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Adaptive Control for Fractional-Integer Order Systems Synchronization

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بسم الله الرحمن الرحيم

شُكْرٌ وَتَقْدِير

الحمد لله على منه وامتنانه والشكر له على نعمه وإنعامه حمدا كثيرا طيبا الذي أنعم علي بنعمة العلم وسهل لي طريقا أبغي فيه علما ووفقني في إنهاء عملي المتواضع هذا، والصلاة والسلام على الحبيب المصطفى الذي بلغ الرسالة وأدى الأمانة ونصح الأمة وعملا بقوله صلى الله عليه وسلم:

(مَنْ لَّمْ يَشْكُرِ القَلِيلْ لَّمْ يَشْكُرِ الكَثِيرْ وَ مَنْ لَّمْ يَشْكُرِ النَّاسْ لَّمْ يَشْكُرِ الله)

رواه أحمد الترمذي.

أتقدم بأسمى عبارات **الشكر والتقدير** إلى كل من علمني ومن أزال غيمة جهل مررت بها برياح العلم الطيبة، إلى كل من علمني علما به أنتفع وأدبا به أرتفع. بدءا من معلمي الابتدائي وصولا إلى أساتذة التعليم العالي والبحث العلمي بقسم الرياضيات و الإعلام الآلي بجامعة الشهيد الشيخ العربي التبسي - تبسة– وأخص بالذكر **الأستاذ المؤطر** حناشي فارح مع كامل تقديري واحترامي لما بذله من مجهودات قيمة وما حضني به من احترام ومعاملة حسنة للوصول إلى إنجاز هذا العمل البسيط. كما أتوجه بالشكر الجزيل إلى **أعضاء لجنة المناقشة** التي شرفتني بقبولها مناقشة مذكرتي، كل من **الأستاذ** زراولية الحاج **رئيسا للجنة،** **والأستاذ** ذياب زهير **ممتحنا**، اللذين لاشك أنهما سيفيضون عليَّ بتوجيهاتهما القيمة وملاحظاتهما السديدة.

لكم مني فائق التقدير و الاحترام.

الإهداء

الحمد لله الذي تتم بفضله النعم.

والصلاة والسلام على سيدنا محمد وعلى آله وصحبه أجمعين.

شيء جميل أن يسعى الإنسان إلى النجاح و يحصل عليه،

و الأجمل أن يذكر من كان السبب في ذلك.

أهدي ثمرة عملي هذا إلى من وضع الله سبحانه وتعالى الجنة تحت قدميها،
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جيهان عويشات

الملخص

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قمنا بالتحقق من النتائج باستخدام المحاكاة العددية من خلال استخدام برنامج ماتلاب.

الكلمات المفتاحية:

نظام ديناميكي، فوضى، مزامنة، نظام برتب صحيحة، نظام برتب كسرية، استقرار ليابونوف، طريقة التحكم التكيفي.

Resumé

Dans ce travail, nous avons utilisé une méthode de contrôle adaptatif pour réaliser la synchronisation entre les systèmes chaotiques d'ordre fractionnaire et entier basée sur la théorie de la stabilité de Lyapunov avec paramètres du système inconnus.
Nous avons vérifié les résultats à l'aide de simulations numériques en Matlab.

Mots-clés :

Système dynamique, Chaos, Synchronisation, Système d'ordre entier, Système d'ordre fractionnaire, Stabilité de Lyapunov, Méthode de contrôle adaptatif.

Abstract

In this work, we used an adaptive control method to achieve synchronization between fractional and integer order chaotic systems based on Lyapunov stability theory with unknown system parameters.
We verified the results using numerical simulations through the use of Matlab.

Keywords:

Dynamical system, Chaos, Synchronization, Integer-order system, Fractional-order system, Lyapunov stability, Adaptive control method.

LIST OF NOTATIONS

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^+$	The set of positive real numbers.
$\mathbb{R}^n$	The set of n-dimensional real vectors.
$\mathbb{R}^m$	The set of m-dimensional real vectors.
$\mathbb{R}^p$	The set of p-dimensional real vectors.
$\mathbb{Z}^+$	The set of positive integers.
$\lim$	The limit.
exp	The exponential function.
$\max$	The maximum.
$\min$	The minimum.
$\det$	The determinant.
$\|.\|$	The norm.
$\lvert . \rvert$	The absolute value.
$\sum$	The sum.
$(.)^T$	Transpose.
$\dot{x}(t) = \frac{dx(t)}{dt}$	The derivative of y(t) with respect to t.
x^*	A fixed point.
L^∞	Space of essentially bounded functions.
Ω_1	An open bounded subset of $\mathbb{R}^n$.
Ω_2	An open bounded subset of $\mathbb{R}^m$.
L	Laplace transform.
L^{-1}	Inverse Laplace transform.
C^1	The set of continuous differentiable functions of degree 1.
$\Gamma(x)$	The gamma function.
$spec(A)$	The spectrum of a matrix A.
$\arg(\lambda)$	The argument or angle of a λ (eigenvalue) .
$diag(\lambda)$	The diagonal matrix formed by placing the eigenvalues λ.
J	The Jacobian matrix.
$V(t)$	The Lyapunov function.
$K!$	The factorial of the non-negative integer K.

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General Introduction

CHAOS theory, pioneered by scientists like Henri Poincaré and Edward Lorenz in the 20th century [30], has been applied across various fields, from meteorology to pandemic crisis management.

Recent research has demonstrated that chaotic systems can exhibit behaviors described by differential equations of integer-order or fractional-order, characterized by unpredictable dynamics and a high sensitivity to initial conditions [31,32]. Interestingly, under certain conditions, these systems can synchronize, prompting a deeper exploration into synchronization phenomena among interconnected systems.

Synchronization between fractional and integer-order systems offers significant benefits [22], including enhanced system stability and improved performance across various applications. There are multiple synchronization methods between these systems, including the adaptive control method [20,24], which allows systems to dynamically adjust and align their behaviors. This can lead to improved stability and performance in complex systems such as communication networks, robotic control systems, and biological systems, ultimately enhancing their efficiency and reliability. Despite these advancements, the synchronization between fractional and integer-order systems remains a largely underexplored area, presenting opportunities for further research.

This work utilizes the adaptive control method, based on Lyapunov stability theory, to achieve synchronization between fractional-integer order chaotic systems with unknown system parameters. The proposed method ensures asymptotic synchronization of the two chaotic systems through Lyapunov stability analysis. The effectiveness of the method is demonstrated through simulation results implemented in MATLAB.

This thesis is organized into three chapters:

- In chapter 1, we recall some basic notions of dynamical systems and the theory of chaos and we discuss the concept of synchronization and learn about one of the methods of synchronization, also basic definitions and properties of fractional derivatives are provided with numerical methods for solving fractional-order systems.

- In chapter 2, we present some examples of integer-order and fractional-order chaotic systems in 3D.

- In chapter 3, we study the synchronization between two 3D fractional-integer order chaotic systems. Numerical simulations are provided using MATLAB to demonstrate the effectiveness of the proposed method.

CHAPTER 1

Preliminaries

Contents

Introduction

In this chapter, we delve into dynamic systems, chaos theory, synchronization, and fractional calculus. We explore the fundamentals of dynamic systems, including continuous and discrete systems, as well as concepts like phase space and phase portraits. Transitioning to chaos theory, we define chaotic systems and discuss their applications. Next, we examine synchronization methods and the adaptive control approach. Finally, we introduce fractional calculus, discussing its applications and numerical methods for solving fractional differential equations.

1.1 Dynamic systems

Dynamical systems are mathematical models that describe how phenomena change over time. These systems can be used to study a wide variety of phenomena, from the motion of planets to the spread of diseases.

Dynamical systems are typically classified into two types: continuous-time systems and discrete-time systems.

1.1.1 Continuous dynamic systems

Definition 1.1 *A continuous dynamical system can be mathematically represented by a set of differential equations [7]:*

$$\frac{dX}{dt} = G(X(t), t) \tag{1.1}$$

Where G function of class $C^1 : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ defines the dynamics of the continuous system.

1.1.2 Discrete dynamic systems

Definition 1.2 *A discrete dynamical system can be defined as a system of recurrent algebraic equations determined by [7]:*

$$X_{k+1} = G(X_k, u), \text{ where } X_k \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^p \tag{1.2}$$

With $G : \mathbb{R}^n \times \mathbb{Z}^+ \to \mathbb{R}^n$, defines the dynamics of the discrete system.

1.1.3 Phase space

The phase space (also called state space), is the collection of all possible states of a dynamical system, where each state is defined by a set of variables representing necessary dimensions to describe the system at a given time [10].

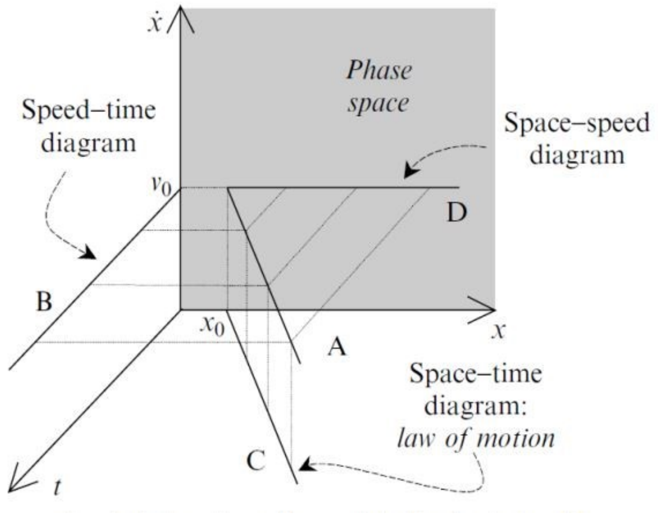

Fig. 1.1: Phase Space.

1.1.4 Phase portrait

A phase portrait illustrates dynamical behaviors in phase space. Resting states appear as single points, while periodic oscillations form closed curves. Trajectories within phase space represent the system's evolution from different initial states. Analyzing phase portraits offers insight into system behavior, particularly for nonlinear equations without explicit solutions [29].

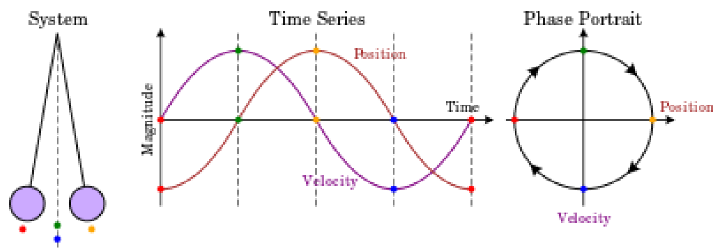

Fig. 1.2: Illustration of how a phase portrait would be constructed for the motion of a simple pendulum.

1.1.5 The Poincaré section

A Poincaré section simplifies the analysis of dynamical systems by reducing continuous-time dynamics to discrete-time ones. It involves intersecting the system's trajectory in phase space with a chosen plane, preserving key properties while facilitating analysis, especially for periodic trajectories.

1.2 Chaos theory

1.2.1 Definition of chaotic systems

Chaotic systems are dynamic systems characterized by intricate and unpredictable behavior that is highly sensitive to initial conditions. In such systems, small changes in initial conditions can lead to significantly different outcomes, making long-term predictions challenging or impossible. Chaotic systems often exhibit complex and irregular evolution, hindering the ability to forecast their behavior over extended periods. These systems are typically described by nonlinear equations and may display features such as chaotic attractors, positive Lyapunov exponents, and sensitive dependence on initial conditions [9, 10].

1.2.2 Characteristics of Chaotic Systems

Chaotic systems possess distinct properties setting them apart from other dynamical systems [10]:

- **Sensitivity to Initial Conditions:** Even small variations in the initial conditions of a chaotic system can result in substantial differences in its behavior over time.
- **Chaotic or strange attractor:** Nonlinear systems can display complex steady-state behavior beyond equilibrium or periodic oscillations, termed chaos. This chaotic motion may appear random despite the system's deterministic nature.
- **Lyapunov Exponent:** The Lyapunov exponent serves as a metric for assessing the rate at which nearby paths within a dynamical system diverge. Named after Alexander

Lyapunov, a Russian mathematician who introduced the concept in the late 19th century, it measures the stability of the system. A positive Lyapunov exponent indicates instability and chaotic behavior, as nearby paths tend to diverge over time. Conversely, a negative Lyapunov exponent indicates stability, as nearby paths tend to converge over time. This metric is frequently employed in analyzing chaotic systems, where sudden and unpredictable divergence among neighboring paths occurs.

- **Fractal dimension:**The fractal dimension quantifies the complexity of chaotic systems' strange attractors, which have non-integer dimensions. It measures the intricate, self-similar patterns across different scales by analyzing the phase space trajectory of the system's state variables over time. This dimension reflects the degree of complexity in the attractor's pattern, aiding in understanding the behavior of chaotic systems.

1.2.3 Chaos applications

This subsection explores the current applications of chaotic systems in mathematics and various real-life fields. Chaos theory, initially inspired by weather patterns, now finds utility in a multitude of areas such as mathematics, geology, biology, economics, and more. These applications include weather modeling, stock market analysis, bird migration studies, boiling water dynamics, neural networks, and quantum phenomena. The theory is based on two key principles: the deterministic nature of systems, governed by underlying equations or principles, and their high sensitivity to initial conditions. Even small changes can lead to significant and unpredictable outcomes [4].
Chaos theory has many applications in various fields. Here are some examples:

1. **Chaos theory in Stock Market:** Chaos theory applied to the stock market reveals randomness with underlying trends, with short-term movements resembling long-term ones. Mandelbrot's prediction of market crashes occurring roughly once a decade has been validated, validated by crashes in 1987, 1998, and 2008. The Fractal Market Hypothesis, derived from Chaos theory, explains financial phenomena beyond the Efficient Market Hypothesis. It utilizes the Hurst exponent to gauge chaos levels and

the Lyapunov exponent for predictability, enabling potential forecasting of market behavior.

2. **Physics:** Chaos theory has been applied in physics to comprehend the behavior of complex systems like celestial mechanics, nonlinear optics, and fluid dynamics.

3. **Engineering:** Employed in engineering, chaos theory aids in optimizing the management and operation of intricate systems like power plants, chemical reactors, and communication networks, enhancing their efficiency and reliability.

4. **Biology:** In the realm of biology, chaos theory sheds light on the dynamics of biological systems, offering valuable perspectives on ecological systems, brain networks, and cardiac rhythms, facilitating deeper comprehension of their functioning.

