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Limit Cycles Bifurcating From A Zero-Hopf Type Equilibrium For Certain Autonomous Differential Systems

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بسم الله الرحمن الرحيم

شكر وتقدير

قال رسول اللّٰه صلى اللّٰه عليه وسلم
(من لم يشكر الناس لم يشكر اللّٰه)
صدق رسول اللّٰه صلى اللّٰه عليه وسلم.
الحمد لله على إحسانه والشكر له على توفيقه وامتنانه ونشهد أن لا إله إلا اللّٰه وحده لا شريك له تعظيما لشأنه ونشهد أن سيدنا ونبينا محمد عبده ورسوله صلى اللّٰه عليه وعلى آله وأصحابه وأتباعه وسلم.
بعد شكر اللّٰه سبحانه وتعالى على توفيقه لنا لإتمام هذه البحث المتواضع أتقدم بجزيل الشكر إلى من شرفني بإشرافه على مذكرة بحثي الأستاذ ذياب زهير الذي لن تكفي حروف هذه المذكرة لإيفائه حقه لمجهوداته الكبيرة معي. و لتوجيهاته العلمية التي لا تقدر بثمن. و التي ساهمت بشكر كبير في إتمام هذا العمل.
كما أتوجه بالشكر لأعضاء اللجنة المناقشة كل من الأستاذ حناشي فارح والأستاذ جدي نذير لتوجيهاتهم القيمة وملاحظاتهم السديدة. وكذلك جميع أساتذة قسم الرياضيات والاعلام الآلي.
كما أتوجه بخالص شكري وتقديري إلى كل من ساعدني من قريب أو من بعيد في إتمام هذا العمل.

"ربي أوزعني أن أشكر نعمتك التي أنعمت عليا
وعلى والدي وأن أعمل صالحا ترضاه
وأدخلني برحمتك في عبادك الصالحين"

الإهداء

الحمد لله حبا و شكرا وامتنانا على البدء والختام
(وآخر دعواهم أن الحمد لله ربي العالمين)
وبكل حب أهدي ثمرة نجاحي وتخرجي إلى الذي أحمل اسمه
بكل فخر. إلى الذي كان ولا زال قدوتي في الحياة. إلى الذي
وهبني كل ما يملك حتى أحقق له آماله. أبي الغالي على
قلبي رحمه اللّٰه.
إلى التي وهبتني كل العطاء والحنان. إلى المرأة التي جعلت مني
فتات طموحة وسهلت عليا بدعائها أمي حفظها اللّٰه.
إلى رفقاء دربي وأصدقائي وقرة عيني والأغلى على قلبي
أخوي ضرار ومحمد حفظهم اللّٰه.
إلى جميع عائلتي وخاصة أعمامي حفظهم اللّٰه.
إلى كل الأصدقاء والزملاء. خصوصا صديقاتي اللواتي عرفتني
عليهن الجامعة وكانوا لي السند ونعم الصحبة الصالحة
(بخوش رحمة, قتال وئام, عويشات جيهان)
أدام اللّٰه صحبتنا.
إلى كل من نساه قلمي ولم ينساه قلبي.
إلى كل من تصفح مذكرتي وإنتفع بها وتذكرنا بدعائه.
إليكم جميعا أهدي هذا العمل

Abstract

The objective of this work is to study the existence of bifurcations of zero-Hopf type at the so-called Chen–Wang differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y - x^2 - xz + 3y^2 + a. \end{cases}$$

The main tool up to now for studying a zero-Hopf bifurcation is to pass the system to the normal form of a zero-Hopf bifurcation. Our analysis of the zero-Hopf bifurcation is different; we study them directly using the averaging theory.

In the second part of this work, we study the existence of zero-Hopf bifurcations of a Lorenz-Haken system in $\mathbb{R}^4$. The main tool used is the averaging theory.

keywords : Zero-Hopf bifurcation, Periodic orbit, Differential system, Averaging theory.

Résumé

L'objectif de ce travail est d'étudier l'existence de bifurcations du type zéro-Hopf d'un système différentiel de Chen–Wang de la forme

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y - x^2 - xz + 3y^2 + a. \end{cases}$$

L'outil principal pour étudier une bifurcation de zéro-Hopf est de passer le système différentiel à la forme normale d'une bifurcation de zéro-Hopf. Notre analyse de la bifurcation de zéro-Hopf est différente; nous les étudions directement en utilisant la méthode de moyennisation.

Dans la deuxième partie de ce travail, nous étudions l'existence de bifurcation de zéro-Hopf d'un système de Lorenz-Haken dans $\mathbb{R}^4$.

L'outil principal utilisé est la méthode de moyennisation.

Mots clés: Bifurcation de zéro-Hopf, Orbite périodique, Système différentiel, Méthode de moyennisation.

ملخص :

الهدف من هذا العمل هو دراسة وجود تشعبـــات من نوع صفر هويف لنظام تشين وانغ المعرف بالشكل الآتي:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y - x^2 - xz + 3y^2 + a, \end{cases}$$

الأداة الرئيسية لدراسة تشعبـــــات صفر هويف هي تحويل النظام إلى الشكل القانوني لتشعب صفر هويف. في هذا العمل طريقة دراستنا لتشعبـــات صفر هويف مختلفة؛ حيث أننا نقوم بدراستها مباشرة باستخدام طريقة المتوســـط.

في الجزء الثاني من هذا العمل، ندرس وجود تشعبـــــات صفر هويف لنظام لورنز-هاكن في $\mathbb{R}^4$. الأداة الرئيسية المستخدمة هي طريقة المتوسط.

الكلمات المفتاحية : تشعب صفر-هويف، حل دوري، نظام تفاضلي، طريقــــــة المتوسط.

Contents

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Introduction

Consider the class of polynomial differential systems of degree n. The maximum number of isolated periodic orbits, the so-called limit cycles, that a polynomial differential system of degree n can have is called Hilbert number, $H(n)$. It is well known that linear systems have no limit cycles, then $H(1) = 0$. For $n = 2$, the problem of estimating $H(2)$ has been studied intensively during the last century. Lower bounds for $H(2)$ can be given by providing concrete examples of polynomial differential systems of degree 2. Up to now, the best result was given by Shi in [21], where he proved the existence of a quadratic system with 4 limit cycles, that is $H(2) \geq 4$. We call by $M(n)$ the maximum number of limit cycles bifurcating from a singular point via a degenerate Hopf bifurcation. Clearly, $M(n)$ is a lower bound for $H(n)$. Bautin showed in [4] that $M(2) = 3$; in [25, 26], Żoladek proved that $M(3) \geq 11$; a simpler proof was provided by Christopher in [8]. For $n = 3$, Li, Liu, and Yang proved in [14] that $H(3) \geq 13$.

This thesis is divided into four chapters as follows:

In the first chapter, we present elementary definitions, techniques and notations about dynamical systems that we need in this work.

In the second chapter, we present the averaging theory for studying the periodic solutions of differential systems.

In the third chapter, we study the zero-Hopf bifurcation of a Chen–Wang differential system in $\mathbb{R}^3$.

In the last chapter, we study the zero-Hopf bifurcations of a Lorenz-Haken differential system in $\mathbb{R}^4$.

Chapter 1

Preliminaries

In this chapter, we present some preliminary concepts, definitions, and results that we shall require throughout this work.

1.1 First order differential equations

Definition 1.1.1 *A first order differential equations is any differential equation of the form*

$$\dot{x} = f(t, x)$$

where f is a continuous function on $I \times U$ with values in $\mathbb{R}^n$, $I \subset \mathbb{R}$ being an open interval and U being an open of $\mathbb{R}^n$.

1.2 Polynomial differential system in the plane

Definition 1.2.1 *Consider a differential system of the form*

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \tag{1.1}$$

where P and Q are polynomials in the variables x and y with real coefficients, system (1.1) is called a planar polynomial differential system. We say that system (1.1) has degree d if $d = \max(\deg P, \deg Q)$.

1.3 Dynamical systems

Definition 1.3.1 *A dynamical system on $\mathbb{R}^n$ is a map $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\begin{aligned} &(1)\ \ \varphi(.,x) : \mathbb{R}^n \to \mathbb{R}^n \text{ is continuous,} \\ &(2)\ \ \varphi(t,.) : \mathbb{R} \to \mathbb{R}^n \text{ is continuous,} \\ &(3)\ \ \varphi(0,x) = x, \\ &(4)\ \ \varphi(t+s,x) = \varphi(t,\varphi(s,x))\ \ \forall t,s \in \mathbb{R}, \forall x \in \mathbb{R}^n. \end{aligned}$$

1.4 Autonomous differential system

Definition 1.4.1 *A differential system in which the independent variable does not appear explicitly, i.e. a system of the form*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

is an autonomous differential system.

1.5 Critical point and hyperbolic critical point

Definition 1.5.1 *A point $x_0 \in \mathbb{R}^n$ is called a critical point, equilibrium point, singular point or fixed point of the nonlinear system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

if $f(x_0) = 0$.

Definition 1.5.2 *Let x_0 be a critical point of (1.2). Consider the linear system*

$$\dot{x} = Ax \text{ with } A = Df(x_0)$$

A is called the linearization of (1.2) at x_0, a critical point x_0 is called a hyperbolic critical point of (1.2) if none of the eigenvalues of the matrix A have zero real part.

1.6 Degenerate and nondegenerate critical points

Definition 1.6.1 *A critical point of the system (1.2) at which $A = Df(x_0)$ has no zero eigenvalues is called a nondegenerate critical point of the system, otherwise, it is called a degenerate critical point of the system.*

1.7 Stability of the solution and its asymptotic stability

Definition 1.7.1 *Consider the non autonomous differential system*

$$\frac{dx}{dt} = f(t,x)\ , x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1.3}$$

We assume that $f(t,x)$ satisfies the conditions of the theorem of existence and uniqueness of solutions. A solution $\phi(t)$ of the system (1.3) such that $\phi(t_0) = \phi_0$ is called stable in the sense of Lyapunov if :
$\forall \varepsilon > 0$, $\exists\ \delta > 0$ *such that for any solution $x(t)$ whose initial value $x(t_0)$ satisfies:*

$$\|x(t_0) - \phi_0\| < \delta \ \Rightarrow\ \|x(t) - \phi(t)\| < \varepsilon, \forall t \geq t_0.$$

If in addition : $\lim\limits_{t\to+\infty} \|x(t) - \phi(t)\| = 0$, *then the solution $\phi(t)$ is asymptotically stable.*

1.8 Flow of the nonlinear differential system

Definition 1.8.1 *Consider the nonlinear differential system*

$$\dot{x} = f(x), \tag{1.4}$$

and the initial value problem

$$\dot{x} = f(x),\ \ \textit{with}\ x(0) = x_0,$$

with $x \in \mathbb{R}^n$, E an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. For $x_0 \in E$ and $\phi(t, x_0)$ the solution of the initial value problem, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0),$$

is called the flow of the differential system (1.4) or the flow defined by the differential system (1.4).

1.9 Flow of the linear differential system

Definition 1.9.1 *Consider the linear differential system*

$$\dot{x} = Ax \tag{1.5}$$

and the initial value problem

$$\dot{x} = Ax,\ \ x(0) = x_0\ \textit{with}\ x \in \mathbb{R}^n, \tag{1.6}$$

where A is a constant matrix $(n \times n)$, the solution of the problem (1.6) is $x(t) = e^{At}x_0$, the set of mappings $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ can be considered as describing the movement of points $x_0 \in \mathbb{R}^n$ along the trajectories of (1.5). This set of mappings is called the flow of the linear system (1.5).

1.10 Phase plane and phase portrait

Definition 1.10.1 *Consider the planar system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{1.7}$$

a phase portrait is the set of trajectories in phase space. In particular, for autonomous systems of differential equations in two variables. The solutions $(x(t), y(t))$ of the system (1.7) represent curves called orbits in the plane (xoy) . The critical points of this system are constant solutions and the complete figure of the orbits of this system as well as these critical points represent the phase portrait and the plane (xoy) is the phase plane..

1.11 Stability conditions for the linear differential system

Consider the linear differential system $x' = Ax$, where $x \in \mathbb{R}^n$ and A is an $(n \times n)$-square matrix with constant real coefficients.

Definition 1.11.1 *If all eigenvalues of matrix A have strictly negative real parts, then all solutions of the system $x' = Ax$ tend to 0 as t tends to $+\infty$ and the origin is a stable equilibrium point.*
If the matrix A has at least one eigenvalue with a positive real part, the origin is an unstable equilibrium point.
If the matrix A is diagonalizable and all its eigenvalues have negative or zero real parts, then the origin is a stable equilibrium point.

1.12 Periodic solution

Definition 1.12.1 *Any solution $\varphi_t(x)$ of the non-linear differential system (1.2) is periodic if there exists a number $T > 0$ such that*

$$\varphi(t + T, x) = \varphi(t, x) \;\; \text{for all } t \in \mathbb{R}.$$

The smallest positive real number $T > 0$ satisfying the above formula is called the period.

1.13 Limit cycle

Definition 1.13.1 *A limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.*

1.14 Gradient and hamiltonian systems

Definition 1.14.1 *Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in \mathcal{C}^2(E)$ where $H = H(x, y)$ with $x, y \in \mathbb{R}^n$.*
A system of the form

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}, \\ \dot{y} = -\dfrac{\partial H}{\partial x}, \end{cases}$$

where

$$\begin{aligned} \frac{\partial H}{\partial x} &= \left(\frac{\partial H}{\partial x_1}, \ldots\ldots, \frac{\partial H}{\partial x_n}\right)^T, \\ \frac{\partial H}{\partial y} &= \left(\frac{\partial H}{\partial y_1}, \ldots\ldots, \frac{\partial H}{\partial y_n}\right)^T, \end{aligned}$$

is called a Hamiltonian system with n degrees of freedom on E.

Definition 1.14.2 *Let E be an open subset of $\mathbb{R}^n$ and let $V \in \mathcal{C}^2(E)$. A system of the form*

$$\dot{x} = -\operatorname{grad} V(x),$$

where

$$\operatorname{grad} V(x) = \left(\frac{\partial V}{\partial x_1}, \ldots\ldots, \frac{\partial V}{\partial x_n}\right)^T,$$

is called a gradient system on E.

