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Periodic orbits in some Zeraoulia-Sprott mapping

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بسم الله الرحمن الرحيم

شكر و عرفان

الحمد لله الذي وفقني وأعانني على إنهاء هذه المذكرة، أشكر الله سبحانه و تعالى على ما أسبغه عليا من نعم وعلى تسيير السبيل ومنحي العزم والصبر لإتمامها. فله الحمد والشكر في كل وقت وكل حين.

كما أتوجه بشكري وامتناني إلى من مهد لي طريق العلم فأعانني وكان لي خير قدوة أستاذي المشرف البروفسور " زراولية الحاج " الذي كانت له بصمات واضحة من خلال توجيهاته ودعمه الدائم لي.

كما أتقدم بالشكر لكل أساتذتي في تكويني خلال مسيرتي الدراسية وإلى كل من قدم لي يد العون من قريب أو من بعيد فله مني خالص الإحترام والتقدير

أسأل الله أن يجازي الجميع كل الخير

إهداء

إلى نفسي الطموحة، المثابرة والمكافحة التي سهرتِ سعياً وإجتهاداً في سبيل العلم وفي سبيل هذه اللحظة كنتي أهلا لكل المصاعب والتحديات. بدأتِها بطموحٍ وأنهيتِها بنجاح وأقول من فرط الطموح أنا لها وإن أبت رغما عنها أتيت بها .

وبكل حب أهدي ثمرة نجاحي وتخرجي:

إلى النور الذي أنار دربي والذي بذل جهد السنين من أجل أن أعتلي سلالم النجاح

"الغالي أبي"

إلى من إحتضنني قلبها قبل يدها، إلى التي سهلت لي الشدائد بدعائها والداعمة الأولى في حياتي واليد الخفية التي أزالت عن طريقي الاشواك والمتاعب

"الحبيبة أمي"

لا يعلو فضل على فضلكم حفظكم الله لي من كل مكروه

إلى ضلعي الثابت وأمان أيامي إلى من شددت عضدي به أخي

"محمد الهادي "

إلى من ساندوني بكل حب عند ضعفي زارعين الثقة والإصرار بداخلي. سندي والكتف الذي أستند عليه دائمًا أخواتي

"شيماء، حنان "

إلى القلوب الطاهرة وملائكة العائلة الأحفاد

"ألين، عبد الرحمان، محمد كنان"

Abstract

This thesis concerns the study of **periodic orbits** in some **Zeraoulia-Sprott** maps in 1 and 2 dimensions. The focus was on the analytical stability of periodic orbits in piecewise smooth maps of one dimension. The thesis was divided into three chapters that dealt with the properties of some 1 and 2 dimensions piecewise smooth maps, the Lyapunov exponent, the study of the periodic orbits of some two-dimensional chaotic attractors, and the study of the periodic orbits in one dimension

Resumé

Cette thèse concerne l'étude des **orbites périodiques** dans certaines cartes de **Zeraoulia-Sprott** en 1 et 2 dimensions. L'accent était mis sur la stabilité analytique des orbites périodiques dans des cartes lisses par morceaux d'une dimension. La thèse était divisée en trois chapitres traitant des propriétés de certaines cartes lisses par morceaux en 1 et 2 dimensions, de l'exposant de Lyapunov, de l'étude des orbites périodiques de certains attracteurs chaotiques bidimensionnels et de l'étude des orbites périodiques en une dimension.

ملخص

تتعلق هذه الأطروحة بدراسة **المدارات الدورية** في بعض خرائط **زراولية-سبروت** في البعدين الأول والثاني. كان التركيز على الاستقرار التحليلي للمدارات الدورية في خرائط سلسة متعددة الأجزاء ذات بعد واحد. وقد قسمت الأطروحة إلى ثلاثة فصول تناولت خواص بعض الخرائط الملساء ذات البعدين الأول والثاني، وأس لابونوف، ودراسة المدارات الدورية لبعض الجاذبات الفوضوية ثنائية الأبعاد، ودراسة المدارات الدورية في البعد الواحد .

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General Introduction

The periodic orbits is a fundamental concept in the analysis of dynamical systems and serving as a critical determinant of their characteristics. This concept was first used at the start of the 20th by the mathematician Henri Poincaré in the field of differential systems. Periodic orbits can be classified according to their period or their stability. It can be found in discrete and continuous dynamical systems.

In this work, we studied periodic orbits in some discrete Zeraoulia-Sprott chaotic maps in 1 and 2 dimensions. We focus on the stability of periodic orbits analytically in one-dimensional piecewise smooth maps. More details we divide this thesis into three chapters as follows:

Chapter 1, is devoted to presenting some properties of one and two-dimensional piecewise smooth maps and Lyapunov exponents.

Chapter 2, is focused only on studying periodic orbits 2-dimensional Zeraoulia-Sprott chaotic attractors.

Chapter 3, is interested in studying periodic orbits of some one-dimensional Zeraoulia-Sprott mappings.

Chapter 1

Notions of dynamical systems and prelimanery concepts

This chapter examines various concepts related to dynamical systems and piecewise smooth maps, including fixed points, periodic orbits, bifurcations, and Lyapunov exponents.

1.1 Fixed pionts and periodic orbits of maps

A discrete time system or map is defined by a difference equation:

$$x_{n+1} = f_{\mu}\left(x_n\right),\ x_n \in \mathbb{R}^n,$$

Definition 1.1 *Fixed points $x_{n+1} = x_n$, that is solutions of $x^* = f\left(x^*\right).$*

Definition 1.2 *Periodic orbits $\left(x_0, ..., x_{p-1}\right)$ with $x_k = f\left(x_{k-1}\right), k = 1, ..., p-1$ and $x_0 = f\left(x_{p-1}\right).$*

Therefore,

$$x_k = f^p\left(x_k\right) = f\left(...\left(f\left(x_k\right)\right)...\right), \quad k = 0, 1, 2, ..., p-1.$$

That is, periodic points are fixed points of the iteration f^p of the map f. The stability of fixed point or periodic orbits can also be studied by linearization. If $n = 1$, then a fixed point x^* is linearly stable if $\left|f'\left(x^*\right)\right| < 1$. The condition for a periodic orbit with period p is:

$$\left|\left(f^p\right)'\left(x_k\right)\right| < 1 \quad k = 0, 1, 2, ...p-1.$$

If fact, we only need to check one k, since

$$\left(f^p\right)'\left(x_0\right) = \left(f^p\right)'\left(x_1\right) = ... = \left(f^p\right)'\left(x_{p-1}\right)$$

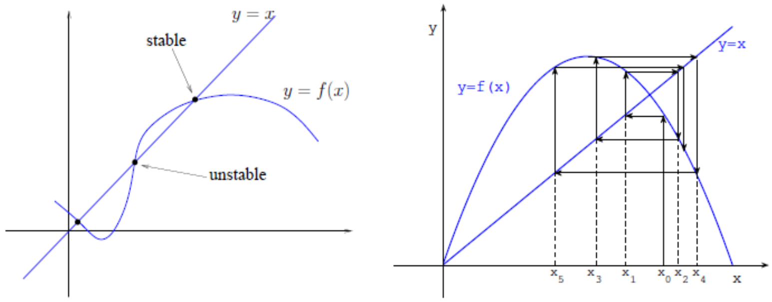


Figure 1.1: Left figure showing graphic of the fixed points at the intersection between $y = x$ and $y = f(x)$, and the cobweb diagram showing the iteration of the map $x^n + 1 = f(x^n)$ in right figure.

We have

$$\begin{aligned}\frac{df^p}{dx}(x_k) &= f'\underbrace{(f(\ldots f(x_k)\ldots))}\times f'\underbrace{(f(\ldots f(x_k)\ldots))}\times\ldots\times f'(x_k)\\ &= f'(x_{k+p-1})\times f'(x_{k+p-2})\times\ldots\times f'(x_k)\\ &= f'(x_{k-1})\times f'(x_{k-2})\times\ldots\times f'(x_k)\\ &= f'(x_0)\times f'(x_1)\times\ldots\times f'(x_{p-1})\end{aligned}$$

So for the linear stability of a periodic orbit there is only one condition: $\prod_{k=0}^{p-1}|f'(x_k)| < 1.$

Example 1.1 *Consider the map*

$$x_{n+1} = \lambda x_n, \quad y_{n+1} = \lambda^2 y_n.$$

with λ is any non-zero constant. An invariant set for this map is the parabola $P = \{(x, y) : y = x^2\}$. In fact, if $(x_n, y_n) \in P$, then $y_n = x_n^2$ and

$$y_{n+1} = \lambda^2 y_n = \lambda^2 x_n^2 = (\lambda x_n)^2 = x_{n+1}^2.$$

That is $(x_{n+1}, y_{n+1}) \in P$ as well. Therefore, the parabola P is invariant.

Due to the discrete nature of the points x_n in the space, special graphical tools are used instead of phase portraits. To visualize fixed points for the one-dimensional map $x_{n+1} = f(x_n)$, we consider the intersection of the line $y = x$ and the curve $y = f(x)$. The stability of a fixed point x^* can be determined graphically by comparing the slope of $f(x)$ at x^* with the slope of the line $y = x$ (see Figure 1.1). The iteration of the trajectory $x_0, x_1, \ldots$ can be viewed from the cobweb diagram (right figure in Figure 1.1):

1. The vertical line $x = x_n$ intersect the curve $y = f(x)$ at $(x_n, f(x_n)) = (x_n, x_{n+1})$.

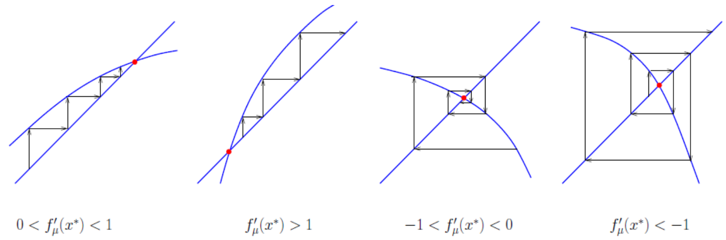


Figure 1.2: The cobweb diagrams description of the behavior near a fixed point.

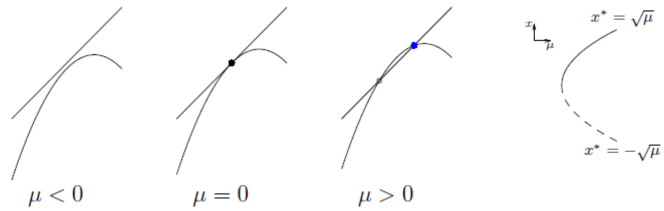


Figure 1.3: Saddle-node bifurcation.

2. The horizontal line through (x_n, x_{n+1}) intersect the line $y = x$ at $(x_{n+1}, f(x_{n+1}))$.

3. The vertical line through (x_{n+1}, x_{n+1}) becomes $x = x_{n+1}$ and the whole process can be continued again.

The behavior of the map $x_{n+1} = f_\mu(x_n)$ near a fixed point can be visualized using a cobweb diagram (see Figure 1.2). The stability (whether it converges towards the fixed point x^* or not) depends on whether the absolute value of the derivative $|f'_\mu(x^*)|$ is greater than one.
The sign of $f'_\mu(x^*)$ determines the appearance of the cobweb diagram. If $f'_\mu(x^*)$ is positive, the diagram looks like stairs. If $f'_\mu(x^*)$ is negative, the diagram looks like spirals.

1.1.1 Bifurcation of maps

Saddle-node (tangential) bifurcation: For the map $x_{n+1} = \mu + x_n - x_n^2$.

- If $\mu > 0$,there are two fixed points $x^*_\pm = \pm\mu^{1/2}$, the fixed point $x^*_+ = \mu^{1/2}$ is stable but $x^*_- = -\mu^{1/2}$ is unstable.

- If $\mu < 0$, there is no fixed point, because bifurcation occurs when the straight line $y = x$ touches the parabola $y = \mu + x_n - x_n^2$ tangentially at $\mu = 0$ (see Figure 1.3).

Transcritical bifurcation: For $x_{n+1} = (1 + \mu) x_n - x_n^2$. There are always two fixed points $x^* = 0$ and $x^* = \mu$.

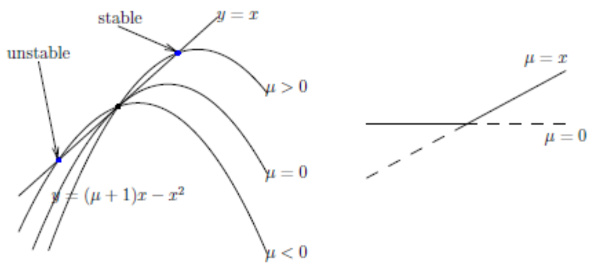


Figure 1.4: Transcritical bifurcation.

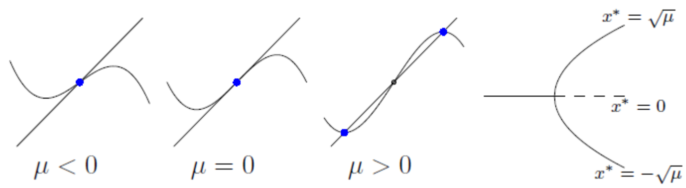


Figure 1.5: Pitchfork bifurcation.

- When $\mu < 0$, the fixed point $x^* = 0$ is stable and $x^* = \mu$ is unstable.
- When $\mu > 0$, $x^* = 0$ becomes unstable and $x^* = \mu$ stable.

Supercritical pitchfork bifurcation: For $x_{n+1} = \mu + x_n - x_n^3$

- If $\mu < 0$, there is only one fixed point $x^* = 0$, and it is stable.
- If $\mu > 0$, there are three fixed points, $x^* = \pm\mu^{1/2}$ and they are stable, but $x = 0$ unstable.

1.1.2 Logistic map

The most basic quadratic family of maps is the *logistic* map

$$f_\mu(x) = \mu x(1 - x), \quad \mu \geq 0, \tag{1.1}$$

In the context of population dynamics, the two terms μx and $-\mu x^2$ in this map can be interpreted as and starvation (density-dependent mortality) respectively.
Fixed Points: There are two fixed points $x^* = 0$ and $x^* = (\mu - 1)/\mu$, provided $\mu \geq 1$.
Linear stability: We have $f'_\mu(x) = \mu - 2x\mu$.

- If $0 \leq \mu < 1$, the fixed point $x^* = 0$ is stable and the fixed point $x^* = (\mu - 1)/\mu$ is not in the range $[0, 1]$.

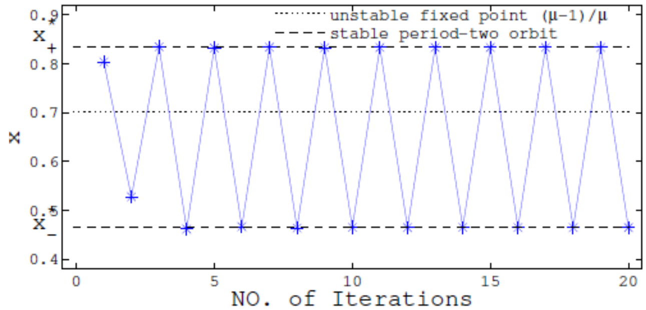


Figure 1.6: The fixed point $x^* = (\mu - 1)/\mu$ becomes unstable as $\mu > 3$, and a period-2 orbit emerges.

