People's democratic republic of Algeria
Ministry of higher education and scientific research
Echahid Cheikh Larbi Tebessi university- Tebessa
Faculty of exact sciences, natural and life
Departement: Mathematics

End of studies Thesis
For obtaining Master's degree
Domain: Mathematics and informatic
Sector: Mathematics
Option: Partial Differential Equations and Applications

Theme

The Orthogonality in B(H)

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Defense date: June 8, 2024

ملخص :

الهدف من هذه المذكرة هو دراسة التعامد في فضاء بناخ، اذا سندرس التعامد في فضاءات مختلفة ، في C_1، $L^1(B(H))$ و $B(H)$ حيث $B(H)$ هو جبر المؤثرات الخطية المحدودة على فضاء هيلبرت H.
سنبدأ أولا بالحد الأدنى الكلي و التعامد فيC_1، في الفصل التالي سندرس تعامد جيمس-بيرخوف في $L^1(B(H))$ ، و أخيرا سنرى تعريف تعامد جيمس بيرخوف في $B(H)$.

Abstract

The objective of this memory is to study the orthogonality in Banch space, So we will study the orthogonality in different spaces, in C_1 classe, $L^1(B(H))$ and $B(H)$, such that $B(H)$ is the algebra of all bounded linear operators in Hilbert space H. we begin first by the global minimum and orthogonality in C_1 class, in the next chapter we will study Birkhof-Jmaes orthogonality in $L^1(B(H))$, Finally we will see the definition of Birkhof-James orthogonality in $B(H)$.

Résumé:

L'objectif de ce mémoire est d'étudier l'orthogonalité dans Banach, alors on va étudier l'orthogonalité dans des espaces différentes la classe C_1,$L^1(B(H))$ et $B(H)$ tel que $B(H)$ est l'algèbre des opérateurs linéaires bornés sur un espace de Hilbert H. On va commencer premièrement par le minimum globale dans la classe C_1 . Puis dans la prochaine chapitre on va étudier l'orthogonalité de Birkhoff-James dans $L^1(B(H))$, finalement on va voir la définition de l'orthogonalité de Birkhoff-James dans $B(H)$.

Acknowledgements:

We thank Allah for giving us the health, will and determination to complete our studies until now. First at all, this work would not have been possible without to help and the supervision of **Dr. Mechri Hacene**, we thank him for the quality of his exceptional supervision, for his patience, his availability, all his suggestions inspired our thoughts and his requirements greatly stimulated during the preparation of this memory. Our thanks go to the members of jury **Dr.Degaichi Nouar** and **Elhafsi Boukhalfa** for accepting evaluate this modest work.

Dedication:

To my dear and adorable father **'Mahmoud'** you have always been by my side to support me and encourage me that this work my graditude and my affection.

To my dear and adorable mother **'Anes'** For supporting me.

To my sisters: **Nour el Houda**, **Rahma**, for her support.

To my brothers: **Ali**, **Nidhale**.

To all my friends.

To all the people I love.

I dedicate this work

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Introduction

Orthogonality is one of the fundamental concepts in geometry. The use of this concept dates back to ancient civilizations, such as the ancient Egyptians and Babylonians. The orthogonality theorem is one of the most important ancient theories that is still applied today in mathematical science. The uses of this theory are not limited to abstract mathematics, geometry, and trigonometry only, but its use extends to the sciences of physics and chemistry, and it has a major role in the sciences of space, marine navigation, charts, and engineering construction. .We have Two primary definition of orthogonality, the usual sense definition in Hilbert space and Birkhof-James orthogonality in Banach space, The Birkhoff-James orthogonality is a generalization of Hilbert space orthogonality to Banach spaces. In this memory we are going to study the orthogonality in $B(H)$, such that $B(H)$ is the the algebra of all bounded linear operators on Hilbert space H.The concept of orthogonality developed through the contributions of many major mathematicians, including John von Neumann, David Hilbert in 1902 his works provided the basic framework for these developments and Marshall H. Stone in 1932 made significant contributions to the theory of operators on Hilbert spaces, including the spectral theorem, which is crucial for understanding orthogonality in B(H) . So we will study the orthogonality in different spaces, C_1, $L^1(B(H))$ and $B(H)$. In the first chapter of this work, we will expose some mathematical notions and complements in relation to this work. We will cite in particular, reminder about Banch space, Hilbert space, orthogonal projections in Hilbert space.

In second chapter we will characterize the global minimum of an arbitrary function defined on a Banach space, we establish several new characterizations of the global minimum of the map F_ψ. Further, we apply these results to characterize the operators which are orthogonal to the range of elementary operators.

In the third chapter, We will establish a new characterization of Birkhoff-James orthogonality of bounded linear operators in $L^1(B(H),\rho)$ also implies best approximation has been proved.

In the last chapter we are going to minimizing the $B(H)-$ norm of suitable affine mappings from $B(H)$ to $B(H)$, using convex and Gâteaux derivative as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The corollary obtained characterize global minima in terms of (Banach space)orthogonality

Chapter 1

Preliminaries:

1.1 Banach space:

1.1.1 Vector space:

Definition 1.1 *E is a nonempty set, E it has an addition operator, E called a vector space if* $\forall(x, y) \in E^2, \forall \lambda \in \mathbb{K}$ *:*

1. $x + y \in E$.
2. $\lambda x \in E$.

1.1.2 Normed vector space:

Definition 1.2 *E is a vector space,* $\| . \|$. *A norm on E is a function:* $\| . \| : E \to \mathbb{R}^+, \forall(x, y) \in E^2, \forall \lambda \in \mathbb{K}$, *satisfying:*

1. $\|x\| = 0 \Rightarrow x = 0$.
2. $\| x + y \| \leq \| x \| + \| y \|$.
3. $\|\lambda x\| = |\lambda| . \| x \|$.

Then $(E, \| . \|)$ *is called normed vector space.*

Definition 1.3 *Let* $(E, \| . \|)$ *be a normed vector space, and let* $x \in E$. *Then x is said to be a unit vector whenever* $\| . \| = 1$.

Proposition 1.1 *Let* $(E, \| . \|)$ *is a normed vector space, and let* $d : E \times E \to \mathbb{R}^+$ *be defined by:* $d(x, y) = \|x - y\|$ *is a metric on E satisfying:*

1. $d(x, y) = 0 \Rightarrow x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y)$.

In other words, every normed vector space is a metric space.

Proposition 1.2 $\forall (x, y) \in E^2$

1. $\|x - y\| = \|y - x\|$.
2. $| \|x - y\| | \leq \|x - y\|$.
3. $\|x - y\| \geq 0$.

Definition 1.4 ***(Cauchy sequence)***
Let E be a vector space, $\{X_n\}$ be a sequence of point in E, We say that $\{X_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m > 0 : \| x_n - x_m \| \leq \varepsilon$$

Definition 1.5 *We say that $\{X_n\}$ converge to a point $x \in X$ if:*

$$\lim_{n \to +\infty} \| x_n - x \| = 0$$

Proposition 1.3 *Let E is a normed vector space, then every convergent sequence in E is a Cauchy sequence.*

Definition 1.6 *An **Banach** space is a complete normed vector space.*

1.2 Hilbert space:

Definition 1.7 *Let E be a vector space in $\mathbb{K}$, An inner product or (scalar product) on E is a map :*
$\langle ... \rangle : E \times E \longrightarrow \mathbb{K}$
such that:

1. $\forall x, y, z \in E, \forall \lambda \in \mathbb{K}, \langle \lambda x + y.z \rangle = \lambda \langle x.z \rangle + \langle y.z \rangle$.
2. $\forall x, y \in E, \langle x.y \rangle = \overline{\langle y.x \rangle}$.
3. $\forall x \in E, \langle x.x \rangle \geq 0$.