1.15 Classification of critical points of linear systems

Consider the linear system of the form

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \text{ with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

We assume det $A \neq 0$; The origin is the only critical point of this linear system. The eigenvalues of A are given by the characteristic equation

$$\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2),$$

Case 1. Real eigenvalues

If A has two negative eigenvalues λ_1,λ_2 with $\lambda_1 \neq \lambda_2$. The origin is stable improper node. (see Figure 1.1)

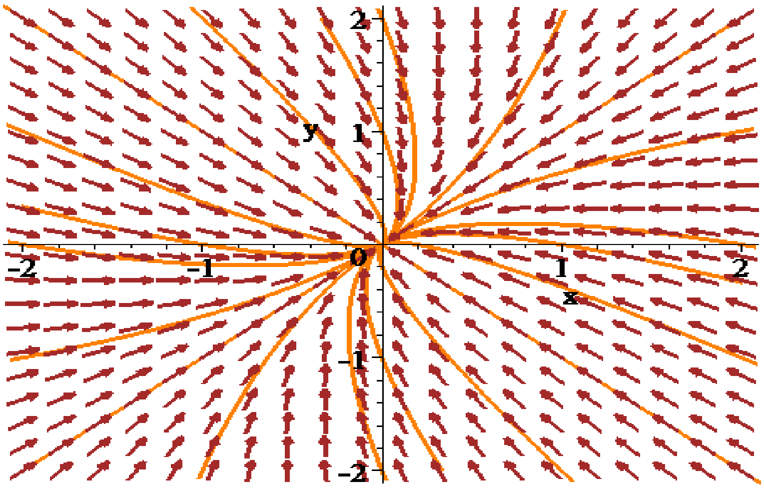

Fig 1.1. Phase portrait with a stable improper node.

If A has two positive eigenvalues λ_1,λ_2 with $\lambda_1 \neq \lambda_2$. The origin is unstable improper node. (see Figure 1.2.)

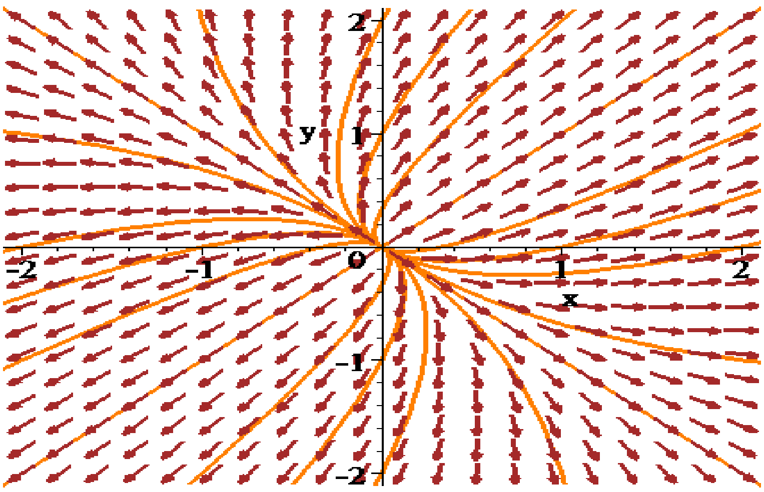

Fig 1.2. Phase portrait with an unstable improper node

If A has two real eigenvalues of opposite sign λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. The origin is saddle point that is always unstable. (See Figure 1.3.)

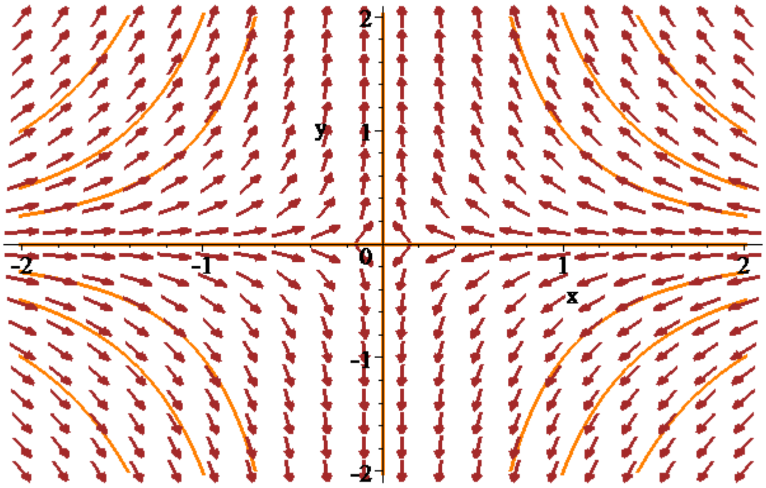

Fig 1.3. Phase portrait with a saddle point

If A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$. Two cases are possible

(**a**) A is diagonalizable. The origin is proper node: stable if $\lambda < 0$ and unstable if $\lambda > 0$. (See Figure 1.4. and 1.5)

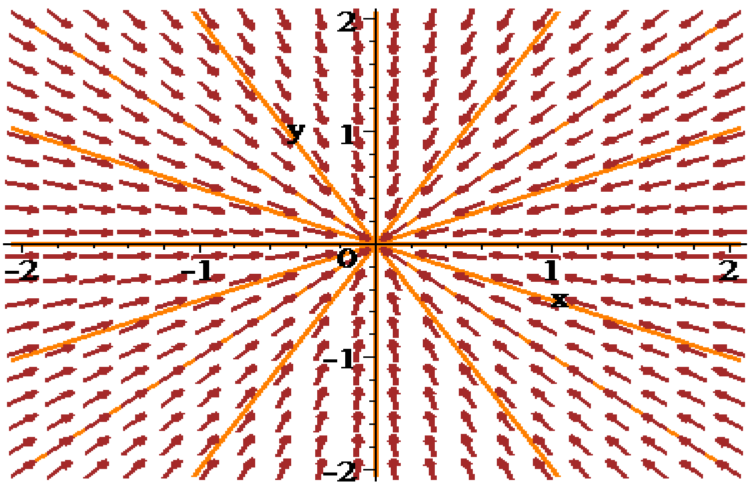

Fig 1.4. Phase portrait with a stable proper node.

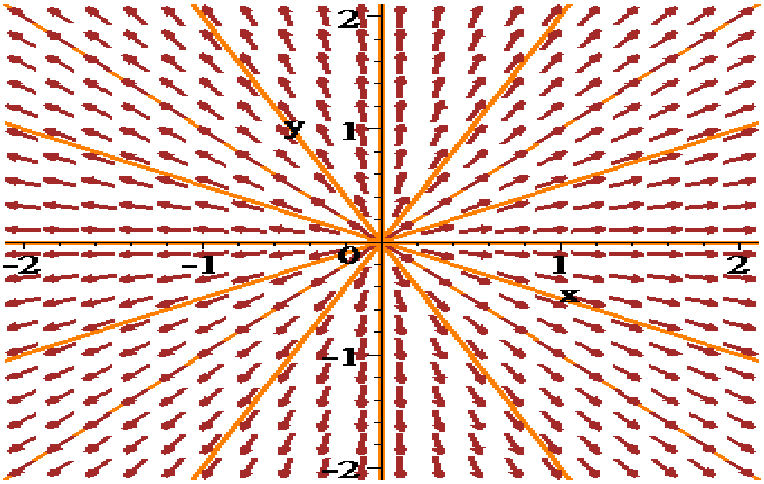

Fig 1.5. Phase portrait with an unstable proper node.

(**b**) A is non-diagonalizable. The origin is degenerate node: stable if $\lambda < 0$ and unstable if $\lambda > 0$. (See Figure 1.6. and 1.7)

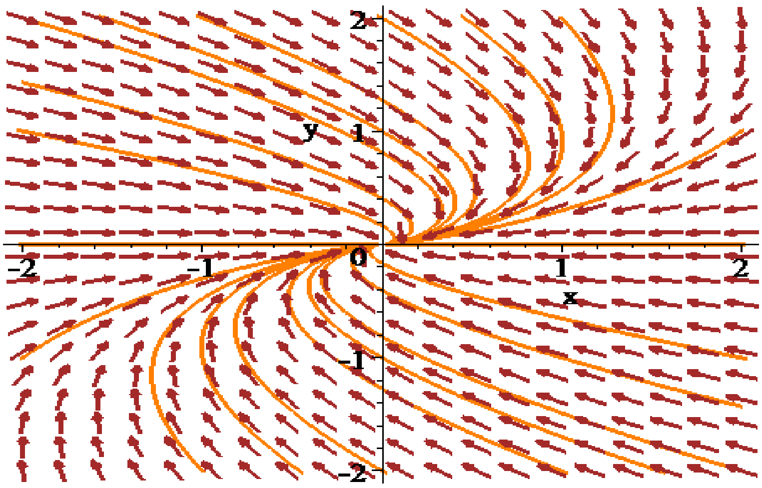

Fig 1.6. Phase portrait with a stable degenerate node.

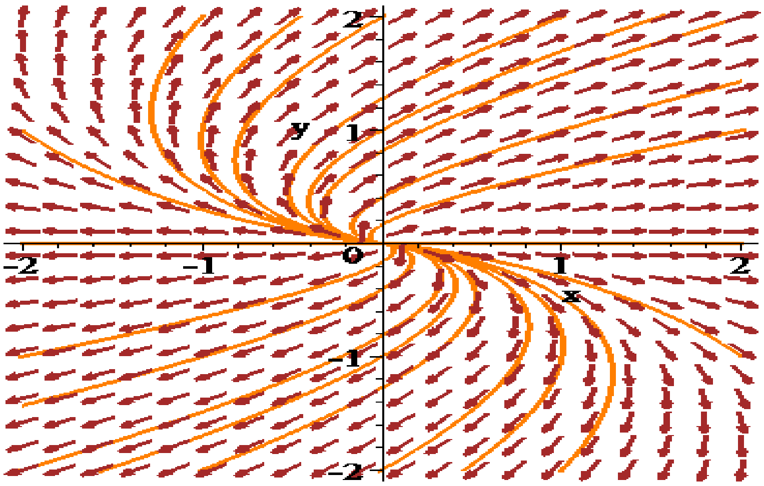

Fig 1.7. Phase portrait with an unstable degenerate node.

Case 2. Complex eigenvalues
(a) If A has a pair of complex conjugate eigenvalues with nonzero real part $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The origin is degenerate node : stable if $\alpha < 0$ and unstable if $\alpha > 0$. (See Figure 1.8. and 1.9).

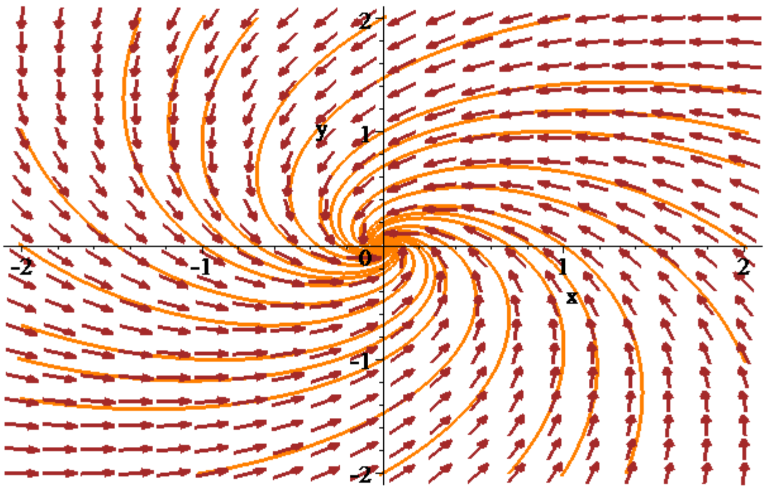

Fig 1.8. Phase portrait with a stable spiral point.

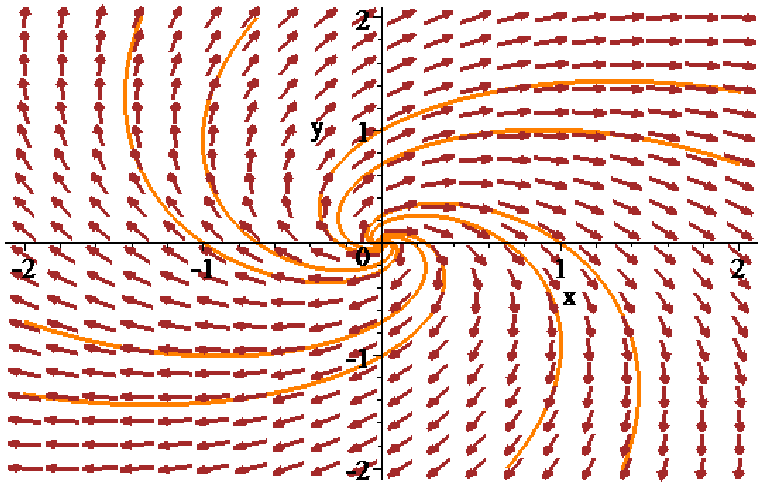

Fig 1.9. Phase portrait with an unstable spiral point

(b) If A has a pair of pure imaginary complex conjugate eigenvalues $\lambda_1 = \pm i\beta$. The origin is center. (See Figure 1.10).

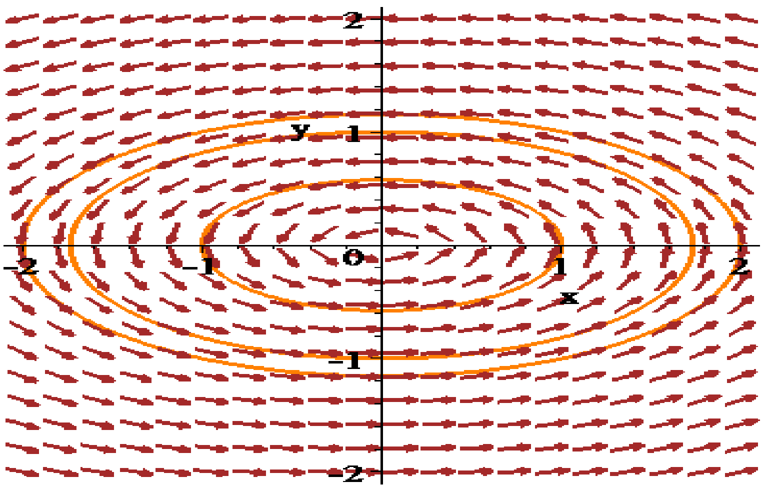

Fig 1.10. Phase portrait with a center.

