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Theme

Chaos Synchronization of Fractional-order systems via Adaptive Control Method

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شكر وتقدير

الحمد الله رب العالمين الذي لا يُضِيعُ أجر العاملين، الحمد الله الذي تَوفِيقُنَا به، الحمد الله الذي مَنَّ عليَ وأَتاحَ لي إنجاز هذا البحث، الحمد الله أولا وأخيرا ودائما.

ثم كل الشكر والتقدير والعِرفان للأستاذ الدكتور حناشي فارح المشرف على رسالة تخرجي الذي كان لي نِعْمَ المُعين خلال إنجازي هذه المذكرة إذْ أنه لم يَدَخِر جهدا في مساعدتي فلا تستطيع الكلمات أن توفيه حقه من الشكر.

كما أوجه شكري لأساتذة اللجنة الأخيار القائمين على مناقشة هذا البحث.

إهــــــداء

إلى كل من ضحى من أجل هذه اللحظة

لحظة تقديم هذا العمل .

إلى والديّ أهدي جهد المقل.

ملخص

في هذه الدراسة، نستخدم طريقة التحكم التكيفي لتزامن أنظمة الفوضى ذات الرتبة الكسرية باستخدام نظرية استقرار ليابونوف، معا معاملات غير محددة. الطريقة المقترحة للتزامن تهدف إلى ضمان تزامن نظمان فوضويين من الرتبة الكسرية بشكل تقاربي من خلال نظرية استقرار ليابونوف. المحاكاة عددية تثبت فعالية طريقتنا والنتائج المتحصل عليها.

كلمات مفتاحية:

نظام ديناميكي، فوضى، نظام فوضوي من النظام الكسري، تحكم تكيفي، تزامن، نظرية استقرار ليابونوف.

Résumé

Dans cette étude, nous utilisons la méthode de contrôle adaptative pour synchroniser des systèmes chaotiques d'ordre fractionnaire en utilisant la théorie de stabilité de Lyapunov, lorsque les paramètres du système sont inconnus. L'approche de synchronisation proposée vise à garantir que les états de deux systèmes chaotiques d'ordre fractionnaire deviennent synchronisés asymptotiquement grâce à la théorie de stabilité de Lyapunov. Des simulations numériques valident l'efficacité de notre méthode et les résultats obtenus.
Mots-clés : .
Système dynamique, chaos, système chaotique d'ordre fractionnaire, contrôle adaptatif, synchronisation, théorie de stabilité de Lyapunov.

Abstract

In this research, we employ the Adaptive Control Method to synchronize fractional-order chaotic systems using Lyapunov stability theory, when system parameters are unknown. The proposed synchronization approach aims to ensure that the states of two fractional-order chaotic systems become asymptotically synchronized through Lyapunov stability theory. Numerical simulations validate the effectiveness of our method and the resulting outcomes.
Keywords:
dynamic system, chaos, chaotic fractional-order system, adaptive control, synchronization, Lyapunov stability theory.

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Introduction

Over the last few decades, chaos, a nonlinear phenomenon crucial in various fields, has been widely studied in engineering, sciences, and applied mathematics. It occurs When a deterministic system displays unpredictable trajectories due to sensitivity to initial conditions or parameters [1].

Recently, several fractional-order systems have been discovered to exhibit chaotic behavior. these systems are valuable for secure communication and advanced encryption due to their complex dynamics. synchronization allows two systems to follow the same path, even if they start from different points. In chaotic systems, synchronization mechanisms play a vital role in designing secure communication schemes. This synchronization can happen between two chaotic systems, where one acts as the master(serves as the transmitter) and the other as the slave(acts as the receiver) . The study of chaotic synchronization initially explored by Yamada and Fujisaka in 1983 [2]. various control methods have been developed to control and synchronize fractional-order chaotic systems, including active control, impulsive control and adaptive control.

This study aims to achieve synchronization between two fractional-order chaotic systems by employing the adaptive control method, which relies on Lyapunov stability theory when system parameters are unknown. The proposed method utilizes Lyapunov stability theory of the zero solution of the errors system to asymptotically synchronize the states of the two chaotic systems. Finally, numerical simulation results are presented in Matlab to demonstrate the effectiveness of the proposed method.

The thesis is structured into the following three chapters:

The first chapter aims to introduce fundamental concepts related to dynamical systems and chaos theory, synchronization, and different synchronization types, as well as an exploration of control methods. Additionally, we offer essential definitions and properties of fractional derivatives, along with numerical methods for solving fractional-order systems.

In the second chapter, we present examples about fractional-order chaotic systems in 3D.

Finally, in the last chapter, we present a study on the synchronization between two 3D and 4D fractional-order chaotic systems.

CHAPTER 1

Preliminaries

Contents

1.1 Introduction

In this chapter, we present some foundational concepts regarding dynamical systems and chaos theory, synchronization, and various types of synchronization. alongside an exploration about adaptive control and other types of control. furthermore, we provide essential definitions and properties of fractional derivatives, along with numerical methods for solving fractional-order systems.

1.2 Dynamic systems

Dynamic systems, found in various fields like physics, engineering, biology, economics, and social sciences, describe how things change over time. They can be categorized as either linear or nonlinear, with linear systems having simple input-output relationships, while nonlinear systems are more intricate. developed in the nineteenth century, dynamic systems offer mathematical tools for analysis, its classified into two categories continuous and discrete [3,4]:

1.2.1 Continuous dynamic systems

Definition 1.1 *In continuous-time systems, dynamics involve continuous evolution of states over time, often represented by the following equation format:*

$$y_t^{\cdot} = k\left(y, t\right); \; y \in \mathbb{R}^n, t \in \mathbb{R}^+$$

with $k : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ *denotes the dynamics of the system.*

1.2.2 Discrete dynamic systems

Definition 1.2 *A discrete-time dynamic system is a mathematical model that describes how a system changes over specific time intervals. It treats time as a sequence of discrete points in time, and it's represented by difference equations, which help understand its behavior.*

$$y\left(k+1\right) = G\left(y\left(k\right), k\right)$$

with $y(k) \in \mathbb{R}^n, k \in \mathbb{N}$ and $G : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$. with k being a discrete moment.

1.2.3 Phase space of a dynamic systems

Definition 1.3 *The phase space [5] of a dynamic system is like a map with many dimensions. Each dimension represents a different aspect of the systems behavior, every point on this map shows where the system is at a specific time, while the path it takes shows how it moves over time.*

Definition 1.4 *(**Phase Portrait**) The phase portrait shows all the paths or curves that represent the different possible outcomes figure 1.1 [6].*

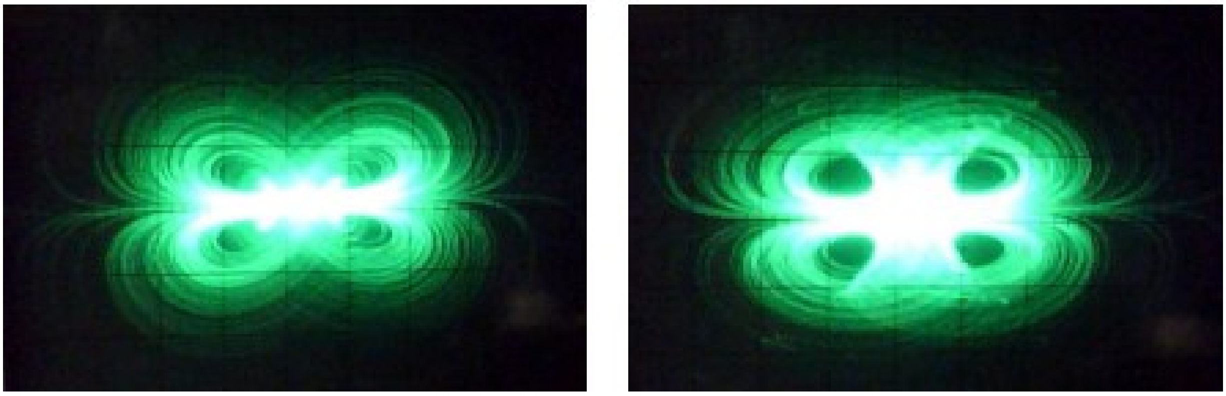

Figure 1.1: Examples about phase Portrait.

1.3 Chaos

The logistic map vividly illustrates a phenomenon that remains only partially understood for most functions: the chaotic behavior of orbits within a dynamical system. Chaos can be defined in various ways, from measure-theoretic concepts of randomness in ergodic theory to the topological approach we will adopt here. before we delve into the definition of chaos, it's essential to establish some preliminary definitions as outlined in [7].

1.3.1 Definition of chaotic system

In dynamic systems, "chaos" denotes a complex behavior that appears to be random or unpredictable. Chaotic systems are highly sensitive to initial conditions, meaning even small changes in the starting conditions can lead to significant differences in outcomes. This renders it extremely challenging, if not impossible, to predict the long-term behavior of chaotic systems.

Definition 1.5 *A dynamic system is considered chaotic if it possesses at least one positive Lyapunov exponent or if it contains at least one chaotic attractor.*

Definition 1.6 *A chaotic attractor is a type of strange attractor found in complex dynamical systems. It exhibits sensitivity to initial conditions, leading to unpredictable behavior. We can observe this phenomenon in the figure 1.2 and figure 1.3 [8]:*

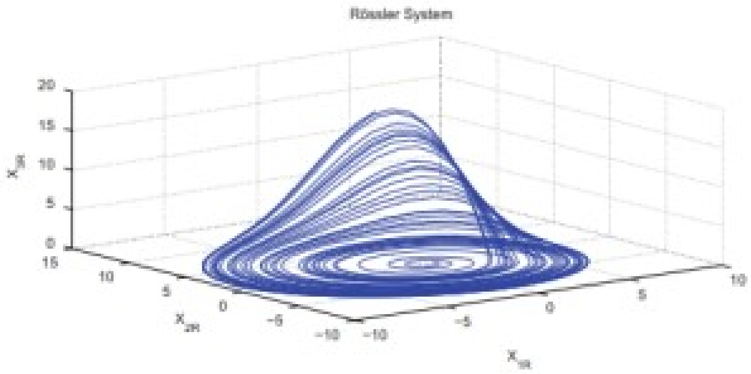

Figure 1.2: Chaotic attractor for the Rössler system.

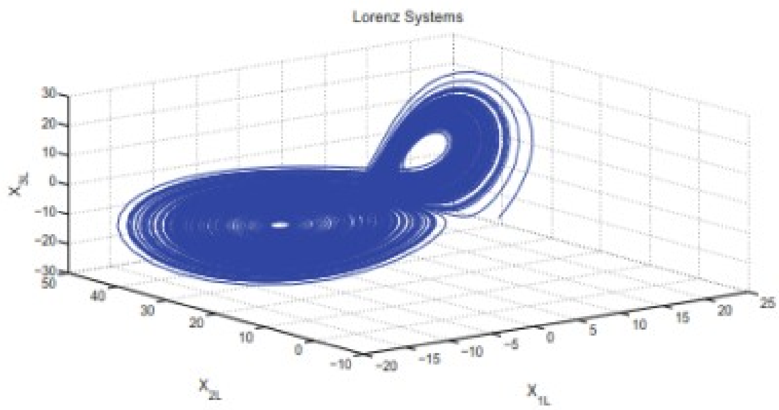

Figure 1.3: Chaotic attractor for the Lorenz system.

1.3.2 Characteristics of chaotic system

Chaotic systems possess several properties that set them apart from other types of dynamical systems:

1- Sensitivity to initial conditions: This refers to the property where small changes in the initial conditions of a system can result in significant differences in its behavior over time.

2-Topological transitivity: refers to a mathematical concept where a point in the phase space possesses an orbit that densely covers the phase space. In this context, the dynamical system is considered topologically transitive, signifying that any point in the phase space can be arbitrarily close to the trajectory of the system it follows.

3-Dense periodic orbits: play a significant role in the study of dynamical systems as they provide a method for approximating the behavior of chaotic systems.

4-Lyapunov exponent: The Lyapunov exponent is a metric used to measure how rapidly nearby trajectories in a dynamical system diverge. Named after the Russian mathematician Alexander Lyapunov, who introduced the concept in the late 19th century, it serves as an indicator of the stability of a dynamical system. A positive Lyapunov exponent signifies instability and chaotic behavior, as nearby trajectories tend to diverge over time. Conversely, a negative Lyapunov exponent indicates stability, as nearby trajectories tend to converge over time. The Lyapunov exponent is commonly employed to study the behavior of chaotic systems, where neighboring trajectories experience sudden and unpredictable divergence.

1.3.3 Chaos application

Chaos theory finds applications in numerous fields. Here are a few examples:

1- Physics: Chaos theory has been utilized to understand the behavior of complex systems such as celestial mechanics, nonlinear optics, and fluid dynamics, providing valuable insights into their intricate dynamics. Additionally, chaos theory has found applications in fields such as weather forecasting, quantum mechanics, and plasma physics.

2-Engineering: Chaos theory has been applied in engineering to optimize the management and operation of complex systems such as power plants, chemical reactors, and communication networks, thereby improving their efficiency and reliability. Additionally, chaos theory has found applications in fields such as robotics, control systems, and structural engineering.

3-Biology: In the realm of biology, chaos theory provides insights into the dynamics of biological systems, offering valuable perspectives on ecological systems, brain networks, and cardiac rhythms. This facilitates a deeper comprehension of their functioning and helps in developing more accurate models and predictions. Additionally, chaos theory has applications in fields such as population dynamics, genetics, and epidemiology.

4-Computer Science: In the realm of computer science, chaos theory guides the creation of algorithms for optimization and data analysis. By using chaotic dynamics, these algorithms improve how computers work and help solve complicated problems. Additionally, chaos theory is useful in areas like cryptography, artificial intelligence, and machine learning, making computer systems even better.