4. $\forall x \in E, \langle x.x\rangle = 0 \Rightarrow x = 0.$

The pair $(E, \langle ...\rangle)$ is called inner product space or pre-Hilbert space.

Lemma 1.1 ***(Cauchy-Schwarz Inequality)***
Let $\langle ...\rangle$ is an inner product space on E, then:

$$|\langle x.y\rangle|^2 \leq \langle x.x\rangle\langle y.y\rangle.$$

Lemma 1.2 *Let $\langle x.x\rangle$ is an inner product space on E, then the map:*

$$x \longrightarrow \|x\| = \sqrt{\langle x.x\rangle}.$$

is an norm on E.

Lemma 1.3 ***(Parallelogram Law)***
Let E be an inner product space, then

$$\forall x, y \in E : \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Lemma 1.4 ***(Polarization Identity)***
Let $(E, \langle ...\rangle)$ be an inner product space. Then $\forall x, y \in E$

$$\langle y.x\rangle = \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2).$$

Definition 1.8 *A **Hilbert** space is a complete pre-hilbert space.*

1.3 Orthogonal projection:

Let H be a Hilbert space. we denoted its inner product by $\langle ...\rangle$ which is another common notation for inner products that is often reserved for Hilbert spaces. The inner product structure of a Hilbert spaces allows us to introduce the concept of orthogonality, which makes it possible to visualize vectors and linear subspaces of a Hilbert space in a geometric way.

Definition 1.9 *Let $(E, \langle ...\rangle)$ be an inner product space, we say that two vectors $x, y \in E$ are orthogonal if $\langle x.y\rangle = 0$ (and write $x \perp y$). We say that subsets A and B are orthogonal if $x \perp y$ for every $x \in A$ and $y \in B$ (and write $A \perp B$).*

Definition 1.10 *The orthogonal complement $S^\perp$ of a subset S is the set of vectors orthogonal to S*

$$S^\perp = \{y \in E : y \perp x, \forall x \in S\}$$

Theorem 1.1 *The orthogonal complement of a subset of a Hilbert space is a closed linear subspace.*

1.3.1 The Pythagorean Theorem:

If $x_1, \ldots x_n \in X$ and $x_j \perp x_k$ for $j \neq k$, then

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \sum_{j=1}^{n} \left\| x_j \right\|^2 .$$

1.3.2 Lemma:"Pythagors"

Let E be an inner product space, if $x \perp y$ then:

$$\left\| x + y \right\|^2 = \|x\|^2 + \left\| y \right\|^2$$

Chapter 2

Global minimum and orthogonality in C_1-classes:

Let $B(H)$ be a complex Banach space. We first define orthogonality in $B(H)$. We say that $b \in B(H)$ is orthogonal to $a \in B(H)$ if for all complex λ there holds

$$\|a+\lambda b\| \geq \|a\|. \tag{2.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a+\lambda b | \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e., iff this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If $B(H)$ is a Hilbert space, then from (2.1) follows $\langle a.b \rangle = 0$, i.e., orthogonality in the usual sense. Next we define the von Neumann–Schatten classes $C_p (1 \leq p < \infty)$. Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq ... \geq 0$ denote the singular values of T , i.e., the eigenvalues of $|T| = (T^* T)^{1/2}$ arranged in their decreasing order. The operator T is said to be belong to the Schatten p-classes C_p if

$$\| T \|_p = \left[\sum_{i=1}^{\infty} s_i(T)^p \right]^{1/p} = \left[tr(T)^p \right]^{1/p}, \quad 1 \leq p < \infty$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert–Schmidt class, and C_∞ corresponds to the class of compact operators with

$$\| T \|_\infty = s_1(T) = \sup_{\|f\|=1} \| Tf \|$$

denoting the usual operator norm. For the general theory of the Schatten p-classes the reader is referred to [1]. Recall (see [1]) that the norm $\| . \|$ of the B-space V is said to be Gâteaux differentiable at non-zero elements $X \in V$ if

$$\lim_{\mathbb{R} \ni t \to 0} \frac{\| X + tY \| - \| X \|}{t} = ReD_X(Y)$$

for all $Y \in V$. Here $\mathbb{R}$ denotes the set of all reals, Re denotes the real part, and D_X is the unique support functional (in the dual space V^*) such that $\| D_X \| = 1$ and $D_X(X) = \| X \|$. The Gâteaux differentiability of the norm at X implies that X is a smooth point of the sphere of radius $\| X \|$. It is well known (see [8] and the references therein) that for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of T is given by

$$D_T(X) = tr\left[\frac{|T|^{p-1}UX^*}{\| T\|_p^{p-1}}\right] \tag{2.2}$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T. The first result concerning the orthogonality in a Banach space was given by Anderson [11] showing that if A is a normal operator on a Hilbert space H, then $AS = SA$ implies that for any bounded linear operator X there holds

$$\|(S + AX - XA)\| \geq \|S\|. \tag{2.3}$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions: by extending the class of elementary mappings

$$E : B(H) \to B(H), \qquad E(X) = \sum_{i=1}^{n} A_i X B_i$$

and

$$\widetilde{E} : B(H) \to B(H), \qquad \widetilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X$$

where $(A_1, A_2, ..., A_n)$ and $(B_1, B_2, ..., B_n)$ are n-tuples of bounded operators on H, and by extending the inequality (2.3) to C_p-classes with $1 < p < \infty$ see [3,9,16]. The Gâteaux derivative concept was used in [2,5,9,17], in order to characterize those operators which are orthogonal to the range of a derivation. In these papers, the attention was directed to C_p-classes for some $p > 1$. The main purpose of this note is to characterize the global minimum of the map

$$X \longmapsto \| S + \phi(X) \|_{C_1},$$

ϕ is a linear map in $B(H)$
in C_1 at points which are not necessarily smooth by using the φ-Gâteaux derivative. These results are then applied to characterize the operators $S \in C_1$ which are orthogonal to the range of elementary operators, where S is not necessarily a smooth point. It is very interesting to point out that this result has been done in C_1-classes with $1 < p < \infty$ but, at least to our knowledge, it was not given, till now, for C_1-classes. Recall that the operator S is a smooth point of the corresponding sphere in C_1 if and only if either S or S^* is injective.

2.1 φ-Gâteaux derivative:

Definition 2.1 *Let $(B(H), \|.\|)$ be an arbitrary Banach space and $F : B(H) \to \mathbb{R}$. We define the φ-Gâteaux derivative of F at a point $x \in B(H)$ in direction $y \in B(H)$ by*

$$D_\varphi F(x; y) = \lim_{t \to 0^+} \frac{F(x + te^{i\varphi} y) - F(x)}{t}$$

Note that when $\varphi = 0$ the φ-Gâteax derivative of F at x in direction y coincides with the usual Gâteaux derivative of F at x in a direction y given by

$$DF(x; y) = \lim_{t \to 0^+} \frac{F(x + ty) - F(x)}{t} \tag{2.4}$$

According to the notation given in [6] we will denote $D_\varphi F(x; y)$ for $F(x) = \|x\|$ by $D_{\varphi,x}(y)$ and for the same function we write $D_x(y)$ for $DF(x; y)$.