1.16 First integral of the differential system

Definition 1.16.1 *We consider the autonomous differential system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

where $f : U \to \mathbb{R}^n$ is $\mathcal{C}^2$, $x \in U$ is an open subset of $\mathbb{R}^n$.
Let $F : U \to \mathbb{R}$ be a non-constant function of class $\mathcal{C}^1$ such that

$$\nabla F(x) \cdot f(x) = 0,$$

where

$$\nabla F(x) = \left(\frac{\partial F(x)}{\partial x_1}, \dots, \frac{\partial F(x)}{\partial x_n} \right).$$

Then F is called a first integral of the differential system $\dot{x} = f(x)$, because F is constant on the solutions this system.

1.17 Roots of a cubic polynomial

We recall that the discriminant Δ of the polynomial

$$ax^3 + bx^2 + cx + d$$

is

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

It is known that
– If $\Delta > 0$, then the equation has three distinct real roots.
– If $\Delta = 0$, then the equation has a root of multiplicity 2 and all its roots are real.
– If $\Delta < 0$, then the equation has one real root and two non–real complex conjugate roots.
For more details see [1].

Chapter 2

The averaging theory

We also present a result from the averaging theory that we shall need for proving theorem 3.1.1 in the chapter 3 and theorem 4.1.1 in the chapter 4, for a general introduction to the averaging theory see the book of Sanders, Verhulst and Murdock [19].

2.1 The averaging theory of first order

We consider the initial value problems

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F(t, x, \varepsilon), \qquad x(0) = x_0, \tag{2.1}$$

and

$$\dot{y} = \varepsilon g(y), \qquad y(0) = x_0, \tag{2.2}$$

with x, y and x_0 in some open subset of $\mathbb{R}^n, t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that F_1 and F_2 are periodic of period T in the variable t, and we set

$$g(y) = \frac{1}{T}\int_0^T F_1(t, y)\, dt. \tag{2.3}$$

We will also use the notation $D_x g$ for all the first derivatives of g, and $D_{xx} g$ for all the second derivatives of g.

For a proof of the next result, see [23].

Theorem 2.1.1 *Assume that F_1, $D_x\ F_1$,$D_{xx}\ F_1$ and $D_x\ F_2$ are continuous and bounded by a constant independent of ε in $[0, \infty) \times \Omega \times (0, \varepsilon_0]$, and that $y(t) \in \Omega$ for*

$t \in \left[0, \frac{1}{\varepsilon}\right]$. *Then, the following statements hold:*

1. *For* $t \in \left[0, \frac{1}{\varepsilon}\right]$, *we have* $x(t) - y(t) = \mathcal{O}(\varepsilon)$ *as* $\varepsilon \to 0$.

2. *If* $p \neq 0$ *is a singular point of system* (2.2) *such that*

$$\det D_y g(p) \neq 0, \tag{2.4}$$

then there exists a periodic solution $x(t, \varepsilon)$ *of period* T *for system* (2.1) *which is close to* p *and such that* $x(0, \varepsilon) - p = \mathcal{O}(\varepsilon)$ *as* $\varepsilon \to 0$.

3. *The stability of the periodic solution* $x(t, \varepsilon)$ *is given by the stability of the singular point.*

Example 2.1.1 *Consider the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\left(3x + 7y - 4x^2y + 5xy^2 + x^4 + y^4\right). \end{cases} \tag{2.5}$$

We write the system (2.5) *in the polar coordinates* (r, θ) *defined by* $x = r\cos\theta$, $y = r\sin\theta$, *and we obtain the differential system*

$$\begin{cases} \dot{r} = -r\sin(\theta)(-2r^3\cos(\theta)^4 - 3\cos(\theta) - 5r^2\cos(\theta) - r^3 + 2r^3\cos(\theta)^2 + 5\cos(\theta)^3r^2 \\ \qquad -7\sin(\theta) + 4r^2\cos(\theta)^2\sin(\theta))\varepsilon, \\ \dot{\theta} = -1 + (7\cos(\theta)\sin(\theta) - 4r^2\cos(\theta)^3\sin(\theta) + 5r^2\cos(\theta)^2 + 2r^3\cos(\theta)^5 + 3\cos(\theta)^2 \\ \qquad -5\cos(\theta)^4r^2 + \cos(\theta)r^3 - 2\cos(\theta)^3r^3)\varepsilon, \end{cases}$$

or equivalently

$$\frac{dr}{d\theta} = r\sin(\theta)(-2r^3\cos(\theta)^4 - 3\cos(\theta) - 5r^2\cos(\theta) - r^3 + 2r^3\cos(\theta)^2 + 5\cos(\theta)^3r^2$$
$$-7\sin(\theta) + 4r^2\cos(\theta)^2\sin(\theta))\varepsilon + \mathcal{O}(\varepsilon^2).$$

We compute the averaged function and we get

$$\begin{aligned} f(r) &= \frac{1}{2\pi}\int_0^{2\pi}(r\sin(\theta)(-2r^3\cos(\theta)^4 - 3\cos(\theta) - 5r^2\cos(\theta) - r^3 + 2r^3\cos(\theta)^2 \\ &\quad +5\cos(\theta)^3r^2 - 7\sin(\theta) + 4r^2\cos(\theta)^2\sin(\theta))d\theta \\ &= \frac{r\left(-7 + r^2\right)}{2}. \end{aligned}$$

The unique positive root of $f(r)$ *is* $r = \sqrt{7}$. *Since* $\left(\frac{df(r)}{dr}\right)(\sqrt{7}) = 7$, *by statement* 2 *of Theorem 2.1.1, it follows that system* (2.5) *has for* $|\varepsilon| \neq 0$ *sufficiently small a limit cycle bifurcating from the periodic orbit of radius* $\sqrt{7}$ *of the unperturbed system* (2.5) *with* $\varepsilon = 0$. *Moreover since* $\left(\frac{df(r)}{dr}\right)(\sqrt{7}) = 7 > 0$, *by statement* 3 *of Theorem 2.1.1, this limit cycle is unstable. (See Figure* 2.1.)

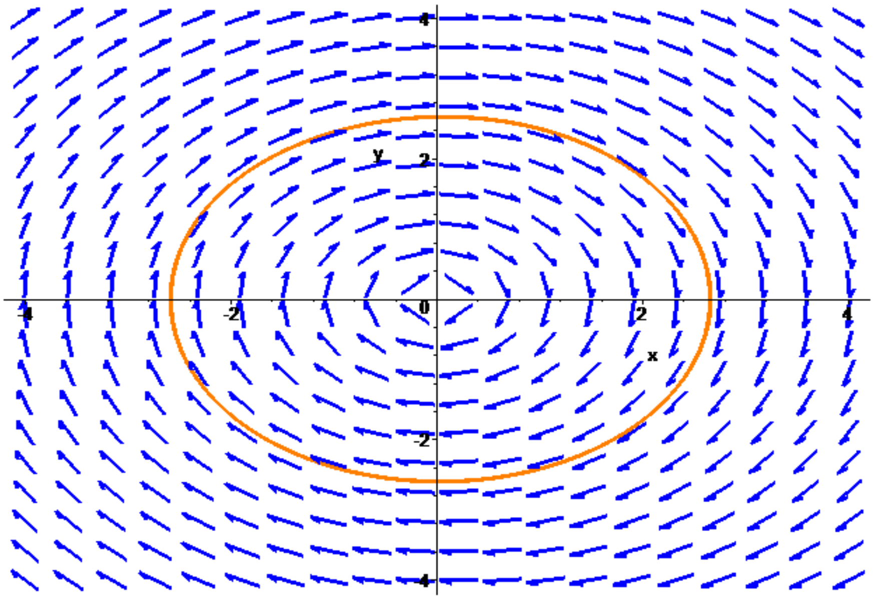

Fig 2.1. The limit cycle for the differential system (2.5) with $\varepsilon = 0.001$.

Chapter 3

On the zero-Hopf bifurcation of a Chen–Wang differential system

The objective of this chapter is to show that Chen–Wang differential system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -y - x^2 - xz + 3y^2 + a,\end{aligned}$$

exhibits two small-amplitude periodic solutions for $a > 0$ sufficiently small that bifurcate from a zero-Hopf equilibrium point localized at the origin of coordinates when $a = 0$.

3.1 Introduction and statement of the main result

In the qualitative theory of differential equations, it is important to know whether a given differential system is chaotic or not. One might think that it is not possible to generate a chaotic system without equilibrium points. The answer to this question was given by Chen and Wang [24] where the authors introduce the following polynomial differential system in $\mathbb{R}^3$

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -y - x^2 - xz + 3y^2 + a,\end{aligned} \tag{3.1}$$

where $a \in \mathbb{R}$ is a parameter. They observe that when $a > 0$ system (3.1) has two equilibria $(\pm\sqrt{a}, 0, 0)$, when $a = 0$ the two equilibria collide at the origin $(0, 0, 0)$

and for $a < 0$ system (3.1) has no equilibria but still generates a chaotic attractor, see for more details again [24]. The Chen–Wang [24] differential system is relevant, because it seems that it is the first example of a differential system in $\mathbb{R}^3$ which exhibits chaotic motion and has no equilibria, as the authors of claimed.

The objective of this work is to study the zero-Hopf bifurcation which exhibits the polynomial differential system (3.1). The main tool up to now for studying a zero-Hopf bifurcation is to pass the system to the normal form of a zero-Hopf bifurcation, later on in this introduction we provide references about this. Our analysis of the zero-Hopf bifurcation is different; we study them directly using the averaging theory, see the chapter 2.

The main objective of this work is to show that system (3.1) exhibits two small amplitude periodic solutions for $a > 0$ sufficiently small that bifurcate from a zero-Hopf equilibrium point localized at the origin of coordinates when $a = 0$. See [15].

We recall that an equilibrium point is a zero-Hopf equilibrium of a 3-dimensional autonomous differential system, if it has a zero real eigenvalue and a pair of purely imaginary eigenvalues. We know that generically a zero-Hopf bifurcation is a two parameter unfolding (or family) of a 3-dimensional autonomous differential system with a zero-Hopf equilibrium. The unfolding can exhibit different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. This theory of zeroHopf bifurcation has been analyzed by Guckenheimer, Han, Holmes, Kuznetsov, Marsden, and Scheurle in [9, 10, 12, 13, 20]. In particular, they show that some complicated invariant sets of the unfolding could bifurcate from the isolated zero-Hopf equilibrium under convenient conditions, showing that in some cases the zero-Hopf bifurcation could imply a local birth of “chaos”, see for instance the articles [2, 3, 5, 7, 20] of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, Scheurle, and Marsden.

Note that the differential system (3.1) only depends on one parameter so it cannot exhibit a complete unfolding of a zero-Hopf bifurcation. For studying the zero-Hopf bifurcation of system (3.1), we shall use the averaging theory in a similar way at it was used in [6] by Castellanos, Llibre and Quilantán.

In the next result, we characterize when the equilibrium points of system (3.1) are zero-Hopf equilibria.

Proposition 3.1.1 *The differential system (3.1) has a unique zero-Hopf equilibrium localized at the origin of coordinates when $a = 0$. The main result of this paper characterizes the Hopf bifurcation of system (3.1). For a precise definition of a classical Hopf bifurcation in $\mathbb{R}^3$ when a pair of complex conjugate eigenvalues cross the imaginary axis and the third real eigenvalue is not zero, see for instance [16].*

Theorem 3.1.1 *The following statements hold for the differential system (3.1)*

(*a*) *This system has no classical Hopf bifurcations.*

(*b*) *This system has a zero-Hopf bifurcation at the equilibrium point localized at the* *origin of coordinates when* $a = 0$ *producing two small periodic solutions for* $a > 0$ *sufficiently small of the form*

$$\begin{aligned} x(t) &= \pm\sqrt{a} + \mathcal{O}(a), \\ y(t) &= \mathcal{O}(t), \\ z(t) &= \mathcal{O}(a). \end{aligned}$$

Both periodic solutions have two invariant manifolds, one stable and one unstable, each of them formed by two cylinders. See Figure 3.1. *for the zero Hopf periodic solution with initial conditions near* $(\sqrt{a}, 0, 0)$ *with* $a = \dfrac{1}{10,000}$. *The other Hopf periodic orbit is symmetric of this under the symmetry* $(x, y, z, t) \to (-x, y, -z, -t)$ *which leaves the differential system* (3.1) *invariant.*

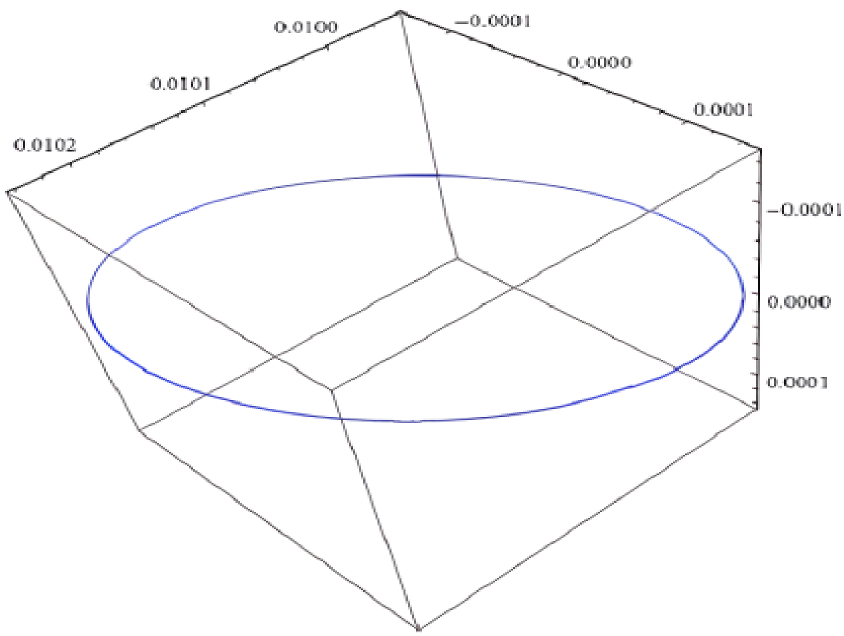

Fig 3.1. The Hopf periodic orbit for with initial conditions near $(\sqrt{a}, 0, 0)$ for $a = \dfrac{1}{10,000}$

In this section, we prove Proposition 3.1.1 and Theorem 3.1.1.