5-Finance: Chaos theory plays a crucial role in analyzing financial markets, enabling the development of models to anticipate market fluctuations and manage risks effectively. This assists investors and financial institutions in decision-making processes, leading to more informed strategies and better outcomes. Additionally, chaos theory has applications in fields such as econometrics, portfolio management, and algorithmic trading.

6-Music and Art: In the creative sphere, chaos theory inspires novel forms of expression in music and art by exploring the interplay between randomness and creativity. This exploration leads to avant-garde compositions in music and visually captivating artworks that challenge traditional notions of form and structure. Additionally, chaos theory provides a framework for understanding the complex dynamics of artistic processes, enriching the creative landscape with innovative ideas and approaches.

1.4 Synchronization

1.4.1 Synchronization methods

This section is dedicated to presenting various methods of synchronization that are highly effective and commonly encountered.

Definition 1.7 *Synchronization between two systems is established when the trajectory of the response system precisely mirrors that of the drive system over time.*

Definition 1.8 *Let* $\dot{w} = W(x;t)$ *be the drive (chaotic, hyperchaotic) system, and* $\dot{x} = F(x;t) + U$ *be the response system, where* $w = (w_1(t),\ w_2(t), ...,\ w_n(t))^T$, $x = (x_1(t),\ x_2(t), ...,\ x_n(t))^T$ *and* $U = (u_1(t),\ u_2(t), ...,\ u_n(t))^T$ *is a controller to be determined later [9,10], [11].*

$\Rightarrow$ *Synchronization is achieved if* $\lim\limits_{t\to+\infty} ||e|| = 0$, $e \in \mathbb{R}^n$ *with* $e = x - w$.

$\Rightarrow$ *Anti-synchronization is occur if* $\lim\limits_{t\to+\infty} ||e|| = 0$, $e \in \mathbb{R}^n$ *with* $e = x + w$.

$\Rightarrow$ *Function projective synchronization is achieved if* $\lim\limits_{t\to+\infty} ||e|| = 0$, $e \in \mathbb{R}^n$ *with* $e = x - h(w)w$, $h(w) = (h_1(w_1),\ h_2(w_2), ...,\ h_n(w_n))$.

$\Rightarrow$ *Inverse function projective synchronization is achieved if* $\lim\limits_{t\to+\infty} ||e|| = 0$, $e \in \mathbb{R}^n$ *with* $e = x + h(x)x$, $h(x) = (h_1(x_1),\ h_2(x_2), ...,\ h_n(x_n))$ *where* h *is a scaling function matrix.*

1.4.2 Control methods

Adaptive control method

When the chaotic system's parameters are unknown, this method is employed. for instance, Wu et al [12]. demonstrated how adaptive controllers can be used to simulate synchronizing two Chua's oscillators by adjusting their settings. On the other hand, Liao created an adaptive control law [13] for synchronizing two Lorenz systems based on the Lyapunov stability theory; the outcomes of the simulations supported the suggested methodology. Behinfaraz et al [14]. created a new fractional-order chaotic system in where they used the Lyapunov stability theory to design adaptive controllers based on the parameters adaptation laws, and they verified the controllers functionality through numerical examples. Wang et al [15]. suggested a nonlinear adaptive system based on the Lyapunov theory to guarantee the synchronization of two Hindmarsh-Rose neuron models. The controllers effectiveness and feasibility were confirmed by the simulation results.

Afterward, the adaptive control involves examining the dynamics of both the drive system (master) and the controlled system (slave).

$$\begin{aligned} D^q X_1 &= aF_1(X_1) \\ D^q X_2 &= aF_2(X_2) + U \end{aligned} \tag{1.1}$$

Where $X_1 \in \mathbb{R}^m$ and $X_2 \in \mathbb{R}^m$ are the states, of the master and slave systems, respectively, q is the fractional order derivatives, and a is the parameter of the systems, $F_1, F_2 : \mathbb{R}^m \to \mathbb{R}^m$and $U = (u_j)_{1\leq j\leq m}$ is a controller to be determined. For the two systems to be synchronized, So the state error is: $e = X_2 - X_1$. Then, the error dynamics can be obtained:

$$D^q e = a(F_2(X_2) - F_1(X_1)) + U \tag{1.2}$$

Let:

$$e_a = a - \hat{a} \tag{1.3}$$

The adaptive controllers U for the synchronization of the proposed system is taken in the following equations:

$$U = -\hat{a}(F_2(X_2) - F_1(X_1)) - ke \tag{1.4}$$

k are strictly positive constant.

Then, the error dynamics can be expressed as follows :

$$D^q e = e_a(F_2(X_2) - F_1(X_1)) - ke \tag{1.5}$$

Definition 1.9 *A Lyapunov function is a scalar function that characterizes the stability of a dynamical system. It assesses whether nearby trajectories remain close to an equilibrium point indefinitely [16].*

If the Lyapunov function is selected as :

$$V(e,\ e_a) = \frac{1}{2}\left(e^2 + e_a^2\right) \tag{1.6}$$

Therefore , the derivative of the function :

$$D^q V = eD^q e + e_a D^q e_a \tag{1.7}$$

We select a control law and an update law to get $D^q V < 0$. As result, the derivative of the Lyapunov is negative. thus, the zero solution of the errors system is asymptotic stable, under any initial conditions, the synchronization state errors exponentially converge to zero over time. Furthermore, the estimated slave parameter exponentially aligns with the equivalent master parameter. Consequently, the asymptotic global stability of the established synchronization technique is ensured.

Active control method:

The active control technique uses adaptive control to achieve a system's desired performance. It involves designing a controller that adjusts system inputs to reach the desired output. This technique manipulates the input signal $(u(t))$ to obtain the required output $(x_2(t))$. The control law is typically expressed as $(u(t) = F_1(x_1(t), x_2(t)))$, where $(x_1(t))$ represents the state vector, $(x_2(t))$ represents the output vector, and (F_1) is a function mapping these signals to the input signal $(u(t))$ [17].

1.5 Fractional calculus

Over time, mathematicians have developed various definitions to describe non-integer order integrals or derivatives, each with its own notation and approach. one widely recognized definition in the field of fractional calculus is the Riemann-Liouville definition.

1.5.1 Specific functions related to fractional calculation

In this section, we will explore the Gamma, Beta and Mittag-Leffler functions, which are fundamental tools in fractional calculus theory.

Definition 1.10 *The Gamma function can be understood as a generalization of the factorial to all real numbers [18], Its definition is given by:*

$$\Gamma(\eta) = \int_0^{\infty} t^{\eta-1} e^{-t} dt, \quad \mathrm{Re}(\eta) \in \mathbb{R}^+.$$

Definition 1.11 *Similar to the Gamma function, the Beta function is also defined by a definite integral [18]. Its definition is as follows:*

$$\beta\left(\eta,\ \alpha\right)=\int_{0}^{1}t^{\eta-1}\left(1-t\right)^{\alpha-1}dt,\ \ \forall\eta,\ \alpha\in\mathbb{R}^{+}.$$

The Beta function can also be expressed using the Gamma function:

$$\beta\left(\eta,\ \alpha\right)=\frac{\Gamma\left(\eta\right)\Gamma\left(\alpha\right)}{\Gamma\left(\eta+\alpha\right)}$$

Definition 1.12 $(The\ Mittag-Leffler\ function)$ *Mittag-Leffler introduced a function that serves as a single-parameter generalization of the exponential function [19], as demonstrated by the following function [20]:*

$$E_{q}\left(\eta\right)=\sum_{k=0}^{\infty}\frac{\eta^{k}}{\Gamma\left(qk+1\right)},\ \ \eta\in\mathbb{R},\ q>0.$$

1.5.2 Fractional Derivatives and Integrals

Definition 1.13 *The Riemann-Liouville derivative of fractional order η of function $h\left(t\right)$, is defined as [18]:*

$$\begin{aligned}{}_{a}^{R}D_{t}^{\eta}h\left(t\right)&=\frac{1}{\Gamma\left(n-\eta\right)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}h\left(\tau\right)\left(t-\tau\right)^{n-\eta-1}d\tau\\&=\frac{d^{n}}{dt^{n}}\left[I^{n-\eta}h\left(t\right)\right],\quad m-1\leq\eta<m.\end{aligned}$$

Definition 1.14 *The Caputo fractional derivative of $h\left(t\right)$ is defined as [18]:*

$${}_{a}^{C}D_{t}^{\eta}h\left(t\right)=\frac{1}{\Gamma\left(n-\eta\right)}\int_{a}^{t}\frac{h^{(n)}\left(\tau\right)}{\left(t-\tau\right)^{\eta-n+1}}d\tau,\ \ \textit{where}\ m-1\leq\eta<m,\ m\in\mathbb{N}^{*}.$$

1.5.3 Laplace transformation

Definition 1.15 *if we have a function g in the space $L^{1}\left(0,\ 1\right)$, we can define a function $G(s)$ in terms of the complex variables using the Laplace transform:*

$$G\left(s\right)=L\left\{g\left(t\right),\ s\right\}=\int_{0}^{+\infty}e^{-st}g\left(t\right)dt \tag{1.8}$$

In this equation, $g(t)$ represents the original function of $G(s)$, and G is often referred to as the Laplace transform of $g(t)$.

A sufficient condition for the existence of the integral (1.8) is that the function $g(t)$ must have exponential order c, which means there exists $N, T > 0$ for that $|g\left(t\right)| \leq Ne^{ct}$ for $t > T, T \in \mathbb{R}_{+}^{*}$.

For example, the Laplace transform formula for the Caputo fractional derivative can be expressed as follows:

$$L\left\{{}_{a}D_{t}^{\eta}g\left(t\right),\ s\right\} = s^{\eta}G\left(s\right) - \sum_{k=0}^{n-1} s^{\eta-k-1}g^{(n)}\left(0\right),\ n-1 < \eta < n,\ \forall\eta > 0. \tag{1.9}$$

1.5.4 Stability of Fractional Order Systems

In stability theory [21], when dealing with linear systems possessing invariant time and derivatives of integer order, stability is determined by the roots of the polynomial characteristic. For such systems, stability is assured if these roots exhibit strictly negative real parts, indicating their placement in the left half of the complex plane. However, when it comes to linear fractional systems with time in-variance, stability criteria differ from those of integer-order systems. These fractional systems can exhibit stable behavior even if their roots are located in the right half of the complex plane. Notably, the renowned stability criterion introduced by Matignon addresses this aspect, particularly concerning commensurable non-integer systems of order η, where η ranges between 0 and 2. This criterion is rooted in the analysis of poles. Let's consider the following differential system:

$$^{c}D^{\eta}x\left(t\right) = f\left(t,\ x\left(t\right)\right);$$

Which $^{c}D^{\eta}$ is the derivative of Caputo, where $0 < \eta < 1$, and $x = \left(x_{1}, \ldots\ldots, x_{n}\right)^{T} \in \mathbb{R}^{n}$; $f : \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ a continuous function.

1.5.5 Numerical method for solving fractional differential equation

Numerical techniques employed for solving ordinary differential equations (ODEs) need adjustments to tackle fractional differential equations (FDEs). One such adjustment is the modification of the Adams-Bashforth-Moulton algorithm proposed by Diethelm for solving

FDEs. $m-1<\eta<m$ the initial value problem (IVP) [22].

$$D^{\eta}x\left(t\right)=f\left(t,\ x\left(t\right)\right),\ 0\leq t\leq T. \tag{1.10}$$

$$x^{(k)}\left(0\right)=x_{0}^{(k)},\ k=0,\ 1,\dots,\ m-1 \tag{1.11}$$

The IVP (1.10) and (1.11) is equivalent to the Volterra integral equation:

$$x\left(t\right)=\sum_{k=0}^{m-1}x_{0}^{(k)}\frac{t^{k}}{k!}+\frac{1}{\Gamma\left(\eta\right)}\int_{a}^{t}f\left(\tau,\ x\left(\tau\right)\right)\left(t-\tau\right)^{\eta-1}d\tau$$

Consider the uniform grid $\{t_{n}=nh,\ n=0,\ 1,\dots,\ N\}$ for some integer N and $h=T/N$ let $x_{h}\left(t_{n}\right)$ be approximation to $x\left(t_{n}\right)$. Assume that we have already calculated approximations $x_{h}\left(t_{j}\right)$, $j=1,2,\dots,n$ and we want to obtain $x_{h}\left(t_{n+1}\right)$ by means of the equation.

$$x_{h}\left(t_{n+1}\right)=\sum_{k=0}^{m-1}x_{0}^{(k)}\frac{t_{n+1}^{k}}{k!}+\frac{h^{\eta}}{\Gamma\left(\eta+2\right)}f\left(t_{n+1},\ x_{h}^{p}\left(t_{n+1}\right)\right)+\frac{h^{\eta}}{\Gamma\left(\eta+2\right)}\sum_{j=0}^{n}b_{j,\,n+1}\,f\left(t_{j},\ x_{n}\left(t_{j}\right)\right), \tag{1.12}$$

Which:

$$b_{j,\,n+1}=\begin{cases}n^{\eta+1}-\left(n-\eta\right)\left(n+1\right)^{\eta}, & \text{if } j=0\\ \left(n-j+2\right)^{\eta+1}+\left(n-j\right)^{\eta+1}-2\left(n-j+1\right)^{\eta+1}, & \text{if } 0\leq j\leq n\\ 1, & \text{if } j=n+1\end{cases} \tag{1.13}$$

The preliminary approximation $x_{h}^{p}\left(t_{n+1}\right)$ is called predictor and is given by:

$$x_{h}^{p}\left(t_{n+1}\right)=\sum_{k=0}^{m-1}x_{0}^{(k)}\frac{t_{n+1}^{k}}{k!}+\frac{1}{\Gamma\left(\eta\right)}\sum_{j=0}^{n}b_{j,\,n+1}\,f\left(t_{j},\ x_{n}\left(t_{j}\right)\right)$$

Which:

$$b_{j,\,n+1}=\frac{h^{\eta}}{\eta}\left(\left(n-j+1\right)^{\eta}-\left(n-j\right)^{\eta}\right).$$

And there for this method error is:

$$\max_{j=0,\,1,\dots,\,N}\left|x\left(t_{j}\right)-x_{h}\left(t_{j}\right)\right|=O\left(h^{p}\right) \tag{1.14}$$

where $p=\min\left(2,\ 1+\eta\right)$.