- If $\mu \geq 1$, the fixed point $x^* = 0$ becomes unstable, but $x^* = (\mu - 1)/\mu$ become stable, as long as $1 < \mu < 3$. Because the fixed points $x^* = 0$ and $x^* = (\mu - 1)/\mu$ exchange stability at $\mu = 1$, this is a transcritical bifurcation.

Period-doublin bifurcation: As μ passes 3, $f'_\mu((\mu - 1)/\mu)$ passes -1 and $x^* = (\mu - 1)/\mu$ becomes unstable (see Figure 1.6 for sample iterations at $\mu = 3.35$). Aperiod-two orbit (x^*_+, x^*_-) appears, such that $x^*_+ = f_\mu(x^*_-), x^*_- = f_\mu(x^*_+)$. In other words, both x^*_+ and x^*_- are fixed points of $x = f_\mu(f_\mu(x))$, but not fixed points of $x = f_\mu(x)$. This is called *period-doubling bifurcation*, signified by $f_{\mu^*}(x) = -1$ at $\mu^* = 3$.
Since

$$x - f_\mu(f_\mu(x)) = x(\mu x - \mu + 1)\left(\mu^2 x^2 - \left(\mu^2 + \mu\right)x + \mu + 1\right),$$

all solutions of $x - f_\mu(f_\mu(x)) = 0$ are

$$x^* = 0, \quad x^* = \frac{\mu - 1}{\mu}, \quad x^*_\pm = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}.$$

The first two are inherited from $x^* = f_\mu(x^*)$, while the last two constitute the periodic two orbits. Solving the quadratic equation $\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1 = 0$, and with a more detailed calculation it's shown that the this period-two orbit loses its stability when the modulus

$$\frac{df_\mu(f_\mu(x))}{dx}|_{x^*_\pm} = f'_\mu(f_\mu(x))\, f'_\mu(x)\,|_{x^*_\pm} = f'_\mu(x^*_+)\, f'_\mu(x^*_-)$$

is greater than unit. From

$$x^*_+ + x^*_- = \frac{\mu - 1}{\mu}, \quad x^*_+ x^*_- = \frac{\mu - 1}{\mu^2},$$

can be simplified the Jacobian $\frac{df_\mu(f_\mu(x))}{dx}|_{x^*_\pm}$ as

$$f'_\mu(x^*_+)\, f'_\mu(x^*_-) = \mu^2\left(1 - 2x^*_-\right)\left(1 - 2x^*_+\right) = \mu^2\left(1 - 2\left(x^*_+ + x^*_-\right) + 4x^*_+ x^*_-\right) = 4 + 2\mu - \mu^2.$$

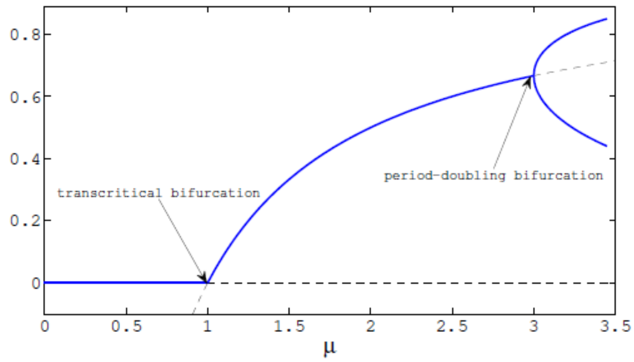


Figure 1.7: Bifurcation diagram for the map (1.1), when μ is not too close to 4.

Resolving

$$\frac{df_{\mu}\left(f_{\mu}\left(x\right)\right)}{dx}|_{x_{\pm}^{*}}=\pm1,$$

we obtain

$$\mu=-1 \text{ or } \mu=3 \quad \text{for} \quad \frac{df_{\mu}\left(f_{\mu}\left(x\right)\right)}{dx}|_{x_{\pm}^{*}}=1,$$

and

$$\mu=1\pm\sqrt{6} \text{ for } \frac{df_{\mu}\left(f_{\mu}\left(x\right)\right)}{dx}|_{x_{\pm}^{*}}=-1.$$

We do not need to consider the negative values of $\mu=-1$ or $\mu=1-\sqrt{6}$, indeed the fixed points $x_{\pm}^{*}$ exists only for $\mu\geq3$. Therefore, the only possible bifurcation occurs at $\mu^{*}=1+\sqrt{6}\approx3.449$, where $\frac{df_{\mu}(f_{\mu}(x))}{dx}|_{x_{\pm}^{*}}=-1$. For equation $x=f_{\mu}(f_{\mu}(x))$, the value -1 suggests an additional period-doubling bifurcation that results in period-four orbits, which are solutions of $x=f_{\mu}\left(f_{\mu}\left(f_{\mu}\left(f_{\mu}\left(x\right)\right)\right)\right).$
In fact, there is an infinite cascade of period-doubling bifurcations. At $\mu_{1}=3$, the system transitions from period 1 to period 2. At $\mu_{2}=1+\sqrt{6}$, the period-doubles again from 2 to 4. This pattern continues: μ_{n} corresponds to a transition from period 2^{n-1} to 2^{n}. Furthermore, as μ_{n} approaches a finite limit (approximately 3.56995), the period-doubling cascade ceases and chaotic behaviors emerge. The limit of the ratio between the lengths of two consecutive bifurcation intervals is a universal *Feigenbaum* constant:

$$\lim_{n\to+\infty}\frac{\mu_{n-1}-\mu_{n}}{\mu-\mu_{n+1}}\approx4.669$$

Many other maps exhibit similar period-doubling bifurcations, and this limiting ratio remains consistent regardless of the specific details of the map. It's worth noting that while the bifurcation diagrams for both pitchfork and period-doubling bifurcations may appear similar, the behaviors of the fixed points differ significantly. In period-doubling bifurcation, the new "fixed points" actually satisfy the equation $x=f_{\mu}(f_{\mu}(x))$, rather than $x=f_{\mu}\left(x\right).$

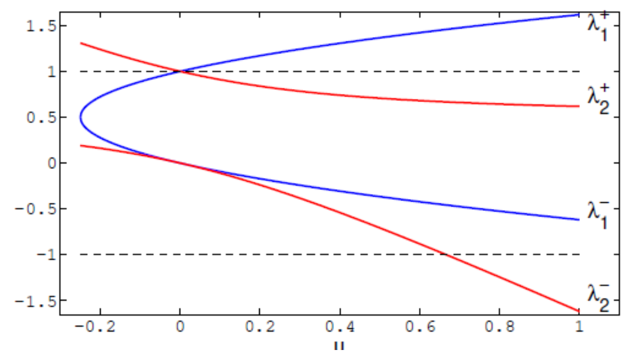


Figure 1.8: The eigenvalues of the Jacobian matrix near (x_1^*, y_1^*) and (x_2^*, y_2^*) .

1.1.3 Bifurcation of two-dimensional maps

By observing that when a parameter changes,it results in eigenvalues of the Jacobian matrix with unit modulus, the same method may be used to investigate the bifurcation of two-dimensional maps.

Example 1.2 *Consider the map* $x_{n+1} = \mu y_n + x_n - x_n^2$, $y_{n+1} = x_n$. *There are two fixed points* $(x_1^*, y_1^*) = (0, 0)\,, (x_2^*, y_2^*) = (\mu, \mu)$. *The Jacobian matrix is given by*

$$J(x, y) = \begin{pmatrix} 1 - 2x & \mu \\ 1 & 0 \end{pmatrix}.$$

we get

$$J(x_1^*, y_1^*) = \begin{pmatrix} 1 & \mu \\ 1 & 0 \end{pmatrix}, J(x_2^*, y_2^*) = \begin{pmatrix} 1 - 2\mu & \mu \\ 1 & 0 \end{pmatrix}.$$

For the fixed point (x_1^*, y_1^*) , *the two eigenvalues are governed by* $\lambda_1^\pm = \frac{1 \pm \sqrt{1+4\mu}}{2}$. *For the fixed point* (x_2^*, y_2^*) , *the two eigenvalues are governed by*

$$\lambda_2^\pm = \frac{1 - 2\mu \pm \sqrt{(1-2\mu)^2 + 4\mu}}{2} = \frac{1 - 2\mu \pm \sqrt{1 + 4\mu^2}}{2}.$$

The stability of the two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) are exchanged, indicating the transcritical bifurcation at $\mu = 0$. The bifurcation is also clear from Figure 1.8. The fixed point $(x_1^*, y_1^*) = (0, 0)$ is stable, for $\mu \in (-1/4, 0)$. The other fixed point $(x_2^*, y_2^*) = (\mu, \mu)$ is stable for $\mu > 0$, but becomes unstable again when $\lambda_2^- = -1$, or $\mu = 2/3$. A period-doubling bifurcation occurs her (associated with eigenvalue -1).

1.2 Piecewise smooth maps

This section examines the piecewise smooth map in one and two dimensions by focusing on three key ideas. Firstly, the map's definition and some of its characteristics. Secondly, the fixed points in both dimensions. Thirdly, periodic orbits. Consider a map $F : \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$x_n + 1 = F(x_n), \quad x_0 \in \mathbb{R} \tag{1.2}$$

Some properties:

- The map (1.2) is a piecewise smooth, if the phase space $\mathbb{R}^m$ can be partitioned into a finite number J of disjoint non-empty open regions $R_i, i = 1, ..., J$, and a boundary Σ, so that $\mathbb{R}^m = \left(\bigcup_{i=1}^{J} R_i\right) \cup \Sigma$.
- The boundary Σ made up of a union of continuously differentiable surfaces which separate the regions R_i.
- F is smooth in each regions R_i.
- Non-smoothness of F occurs on Σ, which is called **switching surface** or switching manifold.
- The map (1.2) is also known as **hybrid system**. For more details see [5]

1.2.1 One-dimensional piecewise smooth maps

Let 1-D piecewise smooth system be defined as follow:

$$x_{n+1} = f(x_n, \mu) = \begin{cases} g(x, \mu), & x < x_b \\ h(x, \mu), & x > x_b \end{cases} \tag{1.3}$$

where μ is the bifurcation parameter, the smooth curve $x = x_b$, the state space was separated into two regions R_L and R_R given by:

$$\begin{cases} R_L = \{x \in \mathbb{R} : x < x_b\} \\ R_R = \{x \in \mathbb{R} : x > x_b\} \end{cases}$$

and the boundary between them is given by:

$$\Sigma = \{x \in \mathbb{R} : x = x_b\}$$

Some properties:

- The map f is continuous, but its derivative is discontinuous at the borderline $x = x_b$.
- The functions g and h are both continuous and they have continuous derivatives in x everywhere except at x_b.
- $x_0(\mu)$ is a possible path of fixed points of f, this path depends continuously on μ.
- The possible fixed point hits the boundary at a critical parameter value $\mu_b : x_0(\mu_b) = x_b$.

This system has a normal form given by:

$$N(x,\mu) = \begin{cases} ax + \mu, & x < 0 \\ bx + \mu, & x > 0 \end{cases} \tag{1.4}$$

where μ is a parameter and a, b are the graph's slopes at its two sides R_L and R_R of the border $x = 0$. The fixed points are as follow: to the left $(x < x_b)$ and right $(x > x_b)$, respectively of the boundary, let x_L^* and x_R^* be the system's possible fixed points. Then, in the normal form (1.4) we have:

$$\begin{cases} x_L^* = \frac{\mu}{1-a} < 0, & \text{if} \quad a < 1 \wedge \mu < 0 \\ x_R^* = \frac{\mu}{1-b} > 0, & \text{if} \quad b < 0 \wedge \mu < 0 \end{cases}$$

Periodic orbits are as follow: The period-2 orbit of the normal form (1.4) is given by:

$$N(N(x,\mu)) - x = \begin{cases} (a^2-1)x + (a+1)\mu, & x < 0 \\ (b^2-1)x + (b+1)\mu, & x > 0 \end{cases}$$

and the period-3 orbit are given by:

$$N(N(N(x,\mu))) - x = \begin{cases} (a^4-1)x + (a^3+a^2+a+1)\mu, & x < 0 \\ (b^4-1)x + (b^3+b^2+b+1)\mu, & x > 0 \end{cases}$$

1.2.2 Two-dimensional piecewise smooth maps

Let 2-D piecewise smooth system defined be as follow:

$$g(\hat{x},\hat{y},\rho) = \begin{cases} g_1 = \begin{pmatrix} f_1(\hat{x},\hat{y},\rho) \\ f_2(\hat{x},\hat{y},\rho) \end{pmatrix}, & if \quad \hat{x} < S(\hat{y},\rho) \\ g_2 = \begin{pmatrix} f_3(\hat{x},\hat{y},\rho) \\ f_4(\hat{x},\hat{y},\rho) \end{pmatrix}, & if \quad \hat{x} > S(\hat{y},\rho) \end{cases} \tag{1.5}$$

where ρ is the bifurcation parameter, the smooth curve $\hat{x} = S(\hat{y},\rho)$ created two regions in the phase plane R_L and R_R given by:

$$\begin{cases} R_L = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2, & \hat{x} < S(\hat{y}, \rho)\} \\ R_R = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2, & \hat{x} > S(\hat{y}, \rho)\} \end{cases}$$

and the boundary between them as:

$$\Sigma = \left\{(\hat{x}, \hat{y}) \in \mathbb{R}^2, \quad \hat{x} = S(\hat{y}, \rho)\right\}$$

Some properties:

- Although the map g is continuous, its derivation discontinues at the borderline $\hat{x} = S(\hat{y}, \rho)$.
- Both g_1 and g_2 are continuous functions with continuous derivatives.
- In each subregion R_L and R_R, the one-sided partial derivatives near the boundary are finite.
- The map (1.5) has one fixed point in R_L and one fixed point in R_R for a value ρ_* of the parameter ρ.

This system has a normal form is given:

$$N(x, y) = \begin{cases} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x < 0 \\ \\ \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x > 0 \end{cases} \tag{1.6}$$

where μ is a parameter and $\tau_{L,R}, \delta_{_{L,R}}$ are the traces and determinants of the corresponding matrices of the linearized map in the two subregions R_L and R_R given by :

$$\begin{cases} R_L = \{(x, y) \in \mathbb{R}^2\}, & x < 0 \\ R_R = \{(x, y) \in \mathbb{R}^2\}, & x > 0 \end{cases}$$

Fixed points: Let P_L and P_R be the possible fixed points of the system near the border to the right: $x < S(\hat{y}, \rho)$ and left: $x > S(\hat{y}, \rho)$ of the border respectively. Then in the normal form (1.6) we have

$$\begin{cases} P_L = \left(\frac{\mu}{1-\tau_L+\delta_L}, \frac{-\delta_L \mu}{1-\tau_L+\delta_L}\right) \in R_L \\ \\ P_R = \left(\frac{\mu}{1-\tau_R+\delta_R}, \frac{-\delta_R \mu}{1-\tau_R+\delta_R}\right) \in R_R \end{cases}$$

with eigenvalues $\lambda_{L\,1.2}$ and $\lambda_{R\,1.2}$ respectively.