3.2 Zero-Hopf bifurcation

Proof of Proposition 3.1.1. System (3.1) has two equilibrium points $e_{\pm} = (\pm\sqrt{a}, 0, 0)$when $a > 0$, which collide at the origin when $a = 0$. The proof is made computing directly the eigenvalues at each equilibrium point. Note that the characteristic polynomial of the linear part of system (3.1) at the equilibrium point $e_{\pm}$ is

$$p(\lambda) = \lambda^3 \pm \sqrt{a}\lambda^2 + \lambda \pm 2\sqrt{a}.$$

As $p(\lambda)$ is a polynomial of degree 3, it has either one, two (then one has multiplicity 2), or three real zeros.
Using the discriminant of $p(\lambda)$, it follows that $p(\lambda)$ has a unique real root, see [1].
Imposing the condition

$$p(\lambda) = (\lambda - \rho)\left(\lambda^2 - \varepsilon - i\beta\right)(\lambda - \varepsilon + i\beta),$$

with $\rho, \varepsilon, \beta \in \mathbb{R}$ and $\beta > 0$, we obtain a system of three equations that correspond to the coefficients of the terms of degree 0, 1 and 2 in λ of the polynomial

$$p(\lambda) = (\lambda - \rho)(\lambda - \varepsilon - i)(\lambda - \varepsilon + i\beta).$$

This system has only two solutions in the variables (a, β, ρ) , which are

$$\left(\frac{1 - 24\varepsilon^2 + 32\varepsilon^4 - \sqrt{1 - 32\varepsilon^2} + 8\varepsilon^2\sqrt{1 - 32\varepsilon^2}}{8\varepsilon^2}, \frac{\sqrt{3 - 2\varepsilon^2 - \sqrt{1 - 32\varepsilon^2}}}{\sqrt{2}},\right.$$
$$\left.-2\varepsilon - \frac{\sqrt{-\dfrac{-8\epsilon^4 + 2\left(6 - 4\epsilon^2 - 2\sqrt{1 - 32\epsilon^2}\right)\varepsilon^2 - \dfrac{1}{2} + \dfrac{\sqrt{1 - 32\varepsilon^2}}{2}}{\varepsilon^2}}}{2}\right),$$
$$= \left(4\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right), 1 + \mathcal{O}\left(\varepsilon^2\right), -4\varepsilon + \mathcal{O}\left(\varepsilon^3\right)\right),$$

and

$$\left(\frac{1 - 24\varepsilon^2 + 32\varepsilon^4 + \sqrt{1 - 32\varepsilon^2} - 8\varepsilon^2\sqrt{1 - 32\varepsilon^2}}{8\varepsilon^2}, \frac{\sqrt{3 - 2\varepsilon^2 - \sqrt{1 - 32\varepsilon^2}}}{\sqrt{2}},\right.$$
$$\left.-2\varepsilon - \frac{\sqrt{-\dfrac{-8\varepsilon^4 + 2\left(6 - 4\varepsilon^2 + 2\sqrt{1 - 32\varepsilon^2}\right)\varepsilon^2 - \dfrac{1}{2} - \dfrac{\sqrt{1 - 32\varepsilon^2}}{2}}{\varepsilon^2}}}{2}\right),$$
$$= \left(\frac{1}{4\varepsilon^2} + \mathcal{O}(1), \sqrt{2} + \mathcal{O}\left(\varepsilon^2\right), -\frac{1}{2\varepsilon} + \mathcal{O}(\varepsilon)\right),$$

Clearly, at $\varepsilon = 0$, only the first solution is well defined and gives $(a, \beta, \rho) = (0, 1, 0)$. Hence, there is a unique zero-Hopf equilibrium point when $a = 0$ at the origin of coordinates with eigenvalues 0 and $\pm i$. This completes the proof of Proposition 3.1.1. ∎

Proof of Theorem 3.1.1. It was proven in Proposition 3.1.1 that when $a = 0$ the origin is zero-Hopf equilibrium point. We want to study if from this equilibrium it bifurcates some periodic orbit moving the parameter a of the system. We shall use the averaging theory of first order described in chapter 2 (see Theorem 2.1.1) for doing this study. But for applying this theory, there are three main steps that we must solve in order that the averaging theory can be applied for studying the periodic solutions of a differential system.

Step 1 Doing convenient changes of variables we must write the differential system (3.1) as a periodic differential system in the independent variable of the system, and the system must depend on a small parameter as it appears in the normal form (2.1) for applying the averaging theory. To find these changes of variables sometimes is the more difficult step.

Step 2 We must compute explicitly the integral (2.2) related with the periodic differential system in order to reduce the problem of finding periodic solutions to a problem of finding the zeros of a function $g(y)$, see (2.2).

Step 3 We must compute explicitly the zeros of the mentioned function, in order to obtain periodic solutions of the initial differential system (3.1).

In order to find the changes of variables for doing the step 1 and write our differential system (3.1) in the normal form for applying the averaging theory, first we write the linear part at the origin of the differential system (3.1) when $a = 0$ into its real Jordan normal form, i.e., into the form

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To do this, we apply the linear change of variables

$$(x, y, z) \to (u, v, w)\,, \text{where } x = -u + w,\ \ y = v,\ \ z = u.$$

In the new variables (u, v, w), the differential system (3.1) becomes

$$\begin{aligned} \dot{u} &= a - v + uw + 3v^2 - w^2, \\ \dot{v} &= u, \\ \dot{w} &= a + uw + 3v^2 - w^2. \end{aligned} \tag{3.2}$$

Now,we write the differential system (3.2) in cylindrical coordinates(r, θ, w) doing the change of variable

$$u = r\cos\theta, \qquad v = r\sin\theta, \qquad w = w,$$

and system (3.2) becomes

$$\begin{aligned} \dot{r} &= \cos\theta \left(a - w^2 + rw\cos\theta + 3r^2\sin^2\theta\right), \\ \dot{\theta} &= 1 + \frac{1}{r}(w^2 - a)\sin\theta - w\cos\theta\sin\theta - 3r\sin^3\theta, \\ \dot{w} &= a - w^2 + rw\cos\theta + 3r^2\sin^2\theta. \end{aligned} \tag{3.3}$$

Now, we do a rescaling of the variables through the change of coordinates

$$(r, \theta, w) \rightarrow (R, \theta, W),$$

where $r = \frac{\sqrt{a}}{2} R, w = \frac{\sqrt{a}}{2} W$.
After this, rescaling system (3.3) becomes

$$\begin{aligned}
\dot{R} &= \frac{\sqrt{a}}{2} \cos\theta \left(4 - w^2 + RW \cos\theta + 3R^2 \sin^2\theta\right), \\
\dot{\theta} &= 1 - \frac{\sqrt{a}}{2R} \sin\theta \left(4 - W^2 + RW \cos\theta + 3R^2 \sin^2\theta\right), \\
\dot{W} &= \frac{\sqrt{a}}{2} \left(4 - W^2 + RW \cos\theta + 3R^2 \sin^2\theta\right).
\end{aligned} \tag{3.4}$$

This system can be written as

$$\begin{aligned}
\frac{dR}{d\theta} &= \frac{\sqrt{a}}{2} F_{11}(\theta, R, W) + \mathcal{O}(a), \\
\frac{dW}{d\theta} &= \frac{\sqrt{a}}{2} F_{12}(\theta, R, W) + \mathcal{O}(a),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
F_{11}(\theta, R, W) &= \cos\theta \left(4 - W^2 + RW \cos\theta + 3R^2 \sin^2\theta\right), \\
F_{12}(\theta, R, W) &= \left(4 - W^2 + RW \cos\theta + 3R^2 \sin^2\theta\right).
\end{aligned}$$

Using the notation of the averaging theory described in chapter 2, we have that if we take $t = \theta, T = 2\pi, \varepsilon = \sqrt{a}, x = (R, W)^T$ and

$$\begin{aligned}
&F_1(t, x) = F_1(\theta, R, W) = \begin{pmatrix} F_{11}(\theta, R, W) \\ F_{12}(\theta, RW) \end{pmatrix}, \\
&\varepsilon^2 F_2(t, x) = \mathcal{O}(a),
\end{aligned}$$

it is immediate to check that the differential system (3.5) is written in the normal form (2.1) for applying the averaging theory and that it satisfies the assumptions of Theorem 2.1.1. This completes the step 1. Now, we compute the integral in (2.3) with $y = (R, W)^T$, and denoting

$$g(y) = g(R, W) = \begin{pmatrix} g_{11}(R, W) \\ g_{12}(R, W) \end{pmatrix},$$

we obtain

$$\begin{aligned}
g_{11}(R, W) &= \frac{1}{4} RW, \\
g_{12}(R, W) &= \frac{1}{4} \left(8 + 3R^2 - 2W^2\right),
\end{aligned}$$

so the step 2 is done. The system $g_{11}(R, W) = g_{12}(R, W) = 0$ has the unique real solutions $(W, R) = (\pm 2, 0)$. The jacobian (2.4) is

$$\begin{vmatrix} \frac{1}{4}W & \frac{1}{4}R \\ \frac{3}{2}R & -W \end{vmatrix} = -\frac{1}{8}\left(3R^2 + 2W^2\right),$$

and evaluated at the solutions $(R, W) = (0, \pm 2)$ takes the value $-1 \neq 0$. Then, by Theorem 2.1.1, it follows that for any $a > 0$ sufficiently small system (3.4) has a periodic solution $x(t, \varepsilon) = (R(\theta, a), W(\theta, a))$ such that $(R(0, a), W(0, a))$ tends to $(0, \pm 2)$ when a tends to zero. We know that the eigenvalues of the Jacobian matrix at the solution $(0, -2)$ are $2, -1/2$ and the eigenvalues of the Jacobian matrix at the solution $(0, 2)$ are $-2, 1/2$. This shows that both periodic orbits are unstable having a stable manifold and an unstable manifold both formed by two cylinders. Going back to the differential system (3.3), we get that such a system for $a > 0$ sufficiently small has two periodic solutions of period approximately 2π of the form

$$\begin{aligned} r(\theta) &= \mathcal{O}(a), \\ w(\theta) &= \pm\sqrt{a} + \mathcal{O}(a), \end{aligned}$$

these two periodic solutions become for the differential system (3.2) into two periodic solutions of period also close to 2π of the form

$$\begin{aligned} u(t) &= \mathcal{O}(a), \\ v(t) &= \mathcal{O}(a), \\ w(t) &= \pm\sqrt{a} + \mathcal{O}(a), \end{aligned}$$

for $a > 0$ sufficiently small. Finally, we get for the differential system (3.1) the two periodic solutions

$$\begin{aligned} x(t) &= \pm\sqrt{a} + \mathcal{O}(a), \\ y(t) &= \mathcal{O}(a), \\ z(t) &= \mathcal{O}(a), \end{aligned}$$

of period near 2π when $a > 0$ is sufficiently small. Clearly, these periodic orbits tend to the origin of coordinates when a tends to zero. Therefore, they are small amplitude periodic solutions starting at the zeroHopf equilibrium point. This concludes the proof of theorem 3.1.1. ■

Chapter 4

Four-dimensional bifurcation for a Lorenz-Haken system

In this chapter, we study the periodic orbits which bifurcate from a zero-Hopf equilibrium point that a Lorenz-Haken system in $\mathbb{R}^4$ can exhibit.

4.1 Introduction and statement of the main results

The Lorenz–Haken equation named after the fluid dynamist Lorenz and laser theorist Haken [11] describe the dynamics of a homogeneously broadened gain medium in an unidirectional ring cavity. In the notation given in the Reference [?], the Lorenz-Haken equations is given by

$$\begin{aligned} \dot{x} &= -\sigma\left(x-y\right)+iqx\left|x\right|^2, \\ \dot{y} &= -\left(1-i\delta\right)y+\left(r-z\right)x, \\ \dot{z} &= -bz+\operatorname{Re}\left(xy\right), \end{aligned} \tag{4.1}$$

where x, y and z are complex variables, and σ,b, q, r, δ are the real parameters. In 2019, Hayder Natiq [17] derived a new $4D$ chaotic laser system with three equilibrium points from (4.1), since both x and z can be chosen to be real and y a complex variable.

In this work, we study a four-dimensional system of differential equations which is a generalization of the system introduced in [17]. We want to study the periodic orbits of the Lorenz-Haken systems of $\mathbb{R}^4$ with five parameters, in which bifurcate in the zero-Hopf bifurcations of the singular points given by

$$\begin{aligned}
\dot{x} &= a\,(y - x)\,,\\
\dot{y} &= -cy - dz + (e - w)\,x,\\
\dot{z} &= dy - cz,\\
\dot{w} &= -bw + xy,
\end{aligned} \tag{4.2}$$

where x, y, z, we are state variables and a, b, c, d and e are real parameters. See [18].

In the first instance we are going to compute the equilibrium points of Lorenz–Haken system (4.2).

Proposition 4.1.1 *Let* $\Delta = \dfrac{(ec - c^2 - d^2)}{c}$ *and* $c \neq 0$. *The following statements are true :*
(1) *If* $\Delta \leq 0$ *and* $b \neq 0$, *system* (4.2) *has an unique equilibrium point*

$$p_0 \neq (0,0,0,0).$$

(2) *If* $\Delta > 0$ *and* $b \neq 0$, *we have two equilibrium points*

$$p\pm = \left(\pm\sqrt{b\Delta}, \pm\sqrt{b\Delta}, \pm\frac{d\sqrt{b\Delta}}{c}, \Delta\right).$$

(3) *If* $b = 0$ *and* $\Delta \neq 0$ *we has a straight line of equilibria*

$$p = (0,0,0,\Delta).$$

Proposition 4.1.1 follows easily by direct computations.

We observe that the two equilibria $p_\pm$ tends to the equilibrium point p when $b \to 0$. In short, the equilibrium point of system (4.2) can be p_+, p_-, p and the origin. Additionally, the system (4.2) has invariance under the coordinate transformation $(x, y, z, w) \to (-x, -y, -z, w)$. Consequently, the system (4.2) has rotational symmetry around the w-axis.

Due to that, in what follows we consider the only equilibrium p_+ in order to verify its possibility of being a zero–Hopf equilibrium for some values of the parameter, and clearly the same will occur for the other equilibrium p_-.

In the next result we characterize when the equilibrium p, $p_\pm$ and the origin are zero–Hopf equilibrium of the system (4.2).

Proposition 4.1.2 *For the hyperchaotic system* (4.2), *the following statements hold:*
(i) p_0 *is a zero-Hopf equilibrium if only if* $a = -2c$, $b = 0$, $d = -\dfrac{\sqrt{c^2+\omega^2}}{\sqrt{3}}$ *and* $e = \dfrac{4c^2+\omega^2}{3c}$.
(ii) p *is a zero-Hopf equilibrium if only if* $a = -2c$, $b = 0$ *and* $3d^2 - c^2 > 0$,
(iii) p_+ *and* p_- *are zero-Hopf equilibrium if only if* $a = -2c$, $b = 0$, $d = -\dfrac{\sqrt{c^2+\omega^2}}{\sqrt{3}}$.