5. **Finance:** Chaos theory is instrumental in analyzing financial markets, enabling the development of models to anticipate market fluctuations and manage risks effectively, thus assisting investors and financial institutions in decision-making processes.

6. **Computer Science:** Within computer science, chaos theory drives the development of algorithms for optimization and data analysis, enhancing computational efficiency and enabling innovative solutions to complex problems.

7. **Music and Art:** In the creative sphere, chaos theory inspires novel forms of expression in music and art, exploring the interplay between randomness and creativity, leading to avant-garde compositions and visually captivating artworks.

In summary, chaos theory emerges as a potent tool, transcending disciplinary boundaries to elucidate the behavior of complex systems. Its ongoing exploration fosters continuous innovation and application across science, engineering, and the arts, enriching diverse fields with its insights and methodologies.

1.2.4 Hyperchaotic systems

Hyperchaotic systems are dynamic systems that exhibit complex chaotic behavior, characterized by having at least two positive Lyapunov exponents, which means they have a higher chaotic dimensionality compared to regular chaotic systems.

1.3 Synchronization

In this section, the topic of synchronization will be explored. It is a vital concept in various fields such as computer science, engineering, and physics. Different synchronization methods, types of synchronization, Lyapunov stability for analyzing the stability of dynamic systems, and the concept of adaptive control, which allows for adjusting the system's behavior according to environmental changes, will be covered.

1.3.1 Synchronisation Methods

The concept behind chaos synchronization is that two chaotic systems, initially evolving on separate attractors, eventually converge to follow a shared trajectory when coupled. This synchronization occurs when either one system adjusts its trajectory to match the other's or both systems adopt a new common trajectory. The technique relies on the notion that although two chaotic systems may initially evolve independently on distinct attractors, they can synchronize to follow the same trajectory over time [3].

1.3.2 Types of Synchronisation

Consider the system dynamics represented by $\dot{x} = F\left(x, t\right)$ as the driving force, exhibiting chaos or hyperchaos. Meanwhile, the response system is denoted by $\dot{y} = G(y, t) + U$ where $x = (x_1(t), x_2(t), ..., x_n(t))^T$, $y = (y_1(t), y_2(t), ..., y_m(t))^T$, and $U = (u_1, u_2, ..., u_n)^T$ represents a controller whose determination is deferred. Several types of synchronization are commonly studied in dynamical systems, and here are some of the most common types [8]:

- Synchronization is achieved if $\lim\limits_{t\to+\infty} \|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y - x$,
- Anti-synchronization is occur if $\lim\limits_{t\to+\infty} \|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y + x$,
- Function projective synchronization is achieved if $\lim\limits_{t\to+\infty} \|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y - h\left(x\right)x$, $h\left(x\right) = \left(h_1\left(x_1\right), h_2\left(x_2\right), ..., h_n\left(x_n\right)\right).$

- Inverse function projective synchronization is achieved if $\lim\limits_{t \to +\infty} \|e\| = 0,\ e \in \mathbb{R}^n$ with $e = y - h(y)\, y,\ h(x) = (h_1(y_1), h_2(y_2), ..., h_n(y_n))$, h is a scaling function matrix.

1.3.3 Lyapunov Stability

Ponder a dynamical system that remains invariant over time:

$$\dot{x}(t) = g(x(t)).$$

Where $g : D \to \mathbb{R}^n$. We say that $x^* \in D$ is an equilibrium point of the system if $g(x^*) = 0$. We assume without loss of generality that $x^* = 0$.

Theorem 1.1 *[26] Let 0 be an equilibrium point for $\dot{x} = g(x)$ where $g : D \to \mathbb{R}^n$. Assume there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that:*

- *$V(0) = 0$ and $V(x) > 0$ for all $x \in D$ not equal to zero.*
- *$\dot{V} = \frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x} = [D_f V](x) \leq 0$ for all $x \in D$.*

Then $x = 0$ is stable in the sense of Lyapunov.

Theorem 1.2 *[26] Under the hypotheses of theorem 1.1, if $\dot{V}(x) < 0$ for all $x \in D - \{0\}$, then the equilibrium is globally asymptotically stable.*

Remark 1.1 *The direct method in analyzing the stability of dynamic systems primarily relies on a Lyapunov function, denoted as V(x). This function decreases over time along the system's trajectories, providing a powerful tool for assessing system stability without the need to explicitly solve the dynamic equations.*

1.3.4 Adaptive Control Method

Adaptive Control Method is an approach in control engineering utilized when system parameters are either unknown or continuously changing. It aims to automatically and dynamically adjust the controller in response to variations in the environment or system parameters, leading to enhanced system performance and long-term stability. This method, which is one of the synchronization methods, has demonstrated effectiveness across various applications, such as smart robots, energy systems, autonomous vehicles, and others.

Mathematical description

Take into consideration the chaotic drive system in the following form [13]:

$$\dot{x} = f(x) + F(x)\alpha. \tag{1.3}$$

In this system, $x \in \Omega_1 \subset R^n$ represents the state vector, $\alpha \in R^m$ is the unknown parameter vector, $f(x)$ is an $n \times 1$ matrix, and $F(x)$ is an $n \times m$ matrix. The elements $F_{ij}(x)$ in the matrix $F(x)$ satisfy $F_{ij}(x) \in L^\infty$ for $x \in \Omega_1 \subset R^n$.

Additionally, the response system is assumed to be described by:

$$\dot{y} = h(y) + H(y)\beta + u. \tag{1.4}$$

In this system, $y \in \Omega_2 \subset R^n$ represents the state vector, $\beta \in R^q$ is the unknown parameter vector, $h(y)$ is an $n \times 1$ matrix, and $H(y)$ is an $n \times q$ matrix. Additionally, $u \in R^n$ represents the control input vector. The elements $H_{ij}(y)$ in matrix $H(y)$ satisfy $H_{ij}(y) \in L^\infty$ for $y \in \Omega_2 \subset R^n$.

Let $e = y - x$ represent the synchronization error vector. Our objective is to design a controller u such that the trajectory of the response system (1.4), with initial condition y_0, asymptotically approaches the drive system (1.3) with initial condition x_0. Our ultimate aim is to achieve synchronization, expressed as:

$$\lim_{t \to +\infty} \|e\| = \lim_{t \to +\infty} \|y(t, y_0) - x(t, x_0)\| = 0, \tag{1.5}$$

Where $\|.\|$ is the Euclidean norm.

Adaptive synchronization controller design

Theorem 1.3 *[13] If the nonlinear control is chosen as:*

$$u = -f(x) - F(x)\widehat{\alpha} - h(y) - H(y)\widehat{\beta} - Ge, \tag{1.6}$$

And adaptive laws for the parameters are formulated as:

$$\begin{aligned} \dot{\widehat{\alpha}} &= [F(x)]^T e, \\ \dot{\widehat{\beta}} &= [H(y)]^T e, \end{aligned} \tag{1.7}$$

Then the response system (1.4) can achieve synchronization with the drive system (1.3), where $G > 0$ is a constant, and $\widehat{\alpha}$ and $\widehat{\beta}$ are estimations of the unknown parameters α and β, respectively, with α and β are constants.

Proof [13] From eqns. (1.3) & (1.4), the error dynamical system is obtained as follows:

$$\dot{e} = F(x)(a - \widehat{\alpha}) + H(y)\left(\beta - \widehat{\beta}\right) - Ge, \tag{1.8}$$

Let $\widetilde{\alpha} = \alpha - \widehat{\alpha}, \widetilde{\beta} = \beta - \widehat{\beta}$. If the Lyapunov function is selected as:

$$V\left(e, \widetilde{\alpha}, \widetilde{\beta}\right) = \frac{1}{2}\left[e^T e + (a - \widehat{\alpha})^T (a - \widehat{\alpha}) + \left(\beta - \widehat{\beta}\right)^T \left(\beta - \widehat{\beta}\right)\right] \tag{1.9}$$

Then the derivative of V along the trajectory of the error dynamical system is as follows :

$$\dot{V}\left(e, \widetilde{\alpha}, \widetilde{\beta}\right) = \dot{e}^T e + (a - \widehat{\alpha})^T \dot{\widetilde{\alpha}} + \left(\beta - \widehat{\beta}\right)^T \dot{\widetilde{\beta}}$$

$$= \left[F(x)(a - \widehat{\alpha}) + H(y)\left(\beta - \widehat{\beta}\right) - Ge\right]^T e - (a - \widehat{\alpha})^T [F(x)]^T e - \left(\beta - \widehat{\beta}\right)^T [H(y)]^T e$$

$$= -Ge^T e < 0.$$

As long as $e \neq 0$, thus, $\frac{dV}{dt} < 0$ for $V > 0$, and the proof follows from the Theorem of Lyapunov stability.

Remark 1.2 *If system (1.3) and system (1.4) satisfies $f(.) = h(.)$ and $F(.) = H(.)$. Then the structure of system (1.3) and system (1.4) are identical. Therefore, Theorem (1.3) is also applicable to the adaptive synchronization of two identical chaotic systems with unknown parameters [13].*

Example 1.1 *We consider the Pan system as the master system and the Chen system as the slave system.*

The chaotic Pan system [13] is described by the following equations:

$$\begin{cases} \dot{x}_1(t) = a(x_2 - x_1), \\ \dot{x}_2(t) = cx_1 - x_1 x_3, \\ \dot{x}_3(t) = x_1 x_2 - bx_3, \end{cases} \tag{1.10}$$

Also, the Chen system [13] is given by:

$$\begin{cases} \dot{y}_1(t) = \alpha(y_2 - y_1) + u_1, \\ \dot{y}_2(t) = \gamma y_1 - \alpha y_1 - y_1 y_3 + \gamma y_2 + u_2, \\ \dot{y}_3(t) = y_1 y_2 - \beta y_3 + u_3, \end{cases} \tag{1.11}$$

Where x_1, x_2, x_3, y_1, y_2 and y_3 are state variables and a, b, c, α, β and γ are constants.

The synchronization error is defined as:

$$e_i = y_i - x_1, i = 1, 2, 3. \tag{1.12}$$

Then the error dynamics from (1.10) and (1.11) is as follows:

$$\begin{cases} \dot{e}_1 = \alpha\left(y_2 - y_1\right) - a\left(x_2 - x_1\right) + u_1, \\ \dot{e}_2 = \gamma y_1 - \alpha y_1 - y_1 y_3 + \gamma y_2 - c x_1 + x_1 x_3 + u_2, \\ \dot{e}_3 = y_1 y_2 - \beta y_3 - x_1 x_2 + b x_3 + u_3, \end{cases} \tag{1.13}$$

Defining the adaptive control function as:

$$\begin{cases} u_1(t) = -\widehat{\alpha}\left(y_2 - y_1\right) + \widehat{a}\left(x_2 - x_1\right) - G_1 e_1, \\ u_2(t) = -\widehat{\gamma} y_1 + \widehat{\alpha} y_1 + y_1 y_3 - \widehat{\gamma} y_2 + \widehat{c} x_1 - x_1 x_3 - G_2 e_2, \\ u_3(t) = -y_1 y_2 + \widehat{\beta} y_3 + x_1 x_2 + \widehat{b} x_3 - G_3 e_3, \end{cases} \tag{1.14}$$

By substituting the values of the adaptive control function (1.14) into the error dynamics (1.13) and simplifying it, and with the parameter estimation error given by $e_a = a - \widehat{a}$, $e_b = b - \widehat{b}$, $e_c = c - \widehat{c}$, $e_\alpha = \alpha - \widehat{\alpha}$, $e_\beta = \beta - \widehat{\beta}$, $e_\gamma = \gamma - \widehat{\gamma}$, *the error dynamics becomes:*

$$\begin{cases} \dot{e}_1 = e_a\left(y_2 - y_1\right) - e_\alpha\left(x_2 - x_1\right) - G_1 e_1, \\ \dot{e}_2 = e_\gamma\left(y_1 + y_2\right) - e_\alpha y_1 - e_c x_1 - G_2 e_2, \\ \dot{e}_3 = -e_\beta y_3 + e_b x_3 - G_3 e_3, \end{cases} \tag{1.15}$$

Chose the Lyapunov function as:

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2\right). \tag{1.16}$$

which is positive definite function, we also have:

$$\begin{aligned} \dot{e}_a = -\dot{\widehat{a}}, \dot{e}_b = -\dot{\widehat{b}}, \dot{e}_c = -\dot{\widehat{c}}, \\ \dot{e}_\alpha = -\dot{\widehat{\alpha}}, \dot{e}_\beta = -\dot{\widehat{\beta}}, \dot{e}_\gamma = -\dot{\widehat{\gamma}}, \end{aligned} \tag{1.17}$$

The parameters estimated update law is defined as:

$$\begin{cases} \dot{\widehat{\alpha}} = e_1\left(y_2 - y_1\right) - e_2 y_1, \\ \dot{\widehat{\beta}} = -e_3 y_3, \\ \dot{\widehat{\gamma}} = y_1 e_2 + y_2 e_2, \\ \dot{\widehat{a}} = -e_1\left(x_2 - x_1\right), \\ \dot{\widehat{b}} = x_3 e_3, \\ \dot{\widehat{c}} = -e_2 x_1, \end{cases} \tag{1.18}$$

Now differentiating (1.16), we get:

$$\dot{V} = -G_1 e_1^2 - G_2 e_2^2 - G_3 e_3^2. \tag{1.19}$$

Which is a negative definite function. Thus, by Lyapunov stability theory, it is evident that the synchronization error e_i $(i = 1, 2, 3)$ *and the parameter estimation error* $e_a, e_b, e_c, e_\alpha, e_\beta,$ *and* e_γ *decay to zero with time.*

Result 1.1 *The chaotic Pan and Chen systems are synchronized using the adaptive control law (1.14), where the parameter estimate update law is given by (1.18) and* G_i $(i = 1, 2, 3)$ *are positive constants.*

Numerical Result 1.1 *To solve the (1.10) and (1.11) with the adaptive nonlinear controller (1.14), by using the mathematica. We take* G_i $(i = 1, 2, 3)$ *. The parameters of the chaotic Pan and Chen systems are chosen as* $a = 10, b = 8/3, c = 16$ *and* $\alpha = 35, \beta = 3, \gamma = 28$*,respectively. The initial values of master and slave systems are chosen as* $x_1(0) = 15, x_2(0) = 12, x_3(0) = 32, y_1(0) = 5, y_2(0) = 18, y_3(0) = 26$*,respectively. Fig.1.3 Shows the synchronization of the Pan and Chen system.*