Remarque 2.1 *In [6] the author used the term φ-Gâteaux derivative instead of the term "φ-directional derivative" that we use here. It seems to us that the most appropriate term is the "φ-directional derivative," because in the classical case when we do not have φ, as in (2.4) the existence of this limit corresponds to the directional differentiability of F at x in the direction y, while the Gâteaux differentiability of F at x corresponds to the existence of the same limit in any direction $y \in E$ and moreover the function $y \mapsto DF(x; y)$ is linear and continuous. We note that the existence of $DF(x; y)$ for any $y \in E$ does not imply the Gâteaux differentiability of F at x. As a simple example of what precedes we take the function $F(x) = \|x\|$. We can easily check that $DF(x; y) = \|y\|$ for any $y \in E$ but the function $y \mapsto DF(x; y)$ is not linear and so the Gâteaux derivative does not exist.*
We recall (see [6], Proposition 6) that the function $y \mapsto D_{\varphi,x}(y)$ is subadditive and

$$| D_{\varphi,x}(y) | \leq \| y \| \tag{2.5}$$

We end this section by establishing a necessary optimality condition in terms of φ-directional derivative for a minimization problem.

Theorem 2.1 *Let $(B(H), \|.\|)$ be an arbitrary Banach space and $F : B(H) \to \mathbb{R}$. If F has a global minimum at $v \in B(H)$, then*

$$\inf_\varphi D_\varphi(v; y) \geq 0 \tag{2.6}$$

for all $y \in B(H)$.

Proof**.** Assume that F has a global minimum at v, i.e.,

$$F(x) \geq F(v) \tag{2.7}$$

for all $v \in B(H)$. Let $t > 0$, φ, and $y \in B(H)$ be taken arbitrarily. Then (2.7) with $x = v + te^{i\varphi}y$ yields

$$F(v + te^{i\varphi}y) - F(v) \geq 0$$

which implies

$$\frac{F(v + te^{i\varphi}y) - F(v)}{t} \geq 0$$

for all $t > 0$. Letting $t \to 0^+$ we obtain

$$\lim_{t \to 0^+} \frac{F(v + te^{i\varphi}y) - F(v)}{t} \geq 0, \quad \forall \varphi, y \in B(H)$$

Thus

$$D_\varphi(v; y) \geq 0, \quad \forall \varphi, y \in B(H)$$

and hence

$$\inf_\varphi D_\varphi(v; y) \geq 0, \quad \forall \varphi, y \in B(H)$$

This completes the proof ■

2.2 The Global Minimum:

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$ for all α, β, X, Y, and let $S \in C_1$. Put

$$U = X \in B(H) : \phi(X) \in C_1.$$

Let $\psi : \mathcal{U} \to C_1$ be defined by

$$\psi(X) = S + \phi(X) \tag{2.8}$$

Define the function $F_\psi : \mathcal{U} \to \mathbb{R}^+$ by $F_\psi(X) = \|\psi(X)\|_{C_1}$. Now we are ready to prove our first result in C_1-classes. It gives a necessary and sufficient optimality condition for minimizing F_ψ.

Theorem 2.2 *The map F_ψ has a global minimum at $V \in \mathcal{U}$ if and only if*

$$\inf_\varphi D_{\varphi,\psi(V)}(\phi(Y)) \geq 0, \quad \forall Y \in \mathcal{U}. \tag{2.9}$$

Before proving this theorem we need the following lemma.

Lemma 2.1 *The following equalities hold for all $V, Y \in \mathcal{U}$*

$$D_\varphi F_\psi(V, Y) = D_\varphi \|\,.\,\|_{C_1} (\psi(V), \phi(Y)) = D_{\varphi,\psi(V)}(\phi(Y)).$$

Proof. we have

$$D_\varphi F_\psi(V,Y) = \lim_{t\to 0^+}\frac{F(V+te^{i\varphi}Y)-F(Y)}{t} \tag{2.10}$$

$$= \lim_{t\to 0^+}\frac{\| \psi(V+te^{i\varphi}Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} \tag{2.11}$$

$$= \lim_{t\to 0^+}\frac{\| S+\phi(V)+te^{i\varphi}\phi(Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} \tag{2.12}$$

$$= \lim_{t\to 0^+}\frac{\| \psi(V)+te^{i\varphi}\phi(Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} \tag{2.13}$$

$$= D_\varphi \|\,.\,\|_{C_1}(\psi(V),\phi(Y)) = D_{\varphi,\psi(V)}(\phi(Y)). \tag{2.14}$$

■ Proof. Proof of Theorem 2.2. For the necessity we have just to combine Theorem 2.1 and Lemma 2.1. Conversely, assume that (2.9) is satisfied. First, observe that

$$D_{\varphi,\psi(V)}(V)(e^{i(\pi-\varphi)}\psi(V)) = \lim_{t\to 0^+}\frac{\| \psi(V)+te^{i\varphi}e^{i(\pi-\varphi)}\psi(V)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} \tag{2.15}$$

$$= \lim_{t\to 0^+}\frac{\| \psi(V)-t\psi(V)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} \tag{2.16}$$

$$=\|\psi(V)\|_{C_1}\lim_{t\to 0^+}\frac{|1-t|-1}{t} = -\|\psi(V)\|_{C_1} \tag{2.17}$$

From this, we have

$$\|\psi(V)\|_{C_1} = -D_{\varphi,\psi(V)}(e^{i(\pi-\varphi)}\psi(V)).$$

Let $Y\in\mathscr{U}$ be arbitrary and put $\widetilde{Y} = Y + e^{i(\pi-\varphi)}V + \phi^{-1}(S+e^{i(\pi-\varphi)}S)$. It is easy to see that $\widetilde{Y}\in\mathscr{U}$. Then by (2.9) we have $D_{\varphi,\psi(V)}(\phi(\widetilde{Y}))\geq 0$ and hence by the subadditivity of $D_{\varphi,\psi(V)}(.)$ and the linearity of ϕ we get

$$\|\psi(V)\|_{C_1} \leq -D_{\varphi,\psi(V)}(e^{i(\pi-\varphi)}\psi(V)) + D_{\varphi,\psi(V)}(\phi(\widetilde{Y})) \tag{2.18}$$

$$= D_{\varphi,\psi(V)}(\phi(\widetilde{Y}) - e^{i(\pi-\varphi)}\psi(V)) \tag{2.19}$$

$$= D_{\varphi,\psi(V)}(\phi(Y) + e^{i(\pi-\varphi)}\phi(V) + S + e^{i(\pi-\varphi)}S - e^{i(\pi-\varphi)}\psi(V)) \tag{2.20}$$

$$= D_{\varphi,\psi(V)}(\psi(Y)). \tag{2.21}$$

By using (2.5) we obtain

$$\|\psi(V)\|_{C_1} \leq D_{\varphi,\psi(V)}(\psi(V)) \leq \|\psi(V)\|_{C_1}$$

Finally as Y is arbitrary in $\mathscr{U}$, then F_ψ has a global minimum at V on $\mathscr{U}$. ■

Note that in our proofs of Theorem 2.2 and Lemma 2.1 we do not use the form of the norm in C_1-classes and we can check that they still hold in any C_p-classes with $1\leq p\leq\infty$.

Now, we restrict our attention on C_1-classes. First, let us recall the following result proved in [6, Theorem 2] for C_1-classes.

Theorem 2.3 *Let $X, Y \in C_1$. Then, there holds*

$$D_X(Y) = Re\left\{tr(U^*Y)\right\} + \| QYP \|_{C_1}.$$

where $X = U|X|$ is the polar decomposition of X, $P = P_{ker X}$, $Q = Q_{ker X^*}$ are projections.
The following corollary establishes a characterization of the φ-Gâteaux derivative of the norm in C_1-classes.

Corollary 2.1 *Let $X, Y \in C_1$. Then, there holds*

$$D_{\varphi,X}(Y) = Re\left\{e^{i\varphi} tr(U^*Y)\right\} + \| QYP \|_{C_1}.$$

for all φ, X, where $X = U|X|$ is the polar decomposition of X, $P = P_{ker X}$, $Q = Q_{ker X^*}$ are projections.