In the rest of this section, we will study the zero-Hopf bifurcation and periodic solutions of the hyperchaotic system (4.2) at all the equilibrium points.

Theorem 4.1.1 *For the hyperchaotic system (4.2). The following statements hold.*
(i) Let

$$(a, b, d, e) = \left(-2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2 + \omega^2}}{3} + \varepsilon d_1, \frac{4c^2 + \omega^2}{3c} + \varepsilon e_1\right)$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameters. If $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$, $\eta = 3ce_1 + 2\sqrt{3}d_1\sqrt{c^2 + \omega^2} \neq 0$ and $\eta_1 = 3a_1\omega^2 - 2c\eta \neq 0$, then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system (4.2) has a zero-Hopf bifurcation at the equilibrium point located at p_0, and at most four periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solutions are stable if $a_1 > 0$, $b_1 > 0$, $16\eta + 3b_1\omega^2 < 0$ and $4\eta_1 + 3b_1\omega^2 < 0$.
(ii) Let

$$(a, b) = (-2c + \varepsilon a_1, \varepsilon b_1),$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameter. If $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$ and $3d^2 - c^2 > 0$, then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system (4.2) has a zero-Hopf bifurcation at the equilibrium point located at p, and at most four periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solutions are unstable if $a_1 < 0$, $b_1(ec - c^2 - d^2) > 0$ and $c > 0$.
(iii) Let

$$(a, b, d) = \left(-2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2 + \omega^2}}{\sqrt{3}} + \varepsilon d_1\right),$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameter. If $c \neq 0$, $a_1 \neq 0$, and $\kappa = b_1(4c^2 - 3ce + 3\omega^2) < 0$, then for $\varepsilon > 0$ sufficiently small, the hyperchaotic system (4.2) has a zero-Hopf bifurcation at the equilibrium point located at $p_\pm$, and at most two periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, the periodic solutions are unstable if $a_1 > 0$ and $\kappa < 0$.

4.2 Proof of results

In this section we will provide the proofs of Proposition 4.1.2 and Theorem 4.1.1.
Proof of Proposition 4.1.2. The characteristic polynomial $P(\lambda)$ of the linear part of the differential systems (4.2) at the equilibrium point $p_0 = (0, 0, 0, 0)$ is

$$P(\lambda) = \lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D, \tag{4.3}$$

where

$$\begin{aligned}
A &= a + b + 2c, \\
B &= 2bc + c^2 + d^2 + a(b + 2c - e), \\
C &= b(c^2 + d^2) + a(2bc + c^2 + d^2 - (b - c)e), \\
D &= ab(c^2 + d^2 - ce).
\end{aligned}$$

The equilibrium point p_0 is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2+\omega^2)$ with $\omega > 0$, the parameter must be satisfied, $a = -2c,\ b = 0,\ d = -\dfrac{\sqrt{c^2+\omega^2}}{\sqrt{3}}$, and $e = \dfrac{4c^2+\omega^2}{3c}$,
(ii) The characteristic polynomial $P(\lambda)$ of the linear part of the differential systems (4.2) at the equilibrium point p is

$$P(\lambda) = \lambda^4 + (a+2c)\,\lambda^3 + \left(c^2 + d^2 + a\left(c - \frac{d^2}{c}\right)\right)\lambda^2 \tag{4.4}$$

The equilibrium point p is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2+\omega^2)$ with $\omega > 0$, the parameter must be satisfied,

$$a = -2c,\ b = 0,$$

in this case, Eq. (4.4) has roots $\lambda_{1,2} = 0,\ \lambda_{3,4} = \pm\sqrt{3d^2 - c^2}i$.
(iii) The Jacobian matrix of systems (4.2) evaluated at p_+ is

$$\begin{pmatrix} -a & a & 0 & 0 \\ c + \dfrac{d^2}{c} & -c & -d & -\dfrac{\sqrt{b(ce-c^2-d^2)}}{\sqrt{c}} \\ 0 & d & -c & 0 \\ \dfrac{\sqrt{b(ce-c^2-d^2)}}{\sqrt{c}} & \dfrac{\sqrt{b(ce-c^2-d^2)}}{\sqrt{c}} & 0 & -b \end{pmatrix}$$

and its characteristic polynomial is

$$(3.3) \qquad P(\lambda) = \lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D, \tag{4.5}$$

where

$$\begin{aligned} A &= a + b + 2c, \\ B &= c^2 + d^2 + a(b + c - \frac{d^2}{c}) + b\left(c - \tfrac{d^2}{c} + e\right), \\ C &= b\left(ce + a\left(-c - \frac{3d^2}{c} + 2e\right)\right), \\ D &= -2ab(c^2 + d^2 - ce). \end{aligned}$$

The equilibrium point p_+ is a zero hopf equilibrium if and only if $P(\lambda) = \lambda^2(\lambda^2+\omega^2)$ with $\omega > 0$, the parameter must be satisfied,

$$a = -2c, b = 0, d = -\frac{\sqrt{c^2+\omega^2}}{\sqrt{3}}.$$

This completes the Proof of Proposition 4.1.2. ■

Proof. of statement (i) **of Theorem 4.1.1.** Let

$$(a,b,d,e) = \left(-2c+\varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2+\omega^2}}{3}+\varepsilon d_1, \frac{4c^2+\omega^2}{3c}+\varepsilon e_1\right)$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameters. Then, the differential systems (4.2) becomes

$$\begin{aligned}
\dot{x} &= 2c(x-y) - a_1(x-y)\varepsilon,\\
\dot{y} &= (e_1 x - d_1 z)\varepsilon - \frac{-4c^2x + 3c\omega x + 3c^2 y - x\omega^2 - \sqrt{3}cz\sqrt{c^2+\omega^2}}{3c},\\
\dot{z} &= d_1 y\varepsilon + \frac{1}{3}(-3cz - \sqrt{3}y\sqrt{c^2+\omega^2}),\\
\dot{w} &= xy - b_1 w\varepsilon.
\end{aligned} \tag{4.6}$$

Performing the rescaling of variables

$$(x,y,z,w) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$$

system (4.6) can be written as

$$\begin{aligned}
\dot{x} &= 2c(x-y) - a_1(x-y)\varepsilon,\\
\dot{y} &= (e_1 x - \omega x - d_1 z)\varepsilon - \frac{-4c^2x + 3c^2 y - x\omega^2 - \sqrt{3}cz\sqrt{c^2+\omega^2}}{3c},\\
\dot{z} &= d_1 y\varepsilon + \frac{1}{3}(-3cz - \sqrt{3}y\sqrt{c^2+\omega^2}),\\
\dot{w} &= (-b_1 w + xy)\varepsilon.
\end{aligned} \tag{4.7}$$

Now we shall write the linear part at the origin of the system (4.7) when $\varepsilon = 0$ into its real Jordan normal form, i.e. as

$$\begin{pmatrix} 0 & -\omega & 0 & 0\\ \omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For doing that we consider the linear change $(x,y,z,w) \mapsto (X,Y,Z,W)$

$$\begin{aligned}
x &= \frac{2c(\sqrt{3}cY\omega + \sqrt{3}X\omega^2 - 3cZ\sqrt{c^2+\omega^2}}{3\omega^2\sqrt{c^2+\omega^2}},\\
y &= \frac{\sqrt{3}cX\omega^2 + \sqrt{3}Y\omega^3 + 2c^2\left(\sqrt{3}Y\omega - 3Z\sqrt{c^2+\omega^2}\right)}{3\omega^2\sqrt{c^2+\omega^2}},\\
z &= \frac{1}{3}\left(X + \frac{c\left(-2Y\omega + 2\sqrt{3}Z\sqrt{c^2+\omega}\right)}{\omega^2}\right),\\
w &= W.
\end{aligned}$$

By using the new variables (X, Y, Z, W), the system (4.7) can be written as follows

$$
\begin{aligned}
\dot{X} &= -Y\omega + \frac{1}{3}\varepsilon\left(a_1\left(-X + \frac{Y\omega}{c}\right) + \frac{d_1}{\omega^2\sqrt{c^2+\omega^2}}\left(-6c^2\sqrt{c^2+\omega}Z\right.\right. \\
&\quad \left.\left. +\sqrt{3}\omega(2c^2Y + cX\omega + Y\omega^2)\right)\right), \\
\dot{Y} &= X\omega + \frac{\varepsilon}{3\omega^3\sqrt{c^2+\omega^2}}\left(6\sqrt{3}c^4\left(-e_1+W\right)Z - 6c^2\left(e_1-W\right)\omega(\sqrt{3}Z\omega\right. \\
&\quad -Y\sqrt{c^2+\omega^2}) - \omega^3\left(\sqrt{3}d_1X\omega + 2a_1Y\sqrt{c^2+\omega^2}\right) + 4c^3d_1(\sqrt{3}Y\omega \\
&\quad -3Z\sqrt{c^2+\omega^2}) + c\omega^2(2\left(a_1+3e_1-3W\right)X\sqrt{c^2+\omega^2} + 3d_1(\sqrt{3}Y\omega \\
&\quad \left. -2Z\sqrt{c^2+\omega^2}))\right), \\
\dot{Z} &= \frac{\varepsilon}{18c^2\omega^2\sqrt{c^2+\omega^2}}\left(-24\sqrt{3}c^5d_1Z - 4\sqrt{3}a_1c^2Y\omega^3 - \sqrt{3}a_1Y\omega^5\right. \\
&\quad +c\omega^3\left(\sqrt{3}a_1X\omega + 6d_1Y\sqrt{c^2+\omega^2}\right) + 4c^3\omega\left(\sqrt{3}((a_1+3e_1\right. \\
&\quad -3W)X - 6d_1Z)\omega + ed_1Y\sqrt{c^2+\omega^2} + 12c^2\left(e_1-W\right)(\sqrt{3}Y\omega \\
&\quad \left.\left. -3Z\sqrt{c^2+\omega^2}\right)\right). \\
\dot{W} &= \varepsilon\left(-b_1W + \frac{2c}{9\omega^4\left(c^2+\omega^2\right)}\left(\sqrt{3}\left(cY\omega + Z\omega^2\right) - 3cZ\sqrt{c^2+\omega^2}\right)\right. \\
&\quad \left.\sqrt{3}(cX\omega^2 + Y\omega^3) + 2c^2\left(\sqrt{3}Y\omega - 3Z\sqrt{c^2+\omega^2}\right)\right)
\end{aligned}
\tag{4.8}
$$

Then we use the cylindrical coordinates $X = r\cos\theta$, $Y = r\sin\theta$, and obtain

$$
\begin{aligned}
\dot{r} &= \frac{\varepsilon}{3c\omega^3\sqrt{c^2+\omega^2}}(cr\omega^3(\sqrt{3}cd_1 - a_1\sqrt{c^2+\omega^2}\cos^2\theta) \\
&\quad +c\sin\theta 6cZ(-(2\sqrt{3})d_1\sqrt{c^2+\omega^2} - \sqrt{3}c\left(e_1-W\right)\left(c^2+\omega^2\right)) \\
&\quad +r\omega(\sqrt{3}\left(4c^3+3c\omega^2\right)d_1 + \left(6c^2\left(e_1-W\right) - 2a_1\omega^2\right)\sqrt{c^2\omega^2}\sin\theta) \\
&\quad +\omega\cos\theta(-6c^3d_1Z\sqrt{c^2+\omega^2} + r\omega(2\sqrt{3}c^3d_1 \\
&\quad +2c^2\left(a_1+3e_1-3W\right)\sqrt{c^2+\omega^2} + a_1\omega^2\sqrt{c^2+\omega^2})\sin\theta)), \\
\dot{\theta} &= \frac{\varepsilon}{3cr\omega^3\sqrt{c^2+\omega^2}}(cr\omega^2\sqrt{c^2+\omega^2}\left(2c\left(a_1+3e_1-3W\right)\varepsilon + 3\omega^2\right)\cos^2\theta \\
&\quad +c\varepsilon\cos\theta(6cZ(-\left(2c^2+\omega\right)d_1\sqrt{c^2+\omega^2} - \sqrt{3}c\left(e_1-W\right)\left(c^2+\omega^2\right)) \\
&\quad +r\omega(4\sqrt{3}c^3d_1 + 6c^2\left(e_1-W\right)\sqrt{c^2+\omega^2} - 2a_1\omega^2\sqrt{c^2+\omega^2})\sin\theta), \\
\dot{Z} &= \frac{\varepsilon}{18c^2\omega^2\sqrt{c^2+\omega^2}}(-12c^3(Z2\sqrt{3}\left(c^2+\omega^2\right)d_1 + 3c\left(e_1-W\right)\sqrt{c^2+\omega^2}) \\
&\quad +\sqrt{3}cr\omega^2\left(4c^2\left(a_1+3e_1-3W\right) + a_1\omega^2\right)\cos\theta + r\omega((24c^3+6c\omega^2)\,d_1\sqrt{c^2+\omega^2} \\
&\quad -\sqrt{3}\left(12c^4\left(e_1+W\right) + 4a_1c^2\omega^2 + a_1\omega^4\right))\sin\theta), \\
\dot{W} &= \frac{\varepsilon}{3^2\omega^4\left(c^2+\omega^2\right)}((12c^4Z^2 + 2c^2r^2\omega^2 - 3b_1W\omega^4)\left(c^2+\omega^2\right) \\
&\quad +cr\omega(-2c^3r\omega\cos 2\theta - 2\sqrt{3}cZ\sqrt{c^2+\omega^2}\left(3c\omega\cos\theta + \left(4c^2+\omega^2\right)\sin\theta\right) \\
&\quad +3c^2r\omega^2\sin 2\theta + r\omega^4\sin 2\theta)).
\end{aligned}
\tag{4.9}
$$