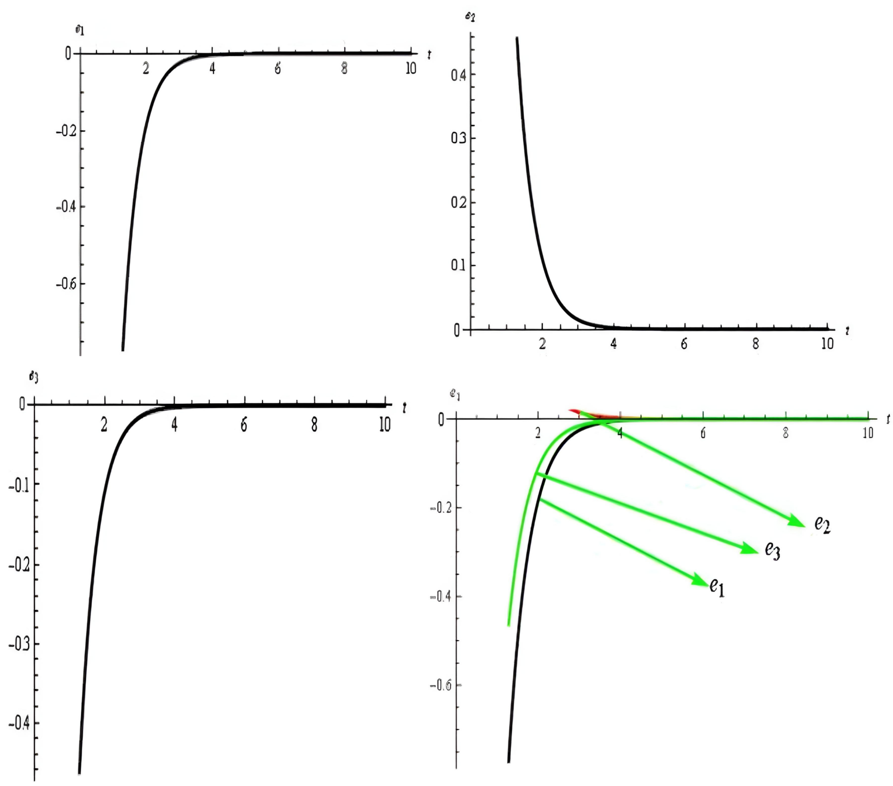

Fig. 1.3: Synchronization of Pan and Chen systems.

1.4 Fractional calculus

In recent decades, fractional calculus has emerged as a powerful tool for characterizing complex systems in diverse scientific and engineering fields, introducing fractional operators that extend beyond integer-order derivatives to accommodate non-integer orders.

1.4.1 Exploring Fundamentals: Fractional Derivatives

This subsection will provide an overview of fundamental definitions [7, 11, 27] and properties [6, 24] of fractional derivatives.

The Gamma Function

Definition 1.3 *The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers. Its definition is given by:*

$$\Gamma\left(t\right) = \int_0^\infty e^{-\tau}\tau^{t-1}dt,\ t \in \mathbb{R}^+. \tag{1.20}$$

The Riemann-Liouville derivative

Definition 1.4 *The Riemann-Liouville derivative of fractional order q of function $g\left(t\right)$, is given by:*

$$^{RL}D_{0,t}^{q}g\left(t\right) = \frac{1}{\Gamma\left(n-q\right)}\frac{d^n}{dt^n}\int_a^t \left(t-\tau\right)^{n-q-1}g\left(\tau\right)d\tau. \tag{1.21}$$

where $n-1 \leqslant q < n$.

The Caputo fractional derivative

Definition 1.5 *The Caputo fractional derivative of $g(t)$ is given as:*

$$^{C}D_{0,t}^{q}g\left(t\right) = \frac{1}{\Gamma\left(n-q\right)}\int_0^t \frac{g^{(n)}\left(\tau\right)}{\left(\tau-n\right)^{q-n+1}}d\tau, \tag{1.22}$$

For $n-1 < q \leq n$, where $n \in N$, $t > 0$. The expression involves the gamma function denoted by $\Gamma(\cdot)$.

The Laplace transform

Definition 1.6 *The Laplace transform formula for the Caputo fractional derivative is as follows:*

$$L\left(^{C}D_t^q g\left(t\right)\right) = s^q G\left(s\right) - \sum_{k=0}^{n-1} s^{q-k-1}g^{(n)}\left(0\right), \left(q>0, n-1<q\leq n\right). \tag{1.23}$$

Particularly, when $0 < q \leq 1$, we have:

$$L\left(D_t^q g\left(t\right)\right) = s^q G\left(s\right) - s^{q-1} g\left(0\right).$$

Additionally, the Laplace transform of the fractional derivative with order q is fulfilled by:

$$L(D_t^{-q} g(t)) = s^{-q} G\left(s\right), (q > 0) \tag{1.24}$$

With $G\left(s\right) = L(g(t))$.

Properties of derivatives

Property 1.1 *Suppose that $0 < q \leq 1$, than*

$$Dg\left(t\right) = D^{1-q} D^q g\left(t\right). \tag{1.25}$$

In which $D = \left(d/dt\right)$.

Property 1.2 *When $q = 0$,*

$$D^0 g\left(t\right) = g\left(t\right). \tag{1.26}$$

Property 1.3 *As in the case of the integer order derivative, the fractional order derivative in the sense of Caputo is a linear operator:*

$$D^q\left(\gamma f\left(t\right) + \delta g\left(t\right)\right) = \gamma D^q f\left(t\right) + \delta D^q g\left(t\right). \tag{1.27}$$

In which γ and δ are real constants.

Property 1.4 *As in the case of the integer order derivative, the fractional order derivative in the sense of Caputo satisfies the additive index law, i. e.,*

$$D^{q_1} D^{q_2} g\left(t\right) = D^{q_2} D^{q_1} g\left(t\right) = D^{q_1+q_2} g\left(t\right). \tag{1.28}$$

With some reasonable constraints on the function $g(t)$.

1.4.2 Stability of Fractional Order Systems

Stability of Linear Fractional Systems

The following linear fractional system:

$${}^c D_t^{\alpha_i} x_i\left(t\right) = \sum_{j=1}^{n} \alpha_{ij} x_j\left(t\right); i = 1, 2, ..., n. \tag{1.29}$$

From where i is a rational number between 0 and 1 and $D_t^{\alpha_i}$ is the Caputo fractional derivative of order i, for $i = 1, 2, ..., n$. Let M be the least common multiple of the denominators of i.

Theorem 1.4 *[10,18] If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ then the system (1.29) is asymptotically stable if $|\arg(spec(A))| > \alpha\frac{\pi}{2}$.*

Theorem 1.5 *[5,10] If α_i are distinct rational numbers, then the system (1.29) is asymptotically stable if all the roots λ of the equation $det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - A\right) = 0$, satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$ with $A = (\alpha_{ij})_{i\leq 1, j\leq n}$.*

The stability of nonlinear fractional systems

Now, let's consider the Caputo fractional nonlinear systems [15,24]:

$$^{c}D_t^{\alpha_i} x_i(t) = g_i(X(t)); i = 1, 2, ..., n. \tag{1.30}$$

Hence, $g_i : \mathbb{R}^n \longmapsto \mathbb{R}$ with continuous second partial derivatives in a ball centered on an equilibrium point $x^* = (x_1, x_2, ..., x_n)$ where $g_i(x^*) = 0$. Let M be the least common multiple of the denominators of i for $i = 1, 2, ..., n$. Then we have the following results:

Theorem 1.6 *[1,10] If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$, then the equilibrium point x^* of the system (1.30) is asymptotically stable if and only if $|arg(spec(J|_{x^*}))| > \alpha\frac{\pi}{2}$, where J is the Jacobian matrix of the system (1.30).*

Theorem 1.7 *[10,19] If α_i are distinct rational numbers, then the equilibrium point x^* of system (1.30) is asymptotically stable if all roots λ of the equation $det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - A\right) = 0$ satisfy $|arg(\lambda)| > \frac{\pi}{2M}$.*

1.4.3 Numerical method for solving fractional differential equation

Numerical techniques for solving dynamic systems of fractional order have gained significant prominence in various disciplines such as physics, engineering, finance, and beyond, owing to their extensive applicability. One such method that stands out is the Adams-Bashforth-Moulton algorithm.

Adams-Bashforth-Moulton algorithm

Considering $\alpha \in (m-1, m]$, the following initial value problem (IVP) is examined [10,16]:

$$D^{\alpha}x\left(t\right)=g\left(t,x\left(t\right)\right),\quad 0\leq t\leq T, \tag{1.31}$$

$$x^{(k)}\left(0\right)=x_{0}^{(k)}\quad k=0,1,...,m-1. \tag{1.32}$$

The IVP (1.31)and (1.32) is equivalent to the Volterra integral equation:

$$x\left(t\right)=\sum_{k=0}^{m-1}x_{0}^{(k)}\frac{t^{k}}{k!}+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}g\left(\tau,x\left(\tau\right)\right)d\tau. \tag{1.33}$$

Let's consider the uniform grid $t_n = nh, n = 0, 1, ..., N$, where N is an integer and $h := T/N$ for some fixed value T. We denote $x_h\left(t_n\right)$ as the approximation to $x\left(t_n\right)$. Assuming that we have already computed approximations $x_h\left(t_j\right)$ for $j = 1, 2, ..., n$, the objective is to determine $x_h\left(t_{n+1}\right)$ using the following equation

$$x_{h}\left(t_{n+1}\right)=\sum_{k=0}^{m-1}\frac{t_{n+1}^{k}}{k!}x_{0}^{(k)}+\frac{h^{\alpha}}{\Gamma\left(\alpha+2\right)}g\left(t_{n+1},x_{h}^{p}\left(t_{n+1}\right)\right)+\frac{h^{\alpha}}{\Gamma\left(\alpha+2\right)}\sum_{j=0}^{n}a_{j,n+1}g\left(t_{j},x_{n}\left(t_{j}\right)\right), \tag{1.34}$$

Where

$$a_{j,n+1}=\begin{cases}n^{\alpha+1}-\left(n-\alpha\right)\left(n+1\right)^{\alpha}, & \text{if}\quad j=0\\ \left(n-j+2\right)^{\alpha+1}+\left(n-j\right)^{\alpha+1}-2\left(n-j+1\right)^{\alpha+1}, & \text{if } 1\leq j\leq n,\\ 1, & \text{if } j=n+1\end{cases}\cdot \tag{1.35}$$

The initial approximation $x_h^p\left(t_{n+1}\right)$, referred to as the predictor, is computed as follows:

$$x_{h}^{p}\left(t_{n+1}\right)=\sum_{k=0}^{m-1}\frac{t_{n+1}^{k}}{k!}x_{0}^{(k)}+\frac{1}{\Gamma\left(\alpha\right)}\sum_{j=0}^{n}b_{j,n+1}g\left(t_{j},x_{n}\left(t_{j}\right)\right)$$

Where

$$b_{j,n+1}=\frac{h^{\alpha}}{\alpha}\left(\left(n+1-j\right)^{\alpha}-\left(n-j\right)^{\alpha}\right). \tag{1.36}$$

The method's error is determined by:

$$\max_{j=0,1,...N}\left|x\left(t_{j}\right)-x_{h}\left(t_{j}\right)\right|=O\left(h^{p}\right), \tag{1.37}$$

Where $p = \min\left(2, 1+\alpha\right)$.

This method entails predicting the solution for the next time step using the Adams-Bashforth method, followed by correcting the solution. Both steps are adapted to accommodate fractional derivatives. While known for its accuracy and efficiency, this method may incur significant computational costs, especially for high-order dynamical systems.

Conclusion

This chapter covers dynamic systems, chaos theory, synchronization, and fractional calculus. It discusses fundamental concepts in dynamic systems, chaotic systems, synchronization methods, and fractional calculus applications, providing a comprehensive overview of their significance in various fields.

CHAPTER 2

Examples of chaotic systems of integer orders and fractional orders

Contents

Introduction

Integer-order chaotic systems and fractional-order chaotic systems represent two distinct classes of dynamical systems characterized by chaotic behavior. Integer-order systems are governed by differential equations featuring integer-order derivatives, showcasing well-established properties such as sensitivity to initial conditions and the presence of strange attractors. On the other hand, fractional-order systems incorporate fractional derivatives or integrals, exhibiting distinctive traits like increased sensitivity to initial conditions and aperiodic long-term behavior. Both types of systems find applications across diverse fields, offering unique insights into the dynamics of complex phenomena.
In this chapter, examples of both integer-order and fractional-order chaotic systems in three dimensions are explored, providing a comprehensive overview of their behavior.

The Nwachioma chaotic system

Example 2.1 ***The Nwachioma chaotic system*** *[21,25], is described by the following mathematical model:*

$$\begin{cases} \dot{x}_1(t) = a_1 x_1(t) + a_2 x_1(t) x_3(t) + a_3 x_2(t) x_3(t), \\ \dot{x}_2(t) = a_4 x_2(t) + a_5 x_1(t) x_3(t) + a_6, \\ \dot{x}_3(t) = a_7 x_3(t) + a_8 x_1^2(t) x_2(t) + a_9, \end{cases} \tag{2.1}$$

The system incorporates constants denoted by a_i (where $i = 1, 2, ..., 9$), with non linear terms: $x_1(t)x_3(t), x_2(t)x_3(t), x_1^2(t)x_2(t)$.
Notably, this autonomous system, meaning it lacks external inputs influencing its dynamics, exhibits purely chaotic behavior under the specific constant values presented in equation (2.1):

$$a_1 = -0.1, a_2 = 0.15, a_3 = 0.18, a_4 = 3.9, a_5 = -1.5, a_6 = -4, a_7 = -4.9, a_8 = 2.5, a_9 = 0,$$

And also when the initial conditions are set to $x_1(0) = 1, x_2(0) = 3$ and $x_3(0) = 8$, (see Figs.2.1–2.4).