Proof. Let $X, Y \in C_1$. Put $\widetilde{Y} = e^{i\varphi}Y$. Applying Theorem 2.3 with φ, X, and $\widetilde{Y}$ we get

$$D_{\varphi,X}(Y) = \lim_{t\to 0^+} \frac{\| X + te^{i\varphi}Y \|_{C_1} - \| X \|_{C_1}}{t} = \lim_{t\to 0^+} \frac{\| X + t\widetilde{Y} \|_{C_1} - \| X \|_{C_1}}{t} = D_X(\widetilde{Y}) \tag{2.22}$$

$$= Re\left\{tr(U^*\widetilde{Y})\right\} + \| Q\widetilde{Y}P \|_{C_1} = Re\left\{tr(U^*e^{i\varphi}Y)\right\} + \| Qe^{i\varphi}YP \|_{C_1} \tag{2.23}$$

$$= Re\left\{e^{i\varphi} tr(U^*Y)\right\} + \| QYP \|_{C_1} \tag{2.24}$$

This completes the proof. ■

In the following theorem we use Theorem 2.2 to give another characterization of the global minimum of F_ψ as global minimum of the function $L_{V,\phi}(Y) : \mathcal{U} \to \mathbb{R}$ defined by

$$L_{V,\phi}(Y) = \| Q\phi(Y)P \|_{C_1} - |\, tr(U^*\phi(Y)) \,|$$

where $\psi(V) = U\,|\,\psi(V)\,|$.

Theorem 2.4

1. *F_ψ has a global minimum on $\mathcal{U}$ at V if and only if*

$$L_{V,\phi}(Y) \geq 0, \quad \forall Y \in \mathcal{U}. \tag{2.25}$$

2. *If $V \in ker\phi$, then F_ψ has a global minimum on $\mathcal{U}$ at V if and only if $L_{V,\phi}(Y)$ has a global minimum on $\mathcal{U}$ at V.*

Proof.

1. We prove the necessity of part (1). Assume that F_ψ has a global minimum on $\mathcal{U}$ at V. Then by Theorem 3.2 we have

$$\inf_\varphi D_{\varphi,\psi(V)}(\phi(Y)) \geq 0, \quad \forall \varphi, Y \in \mathcal{U},$$

which ensures by Corollary 2.1 that

$$\inf_{\varphi} Re\left\{e^{i\varphi} tr(U^* \phi(Y))\right\} + \| Q\phi(Y)P \|_{C_1} \geq 0$$

with $\psi(V) = U|\psi(V)|$ is the polar decomposition of $\psi(V)$ and $P = P_{ker\psi(V)}, Q = Q_{ker\psi(V)^*}$ or equivalently

$$\| Q\phi(Y)P \|_{C_1} \geq -\inf_{\varphi} Re\left\{e^{i\varphi} tr(U^* \phi(Y))\right\}$$

By choosing the most suitable φ we get

$$\| Q\phi(Y)P \|_{C_1} \geq |tr(U^*)\phi(Y)|, \quad \forall Y \in \mathcal{U}, \tag{2.26}$$

and so $L_{V,\phi}(Y) \geq 0$ for all $Y \in \mathcal{U}$

Conversely, assume that (2.25) is satisfied. Let φ be arbitrary and $Y \in \mathcal{U}$. By (2.25) we have

$$\| Q\phi(\widetilde{Y})P \|_{C_1} \geq |tr(U^* \phi(\widetilde{Y})| \geq -Re(tr(U^* \phi(\widetilde{Y})))$$

with $\widetilde{Y} = e^{i\varphi} Y \in \mathcal{U}$. Hence, by the linearity of ϕ we obtain

$$\| Q\phi(Y)P \|_{C_1} \geq -Re(e^{i\varphi}(tr(U^* \phi(Y)))$$

for $Y \in \mathcal{U}$ and all φ and so

$$\inf_{\varphi} \left[\| Q\phi(Y)P \|_{C_1} + Re(e^{i\varphi} tr(U^* \phi(Y))\right] \geq 0$$

for $Y \in \mathcal{U}$ and all φ. Thus Theorem 2.2 and Lemma 2.1 complete the proof of part (1)

2. Assume that $V \in ker\phi$, that is, $\phi(V) = 0$; then $L_{V,\phi}(V) = 0$ and so (2.25) is equivalent to

$$L_{V,\phi}(Y) \geq L_{V,\phi}(V), \quad \forall Y \in U.$$

This means that $L_{V,\phi}$ has a global minimum at V. Therefore part (1) ends the proof.

■

Now we characterize the global minimum of F_ψ on C_1, when ϕ is a linear map satisfying the following useful condition:

$$tr(X\phi(Y)) = tr(\phi^*(X)Y), \quad \forall X, Y \in C_1, \tag{2.27}$$

where ϕ^* is an appropriate conjugate of the linear map ϕ. We state some examples of ϕ and ϕ^* which satisfy condition (2.27).

1. The elementary operator $E : \mathcal{I} \to \mathcal{I}$ defined by

$$E(X) = \sum_{i=1}^{n} A_i X B_i$$

where $(A_1, A_2, ..., A_n)$ and $(B_1, B_2, ..., B_n)$ are n-tuples of bounded Hilbert space operators and $\mathcal{I}$ is a separable ideal of compact operators associated with some unitarily invariant norm. In [6, Proposition 8] the author showed that the conjugate operator $E^* : \mathcal{I}^* \to \mathcal{I}^*$ of E has the form

$$E^*(X) = \sum_{i=1}^{n} A_i X B_i$$

and that the operators E and E^* satisfy condition (2.27).

2. The elementary operator $\widetilde{E} : \mathcal{I} \to \mathcal{I}$ defined by

$$\widetilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X$$

where $(A_1, A_2, ..., A_n)$ and $(B_1, B_2, ..., B_n)$ are n-tuples of bounded Hilbert space operators and $\mathcal{I}$ is a separable ideal of compact operators associated with some unitarily invariant norm. Using the same ideas of the proof of [6, Proposition 8] we can check that the conjugate operator $\widetilde{E}^* : \mathcal{I}^* \to \mathcal{I}^*$ of $\widetilde{E}$ has the form

and that the operators $\widetilde{E}$ and $\widetilde{E}^*$ satisfy condition (2.27).
Now, we are in position to prove the following theorem.

Theorem 2.5 *Let $V \in C_1$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U|\psi(V)|$. Then F_ψ has a global minimum on C_1 at V if and only if $U^* \in ker\phi^*$.*

Proof**.** Assume that F_ψ has a global minimum on C_1 at V. Then

$$\inf_{\varphi} D_{\varphi,\psi(V)}(\phi(Y)) \geqslant 0 \tag{2.28}$$

for all $Y \in C_1$. That is,

$$\inf_{\varphi} Re\left\{e^{i\varphi} tr(U^*\phi(Y))\right\} + \|Q\phi(Y)P\|_{C_1} \geqslant 0, \quad \forall Y \in C_1.$$