Periodic orbits: The period-2 orbit of the normal form (1.6) is given by:

$$N(N(x,y)) - (x,y) = \begin{cases} \begin{pmatrix} N_1 - x \\ N_2 - y \end{pmatrix}, & x < 0 \\ \\ \begin{pmatrix} N_3 - x \\ N_4 - y \end{pmatrix}, & x > 0 \end{cases}$$

where

$$\begin{cases} \begin{cases} N_1 = y + 2\mu + x\tau_L + y\tau_L + \mu\tau_L + x\tau_L^2 \\ \\ N_2 = -\delta_L x \end{cases}, & x < 0 \\ \\ \begin{cases} N_3 = y + 2\mu + x\tau_R + y\tau_R + \mu\tau_R + x\tau_R^2 \\ \\ N_4 = -\delta_R x \end{cases}, & x > 0 \end{cases}$$

and the period-3 orbit given by:

$$N(N(N(x,y))) - (x,y) = \begin{cases} \begin{pmatrix} N_5 - x \\ N_6 - y \end{pmatrix}, & x < 0 \\ \\ \begin{pmatrix} N_7 - x \\ N_8 - y \end{pmatrix}, & x > 0 \end{cases}$$

where

$$\begin{cases} \begin{cases} N_5 = y + 3\mu + x\tau_L + 2y\tau_L + 3\mu\tau_L + 2x\tau_L^2 + x\tau_L^3 + y\tau_L^2 + \mu\tau_L^2 \\ \\ N_6 = -\delta_L x \end{cases}, & x < 0 \\ \begin{cases} N_7 = y + 3\mu + x\tau_R + 2y\tau_R + 3\mu\tau_R + 2x\tau_R^2 + x\tau_R^3 + y\tau_R^2 + \mu\tau_R^2 \\ \\ N_8 = -\delta_R x \end{cases}, & x > 0 \end{cases}$$

1.3 Lyapunov Exponent

In this section, we present an analytical metric that can be applied to periodic orbits and chaos analysis. The rate at which an infinitesimally small gap between two initially close states increases over time is known as the *Lyapunov exponent*:

$$F^t(x_0 + \varepsilon) - F^t(x_0) \approx \varepsilon e^{\lambda t} \tag{1.7}$$

The distance between two initially near states after t steps is represented on the left, while the assumption that the distance increases exponentially with time is represented on the right. The Lyapunov exponent is the exponent λ that is measured over an extended period of time (preferably $t \to \infty$). Small distances extend infinitely over time if $\lambda > 0$, indicating the presence of the stretching mechanism. Alternatively, in the case when $\lambda < 0$, the system finally settles into a periodic trajectory, meaning that short distances don't expand endlessly. Keep in mind that while stretching is the only mechanism of chaos that the Lyapunov exponent characterizes, this is not the only mechanism. Remember that this Lyapunov exponent does not capture the folding mechanism. We can do a little bit of mathematical derivation to transform Equation (1.7) and make it easier to compute:

$$\begin{aligned} e^{\lambda t} &\approx \frac{|F^t(x_0+\varepsilon) - F^t(x_0)|}{\varepsilon} \\ \lambda &= \lim_{t\to\infty,\varepsilon\to 0} \frac{1}{t} \log \frac{|F^t(x_0+\varepsilon) - F^t(x_0)|}{\varepsilon} \\ &= \lim_{t\to\infty,\varepsilon\to 0} \frac{1}{t} \log \left|\frac{dF}{dx}\right|_{x=x_0} \end{aligned}$$

(using the chain rule of differentiation)

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \left|\frac{dF}{dx}\right|_{x=F^{t-1}(x_0)=x_{t-1}} \cdot \frac{dF}{dx}|_{x=F^{t-2}(x_0)=x_{t-2}} \cdots \frac{dF}{dx}|_{x=x_0}$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} \log \left|\frac{dF}{dx}\right|_{x=x_i}$$

Hence, the Lyapunov exponent is a time average of $log|\frac{dF}{dx}|$ at every state the system visits over the course of the simulation.

By comparing this figure with the bifurcation diagram (Fig.1.9), we will notice that the parameter range where the Lyapunov exponent takes positive values nicely matches the range where the system shows chaotic behaviors. Additionally, the Lyapunov exponent meets the $\lambda = 0$ line whenever a bifurcation takes place (e.g $r = 1, 1.5$, etc), showing the criticality of such parameter values. Finally, there are several locations in the plot where the Lyapunov exponent diverges to negative infinity. Such values occur when the system converges to an extremely stable equilibrium point with $\frac{dF^t}{dx}|_{x=x0} \approx 0$ for certain t.

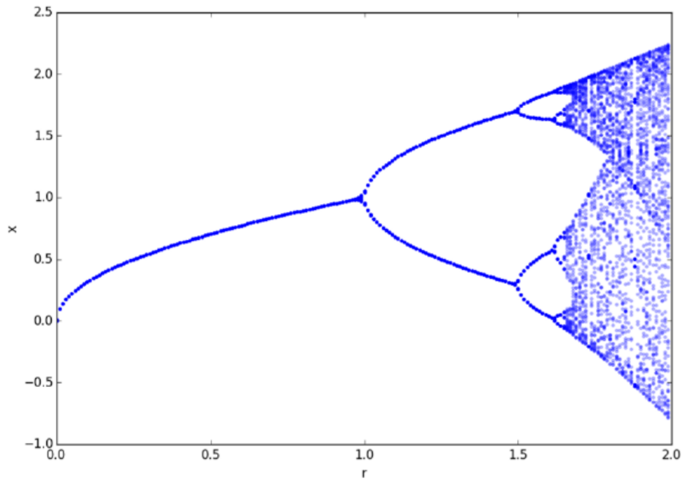


Figure 1.9: The numerically constructed bifurcation diagram of equation $x_t = x_{t-1} + r - x_{t-1}^2$.

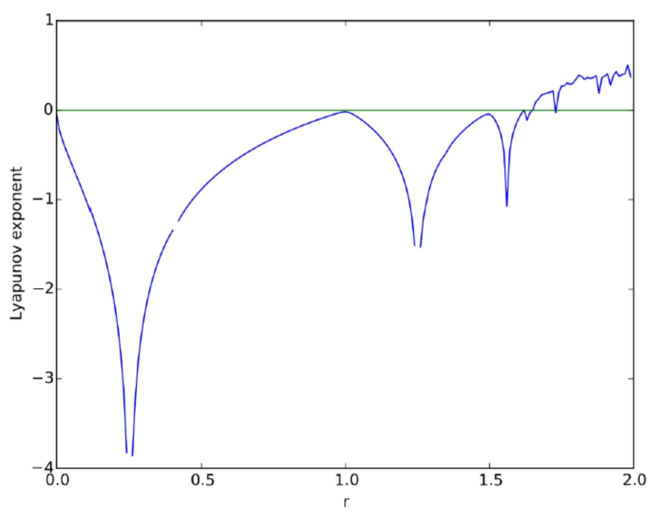


Figure 1.10: The Lyapunov exponent of equation $x_t = x_{t-1} + r - x_{t-1}^2$.

Chapter 2

Periodic orbits of some 2-D Zeraoulia-Sprott chaotic attractors

This chapter presents some characteristics of periodic orbits of some two dimensions Zeraoulia-Sprott chaotic maps. It also touches on fixed points their stability. In addition, we give some graphical representations of their dynamical behaviors in parameter space.

2.1 A minimal 2-D quadratic map

This section describes and analyse a simple minimal 2-dimensional quadratic map. Indeed, the Hénom map [7] given by:

$$H(x,y) = \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}$$

is the simplest example of a dissipative map with chaotic solutions and it has been widely studied. Its area contraction is constant over the orbit in the ab-plane and depends solely on b, resulting in a single quadratic nonlinearity. An alternative way to express it is as a one-dimensional time-delayed map $x_{n+1} = 1 - ax_n^2 + bx_{n-1}$. Here we present and analyze an equally simple two-dimensional quadratic map given by:

$$f(x,y) = \begin{pmatrix} 1 - ay^2 + bx \\ x \end{pmatrix} \tag{2.1}$$

where a and b are bifurcation parameters.

2.1.1 Some properties

- Equation (2.1) is an interesting minimal system, similar to the Hénom map, but with the time delay in the non-linear rather than the linear term as evidence by writing it in the time-delayed form $x_{n+1} = 1 - ax_{n-1}^2 + bx_n$.
- It differs from the Hénon map, despite its similarity and simplicity, in that it contains irregular dissipation, a richer and more diverse path to chaos and a variety of attractors.
- These attractors covering the entire range of dimensions from 1 to 2 with basin of attraction that are often much more complicated than for the Hénon map.
- This system is special case of general 2-D quadratic maps and differs from other well-known 2-D maps such as the delayed logistic map [8] given by $g(x, y) = (ax(1 - y), x)$.

2.1.2 Fixed points

We starting by studying the existence of the fixed points of the map (2.1), then determine their stability. We have

$$f(x, y) = (x, y) \Leftrightarrow \begin{pmatrix} 1 - ay^2 + bx \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

we get the following equations:

$$\begin{cases} 1 - ay^2 + bx = x \\ x = y \end{cases} \Rightarrow 1 - ax^2 - (1 - b)x = 0$$

we calculate the discriminant of the last equation $\Delta = (1 - b)^2 + 4a$ and we have two cases:

- If $\Delta = (1 - b)^2 + 4a \geq 0$, the map (2.1) has two fixed points

$$\begin{cases} P_1 = \left(\frac{b-1-\sqrt{4a-2b+b^2+1}}{2a}, \frac{b-1-\sqrt{4a-2b+b^2+1}}{2a}\right) \\ \\ P_2 = \left(\frac{b-1+\sqrt{4a-2b+b^2+1}}{2a}, \frac{b-1+\sqrt{4a-2b+b^2+1}}{2a}\right) \end{cases}$$

- If $\Delta = (1 - b)^2 + 4a < 0$, the map (2.1) has no fixed points.

Now, we determine their stability:

1. The Jacobian matrix of the map (2.1) is $J(x,y) = \begin{pmatrix} b & -2ay \\ 1 & 0 \end{pmatrix}$ and at a fixed point (x,x), its characteristic polynamials are given by:

$$\lambda^2 - b\lambda + 2ax = 0 \tag{2.2}$$

2. The local stability of the two equilibria is studied by evaluating the roots of equation (2.2). So, after some of calculate we obtained the following results:

P_1 is unstable in the following cases:

1. $a \geq -((-b+1)/2)^2, b < 0.$
2. $a \geq -((-b+1)/2)^2, a > (1/2)b + (3/4)b^2 - (1/4), b > 0.$

P_1 is a saddle point in the following case:

1. $a \geq -((-b+1)/2)^2, a < (1/2)b + (3/4)b^2 - (1/4), b > 0.$

On the other hand, P_2 is unstable in the following cases:

1. $a \geq -((-b+1)/2)^2, a > (1/8)b^2 - (1/8)b^3 + (1/64)b^4, b \geq 2.$
2. $a \geq -((-b+1)/2)^2, a > -(1/2)b + (3/4), b < 2.$
3. $a \geq -((-b+1)/2)^2, a \leq (1/8)b^2 - (1/8)b^3 + (1/64)b^4, b > 2.$

P_2 is stable in the following cases:

1. $a \geq -((-b+1)/2)^2, a > (1/8)b^2 - (1/8)b^3 + (1/64)b^4, a \geq -((-b+1)/2)^2,$
   $a < -(1/2)b + (3/4), b < 2.$
2. $a \geq -((-b+1)/2)^2, a \leq (1/8)b^2 - (1/8)b^3 + (1/64)b^4, 0 \leq b \leq 2.$
3. $a \geq -((-b+1)/2)^2, a \leq (1/8)b^2 - (1/8)b^3 + (1/64)b^4,$
   $a > (1/2)b + (3/4)b - (1/4), -2 < b < 0$

P_2 is a saddle point in the following cases:

1. $a \geq -((-b+1)/2)^2, a \leq (1/8)b^2 - (1/8)b^3 + (1/64)b^4,$
   $a < (1/2)b + (3/4)b^2 - (1/4), -2 \leq b < 2.$

A shematic representation of these results is given in Figure 2.1.

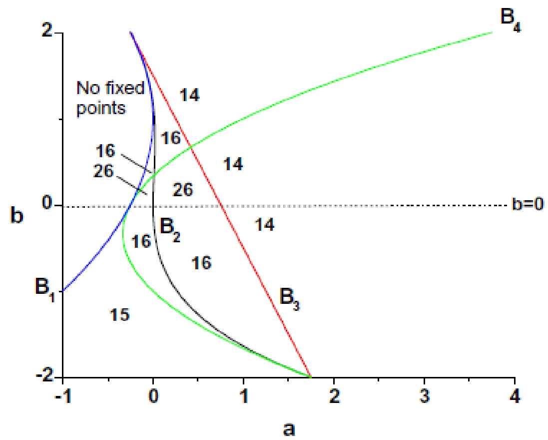


Figure 2.1: Stability of the fixed points of the map (2.1) in the ab-plane, where the numbers on the figure are as follows: 1: P_1 is unstable, 2: P_1 is a saddle, 3: P_1 is stable, 4: P_2 is unstable, 5: P_1 is a saddle, 6: P_1 is stable.

2.1.3 Priodic orbits

We have

$$f\left(f\left(x,y\right)\right)=\begin{pmatrix} f_1\left(x,y\right) \\ f_2\left(x,y\right) \end{pmatrix}$$

where

$$\begin{cases} f_1\left(x,y\right)=a^3y^4+2a^2bxy^2+2a^2y^2 \\ -ab^2x^2-2abx-aby^2-a+b^2x+b+1 \\ \\ f_2\left(x,y\right)=x \end{cases}$$

The period-2 orbit of the map (2.1) is given by:

$$f\left(f\left(x,y\right)\right)-\left(x,y\right)=0\Leftrightarrow\begin{pmatrix} f_1\left(x,y\right)-x \\ f_2\left(x,y\right)-y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the period-3 orbit given by:

$$f\left(f\left(f\left(x,y\right)\right)\right)-\left(x,y\right)=0\Leftrightarrow\begin{pmatrix} f_3\left(x,y\right)-x \\ f_4\left(x,y\right)-y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{cases} f_3\left(x,y\right)=1-a\left(f_1\left(x,y\right)\right)^2+b\left(f_1\left(x,y\right)\right) \\ f_4\left(x,y\right)=x \end{cases}$$

The periodic orbits are shown respectively in Figure.2.2 along with their basins of attraction in white and their absence in Figure.2.3.