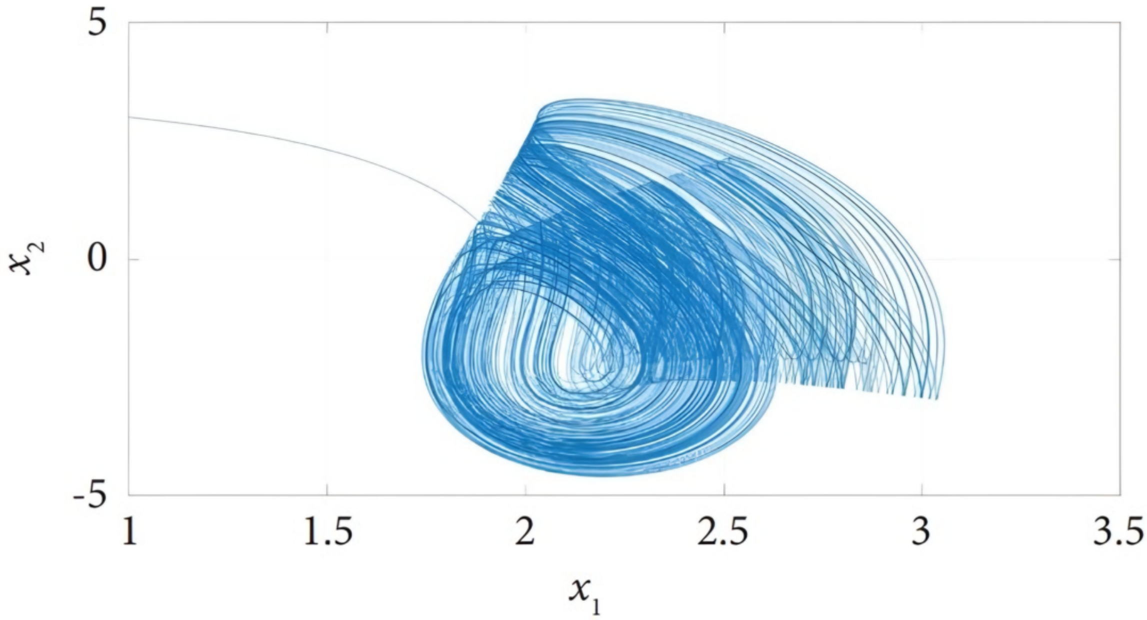

Fig. 2.1: Chaotic attractor of system (2.1) in $(x_1 - x_2)$ plan.

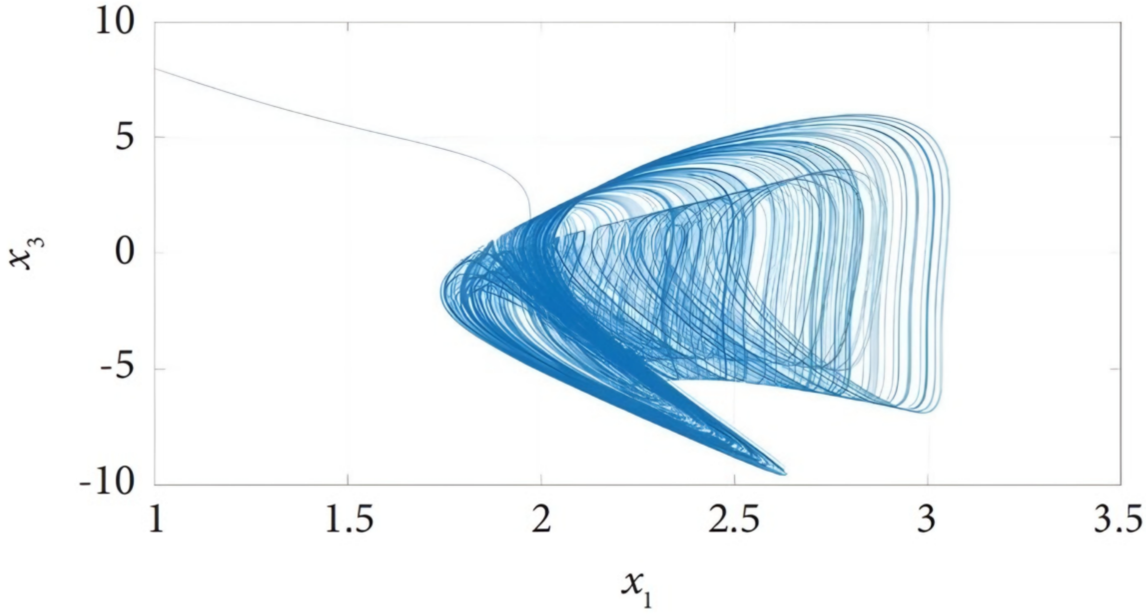

Fig. 2.2: Chaotic attractor of system (2.1) in $(x_1 - x_3)$ plan.

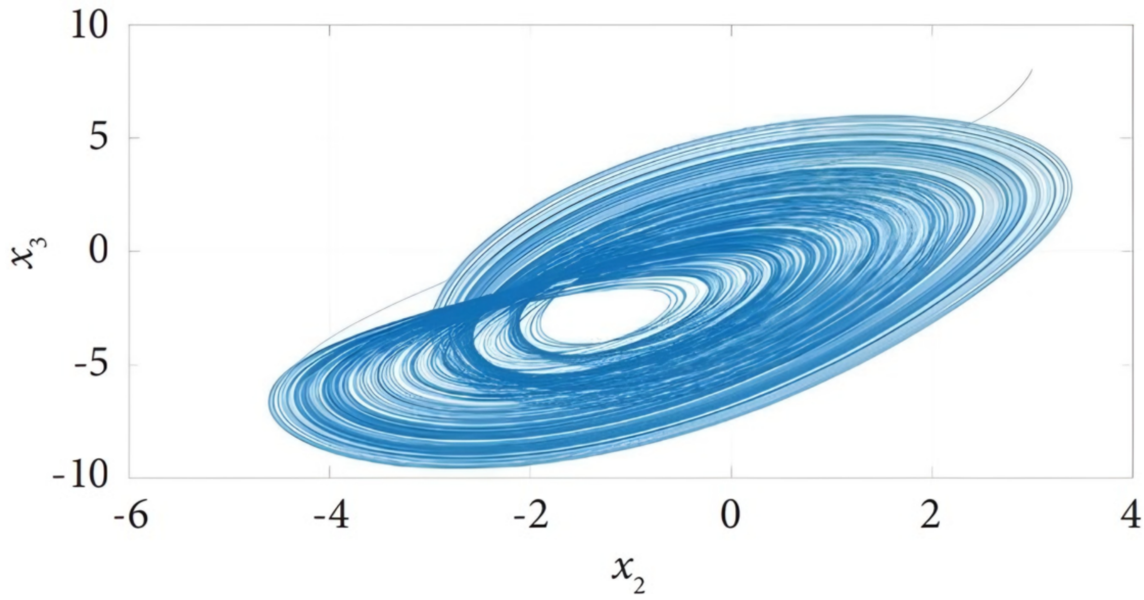

Fig. 2.3: Chaotic attractor of system (2.1) in $(x_2 - x_3)$ plan.

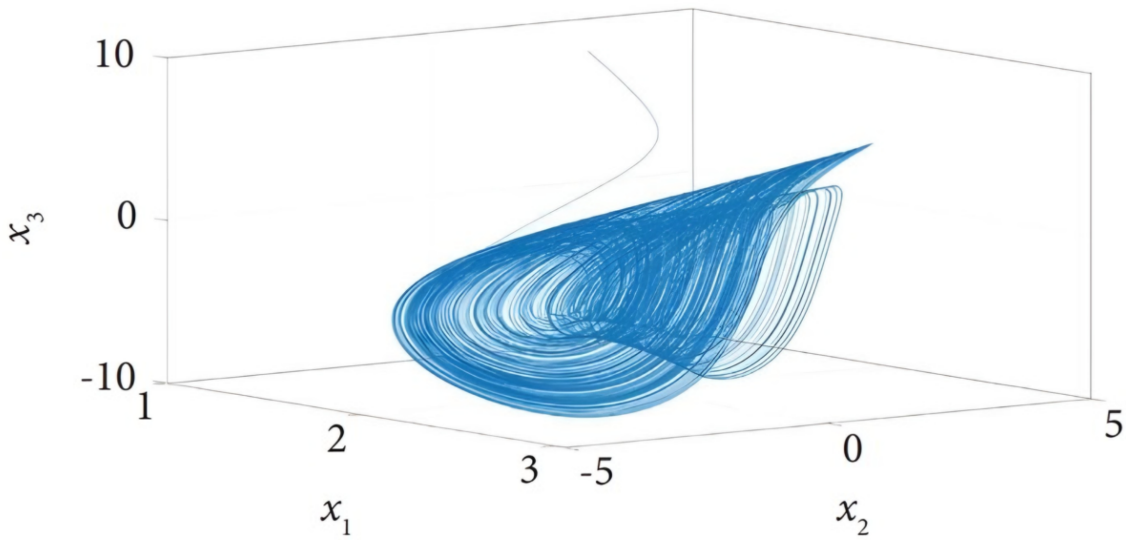

Fig. 2.4: Chaotic attractor of system (2.1) in $(x_1 - x_2 - x_3)$ space.

The Cai chaotic system

Example 2.2 ***The chaotic Cai system*** *[12, 33], with the following system equations:*

$$\begin{cases} \dot{x}_1(t) = a\left(x_2(t) - x_1(t)\right), \\ \dot{x}_2(t) = bx_1(t) + cx_2(t) - x_1(t)x_3(t), \\ \dot{x}_3(t) = x_1^2(t) - dx_3(t), \end{cases} \tag{2.2}$$

Where $x_1(t), x_2(t), x_3(t)$ are the state variables, a, b and c, d are the parameters.

The systems (2.2) are chaotic and hyperchaotic. The projection of attractor of chaotic Cai system is shown in Figs.(2.8—2.7) when the parameters are taken as $a = 20, b = 14, c = 10.6, d = 28$, and the initial values $(x_1(0), x_2(0), x_3(0)) = (4, -3, 4)$.

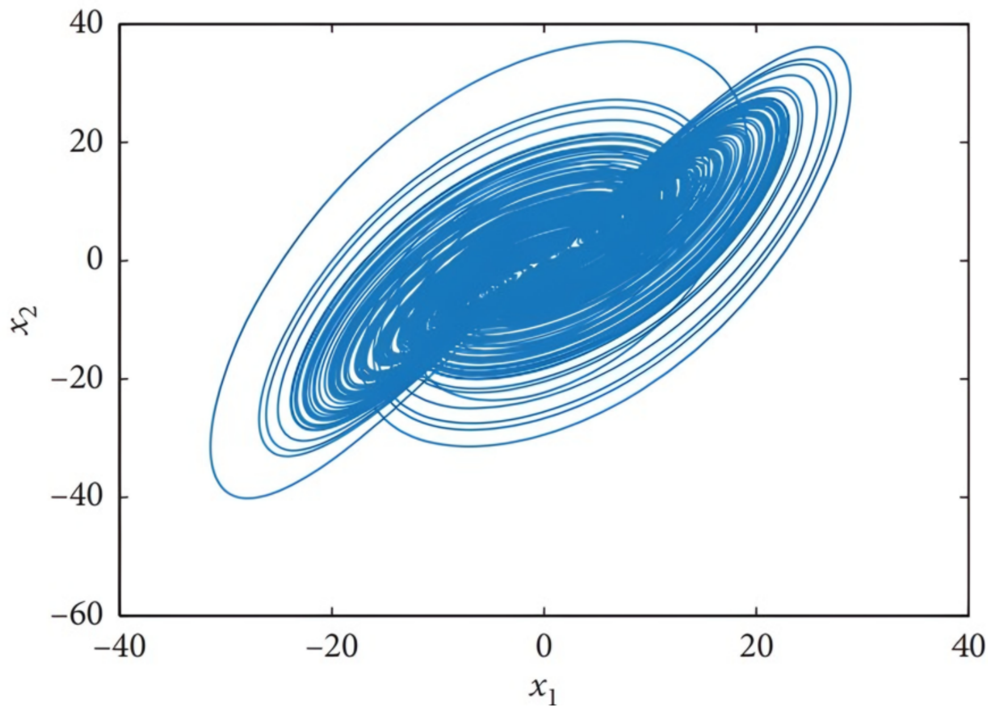

Fig. 2.5: Chaotic attractor of system (2.2) in $(x_1 - x_2)$ plan.

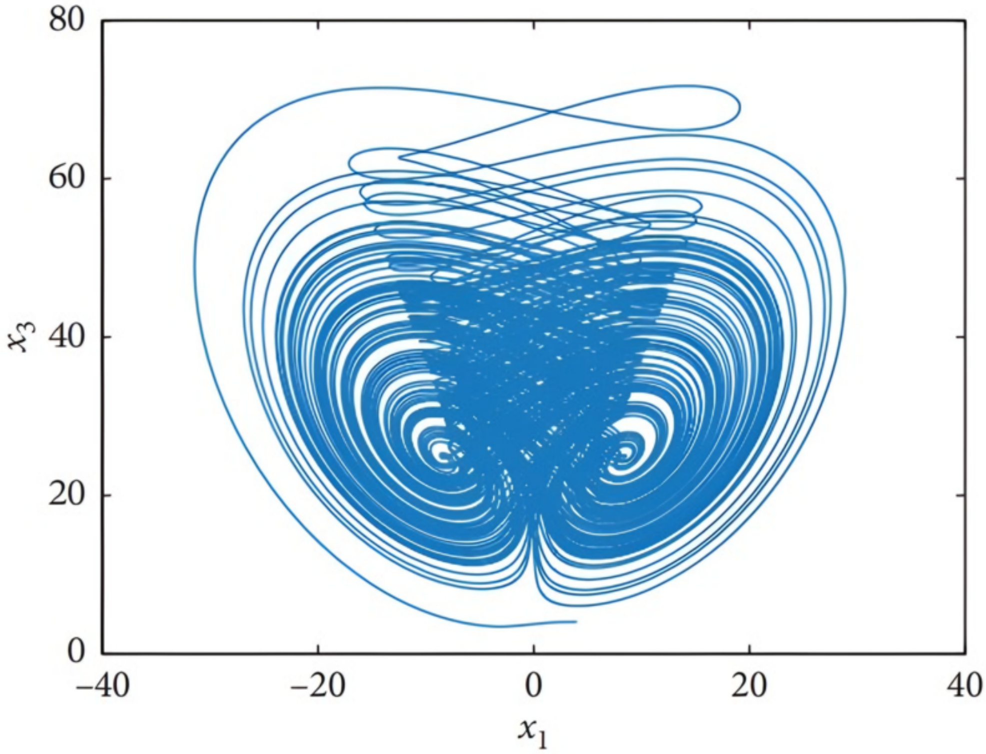

Fig. 2.6: Chaotic attractor of system (2.2) in $(x_1 - x_3)$ plan.