1.6 Conclusion

This chapter covers the basics of dynamical systems and chaos theory, synchronization, and different synchronization types. It also explores about adaptive control and active control theory, and presents essential definitions and properties of fractional derivatives, along with numerical methods for solving fractional-order systems.

CHAPTER 2

Examples of fractional-order chaotic systems

Contents

2.1 Introduction

In this chapter, we present examples about fractional-order chaotic systems in 3D.

2.2 Fractional liu system

The fractional-order Liu system [23], is described as follows :

$$\begin{cases} D^{q_1}x_1 = -ax_1 + \alpha y_1 \\ D^{q_2}y_1 = bx_1 - \beta x_1 z_1 \\ D^{q_3}z_1 = -cz_1 + \gamma x_1^2 \end{cases} \tag{2.1}$$

which, x_1, y_1, and z_1 are the state variables, and the fractional order derivatives $q_i = 0.95$: $0 < q_i < 1$, $i = \overline{1,3}$, and the parameters are a, α, b, β, c and γ.

The system displays chaotic behavior for this parameter values $a = 10$, $\alpha = 1$, $b = 40$, $\beta = 1$, $c = 2.5$ and $\gamma = 4$. With the following initial condition $(x_1(0)\ , y_1(0)\ , z_1(0)) = (0.2\ , 0\ , 0.5)$ with $h = 0.005$ and $TSim = 100s$, a two scroll attractor existsand, and the numerical simulation of the liu fractional chaotic system is given in Figures 2.1, 2.2, 2.3 and 2.4.

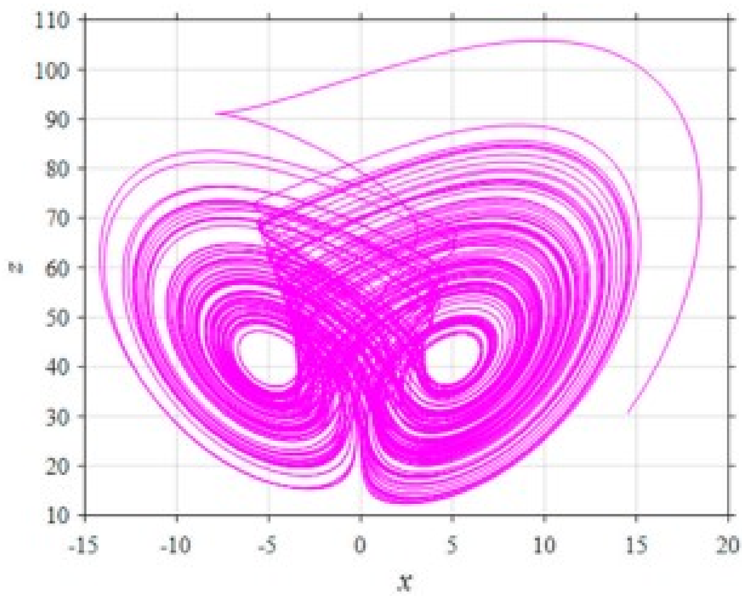

Figure 2.1: Plane of chaotic attractor of system (2.1) between x_1 and z_1.

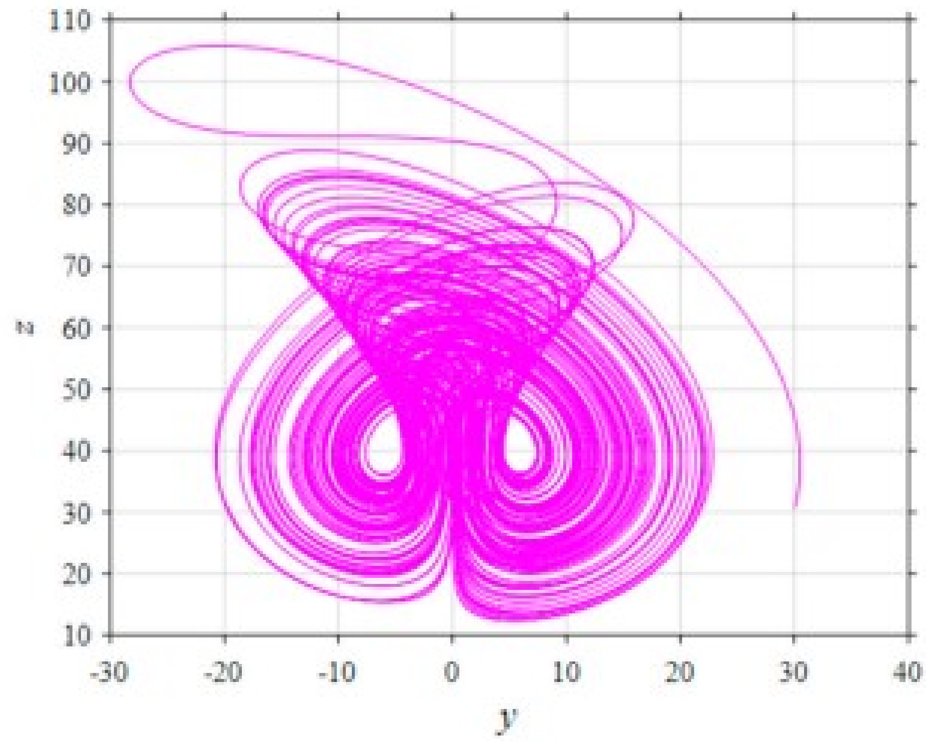

Figure 2.2: Plane of chaotic attractor of system (2.1) between y_1 and z_1.

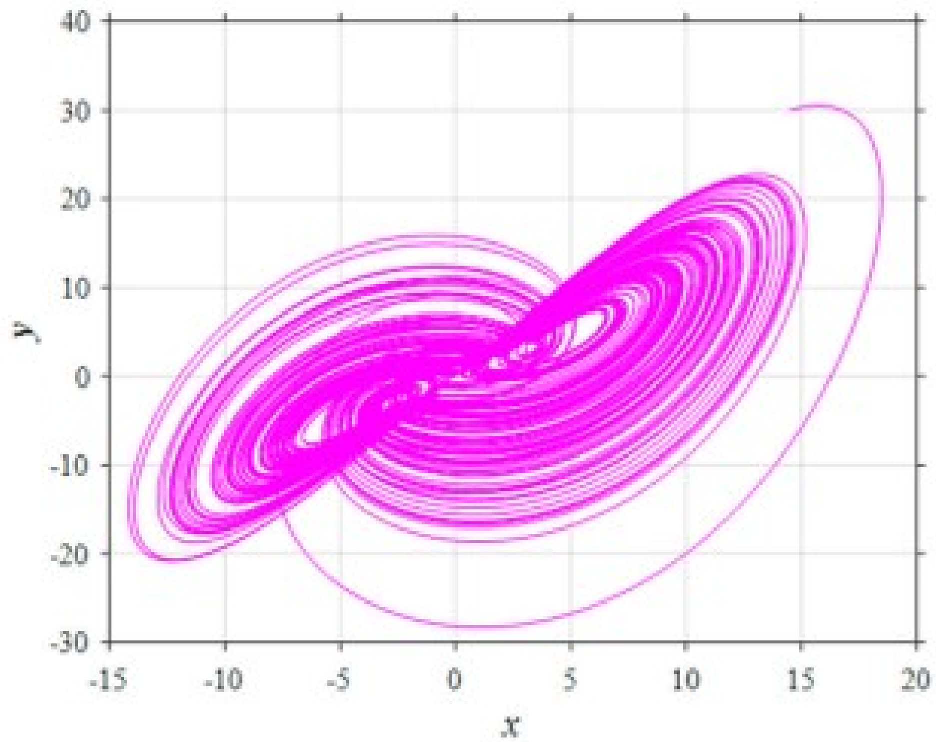

Figure 2.3: Plane of chaotic attractor of system (2.1) between x_1 and y_1.

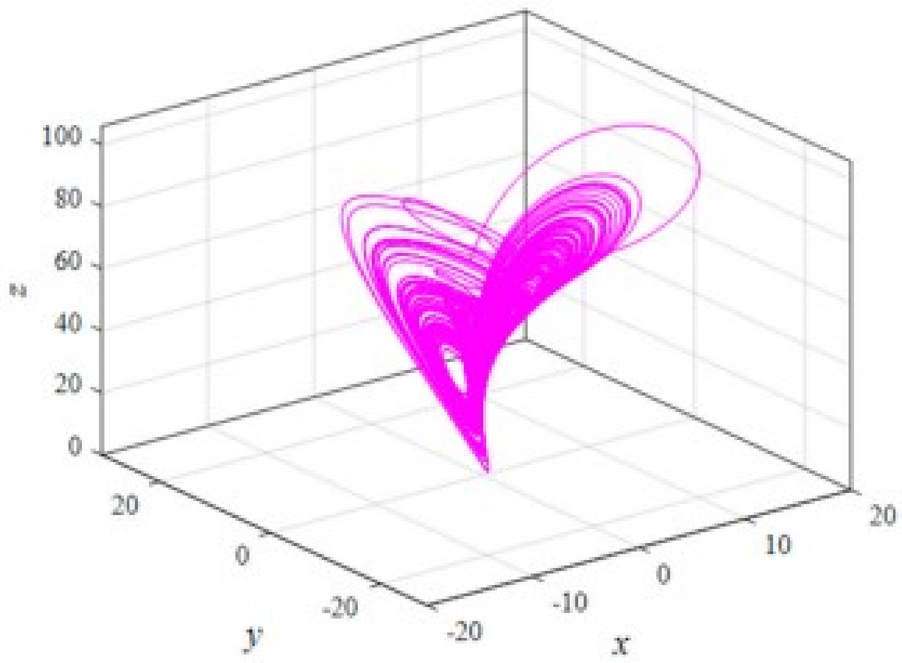

Figure 2.4: 3D view of the chaotic attractor for the system (2.1) in $(x_1,\ y_1,\ z_1)$ space.

2.3 Fractional Chen system

In 1999, Chen and Ueta introduced the Chen system [24], similar to the Lorenz system but not topologically equivalent to it. The Chen system, a chaotic system, exhibits a double-scroll attractor. The fractional-order chen system [25] and [26] , is described as follows :

$$\begin{cases} D^{q_1} x_2 = -a\left(y_2 - x_2\right) \\ D^{q_2} y_2 = c y_2 - x_2 z_2 + (c-a)\, x_2 \\ D^{q_3} z_2 = x_2 y_2 - b z_2 \end{cases} \tag{2.2}$$

where, x_2, y_2, and z_2 are the state variables, and the fractional order derivatives $q_i = 0.9$: $0 < q_i < 1$, $i = \overline{1,3}$, and the parameters are a, b and c.

The system displays chaotic attractor for this parameter values $a = 35$, $b = 3$ and $c = 28$. With the following initial condition $(x_2\,(0)\ ,y_2\,(0)\ ,z_2\,(0)) = (-9\,,\,-5\,,14)$ with $h = 0.005$ and $TSim = 100s$, and the numerical simulation of the Chen fractional chaotic system is given in the Figures 2.5, 2.6, 2.7 and 2.8.

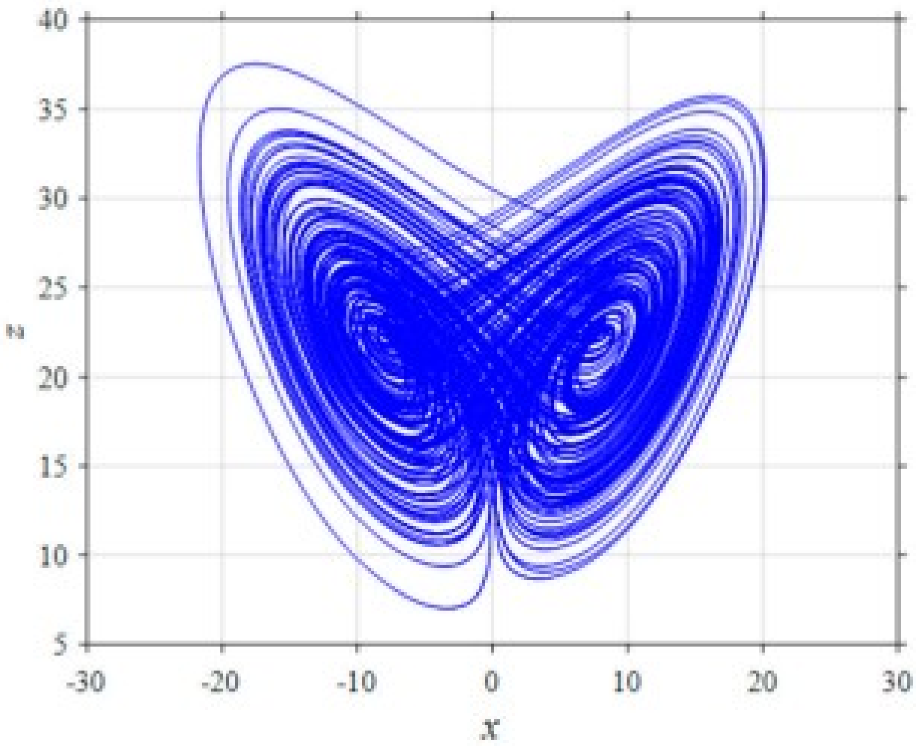

Figure 2.5: Plane of chaotic attractor of system (2.2) between x_2 and z_2.