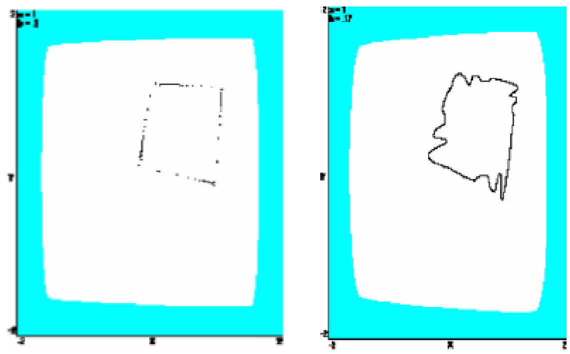


Figure 2.2: Left figure represent a periodic orbit of the map (2.1) with its basin of attraction (white) obtained for $a = 1$and $b = 0.1$, right figure represent a quasi-periodic orbit with its basin of attraction (white) for $a = 1$ and $b = 0.17$.

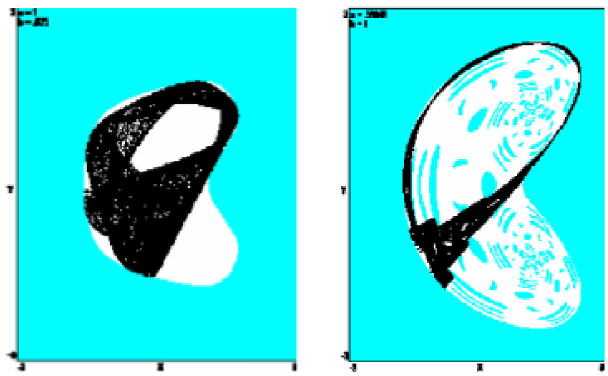


Figure 2.3: Left figure represent the chaotic attractor with its basin of attraction (white) for $a = 1$ and $b = 0.675$, and the right figure represent another chaotic attractor with its basin of attraction (white) for $a = 0.59948$ and $b = 1$.

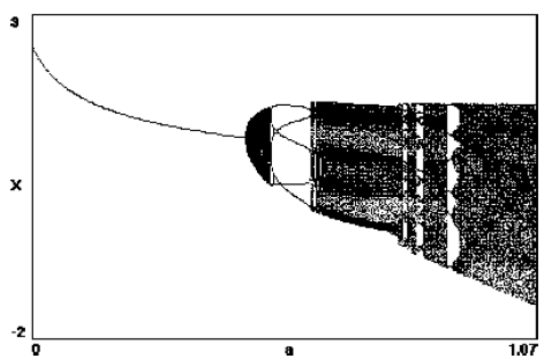
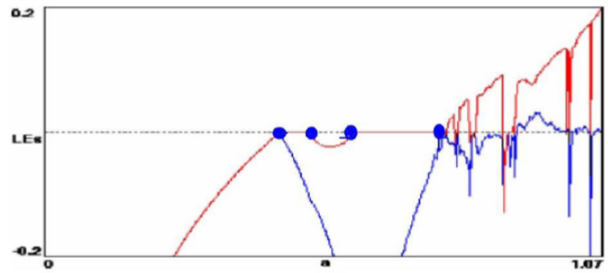


Figure 2.4: The quasi-periodic route to chaos for the map (2.1) obtained versus the parameter $0 < a \leq 1.07$ with $b = 0.6$.



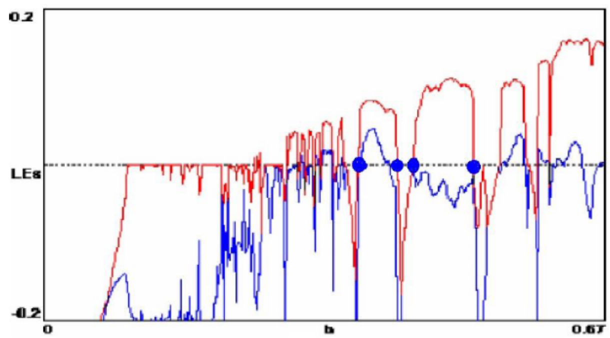


Figure 2.5: The Lyapunov exponents of map (2.1) versus the parameter $0 < a \leq 1.07$ with $b = 0.6$ and $0 < b \leq 0.67$, with $a = 1$.

Figure 2.4 shows a the periodic orbits of periods 5 and 6 for map (2.1)
Figure 2.5 shows big dots that indicate the existence periodic orbits and the values for which Lyapunov exponents are zero.
Figure 2.6 shows regions of unbounded (white), fixed point (gray), periodic (blue), quasiperiodic (green), and chaotic (red) solutions in the ab-plane for the map (2.1), where we use $|LE| < 0,0001$ as the crirterion for quasi-periodic orbits with 10^6 iterations for each point.
Figure 2.7 shows a similar plot for the Hénom map.

2.2 A simple 2-D rational discrete mapping

This section introduces a simple rational map along with some of its dynamical properties. In [9], a 1-D discrete iterative system featuring a rational fraction was discovered during a study of evolutionary algorithms:

$$g(x) = \frac{1}{0.1 + x^2} - ax, \tag{2.3}$$

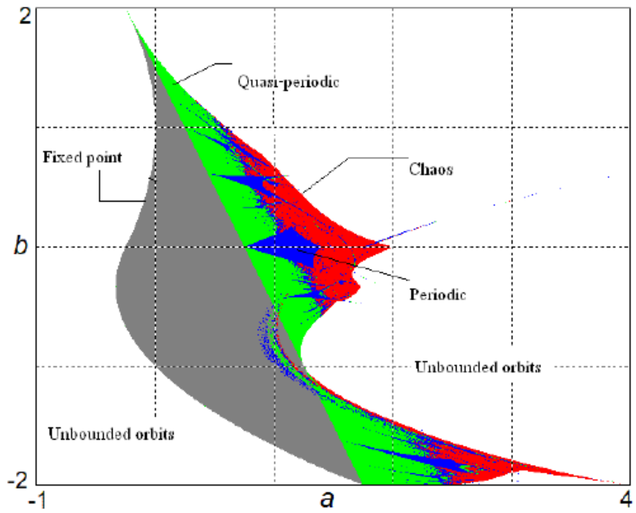


Figure 2.6: Dynamical behaviors regions in the ab-plane of the map (2.1).

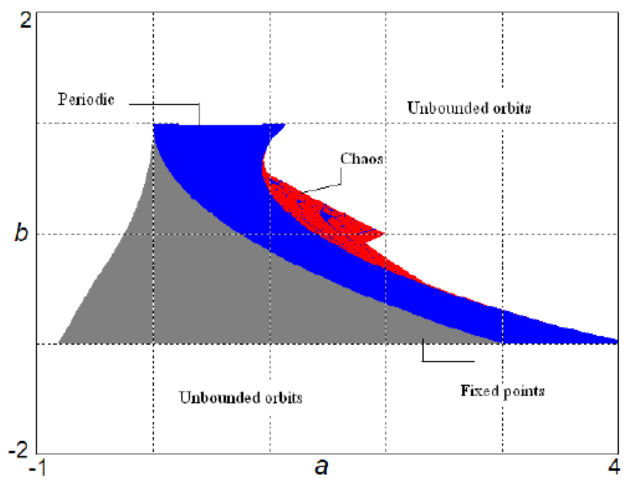


Figure 2.7: Hénon map's dynamical behaviors regions in the ab-plane.

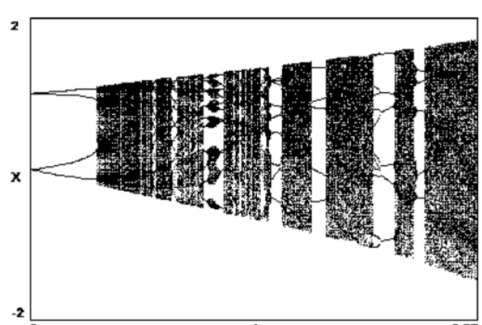


Figure 2.8: The bifurcation diagram for the map (2.1) obtained for $a = 1$and $0 < b \leq 0.67$.

where a is a parameter, the map (2.3) delineates diverse random evolutionary processes, presenting a complexity surpassing that of the logistic system. In [10], an extended version of the initial one-dimensional discrete chaotic system from [9] is introduced, broadening it into two dimensions as follows:

$$h(x,y) = \begin{pmatrix} \frac{1}{0.1+x^2} - ay \\ \frac{1}{0.1+y^2} - ax \end{pmatrix}, \tag{2.4}$$

where a and b are parameters. The map (2.4) has more complicated dynamical behavior than the one-dimensional map (2.3). A novel and very simple 2-D map is created by Zeraoulia and Sprott [3] based on concepts studied in [9,10], which is defined by the existence of only one rational fraction without a vanishing denominator:

$$f(x,y) = \begin{pmatrix} \frac{-ax}{1+y^2} \\ x + by \end{pmatrix}, \tag{2.5}$$

where a and b are bifurcation parameters.

2.2.1 Some properties

- The map (2.5) is algebraically simpler but with more complicated behavior than map (2.4).
- It produces several new chaotic attractors obtained via the quasi-periodic route to chaos.
- The map (2.5) is defined for all points in the plane.
- The associated function $f(x,y)$ of the map (2.5) is of class $C^\infty(\mathbb{R}^2)$, and it has no vanishing denominator.
- The chaotic map (2.5) is symmetric under the coordinate transformation $(x,y) \to (-x,-y)$, and this transformation persists for all values of the map parameters.

2.2.2 Fixed points

The fixed points of the map (2.5) are the real solution of

$$f(x,y) = (x,y) \Leftrightarrow \begin{pmatrix} \frac{-ax}{1+y^2} \\ x + by \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So, we may easily obtain the equations:

$$\begin{cases} (a + 1 + y^2)\, x = 0 \\ (1-b)\, y = x \end{cases}$$

Let $-1 \leq a \leq 4$. Then, if $b \neq 1$, the map (2.5) has only fixed point $P = (0,0)$. If $b = 1$, then the y-axis is invariant by iteration of the map (2.5). The Jacobian matrix of map (2.5) evaluated at a point (x, y) is given by:

$$Df(x,y) = \begin{pmatrix} \frac{-a}{1+y^2} & \frac{2axy}{(1+y^2)^2} \\ 1 & b \end{pmatrix},$$

and at the fixed point $p = (0,0)$, the Jacobian matrix is given by

$$Df(x,y) = \begin{pmatrix} -a & 0 \\ 1 & b \end{pmatrix},$$

The local stability of P is studied by evaluating the eigenvalues of the Jacobian $Df(P)$. The eigenvalues of $Df(P)$ are: $\lambda_1 = -a$ and $\lambda_2 = b$. Then one has the following results:
If $|a| < 1$ and $|b| < 1$, then P is asymptotically stable. If $|a| > 1$ or $|b| > 1$, then P is an unstable fixed point. If $|a| < 1$ and $|b| > 1$, or $|a| > 1$ and $|b| < 1$, then P is saddle point. If $|a| = 1$ or $|b| = 1$, then P is a non-hyperbolic fixed point.

2.2.3 Periodic orbits

The period-2 orbit of the map (2.5) is given by:

$$f(f(x,y)) - (x,y) = 0 \Leftrightarrow \begin{pmatrix} f_1(x,y) - x \\ f_2(x,y) - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{cases} f_1(x,y) = \frac{xa^2y^2 + xa^2}{a^2x^2 + y^4 + 2y^2 + 1} \\ \\ f_2(x,y) = x + b^2y + bx + by \end{cases}$$

and the period-3 orbit given by:

$$f(f(f(x,y))) - (x,y) = 0 \Leftrightarrow \begin{pmatrix} f_3(x,y) - x \\ f_4(x,y) - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{cases} f_3(x,y) = \frac{-a\left(a^2x\frac{y^2+1}{a^2x^2+y^4+2y^2+1}\right)}{1+\left(a^2x\frac{y^2+1}{a^2x^2+y^4+2y^2+1}\right)^2} \\ \\ f_4(x,y) = x + b^2x + 2b^2y + b^3y + 2bx + by \end{cases}$$

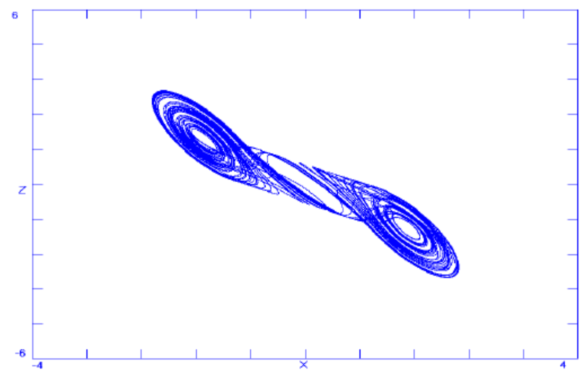


Figure 2.9: The classic double scroll attractor obtained for $\alpha = 9.35$, $\beta = 14.79$, $m_0 = -\frac{1}{7}$, $m_1 = \frac{2}{7}$.