We take θ as a new independent variable and obtain the system

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon}{3c\omega^4\sqrt{c^2+\omega^2}}(cr\omega^3(\sqrt{3}cd_1 - a_1\sqrt{c^2+\omega^2})\cos^2\theta \\
&+\omega\sqrt{c^2+\omega^2}\cos\theta\left(-6c^3d_1Z + r\omega\left(6c^2\left(e_1-W\right)+a_1\omega^2\right)\sin\theta\right) \\
&+c\sin\theta((6cz(-\left(2c^2+\omega^2\right)d_1\sqrt{c^2+\omega^2} - \sqrt{3}c\left(e_1-W\right)\left(c^2+\omega^2\right)) \\
&+r\omega(2c\omega(\sqrt{3}cd_1 + a_1\sqrt{c^2+\omega^2})\cos\theta + (\left(4c^3+3c\omega^2\right)\sqrt{3}d_1 \\
&+(6c^2\left(e_1-W\right) - 2a_1\omega^2)\sqrt{c^2+\omega^2})\sin\theta))) + \mathcal{O}\left(\varepsilon^2\right) \\
&= \varepsilon F_1\left(\theta,r,Z,W\right) + \mathcal{O}\left(\varepsilon^2\right). \\
\frac{dZ}{d\theta} &= \frac{\varepsilon}{18c^2\omega^2\sqrt{c^2+\omega^2}}(-12c^3Z(2\sqrt{3}\left(c^2+\omega^2\right)d_1 \\
&+3c\left(e_1-W\right)\sqrt{c^2+\omega^2}) + \sqrt{3}cr\omega^2(4c^2\left(a_1+3e_1-3W\right) \\
&+a_1\omega^2)\cos\theta + r\omega(6\left(4c^3+c\omega^2\right)d_1\sqrt{c^2+\omega^2} \\
&-\sqrt{3}(12c^4\left(-e_1+W\right) + 4a_1c^2\omega^2 + a_1\omega^4))\sin\theta) + \mathcal{O}\left(\varepsilon^2\right), \\
&= \varepsilon F_2\left(\theta,r,Z,W\right) + \mathcal{O}\left(\varepsilon^2\right). \\
\frac{dW}{d\theta} &= \frac{\varepsilon}{3\omega^5\left(c^2+\omega^2\right)}(\left(c^2+\omega^2\right)\left(12c^4Z^2+2c^2r^2\omega^2-3b_1W\omega^2\right) \\
&+cr\omega(-2c^3r\omega\cos 2\theta - 2\sqrt{3}cZ\sqrt{c^2+\omega^2}(3c\omega\cos\theta \\
&+\left(4c^2+\omega^2\right)\sin\theta) + 3c^2r\omega^2\sin 2\theta + r\omega^4\sin 2\theta)) + O\left(\varepsilon^2\right), \\
&= \varepsilon F_3\left(\theta,r,Z,W\right) + \mathcal{O}\left(\varepsilon^2\right).
\end{aligned}
\tag{4.10}
$$

Using the notation of averaging theory introduced in chapter 2, we get $t=\theta$, $T=2\pi$, $x=(r,Z,W)$ and

$$
F\left(\theta,r,Z,W\right) = \begin{pmatrix} F_1\left(\theta,r,Z,W\right) \\ F_2\left(\theta,r,Z,W\right) \\ F_3\left(\theta,r,Z,W\right) \end{pmatrix}, \quad \text{and} \quad f\left(r,Z,W\right) = \begin{pmatrix} f_1\left(r,Z,W\right) \\ f_2\left(r,Z,W\right) \\ f_3\left(r,Z,W\right) \end{pmatrix}.
$$

Then we compute the integrals, i.e.

$$
\begin{aligned}
f_1\left(r,Z,W\right) &= \frac{1}{2\pi}\int_0^{2\pi} F_1\left(\theta,r,Z,W\right)d\theta \\
&= \frac{r\left(6c^2\left(e_1-W\right) - 3a_1\omega^2 + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2}\right)}{6\omega^3}, \\
f_2\left(r,Z,W\right) &= \frac{1}{2\pi}\int_0^{2\pi} F_1\left(\theta,r,Z,W\right)d\theta \\
&= -\frac{2cZ\left(3c\left(e_1-W\right) + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)}{3\omega^3}, \\
f_3\left(r,Z,W\right) &= \frac{1}{2\pi}\int_0^{2\pi} F_1\left(\theta,r,Z,W\right)d\theta \\
&= \frac{12c^2Z^2 + 2c^2r^2\omega^2 - 3b_1W\omega^4}{3\omega^5}.
\end{aligned}
$$

Solving the equations $f_1(r, Z, W) = f_2(r, Z, W) = f_3(r, Z, W) = 0$, we can get the following five solutions :

$$
\begin{aligned}
s_0 &= (0, 0, 0),\\
s_{1,2} &= \left(0, \pm\frac{\sqrt{b_1\omega^4\left(3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)}}{2\sqrt{3}c^{5/2}}, e_1 + \frac{2d_1\sqrt{c^2+\omega^2}}{\sqrt{3}c}\right),\\
s_{3,4} &= \left(\pm\frac{\sqrt{b_1\omega^2\left(6c^2e_1 - 3a_1\omega^2 + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2}\right)}}{2c^2}, 0,\right.\\
&\quad \left. +\frac{1}{6c^2}\left(-3a_1\omega^2 + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2}\right)\right).
\end{aligned}
$$

The first solution s_0 corresponds to the equilibrium at the origin. For other four solutions, we get (I) For the solution s_1 and s_2 when $c \neq 0$, $s_{1,2}$ are real solutions. The Jacobian of solution $s_{1,2}$ is

$$
\det\left(\frac{\partial f}{\partial x}(s_1)\right) = \det\left(\frac{\partial f}{\partial x}(s_2)\right) = \frac{2a_1b_1c\left(3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)}{3\omega^5}.
$$

(II) For the solution s_3 and s_4 when $c \neq 0$, $s_{3,4}$ are real solutions. The Jacobian of solution $s_{3,4}$ is

$$
\det\left(\frac{\partial f}{\partial x}(s_3)\right) = \det\left(\frac{\partial f}{\partial x}(s_4)\right) = \frac{a_1b_1\left(-6c^2e_1 + 3a_1\omega^2 - 4\sqrt{3}cd_1\sqrt{c^2+\omega^2}\right)}{3\omega^5}.
$$

When $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$, $\eta = 3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2} \neq 0$ and $\eta_1 = 3a_1\omega^2 - 2c\eta \neq 0$, then

$$
\det\left(\frac{\partial f}{\partial x}(s_j)\right) \neq 0, j = 1, ..., 4.
$$

Then according to Theorem 2.1.1, we see that the system (4.10) has one periodic solution $x_j(\theta, \varepsilon)$ such that $x_j(\theta, \varepsilon) = s_j + \mathcal{O}(\varepsilon)$, $j = 1, ..., 4$. Bring the solution back to the system , and we have one periodic solution

$$
\Phi_j(\theta, \varepsilon) = \left(X_j(\theta, \varepsilon), Y_j(\theta, \varepsilon), Z_j(\theta, \varepsilon), W_j(\theta, \varepsilon)\right).
$$

Then the system (4.6) has the periodic solution $\varepsilon\Phi_j(\theta, \varepsilon)$, $j = 1, ..., 4$.
To determine the stability of the periodic solution $\varepsilon\Phi_j(\theta, \varepsilon)$, $j = 1, ..., 4$, one needs to calculate eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}(s_{2,3})$

$$
P(\lambda) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3, \tag{4.11}
$$

where c_0, c_1,c_2 and c_3 are

$$\begin{aligned}
c_0 &= -1,\\
c_1 &= -\frac{a_1+2b_1}{2\omega},\\
c_2 &= -\frac{b_1\left(-3a_1\omega^2+8c\left(3ce_1+2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)\right)}{6\omega^4},\\
c_3 &= \frac{2a_1b_1c\left(3ce_1+2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)}{3\omega^5}.
\end{aligned}$$

The eigenvalues are given as follows

$$\lambda_1 = -\frac{a_1}{2\omega},\ \lambda_{2,3} = -\frac{3b_1 \pm \sqrt{\frac{b_1\left(48c^2e_1+3b_1\omega^2+32\sqrt{3}cd_1\sqrt{c^2+\omega^2}\right)}{\omega^4}}}{6\omega^3}.$$

On the other hand the characteristic polynomial and its eigenvalues of the jacobian matrix $\dfrac{\partial f}{\partial x}(s_{3,4})$ are

$$P(s_{3,4}) = c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 \tag{4.12}$$

where c_0, c_1, c_2 and c_3 are

$$\begin{aligned}
c_0 &= -1,\\
c_1 &= -\frac{a_1+2b_1}{2\omega},\\
c_2 &= -\frac{b_1\left(-3ce_1+2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)}{3\omega^4},\\
c_3 &= \frac{a_1b_1\left(-6c^2e_1+3a_1\omega^2-4\sqrt{3c}d_1\sqrt{c^2+\omega^2}\right)}{3\omega^5}.
\end{aligned}$$

The eigenvalues are given as follows

$$\tilde{\lambda}_1 = -\frac{a_1}{\omega},\ \tilde{\lambda}_{2,3} = -\frac{3b_1\omega^3 \pm \sqrt{3}\sqrt{b_1\omega^4\left(3\left(4a_1+b_1\right)\omega^2-8c\left(3ce_1+2\sqrt{3}d_1\sqrt{c^2+\omega^2}\right)\right)}}{6\omega^3}.$$

We have that λ_1, $\tilde{\lambda}_1$ is real and $\lambda_{2,3}$, $\tilde{\lambda}_{2,3}$ are complex numbers if $16\eta + 3b_1\omega^2 < 0$ and $4\eta_1 + 3b_1\omega^2 < 0$. In this case, the periodic solution $\varepsilon\Phi_j(\theta,\varepsilon)$ is stable if $a_1 > 0$, $b_1 > 0$. ∎

Proof. of statement (ii) of Theorem 4.1.1.

Let

$$(a,b) = (-2c+\varepsilon a_1, \varepsilon^2 b_1),$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameter. Them, we translate p to the origin the coordinates doing system (4.2) becomes $(x,y,z,w) = (\bar{x},\bar{y},\bar{z},\bar{w}) + p$, then we introduce the scaling of variables $(x, y, z, w) = (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$, with these changes of variables system (4.2) can be written as

$$\begin{aligned}
\dot{x} &= 2c\,(x-y) - a_1\,(x-y)\,\varepsilon,\\
\dot{y} &= \frac{c^2x + d^2x - c^2y - cdz}{c} - wx\varepsilon,\\
\dot{z} &= dy - cz,\\
\dot{w} &= \frac{\varepsilon b_1\,(c^2+d^2-ce)}{c} + (-\varepsilon b_1 w + xy)\,\varepsilon,
\end{aligned} \tag{4.13}$$

After the linear change in variables $(x, y, z, w) \mapsto (X, Y, Z, W)$,

$$\begin{aligned}
x &= \frac{(-6d^2+2c^2)\,X}{3c^2+9d^2} - \frac{2c\sqrt{-c^2+3d^2}Y}{3c^2+9d^2} + \frac{6c^2W}{3c^2-9d^2},\\
y &= \frac{(-3\,d^2+c^2)\,X}{3c^2+9d^2} - \frac{\sqrt{-c^2+3d^2}Y}{3c} + \frac{6c^3W}{3c^3-9cd^2},\\
z &= \frac{d\,(3d^2-c^2)\,X}{3c\,(c^2+3d^2)} - \frac{2d\sqrt{-c^2+3d^2}Y}{3c^2+9d^2} + \frac{6dc^2W}{3\,c^3-9cd^2},\\
w &= Z.
\end{aligned} \tag{4.14}$$

the linear part at the origin of system (4.13) for $\varepsilon = 0$ can be transformed into its real Jordan normal form,

$$\begin{pmatrix} 0 & -\sqrt{3d^2-c^2} & 0 & 0\\ \sqrt{3d^2-c^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Under the change in variable (4.14), the system (4.13) can be written as

$$\begin{aligned}
\dot{X} &= -\frac{1}{3c}\left(cXa_1 - \sqrt{-c^2+3d^2}Ya_1\right)\varepsilon - \sqrt{-c^2+3d^2}Y,\\
\dot{Y} &= \frac{1}{3}(-18Zc^2\sqrt{-c^2+3d^2}Yd^2 - 54ZcXd^4 + 36Zc^3Xd^2\\
&\quad +6Zc^4\sqrt{-c^2+3d^2}Y - 54Zc^3Wd^2 - 18Zc^5W - 6Zc^5W - 6Zc^5X\\
&\quad +2Xc^5a_1 + 12c^2\sqrt{-c^2+3d^2}Ya_1d^2 - 12Xc^3d^2a_1 + 18Xcd^4a_1\\
&\quad -2c^4\sqrt{-c^2+3d^2}Ya_1 - 18\sqrt{-c^2+3d^2}Yd^4a_1)\varepsilon/\left(-c^2+3d^2\right)^{(5/2)}\\
&\quad +\sqrt{-c^2+3d^2}X,\\
\dot{Z} &= -\frac{1}{9}\frac{(9b_1Zc^9 - 162b_1Zc^5d^4 + 729b_1Zcd^8)\,\varepsilon^2}{(c^2+3d^2)^2\,(-3d^2+c^2)\,c} - \frac{1}{9}(162b_1c^5ed^4\\
&\quad -162c^2Wd^6\sqrt{-c^2+3d^2}Y + 2c^9Y^2 - 30Xd^2c^2\sqrt{-c^2+3d^2}Y
\end{aligned}$$

$$
\begin{aligned}
&+54Xd^4\sqrt{-c^2+3d^2}Yc^4+54Xd^6c^2\sqrt{-c^2+3d^2}Y\\
&+18c^6\sqrt{-c^2+3d^2}Yd^2W-162c^4\sqrt{-c^2+3d^2}Yd^4W\\
&-12c^7Y^2d^2+108c^3Y^2d^6-162cY^2d^8+729b_1ced^8\\
&-162Xd^8\sqrt{-c^2+3d^2}Y+4Xc^8\sqrt{-c^2+3d^2}Y\\
&+18c^8\sqrt{-c^2+3d^2}YW+162Xd^4c^5W-486Xd^6c^3W\\
&+54Xd^2c^7W-2X^2c^9-36c^9W^2-9b_1c^{10}-729b_1d^{10}\\
&+162b_1c^4d^6-9b_1c^8d^2+162b_1c^6d^4+9b_1c^9e-729b_1c^2d^8\\
&-162X^2d^8c+216X^2d^6c^3-108X^2d^4c^5+24X^2d^2c^7-18Xc^9W\\
&-216c^7W^2d^2-324c^5W^2d^4)\varepsilon/\left(\left(c^2+3d^2\right)^2\left(-3d^2+c^2\right)^2c\right),\\
\dot W &= \frac{1}{6}\varepsilon(-5Xc^5d^2a_1-12Zc^4\sqrt{-c^2+3d^2}Yd^2+3Xc^3d^2a_1+9Xcd^6a_1\\
&+Xc^7a_1-\sqrt{-c^2+3d^2}Yc^6a_1-9\sqrt{-c^2+3d^2}Yd^6a_1\\
&-3c^2\sqrt{-c^2+3d^2}Yd^4a_1+5\sqrt{-c^2+3d^2}Yc^4a_1d^2-12Zc^7W\\
&-4Zc^7X+4Zc^6\sqrt{-c^2+3d^2}Y+24Zc^5Xd^2-36Zc^3Xd^4\\
&-36Zc^5Wd^2)/\left(c^3\left(c^4-9d^4\right)\right),
\end{aligned}
\tag{4.15}
$$