Take φ so that

$$Re\left\{tr(U^*\phi(Y))\right\} \geqslant 0. \tag{2.29}$$

Let $f \otimes g$ be the rank one operator defined by $x \mapsto \langle x, f\rangle g$, where f, g are arbitrary vectors in the Hilbert space H. Take $Y = f \otimes g$, since the map ϕ satisfies (2.27) one has

$$tr(U^*\phi(Y)) = tr(\phi^*(U^*)Y).$$

Then (2.29) is equivalent to $Re\{tr(\phi^*(U^*)Y)\} \geqslant 0$ for all $Y \in C_1$, or equivalently

$$Re\{\langle \phi^*(U^*)g, f \rangle\} \geqslant 0, \quad \forall f, g \in H.$$

As f, g are arbitrary we can easily check that

$$Re\{\langle \phi^*(U^*)g, f \rangle\} = 0, \quad \forall f, g \in H.$$

Thus $\phi^*(U^*) = 0$, i.e., $U^* \in ker\phi^*$.
Conversely, let φ be arbitrary. If $U^* \in ker\phi^*$, then $e^{i\varphi}U^* \in ker\phi^*$. It is easily seen (using the same arguments above) that

$$Re\{e^{i\varphi} tr(U^*\phi(Y))\} + \|Q\phi(Y)P\|_{C_1} \geqslant 0, \quad \forall Y \in C_1$$

Now as φ is taken arbitrary, we get (2.28).
We state our first corollary of Theorem 2.5. Let $\phi = \delta_{A,B}$, where $\delta_{A,B} : B(H) \to B(H)$ is the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$. ■

Corollary 2.2 *Let $V \in C_1$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U|\psi(V)|$. Then F_ψ has a global minimum on C_1 at V, if and only if $U^* \in ker\delta^*_{A,B} = ker\delta_{A,B}$*

Proof**.** It is a direct consequence of Theorem 2.5. ■
This result may be reformulated in the following form where the global minimum V does not appear. It characterizes the operators S in C_1 which are orthogonal to the range of a derivation.

Theorem 2.6 *Let $S \in C_1$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U|\psi(S)|$.*

$$\|S + (AX - XB)\|_{C_1} \geqslant \|\psi(S)\|_{C_1}$$

for all $X \in C_1$ if and only if $U^* \in ker\delta_{B,A}$.
As a corollary of this theorem we have

Corollary 2.3 *Let $S \in C_1$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U|\psi(S)|$. Then the two following assertions are equivalent:*

1. $\|S + (AX - XB)\|_{C_1} \geqslant \|S\|_{C_1}, \quad \forall X \in C_1$
2. $U^* \in ker\delta_{B,A}$.

Chapter 3

The Orthogonality In $L^1(B(H))$:

Let $B(H)$ be a complex Banach space and let $(B(H),\rho)$ be a positive measure space. M denote a closed subspace of X. Let $f \in L^1(B(H)) \setminus \overline{M}$. Then there exists a unique best approximant g to f from M if and only if

$$\|f-g\| \leq \|f-h\|, \quad \forall h \in M. \tag{3.1}$$

We recall that f is said to be orthogonal to M, written $f \perp M$, if and only if

$$\forall \lambda \in \mathbb{C} : \|f\| \leq \|f+\lambda g\|, \quad \forall g \in M. \tag{3.2}$$

we are going to to establish a new characterization of Birkhoff-James orthogonality of bounded linear operators in $L^1(B(H),\rho)$ also implies best approximation has been proved.

3.1 Birkhoff-James Orthogonality: A New Characterization in $L^1(B(H))$

Definition 3.1 *Let $L^p(B(H)), 1 < p < \infty$, and $L^q(B(H))$ the Dual space. Let M be a closed subspace of $L^p(B(H))$,we recall that $f \in L^p(B(H))$ is orthogonal to M, written $f \perp M$, if and only if*

$$\|f\|_p \leq \|f+g\|_p, \quad \forall g \in M. \tag{3.3}$$

Theorem 3.1 *Let M be a closed subspace of $L^p(\Omega), 1 < p < \infty, f \in L^p(B(H))$ is orthogonal to M if and only if*

$$\int_X g \mid f \mid^{p-1} sign(f) dX = 0, \quad \forall g \in M. \tag{3.4}$$

Proof. *See [19].* ■

Definition 3.2 *Let $(B(H), \| . \|)$ be an arbitrary Banach space. Then the φ-Gâteaux derivative of the norm at f in the direction g is defined as*

$$D_{\varphi,f}(g) = \lim_{t\to 0^+} \frac{\| f + te^{i\varphi} g \| - \| f \|}{t}. \tag{3.5}$$

3.2 Best Approximation in $L^1(B(H))$

Proposition 3.1 *If the function $H_{f,g}(t) = \| f + te^{i\varphi} g \|$ is convex, then the following statements are hold :*

1. $D_{\varphi,f}(g)$ *is subadditive and positively homogeneous functional on* $B(H)$.
2. $D_{\varphi,f}(g) \leq \| g \|$.
3. $D_{\varphi,f}(e^{i\theta} g) = D_{\varphi+\theta,f}(g)$.

Proof.

1. We have
$$\| f + te^{i\varphi}(g+h) \| \leq \left\| \frac{f}{2} + te^{i\varphi} g \right\| + \left\| \frac{f}{2} + te^{i\varphi} h \right\|.$$
Taking the limit as $t \to 0^+$, we obtain
$$D_{\varphi,f}(g+h) = \lim_{t\to 0^+} \frac{\left\| f + te^{i\varphi}(g+h) \right\| - \left\| f \right\|}{t} \leq \lim_{t\to 0^+} \frac{\left\| f + 2te^{i\varphi} g \right\| + \left\| f + 2te^{i\varphi} h \right\| - 2\left\| f \right\|}{2t} \tag{3.6}$$
$$= D_{\varphi,f}(g) + D_{\varphi,f}(h) \tag{3.7}$$
Positive homogeneity is obvious.

2. It is easy to see that
$$\Big| \| f + te^{i\varphi} g \| - \| f \| \Big| \leq \| f + te^{i\varphi} g - f \| = t \| g \|.$$
Taking the limit as $t \to 0^+$, we get
$$D_{\varphi,f}(g) = \lim_{t\to 0^+} \frac{\| f + te^{i\varphi}(g) \| - \| f \|}{t} \leq \| g \|.$$

3. The proof is obvious.

■

Theorem 3.2 *Let $(B(H), \| . \|)$ be an arbitrary Banach space. If the function $f \in B(H)$ is orthogonal to $g \in B(H)$, then*
$$\inf_{\varphi} D_{\varphi,f}\left(g\right) \geq 0. \tag{3.8}$$

Proof. Let f be orthogonal to g, i.e.

$$\forall \lambda \in \mathbb{C} \; \| f \| \leq \| f + \lambda g \|, \quad \forall g \in M.$$

Then

$$\frac{\| f + te^{i\varphi} g \| - \| f \|}{t} \geq 0, \quad \forall t > 0,$$

and passing to the limit as $t \to 0^+$, we obtain

$$\inf_{\varphi} D_{\varphi,f}(g) \geq 0.$$

■

Theorem 3.3 *Let M be linear subspace of $L^1(B(H))$ and $f \in L^1(B(H)) \setminus \overline{M}$, then g is a best $L^1(B(H))$ approximant to f from M if and only if*

$$\left| \int_{f(x)=g(x)} e^{-i\theta(x)} h(x) d\rho(x) \right| \leq \int_{f(x)\neq g(x)} |h(x)|\, d\rho(x), \quad \forall h \in M, \tag{3.9}$$

where

$$(f-h)(x) = \left|f-h\right| e^{-i\theta(x)}.$$

Proof. *See [20].* ■

Definition 3.3 *Let M be a linear closed subspace of $L^1(B(H))$ and let $S(B(H)) = \{\varphi \in B(H) / \| \varphi \| \leq 1\}$. $f \in L^1(B(H))$ is orthogonal to M if and only if, there exists a function $\varphi \in S(B(H))$, such that*

1. $\int_{B(H)} f \varphi dx = \int_{B(H)} | f | dx.$
2. $\int_{B(H)} \varphi h dx = 0, \quad \forall h \in M.$