2.3 The discrete hyperchaotic double scroll map

This section introduces a 2-D discrete piecewise linear chaotic map and we focusing on three key ideas. Firstly, the map's definition and some of its characteristics. Secondly, the fixed points and the Jacobian matrix. Thirdly, periodic orbits. This map is called *the discrete hyperchaotic double-scroll*, capable of generating a hyperchaotic solutions similar to the calssical double-scroll attractor generated by the Chua circuit [11] given by:

$$\begin{aligned} x' &= \alpha\left(y - h\left(x\right)\right) \\ y' &= x - y + z \\ z' &= -\beta y \end{aligned}$$

where

$$h\left(x\right) = \frac{2m_1 x + \left(m_0 - m_1\right)\left(|x+1| - |x-1|\right)}{2}.$$

The double scroll attractor for this case is shown in Figure 2.9.
Consider the following 2-D piecewise linear map:

$$f\left(x,y\right) = \begin{pmatrix} x - ah\left(y\right) \\ bx \end{pmatrix} \tag{2.6}$$

where a and b are the bifurcation parameters, h is given above and m_0 and m_1 are respectively the slopes of the original Chua circuit's inner and outer sets. So, the discrete hyper-chaotic double

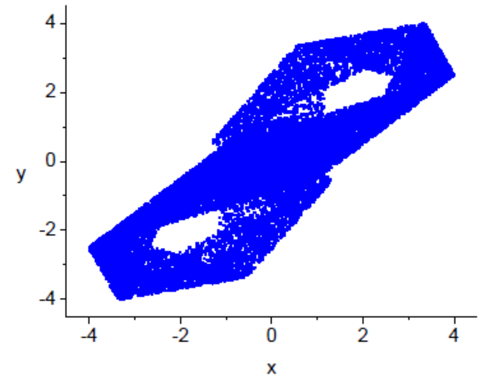


Figure 2.10: The discrete hyperchaotic double scroll attractor obtained from the map (2.6) for $a = 3.36$, $b = 1.4$, $m_0 = -0.43$, and $m1 = 0.41$ with initial conditions $x = y = 0.1$.

scroll map can be given by:

$$f(x,y) = \begin{cases} A_1 \times \begin{pmatrix} x \\ y \end{pmatrix} + b_1, & y \geq 1 \\ A_2 \times \begin{pmatrix} x \\ y \end{pmatrix} + b_2, & |y| \leq 1 \\ A_3 \times \begin{pmatrix} x \\ y \end{pmatrix} + b_3, & y \leq -1 \end{cases}$$

where

$$\begin{cases} A_1 = A_3 = \begin{pmatrix} 1 & -am_1 \\ b & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -am_0 \\ b & 0 \end{pmatrix}, \\ b_1 = \begin{pmatrix} a(m_1 - m_0) \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} a(m_0 - m_1) \\ 0 \end{pmatrix}. \end{cases}$$

Due to the shape of the vector field f of the map (2.6), the plane can be devided into three linear regions denoted by:

$$\begin{aligned} R_1 &= \left\{(x,y) \in \mathbb{R}^2 / \ y \geq 1\right\}, \\ R_2 &= \left\{(x,y) \in \mathbb{R}^2 / \ |y| \leq 1\right\}, \\ R_3 &= \left\{(x,y) \in \mathbb{R}^2 / \ y \leq -1\right\}. \end{aligned}$$

where in each of these regions the map (2.6) is linear

2.3.1 Some properties

- The associated function $f(x,y)$ is continuous in $\mathbb{R}^2$, but it is not differentiable at the points $(x,-1)$ and $(x,1)$ for all $x \in \mathbb{R}$.
- The map (2.6) is a diffeomorphism except at points $(x,-1)$ and $(x,1)$ when $abm_1m_0 \neq 0$, since the determinant of its Jacobian is non zero if and only if $abm_0 \neq 0$ or $abm_1 \neq 0$, but it does not preserve area and it is not a reversing twist map for all values of the system parameters.
- The map (2.6) is symmmetric under the coordinate transformion $(x,y) \to (-x,-y)$, and this transformation persists for all values of the system parameters.

2.3.2 Fixed points

The fixed points of the map (2.6) are the real solutions of the system:

$$f(x,y) = (x,y) \Leftrightarrow \begin{pmatrix} x - ah(y) \\ bx \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We get the following equation:

$$\begin{cases} \begin{cases} f_1(x,y) = 0, & \text{if } y \geq 1 \\ f_2(x,y) = 0, & \text{if } |y| \leq 1 \\ f_3(x,y) = 0, & \text{if } y \leq -1 \end{cases} \\ bx = y \end{cases}$$

where

$$\begin{cases} f_1(x,y) = am_1x + a(m_0 - m_1) \\ f_2(x,y) = am_0x \\ f_3(x,y) = am_1x - a(m_0 - m_1) \end{cases}$$

So, we have three cases from the existance of the fixed points:

Case 1: For $y \geq 1$

$$x_1 = \frac{m_1 - m_0}{bm_1} \Rightarrow y_1 = \frac{m_1 - m_0}{m_1}, abm_1 \neq 0$$

such that

$$y_1 = \frac{m_1 - m_0}{m_1} \geq 1 \Leftrightarrow m_1 - m_0 \geq m_1 \Leftrightarrow m_0 < 0 \text{ and } m_1 > 0$$

So, $P_1 = \left(\frac{m_1-m_0}{bm_1}, \frac{m_1-m_0}{m_1}\right)$ exist in R_1 if $m_0m_1 < 0$.

Case 2: For $|y| \leq 1$

$$x_2 = 0 \Rightarrow y_2 = 0, abm_0 \neq 0$$

So, $P_2 = (0,0)$ exist in R_2 if $m_0 \neq 0$.

Case 3: For $y \leq -1$

$$x_3 = \frac{m_0 - m_1}{bm_1} \Rightarrow y_1 = \frac{m_0 - m_1}{m_1}, abm_1 \neq 0$$

such that

$$y_1 = \frac{m_1 - m_0}{m_1} \leq -1 \Leftrightarrow m_1 - m_0 \leq -m_1 \Leftrightarrow m_0 < 0 \text{ and } m_1 > 0$$

So, $P_3 = \left(\frac{m_0-m_1}{bm_1}, \frac{m_0-m_1}{bm_1}\right)$ exist in R_3 if $m_0 m_1 \neq 0$. Then, can be written the fixed points as follows:

$$\begin{cases} \begin{cases} P_1 = \left(\frac{m_1-m_0}{bm_1}, \frac{m_1-m_0}{m_1}\right), \\ P_2 = (0,0) & \text{if } m_0 m_1 < 0 \\ P_3 = \left(\frac{m_0-m_1}{bm_1}, \frac{m_0-m_1}{bm_1}\right). \end{cases} \\ P = (0,0), & \text{if } m_0 m_1 > 0 \end{cases}$$

Jacobian matrix: The Jacobian matrix of the map (2.6) evaluated at the fixed points P_1, P_2 and P_3 given by:

$$J_{1,3} = \begin{pmatrix} 1 & -abm_1 \\ 1 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 1 & -abm_0 \\ 1 & 0 \end{pmatrix}$$

We note here that P_1 and P_3 is the same, therefore the two equilibrium points P_1 and P_3 have the same stability type. The eigenvalues of the corresponding Jacobian matrices (2.6) is given by the solutions of their characteristic polynomials given respectively by:

$$\begin{cases} \lambda^2 - \lambda + abm_1 = 0 \\ \lambda^2 - \lambda + abm_0 = 0 \end{cases}$$

To study the stability of the fixed points, we perform three main steps:

1. We assess the Jacobian matrix at the fixed point.
2. We calculate the eigenvalues from the solution of the characteristic polynomial.
3. We compare the resulting eigenvalues with the unit disk (if $|\lambda| < 1$ the fixed point is stable, if $|\lambda| > 1$ the fixed point is unstable).

2.3.3 Periodic orbits

The period-2 orbit of the map (2.6) is given by:

$$f\left(f\left(x,y\right)\right)-\left(x,y\right)=0\Leftrightarrow\left(\begin{array}{c}\left\{\begin{array}{lll}f_4\left(x,y\right) & \text{if} & y\geq 1\\ f_5\left(x,y\right) & \text{if} & \left|y\right|\leq 1\\ f_6\left(x,y\right) & \text{if} & y\leq -1\end{array}\right.\\ bx=y\end{array}\right)=\left(\begin{array}{c}0\\0\end{array}\right)$$

where

$$\left\{\begin{array}{c}f_4\left(x,y\right)=2am_1-2am_0-a^2m_1^2+a^2m_0m_1+a^2ym_1^2-axm_1-aym_1\\ f_5\left(x,y\right)=a^2ym_0^2-axm_0-aym_0\\ f_6\left(x,y\right)=a^2m_1^2+2am_0-2am_1-a^2m_0m_1+a^2ym_1^2-axm_1-aym_1\end{array}\right.$$

and the period-3 given by:

$$f\left(f\left(f\left(x\right)\right)\right)-\left(x,y\right)=0\Leftrightarrow\left(\begin{array}{c}\left\{\begin{array}{lll}f_7\left(x,y\right), & \text{if} & y\geq 1\\ f_8\left(x,y\right), & \text{if} & \left|y\right|\leq 1\\ f_9\left(x,y\right), & \text{if} & y\leq -1\end{array}\right.\\ bx-y\end{array}\right)=\left(\begin{array}{c}0\\0\end{array}\right)\qquad(2.7)$$

where

$$\left\{\begin{array}{c}f_7\left(x,y\right)=-3m_0+3m_1+a^2m_1^3-3am_1^2+a^2xm_1^2+2a^2ym_1^2-a^3ym_1^3-2axm_1-aym_1\\ -a^2m_0m_1^2+3am_0m_1\\ \\ f_8\left(x,y\right)=a^2xm_0^2+2a^2ym_0^2-a^3ym_0^3-2axm_0-aym_0\\ \\ f_9\left(x,y\right)=3a^2m_1^2-a^3m_1^3+3am_0-3am_1-3a^2m_0m_1+a^2xm_1^2+2a^2ym_1^2-a^3ym_1^3\\ -2axm_1-aym_1+a^3m_0m_1^2\end{array}\right.$$

Big dots that indicate the periodic orbits for the map (2.6) and the Lyapunov exponent spectrum for $m_0=-0.43$ and $m_1=0.41$, $b=1.4$, and $-3.365\leq a\leq 3.365$ as shown in Figure 2.11.
If we fix parameters $b=1.4$, $m_0=-0.43$, and $m_1=0.41$ and vary $a\in\mathbb{R}$, the map (2.6) exhibits the following dynamical behaviors as shown in Fig 2.12. For $a=-1.8$ and $a=1.8$, the map (2.6) has a stable period-3 orbit with equation (2.7). In the interval $-1.8\leq a\leq 0.1$, and $0\leq a\leq 1.8$,

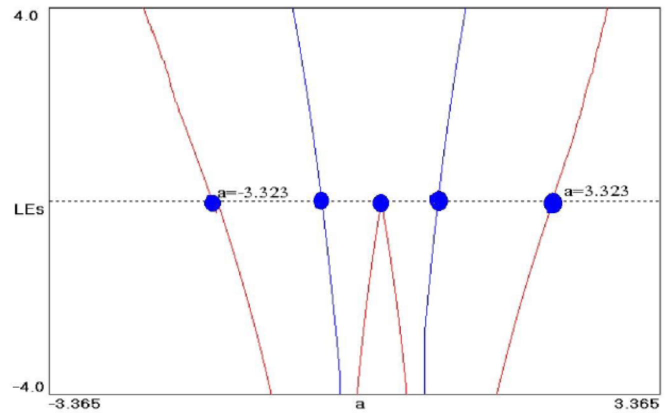


Figure 2.11: The Lyaponov exponents of the map (2.6) versus the parameter $-3.365 \leq a \leq 3.365$ with $b = 1.4$, $m_0 = -0.43$,and $m_1 = 0.41$.

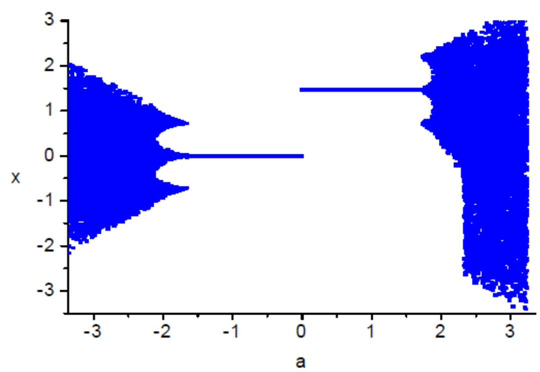


Figure 2.12: The border collision bifurcation route to chaos of the map (2.6) for $b = 1.4$ with $-3.365 \leq a \leq 3.365$, $m_0 = -0.43$,and $m_1 = 0.41$.

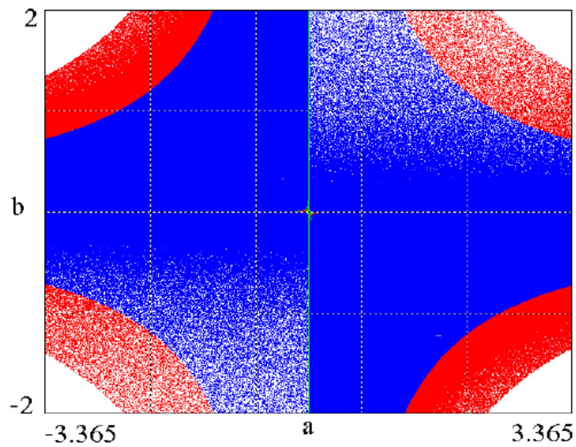


Figure 2.13: Regions of dynamical behaviors in the ab-plane for the map (2.6).

the map (2.6) has a period-1 orbit and its equation is of the form:

$$
\begin{aligned}
f(x,y)-(x,y) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Leftrightarrow &\left\{ \begin{array}{ll}
\begin{pmatrix} -am_1 y + a(m_0 - m_1) \\ bx - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{if } y \leq -1 \\
\begin{pmatrix} x - am_0 y \\ bx - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{if } |y| \leq 1 \\
\begin{pmatrix} -am_1 y + a(m_1 - m_0) \\ bx - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{if } y \geq 1
\end{array} \right.
\end{aligned}
$$

Figure 2.13 shows regions in the ab-plane given by $(a,b) \in [-3.365, 3.365] \times [-2, 2]$ of unbounded (white), periodic orbits of periods 1 and 3 (blue) in the ab-plane for the map (2.6), with 10^6 iterations for each point.

Chapter 3

Periodic orbits of some 1-D Zeraoulia-Sprott mpaping

This chapter presents the mathematical analysis of some one-dimensional Zeraoulia-Sprott mpaping, with special emphasis on the analysis of the stability of fixed points and periodic orbits.

3.1 One-dimensional discrete mapping

Consider the arbitrary 1-D discrete mappings given by:

$$x_{k+1} = g_{\upsilon}(x_k), \quad \upsilon \in (\upsilon_{\min}, \upsilon_{\max}). \tag{3.1}$$

where $g_{\upsilon} : [a, b] \to \mathbb{R}$ with $a < b$ be two real numbers, and g_{υ} is of class C^3. Let us consider the controlled 1-D mapping given by:

$$\begin{aligned} x_{k+1} &= g_{\upsilon}(x_k) + u(x_k) = \varphi_{\upsilon}(x_k), \\ \upsilon &\in (\upsilon_{\min}, \upsilon_{\max}). \end{aligned} \tag{3.2}$$

where $u : [a, b] \to [a, b]$ is the unknown controller to be chosen. Define the controller $u : [a, b] \to [a, b]$ by the following conditios:

(A1) The controller $u(x)$ is of class C^3.

(A2) The controller $u(x)$ has the following special values:

$$\begin{cases} u(a) = a - g_{\upsilon}(a) \\ u(b) = a - g_{\upsilon}(b) \\ \text{there exist a piont } c \in (a, b) : u'(c) = -g'_{\upsilon}(c) \end{cases}$$

(A3)

$$a - g_{\upsilon}(x) \leq u(x) \leq b - g_{\upsilon}(x)$$

(A4)

$$u'(x) > -g'_{\upsilon}(x), \quad \text{for all } x \in [a, c).$$

(A5)

$$u'(x) < -g'_{\upsilon}(x), \quad \text{for all } x \in (c, b].$$

(A6)

$$2\left(g'_{\nu}(x) + u'(x)\right)\left(g'''_{\nu}(x) + u'''(x)\right) - 3\left(g''_{\nu}(x) + u''(x)\right)^2 < 0 \text{ for all } x \in [a, b]$$

More generally, take $g_{\upsilon}(x) = \upsilon x$, with $x \in [0, 1]$. Define the controller

$$u(x) = -(\upsilon + \beta + \gamma)x^3 + \beta x^2 + \gamma x,$$

where

$$\begin{cases} 0 \leq \upsilon < \frac{1}{2}, \\ 0 \leq \upsilon < \gamma < 1 - \upsilon, \\ \upsilon + \beta + \gamma < 1, \\ \beta < \min\left\{1 - (\gamma + \upsilon), \frac{1}{6}\sqrt{13\upsilon^2 + 30\upsilon\gamma + 21\gamma^2} - \frac{1}{2}\gamma - \frac{5}{6}\upsilon\right\}. \end{cases}$$

Hence

$$\varphi_{\upsilon}(x) = (-\nu - \beta - \gamma)x^3 + \beta x^2 + (\nu + \gamma)x. \tag{3.3}$$

The conditions (A1)-(A6) are satisfied.