Performing the cylindrical change of variables

$$
(x,y,Z,W)\mapsto(r\cos\theta, r\sin\theta, Z, W) \tag{4.16}
$$

the system (4.15) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{1}{3}(-27r\cos\theta a_1\sin\theta d^6+54\sin\theta c^2Zr\cos\theta d^4-36\sin\theta c^4Zr\cos\theta d^2\\
&-\sin\theta c^2r\cos\theta a_1+54\sin\theta c^4ZW+6\sin\theta c^6Zr\cos\theta+3\sin\theta c^4r\cos\theta d^2a_1\\
&+9\sin\theta c^2r\cos\theta d^4a_1-c^5\sqrt{-c^2+3d^2}ra_1\cos^2\theta\\
&+6c^3\sqrt{-c^2+3d^2}ra_1d^2\cos^2\theta-9c\sqrt{-c^2+3d^2}rd^4a_1\cos^2\theta\\
&+6c^5Z\sqrt{-c^2+3d^2}r\cos^2\theta-6c^5Z\sqrt{-c^2+3d^2}r-12c^3\sqrt{-c^2+3d^2}ra_1d^2\\
&+2c^5\sqrt{-c^2+3d^2}ra_1+18c\sqrt{-c^2+3d^2}rd^4a_1)-18c^3Z\sqrt{-c^2+3d^2}rd^2\cos^2\theta\\
&+18c^3Z\sqrt{-c^2+3d^2}rd^2/\left(c\left(-9c^4d^2+27c^2d^4-27d^6+c^6\right)\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\\
&=\varepsilon F_1\left(\theta,r,z,\omega\right)+\mathcal{O}\left(\varepsilon^2\right),\\
\frac{dZ}{d\theta} &= \frac{1}{9}\sqrt{-c^2+3d^2}(-162b_1c^5ed^4-4r^2\cos^2\theta c^9+162r\cos\theta d^4c^5W\\
&-186r\cos\theta d^6c^3W+54r\cos\theta d^2c^7W-162c^2Wd^6\sqrt{-c^2+3d^2}r\sin\theta\\
&+54r^2\cos\theta d^4\sqrt{-c^2+3d^2}\sin\theta c^4-30r^2\cos\theta d^2c^2\sqrt{-c^2+3d^2}\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
&+54r^2\cos\theta d^6c^2\sqrt{-c^2+3d^2}\sin\theta-162r^2\cos\theta d^8\sqrt{-c^2+3d^2}\sin\theta\\
&+4r^2\cos\theta c^8\sqrt{-c^2+3d^2}\sin\theta+18c^6\sqrt{-c^2+3d^2}r\sin\theta d^2W\\
&+36r^2\cos^2\theta d^2c^7-108r^2\cos^2\theta d^4c^5+108r^2\cos^2\theta d^6c^3-18r\cos\theta c^9W\\
&-162c^4\sqrt{-c^2+3d^2}r\sin\theta d^4W+18c^8\sqrt{-c^2+3d^2}r\sin\theta W+729b_1ced^8\\
&-36c^9W^2-9b_1c^{10}-729b_1d^{10}+2c^9r^2+162b_1c^4d^6-9b_1c^8d^2+162b_1c^6d^4\\
&+9b_1c^9e-729b_1c^2d^8-216c^7W^2d^2-324c^5W^2d^4-12c^7r^2d^2-162cr^2d^8\\
&+108c^3r^2d^6)/\left(\left(-9c^4d^4+27c^2d^4-27d^6+c^6\right)c\left(c^2+3d^2\right)^2\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right),\\
&=\varepsilon F_2\left(\theta,r,z,\omega\right)+\mathcal{O}\left(\varepsilon^2\right),\\
\frac{dW}{d\theta}&=-\frac{1}{6}(-5r\cos\theta c^5d^2a_1-12Zc^4\sqrt{-c^2+3d^2}r\sin\theta d^2+3r\cos\theta c^3d^4a_1\\
&+9r\cos\theta cd^9a_1+r\cos\theta c^7a_1-\sqrt{-c^2+3d^2}r\sin\theta c^6a_1-9\sqrt{-c^2+3d^2}r\sin\theta d^6a_1\\
&-3c^2\sqrt{-c^2+3d^2}r\sin\theta d^4a_1+5\sqrt{-c^2+3d^2}r\sin\theta c^4a_1-12Zc^7W-4Zc^7r\cos\theta\\
&+4Zc^6\sqrt{-c^2+3d^2}r\sin\theta+24Zc^5r\cos\theta d^2-36Zc^3r\cos\theta d^4\\
&-36Zc^5Wd^2)\sqrt{-c^2+3d^2}/\left(\left(c^4-6c^2d^2+9d^4\right)\left(c^2+3d^2\right)c^3\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right),\\
&=\varepsilon F_3\left(\theta,r,z,\omega\right)+\mathcal{O}\left(\varepsilon^2\right),
\end{aligned}
\tag{4.17}
$$

System (4.17) is written in the normal form (2.1) for applying the averaging theory and satisfies all the assumptions of Theorem 2.1.1. Then, using the notations of the averaging theory described in Chapte 2, we have $t=\theta$, $T=2\pi$, $x=(r,Z,W)$,

$$
F\left(\theta,r,Z,W\right)=\begin{pmatrix}F_1\left(\theta,r,Z,W\right)\\F_2\left(\theta,r,Z,W\right)\\F_3\left(\theta,r,Z,W\right)\end{pmatrix},\ \text{and}\ \ f(r,z,w)=\begin{pmatrix}f_1(r,Z,W)\\f_2(r,Z,W\\f_3(r,Z,W)\end{pmatrix}
$$

Then we compute the integrals, i.e.

$$
\begin{aligned}
f_1(r,Z,W)&=\frac{1}{2\pi}\int_0^{2\pi}F_1\left(\theta,r,Z,W\right)d\theta\\
&=-\frac{\left(-c^2a_1+2c^2Z+3c^2Z+3d^2a_1\right)\sqrt{-c^2+3d^2}r}{2\left(c^4-6c^2d^2+9d^4\right)},\\
f_2(r,Z,W)&=\frac{1}{2\pi}\int_0^{2\pi}F_2\theta,r,Z,W)d\theta\\
&=-\frac{1}{3}(3b_1c^{10}+12c^9W^2-3b_1c^9e+3b_1c^8d^2-2c^7r^2d^2\\
&+72c^7W^2d^2-54b_1c^6d^4+108c^5W^2d^4+54b_1c^5ed^4\\
&+18r^2d^4c^5-54b_1c^4d^6-54c^3r^2d^6+243b_1c^2d^8\\
&-243b_1ced^8+54cr^2d^8+243b_1d^{10})\sqrt{-c^2+3d^2}/(c(-3c^8d^2\\
&-18c^6d^4+54c^4d^6+81c^2d^8-243d^{10}+c^{10})),
\end{aligned}
$$

$$
\begin{aligned}
f_3(r,Z,W) &= \frac{1}{2\pi}\int_0^{2\pi} F_3\left(\theta,r,Z,W\right)d\theta \\
&= \frac{2\sqrt{-c^2+3d^2}c^2ZW}{c^4-6c^2d^2+9d^4},
\end{aligned}
$$

Solving the equations $f_1(r,Z,W) = f_2(r,Z,W) = f_3(r,Z,W) = 0$, we can get the following four solutions :

$$
s_{1,2} = \left(0,0,\pm\frac{\sqrt{\dfrac{b_1\left(-c^2+ec-d^2\right)}{4c}}\left(-3d^2+c^2\right)}{c^2}\right),
$$

$$
s_{3,4} = \left(\pm\frac{\sqrt{\dfrac{3b_1\left(-c^2+ec-d^2\right)}{6d^2c-2c^3}}\left(c^2+3d^2\right)}{d},\frac{1}{2}\frac{a_1\left(-3d^2+c^2\right)}{c^2},0\right).
$$

The solution s_j, $j = 1,...,4$ exist if only if $c \neq 0$, $d \neq 0$. On the other hand, the solutions $s_{1,2}$ and $s_{3,4}$ are real if only if $b_1\left(ec-c^2-d^2\right) > 0$, $c > 0$ and $3d^2-2c^2 > 0$. For the four solutions, we get

$$
\begin{aligned}
\det\left(\frac{df}{dx}\left(s_1\right)\right) &= \det\left(\frac{df}{dx}\left(s_2\right)\right) \\
&= -\frac{2a_1cb_1\left(c^2+d^2-ce\right)}{\left(3d^2-c^2\right)^{\frac{5}{2}}}, \\
\det\left(\frac{df}{dx}\left(s_3\right)\right) &= \det\left(\frac{df}{dx}\left(s_4\right)\right) \\
&= \frac{2a_1cb_1\left(c^2+d^2-ce\right)}{\left(3d^2-c^2\right)^{\frac{5}{2}}}.
\end{aligned}
$$

When $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$ and $3d^2-c^2 > 0$ then $\det\left(\frac{df}{dx}\left(s_j\right)\right) \neq 0$, for each $j = 1,...,4$. Then according to Theorem 2.1.1, we see that the system (4.17) has one periodic solution $x_j(\theta,\varepsilon)$ such that $x_j(0,\varepsilon) = s_j + \mathcal{O}(\varepsilon)$, for each $j = 1,...,4$.
Bring the solution back to the system (4.15), and we have one periodic solution $\Phi_j(t,\varepsilon) = X_j(t,\varepsilon), Y_j(t,\varepsilon), Z_j(t,\varepsilon), W_j(t,\varepsilon)$. Then the system (4.13) has the periodic solution $\varepsilon\Phi_j(t,\varepsilon)$, $j = 1,...,4$.
To determine the stability of the periodic solution one needs to calculate eigenvalues of the Jacobian matrix $\partial F\left(s_j\right)/\partial x$, $j = 1,...,4$.
The Jacobian matrices $\partial F\left(s_1\right)/\partial x$ and $\partial F\left(s_2\right)/\partial x$ have the same characteristic equation

$$
\lambda^3 + \Theta_1\lambda^2 + \Theta_2\lambda + \Theta_3
$$

where Θ_1, Θ_2 and Θ_3 are

$$\Theta_1 = \frac{a_1}{2\sqrt{-c^2+3\,d^2}},$$
$$\Theta_2 = \frac{4cb_1\left(c^2+d^2-ce\right)}{c^4-6\,c^2d^2+9d^4},$$
$$\Theta_3 = \frac{2b_1ca_1\left(c^2+d^2-ce\right)}{\left(3\,d^2-c^2\right)^{\frac{5}{2}}},$$

The eigenvalues are given as follows

$$\lambda_1 = \frac{a_1}{2\sqrt{-c^2+3d^2}},$$
$$\lambda_2 = -\frac{2\sqrt{-cb_1\left(c^2+d^2-ce\right)}}{c^2-3\,d^2},$$
$$\lambda_3 = \frac{2\sqrt{-cb_1\left(c^2+d^2-ce\right)}}{c^2-3\,d^2}.$$

The Jacobian matrices $\partial F\left(s_3\right)/\partial x$ and $\partial F\left(s_4\right)/\partial x$ have the same characteristic equation,

$$\lambda^3+\Gamma_1\lambda^2+\Gamma_2\lambda+\Gamma_3,$$

where Γ_1, Γ_2 and Γ_3 are

$$\Gamma_1 = \frac{a_1}{\sqrt{3\,d^2-c^2}},$$
$$\Gamma_2 = -\frac{2cb_1\left(c^2+d^2-ce\right)c}{\left(3\,d^2-c^2\right)^2},$$
$$\Gamma_3 = -\frac{2b_1a_1\left(c^2+d^2-ce\right)c}{\left(3\,d^2-c^2\right)^{\frac{5}{2}}},$$

The eigenvalues are given as follows

$$\hat{\lambda}_1 = -\frac{a_1}{\sqrt{-c^2+3d^2}},$$
$$\hat{\lambda}_2 = -\frac{\sqrt{2b_1\left(c^2+d^2-ce\right)c}}{c^2-3\,d^2},$$
$$\hat{\lambda}_3 = \frac{\sqrt{2b_1\left(c^2+d^2-ce\right)c}}{c^2-3\,d^2}$$

We have that λ_1, $\hat{\lambda}_1$and $\lambda_{2,3}$ are real numbers and $\hat{\lambda}_{2,3}$ are complex numbers. In this case, since that $a_1<0$, $b_1\left(ec-c^2-d^2\right)>0$ and $c>0$, then this implies that the periodic orbits $\varepsilon\Phi(t,\varepsilon)$, $j\in\{1,...,4\}$ are unstable. ∎

Proof. of statement (iii) of Theorem 4.1.1.

