x)}\right] dx \geq 0 \end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we see:

$$\begin{aligned} F'(t) &\geq 2\int\limits_{\Omega} p(x)\left[\frac{1}{r(x)}-\frac{1}{p(x)}\right]|\nabla v|^{r(x)}\,dx \\ &\geq 2\int\limits_{\Omega} p_-\left[\frac{1}{r_+}-\frac{1}{p_-}\right]|\nabla v|^{r(x)}\,dx = C_0\int\limits_{\Omega}|\nabla v|^{r(x)}\,dx, \end{aligned} \tag{3.16}$$

where $C_0 = 2p_-\left[\frac{1}{r_+}-\frac{1}{p_-}\right]$.

We define the sets $\Omega_+ = \{x\in\Omega, |\nabla v|\geq 1\}$ and $\Omega_- = \{x\in\Omega, |\nabla v|<1\}$, so we get

$$F'(t) \geq C_0\left(\int\limits_{\Omega_-}|\nabla v|^{r_+}\,dx + \int\limits_{\Omega_{+-}}|\nabla v|^{r_-}\,dx\right)$$

$$F'(t) \geq C_1\left(\int\limits_{\Omega_-}\left(|\nabla v|^2\right)^{\frac{r_+}{2}}\,dx + \int\limits_{\Omega_{+-}}\left(|\nabla v|^2\right)^{\frac{r_-}{2}}\,dx\right)$$

according to $\|\nabla v\|_2 \leq C\|\nabla v\|_r$ for all $r\geq 2$.

This implies that

$$(F'(t))^{\frac{2}{r_+}} \geq C_2\int\limits_{\Omega_-}|\nabla v|^2\,dx \geq 0 \text{ and } (F'(t))^{\frac{2}{r_-}} \geq C_3\int\limits_{\Omega_+}|\nabla v|^2\,dx \geq 0 \tag{3.17}$$

The Poincaré inequality gives $\|\nabla v\|^2 \geq \lambda_1\|v\|^2$, where $\lambda_1$is the first eigenvalue of the problem

$$\begin{cases} \Delta w + \lambda w = 0, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega \end{cases}$$

Hence, we have:

$$\begin{aligned} \|\nabla v\|^2 &= \frac{1}{1+\lambda_1}\|\nabla v\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|^2 \geq \frac{\lambda_1}{1+\lambda_1}\|v\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|^2 \\ &\geq \frac{\lambda_1}{1+\lambda_1}\|v\|^2_{H_0^1} \end{aligned} \tag{3.18}$$

It follows from (3.17) and (3.18) that:

$$\begin{aligned} (F'(t))^{\frac{2}{r_+}} + (F'(t))^{\frac{2}{r_-}} &\geq C_2\int\limits_{\Omega_-}|\nabla v|^2\,dx + C_3\int\limits_{\Omega_+}|\nabla v|^2\,dx \\ &\geq (C_2+C_3)\int\limits_{\Omega}|\nabla v|^2\,dx \\ &\geq (C_2+C_3)\|\nabla v\|^2 \\ &\geq \frac{\lambda_1}{1+\lambda_1}(C_2+C_3)\|v\|^2_{H_0^1} \\ &\geq C_4F(t) \end{aligned} \tag{3.19}$$

or

$$(F'(t))^{\frac{2}{r_-}}\left(1+(F'(t))^{\left(\frac{2}{r_+}-\frac{2}{r_-}\right)}\right)\geq C_4F(t) \tag{3.20}$$

From (3.19) and the fact that $F(t)\geq F(0)>0$ $(F'(t)\geq 0)$, we deduce either

$$2\,(F'(t))^{\frac{2}{r_+}}\geq C_4F(t) \quad \text{or} \quad 2\,(F'(t))^{\frac{2}{r_-}}\geq C_4F(t)$$

so

$$(F'(t))^{\frac{2}{r_+}}\geq \frac{C_4}{2}F(t)\geq \frac{C_4}{2}F(0) \text{ or } (F'(t))^{\frac{2}{r_-}}\geq \frac{C_4}{2}F(t)\geq \frac{C_4}{2}F(0) \tag{3.21}$$

which implies that

$$F'(t)\geq C_5\,(F(0))^{\frac{r_+}{2}} \text{ or } F'(t)\geq C_5\,(F(0))^{\frac{r_-}{2}}$$

