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Study Of the Stability Of Fractional Chaotic Systems With Sinusoidal Term

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بسم الله الرحمن الرحيم

شكر وعرفان

الحمد لله الذي وفقني وأعانني على إنهاء هذه المذكرة، أشكر الله سبحانه وتعالى على ما أسبغه عليا من نعم وعلى تسيير السبيل في كل وقت وكل ومنحي العزم والصبر لإتمامها. فله الحمد والشكر حين.

كما أتوجه بشكري وامتناني إلى من مهد لي طريق العلم فأعانني وكان لي خير قدوة أستاذي المشرف البروفسور " قصري احلام " الذي كانت لها بصمات واضحة من خلال توجيهاتها ودعمها الدائم لي.

كما أشكر أعضاء لجنة المناقشة التي شرفتني بقبولها مناقشة مذكرتي، الاستاذ "جدي نذير" رئيسا والاستاذة " مزهود رشيدة" ممتحنا اللذين لاشك أنهما سيفيضون لي بتوجيهاتهما القيمة وملاحظاتهما السديدة.

كما أتقدم بالشكر لكل أساتذتي في تكويني خلال مسيرتي الدراسية وإلى كل من قدم لي يد العون من قريب أو من بعيد فله مني خالص الاحترام والتقدير

أسأل الله أن يجازي الجميع كل الخير

إهداء

من قال انا لها ...نالها

وانا لها وان ابت رغما عنها اتيت بها.

نلتها وعانقت اليوم مجدا عظيما لم يكن الحلم قريبا ولا طريق سهلا ولكن وصلت والحمد لله حبا وشكرا وامتنانا، الحمدالله الذي بفضله أدركت أسمى الغايات

اهدي بكل حب مذكرة تخرجي الى نفسي العظيمة التي تحملت كل العثرات واكملت رغم الصعوبات الى أعظم الأشخاص واعز الناس على روحي

"جدي، جدتي، امي، ابي"

الى من دامت لي اياديهم وقت ضعفي الى ضلعي الثابت وامان قلبي اخوالي

"لطفي، سيف الدين، حمزة"

الى من اعطاني يد العون خالاتي

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الى من ساندوني بكل حب عند ضعفي زارعين الثقة والإصرار بداخلي سندي والكتف الذي استند علية دائما اخواتي وأخي العزيز

"كوثر، نور الهدى، تقوى، وصال، روان، رفيق"

الى القلوب الطاهرة وملائكة العائلة الاحفاد

"هبة الرحمان، ريتال، ساجد، اسيد"

الى رفاق الخطوة الأولى والخطوة ما قبل الأخيرة

" يسرى، فدوى"

Abstract

In this memory, two typical chaotic maps with sine terms serve as the basis for studying the dynamics of two fractional-order chaotic maps. Using numerical methods including phase plots, bifurcation diagrams, Lyapunov exponents, and 0–1 test, the dynamic behavior of this map is examined. It is demonstrated that the suggested fractional maps display a variety of distinct dynamical behaviors, including coexisting attractors, with a change in fractional order. The charting of a bifurcation diagram for two symmetric beginning conditions illustrates the existence of coexistence attractors. Furthermore, three control strategies are presented. The suggested maps' states are stabilized, and their convergence to zero is guaranteed by the first two controllers. while the last synchronizes two non-identical fractional maps asymptotically. The conclusions are validated using numerical outcomes.

Keywords: Chaos, discrete fractional calculus, sine maps.

Résumé

Dans ce mémoire, deux cartes chaotiques typiques avec des termes sinus servent de base pour l'étude de la dynamique de deux cartes chaotiques d'ordre fractionnel. En utilisant des méthodes numériques, y compris les plans de phase, les diagrammes de bifurcation, les exponents de Lyapunov et le test 0-1, le comportement dynamique de ces cartes est examiné. Il est démontré que les cartes fractionnelles suggérées montrent une variété de comportements dynamiques distincts, compris les attracteurs qui existent ensemble en même temps, avec un changement d'ordre fractionnaire.Dessiner un diagramme de bifurcation pour deux conditions initiales identiques illustre que les attracteurs sont existent ensemble en même temps. En outre, trois observations de contrôle sont affichées. Les états des cartes suggérées sont stabilisés, et leur convergence vers zéro est garantie par les deux premiers contrôleurs. Tandis que cette dernière synchronise asymptotiquement deux cartes fractionnaires non identiques, les conclusions sont validées en utilisant de résultats numériques.