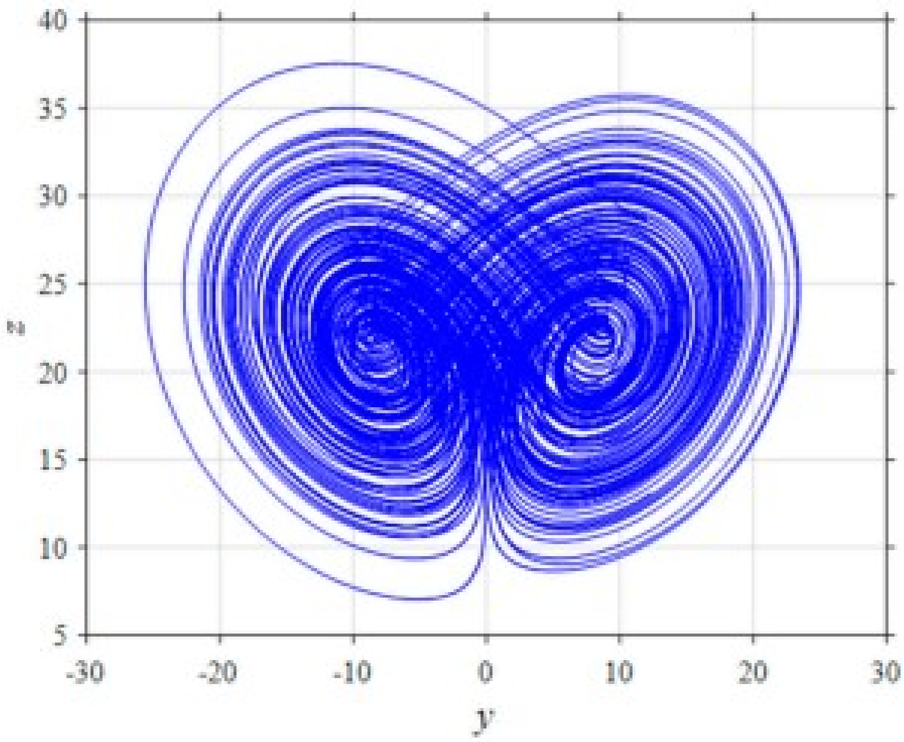

Figure 2.6: Plane of chaotic attractor of system (2.2) between y_2 and z_2.

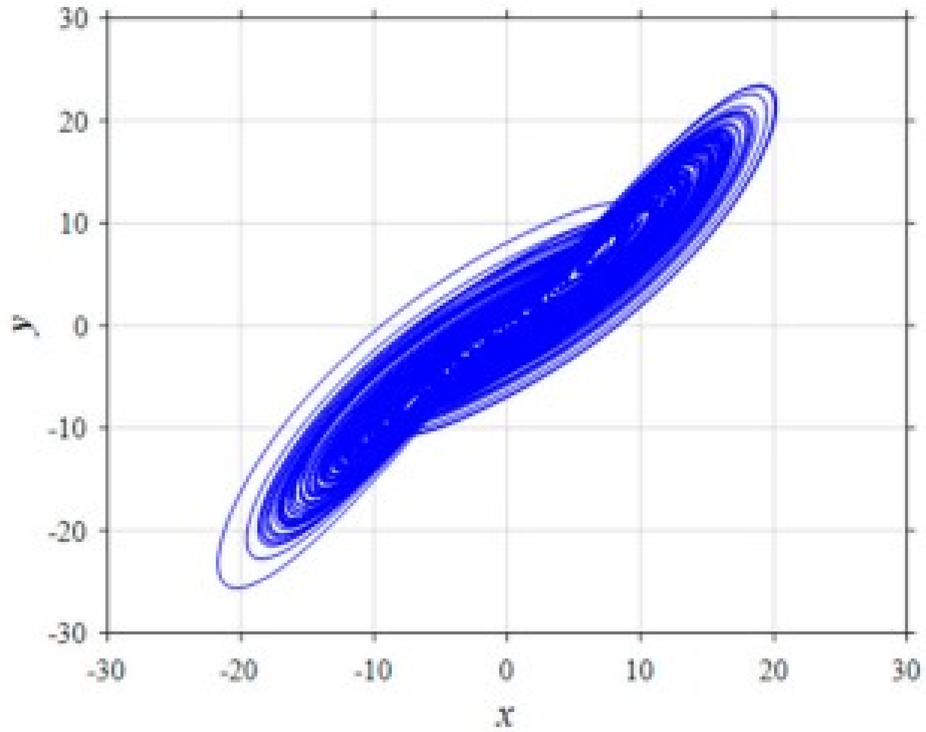

Figure 2.7: Plane of chaotic attractor of system (2.2) between x_2 and y_2.

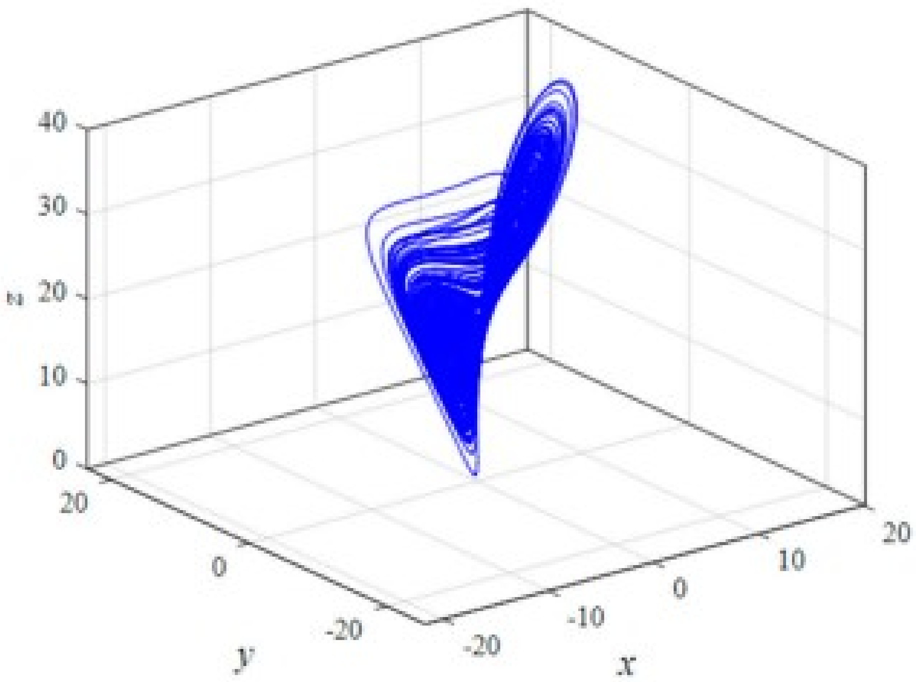

Figure 2.8: 3D view of the chaotic attractor for the system (2.2) in $(x_2,\ y_2,\ z_2)$ space.

2.4 Fractional Shimizu-Morioka system

The Shimizu-Morioka system [27] can be written in fractional-order as follows :

$$\begin{cases} D^{q_1} x_3 = c y_3 \\ D^{q_2} y_3 = (1 - z_3)\, x_3 - b y_3 \\ D^{q_3} z_3 = x_3^2 - a z_3 \end{cases} \tag{2.3}$$

where, x_3, y_3, and z_3 are the state variables, and the fractional order derivatives $q_i = 0.9 : 0 < q_i < 1$, $i = \overline{1,3}$, and the parameters are a, b and c.

The system displays chaotic attractor for this parameter values $a = 0.375$, $b = 0.8$ and $c = 1$. With the following initial condition $(x_3\,(0)\,, y_3\,(0)\,, z_3\,(0)) = (0.1, 0.1, 0.1)$ with $h = 0.005$ and $TSim = 100s$, and the numerical simulation of the Shimizu-Morioka fractional chaotic system is given in the Figures 2.9, 2.10, 2.11 and 2.12.

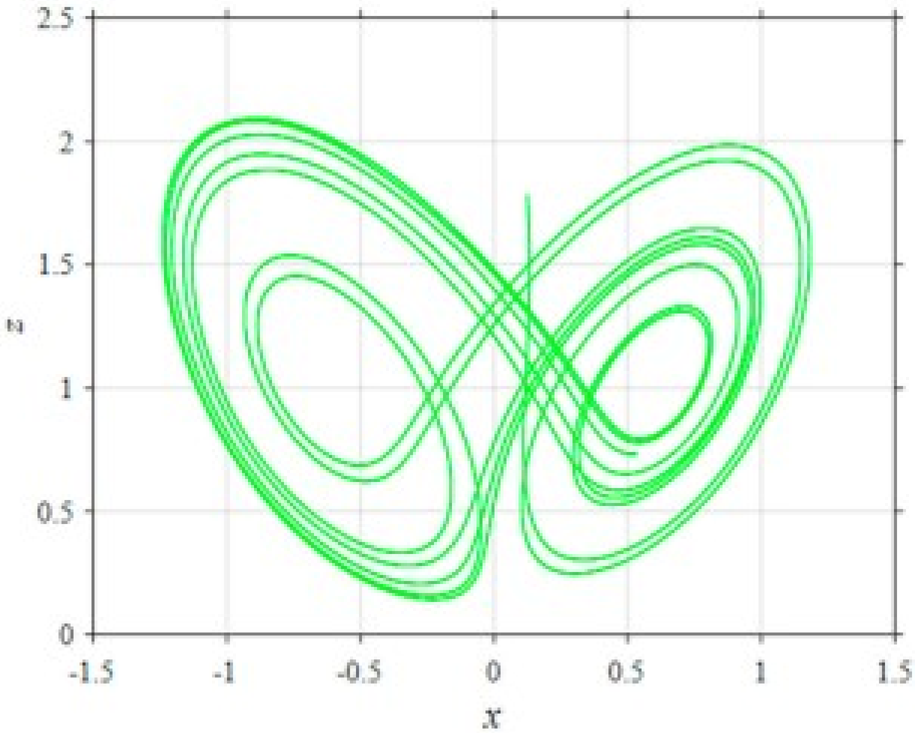

Figure 2.9: Plane of chaotic attractor of system (2.3) between x_3 and z_3.

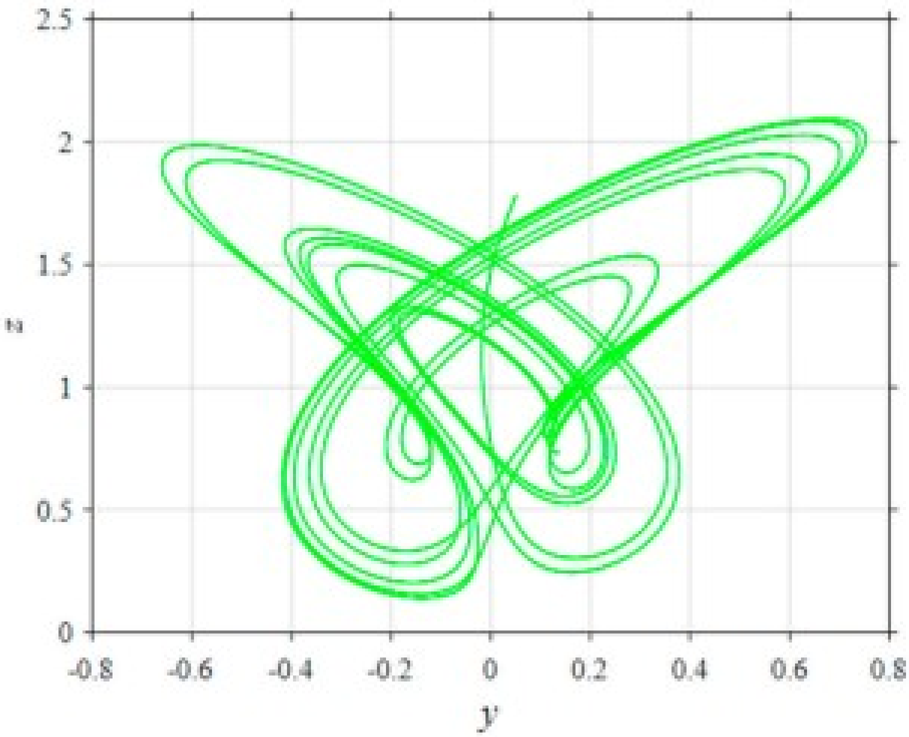

Figure 2.10: Plane of chaotic attractor of system (2.3) between y_3 and z_3.