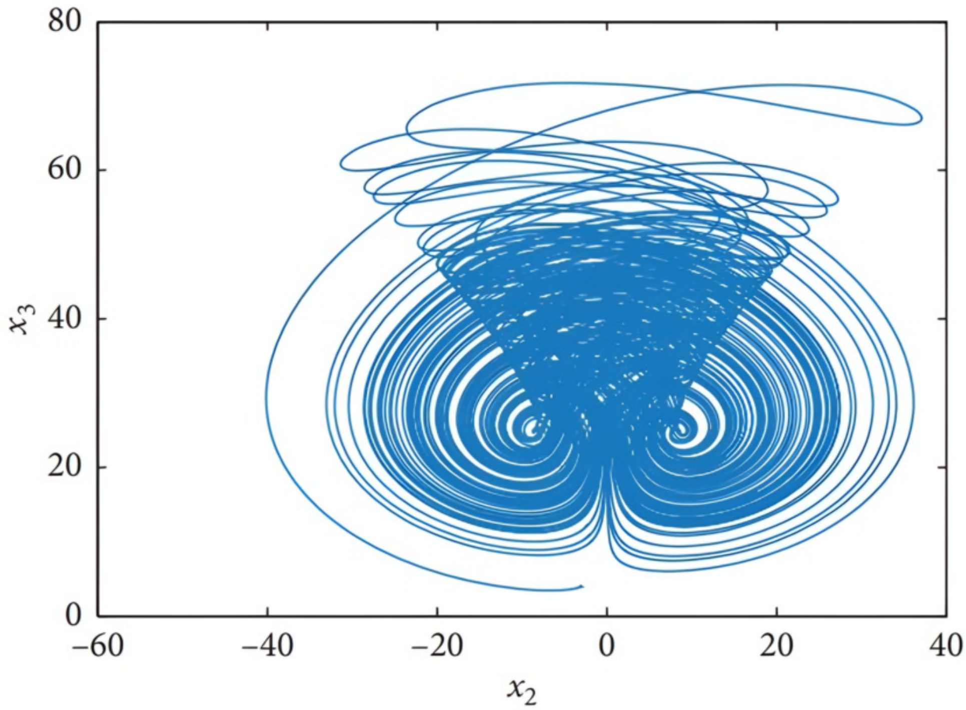

Fig. 2.7: Chaotic attractor of system (2.2) in $(x_2 - x_3)$ plan.

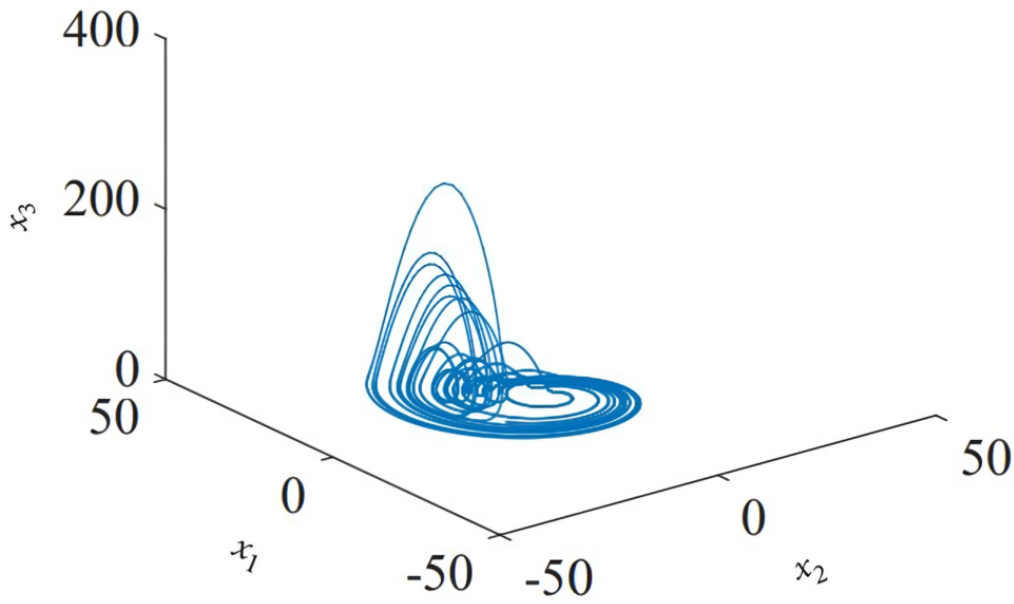

Fig. 2.8: Chaotic attractor of system (2.2) in $(x_1 - x_2 - x_3)$ space.

The fractional-order Newton–Leipnik system

Example 2.3 ***The fractional-order Newton–Leipnik system*** *is described as follows [2, 28]:*

$$\begin{cases} D_t^{q_1} x_1(t) = -ax_1(t) + x_2(t) + 10x_2(t)x_3(t), \\ D_t^{q_2} x_2(t) = -x_1(t) - 0.4x_2(t) + 5x_1(t)x_3(t), \\ D_t^{q_3} x_3(t) = bx_3(t) - 5x_1(t)x_2(t), \end{cases} \tag{2.3}$$

Where q_1, q_2, and q_3 are real derivative orders and where $0 < q_1, q_2, q_3 \leq 1$, and a and b are variable parameters. In Figs.(2.12—2.11) is depicted the simulation result (double-scroll attractor) of the fractional order Chen system (2.3) with the parameters $a = 0.4, b = 0.175$, orders $q_1 = q_2 = q_3 = 0.95$, and initial conditions $(x_1(0), x_2(0), x_3(0)) = (0.19, 0, -0.18)$.

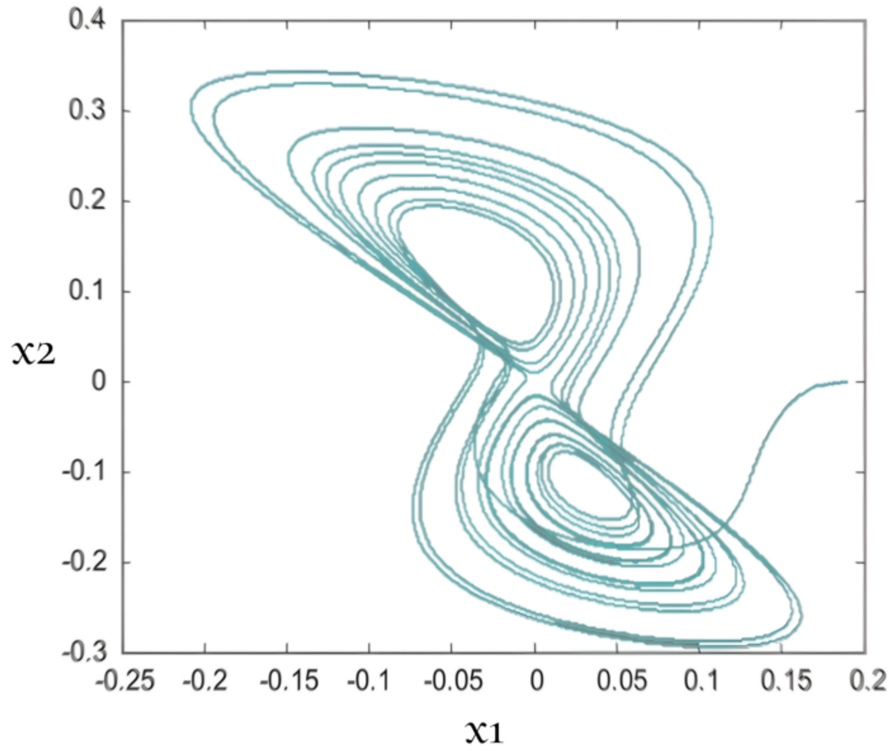

Fig. 2.9: Chaotic attractor of system (2.3) in $x_1 - x_2$ plane.

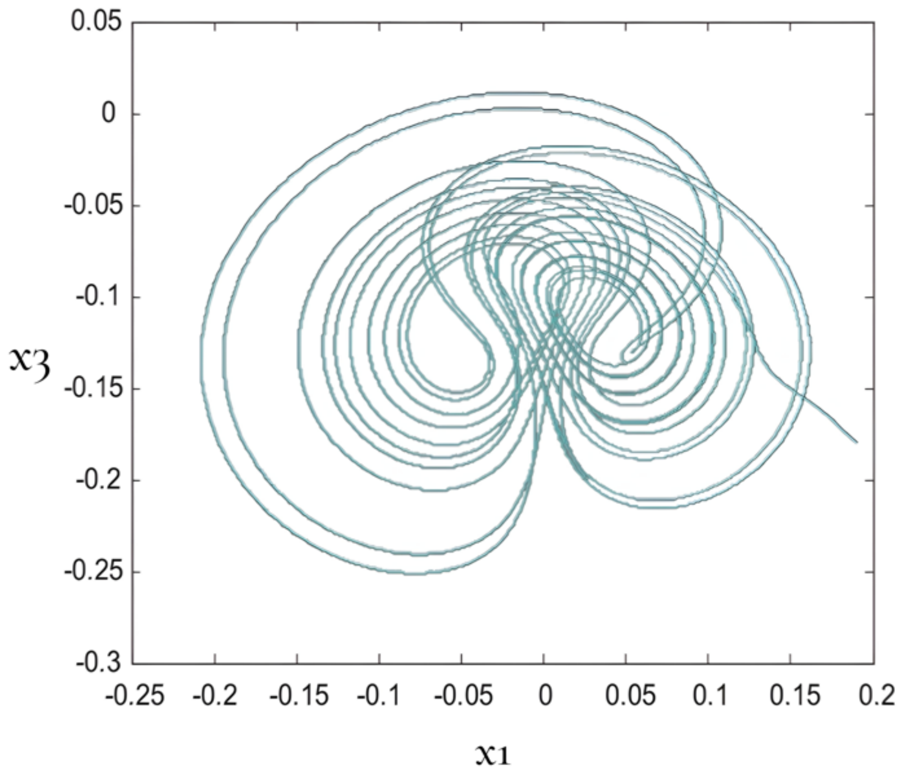

Fig. 2.10: Chaotic attractor of system (2.3) in $x_1 - x_3$ plane.

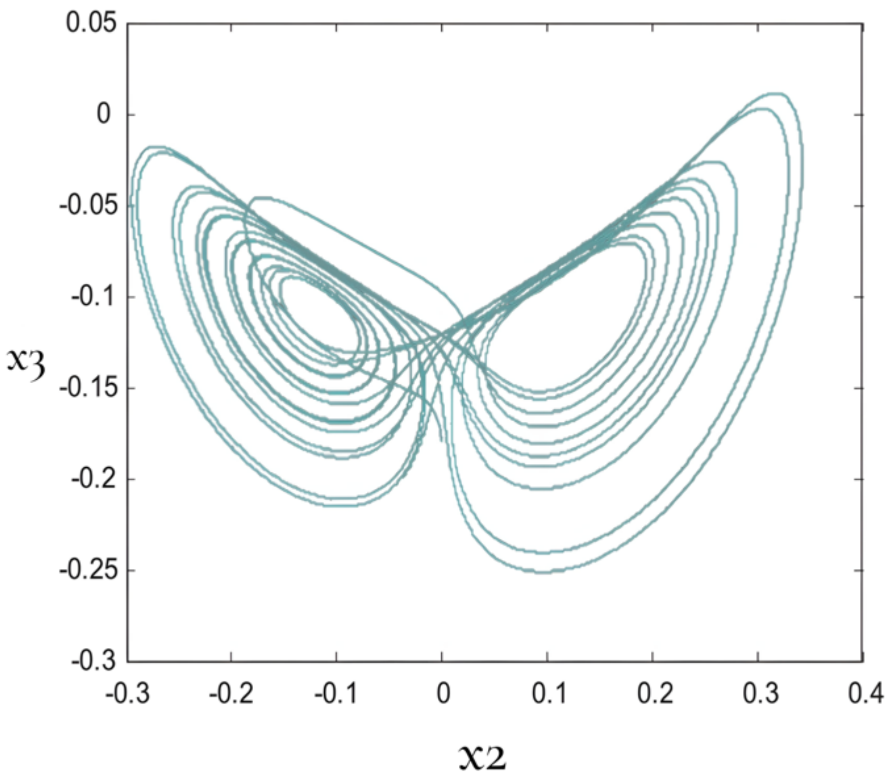

Fig. 2.11: Chaotic attractor of system (2.3) in $x_2 - x_3$ plane.

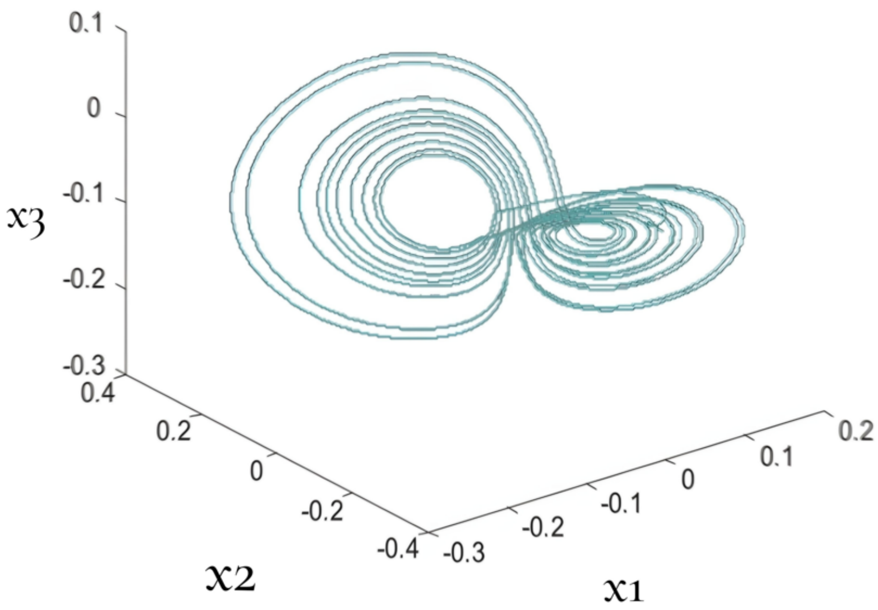

Fig. 2.12: Chaotic attractor of system (2.3) in $x_1 - x_2 - x_3$ space.