Therefore,we have that $F'(t)\geq\alpha$, where $\alpha=\min\left\{C_5\,(F(0))^{\frac{r_+}{2}},C_5\,(F(0))^{\frac{r_-}{2}}\right\}$. Moreover, utilizing $\frac{1}{r_+}-\frac{1}{r_-}\leq 0$ and (3.20), we get:

$$(F'(t))^{\frac{2}{m_-}}\left(1+(\alpha)^{2\left(\frac{1}{r_+}-\frac{1}{r_-}\right)}\right)\geq C_4F(t)$$

then

$$F'(t)\geq\beta\,(F(t))^{\frac{r_-}{2}} \tag{3.22}$$

Where the constant $\beta=\left(\frac{C_4}{\left(1+(\alpha)^{2\left(\frac{1}{r_+}-\frac{1}{r_-}\right)}\right)}\right)^{\frac{r_-}{2}}$.

Integrating the inequality (3.22) from 0 to t, we observe

$$\int_0^t \frac{dF(\eta)}{(F(\eta))^{\frac{r_-}{2}}}\geq\int_0^t\beta d\eta$$

then

$$F(t)^{1-\frac{r_-}{2}}\geq F(0)^{1-\frac{r_-}{2}}+\frac{(2-r_-)\beta t}{2}$$

so

$$F(t)\leq\frac{1}{\left(F(0)^{1-\frac{r_-}{2}}+\frac{(2-r_-)\beta t}{2}\right)^{\frac{2}{r_--2}}} \tag{3.23}$$

Thus, (3.23) shows that $F(t)$ blows up at some finite time $T^*\leq\frac{2(F(0))^{1-\frac{r_-}{2}}}{(2-r_-)\beta}$, so the solution v blows up in $H_0^1(\Omega)$-norm in finite time.

Remark 3.1 *Considering the time estimate (3.11), it becomes evident that the larger $F(0)$ is, the faster the blow-up phenomenon occurs.*

■

3.3.2 Lower bound for blow-up time

Theorem 3.3 *[14]Suppose that (3.2) and (3.3) hold. Furthermore assume that $2 < p_+ < \infty$ if $n \leq 2$, $2 < p_+ < \frac{2n}{n-2}$ if $n \geq 3$, $\upsilon_0 \in W_0^{1,r(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and the solution u of the problem (3.1) becomes unbounded at finite time T^* in $H^1(\Omega)$-norm, then a lower bound T^* for blow up time is given by*

$$T^* \geq \int_{F(0)}^{\infty} \frac{d\eta}{2B_+^{p_+}\eta^{\frac{p_+}{2}} + 2B_-^{p_-}\eta^{\frac{p_-}{2}}} \tag{3.24}$$

where B_+,B_- are the Sobolev embedding constants for $H_0^1(\Omega) \hookrightarrow L^{p_+}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^{p_-}(\Omega)$, respectively and $F(0) = \|\upsilon_0\|_{H_0^1}^2$

Proof. We define the function $F(t)$ as in equation (3.12), and perform calculations as in the previous section.

$$F'(t) = 2\int_{\Omega} \upsilon\upsilon_t dx + 2\int_{\Omega} \nabla\upsilon.\nabla\upsilon_t dx$$

according to (3.13)

$$F'(t) = 2\int_{\Omega} |\upsilon|^{p(x)}\, dx - 2\int_{\Omega} |\nabla\upsilon|^{r(x)}\, dx \leq 2\int_{\Omega} |\upsilon|^{p(x)}\, dx \tag{3.25}$$

By defining the sets $\Omega_+ = \{x \in \Omega \mid |\upsilon| \geq 1\}$ and $\Omega_- = \{x \in \Omega \mid |\upsilon| < 1\}$, we get:

$$\begin{aligned} \int_{\Omega} |\upsilon|^{p(x)}\, dx &\leq \int_{\Omega_+} |\upsilon|^{p_+}\, dx + \int_{\Omega_-} |\upsilon|^{p_-}\, dx \\ &\leq \int_{\Omega} |\upsilon|^{p_+}\, dx + \int_{\Omega} |\upsilon|^{p_-}\, dx \end{aligned}$$