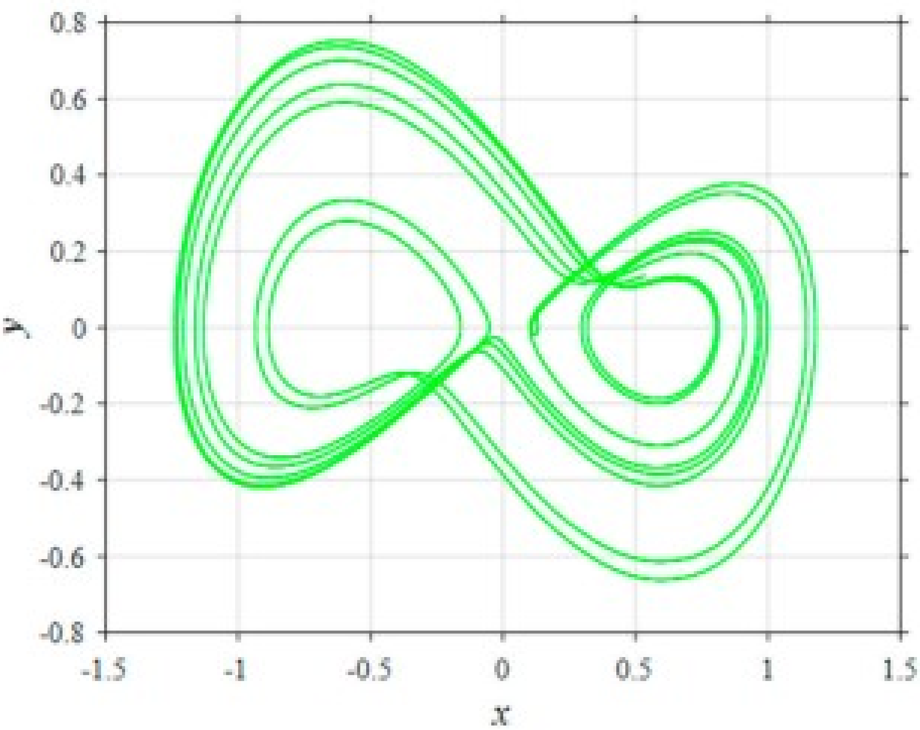

Figure 2.11: Plane of chaotic attractor of system (2.3) between x_3 and y_3.

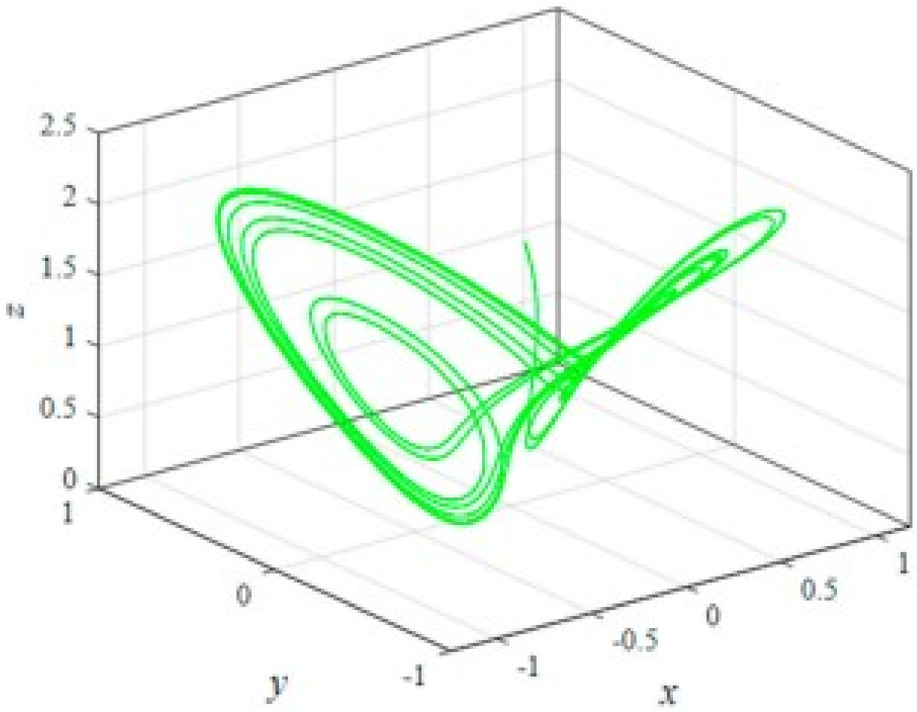

Figure 2.12: 3D view of the chaotic attractor for the system (2.3) in $(x_3,\ y_3,\ z_3)$ space.

2.5 Fractional Burke-Shaw system

The Burke-Shaw system [28]can be written in fractional-order as follows :

$$\begin{cases} D^{q_1}x_3 = \alpha\left(x_4 + y_4\right) \\ D^{q_2}y_3 = -\gamma x_4 z_4 - y_4 \\ D^{q_3}z_3 = \beta x_4 y_4 + k \end{cases} \tag{2.4}$$

which, x_4, y_4 and z_4 are the state variables, and the fractional order derivatives $q_i = 0.97$: $0 < q_i < 1$, $i = \overline{1,3}$, and the parameters are α, β, γ and k.

The system displays chaotic attractor for this parameter values $\alpha = \beta = \gamma = 10$ and $k = 13$. With the following initial condition $x_3\left(0\right) = y_3\left(0\right) = z_3\left(0\right) = 0.1$ with $h = 0.005$ and $TSim = 100s$, and the numerical simulation of the Burke-Shaw fractional chaotic system is given in the Figures 2.13, 2.14, 2.15 and 2.16.

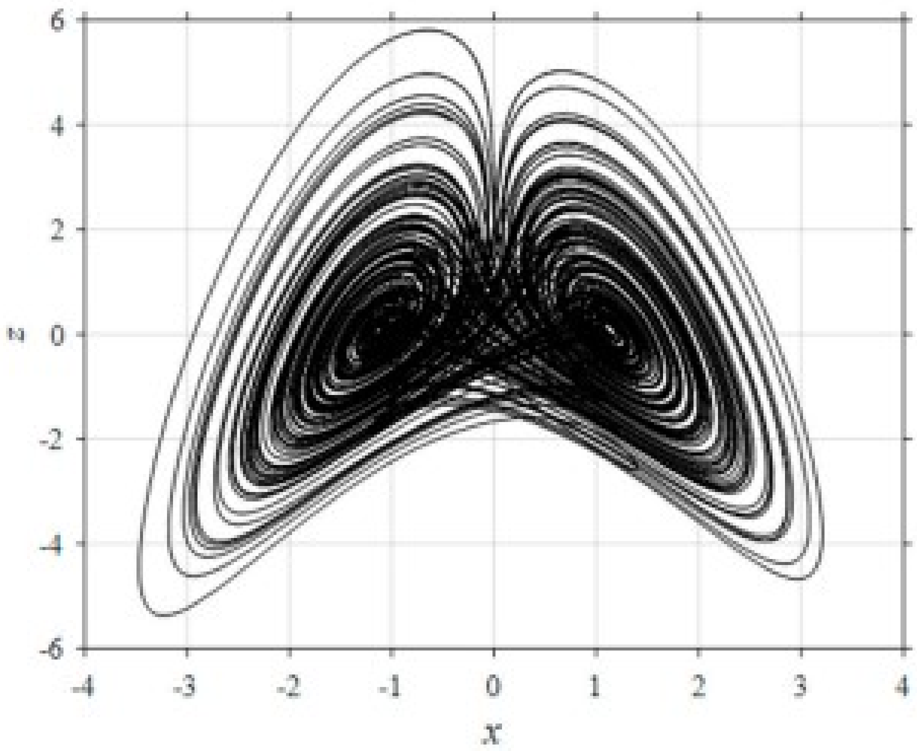

Figure 2.13: Plane of chaotic attractor of system (2.4) between x_3 and z_3.

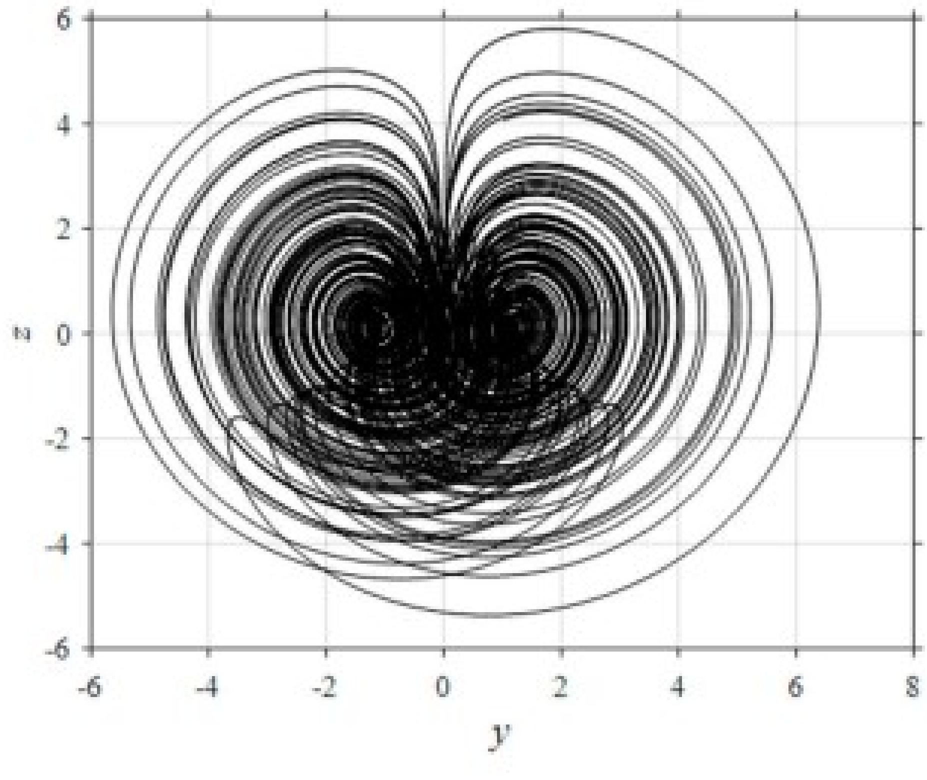

Figure 2.14: Plane of chaotic attractor of system (2.4) between y_3 and z_3.

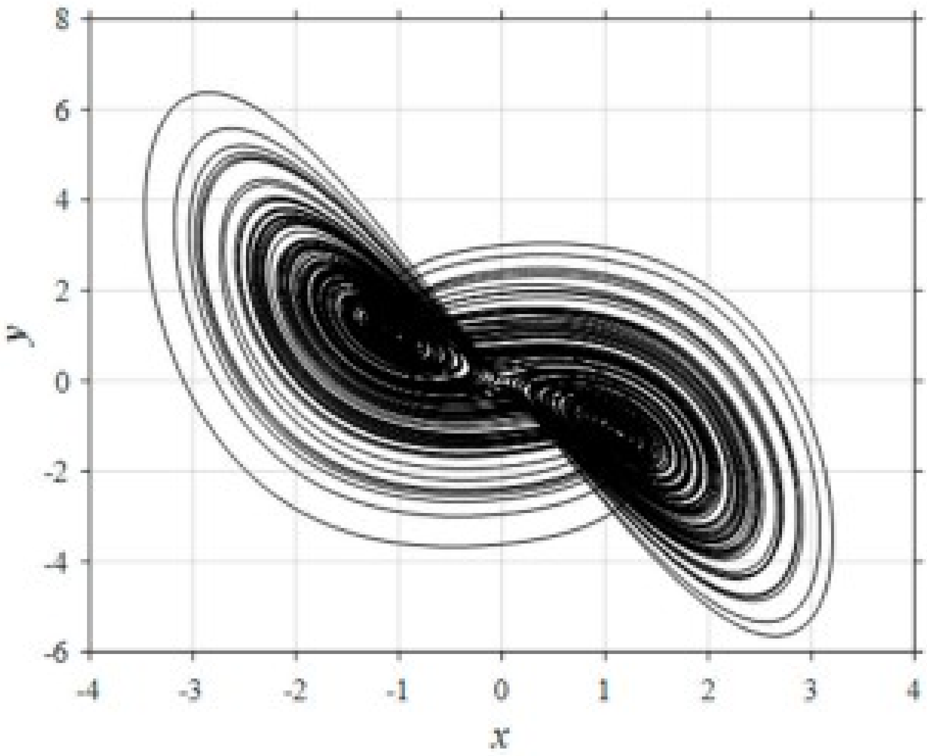

Figure 2.15: Plane of chaotic attractor of system (2.4) between x_3 and y_3.

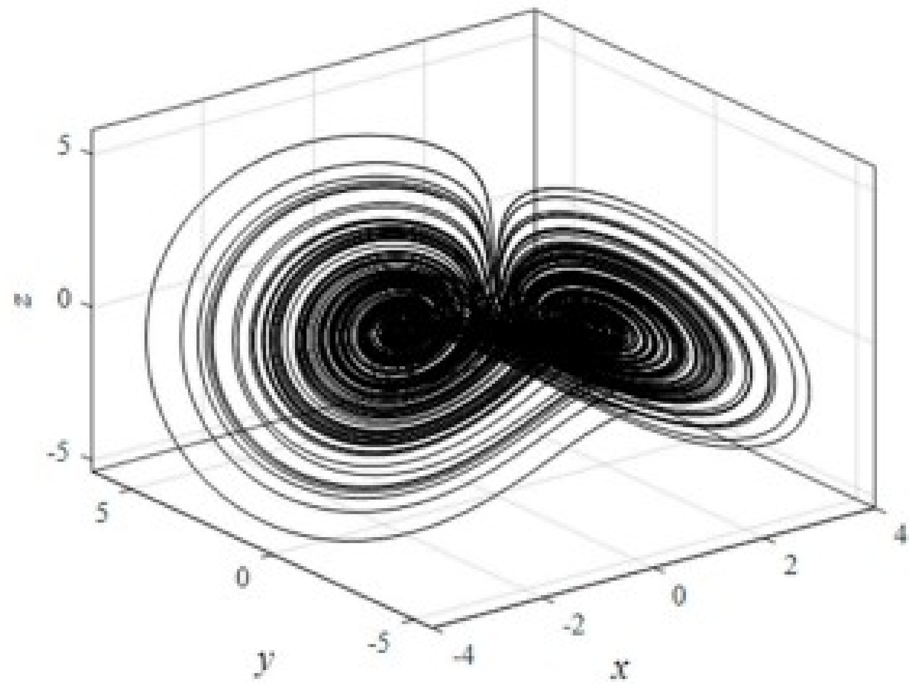

Figure 2.16: 3D view of the chaotic attractor for the system (2.4) in $(x_3,\ y_3,\ z_3)$ space.