3.1.1 Fixed pionts

The fixed points of map (3.3) are:

$$\begin{aligned} \varphi_{\upsilon}(x) &= x \Leftrightarrow (-\nu - \beta - \gamma)x^3 + \beta x^2 + (\nu + \gamma)x = x \\ &\Rightarrow (-\nu - \beta - \gamma)x^3 + \beta x^2 + (\nu + \gamma - 1)x = 0 \\ &\Rightarrow \left((-\nu - \beta - \gamma)x^2 + \beta x + (\nu + \gamma - 1)\right)x = 0 \end{aligned}$$

$$\begin{cases} x_1 = 0 \\ \\ (-\nu - \beta - \gamma)\, x^2 + \beta x + (\nu + \gamma - 1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ \\ x_{2,3} = \pm\frac{1}{2}\frac{\beta+\sqrt{-4\beta-4\gamma-4\nu+\beta^2+4\gamma^2+4\nu^2+4\beta\gamma+4\beta\nu+8\gamma\nu}}{\beta+\gamma+\nu}, \beta+\gamma+\nu \neq 0 \end{cases}$$

So, there are three fixed points. Now, we study their stability, we have

$$\varphi_v'(x) = 3(-\nu - \beta - \gamma)\, x^2 + 2\beta x + (\nu + \gamma)$$

At x_1 we have:

$$\varphi_v'(x_1) = \varphi_v'(0) = \nu + \gamma = m_1$$

Thus, we get

- If $|m_1| < 1$, x_1 stable.
- If $|m_1| > 1$, x_1 unstable.
- If $|m_1| = 1$, we cannot conclude their stability.

At x_2 we have:

$$\varphi_v'(x_2) = \varphi_v'\left(\frac{1}{2}\frac{\beta+\sqrt{-4\beta-4\gamma-4\nu+\beta^2+4\gamma^2+4\nu^2+4\beta\gamma+4\beta\nu+8\gamma\nu}}{\beta+\gamma+\nu}\right) = m_2$$

Thus, we get

- If $|m_2| < 1$, x_2 stable
- If $|m_2| > 1$, x_2 unstable
- If $|m_2| = 1$,we cannot conclude their stability.

At x_3 we have:

$$\varphi_v'(x3) = \varphi_v'\left(-\frac{1}{2}\frac{\beta+\sqrt{-4\beta-4\gamma-4\nu+\beta^2+4\gamma^2+4\nu^2+4\beta\gamma+4\beta\nu+8\gamma\nu}}{\beta+\gamma+\nu}\right) = m_3$$

Thus, we get

- If $|m_3| < 1$, x_3 stable.
- If $|m_3| > 1$, x_3 unstable.
- If $|m_3| = 1$,we cannot conclude their stability.

3.1.2 Periodic orbits

The period-2 orbit of the map (3.3) is given by $\varphi_v(\varphi_v(x)) - x = 0$, i.e.,

$$a_0x^9 + a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + a_5x^4 + a_6x^3 + a_7x^2 + (a_8 - 1)x = 0$$

where

$$\begin{cases} a_0 = \beta^4 + 4\beta^3\gamma + 4\beta^3\nu + 6\beta^2\gamma^2 + 12\beta^2\gamma\nu + 6\beta^2\nu^2 + 4\beta\gamma^3 + 12\beta\gamma^2\nu \\ \quad +12\beta\gamma\nu^2 + 4\beta\nu^3 + \gamma^4 + 4\gamma^3\nu + 6\gamma^2\nu^2 + 4\gamma\nu^3 + \nu^4 \\ \\ a_1 = -3\beta^4 - 9\beta^3\gamma - 9\beta^3\nu - 9\beta^2\gamma^2 - 18\beta^2\gamma\nu - 9\beta^2\nu^2 - 3\beta\gamma^3 - 9\beta\gamma^2\nu \\ \quad -9\beta\gamma\nu^2 - 3\beta\nu^3 \\ \\ a_2 = 3\beta^4 + 3\beta^3\gamma + 3\beta^3\nu - 6\beta^2\gamma^2 - 12\beta^2\gamma\nu - 6\beta^2\nu^2 - 9\beta\gamma^3 - 27\beta\gamma^2\nu \\ \quad -27\beta\gamma\nu^2 - 9\beta\nu^3 - 3\gamma^4 - 12\gamma^3\nu - 18\gamma^2\nu^2 - 12\gamma\nu^3 - 3\nu^4 \end{cases}$$

$$\begin{cases} a_3 = 5\beta^3\gamma - \beta^4 + 5\beta^3\nu + \beta^3 + 12\beta^2\gamma^2 + 24\beta^2\gamma\nu + 2\beta^2\gamma \\ \quad +12\beta^2\nu^2 + 2\beta^2\nu + 6\beta\gamma^3 + 18\beta\gamma^2\nu + \beta\gamma^2 \\ \quad +18\beta\gamma\nu^2 + 2\beta\gamma\nu + 6\beta\nu^3 + \beta\nu^2 \\ \\ a_4 = 6\beta\gamma^3 - 3\beta^3\nu - 2\beta^3 - 2\beta^2\gamma - 2\beta^2\nu - 3\beta^3\gamma + 18\beta\gamma^2\nu \\ \quad +18\beta\gamma\nu^2 + 6\beta\nu^3 + 3\gamma^4 + 12\gamma^3\nu + 18\gamma^2\nu^2 \\ \quad +12\gamma\nu^3 + 3\nu^4 \\ \\ a_5 = \beta^3 - 3\beta^2\gamma^2 - 6\beta^2\gamma\nu - 2\beta^2\gamma - 3\beta^2\nu^2 - 2\beta^2\nu - 3\beta\gamma^3 \\ \quad -9\beta\gamma^2\nu - 2\beta\gamma^2 - 9\beta\gamma\nu^2 - 4\beta\gamma\nu - 3\beta\nu^3 - 2\beta\nu^2 \end{cases}$$

$$\begin{cases} a_6 = 2\beta^2\gamma + 2\beta^2\nu - \beta\gamma^3 - 3\beta\gamma^2\nu - 3\beta\gamma\nu^2 - \beta\gamma - \beta\nu^3 - \beta\nu - \gamma^4 \\ \quad -4\gamma^3\nu - 6\gamma^2\nu^2 - \gamma^2 - 4\gamma\nu^3 - 2\gamma\nu - \nu^4 - \nu^2 \\ \\ a_7 = \beta\gamma^2 + 2\beta\gamma\nu + \beta\gamma + \beta\nu^2 + \beta\nu \\ \\ a_8 = \gamma^2 + 2\gamma\nu + \nu^2 \end{cases}$$

and the period-3 orbit given by $\varphi_v(\varphi_v(\varphi_v(x))) - x = 0$, i.e.,

$$\begin{aligned} & b_0x^{27} + b_1x^{26} + b_2x^{25} + b_3x^{24} + b_4x^{23} + b_5x^{22} + b_6x^{21} + b_7x^{20} + b_8x^{19} \\ & + b_9x^{18} + b_{10}x^{17} + b_{11}x^{16} + b_{12}x^{15} + b_{13}x^{14} + b_{14}x^{13} + b_{15}x^{12} + b_{16}x^{11} \\ & + b_{17}x^{10} + b_{18}x^9 + b_{19}x^8 + b_{20}x^7 + b_{21}x^6 + b_{22}x^5 + b_{23}x^4 + b_{24}x^3 \\ & + b_{25}x^2 + (b_{26} - 1)x = 0 \end{aligned}$$

where

$$\begin{cases} b_0 = -\beta a_0^3 - \gamma a_0^3 - \nu a_0^3 \\ \\ b_1 = -3\beta a_0^2 a_1 - 3\gamma a_0^2 a_1 - 3\nu a_0^2 a_1 \\ \\ b_2 = -3\beta a_0 a_1^2 - 3\beta a_0^2 a_2 - 3\gamma a_0 a_1^2 - 3\gamma a_0^2 a_2 - 3\nu a_0 a_1^2 - 3\nu a_0^2 a_2 \\ \\ b_3 = -\beta a_1^3 - \gamma a_1^3 - \nu a_1^3 - 3\beta a_0^2 a_3 - 3\gamma a_0^2 a_3 - 3\nu a_0^2 a_3 - 6\beta a_0 a_1 a_2 - 6\gamma a_0 a_1 a_2 - 6\nu a_0 a_1 a_2 \end{cases}$$

$$\begin{cases} b_4 = -3\beta a_0 a_2^2 - 3\beta a_1^2 a_2 - 3\gamma a_0 a_2^2 - 3\beta a_0^2 a_4 - 3\gamma a_1^2 a_2 - 3\nu a_0 a_2^2 - 3\gamma a_0^2 a_4 \\ \quad -3\nu a_1^2 a_2 - 3\nu a_0^2 a_4 - 6\beta a_0 a_1 a_3 - 6\gamma a_0 a_1 a_3 - 6\nu a_0 a_1 a_3 \\ \\ b_5 = -3\beta a_1 a_2^2 - 3\beta a_1^2 a_3 - 3\gamma a_1 a_2^2 - 3\beta a_0^2 a_5 - 3\gamma a_1^2 a_3 - 3\nu a_1 a_2^2 - 3\gamma a_0^2 a_5 \\ \quad -3\nu a_1^2 a_3 - 3\nu a_0^2 a_5 - 6\beta a_0 a_1 a_4 - 6\beta a_0 a_2 a_3 - 6\gamma a_0 a_1 a_4 - 6\gamma a_0 a_2 a_3 \\ \quad -6\nu a_0 a_1 a_4 - 6\nu a_0 a_2 a_3 \\ \\ b_6 = -\beta a_2^3 - \gamma a_2^3 - \nu a_2^3 - 3\beta a_0 a_3^2 - 3\gamma a_0 a_3^2 - 3\beta a_1^2 a_4 - 3\nu a_0 a_3^2 - 3\beta a_0^2 a_6 \\ \quad -3\gamma a_1^2 a_4 - 3\gamma a_0^2 a_6 - 3\nu a_1^2 a_4 - 3\nu a_0^2 a_6 - 6\beta a_0 a_1 a_5 - 6\beta a_0 a_2 a_4 \\ \quad -6\beta a_1 a_2 a_3 - 6\gamma a_0 a_1 a_5 - 6\gamma a_0 a_2 a_4 - 6\gamma a_1 a_2 a_3 - 6\nu a_0 a_1 a_5 \\ \quad -6\nu a_0 a_2 a_4 - 6\nu a_1 a_2 a_3 \end{cases}$$

$$
\begin{cases}
b_7 = -3\beta a_1 a_3^2 - 3\beta a_2^2 a_3 - 3\gamma a_1 a_3^2 - 3\beta a_1^2 a_5 - 3\gamma a_2^2 a_3 - 3\nu a_1 a_3^2 - 3\beta a_0^2 a_7 \\
\quad -3\gamma a_1^2 a_5 - 3\nu a_2^2 a_3 - 3\gamma a_0^2 a_7 - 3\nu a_1^2 a_5 - 3\nu a_0^2 a_7 - 6\beta a_0 a_1 a_6 \\
\quad -6\beta a_0 a_2 a_5 - 6\beta a_0 a_3 a_4 - 6\beta a_1 a_2 a_4 - 6\gamma a_0 a_1 a_6 - 6\gamma a_0 a_2 a_5 \\
\quad -6\gamma a_0 a_3 a_4 - 6\gamma a_1 a_2 a_4 - 6\nu a_0 a_1 a_6 - 6\nu a_0 a_2 a_5 \\
\quad -6\nu a_0 a_3 a_4 - 6\nu a_1 a_2 a_4 \\
\\
b_8 = -3\beta a_0 a_4^2 - 3\beta a_2 a_3^2 - 3\gamma a_0 a_4^2 - 3\beta a_2^2 a_4 - 3\gamma a_2 a_3^2 - 3\nu a_0 a_4^2 - 3\beta a_1^2 a_6 \\
\quad -3\gamma a_2^2 a_4 - 3\nu a_2 a_3^2 - 3\beta a_0^2 a_8 - 3\gamma a_1^2 a_6 - 3\nu a_2^2 a_4 - 3\gamma a_0^2 a_8 \\
\quad -3\nu a_1^2 a_6 - 3\nu a_0^2 a_8 - 6\beta a_0 a_1 a_7 - 6\beta a_0 a_2 a_6 - 6\beta a_0 a_3 a_5 \\
\quad -6\beta a_1 a_2 a_5 - 6\beta a_1 a_3 a_4 - 6\gamma a_0 a_1 a_7 - 6\gamma a_0 a_2 a_6 - 6\gamma a_0 a_3 a_5 \\
\quad -6\gamma a_1 a_2 a_5 - 6\gamma a_1 a_3 a_4 - 6\nu a_0 a_1 a_7 - 6\nu a_0 a_2 a_6 - 6\nu a_0 a_3 a_5 \\
\quad -6\nu a_1 a_2 a_5 - 6\nu a_1 a_3 a_4 \\
\\
b_9 = \beta a_0^2 - \beta a_3^3 - \gamma a_3^3 - \nu a_3^3 - 3\beta a_1 a_4^2 - 3\gamma a_1 a_4^2 - 3\beta a_2^2 a_5 - 3\nu a_1 a_4^2 \\
\quad -3\beta a_1^2 a_7 - 3\gamma a_2^2 a_5 - 3\gamma a_1^2 a_7 - 3\nu a_2^2 a_5 - 3\nu a_1^2 a_7 - 6\beta a_0 a_1 a_8 \\
\quad -6\beta a_0 a_2 a_7 - 6\beta a_0 a_3 a_6 - 6\beta a_0 a_4 a_5 - 6\beta a_1 a_2 a_6 - 6\beta a_1 a_3 a_5 \\
\quad -6\beta a_2 a_3 a_4 - 6\gamma a_0 a_1 a_8 - 6\gamma a_0 a_2 a_7 - 6\gamma a_0 a_3 a_6 - 6\gamma a_0 a_4 a_5 \\
\quad -6\gamma a_1 a_2 a_6 - 6\gamma a_1 a_3 a_5 - 6\gamma a_2 a_3 a_4 - 6\nu a_0 a_1 a_8 - 6\nu a_0 a_2 a_7 \\
\quad -6\nu a_0 a_3 a_6 - 6\nu a_0 a_4 a_5 - 6\nu a_1 a_2 a_6 - 6\nu a_1 a_3 a_5 - 6\nu a_2 a_3 a_4
\end{cases}
$$