By using Sobolev embedding inequalities, we obtain :

$$\int_{\Omega} |\upsilon|^{p(x)}\, dx \leq B_+^{p_+}\left(\int_{\Omega} |\nabla\upsilon|^2\, dx\right)^{\frac{p_+}{2}} + B_-^{p_-}\left(\int_{\Omega} |\nabla\upsilon|^2\, dx\right)^{\frac{p_-}{2}} \tag{3.26}$$

where B_+,B_- are the optimal constants satisfying theSobolev embedding inequalities $\|\upsilon\|_{L^{p_+}} \leq \|\nabla\upsilon\|_2$ and $\|\upsilon\|_{L^{p_-}} \leq \|\nabla\upsilon\|_2$ respectively. Therefore, the combination of (3.25) and (3.26) implies that

$$F'(t) \leq 2B_+^{p_+}\left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{p_+}{2}} + 2B_-^{p_-}\left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{p_-}{2}}$$

$$\begin{aligned} F'(t) &\leq 2B_+^{p_+}\left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{p_+}{2}} + 2B_-^{p_-}\left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{p_-}{2}} \\ &\leq 2B_+^{p_+}(F(t))^{\frac{p_+}{2}} + 2B_-^{p_-}(F(t))^{\frac{p_-}{2}} \end{aligned} \tag{3.27}$$

Integrating the inequality (3.27) from 0 to t, we get:

$$\int_{F(0)}^{F(t)} \frac{d\eta}{2B_+^{p_+}(\eta)^{\frac{p_+}{2}} + 2B_-^{p_-}(\eta)^{\frac{p_-}{2}}} \leq t \tag{3.28}$$

If u blows up in the $H_0^1(\Omega)$-norm, then we derive a lower bound for T^* given by:

$$T^* \geq \int_{F(0)}^{\infty} \frac{d\eta}{2B_+^{p_+}(\eta)^{\frac{p_+}{2}} + 2B_-^{p_-}(\eta)^{\frac{p_-}{2}}} \tag{3.29}$$

Obviously, the integral is bounded since the exponents $p_+ \geq p_- > 2$. This completes the proof of Theorem 3.3 ■

Chapter 4

Blowing up solution to a nonlinear pseudo-parabolic equation with presence of a matrix with variable entries in the divergence operator.

- Introduction - Main tools in the study of blow-up - Blow-up result

4.1 Introduction

This chapter is the subject of an article written by Toualbia. A (University of Tebessa). This article [45] will published in Applied Mathematics E-Notes. The novelties are the presence of the matrix with variable entries in the divergence operator.

In this chapter, we consider the following problem:

$$\begin{cases} v_t - \Delta v_t - \operatorname{div}(A(x,t)\,|\nabla v|^{r(x)-2}\,\nabla v) = |v|^{p(x)-2}\,v, & \text{in } \Omega \times (0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$,$n > 1$, with a smooth boundary $\partial\Omega$.
The matrix $A = (a_{ij}(x,t))_{i,j}$, where a_{ij} is a function of class $C^1(\bar{\Omega} \times [0,\infty[)$, and there exists a constant $a_0 > 0$ such that, for all $(x,t) \in \Omega \times [0,\infty[$ and $\xi \in \mathbb{R}^n$, we obtaine :

$$A\xi.\xi \geq a_0\,|\xi|^2 \tag{4.2}$$

and

$$A'\xi.\xi \leq 0 \tag{4.3}$$

where $A' = \frac{\partial A}{\partial t}\,(.,t)$ .The exponents $r(.)$ and $p(.)$ are given continuous functions defined on Ω and satisfy

$$2 < r_- \leq r(x) \leq r_+ < p_- \leq p(x) \leq p_+ < \infty \tag{4.4}$$

where

$$\begin{aligned} r_- &= ess\ \inf r(x),\ r_+ = ess\ \sup r(x) \\ p_- &= ess\ \inf p(x),\ p_+ = ess\ \sup p(x) \end{aligned}$$

Obviously, if $A = I_n$, then Eq (4.1) reduces to the following pseudo-parabolic equation

$$v_t - \Delta v_t - \operatorname{div}(|\nabla v|^{r(x)-2}\,\nabla v) = |v|^{p(x)-2}\,v, \quad \text{in } \Omega \times (0,T). \tag{4.5}$$

We proved in the **precedent chapter** that the solutions to this problem blow up in a finite time in $H_0^1(\Omega)$-norm, and we obtained an upper bound and a lower bound of the blowup rate.