2.6 Conclusion

This chapter contains a collection of both integer-order and fractional-order chaotic systems 3d.

CHAPTER 3

Synchronization of identical Fractional-order systems using the Adaptive Control Method

Contents

3.1 Introduction

In this chapter, we explore the synchronization between two fractional-order chaotic systems using the adaptive control method, and Lyapunov stability theory.
We propose a method to synchronize two chaotic systems. By design of the control law and an update law we ensure stability using only adaptive control. We apply the control law and an update law to synchronize two fractional-order chaotic system. The adaptive synchronization of the identical fractional-order chaotic systems is achieved using the master-slave systems. We prove their effectiveness with Lyapunov theory. Furthermore, the effectiveness of this approach is further validated through numerical simulations.

3.2 3d Fractional-Order Chaotic Lü System

The fractional-order Lü system [29], is described as follows:

$$\begin{cases} D^q x(t) = a(y(t) - x(t)) \\ D^q y(t) = -x(t)z(t) + cy(t) \\ D^q z(t) = x(t)y(t) - bz(t) \end{cases} \tag{3.1}$$

Which, x, y, and z are the state variables, and the fractional order derivatives $q = 0.95$, and the parameters are a, b and c.

The system displays chaotic behavior for this parameter values $a = 36$, $b = 3$ and $c = 20$. With the following initial condition $(x(0)\ , y(0)\ , z(0)) = (0.2, 0.5, 0.3)$ with $h = 0.005$ and $TSim = 200s$, a two scroll attractor existsand, and the numerical simulation of the Lü fractional chaotic system is given in the Figure 3.1.

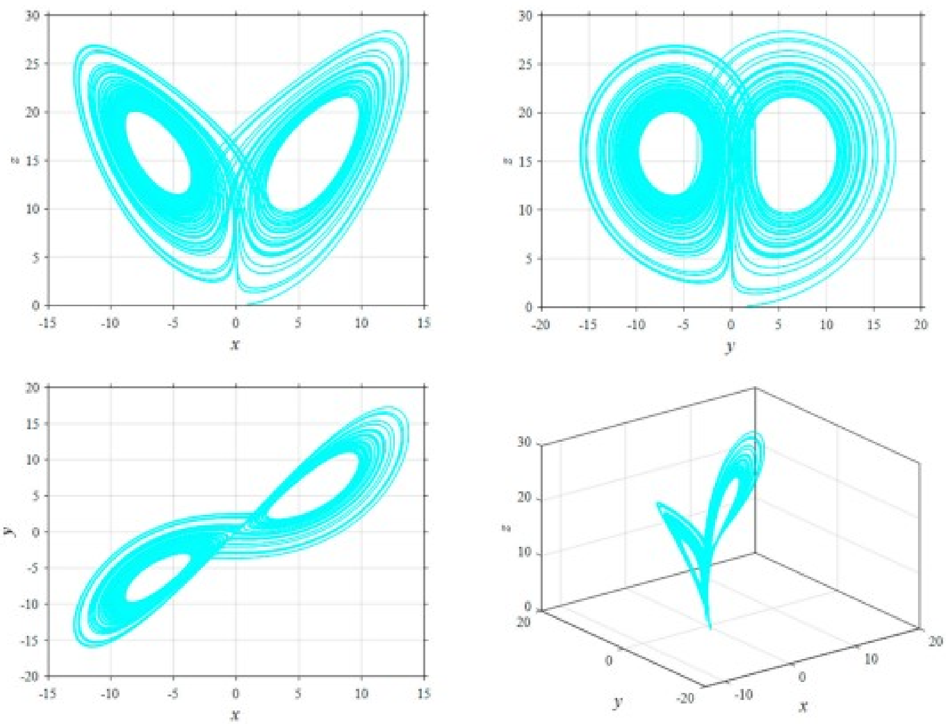

Figure 3.1: Fractional-order Lü system chaotic behavior for the parameters and the state variables with time $T = 200$.

3.2.1 Adaptive Control Law Design

In this section to achieve the synchronization we use the adaptive control method. This involves examining the dynamics of both the drive system (master) and the controlled system (slave).

The master system Lü fractional system is given by:

$$\begin{cases} D^q x_m = a\left(y_m - x_m\right) \\ D^q y_m = -x_m z_m + c y_m \\ D^q z_m = x_m y_m - b z_m \end{cases} \tag{3.2}$$

The slave system of Lü fractional system has the following form:

$$\begin{cases} D^q x_s = a\left(y_s - x_s\right) + u_1 \\ D^q y_s = -x_s z_s + c y_s + u_2 \\ D^q z_s = x_s y_s - b z_s + u_3 \end{cases} \tag{3.3}$$

So the state errors are: $e_1 = x_s - x_m$, $e_2 = y_s - y_m$ and $e_3 = z_s - z_m$.
Then for that, the error dynamics can be obtained:

$$\begin{cases} D^q e_1 = a\left(y_s - x_s - y_m + x_m\right) + u_1 \\ D^q e_2 = -x_s z_s + c y_s + x_m z_m - c y_m + u_2 \\ D^q e_3 = x_s y_s - b z_s - x_m y_m + b z_m + u_3 \end{cases}$$

Then

$$\begin{cases} D^q e_1 = a\left(e_2 - e_1\right) + u_1 \\ D^q e_2 = \left(x_m z_m - x_s z_s\right) + c e_2 + u_2 \\ D^q e_3 = \left(x_s y_s - x_m y_m\right) - b e_3 + u_3 \end{cases} \tag{3.4}$$

The adaptive controllers $[u_1, u_2, u_3]$ for the synchronization of the proposed system is taken in the following equations:

$$\begin{cases} u_1 = -\hat{a}\left(e_2 - e_1\right) - k_1 e_1 \\ u_2 = -\left(x_m z_m - x_s z_s\right) - \hat{c} e_2 - k_2 e_2 \\ u_3 = -\left(x_s y_s - x_m y_m\right) + \hat{b} e_3 - k_3 e_3 \end{cases} \tag{3.5}$$

k_1, k_2 and k_3 are strictly positive constants.

Let:

$$\begin{cases} e_a = a - \hat{a} \\ e_b = b - \hat{b} \\ e_c = c - \hat{c} \end{cases}$$

Thus:

$$\begin{cases} D^q e_a = -D^q \hat{a} \\ D^q e_b = -D^q \hat{b} \\ D^q e_c = -D^q \hat{c} \end{cases} \tag{3.6}$$

After substituting into the equation of the error dynamics, we obtain:

$$\begin{cases} D^q e_1 = e_a \left(e_2 - e_1\right) - k_1 e_1 \\ D^q e_2 = e_c e_2 - k_2 e_2 \\ D^q e_3 = -e_b e_3 - k_3 e_3 \end{cases} \tag{3.7}$$

The Lyapunov function V, used to derive the update law that is responsible for estimating the slave parameter corresponding to the equivalent parameter of the master, is obtained in the following equations

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2\right) \tag{3.8}$$

Therefore, the derivative of the function:

$$D^q V = e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_a D^q e_a + e_b D^q e_b + e_c D^q e_c \tag{3.9}$$

By substituting the dynamic error into the equation (3.9), we get:

$$D^q V = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a \left[e_1 \left(e_2 - e_1\right) - D^q \hat{a}\right] + e_b \left[-e_3^2 - D^q \hat{b}\right] + e_c \left[e_2^2 - D^q \hat{c}\right] \tag{3.10}$$

In view of (3.10), the update law for the system parameters can be set as:

$$\begin{cases} D^q \hat{a} = e_1 \left(e_2 - e_1\right) \\ D^q \hat{b} = -e_3^2 \\ D^q \hat{c} = e_2^2 \end{cases} \tag{3.11}$$

Finally, we get from the above that the time derivative of the Lyapunov function is:

$$D^qV = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 \tag{3.12}$$

We see that the derivative of the Lyapunov function is negative definite. Consequently, under any initial conditions, the synchronization state errors exponentially converge to zero over time. Furthermore, the estimated slave parameter exponentially aligns with the equivalent master parameter. Consequently, the asymptotic global stability of the established synchronization technique is ensured.

3.2.2 Numerical Simulation

For numerical simulations, we utilized MATLAB software to simulate the adaptive synchronization mechanism between two identical fractional-order chaotic systems (3.2) and (3.3). During the synchronization simulation, the parameters of the fractional-order Lü chaotic systems were chosen as follows: $a = 36$, $b = 3$ and $c = 20$, and the values of the slave parameters are: $\hat{a} = 35$, $\hat{b} = 4$ and $\hat{c} = 18$. Where q represents the fractional order derivative $q = 0.95$. Initial conditions for the master system were set to $x_m(0) = 0.2$, $y_m(0) = 0.5$ and $z_m(0) = 0.3$, while for the slave system they were $x_s(0) = 8.2$, $y_s(0) = 3.5$ and $z_s(0) = -6.2$. The synchronization of state variables for the identical fractional-order Lü chaotic systems is illustrated in Figure 3.2, 3.3 and 3.4, while Figure 3.5 demonstrates the convergence of the synchronization errors e_1 , e_2 and e_3 , which exponentially approach zero over time.

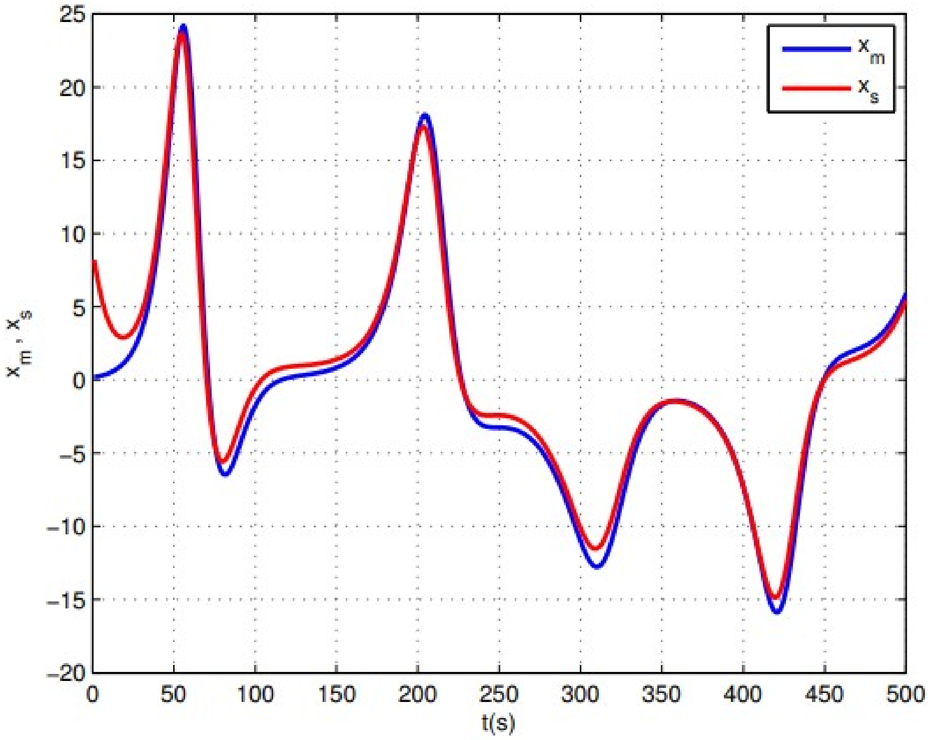

Figure 3.2: Synchronization between x_m and x_s.