The fractional-order Rössler system

Example 2.4 ***The Rössler system*** *is a nonlinear system with the potential to exhibit a chaotic attractor consisting of a single scroll [14,17], its* ***fractional*** *version can be expressed as :*

$$\begin{cases} D_t^{q_1} x_1(t) = -\left(x_2(t) + x_3(t)\right), \\ D_t^{q_2} x_2(t) = x_1(t) + a x_2(t), \\ D_t^{q_3} x_3(t) = b + x_3(t)\left(x_1(t) - c\right), \end{cases} \tag{2.4}$$

Where $x_1(t), x_2(t), x_3(t)$ are the state variables, a, b and c are the parameters, q_1, q_2, q_3 are derivative orders, $0 < q_1, q_2, q_3 < 1$. Where $a = b = 0.2, c = 0.5$. In Figs.(2.14—2.16) is depicted the simulation result (double-scroll attractor) of the fractional order Rössler system (2.4) with the parameters $a = 0.63, b = 10, c = 0.2$, orders $q_1 = q_2 = q_3 = 0.95$, and initial conditions $(x_1(0), x_2(0), x_3(0)) = (0, 0, 0)$ with $h = 0, 005$.

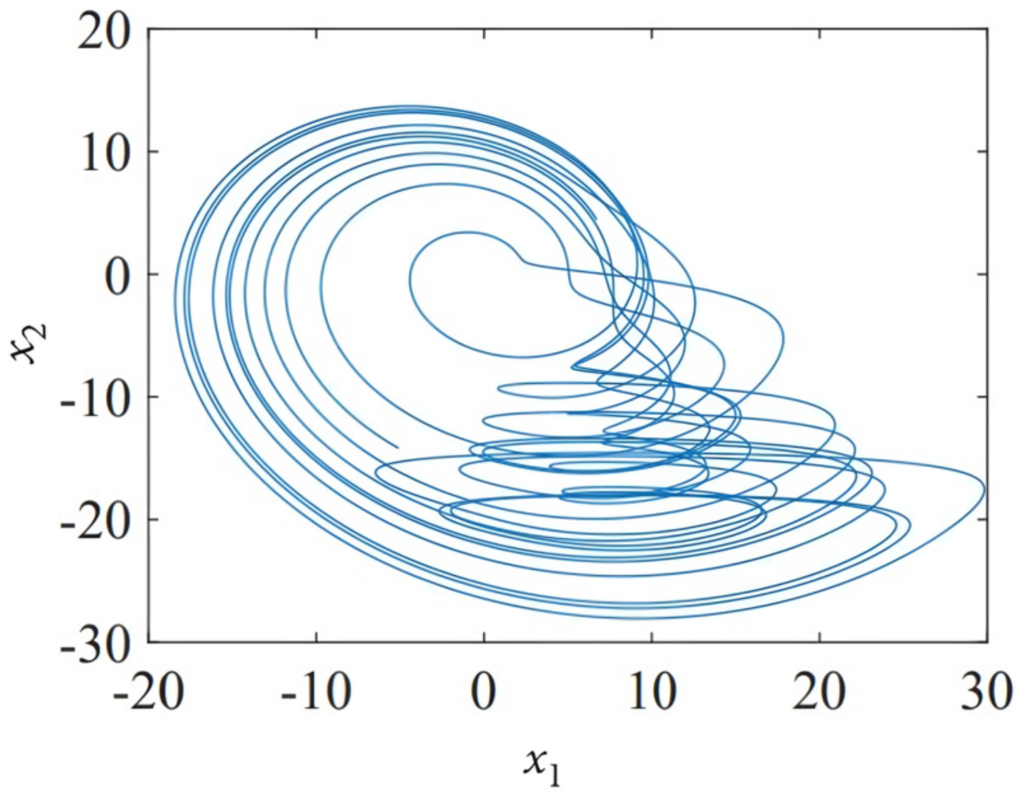

Fig. 2.13: Chaotic attractor of system (2.4) in $x_1 - x_2$ plane.

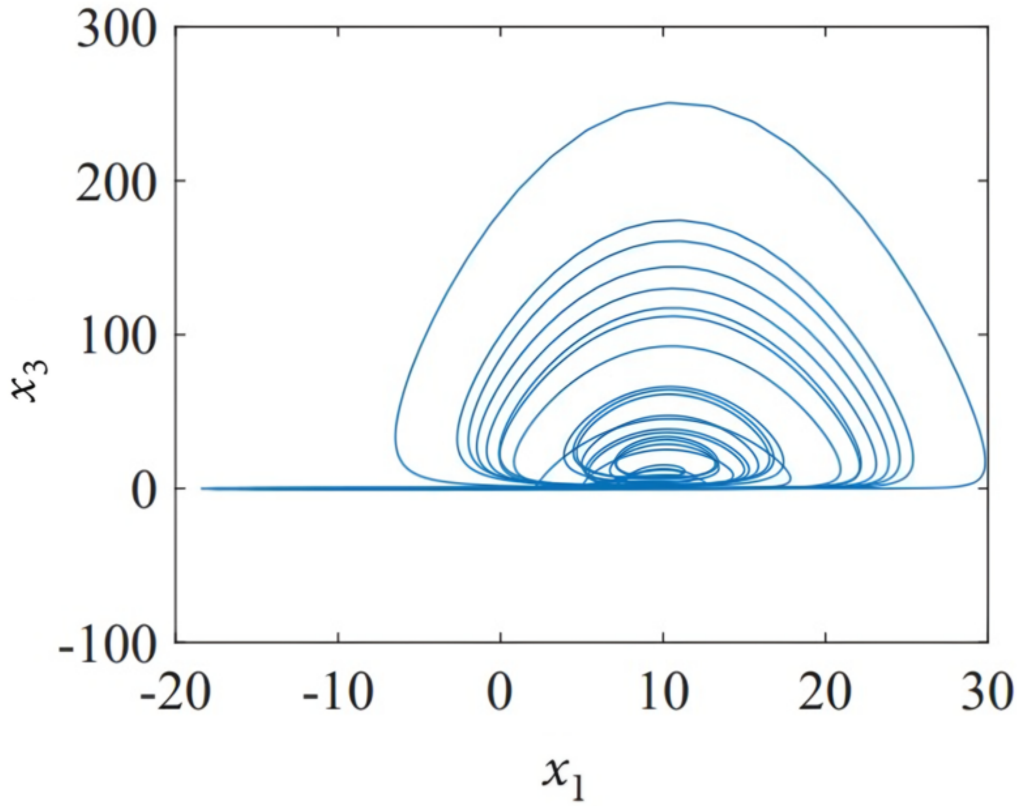

Fig. 2.14: Chaotic attractor of system (2.4) in $x_1 - x_3$ plane.

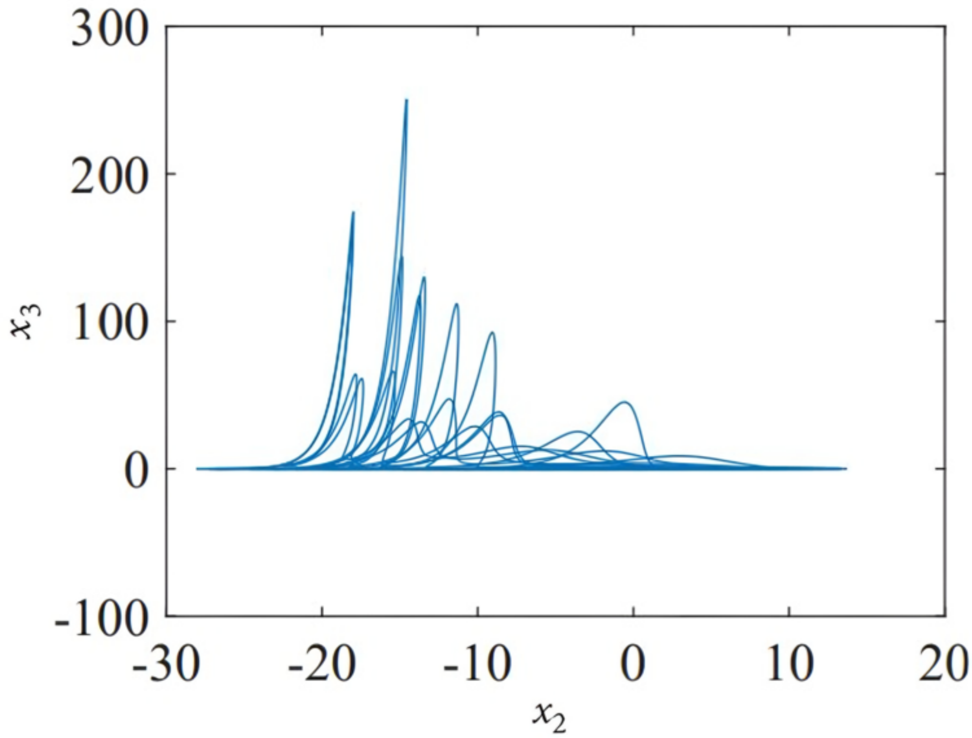

Fig. 2.15: Chaotic attractor of system (2.4) in $x_2 - x_3$ plane.

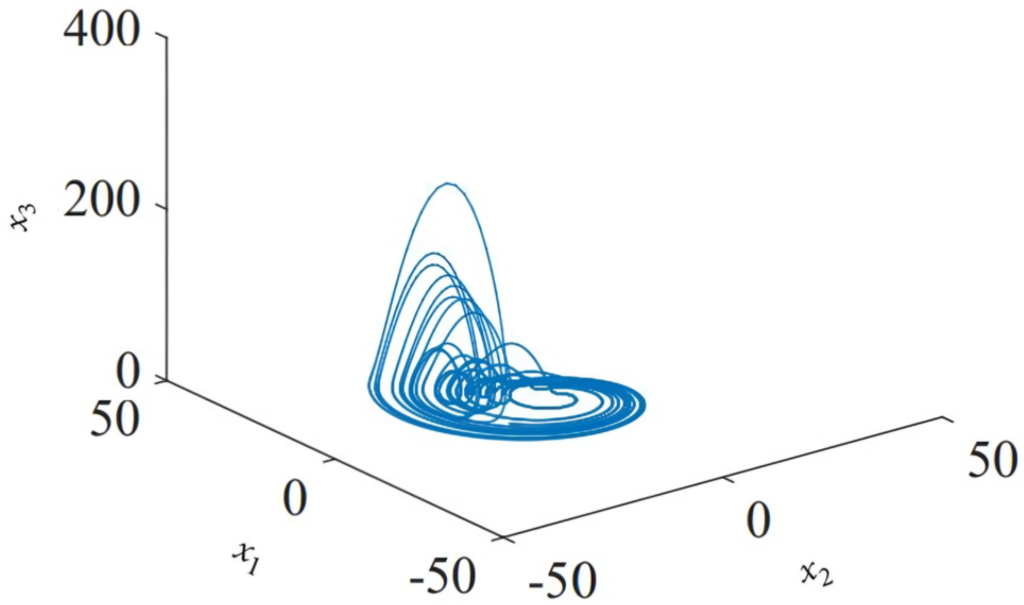

Fig. 2.16: Chaotic attractror of system (2.4) in $x_2 - x_1 - x_3$ space.

Conclusion

In this chapter, we have presented some examples of 3D integer-order and fractional-order chaotic systems, which illustrate the potential of this approach for modeling complex systems and designing control strategies.

CHAPTER 3

Synchronization between fractional and integer order chaotic systems

Contents

Introduction

In this chapter, the synchronization between a fractional-order chaotic system and an integer-order chaotic system is examined in detail. The adaptive synchronization of identical new chaotic systems is achieved using the master-slave adaptive feedback control method, with verification conducted through the Lyapunov stability theory. The effectiveness of this approach is further validated through numerical simulations.

3.1 Problem formulation

We consider the drive system given by [8]:

$$\dot{x}_i(t) = f_i(X(t)), \ i = 1, ..., n. \tag{3.1}$$

Where: $X(t) = (x_1, x_2, ..., x_n)^T$ is the state vector of the system (3.1) , $f_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, ..., n$ are nonlinear functions, and as response system the system given by [8]:

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + V_i, \ i = 1, ..., n. \tag{3.2}$$

Where: $Y(t) = (y_1, y_2, ..., y_n)^T$ is the state vector of the system (3.2), $g_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, ..., n$ are nonlinear functions, $0 < q_i < 1$, $D_t^{q_i}$ is the Caputo fractionnal derivative of order q_i for $i = 1, ..., n$, V_i are controllers to be designed such as the system (3.1) and the system (3.2) to be synchronized.