$$
\begin{cases}
b_{10} = 2\beta a_0a_1 - 3\beta a_2a_4^2 - 3\gamma a_0a_5^2 - 3\beta a_3^2a_4 - 3\gamma a_2a_4^2 - 3\nu a_0a_5^2 - 3\beta a_2^2a_6 - 3\gamma a_3^2a_4 \\
\quad -3\nu a_2a_4^2 - 3\beta a_1^2a_8 - 3\gamma a_2^2a_6 - 3\nu a_3^2a_4 - 3\gamma a_1^2a_8 - 3\nu a_2^2a_6 - 3\nu a_1^2a_8 \\
\quad -3\beta a_0a_5^2 - 6\beta a_0a_2a_8 - 6\beta a_0a_3a_7 - 6\beta a_0a_4a_6 - 6\beta a_1a_2a_7 - 6\beta a_1a_3a_6 \\
\quad -6\beta a_1a_4a_5 - 6\beta a_2a_3a_5 - 6\gamma a_0a_2a_8 - 6\gamma a_0a_3a_7 - 6\gamma a_0a_4a_6 - 6\gamma a_1a_2a_7 \\
\quad -6\gamma a_1a_3a_6 - 6\gamma a_1a_4a_5 - 6\gamma a_2a_3a_5 - 6\nu a_0a_2a_8 - 6\nu a_0a_3a_7 - 6\nu a_0a_4a_6 \\
\quad -6\nu a_1a_2a_7 - 6\nu a_1a_3a_6 - 6\nu a_1a_4a_5 - 6\nu a_2a_3a_5 \\
\\
b_{11} = \beta a_1^2 - 3\beta a_1a_5^2 - 3\beta a_3a_4^2 - 3\gamma a_1a_5^2 - 3\beta a_3^2a_5 - 3\gamma a_3a_4^2 - 3\nu a_1a_5^2 - 3\beta a_2^2a_7 \\
\quad -3\gamma a_3^2a_5 - 3\nu a_3a_4^2 - 3\gamma a_2^2a_7 - 3\nu a_3^2a_5 - 3\nu a_2^2a_7 + 2\beta a_0a_2 - 6\beta a_0a_3a_8 \\
\quad -6\beta a_0a_4a_7 - 6\beta a_0a_5a_6 - 6\beta a_1a_2a_8 - 6\beta a_1a_3a_7 - 6\beta a_1a_4a_6 - 6\beta a_2a_3a_6 \\
\quad -6\beta a_2a_4a_5 - 6\gamma a_0a_3a_8 - 6\gamma a_0a_4a_7 - 6\gamma a_0a_5a_6 - 6\gamma a_1a_2a_8 - 6\gamma a_1a_3a_7 \\
\quad -6\gamma a_1a_4a_6 - 6\gamma a_2a_3a_6 - 6\gamma a_2a_4a_5 - 6\nu a_0a_3a_8 - 6\nu a_0a_4a_7 - 6\nu a_0a_5a_6 \\
\quad -6\nu a_1a_2a_8 - 6\nu a_1a_3a_7 - 6\nu a_1a_4a_6 - 6\nu a_2a_3a_6 - 6\nu a_2a_4a_5 \\
\\
b_{12} = 2\beta a_0a_3 - \gamma a_4^3 - \nu a_4^3 - 3\beta a_0a_6^2 - 3\beta a_2a_5^2 - 3\gamma a_0a_6^2 - 3\gamma a_2a_5^2 - 3\nu a_0a_6^2 \\
\quad -3\beta a_3^2a_6 - 3\nu a_2a_5^2 - 3\beta a_2^2a_8 - 3\gamma a_3^2a_6 - 3\gamma a_2^2a_8 - 3\nu a_3^2a_6 - 3\nu a_2^2a_8 \\
\quad -\beta a_4^3 + 2\beta a_1a_2 - 6\beta a_0a_4a_8 - 6\beta a_0a_5a_7 - 6\beta a_1a_3a_8 - 6\beta a_1a_4a_7 \\
\quad -6\beta a_1a_5a_6 - 6\beta a_2a_3a_7 - 6\beta a_2a_4a_6 - 6\beta a_3a_4a_5 - 6\gamma a_0a_4a_8 - 6\gamma a_0a_5a_7 \\
\quad -6\gamma a_1a_3a_8 - 6\gamma a_1a_4a_7 - 6\gamma a_1a_5a_6 - 6\gamma a_2a_3a_7 - 6\gamma a_2a_4a_6 - 6\gamma a_3a_4a_5 \\
\quad -6\nu a_0a_4a_8 - 6\nu a_0a_5a_7 - 6\nu a_1a_3a_8 - 6\nu a_1a_4a_7 - 6\nu a_1a_5a_6 - 6\nu a_2a_3a_7 \\
\quad -6\nu a_2a_4a_6 - 6\nu a_3a_4a_5
\end{cases}
$$

$$\begin{cases} b_{13} = \beta a_2^2 - 3\beta a_1 a_6^2 - 3\beta a_3 a_5^2 - 3\gamma a_1 a_6^2 - 3\beta a_4^2 a_5 - 3\gamma a_3 a_5^2 - 3\nu a_1 a_6^2 - 3\beta a_3^2 a_7 \\ \quad -3\gamma a_4^2 a_5 - 3\nu a_3 a_5^2 - 3\gamma a_3^2 a_7 - 3\nu a_4^2 a_5 - 3\nu a_3^2 a_7 + 2\beta a_0 a_4 + 2\beta a_1 a_3 \\ \quad -6\beta a_0 a_5 a_8 - 6\beta a_0 a_6 a_7 - 6\beta a_1 a_4 a_8 - 6\beta a_1 a_5 a_7 - 6\beta a_2 a_3 a_8 - 6\beta a_2 a_4 a_7 \\ \quad -6\beta a_2 a_5 a_6 - 6\beta a_3 a_4 a_6 - 6\gamma a_0 a_5 a_8 - 6\gamma a_0 a_6 a_7 - 6\gamma a_1 a_4 a_8 - 6\gamma a_1 a_5 a_7 \\ \quad -6\gamma a_2 a_3 a_8 - 6\gamma a_2 a_4 a_7 - 6\gamma a_2 a_5 a_6 - 6\gamma a_3 a_4 a_6 - 6\nu a_0 a_5 a_8 - 6\nu a_0 a_6 a_7 \\ \quad -6\nu a_1 a_4 a_8 - 6\nu a_1 a_5 a_7 - 6\nu a_2 a_3 a_8 - 6\nu a_2 a_4 a_7 - 6\nu a_2 a_5 a_6 - 6\nu a_3 a_4 a_6 \\ \\ b_{14} = 2\beta a_0 a_5 - 3\beta a_2 a_6^2 - 3\gamma a_0 a_7^2 - 3\beta a_4 a_5^2 - 3\gamma a_2 a_6^2 - 3\nu a_0 a_7^2 - 3\beta a_4^2 a_6 \\ \quad -3\gamma a_4 a_5^2 - 3\nu a_2 a_6^2 - 3\beta a_3^2 a_8 - 3\gamma a_4^2 a_6 - 3\nu a_4 a_5^2 - 3\gamma a_3^2 a_8 - 3\nu a_4^2 a_6 \\ \quad -3\nu a_3^2 a_8 - 3\beta a_0 a_7^2 + 2\beta a_1 a_4 + 2\beta a_2 a_3 - 6\beta a_0 a_6 a_8 - 6\beta a_1 a_5 a_8 \\ \quad -6\beta a_1 a_6 a_7 - 6\beta a_2 a_4 a_8 - 6\beta a_2 a_5 a_7 - 6\beta a_3 a_4 a_7 - 6\beta a_3 a_5 a_6 \\ \quad -6\gamma a_0 a_6 a_8 - 6\gamma a_1 a_5 a_8 - 6\gamma a_1 a_6 a_7 - 6\gamma a_2 a_4 a_8 - 6\gamma a_2 a_5 a_7 \\ \quad -6\gamma a_3 a_4 a_7 - 6\gamma a_3 a_5 a_6 - 6\nu a_0 a_6 a_8 - 6\nu a_1 a_5 a_8 - 6\nu a_1 a_6 a_7 \\ \quad -6\nu a_2 a_4 a_8 - 6\nu a_2 a_5 a_7 - 6\nu a_3 a_4 a_7 - 6\nu a_3 a_5 a_6 \\ \\ b_{15} = \beta a_3^2 - \beta a_5^3 - \gamma a_5^3 - \nu a_5^3 - 3\beta a_1 a_7^2 - 3\beta a_3 a_6^2 - 3\gamma a_1 a_7^2 - 3\gamma a_3 a_6^2 - 3\nu a_1 a_7^2 \\ \quad -3\beta a_4^2 a_7 - 3\nu a_3 a_6^2 - 3\gamma a_4^2 a_7 - 3\nu a_4^2 a_7 + 2\beta a_0 a_6 + 2\beta a_1 a_5 + 2\beta a_2 a_4 \\ \quad -6\beta a_0 a_7 a_8 - 6\beta a_1 a_6 a_8 - 6\beta a_2 a_5 a_8 - 6\beta a_2 a_6 a_7 - 6\beta a_3 a_4 a_8 - 6\beta a_3 a_5 a_7 \\ \quad -6\beta a_4 a_5 a_6 - 6\gamma a_0 a_7 a_8 - 6\gamma a_1 a_6 a_8 - 6\gamma a_2 a_5 a_8 - 6\gamma a_2 a_6 a_7 - 6\gamma a_3 a_4 a_8 \\ \quad -6\gamma a_3 a_5 a_7 - 6\gamma a_4 a_5 a_6 - 6\nu a_0 a_7 a_8 - 6\nu a_1 a_6 a_8 - 6\nu a_2 a_5 a_8 - 6\nu a_2 a_6 a_7 \\ \quad -6\nu a_3 a_4 a_8 - 6\nu a_3 a_5 a_7 - 6\nu a_4 a_5 a_6 \end{cases}$$

$$\begin{cases} b_{16} = 2\beta a_0 a_7 - 3\beta a_2 a_7^2 - 3\gamma a_0 a_8^2 - 3\beta a_4 a_6^2 - 3\gamma a_2 a_7^2 - 3\nu a_0 a_8^2 \\ -3\beta a_5^2 a_6 - 3\gamma a_4 a_6^2 - 3\nu a_2 a_7^2 - 3\beta a_4^2 a_8 - 3\gamma a_5^2 a_6 - 3\nu a_4 a_6^2 \\ -3\gamma a_4^2 a_8 - 3\nu a_5^2 a_6 - 3\nu a_4^2 a_8 - 3\beta a_0 a_8^2 + 2\beta a_1 a_6 + 2\beta a_2 a_5 \\ +2\beta a_3 a_4 - 6\beta a_1 a_7 a_8 - 6\beta a_2 a_6 a_8 - 6\beta a_3 a_5 a_8 - 6\beta a_3 a_6 a_7 \\ -6\beta a_4 a_5 a_7 - 6\gamma a_1 a_7 a_8 - 6\gamma a_2 a_6 a_8 - 6\gamma a_3 a_5 a_8 - 6\gamma a_3 a_6 a_7 \\ -6\gamma a_4 a_5 a_7 - 6\nu a_1 a_7 a_8 - 6\nu a_2 a_6 a_8 - 6\nu a_3 a_5 a_8 - 6\nu a_3 a_6 a_7 \\ -6\nu a_4 a_5 a_7 \\ \\ b_{17} = \beta a_4^2 - 3\beta a_1 a_8^2 - 3\beta a_3 a_7^2 - 3\gamma a_1 a_8^2 - 3\beta a_5 a_6^2 - 3\gamma a_3 a_7^2 \\ -3\nu a_1 a_8^2 - 3\beta a_5^2 a_7 - 3\gamma a_5 a_6^2 - 3\nu a_3 a_7^2 - 3\gamma a_5^2 a_7 \\ -3\nu a_5 a_6^2 - 3\nu a_5^2 a_7 + 2\beta a_0 a_8 + 2\beta a_1 a_7 + 2\beta a_2 a_6 \\ +2\beta a_3 a_5 - 6\beta a_2 a_7 a_8 - 6\beta a_3 a_6 a_8 - 6\beta a_4 a_5 a_8 \\ -6\beta a_4 a_6 a_7 - 6\gamma a_2 a_7 a_8 - 6\gamma a_3 a_6 a_8 - 6\gamma a_4 a_5 a_8 \\ -6\gamma a_4 a_6 a_7 - 6\nu a_2 a_7 a_8 - 6\nu a_3 a_6 a_8 - 6\nu a_4 a_5 a_8 \\ -6\nu a_4 a_6 a_7 \\ \\ b_{18} = \gamma a_0 + \nu a_0 - \beta a_6^3 - \gamma a_6^3 - \nu a_6^3 - 3\beta a_2 a_8^2 - 3\beta a_4 a_7^2 - 3\gamma a_2 a_8^2 \\ -3\gamma a_4 a_7^2 - 3\nu a_2 a_8^2 - 3\beta a_5^2 a_8 - 3\nu a_4 a_7^2 - 3\gamma a_5^2 a_8 - 3\nu a_5^2 a_8 \\ +2\beta a_1 a_8 + 2\beta a_2 a_7 + 2\beta a_3 a_6 + 2\beta a_4 a_5 - 6\beta a_3 a_7 a_8 \\ -6\beta a_4 a_6 a_8 - 6\beta a_5 a_6 a_7 - 6\gamma a_3 a_7 a_8 - 6\gamma a_4 a_6 a_8 \\ -6\gamma a_5 a_6 a_7 - 6\nu a_3 a_7 a_8 - 6\nu a_4 a_6 a_8 - 6\nu a_5 a_6 a_7 \end{cases}$$

$$\begin{cases} b_{19} = \gamma a_1 + \nu a_1 + \beta a_5^2 - 3\beta a_3 a_8^2 - 3\beta a_5 a_7^2 - 3\gamma a_3 a_8^2 - 3\beta a_6^2 a_7 - 3\gamma a_5 a_7^2 \\ \quad -3\nu a_3 a_8^2 - 3\gamma a_6^2 a_7 - 3\nu a_5 a_7^2 - 3\nu a_6^2 a_7 + 2\beta a_2 a_8 + 2\beta a_3 a_7 \\ \quad +2\beta a_4 a_6 - 6\beta a_4 a_7 a_8 - 6\beta a_5 a_6 a_8 - 6\gamma a_4 a_7 a_8 - 6\gamma a_5 a_6 a_8 \\ \quad -6\nu a_4 a_7 a_8 - 6\nu a_5 a_6 a_8 \\ \\ b_{20} = \gamma a_2 + \nu a_2 - 3\beta a_4 a_8^2 - 3\beta a_6 a_7^2 - 3\gamma a_4 a_8^2 - 3\beta a_6^2 a_8 - 3\gamma a_6 a_7^2 - 3\nu a_4 a_8^2 \\ \quad -3\gamma a_6^2 a_8 - 3\nu a_6 a_7^2 - 3\nu a_6^2 a_8 + 2\beta a_3 a_8 + 2\beta a_4 a_7 + 2\beta a_5 a_6 \\ \quad -6\beta a_5 a_7 a_8 - 6\gamma a_5 a_7 a_8 - 6\nu a_5 a_7 a_8 \\ \\ b_{21} = \gamma a_3 + \nu a_3 + \beta a_6^2 - \beta a_7^3 - \gamma a_7^3 - \nu a_7^3 - 3\beta a_5 a_8^2 - 3\gamma a_5 a_8^2 - 3\nu a_5 a_8^2 \\ \quad +2\beta a_4 a_8 + 2\beta a_5 a_7 - 6\beta a_6 a_7 a_8 - 6\gamma a_6 a_7 a_8 - 6\nu a_6 a_7 a_8 \\ \\ b_{22} = \gamma a_4 + \nu a_4 - 3\beta a_6 a_8^2 - 3\beta a_7^2 a_8 - 3\gamma a_6 a_8^2 - 3\gamma a_7^2 a_8 - 3\nu a_6 a_8^2 \\ \quad -3\nu a_7^2 a_8 + 2\beta a_5 a_8 + 2\beta a_6 a_7 \end{cases}$$

$$\begin{cases} b_{23} = \gamma a_5 + \nu a_5 + \beta a_7^2 - 3\beta a_7 a_8^2 - 3\gamma a_7 a_8^2 - 3\nu a_7 a_8^2 + 2\beta a_6 a_8 \\ \\ b_{24} = \gamma a_6 + \nu a_6 - \beta a_8^3 - \gamma a_8^3 - \nu a_8^3 + 2\beta a_7 a_8 \\ \\ b_{25} = \beta a_8^2 + \gamma a_7 + \nu a_7 \\ \\ b_{26} = \gamma a_8 + \nu a_8 \end{cases}$$