Remarque 3.1 *In the particular case, for $x \in \ker f$ has measure zero. The function $f \in L^1(B(H))$ is orthogonal to M if and only if*

$$\int_{B(H)} (signf) h(x) d\rho(x) = 0, \quad \forall h \in M. \tag{3.10}$$

Theorem 3.4 *Let M be linear closed subspace of $L^1(B(H))$ .The function $f \in L^1(B(H))$ is orthogonal to $g \in M$ if and only if*

$$\left| \int_{\{g\neq 0\}} e^{-i\theta(x)} f(x) d\rho(x) \right| \leq \int_{\{g=0\}} \left|f(x)\right| d\rho(x), \tag{3.11}$$

where

$$f(x) = \left|f(x)\right| e^{i\theta(x)}.$$

Proof. We have, in $L^1(B(H))$

$$D_{\varphi,g}(f) = Re\left\{ \int_{g\neq 0} e^{i\varphi} e^{-i\theta(x)} f(x) d\rho(x) \right\} + \int_{g=0} \left|f(x)\right| d\rho(x).$$

Since

$$\lim_{\rho\to 0}\frac{\left|g(x)+\rho e^{i\varphi}f(x)\right|-\left|g(x)\right|}{\rho}$$

$$\begin{cases} \cos(\varphi-\theta(x))+\psi(x)|f(x)|, & g(x)\neq 0.\\ \left|f(x)\right|, & g(x)=0.\end{cases}$$

and also

$$\frac{\left|g(x)+\rho e^{i\varphi}f(x)\right|-\left|g(x)\right|}{\rho}\leq\left|f(x)\right|.$$

Thus, we get $f\perp g$ if and only if

$$infRe\left\{\int_{f\neq 0}e^{i\varphi}e^{-i\theta(x)}f(x)d\rho(x)\right\}+\int_{f=0}|f(x)|d(x)\geq 0.$$

However, the infimum will be attained for that φ, for which

$$e^{i\varphi}\int_{f\neq 0}e^{-i\theta(x)}f(x)d\rho(x)=-\left|\int_{f\neq 0}e^{-i\theta(x)}f(x)d\rho(x)\right|,$$

and the result follows . ∎

Corollary 3.1 *Let M be a linear closed subspace of $L^1(B(H))$, then the following assertions are equivalent*

1. *The function $f\in L^1(B(H))$ is orthogonal to $g\in M$ if and only if*

$$\left|\int_{g\neq 0}e^{-i\theta(x)}f(x)d\rho(x)\right|\leq\int_{g=0}\left|f(x)\right|d\rho(x).$$

2. *g is a best $L^1(B(H))$ approximant to f from M if and only if*

$$\left|\int_{f(x)=g(x)}e^{-i\theta(x)}h(x)d\rho(x)\right|\leq\int_{f(x)\neq g(x)}|h(x)|\,d\rho(x),\quad\forall h\in M,$$

where

$$(f-h)(x)=\left|f-h\right|e^{-i\theta(x)}$$

3. *The function $f\in L^1(B(H))$ is orthogonal to $g\in M$ if and only if*

$$e^{i\varphi}\int_{f\neq 0}e^{-i\theta(x)}f(x)d\rho(x)=-\left|\int_{f\neq 0}e^{-i\theta(x)}f(x)d\rho(x)\right|.$$

Proof. 1)$\Longrightarrow$ 2)
Let $f\in L^1(B(H))$ is orthogonal to $g\in M$, then

$$\left|\int_{g\neq 0}e^{-i\theta(x)}f(x)d\rho(x)\right|\leq\int_{g=0}\left|f(x)\right|d\rho(x).$$

Taking

$$g(x) = f_1(x) - h_1(x)$$

Then f_1 is a best $L^1(B(H))$ approximant to h_1 from M.

3)$\Longrightarrow$ 1)

The function $f \in L^1(B(H))$ is orthogonal to $g \in M$ implies

$$e^{i\varphi}\int_{f\neq 0} e^{-i\theta(x)} f(x)d\rho(x) = -\left|\int_{f\neq 0} e^{-i\theta(x)} f(x)d\rho(x)\right|.$$

Taking $\varphi = 0$, then

$$\left|\int_{g\neq 0} e^{-i\theta(x)} f(x)d\rho(x)\right| \leq \int_{g=0} \left|f(x)\right| d\rho(x).$$

3)$\Longrightarrow$ 1)$\Longrightarrow$ 2)

1)$\Longrightarrow$ 3),using Theorem 3.4. ■

Chapter 4

Jemes and Birkhoff orthogonality in $B(H)$:

Let $B(H)$ be a complex Banach space, and $X, Y \in B(H), X$ we first define orthogonality in $B(H)$. We say that $Y \in B(H)$ is orthogonal to $X \in B(H)$ if for all complex λ there holds,

$$\| X + \lambda Y \|_{B(H)} \geq \| X \|_{B(H)} . \tag{4.1}$$

This definition has a natural geometric interpretation. Namely $Y \perp X$ if and only if the complex line $\{X + \lambda Y | \lambda \in \mathbb{C}\}$ is orthogonal in $B(H)$, i.e, if and only if this complex line is a tangent one.

Definition 4.1 *Let $B(H)$ be a Banach space, and $X, Y \in B(H), X$ is smooth point of the boundary of K in $B(H)$ if there exists a unique functional F_X, called the support functional, such that*
$\| F_X \| = 1$ and $F_X = \| X \|$.

Remarque 4.1 *If $B(H)$ is a Hilbert space from (4.1) We can easily derive $\langle X, Y \rangle = 0$, i.e, orthogonality in the usual sense. In general, such orthogonality is not symmetric in Banach space We can take as example the following vectors $(-1, 0)$ and $(1, 1)$, which are in the Hilbert-Schmidt classes C_2, with the max-norm. Joel Anderson has proved that, every non-zero $X \in B(H)$, is a smooth point if and only if $X \in B(H)$ attains its norm, $e \in B(H)$, $\|Xe\| = \|X\|$, and in this case the support functional of X is given by*

$$D_T(X) = Retr\left[\frac{e \otimes Te}{\|T\|} X\right] = Re\left\langle Xe, \frac{Te}{\| T \|}\right\rangle, \quad \forall X \in B(H). \tag{4.2}$$

Here, Re denotes the real part and $D_T(X)$ is the unique support functional (in the dual space $B(H)^*$), recall that the rank one operator, $e \otimes Te$ is defined by

$$(e \otimes Te)X = \langle X, Te\rangle\, e, \quad \forall X \in B(H). \tag{4.3}$$

and

$$tr(\frac{(e \otimes Te)}{\|T\|} X) = \left\langle Xe, \frac{Te}{\| T \|}\right\rangle, \quad \forall X \in B(H). \tag{4.4}$$

The first result concerning the orthogonality in Banach space was given by Anderson [11] ,showing that if A is a normal operator on a Hilbert space H and $S \in B(H)$ then $AS = SA$ implies that for any bounded linear operator x there holds

$$\| S + AX - XA \| \geq \| S \| . \tag{4.5}$$

This means that the range of the derivation

$$\delta_A : B(H) \to B(H),$$

defined by

$$\delta_A(X) = AX - XA, \tag{4.6}$$

is orthogonal to its kernel. This result has been generalized in two directions: by extending to the class of elementary mappings

$$E : B(H) \to B(H).$$

$$E(X) = \sum_{i=1}^{i=n} A_i X B_i,$$

and

$$\widetilde{E} : B(H) \to B(H).$$

$$\widetilde{E}(X) = \sum_{i=1}^{i=n} A_i X B_i - X,$$

where $(A_1, A_2, ... A_n)$ and $(B_1, B_2, ... B_n)$ are $n-$ tuples of bounded operators on H, and by extending the equality (4.2) to C_p the Schatten p-classes with $1 < p < \infty$ see [15], [18].The Gâteaux derivative concept was used in [12, 13, 14] and [5]. In order to characterize those operators which are orthogonal to the range of a derivation in C_p. First we characterize the global minimum of the map

$$X \to \| S + \Phi(X) \|,$$

where Φ is a linear map in $B(H)$, by using the Gâteaux derivative. These results are then applied to characterize the operators $S \in B(H)$ which are orthogonal to the range of elementary operators.