In this chapter, we use some assumptions for the variable exponents $r(.), p(.)$, initial data, and matrix $A(.,t)$ to prove the blow-up of the solution to the problem 4.1. By means of a differential inequality technique, we prove that the solutions become unbounded at a finite time T, and find an upper bound for this time with a negative initial energy. Also, a lower bound for blow-up time is determined.

The novelty in this chapter is the presence of the matrix with variable entries in the divergence operator.

4.2 Main tools in the study of blow-up

We first start with the existence and uniqueness of a local solution for the problem (4.1), which can be obtained by using Faedo-Galerkin methods as in ([2] [17] [27]). Here, the proof is thus omitted.

Theorem 4.1 *Let $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)$ be given. Assume that the conditions on $p(x), r(x)$, and A, given in Section 4.1, hold. Then, problem (4.1) has a unique local solution υ on $[0, T_0)$*

$$\upsilon \in L^\infty([0,T_0]; W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)), \quad \upsilon_t \in L^2([0,T_0]; W_0^{1,2}(\Omega))$$

for some $T_0 > 0$, satisfying

$$(\upsilon_t, w) + (\nabla \upsilon_t, \nabla w) + (A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon, \nabla w) = (|\upsilon|^{p(x)-2} \upsilon, w), \text{ for all } w \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega) \tag{4.6}$$

Moreover, the following alternatives hold

$$i)\ T_0 = +\infty \text{ or}$$

$$ii)\ T_0 < +\infty \text{ and } \lim_{t \longrightarrow T} \|\nabla u\|_2^2 + \|u\|_2^2 = +\infty$$

Remark 4.1 *It is easy to see, under the condition (4.4) that $|\upsilon|^{p(x)-2} \upsilon$, $A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon \in L^2(\Omega)$; hence $(|\upsilon|^{p(x)-2} \upsilon, w)$ and $(A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon, \nabla w)$ make sense in formula (4.6).*

The decay of the energy of the system (4.1) is given in the following lemma:

Lemma 4.1 *The energy functional E of the problem (4.1) is a decreasing function. Here*

$$E(t) = \int_\Omega \frac{1}{r(x)} A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon . \nabla \upsilon dx - \int_\Omega \frac{1}{p(x)} |\upsilon|^{p(x)} dx \tag{4.7}$$

Proof. It is enough to multiply the first equation in (4.1) by υ_t and integrate over Ω, to obtain

$$\int_\Omega \upsilon_t \upsilon_t dx - \int_\Omega \Delta \upsilon_t \upsilon_t dx - \int_\Omega \operatorname{div}\left(A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon\right) \upsilon_t dx = \int_\Omega |\upsilon|^{p(x)-2} \upsilon \upsilon_t dx$$

Then, we use the generalized Green formula and the boundary conditions, to find

$$\int_\Omega \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2\right) dx + \int_\Omega A |\nabla \upsilon|^{r(x)-2} \nabla \upsilon . \nabla \upsilon_t dx = \frac{d}{dt} \int_\Omega \frac{1}{p(x)} |\upsilon|^{p(x)} dx.$$

This implies that

$$\int_{\Omega}\left(|\upsilon_t|^2+|\nabla\upsilon_t|^2\right)dx+\frac{d}{dt}\int_{\Omega}\frac{1}{r(x)}A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx-\int_{\Omega}\frac{1}{r(x)}A'\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx$$
$$=\frac{d}{dt}\int_{\Omega}\frac{1}{p(x)}\,|\upsilon|^{p(x)}\,dx.$$

So

$$E'(t)=-\int_{\Omega}\left(|\upsilon|^2+|\nabla\upsilon_t|^2\right)dx+\int_{\Omega}\frac{1}{r(x)}A'\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx. \tag{4.8}$$

Taking into account condition (4.3) on A', we find

$$E'(t)\leq-\int_{\Omega}\left(|\upsilon_t|^2+|\nabla\upsilon_t|^2\right)dx\leq 0 \tag{4.9}$$