3.2 About periodic orbits in 1-D linear piecewise smooth maps

In this section we shall study stable periodic orbits of piecewise-smooth systems analytically. The map is defined by:

$$x_{n+1} = f(x_n, a, b, \mu, l) = \begin{cases} a x_n + \mu & \text{for } x_n \leq 0 \\ b x_n + \mu + l & \text{for } x_n > 0 \end{cases}, \tag{3.4}$$

where $a, b \in (0,1)$, and l is denoted for height of the discontinuity. With these parameters, it turns out that stable periodic orbits exist for $\mu \in (0,1]$. Let us consider $l > 0$ in equation (3.4). There are three cases as shown in Figure.3.1.

Case 1: For $\mu > 0$, there is a stable fixed point $x_R = \frac{\mu+1}{1-b}$ on the the right-half plane.

Case 2: For $0 > \mu > -l$, there are two stable fixed points on both sides of the discontinuity.

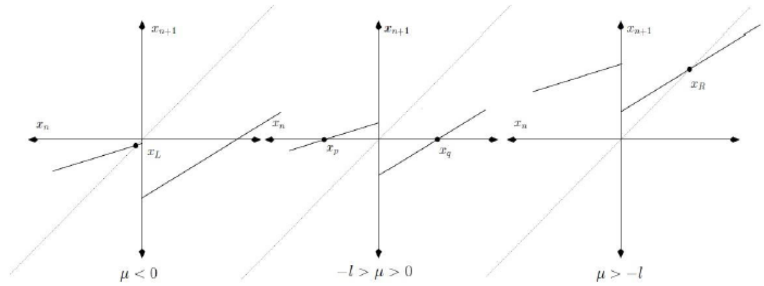


Figure 3.1: Graph of the map for $0 < a < 1$ and $0 < b < 1$, and $l > 0$

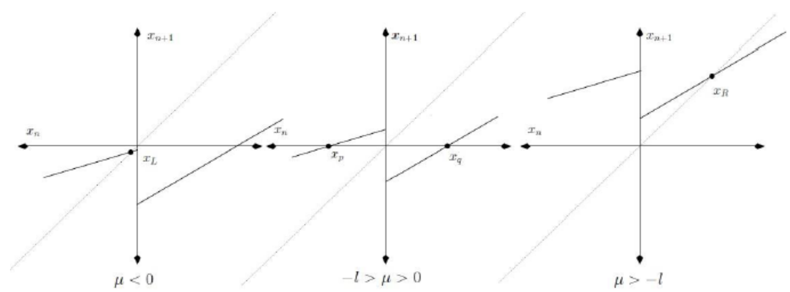


Figure 3.2: Graph of the map for $0 < a < 1$ and $0 < b < 1$, and $-l > 0$.

Case 3: For $\mu < -l$, there is a stable fixed point in the left half plane and it is given by $x_L = \frac{\mu}{1-a}$.
Three additional cases may be observed when $l < 0$, as shown in Figure.3.2.

Case 4: For $\mu < 0$, there is a stable fixed point in the left half plane and it is given by $x_L = \frac{\mu}{1-a}$.

Case 5: For $-l > \mu > 0$, there is no fixed point.

Case 6: For $\mu < -l$, there is another stable fixed point in the right half plane $x_R = \frac{\mu+1}{1-b}$.
Assume that the left half plane is $L = (-\infty, 0]$ and the right half plane is $R = (0, \infty)$. By designating which of the two sets (L or R) the corresponding point belongs to, one can transform (code) a given sequence of points $\{x_n\}_{n\geq 0}$ through which the system evolves into a sequence of Ls and Rs. It is obvious that a periodic orbit has a repeating string of Ls and Rs. We designate this repeated string with the symbol σ as a pattern. The length of the string σ is denoted by $|\sigma|$ and gives the number of symbols in the pattern i.e., the period of the orbit. A period orbit with a pattern σ is denote as O_σ. P_σ denote the interval of paremeter μ for which orbit O_σ exists. The sum of geometric series $1 + k + k^2 + ... + k^n$ is denoted by S_n^k.

Definition 3.1 *A periodic orbit O_σ is termed as admissible if $P_\sigma \neq \emptyset$. An admissible pattern is the pattern of an admissible orbit.*

Definition 3.2 *If a pattern of a periodic orbit O_σ consists of only one R and multiple Ls or vica-*

versa, it is called an atomic pattern.

Thus, there are two types of atomic patterns, thos with pattern $\overbrace{LLL........LL}R$, abbreviated as $L^n R$ (termed as L-atomic pattern) and those with pattern $L\overbrace{RRR........RR}$, abbreviated as LR^n (termed as R-atomic pattern). Both L and R-atomic form the pattern LR.

Definition 3.3 *A pattern is called a molecullar pattern if it is made up of a combination of atomic patterns.*

Example 3.1 *$LLRLLRLR$ is a molecular pattern. The atomic patterns LLR and LR are combined to create it.*

Lemma 3.1 *An atomic pattern of any period is admissible.*

Proof. Consider an atomic orbit $O_{L^n R}$ with period $n+1$. We write down the inequalities as: ■

$$\begin{aligned}
x_0 &\leq 0, \\
x_1 &= ax_0 + \mu \leq 0, \\
x_2 &= ax_0 + \mu \leq 0, \\
&= a^2 x_0 + (a+1)\mu \leq 0, \\
&\quad ..... \\
x_{n-1} &= a^{n-1} x_0 + \mu S^a_{n-2} \leq 0, \\
x_n &= a^n x_0 + \mu S^a_{n-1} > 0, \\
x_{n+1} &= x_0 = bx_n + \mu - 1 \leq 0, \\
x_0 &= \frac{\left(a^{n-1}b + a^{n-2}b + ... + ab + b + 1\right)\mu - 1}{1 - a^n b}.
\end{aligned}$$

Substituting the value of x_0 into the list of inequalities above, would yield a list of upper bounds for μ (whenever the point x_i is in L) and lower bounds for μ (when the point x_i is in R). We denote upper bounds by μ_i^{upper} and lower boundes by μ_i^{lawer}. We define $\mu_2 = \min_i \left(\mu_i^{upper}\right)$ and $\mu_1 = \max_i \mu_i^{lawer}$. Therefore, $P_\sigma = (\mu_1, \mu_2]$. Some simple algebraic manipulation of the inequalities abouve gives:

$$P_{L^n R} = \left(\frac{a^n}{S^a_n}, \frac{a^{n-1}}{a^{n-1}b + S^a_{n-1}} \right].$$

Let us assume $P_{L^n R} = \emptyset$, then

$$\frac{a^n}{S^a_n} > \frac{a^{n-1}}{a^{n-1} + S^a_{n-1}}$$

$$a^n \times \left(a^{n-1}b + S^a_{n-1}\right) - a^{n-1} \times \left(S^a_n\right) > 0$$

$$a^{n-1}\left[a^n b + aS^a_{n-1} - S^a_n\right] > 0$$

$$-a^{n-1}\left(1 - a^n b\right) > 0$$

which is a contradiction since $a, b \in (0, 1)$. Hence $P_{L^n R} \neq \emptyset$. We write down the inequalities as:

$$\begin{aligned} x_0 &\leq 0, \\ x_1 &> 0, \\ &\ldots. \\ x_{n+1} &= x_0 \leq 0 \\ x_0 &= \frac{\left(b^{n-1} + b^{n-2} + \ldots + b + 1\right)\left(\mu - 1\right) + b^n \mu}{1 - b^n a} \end{aligned}$$

Finding μ_1 and μ_2 in the way as explained above, we get

$$P_{L^n R} = \left(\frac{ab^{n-1} + S^b_{n-2}}{ab^{n-1} + S^b_{n-1}}, \frac{S^b_{n-1}}{S^b_n}\right].$$

Further, it can be easily checked that $P_{L^n R} \neq \emptyset$.

Example 3.2 *Let us consider an orbit O_{LR^n}. Here $x_0 \leq 0, x_1 > 0$ and $x_2 = x_0$. From equation (3.4) we get*

$$\begin{aligned} x_1 &= ax_0 + \mu. \\ x_2 &= bx_1 + \mu - 1, \\ &= abx_0 + (b+1)\mu - 1, \\ &= x_0. \\ x_0 &= \frac{(b+1)\mu - 1}{1 - ab} \leq 0. \\ \mu &\leq \frac{1}{b+1}. \end{aligned}$$

Substituting the value of x_0 in x_1 we get:

$$\begin{aligned} x_1 &= a\frac{(b+1)\mu - 1}{1 - ab} + \mu > 0. \\ \mu &> \frac{a}{1+a}. \end{aligned}$$

Hence $P_{L^n R} = \left(\frac{a}{1+a}, \frac{1}{b+1}\right]$.

Example 3.3 *Consider a pattern* $LLRLLRLR\ LLRLLRLR\ LLRLR$. *This pattern corresponds to a molecular orbit of period-*21. LLR *is represented by* L' *and* LR *is represented by* R'. *Then the above pattern becomes* $L'L'R'L'L'R'L'R'$. *Additionally,* $L'L'R' \equiv LLRLLRLR$ *is now designated as* L'' *and* $L'R' \equiv LLRLR$ *is designated as* R''. *Therefore, the above pattern can be written as* $L''L''R''$, *which is atomic in symbols* L'' *and* R''. *So this is an admissible pattern. Now consider another pattern like* $LLRLLRLR\ LLRLR\ LLRLR\ LLRLLRLR$. *It can be expressed as* $L''R''R''L''$. *This pattern does not correspond to an admissible orbit since it is neither atomic or molecular in the new notation.*

3.3 Conclusion

In conclusion, this study focused on finding periodic orbits of some Zeraoulia-Sprott maps. The goal of this study is to understand the behaviors of dynamic systems by providing information about the stability of these periodic orbits. This study also revealed the presence of chaotic orbits in the Zeraoulia-Sprott maps that are very sensitive to the starting conditions.

Bibliography

[1] E. Zeraoulia, *Chaotifying one-dimensional discrete mappings using S-unimodality and Collet-Eckmann conditio*n, International Journal of Bifurcations and Chaos, **29** (4), 1950050, 2019.

[2] E. Zeraoulia and J. C. Sprott, *A minimal 2-D quadratic map with quasi-periodic route to chaos*, International Journal of Bifurcations and Chaos, **18** (5), 1567-1577, 2008.

[3] E. Zeraoulia, *On the dynamics of a new simple 2-D rational discrete mapping,* International Journal of Bifurcations and Chaos, **21** (1), 155-160, 2011.

[4] E. Zeraoulia and J. C. Sprott, *The discrete hyperchaotic double scroll.* International Journal of Bifurcations and Chaos, **19** (3), 1023-1027, 2009.

[5] C. H. Wong, *Border collision bifurcations in piecewise smooth systems*, A thesis for the degree of Doctor of Philosophy, University of Manchester, 2011.

[6] E. Zeraoulia, *Dynamical systems*, Theory and Applications, Science Publishers, 2018.

[7] Hénom, M. *A two dimensional mapping with a strange attractor,* Commun. Math, Phys. **50**, 69-77, 1976.

[8] Aronson, D. G.,Chory, M. A., Hall, G. R. & McGe- hee, R. P. *Bifurcations from an invariant cir-cle for two-parameter families of maps of the plane: A computer-assisted study*, Commun. Math. Phys. **83**, 303–354, 1982.

[9] Lu. J, Wu. X., Lu. J., Kang., *A new discrete chaotic system with rational fraction and its dynamical behaviors,* Chaos, Solitons & Fractals **22,** 311-319, 2004.

[10] Chang. L., Lu. J., Deng. X., *A new two-dimensional discrete chaotic system with rational fraction and its tracking and synchronization,* Chaos, Solitons & Fractals **24**, 1135-1143, 2005.

[11] Chua. L. O, Komuro. M, Matsumoto. T, *The double scroll family*, Part I and II, IEEE Trans. Circuits. Syst. CAS-33, 1073—1118, 1986.

[12] Aronson, D. G.,Chory, M. A., Hall, G. R. & McGe- hee, R. P. *Bifurcations from an invariant cir-cle for two-parameter families of maps of the plane: A computer-assisted study*, Commun. Math. Phys. **83**, 303–354, 1982.

[13] Lu. J, Wu. X., Lu. J., Kang., *A new discrete chaotic system with rational fraction and its dynamical behaviors,* Chaos, Solitons & Fractals **22,** 311-319, 2004.

[14] Chang. L., Lu. J., Deng. X., *A new two-dimensional discrete chaotic system with rational fraction and its tracking and synchronization,* Chaos, Solitons & Fractals **24**, 1135-1143, 2005.

[15] Chua. L. O, Komuro. M, Matsumoto. T, *The double scroll family*, Part I and II, IEEE Trans. Circuits. Syst. CAS-33, 1073—1118, 1986.