4.1 φ-Gâteaux derivative:

Proposition 4.1 *1)Let $B(H)$ be a Banach space $X, Y \in B(H)$, and $\varphi \in [0, 2\pi)$.The function*

$$\gamma : \mathbb{R} \to \mathbb{R},$$

$$\gamma(t) = \| X + e^{it} Y \|, \tag{4.7}$$

is convex.

The limit

$$D_{\varphi,X}(Y) = \lim_{t\to 0^+} \frac{\| X + te^{i\varphi} Y \| - \| X \|}{t}, \tag{4.8}$$

always exists.The number $D_{\varphi,X}(Y)$ we shall call the φ-Gâteaux derivative of the norm at the vector X, in the Y and φ directions.

2)The vector Y is orthogonal to X in the sense of James if and only if the inequality

$$\inf_{\varphi} D_{\varphi,X}(Y) \geq 0, \tag{4.9}$$

holds.

Theorem 4.1 *Let $(B(H))$ be an arbitrary Banach space we define the function*

$$F : B(H) \to \mathbb{R},$$

$$F(X) = \| X \| .$$

If F has a global minimum at $X \in B(H)$, then

$$D_{F(X)}(Y) \geq 0, \forall Y \in B(H). \tag{4.10}$$

4.2 The Global Minimum in B(H):

Let φ be a linear map :

$$\varphi : B(H) \to B(H),$$

and let the map ψ defined by

$$\psi(X) = \varphi(X) + S, \tag{4.11}$$

for some element $S \in B(H)$.

$$D_X(Y) = \lim_{t\to 0^+} \frac{\| X + tY \| - \| X \|}{t}, \tag{4.12}$$

such as

$$D_X(Y) \leq \| Y \|, \tag{4.13}$$

$$D_X(X) = \| X \|, \tag{4.14}$$

$$D_X(-X) = - \| X \| \tag{4.15}$$

Theorem 4.2 *The map $F_\psi(X) = \| \psi(X) \|$ has a global minimum at $X \in B(H)$ if and only if*

$$D_{\psi(X)}(\varphi(Y)) \geq 0, \quad \forall Y \in B(H), \tag{4.16}$$

it is clear to see that

$$\psi(X) + t\varphi(Y) = \psi(X + tY), \tag{4.17}$$

we choose t such that

$$\varphi(Y - X) = \psi(Y) - \psi(X), \tag{4.18}$$

$$D_{\psi(X)} = L,$$

then

$$\| \psi(X) \| = -L(-\psi(X)) \leq -L(-\psi(X)) + L(\psi(Y)) - \psi(Y)), \tag{4.19}$$

from where

$$\| \psi(X) \| \leq L(\psi(Y)), \tag{4.20}$$

and by sub additivity we get

$$\| \psi(X) \| \leq \| \psi(Y) \|, \tag{4.21}$$

In the following theorem we characterize the global minimum of the map F_ψ on $B(H)$ at V when φ is a linear map .

Theorem 4.3 *Let $V \in B(H)$ be a smooth point and f is a unique vector for which V attains its norm, then the map F_ψ has a global minimum at $V \in B(H)$, if and only if*

$$tr((f \otimes V)\varphi(Y)) = 0, \quad \forall Y \in B(H). \tag{4.22}$$

Proof.

1. Let $V \in B(H)$ be a smooth point
   F_ψ has a global minimum on $B(H)$ at V, then

   $$D_{\psi(V)}(\varphi(Y)) \geq 0, \quad \forall Y \in B(H).$$

   By (4.2) we get

   $$Re(\langle (\varphi(Y))f, Vf \rangle) \geq 0, \quad \forall Y \in B(H).$$

   Let Γ is the subspace of $B(H)$ in which $V \in B(H)$ attains its norm the set

   $$\{\langle V^*\varphi(Y)f, f \rangle \ / f \in \Gamma, \| f \| = 1\}.$$

is numerical range of $V^*\varphi(Y)$ on the subspace Γ. is convex and closed. By (4.16), it must contain a value whose real part is positive, under all rotations around the origin, it must contain the origin, and we will have a vectors $f \in \Gamma$ such that

$$\langle V^*\varphi(Y)f, f\rangle < \delta, \quad \forall Y \in B(H).$$

Where $\delta > 0$, as δ is arbitrary we can easily check that

$$\langle V^*\varphi(Y)f, f\rangle = 0, \quad \forall Y \in B(H).$$

Then

$$tr((f \otimes V)\varphi(Y)) = 0, \quad \forall Y \in B(H).$$

2. Suppose that

$$tr((f \otimes V)\varphi(Y)) = 0, \quad \forall Y \in B(H).$$

Then we use the arguments of least proof 1) we get

$$Re(\langle \varphi(Y)f, Vf\rangle) \geq 0, \quad \forall Y \in B(H),$$

which completes the proof of the second part of the theorem.

■

Let $\varphi = \delta_{A,B}$.

$$\delta_{A,B} : B(H) \to B(H),$$

is the generalized derivation defined by

$$\delta_{A,B}(X) = AX - XB. \tag{4.23}$$

Corollary 4.1 *Let $V \in B(H)$ be a smooth point, and f is the unitary vector in which $V \in B(H)$ attains its norm, then F_ψ has a global minimum at $V \in B(H)$,if and only if*

$$f \otimes \psi(V)f \in Ker\delta_{B,A}. \tag{4.24}$$

Proof. It is easily seen that

$$f \otimes \psi(V)f \in Ker\delta_{A,B} \Leftrightarrow tr((f \otimes V)\delta_{A,B}) = 0.$$

■

Theorem 4.4 *Let $S \in B(H)$ be a smooth point, then*

$$\| S+(AX-XB) \|_{B(H)} \geq \| \psi(S) \|_{B(H)}, \qquad \forall X \in B(H), \tag{4.25}$$

if and only if $\exists f \in \Gamma, \| f \| = 1$, *such that*

$$f \otimes \psi(S) f \in Ker\delta_{B,A} \tag{4.26}$$

Proof. This theorem is a particular case of the previous one. its proof is trivial. ■

Corollary 4.2 *Let $V \in B(H)$ be a smooth point, f is the unitary vector in which $V \in B(H)$ attains its norm, if $S \in Ker\delta_{A,B}$, then the following assertions are equivalent*

1. $$\| S+(AX-XB) \|_{B(H)} \geq \| \psi(S) \|_{B(H)}, \quad \forall X \in B(H). \tag{4.27}$$

2. $$f \otimes \psi(S) f \in Ker\delta_{B,A} \tag{4.28}$$

Remarque 4.2 *We point out that, thanks to our general results given previously with more general linear maps ψ.Theorem 5.4 and its Corollary 4.1 are still true for more general classes of operators than $\delta_{A,B}$ such as the elementary operators $E(X)$ and $\widetilde{E}(X)$. Note that Theorem 4.4 and Corollary 4.2 generalize the results given in [7].*
us some applications of the previous corollary let $\Delta_{A,B}$ the elementary operator defined by

$$\Delta_{A,B} : B(H) \rightarrow B(H)$$

$$\Delta_{A,B} = AXB - X. \tag{4.29}$$

Theorem 4.5 *Let $S \in B(H)$ be a smooth point, and $A, B \in B(H)$, are contractions, that verify*

$$\Delta_{A,B}(S) = 0. \tag{4.30}$$

Then $\exists \widetilde{S} \in B(H)$, verify

$$\Delta_{A,B}(\widetilde{S}) = 0 = \Delta_{A^*,B^*}(\widetilde{S}). \tag{4.31}$$