■

4.3 Blow-up result

4.3.1 Upper bound for blow-up time

Theorem 4.2 *([45]) Assume that (4.2), (4.3), (4.4), and (3.3) hold. Let υ be a solution of (4.1) and assume that $\upsilon_0\in W_0^{1,r(.)}(\Omega)\cap L^{p(.)}(\Omega)$ satisfies*

$$\int_{\Omega}\frac{1}{p(x)}\,|\upsilon_0|^{p(x)}\,dx-\int_{\Omega}\frac{1}{r(x)}A(x,0)\,|\nabla\upsilon_0|^{r(x)-2}\,\nabla\upsilon_0.\nabla\upsilon_0 dx\geq 0, \tag{4.10}$$

then the solution υ blow up at finite time $T_{\max}>0$ in $H_0^1(\Omega)$-norm. In addition, there exists an upper bound for the time as given by

$$T_{\max}\leq\frac{2\,(G(0))^{\left(\frac{2-r_-}{2}\right)}}{(r_--2)\,K} \tag{4.11}$$

where K is a suitable positive constant is given later and the constant $G(0)=\|\upsilon_0\|^2_{H_0^1(\Omega)}$.

Proof. Let us define the auxiliary function

$$G(t)=\|\upsilon\|^2_{H_0^1(\Omega)}=\int_{\Omega}\upsilon^2dx+\int_{\Omega}|\nabla\upsilon|^2\,dx \tag{4.12}$$

Our goal is to show that G satisfies a differential inequality which leads to blow up in finite time. Multiply (4.1) by υ and integrate over Ω to get

$$\int_{\Omega} \upsilon\upsilon_t dx + \int_{\Omega} \nabla\upsilon\nabla\upsilon_t dx = \int_{\Omega} |\upsilon|^{p(x)}\, dx - \int_{\Omega} A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx \tag{4.13}$$

Now differentiate $G(t)$ with respect to t to obtain

$$G'(t) = 2\int_{\Omega} (\upsilon\upsilon_t dx + \nabla\upsilon\nabla\upsilon_t)\, dx = 2\int_{\Omega} \left(|\upsilon|^{p(x)} - A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon\right) dx$$

$$= 2\int_{\Omega} \left(p(x)\left(\frac{|\upsilon|^{p(x)}}{p(x)} - \frac{A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon}{r(x)}\right) + p(x)\left(\frac{1}{r(x)} - \frac{1}{p(x)}\right) A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon\right) dx \tag{4.14}$$

By (4.10) and the fact that $E(t) \leq E(0)$ $(E'(t) \leq 0)$, we have

$$\int_{\Omega} p(x)\left[\frac{|\upsilon|^{p(x)}}{p(x)} - \frac{A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon}{r(x)}\right] dx \geq \int_{\Omega} p(x)\left[\frac{|\upsilon_0|^{p(x)}}{p(x)} - \frac{A(x,0)\,|\nabla\upsilon_0|^{r(x)-2}\,\nabla\upsilon_0.\nabla\upsilon_0}{r(x)}\right] dx$$

$$\geq\ p_- \int_{\Omega} \left[\frac{|\upsilon_0|^{p(x)}}{p(x)} - \frac{A(x,0)\,|\nabla\upsilon_0|^{r(x)-2}\,\nabla\upsilon_0.\nabla\upsilon_0}{r(x)}\right] dx \geq 0. \tag{4.15}$$

By (4.14) and (4.15), we see

$$G'(t) \geq 2\int_{\Omega} p_-\left(\frac{1}{r_+} - \frac{1}{p_-}\right) A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx$$

Using condition (4.2) on A, we obtain

$$G'(t) \geq a_0 C_0 \int_{\Omega} |\nabla\upsilon|^{r(x)}\, dx, \tag{4.16}$$

where $C_0 = 2\,p_-\left(\frac{1}{r_+} - \frac{1}{p_-}\right) > 0.$

Now we define the sets $\Omega_+ = \{x \in \Omega : |\nabla\upsilon| \geq 1\}$ and $\Omega_- = \{x \in \Omega : |\nabla\upsilon| < 1\}$. By using the fact that $\|\upsilon\|_2 \leq C\,\|\upsilon\|_r$ for all $r > 2$, we get

$$\begin{aligned} G'(t) &\geq a_0 C_0\left(\int_{\Omega_-} |\nabla\upsilon|^{r_+}\, dx + \int_{\Omega_+} |\nabla\upsilon|^{r_-}\, dx\right) \\ &\geq C_1\left(\left(\int_{\Omega_-} |\nabla\upsilon|^2\, dx\right)^{\frac{r_+}{2}} + \left(\int_{\Omega_+} |\nabla\upsilon|^2\, dx\right)^{\frac{r_-}{2}}\right). \end{aligned}$$