3.2 Synchronization of fractionel-integer order systems

We decompose the controller V_i, $i = 1, ..., n$, into two sub-controllers U_i and U_{ii} i.e., $V_i = U_i + U_{ii}$ for $i = 1, ..., n$, and propose the following form for the sub-controller U_i given by:

$$U_i = \left(D_t^{q_i - 1} - 1\right)\left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)), \ i = 1, ..., n. \tag{3.3}$$

So we have:

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + U_i + U_{ii},\ i = 1, ..., n. \tag{3.4}$$

By using (3.3), we can rewrite the slave system (3.4) as follows:

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + \left(D_t^{q_i - 1} - 1\right) (\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + U_{ii},\ i = 1, ..., n. \tag{3.5}$$

By simplifying, we find:

$$D_t^{q_i} y_i(t) = D_t^{q_i - 1} \left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right) + U_{ii},\ i = 1, ..., n. \tag{3.6}$$

Applying a Laplace transform :

$$s^{q_i} Y_i(s) - s^{q_i - 1} y_i(0) = s^{q_i - 1} L \left[\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right] + L[U_{ii}],\ i = 1, ..., n. \tag{3.7}$$

By multiplying all sides by $s^{-(q_i - 1)}$, we get:

$$s Y_i(s) - y_i(0) = L \left[\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right] + s^{-(q_i - 1)} L[U_{ii}],\ i = 1, ..., n. \tag{3.8}$$

Appling the inverse Laplace transform :

$$L^{-1}[s Y_i(s) - y_i(0)] = L^{-1} \left[\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right] + L^{-1}\left[s^{-(q_i - 1)} [U_{ii}] \right]. \tag{3.9}$$

Then

$$D_t(y_i(t)) = \left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right) + D_t^{1 - q_i}(U_{ii}), i = 1, ..., n. \tag{3.10}$$

This is equal to:

$$\dot{y}_i(t) = \left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right) + D_t^{1 - q_i}(U_{ii}), i = 1, ..., n. \tag{3.11}$$

Then, the problem of synchronization (of any type) between the fractional-integer order systems (3.1) and (3.2) is reduced to another problem of synchronization between the integer-order systems (3.1) and (3.11). The state errors for systems (3.1) and (3.11) are:

$$e_i(t) = y_i(t) - x_i(t), i = 1, ..., n. \tag{3.12}$$

Therefore, the error dynamic system is given by:

$$\dot{e}_i(t) = \dot{y}_i(t) - \dot{x}_i(t), \; i = 1, ..., n. \tag{3.13}$$

$$\left\{ \dot{e}_i(t) = \left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right) + D_t^{1-q_i}(U_{ii}) - f_i(X(t)), \; i = 1, ..., n. \right. \tag{3.14}$$

In view of (3.14), we propose the sub-controller U_{ii} in the form:

$$U_{ii} = D_t^{q_i - 1} \left(f_i(X(t)) - \sum_{j=1}^{n} b_{ij} x_j(t) - g_i(Y(t)) - G_i e_i(t) \right), \; i = 1, ..., n. \tag{3.15}$$

Using the Adaptive control method given in (theorem 1.3.4), we can Achieve synchronization between systems (3.1) and (3.11), This implies that for any initial conditions $y(0) \neq x(0)$, we have :

$$\lim_{t \to +\infty} \|e_i\| = 0. \tag{3.16}$$

3.3 Exemple in 3D

We use the following *integer-order Yang chaotic system* [23] as a *driving* system in this study:

$$\begin{cases} \dot{x}_1(t) = a(x_2 - x_1) \\ \dot{x}_2(t) = cx_1 - x_1 x_3 \\ \dot{x}_3(t) = x_1 x_2 - bx_3 \end{cases} \tag{3.17}$$

Where a, b, c are real parameters with $a, b > 0$ and $c \in R$.

Let us define *the fractional order Yang system* [23] (*the controlled*) described by (3.17), which has the following form:

$$\begin{cases} D_t^{q_1} y_1(t) = a(y_2 - y_1) + V_1 \\ D_t^{q_2} y_2(t) = cy_1 - y_1 y_3 + V_2 \\ D_t^{q_3} y_3(t) = y_1 y_2 - by_3 + V_3 \end{cases} \tag{3.18}$$

Where q_1, q_2, q_3 are derivative orders, $0 < q_1, q_2, q_3 \leq 1$, and a, b and c are the parameters. In Figs.(3.1) is depicted the simulation result (double scroll-attractor) of the fractional order Yang chotic system (3.18) with parameters $a = 10, b = 8/3$, and $c = 16$ orders $q_1 = q_2 = q_3 = 0.99$, and initial conditions $y_1(0), y_2(0), y_3(0)) = (0.1, 0.1, 0.1)$.

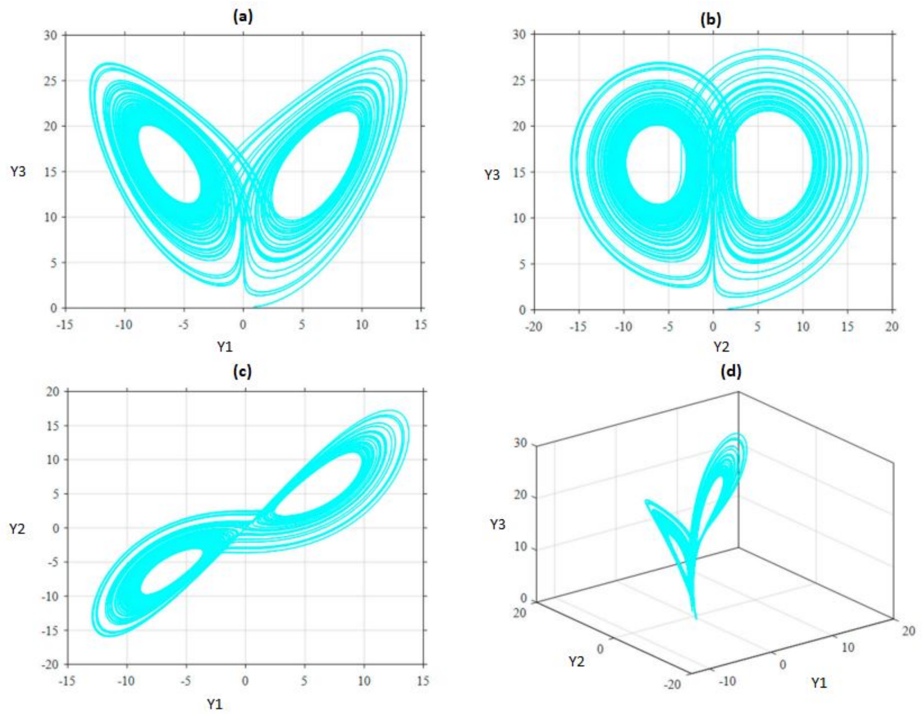

Fig. 3.1: Strange attractor of the fractional-order yang system (3.18) in state space with orders $q_1 = q_2 = q_3 = 0.99$: (a) $y_1 - y_3$ plane; (b) $y_2 - y_3$ plane; (c)$y_1 - y_2$ plane; and (d) $y_1 - y_2 - y_3$ space.

We decompose the controller V_i, $i = 1, 2, 3$ into two sub-controllers U_i and U_{ii} i.e;

$$\begin{cases} V_1 = U_1 + U_{11} \\ V_2 = U_2 + U_{22} \\ V_3 = U_3 + U_{33} \end{cases} \tag{3.19}$$

So that it is:

$$\begin{cases} U_1 = (D_t^{q_1-1} - 1)(a(y_2 - y_1)) \\ U_2 = (D_t^{q_2-1} - 1)(cy_1 - y_1y_3) \\ U_3 = (D_t^{q_3-1} - 1)(y_1y_2 - by_3) \end{cases} \tag{3.20}$$

So the response system (3.18) can be rwitten as followes :

$$\begin{cases} D_t^{q_1} y_1 = a(y_2 - y_1) + U_1 + U_{11} \\ D_t^{q_2} y_2 = cy_1 - y_1y_3 + U_2 + U_{22} \\ D_t^{q_3} y_3 = y_1y_2 - by_3 + U_3 + U_{33} \end{cases} \tag{3.21}$$

Substituting the values of U_i, $i = 1, 2, 3$ we find:

$$\begin{cases} D_t^{q_1} y_1 = a(y_2 - y_1) + D_t^{q_1-1}(a(y_2 - y_1)) - (a(y_2 - y_1) + U_{11} \\ D_t^{q_2} y_2 = cy_1 - y_1y_3 + D_t^{q_2-1}(cy_1 - y_1y_3) - (cy_1 - y_1y_3) + U_{22} \\ D_t^{q_3} y_3 = y_1y_2 - by_3 + D_t^{q_3-1}(y_1y_2 - by_3) - (y_1y_2 - by_3) + U_{33} \end{cases} \tag{3.22}$$

From that we obtiend:

$$\begin{cases} D_t^{q_1} y_1 = D_t^{q_1-1}(a(y_2 - y_1)) + U_{11} \\ D_t^{q_2} y_2 = D_t^{q_2-1}(cy_1 - y_1y_3) + U_{22} \\ D_t^{q_3} y_3 = D_t^{q_3-1}(y_1y_2 - by_3) + U_{33} \end{cases} \tag{3.23}$$

Appling a laplace transform:

$$\begin{cases} L\left[D_t^{q_1} y_1\right] = L\left[D_t^{q_1-1}(a(y_2 - y_1))\right] + L\left[U_{11}\right] \\ L\left[D_t^{q_2} y_2\right] = L\left[D_t^{q_2-1}(cy_1 - y_1y_3)\right] + L\left[U_{22}\right] \\ L\left[D_t^{q_3} y_3\right] = L\left[D_t^{q_3-1}(y_1y_2 - by_3)\right] + L\left[U_{33}\right] \end{cases} \tag{3.24}$$

For that we have:

$$\begin{cases} S^{q_1}y_1(s) - S^{q_1-1}y_1(0) = S^{q_1-1}L\left[a(y_2-y_1)\right] + L\left[U_{11}\right] \\ S^{q_2}y_2(s) - S^{q_2-1}y_2(0) = S^{q_2-1}L\left[cy_1-y_1y_3\right] + L\left[U_{22}\right] \\ S^{q_3}y_3(s) - S^{q_3-1}y_3(0) = S^{q_3-1}L\left[y_1y_2-by_3\right] + L\left[U_{33}\right] \end{cases} \tag{3.25}$$

Multiply both sides by : $S^{-q_i+1}, i=\overline{1,3}$:

$$\begin{cases} S^{-q_1+1}S^{q_1}y_1(s) - S^{-q_1+1}S^{q_1-1}y_1(0) = S^{-q_1+1}S^{q_1-1}L\left[a(y_2-y_1)\right] + S^{-q_1+1}L\left[U_{11}\right] \\ S^{-q_2+1}S^{q_2}y_2(s) - S^{-q_2+1}S^{q_2-1}y_2(0) = S^{-q_2+1}S^{q_2-1}L\left[cy_1-y_1y_3\right] + S^{-q_2+1}L\left[U_{22}\right] \\ S^{-q_3+1}S^{q_3}y_3(s) - S^{-q_3+1}S^{q_3-1}y_3(0) = S^{-q_3+1}S^{q_3-1}L\left[y_1y_2-by_3\right] + S^{-q_3+1}L\left[U_{33}\right] \end{cases} \tag{3.26}$$

For that we obtien:

$$\begin{cases} Sy_1(s) - y_1(0) = L\left[a(y_2-y_1)\right] + S^{-q_1+1}L\left[U_{11}\right] \\ Sy_2(s) - y_2(0) = L\left[cy_1-y_1y_3\right] + S^{-q_2+1}L\left[U_{22}\right] \\ Sy_3(s) - y_3(0) = L\left[y_1y_2-by_3\right] + S^{-q_3+1}L\left[U_{33}\right] \end{cases} \tag{3.27}$$

Appling a inverse laplace transforme:

$$\begin{cases} L^{-1}\left[Sy_1(s) - y_1(0)\right] = L^{-1}L\left[a(y_2-y_1)\right] + L^{-1}\left[S^{-q_1+1}L\left[U_{11}\right]\right] \\ L^{-1}\left[Sy_2(s) - y_2(0)\right] = L^{-1}L\left[cy_1-y_1y_3\right] + L^{-1}\left[S^{-q_2+1}L\left[U_{22}\right]\right] \\ L^{-1}\left[Sy_3(s) - y_3(0)\right] = L^{-1}L\left[y_1y_2-by_3\right] + L^{-1}\left[S^{-q_3+1}L\left[U_{33}\right]\right] \end{cases} \tag{3.28}$$

Which be can writing:

$$\begin{cases} D_t(y_1(t)) = a(y_2-y_1) + D^{1-q_1}(U_{11}) \\ D_t(y_2(t)) = cy_1 - y_1y_3 + D^{1-q_2}(U_{22}) \\ D_t(y_3(t)) = y_1y_2 - by_3 + D^{1-q_3}(U_{33}) \end{cases} \tag{3.29}$$

This is the same as:

$$\begin{cases} \dot{y}_1(t) = a(y_2-y_1) + D^{1-q_1}(U_{11}) \\ \dot{y}_2(t) = cy_1 - y_1y_3 + D^{1-q_2}(U_{22}) \\ \dot{y}_3(t) = y_1y_2 - by_3 + D^{1-q_3}(U_{33}) \end{cases} \tag{3.30}$$

So the state errors are:

$$\begin{cases} \dot{e}_1\left(t\right) = \dot{y}_1\left(t\right) - \dot{x}_1\left(t\right) \\ \dot{e}_2\left(t\right) = \dot{y}_2\left(t\right) - \dot{x}_2\left(t\right) \\ \dot{e}_3\left(t\right) = \dot{y}_3\left(t\right) - \dot{x}_3\left(t\right) \end{cases} \tag{3.31}$$

For that we have:

$$\begin{cases} \dot{e}_1\left(t\right) = a(y_2 - y_1 - x_2 + x_1) + D^{1-q_1}(U_{11}) \\ \dot{e}_2\left(t\right) = c\left(y_1 - x_1\right) + \left(-y_1 y_3 + x_1 x_3\right) + D^{1-q_2}(U_{22}) \\ \dot{e}_3\left(t\right) = b\left(-y_3 + x_3\right) + \left(y_1 y_2 - x_1 x_2\right) + D^{1-q_3}(U_{33}) \end{cases} \tag{3.32}$$

Knowing that:

$$\begin{cases} U_{11} = D^{q_1-1}\left(-\widehat{a}(y_2 - y_1 - x_2 + x_1) - G_1 e_1\right) \\ U_{22} = D^{q_2-1}\left(-\widehat{c}\left(y_1 - x_1\right) - \left(-y_1 y_3 + x_1 x_3\right) - G_2 e_2\right) \\ U_{33} = D^{q_3-1}\left(-\widehat{b}\left(-y_3 + x_3\right) - \left(y_1 y_2 - x_1 x_2\right) - G_3 e_3\right) \end{cases} \tag{3.33}$$

By substituting $D^{1-q_i}(U_{ii}) = W_i$, $i = \overline{1.3}$, in (3.32):

$$\begin{cases} \dot{e}_1\left(t\right) = a(y_2 - y_1 - x_2 + x_1) + W_1 \\ \dot{e}_2\left(t\right) = c\left(y_1 - x_1\right) + \left(-y_1 y_3 + x_1 x_3\right) + W_2 \\ \dot{e}_3\left(t\right) = b\left(-y_3 + x_3\right) + \left(y_1 y_2 - x_1 x_2\right) + W_3 \end{cases} \tag{3.34}$$

Now, the adaptive controllers $[W_1, W_2, W_3]$ for the synchronization of the proposed system can be obtained with the following equations:

$$\begin{cases} W_1 = -\widehat{a}(y_2 - y_1 - x_2 + x_1) - G_1 e_1 \\ W_2 = -\widehat{c}\left(y_1 - x_1\right) - \left(-y_1 y_3 + x_1 x_3\right) - G_2 e_2 \\ W_3 = -\widehat{b}\left(-y_3 + x_3\right) - \left(y_1 y_2 - x_1 x_2\right) - G_3 e_3 \end{cases} \tag{3.35}$$