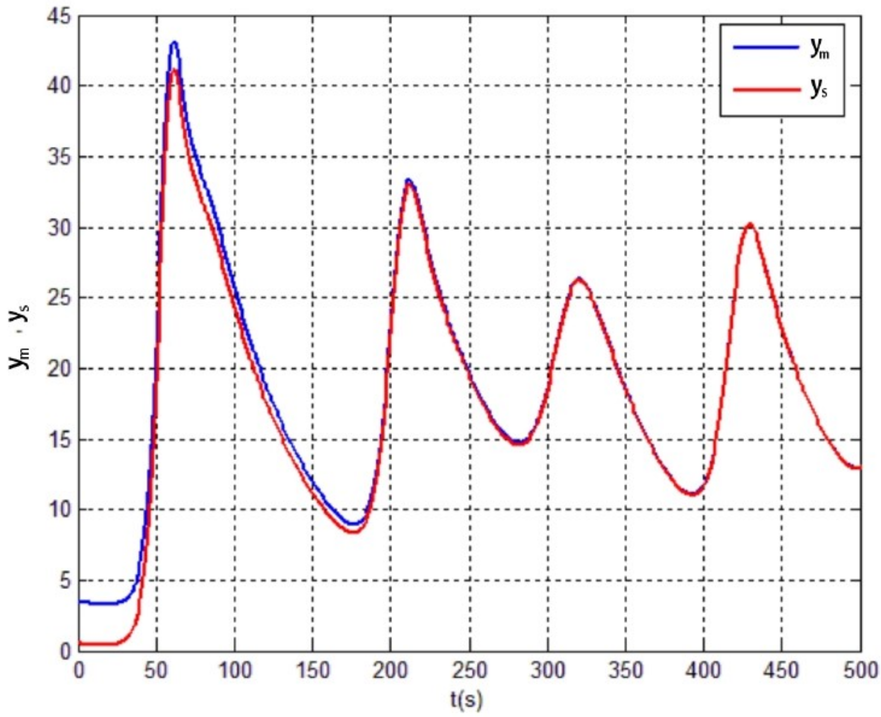

Figure 3.3: Synchronization between y_m and y_s

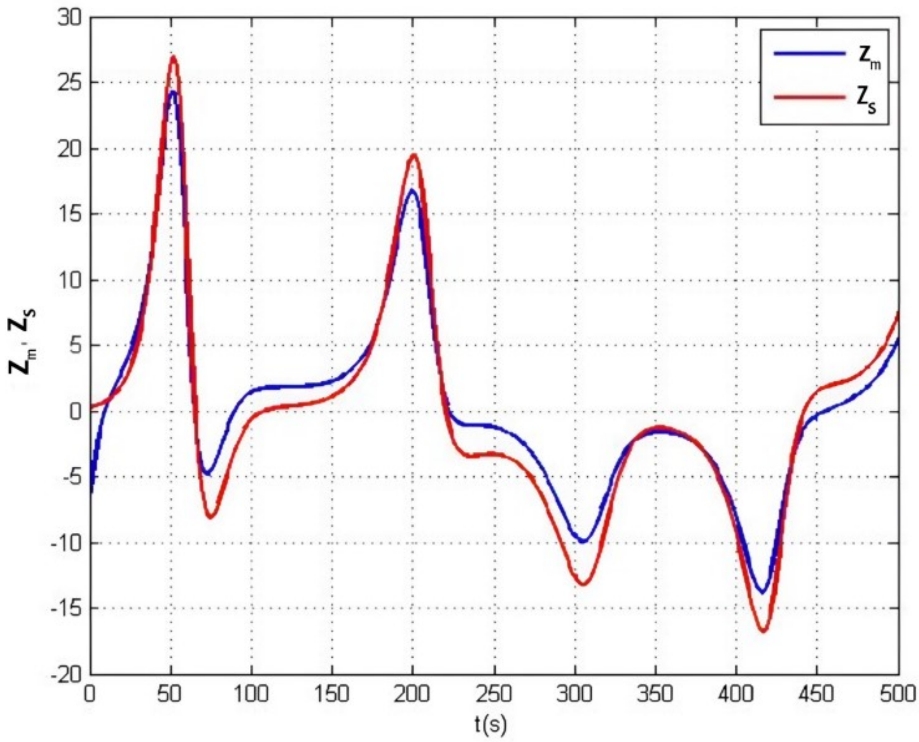

Figure 3.4: Synchronization between z_m and z_s.

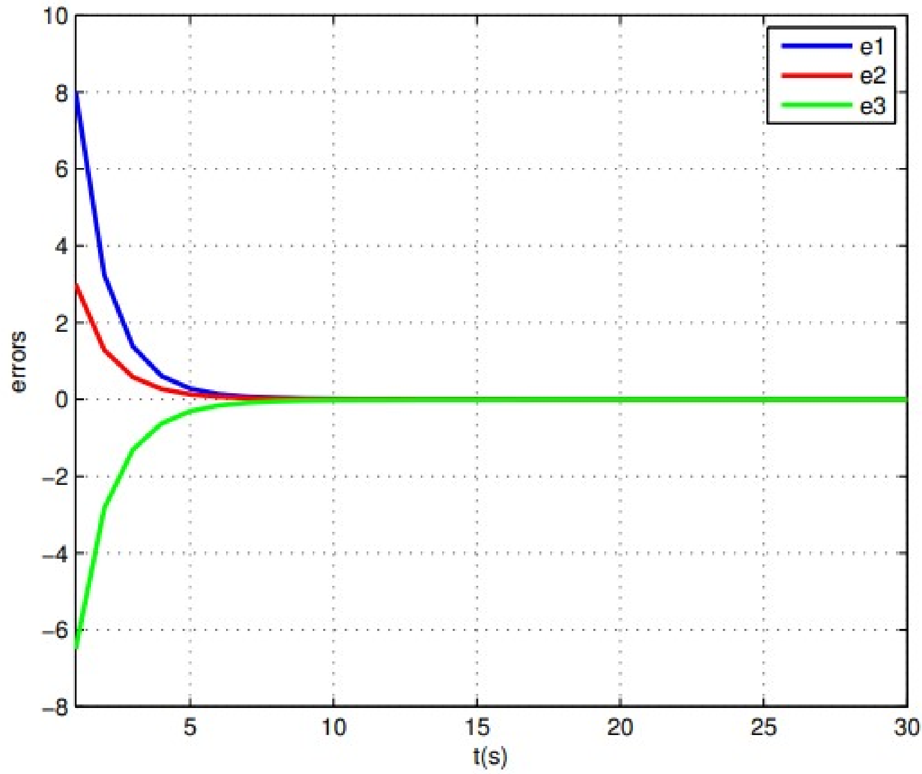

Figure 3.5: Time-evolution of the synchronization errors e_1, e_2 and e_3. where, $e_1(0) = 8$, $e_2(0) = 3$ and $e_3(0) = -6.5$.

3.3 Fractional-Order Chaotic System in 4D

A new fractional-order system [30], is described as follows:

$$\begin{cases} D^{q}x\left(t\right)=-by\left(t\right)z\left(t\right)\\ D^{q}y\left(t\right)=cx\left(t\right)z\left(t\right)-k\\ D^{q}z\left(t\right)=x\left(t\right)-dz\left(t\right)\\ D^{q}w\left(t\right)=ax^{3}\left(t\right)-w\left(t\right)\end{cases} \tag{3.13}$$

where, x, y, z and w are the state variables, and the fractional order derivatives $q = 0.9$, and the parameters are a, b, c, d and k.

The system displays chaotic behavior for this parameter values $a = 0.001$, $b = 2.5$, $c = 0.05$, $d = 2$ and $k = 1.2$. With the following initial condition $(x(0)\ ,y(0)\ ,z(0),w(0)) = (0.1, 0.2, -1, 0.3)$ with $h = 0.005$ and $TSim = 200s$, a two scroll attractor existsand, and the numerical simulation of the Lü fractional chaotic system is given in the Figure 3.6.

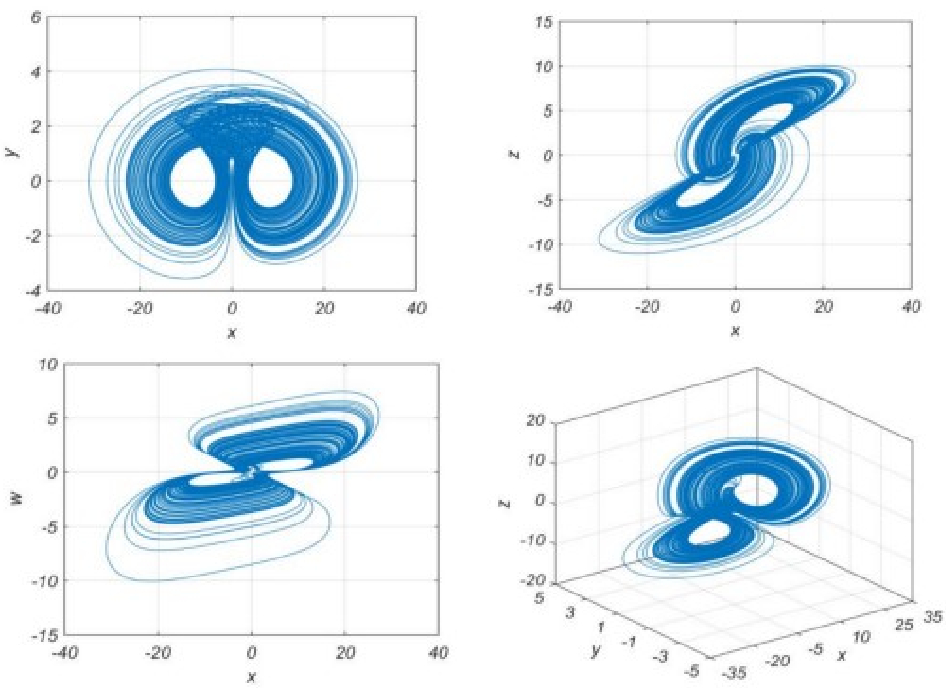

Figure 3.6: The fractional-order system chaotic behavior for the parameters and the state variables with time $T = 200$.

3.3.1 Lyapunov Exponents

The Lyapunov exponents of the system (3.13) have been calculated, strongly indicating that the new system exhibits chaotic behavior. The presence of at least one positive Lyapunov exponent in nonlinear dynamic systems ensures chaos [31] and [32]. For the suggested system, the Lyapunov exponents are numerically determined using Matlab, as shown in Figure 3.7. They are obtained as follows: $Le1 = 0.1558$, $Le2 = -0.0001$, $Le3 = -1.0366$ and $Le4 = -2.2034$. Thus, the proposed FOCS exhibits chaotic behavior because one of the Lyapunov exponents is positive.

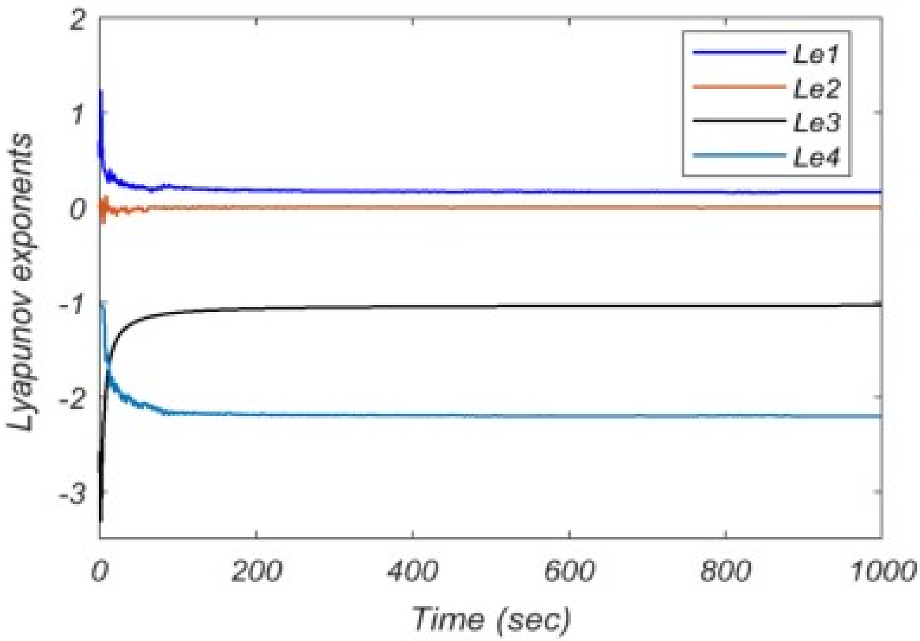

Figure 3.7: Lyapunov exponents with parameters $a = 0.001$, $b = 2.5$, $c = 0.05$, $d = 2$ and $k = 1.2$. With initial condition$x\,(0) = 0.1$, $y\,(0) = 0.2$, $z\,(0) = 1$, and $w\,(0) = 0.3$, and the fractional order derivatives $q = 0.9$.

3.3.2 Adaptive Control Law Design

In this section to achieve the synchronization we apply the adaptive control method.

The master system for the 4d fractional system is given by:

$$\begin{cases} D^q x_m = -b y_m z_m \\ D^q y_m = c x_m z_m - k \\ D^q z_m = x_m - d z_m \\ D^q w_m = a x_m^3 - w_m \end{cases} \tag{3.14}$$

The slave system of Focs has the following form:

$$\begin{cases} D^q x_s = -b y_s z_s + u_1 \\ D^q y_s = c x_s z_s - k + u_2 \\ D^q z_s = x_s - \hat{d} z_s + u_3 \\ D^q w_s = a x_s^3 - w_s + u_4 \end{cases} \tag{3.15}$$

So the state errors are: $e_1 = x_s - x_m$, $e_2 = y_s - y_m$, $e_3 = z_s - z_m$ and $e_4 = w_s - w_m$.

For that, the error dynamics can be obtained:

$$\begin{cases} D^q e_1 = b\left(y_m z_m - y_s z_s\right) + u_1 \\ D^q e_2 = c\left(x_s z_s - x_m z_m\right) + u_2 \\ D^q e_3 = e_1 - \hat{d} z_s + d z_m + u_3 \\ D^q e_3 = a\left(x_s^3 - x_m^3\right) - e_4 + u_4 \end{cases} \tag{3.16}$$

Let:

$$e_d = \hat{d} - d$$

For that we have

$$D^q e_d = D^q \hat{d} \tag{3.17}$$

Then, the error dynamics can be obtained:

$$\begin{cases} D^q e_1 = b\left(y_m z_m - y_s z_s\right) + u_1 \\ D^q e_2 = c\left(x_s z_s - x_m z_m\right) + u_2 \\ D^q e_3 = e_1 - \hat{d} z_s + d z_m + \hat{d} z_m - \hat{d} z_m + u_3 \\ D^q e_3 = a\left(x_s^3 - x_m^3\right) - e_4 + u_4 \end{cases}$$

Finally we get that the error dynamics is:

$$\begin{cases} D^q e_1 = b\left(y_m z_m - y_s z_s\right) + u_1 \\ D^q e_2 = c\left(x_s z_s - x_m z_m\right) + u_2 \\ D^q e_3 = e_1 - \hat{d} e_3 - e_d z_m + u_3 \\ D^q e_3 = a\left(x_s^3 - x_m^3\right) - e_4 + u_4 \end{cases} \tag{3.18}$$

The Lyapunov function V, used to derive the update law that is responsible for estimating the slave parameter corresponding to the equivalent parameter of the master, is obtained in the following equations:

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_d^2\right) \tag{3.19}$$

Therefore, the derivative of the function:

$$D^q V = e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_4 D^q e_4 + e_d D^q e_d$$

By substituting the dynamic error into the equation, we get:

$$D^q V = e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_4 D^q e_4 + e_d D^q \hat{d} \tag{3.20}$$

The adaptive controllers $[u_1, u_2, u_3, u_4]$ for the synchronization of the proposed system is taken in the equations (3.21), based on Lyapunov theory for stability. Various methods can be employed for designing similar controllers, as described in [33] and [34].

$$\begin{cases} u_1 = -b\left(y_m z_m - y_s z_s\right) - k_1 e_1 \\ u_2 = -c\left(x_s z_s - x_m z_m\right) - k_2 e_2 \\ u_3 = -e_1 + \hat{d} e_3 - k_3 e_3 \\ u_4 = -a\left(x_s^3 - x_m^3\right) + e_4 - k_4 e_4 \end{cases} \tag{3.21}$$

k_1, k_2, k_3 and k_4 are strictly positive constants.