This implies that

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} \geq C_2 \left(\int_{\Omega_-} |\nabla v|^2 \, dx\right) \quad \text{and} \quad \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq C_3 \left(\int_{\Omega_+} |\nabla v|^2 \, dx\right). \tag{4.17}$$

The Poincare inequality gives $\|\nabla v\|_2^2 \geq \lambda \|v\|_2^2$, where λ is the first eigenvalue of $(-\Delta)$. Therefore, we get

$$\|\nabla v\|_2^2 = \frac{1}{1+\lambda} \|\nabla v\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla v\|_2^2 \geq \frac{\lambda}{1+\lambda} \|v\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla v\|_2^2 = \frac{\lambda}{1+\lambda} \|v\|_{H_0^1(\Omega)}^2 \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} + \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq (C_2 + C_3) \|\nabla v\|_2^2 \geq \frac{(C_2 + C_3)\lambda}{1+\lambda} \|v\|_{H_0^1(\Omega)}^2 = C_4 G(t) \tag{4.19}$$

Since we have $G(t) \geq G(0) > 0$ (because $G^{'}(t) \geq 0$), and from (4.19), we get

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0) \quad \text{or} \quad \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0). \tag{4.20}$$

This implies that

$$G^{'}(t) \geq C_5 \left(G(0)\right)^{\frac{r_+}{2}} \quad \text{or} \quad G^{'}(t) \geq C_5 \left(G(0)\right)^{\frac{r_-}{2}}.$$

Now put $\beta = \min\left\{C_5 \left(G(0)\right)^{\frac{r_+}{2}}, \; C_5 \left(G(0)\right)^{\frac{r_-}{2}}\right\}$, then we get

$$G^{'}(t) \geq \beta \tag{4.21}$$

(4.19) implies that

$$\left(G^{'}(t)\right)^{\frac{2}{r_-}} \left(1 + \left(G^{'}(t)\right)^{2\left(\frac{1}{r_+} - \frac{1}{r_-}\right)}\right) \geq C_4 G(t) \tag{4.22}$$

From (4.2), we observe that $2\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \leq 0$. Making use (4.21), we get

$$G^{'}(t) \geq K \left(G(t)\right)^{\frac{r_-}{2}} \tag{4.23}$$

where $K = \left(\frac{C_4}{1 + \beta^{2\left(\frac{1}{r_+} - \frac{1}{r_-}\right)}}\right)^{\frac{r_-}{2}}$ is a positive constant.

Integrating (4.23) from 0 to t we get

$$G(t) \geq \frac{1}{\left((G(0))^{1-\frac{r_-}{2}} + \frac{(2-r_-)Kt}{2}\right)^{\frac{2}{r_--2}}}$$

which implies that $G(t) \longrightarrow \infty$ as $t \longrightarrow T_{\max}$ in $H_0^1(\Omega)$, where

$$T_{\max} \leq \frac{2\,(G(0))^{\left(\frac{2-r_-}{2}\right)}}{(r_- - 2)\,K}.$$

Consequently, the solution to the problem (4.1) blows up in finite time. Hence the proof is completed. ■

4.3.2 Lower bound for blow-up time

In this section, we determine a lower bound for the blow-up time of the problem (2.19).

Theorem 4.3 *[45]Suppose that the conditions on $p(x), r(x)$, and A, given in section 1, hold. Furthermore assume that $2 < p_+ < \infty$ if $n \leq 2$, $2 < p_+ < \frac{2n}{n-2}$ if $n > 2$, $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{p(.)}(\Omega)$ and υ be a blow-up solution of problem (2.19), then a lower bound for blow-up time $T_{\min}$ can be estimated in the form*

$$T_{\min} \geq \int_{G(0)}^{\infty} \frac{d\xi}{2\max(C_-^{p_+}, C_+^{p_-})\left(\xi^{\frac{p_+}{2}} + \xi^{\frac{p_-}{2}}\right)}, \tag{4.24}$$

where C_-, C_+ are the optimal constants satisfying the Sobolev embedding inequalities $\|\upsilon\|_{L^{p_-}} \leq C_- \|\nabla\upsilon\|_2$ and $\|\upsilon\|_{L^{p_+}} \leq C_+ \|\nabla\upsilon\|_2$, respectively.