Where G_1, G_2 and G_3 represent the positive gains and $\widehat{a}, \widehat{b}$ and $\widehat{c}$ are the estimates of the unknown parameters $a, b,$ and c, respectively, substituting (3.35) into (3.34) leads to the following closed-loop error dynamics:

$$\begin{cases} \dot{e}_1(t) = a(y_2 - y_1 - x_2 + x_1) - \widehat{a}(y_2 - y_1 - x_2 + x_1) - G_1 e_1 \\ \dot{e}_2(t) = c(y_1 - x_1) + (-y_1 y_3 + x_1 x_3) - \widehat{c}(y_1 - x_1) - (-y_1 y_3 + x_1 x_3) - G_2 e_2 \\ \dot{e}_3(t) = b(-y_3 + x_3) + (y_1 y_2 - x_1 x_2) - \widehat{b}(-y_3 + x_3) - (y_1 y_2 - x_1 x_2) - G_3 e_3 \end{cases} \tag{3.36}$$

By simplifying we find:

$$\begin{cases} \dot{e}_1(t) = (a - \widehat{a})(y_2 - y_1 - x_2 + x_1) - G_1 e_1 \\ \dot{e}_2(t) = (c - \widehat{c})(y_1 - x_1) - G_2 e_2 \\ \dot{e}_3(t) = \left(b - \widehat{b}\right)(-y_3 + x_3) - G_3 e_3 \end{cases} \tag{3.37}$$

Here $e_a = a - \widehat{a}, e_b = b - \widehat{b}, e_c = c - \widehat{c}$, become:

$$\begin{cases} \dot{e}_1(t) = e_a(y_2 - y_1 - x_2 + x_1) - G_1 e_1 \\ \dot{e}_2(t) = e_c(y_1 - x_1) - G_2 e_2 \\ \dot{e}_3(t) = e_b(-y_3 + x_3) - G_3 e_3 \end{cases} \tag{3.38}$$

Now, consider the Lyapunov stability function $V(t)$:

$$V(t) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2\right). \tag{3.39}$$

The derivative of the function is:

$$\dot{V}(t) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_a\dot{e}_a + e_b\dot{e}_b + e_c\dot{e}_c. \tag{3.40}$$

Here $\dot{e}_a = -\dot{\widehat{a}}, \dot{e}_b = -\dot{\widehat{b}}, \dot{e}_c = -\dot{\widehat{c}}$, and by substituting (3.38) , we obtained:

$$\begin{aligned} \dot{V}(t) = {} & e_1(e_a(y_2 - y_1 - x_2 + x_1) - G_1 e_1) + e_2(e_c(y_1 - x_1) - G_2 e_2) \\ & + e_3(e_b(-y_3 + x_3) - G_3 e_3) + e_a\left(-\dot{\widehat{a}}\right) + e_b\left(-\dot{\widehat{b}}\right) + e_c\left(-\dot{\widehat{c}}\right). \end{aligned}$$

In simple terms, we find:

$$\begin{aligned}\dot{V}(t) = &-\left[G_1 e_1^2 + G_2 e_2^2 + G_3 e_3^2\right] + e_a\left[e_1(y_2 - y_1 - x_2 + x_1) - \dot{\widehat{a}}\right] \\ &+ e_b\left[e_3\left(-y_3 + x_3\right) - \dot{\widehat{b}}\right] + e_c\left[e_2\left(y_1 - x_1\right) - \dot{\widehat{c}}\right].\end{aligned} \tag{3.41}$$

The parameter update law can be selected as:

$$\begin{cases} \dot{\widehat{a}} = e_1(y_2 - y_1 - x_2 + x_1) \\ \dot{\widehat{b}} = e_3\left(-y_3 + x_3\right) \\ \dot{\widehat{c}} = e_2\left(y_1 - x_1\right) \end{cases} \tag{3.42}$$

The time derivative of the Lyapunov function can be expressed as follows:

$$\dot{V}(t) = -\left[G_1 e_1^2 + G_2 e_2^2 + G_3 e_3^2\right]. \tag{3.43}$$

Which is a negative definite function.

Thus, by applying Lyapunov stability theory, we demonstrate that closed-loop system (3.38) achieves global asymptotic stability for all initial conditions of the error signals e_1, e_2, and e_3, using the adaptive controller (3.35) and update parameter law (3.42).

Result 3.1 *The identical chaotic Yang systems are synchronized by adaptive control law (3.35), where the update law for the parameter estimates is given by (3.42) and $G_i (i = 1, 2, 3)$ are positive constants.*

3.4 Numerical simulation

MATLAB software was utilized for conducting numerical simulations to implement the adaptive synchronization mechanism between two identical integer-fractional order chaotic Yang systems (3.17) and (3.18). The parameters chosen for the synchronization simulation of the new integer-order chaotic systems were are $a = 10, b = 8/3$, and $c = 16$. We take $G = 5$ and orders $q_1 = q_2 = q_3 = 0.99$. Initial conditions were set as follows: for the master system, $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.3$, and for the slave system, $y_1(0) = -5$, $y_2(0) = 2.2$, $y_3(0) = 4.3$.

In Figs.(3.2—3.4) illustrates the synchronization of state variables between systems (3.17) and (3.30).

In Fig.(3.5) provides evidence of the convergence of synchronization errors e_1, e_2, and e_3, which exponentially approach zero over time.

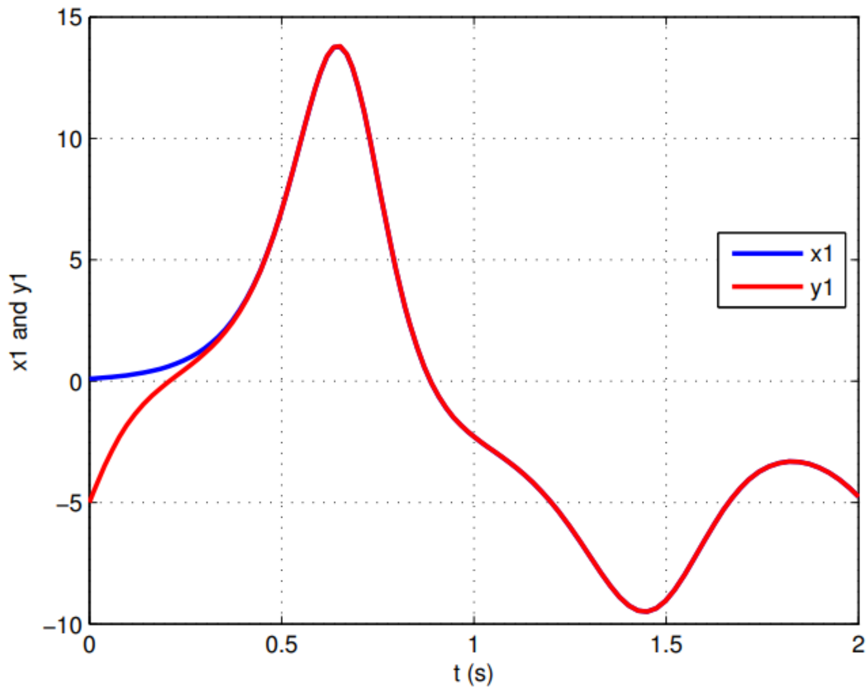

Fig. 3.2: Synchronization of the states x_1 and y_1.

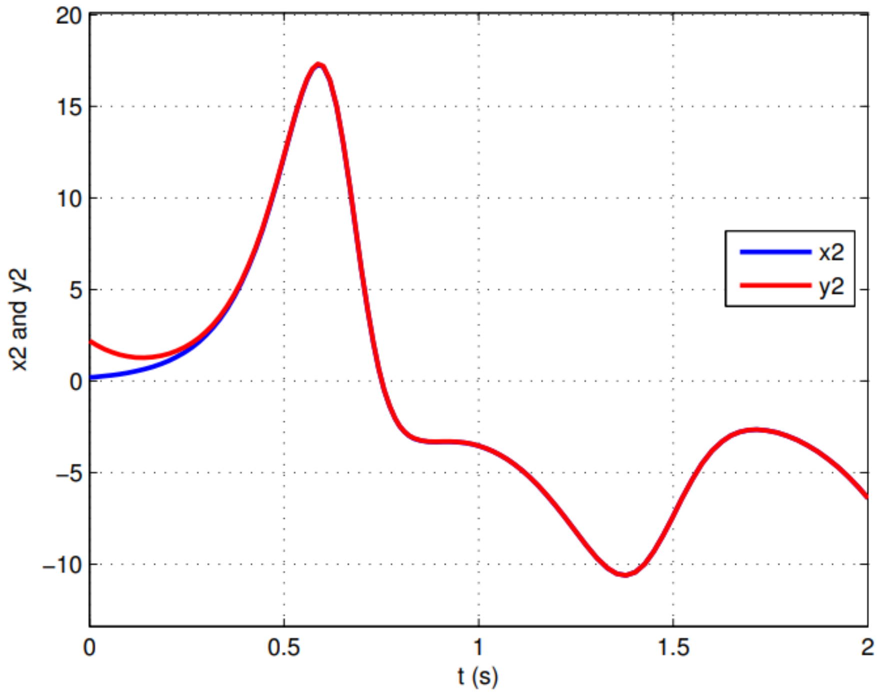

Fig. 3.3: Synchronization of the states x_2 and y_2.

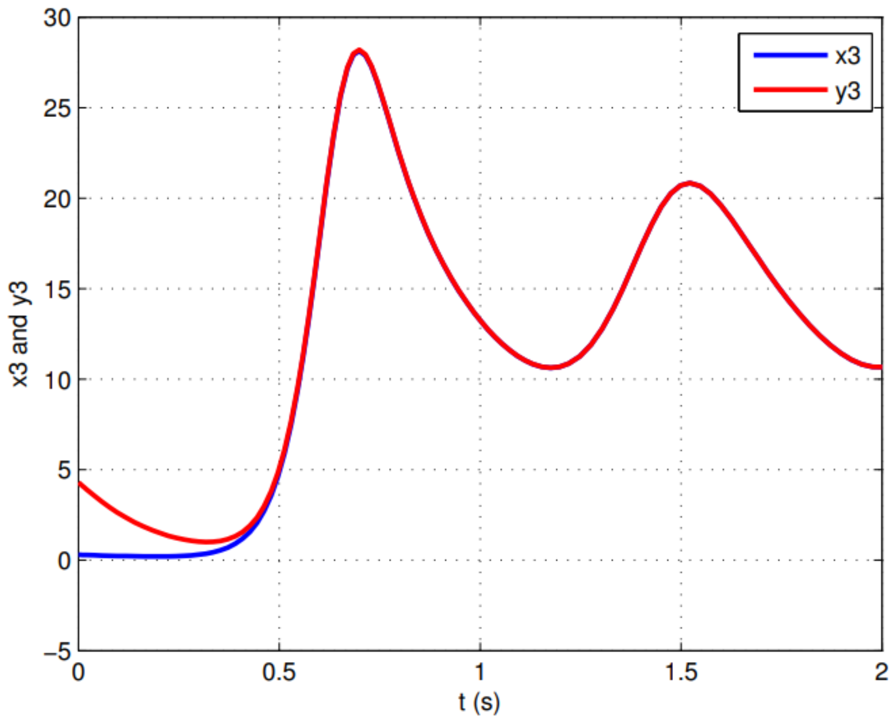

Fig. 3.4: Synchronization of the states x_3 and y_3.

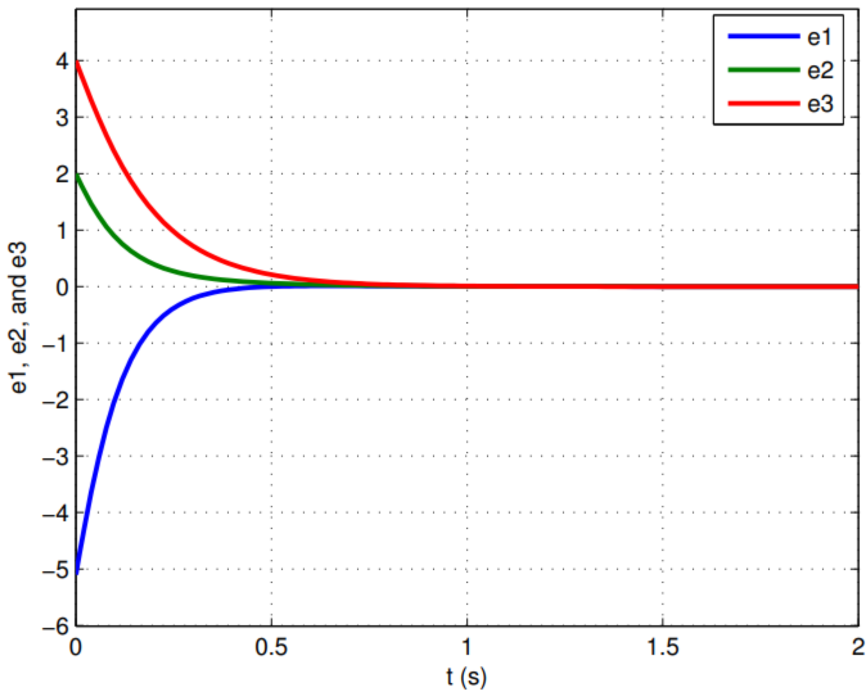

Fig. 3.5: Time-History of the synchronization errors e_1, e_2, and e_3.

conclusion

This chapter explores the application of an adaptive nonlinear control method to synchronize identical integer-fractional order chaotic Yang systems. Lyapunov stability theory is employed to establish the convergence of the proposed synchronization scheme. The effectiveness of this approach is further validated through numerical simulations.

General Conclusion

Synchronization between fractional and integer-order chaotic systems has the potential to revolutionize fields such as predictive weather modeling, secure digital communications, and adaptive neurological treatments. Further exploration in this area is essential for unlocking transformative advancements across diverse scientific and engineering disciplines.

In this work, I mentioned some basic concepts of dynamical systems and chaos theory and synchronization , and also provided basic definitions and properties of fractional derivatives by numerical methods to solve fractional-order systems, and then provided some examples of integer-order and fractional-order chaotic systems in 3D. I also presented a study on the synchronization between two 3D fractional-integer order chaotic systems, supported by numerical simulations conducted using MATLAB to demonstrate the effectiveness of the proposed method.

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