By substituting the dynamic error into the Lyapunov function V, we get:

$$D^q V = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_d\left(-e_3 z_m + D^q \hat{d}\right)$$

The update law for the system parameter can be set as:

$$D^q \hat{d} = e_3 z_m \tag{3.22}$$

Finally, we get from the above that the time derivative of the Lyapunov function is:

$$D^q V = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \tag{3.23}$$

We see that the derivative of the Lyapunov function is negative-definite. Consequently, under any initial conditions, the synchronization state errors exponentially converge to zero over time. Furthermore, the estimated slave parameter exponentially aligns with the equivalent master parameter. Consequently, the asymptotic global stability of the established synchronization technique is ensured.

3.3.3 Numerical Simulation

For numerical simulations, MATLAB software is utilized to simulate the adaptive synchronization mechanism between the two identical fractional-order chaotic systems (3.14) and (3.15). In the synchronization simulation, the parameters of the new fractional-order chaotic systems were selected as follows: $a = 0.001$, $b = 2.5$, $c = 0.05$, $d = 2$ and $k = 1.2$, the slave parameters value are uncertain, where q are the fractional order derivative $q = 0.9$. With the following initial condition $(x_m(0)\ , y_m(0)\ , z_m(0), w_m(0)) = (0.1, 0.2, -1, 0.3)$ for the master system, and the slave system are $(x_s(0)\ , y_s(0)\ , z_s(0), w_s(0)) = (0.2, 0, 0.1, -1)$.
The synchronization of state variables for the identical New fractional-order chaotic systems is depicted in Figures 3.8, 3.9, 3.10 and 3.11 and Figure 3.12 demonstrates the convergence of the synchronization errors e_1, e_2, e_3 and e_4, which exponentially tend to zero over time. Where the parameters on the slave side exponentially converge to the equivalent parameter values on the master side.

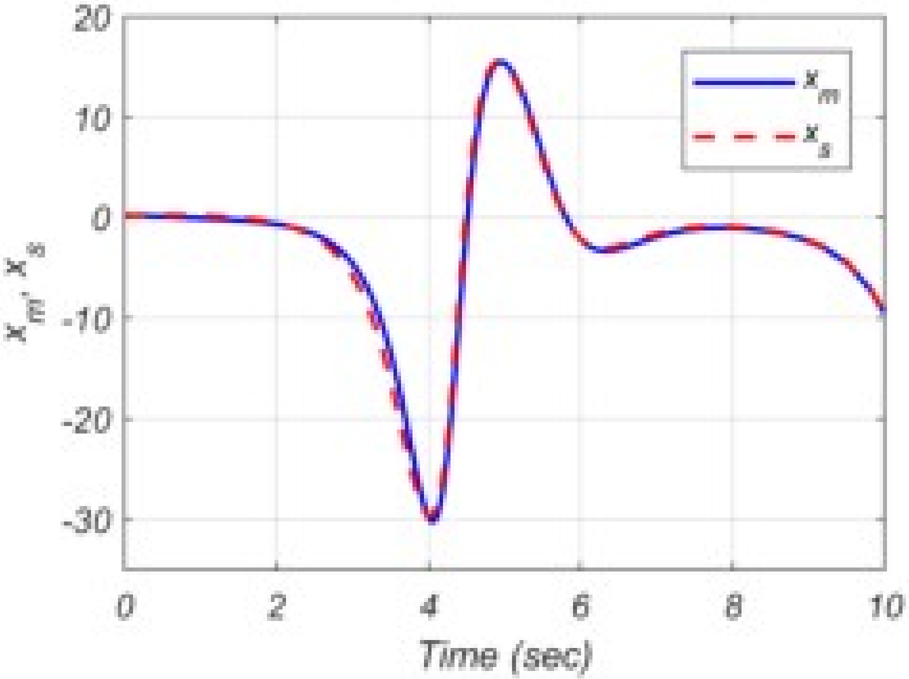

Figure 3.8: Synchronization between x_m and x_s.

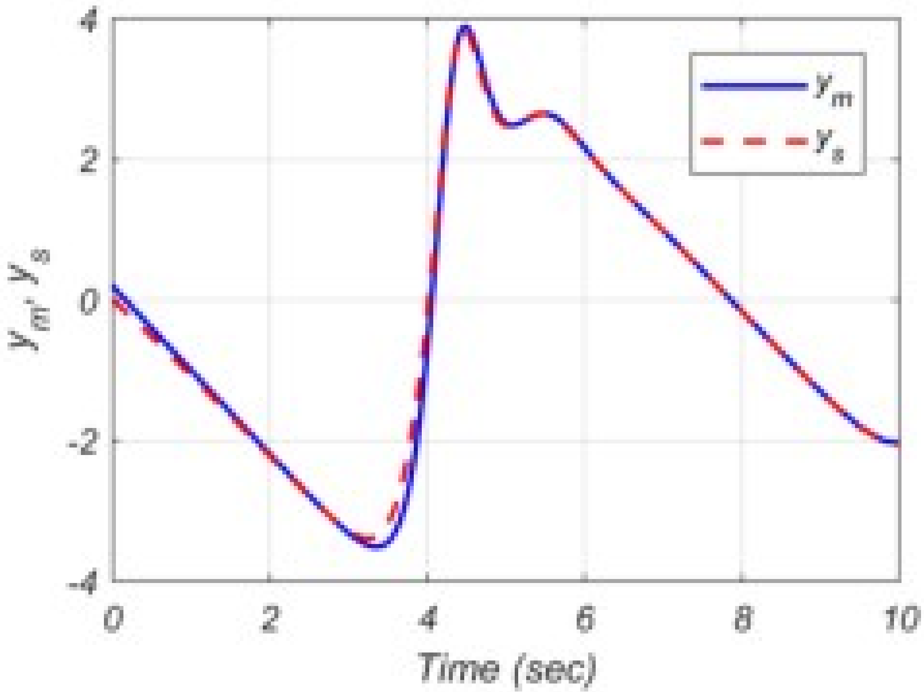

Figure 3.9: Synchronization between y_m and y_s.

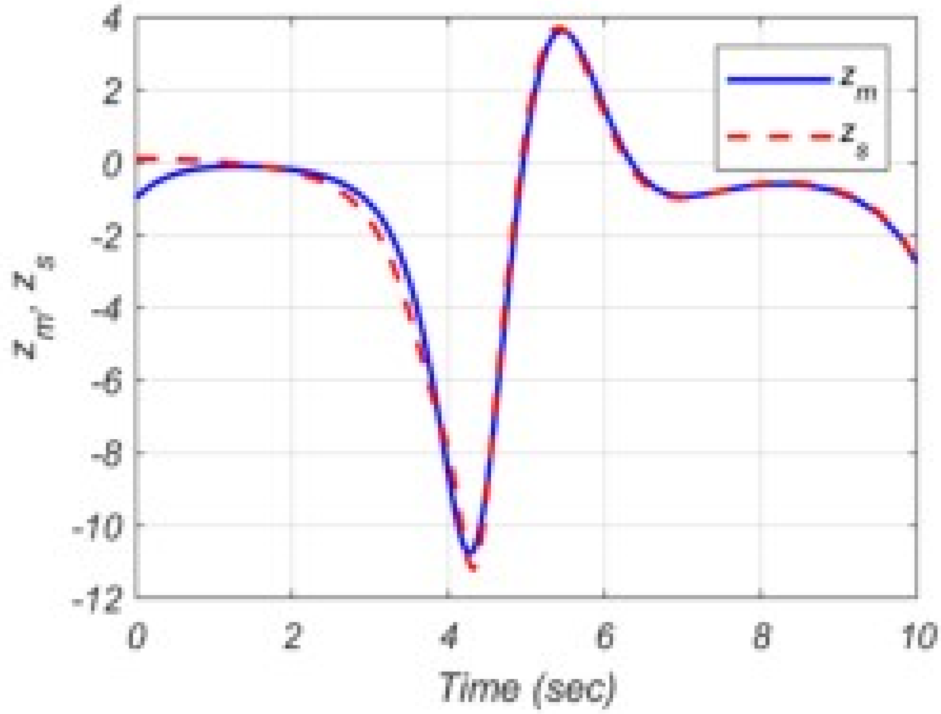

Figure 3.10: Synchronization between z_m and z_s.

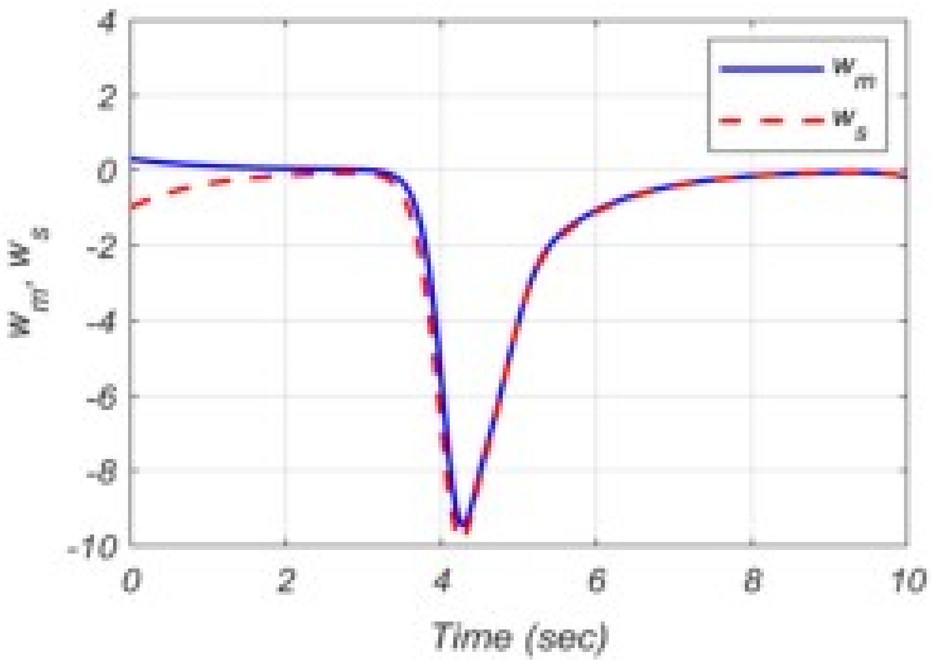

Figure 3.11: Synchronization between w_m and w_s.

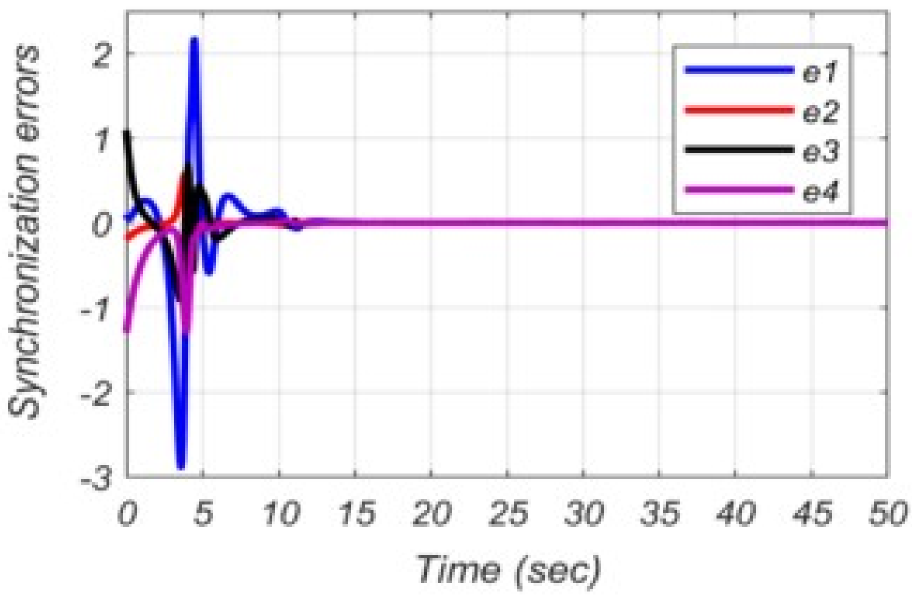

Figure 3.12: Time-evolution of the synchronization errors.

3.4 Conclusion

In this chapter, we investigated synchronization between two fractional-order chaotic systems by using the adaptive control method. As a result, we found that the asymptotic global stability of the established synchronization technique is guaranteed.

Conclusion

Chaos, a complex phenomenon studied across various fields, has practical applications in industries and secure communication. Researchers have developed numerous synchronization methods for chaotic systems, including the OGY method, back-stepping design, sliding mode control, and others. These methods play a crucial role in making chaotic systems work together effectively.

This work aims to synchronize two fractional-order chaotic systems using the adaptive control method, which relies on Lyapunov stability theory when system parameters are unknown. The proposed method utilizes Lyapunov stability theory of the zero solution of the errors system to achieve asymptotic synchronization of the two chaotic systems. Numerical simulation results in Matlab demonstrate the effectiveness of the proposed method.

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