Let

$$(a,b,d)=\left(-2c+\varepsilon a_1,\varepsilon b_1,-\frac{\sqrt{c^2+\omega^2}}{\sqrt{3}}+\varepsilon d_1\right),$$

where $\omega>0$ and $\varepsilon>0$ are sufficiently small parameter. Them, we translate $p_\pm$ to the origin of the coordinates doing system (4.2) becomes $(x,y,z,w)=(\bar{x},\bar{y},\bar{z},\bar{w})+p_\pm$, then we introduce the scaling of variables $(x,y,z,w)=(\varepsilon x,\varepsilon y,\varepsilon z,\varepsilon w)$, with these changes of variables system (4.2) can be written as

$$\begin{aligned}
\dot{x}&=(x-y)\left(2c-a_1\varepsilon\right),\\
\dot{y}&=\frac{1}{3}\left(-\varepsilon 3x-\sqrt{3}\sqrt{\frac{\varepsilon b_1\left(-4\,c^2-\omega^2+2\,\sqrt{c^2+\omega^2}\sqrt{3}\varepsilon\,d_1-3\varepsilon^2{d_1}^2+3\,ec\right)}{c}}\right)w\\
&+\frac{\left(c\sqrt{c^2+\omega^2}\sqrt{3}-3\,c\varepsilon\,d_1\right)z}{3c}+\frac{\left(3\,\varepsilon^2{d_1}^2+4\,c^2+\omega^2-2\,\sqrt{c^2+\omega^2}\sqrt{3}\varepsilon d_1\right)x}{3c}-cy,\\
\dot{z}&=-cz+d_1y\varepsilon-\frac{y\sqrt{c^2+\omega^2}}{\sqrt{3}},\\
\dot{w}&=-\varepsilon\,b_1w+\varepsilon xy+\frac{\sqrt{3}x}{3}\sqrt{\frac{\varepsilon b_1\left(-4\,c^2-\omega^2+2\,\sqrt{c^2+\omega^2}\sqrt{3}\varepsilon d_1-3\varepsilon^2{d_1}^2+3\,ec\right)}{c}}\\
&+\frac{\sqrt{3}}{3}\sqrt{\frac{\varepsilon\,b_1\left(-4\,c^2-\omega^2+2\,\sqrt{c^2+\omega^2}\sqrt{3}\varepsilon\,d_1-3\,\varepsilon^2{d_1}^2+3\,ec\right)}{c}}y,
\end{aligned}\tag{4.18}$$

After the linear change in variables $(x,y,z,w)\mapsto(X,Y,Z,W)$,

$$\begin{aligned}
x&=\frac{2\,c\sqrt{3}X}{3}+\frac{2c^2\sqrt{3}Y}{3\omega}-\frac{2c^2Z}{\omega^2},\\
y&=\frac{c\sqrt{3}X}{3}+\frac{\left(\omega^3+2\,\omega\,c^2\right)\sqrt{3}Y}{3\omega^2}-\frac{2c^2Z}{\omega^2},\\
z&=\frac{\sqrt{c^2+\omega^2}X}{3}-\frac{2c\sqrt{c^2+\omega^2}Y}{3\omega}+\frac{2c\sqrt{c^2+\omega^2}\sqrt{3}Z}{3\omega^2},\\
w&=W,
\end{aligned}\tag{4.19}$$

the linear part at the origin of system (4.18) for $\varepsilon=0$ can be transformed into its real Jordan normal form,

$$\begin{pmatrix}0&-\omega&0&0\\ \omega&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}.$$

Under the change in variable (4.19), the system (4.18) can be written as

$$
\begin{aligned}
\dot{X} &= -Y\omega + \frac{1}{3}\left(a_1\left(-X+\frac{Y\omega}{c}\right) + \frac{-6c^2d_1Z+\sqrt{3}d_1\omega\left(2c^2Y+cX\omega+Y\omega^2\right)}{\omega^2\sqrt{c^2+\omega^2}}\right),\\
\dot{Y} &= \frac{1}{3\omega^3}(3(-2cW+\omega^2)X\omega^2+6c^2W(\sqrt{3}Z-Y\omega)+6d_1^2\varepsilon^2(-\sqrt{3}cZ+cY\omega+X\omega^2)\\
&\quad -\frac{3W\omega^2}{\sqrt{c}}\sqrt{-b_1\varepsilon(4c^2-3ce+\omega^2+d_1(3d_1\varepsilon-2\sqrt{3}\sqrt{c^2+\omega^2}))}\\
&\quad -\frac{\varepsilon\omega^2}{\sqrt{c^2+\omega^2}}(4\sqrt{3}c^2d_1X+c(-6d_1Z+\sqrt{3}d_1Y\omega-2a_1X\sqrt{c^2+\omega^2})\\
&\quad +\omega(5\sqrt{3}d_1X\omega+2a_1Y\sqrt{c^2+\omega^2}))),\\
\dot{Z} &= -\frac{2cWX}{\sqrt{3}}+\frac{2c^2WZ}{\omega^2}+\frac{2c^2WY}{\sqrt{3}\omega}+\frac{2d_1^2\varepsilon\left(-3cZ+\sqrt{3}\omega\left(cY+X\omega\right)\right)}{3\omega^2}\\
&\quad +\frac{\varepsilon}{18c^2}\left(6cd_1\left(-4cX+Y\omega\right)\sqrt{c^2+\omega^2}+\sqrt{3}a_1\left(cX-Y\omega\right)\left(4c^2+\omega^2\right)\right)\\
&\quad -\frac{W}{\sqrt{c}}\sqrt{-b_1\varepsilon\left(c^2-ce+\left(d_1\varepsilon-\frac{\sqrt{c^2+\omega^2}}{\sqrt{3}}\right)^2\right)},\\
\dot{W} &= -b_1W\varepsilon+\frac{1}{3\sqrt{c}}\left(2c^{5/2}\left(X^2+Y\left(Y-\frac{\sqrt{3}Z}{\omega}\right)\right)+2c^{3/2}XY\omega\right.\\
&\quad +\frac{6c^{7/2}X\left(-\sqrt{3}Z+Y\omega\right)}{\omega^2}+\frac{4c^{9/2}}{\omega^4}\left(3Z^2-2\sqrt{3}YZ\omega+Y^2\omega^2\right)\\
&\quad +3cX\sqrt{-b_1\varepsilon\left(4c^2-3ce+\omega^2+d_1\varepsilon\left(3d_1\varepsilon-2\sqrt{3}\sqrt{c^2+\omega^2}\right)\right)}\\
&\quad +Y\omega\sqrt{-b_1\varepsilon\left(4c^2-3ce+\omega^2+d_1\varepsilon\left(3d_1\varepsilon-2\sqrt{3}\sqrt{c^2+\omega^2}\right)\right)}\\
&\quad \left.+\frac{4c^2\left(-\sqrt{3}Z+Y\omega\right)}{\omega^2}\sqrt{-b_1\varepsilon\left(4c^2-3ce+\omega^2+d_1\varepsilon\left(3d_1\varepsilon-2\sqrt{3}\sqrt{c^2+\omega^2}\right)\right)}\right).
\end{aligned}
\tag{4.20}
$$

Performing the cylindrical change of variables,

$$(X,Y,Z,W)\mapsto(r\cos\theta,r\sin\theta,Z,W)$$

system (4.20) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{1}{3cr\omega^4\sqrt{c^2+\omega^2}}(-a_1cr^2\sqrt{c^2+\omega^2}\cos^2\theta+\cos\theta(-6c^3d_1rZ\omega+(-2\sqrt{3}d_1r^2\omega^2\\
&+\omega^2\sqrt{c^2+\omega^2}\left(3b_1W^2+a_1r^2\omega^2\right)+2c^2\sqrt{c^2+\omega^2}\left(6b_1W^2+r^2\left(a_1-3W\right)\omega^2\right)\\
&-c(4\sqrt{3}d_1r^2\omega^2+9b_1eW^2\sqrt{c^2+\omega^2}))\sin\theta)+cr(\sqrt{3}cd_1r\omega^3\cos 2\theta+6cZ(d_1\omega^2\\
&+\sqrt{3}cW\sqrt{c^2+\omega^2})\sin\theta-2r\omega\sqrt{c^2+\omega^2}\left(3c^2W+a_1\omega^2\right)\sin^2\theta))\varepsilon+\mathcal{O}\left(\varepsilon^2\right),\\
\frac{dZ}{d\theta} &= \frac{1}{18c^2r\omega^3}(c(6\sqrt{3}b_1\omega^2\left(4c^2-3ce+\omega^2\right)+r^2\omega^2(-24cd_1\sqrt{c^2+\omega^2}\\
&+\sqrt{3}\left(2c^2\left(a_1-3W\right)+a_1\omega^2\right)))\cos\theta+r(36c^4WZ+r\omega(6cd_1\omega^2\sqrt{c^2+\omega^2}\\
&-\sqrt{3}\left(12c^4W+4a_1c^2\omega^2+a_1\omega^4\right))\sin\theta))\varepsilon+\mathcal{O}\left(\varepsilon^2\right),\\
\frac{dW}{d\theta} &= \frac{1}{3cr\omega^5}(12c^5rZ^2+cr\left(2c^2r^2-3b_1W\right)\omega^4+c^2r^2\omega(6c^2\omega\cos\theta(-\sqrt{3}Z\\
&+r\omega\sin\theta)cZ\left(4c^2+\omega^2\right)+r\omega^4\cos\theta+2c^3r\omega\sin\theta)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\\
&-b_1W\left(4c^2-3ce+\omega^2\right)\cos\theta(-4\sqrt{3}c^2Z+r\omega\left(3c\omega\cos\theta+\left(4c^2+\omega^2\right)\sin\theta\right))).
\end{aligned}
\tag{4.21}
$$

System (4.21) is written in the normal form (2.1) for applying the averaging theory and satisfies all the assumptions of Theorem 2.1.1. Then, using the notations of the averaging theory described in chapter 2, we have $t=\theta$, $T=2\pi$, $x=(r,Z,W)$,

$$
F\left(\theta,r,Z,W\right)=\begin{pmatrix} F_1\left(\theta,r,Z,W\right)\\ F_2\left(\theta,r,Z,W\right)\\ F_3\left(\theta,r,Z,W\right)\end{pmatrix},\ \text{and}\ \ f(r,z,w)=\begin{pmatrix} f_1(r,Z,W)\\ f_2(r,Z,W)\\ f_3(r,Z,W)\end{pmatrix}.
$$

Then we compute the integrals, i.e.

$$
\begin{aligned}
f_1(r,Z,W) &= \frac{1}{2\pi}\int_0^{2\pi}F_1\left(\theta,r,Z,W\right)d\theta=-\frac{r\left(2c^2W+a_1\omega\right)}{2\omega^3},\\
f_2(r,Z,W) &= \frac{1}{2\pi}\int_0^{2\pi}F_2\left(\theta,r,Z,W\right)d\theta=\frac{2c^2WZ}{2\omega^3},\\
f_3(r,Z,W) &= \frac{1}{2\pi}\int_0^{2\pi}F_3\left(\theta,r,Z,W\right)d\theta\\
&= \frac{24c^2Z^2+c\left(4c^3r^2-12b_1cW+9b_1eW\right)\omega^2+\left(4c^2r^2-9b_1W\right)\omega}{6\omega^5}.
\end{aligned}
$$

Solving the equations $f_1(r,Z,W)=f_2(r,Z,W)=f_3(r,Z,W)=0$, we can get the following three solutions

$$
\begin{aligned}
s_0 &= (0,0,0)\,,\\
s_{1,2} &= \left(\pm\frac{1}{2c^2}\sqrt{\frac{3}{2}}\sqrt{a_1}\varepsilon\sqrt{\frac{b_1\left(-4c^2+3ce-3\omega^2\right)}{c^2+\omega^2}},0,-\frac{a_1\omega^2}{2c^2}\right).
\end{aligned}
$$

For two solutions, we get

$$\det\left(\frac{df}{dx}(s_1)\right) = \det\left(\frac{df}{dx}(s_2)\right) = \frac{a_1^2 b_1\left(4c^2 - 3ce + 3\omega^2\right)}{2\omega^5}.$$

When $c \neq 0$, $a_1 \neq 0$, and $\kappa = b_1\left(4c^2 - 3ce + 3\omega^2\right) < 0$ then

$$\det\left(\frac{\partial f}{\partial x}(s_j)\right) \neq 0, j = 1, 2.$$

Then according to Theorem 2.1.1, we see that the system (4.21) has one periodic solution $x_j(\theta, \varepsilon)$ such that $x_j(0, \varepsilon) = s_j + \mathcal{O}(\varepsilon)$, $j = 1, 2$. Bring the solution back to the system (4.20), and we have one periodic solution

$$\Phi_j(\theta, \varepsilon) = X_j(t, \varepsilon), Y_j(t, \varepsilon), Z_j(t, \varepsilon), W_j(t, \varepsilon).$$

Then the system (4.18) has the periodic solution $\varepsilon\Phi_j(\theta, \varepsilon)$, $j = 1, 2$.
The Jacobian matrices $\partial F(s_1)/\partial x$ have the same characteristic equation,

$$\lambda^3 + \frac{b_1 c(4c - 3e) + (2a_1 + 3b_1)\,\omega^2}{2\omega^3}\lambda^2 - \frac{a_1^2 b_1\left(4c^2 - 3ce + 3\omega^2\right)}{2\omega^5}.$$

The eigenvalues are given as follows

$$\begin{aligned}
\lambda_1 &= -\frac{a_1}{\omega}, \\
\lambda_2 &= -\frac{1}{4\omega^3}\Big(b_1\left(4c^2 - 3ce + 3\omega^2\right) \\
&\quad + \sqrt{b_1\left(4c^2 - 3ce + 3\omega^2\right)\left(b_1 c(4c - 3e) + (8a_1 + 3b_1)\,\omega^2\right)}\Big), \\
\lambda_3 &= -\frac{1}{4\omega^3}\Big(b_1\left(4c^2 - 3ce + 3\omega^2\right) \\
&\quad - \sqrt{b_1\left(4c^2 - 3ce + 3\omega^2\right)\left(b_1 c(4c - 3e) + (8a_1 + 3b_1)\,\omega^2\right)}\Big).
\end{aligned}$$

We have that λ_1 and $\lambda_{2,3} = -\frac{1}{4\omega^3}\left(\kappa \pm \sqrt{\kappa\left(\kappa + 8a_1\omega^2\right)}\right)$ are reals, if $a_1 > 0, \kappa < 0$ and regardless of the sign assumed by $\kappa\left(\kappa + 8a_1\omega^2\right)$, at least one of the eigenvalues has a positive real part. In this case, the periodic solution $\varepsilon\Phi_j(t, \varepsilon), j = 1, 2$ is unstable. ∎

Conclusion

The averaging theory is one a well known and important perturbation method to study the existence and stability of periodic solutions for some ordinary differential equation systems. It is a powerful tool and it has been proven its effectiveness many times in the literature by examining the existence and stability of isolated periodic orbits of dynamical systems with applications to physics and engineering sciences.

The aim of the present work has been to perform an analytical analysis of the periodic solutions of the so called Chen–Wang differential system in $\mathbb{R}^3$ and the Lorenz-Haken system in $\mathbb{R}^4$. These two systems exhibit some small–amplitude periodic solutions that bifurcate from a zero–Hopf equilibrium point.

Our future work plan will consist to study zero–Hope bifurcations of some differential systems in $\mathbb{R}^n$, where $n \geq 4$.

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