Proof. if $A, B \in B(H)$, are contractions, that verify

$$\Delta_{A,B}(S) = 0.$$

■

$$\|\Delta_{A,B}(\widetilde{S}) + S\|_{B(H)} \geq \|S\|_{B(H)}, \quad \forall \widetilde{S} \in B(H). \tag{4.32}$$

Suppose that $S \in B(H)$ be a smooth point, we use the Corollary 4.2 applied to $\Delta_{A,B}$, $\exists f \in H$, hold's

$$\Delta_{A,B}(S)(f \otimes Sf) = 0 = \Delta_{A^*,B^*}(f \otimes Sf), \tag{4.33}$$

taking

$$\widetilde{S} = f \otimes Sf. \tag{4.34}$$

completes the proof. On the other hand, we applied the Corollary 4.2 we obtain

$$\Delta_{A,B}(\widetilde{S}) = 0 = \Delta_{A,B}(s) \Leftrightarrow \|\Delta_{A,B}(\widetilde{S}) + S\|_{B(H)} \geq \|S\|_{B(H)}, \quad \forall S \in B(H). \tag{4.35}$$

Now we will present an other characterization of the orthogonality in the sense of Birkhoff

Theorem 4.6 *Let $S \in B(H)$ be a smooth point, and $Y \in B(H)$, then the following assertions are equivalent.*

1. *The map F_ψ has a global minimum at $S \in B(H)$.*
2. *There exists unitary vector $f \in \Gamma$, such that*

$$Re(<(\varphi(Y)f.Sf)>) \geq 0. \tag{4.36}$$

3. *There exists unitary vector $f \in \Gamma$, such that*

$$tr((f \otimes Sf)\varphi(Y)) = 0, \quad \forall Y \in B(H). \tag{4.37}$$

4. *there exists unitary vectors $f_n \in \Gamma$, such that*

$$\| Sf_n \|_{B(H)} \underset{n \to +\infty}{\to} \| S \|_{B(H)} \tag{4.38}$$

and

$$\langle \varphi(Y)f_n, Sf_n \rangle \underset{n \to +\infty}{\to} 0 \tag{4.39}$$

Proof.

$$1) \to 2)$$

We use the Theorem 4.2 and Theorem 4.3

$$2) \to 3)$$

see Theorem 4.3

$$4) \to 1)$$

Its easily to see that $X \perp Y \in B(H)$, in the sense of Birkhoff if and only if

$$D_X(Y) \geq 0, \quad \forall Y \in B(H).$$

■We prove that

$$\| S+\lambda\varphi(Y) \|_{B(H)} \geq \| S \|_{B(H)}, \quad \forall \lambda \in \mathbb{C}. \tag{4.40}$$

There exists a sequence of unitary vectors $f_n \in \Gamma$, such that

$$\| Sf_n \|_{B(H)} \underset{n\to+\infty}{\to} \| S \|_{B(H)}, \tag{4.41}$$

$$\langle \varphi(Y)f_n, Sf_n \rangle \underset{n\to+\infty}{\to} 0. \tag{4.42}$$

Then

$$\| S+\lambda\varphi(Y) \|^2_{B(H)} \geq \| S+\lambda\varphi(Y)f_n \|^2_{B(H)} \tag{4.43}$$

$$\geq \| Sf_n \|^2_{B(H)} + 2Re\lambda < \varphi(Y)f_n.Sf_n > + \| \varphi(Y)f_n \|^2_{B(H)}$$

$$\geq \| Sf_n \|^2_{B(H)} + 2Re(< \varphi(Y)f_n.Sf_n >) \underset{n\to+\infty}{\to} \| S \|^2_{B(H)}. \tag{4.44}$$

$$3) \to 4)$$

On the other hand, in the case of the proof of Theorem 4.7 we obtain unitary vector f such that

$$|< \varphi(Y)f, Sf >| < \delta. \tag{4.45}$$

let $N \in \mathbb{N}^*$,if we take $\delta \to \frac{1}{N}$ we get the result

Corollary 4.3 *Let $\varphi(Y) = \delta_{A,B}(Y) = AY - YB$,and $S, Y \in B(H)$ where S in a smooth point, then the following conditions are equivalent.*

1. *The map $\| S + AY - YB \|^2_{B(H)}$ has a global minimum at $S \in B(H)$*

2. *There exist unitary vector $f \in \Gamma$, such that*

$$Re\langle (AY - YB)f, Sf \rangle \geq 0. \tag{4.46}$$

3. *There exist unitary vector $f \in \Gamma$, such that*

$$tr((f \otimes Sf)(AY - YB) = 0, \quad \forall Y \in B(H), \tag{4.47}$$

4. *There exists a sequence of unitary vectors$f_n \in \Gamma$, such that*

$$\| Sf_n \|_{B(H)} \underset{n\to+\infty}{\to} \| S \|_{B(H)} \tag{4.48}$$

and

$$\langle (AY - YB)f_n, Sf_n \rangle \underset{n\to+\infty}{\to} 0. \tag{4.49}$$

If $S \in Ker\delta_{A,B}$, we obtain the following corollary

Corollary 4.4 *Let $\varphi(Y) = \delta_{A,B}(Y) = AY - YB$ and $S, Y \in B(H)$, where S is a smooth point, then the following assertions are equivalent.*

1.

$$\| S + AY - YB \|_{B(H)}^{2} \geq \| S \|_{B(H)}, \quad \forall S \in Ker\delta_{A,B}. \tag{4.50}$$

2. *There exist unitary vector $f \in \Gamma$, such that*

$$Re\langle (AY - YB)f, Sf \rangle \geq 0. \tag{4.51}$$

3. *There exist unitary vector $f \in \Gamma$, such that*

$$tr((f \otimes Sf)(AY - YB) = 0, \quad \forall Y \in B(H). \tag{4.52}$$

4. *There exists a sequence of unitary vectors$f_n \in \Gamma$, such that*

$$\| Sf_n \|_{B(H)} \underset{n \to +\infty}{\to} \| S \|_{B(H)}, \tag{4.53}$$

and

$$\langle (AY - YB)f_n, Sf_n \rangle \underset{n \to +\infty}{\to} 0. \tag{4.54}$$

If we put

$$\varphi(Y) = Y.$$

Then we obtain the following corollary

Corollary 4.5 *Let $\varphi(Y) = Y$ and $S, Y \in B(H)$, where S is a smooth point, then the following assertions are equivalent.*

1. *$Y \perp S$, in the sense of Birkhoff.*

2. *There exist unitary vector $f \in \Gamma$, such that*

$$Re\langle Yf.Sf \rangle \geq 0. \tag{4.55}$$

3. *There exist unitary vector $f \in \Gamma$, such that*

$$tr((f \otimes Sf)Y) = 0, \quad \forall Y \in B(H). \tag{4.56}$$

4. *There exists a sequence of unitary vectors$f_n \in \Gamma$, such that*

$$\| Sf_n \|_{B(H)} \underset{n \to +\infty}{\to} \| S \|_{B(H)} . \tag{4.57}$$

$$\langle (AY - YB)f_n, Sf_n \rangle \underset{n \to +\infty}{\to} 0. \tag{4.58}$$

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