Proof. Consider $G(t)$ as in (4.12)

$$G(t) = \|\upsilon\|^2_{H_0^1(\Omega)}.$$

Multiply (4.1) by υ and integrate over Ω to get

$$\int_\Omega \upsilon\upsilon_t dx + \int_\Omega \nabla\upsilon\nabla\upsilon_t dx = \int_\Omega |\upsilon|^{p(x)}\, dx - \int_\Omega A\,|\nabla\upsilon|^{r(x)-2}\,\nabla\upsilon.\nabla\upsilon dx$$

A direct differentiation of $G(t)$ yields

$$G'(t) = 2\int_\Omega (\upsilon\upsilon_t + \nabla\upsilon\nabla\upsilon_t)\, dx,$$

then

$$G^{'}(t) = 2\left[\int_\Omega |v|^{p(x)}\,dx - \int_\Omega A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx\right].$$

Taking into account condition (4.2) on A, we find

$$G'(t) \le 2\int_\Omega |v|^{p(x)}\,dx. \tag{4.25}$$

Defining the sets

$$\Omega_+ = \{x\in\Omega : |v|\ge 1\} \text{ and } \Omega_- = \{x\in\Omega : |v|<1\}.$$

Thus, we have

$$\begin{aligned}\int_\Omega |v|^{p(x)}\,dx &= \int_{\Omega_+} |v|^{p(x)}\,dx + \int_{\Omega_-} |v|^{p(x)}\,dx\\ &\le \int_{\Omega_+} |v|^{p_+}\,dx + \int_{\Omega_-} |v|^{p_-}\,dx\\ &\le \int_{\Omega} |v|^{p_+}\,dx + \int_{\Omega} |v|^{p_-}\,dx.\end{aligned}$$

By the Sobolev embeddings (Lemma 1.7), we have

$$\begin{aligned}\int_\Omega |v|^{p(x)}\,dx &\le C_+^{p_+}\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p_+}{2}} + C_-^{s_-}\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p_-}{2}}\\ &\le \max(C_-^{p_+}, C_+^{p_-})\left(\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p_+}{2}} + \left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p_-}{2}}\right)\\ &\le \max(C_-^{p_+}, C_+^{p_-})\left((G(t))^{\frac{p_+}{2}} + (G(t))^{\frac{p_-}{2}}\right)\end{aligned} \tag{4.26}$$

where C_- and C_+ are the corresponding embedding constants. Therefore, (4.25) becomes

$$G^{'}(t) \le 2\max(C_-^{p_+}, C_+^{p_-})\left((G(t))^{\frac{p_+}{2}} + (G(t))^{\frac{p_-}{2}}\right) \tag{4.27}$$

By integrating both sides of the last inequality over $(0,T)$, we obtain

$$\int_{G(0)}^{G(t)} \frac{d\xi}{2\max(C_-^{p_+}, C_+^{p_-})\left(\xi^{\frac{p_+}{2}} + \xi^{\frac{p_-}{2}}\right)} \le T.$$

If v blow-up in H_0^1-norm, then we establish a lower bound for $T_{\min}$ by the form

$$T_{\min} \ge \int_{G(0)}^{\infty} \frac{d\xi}{2\max(C_-^{p_+}, C_+^{p_-})\left(\xi^{\frac{p_+}{2}} + \xi^{\frac{p_-}{2}}\right)},$$

which is the desired result. ■

4.4 Conclusion

In this study, we aim to investigate the possibility that solutions to certain evolution problems experience blow-up in finite time. This implies dealing with the presence of local, but not global, time. We concentrate here on the case in which the singularity occurs because the solution becomes unbounded in a specific region, causing the equation in question to lose its meaning. This is what we refer to as blow-up.

The blow-up phenomenon is observed in various types of nonlinear evolution equations, including Schrödinger equations, hyperbolic equations, parabolic equations, and pseudo-parabolic equations. In this work, we will specifically address pseudo-parabolic equations.

In this work, we have answered the questions that are usually posed in the study of the blow-up phenomenon, which include which solutions blow up, where, and how this occurs.

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