Mots-clés : Chaos, calcul fractionnel discrète, cartes sinus.

ملخص

في هذه المذكرة، نُستخدم خريطتان فوضويتان نموذجيتان تحتويان على مصطلحات جيبية كقاعدة لدراسة ديناميات خريطتين فوضويتين من الرتبة الكسرية. باستخدام الطرق العددية، بما في ذلك مخططات الطور، ورسوم التشعب، ومُرَكَّبات ليابونوف، واختبار 0-1، يتم فحص السلوك الديناميكي لهذه الخريطة. تم إثبات أن الخرائط الكسرية المقترحة تُظهر مجموعة متنوعة من السلوكيات الديناميكية المميزة، بما في ذلك الجاذبات المتواجدة معا في نفس الوقت، مع تغيير في الرتبة الكسرية. توضح رسم تشعب لشرطين ابتدائيين متماثلين وجود جاذبات متواجدة معا في نفس الوقت. بالإضافة إلى ذلك، تُعرض ثلاث استراتيجيات للتحكم. يتم استقرار حالات الخرائط المقترحة، ويتم ضمان تقاربها إلى الصفر بواسطة أول متحكمين، بينما يقوم المتحكم الأخير بمزامنة خريطتين كسريتين غير متطابقتين بشكل غير متماثل. يتم التحقق من الاستنتاجات باستخدام النتائج العددية.
الكلمات المفتاحية: الفوضى، الحساب الكسري المتقطع، الخرائط الجيبية.

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General Introduction

The Chaotic dynamical systems which originate from both natural and engineered occurrences, have trajectory patterns that resemble randomness, significant sensitivity to initial conditions, and ergodicity, with attractors that are distinguished by intricate fractal configurations. Chaos maps in discrete-time systems, extensively scrutinized for their applications in science and engineering, are acknowledged. Various chaotic maps have been suggested and utilized in diverse domains, such as the Lozi[22] system, generalized Hanon map, Lorenz map, Stefanski map, Rossler map, Arnold map, and Grassi-Miller map. Discrete-time chaotic dynamical systems have been a focal point of exploration in chaos control, synchronization, and practical uses such as secure communications for encryption and synchronization using chaos. Fractional calculus serves as a potent tool for mathematical modeling, providing memory and non-locality to accurately delineate complex non-linear phenomena. Recent research endeavors have focused on the stability analysis of linear and nonlinear fractional discrete systems, employing techniques such as eigenvalue evaluation and extensions of the direct Lyapunov method.Fractional-order discrete maps provide a novel level of freedom and can be utilized to capture concealed facets of real-world phenomena in ecology. These maps exhibit sensitivity to minute perturbations in parameters and initial conditions, as well as variations in fractional orders. The investigation of the dynamics of fractional-order discrete maps is conducted due to their uncomplicated forms and diverse dynamics, which render them suitable for system analysis and computational purposes. The focal point of the dissertation revolves around distinct fractional-order discrete time systems characterized by the Caputo-like difference operator, with an examination of their dynamics carried out numerically.To assess the advantages of fractional-order maps, the dissertation evaluates their asymptotic convergence through stability analysis of linear fractional discrete systems. Additionally, the exploration of control aspects of fractional maps, encompassing stabilization and synchronization techniques,is undertaken to unveil novel intricate dynamical behaviors. The dynamics of discrete systems in a state of chaos and fractional discrete systems are provided. additionally attempts to investigate synchronization in fractional discrete systems.For that reason, this memory is divided into four chapters.

Chapter 1:explains the key elements of the chaos theory as well as techniques for identifying chaos.

Chapter 2:devoted to explaining the fundamental ideas of discrete fractional calculus, including the theory of the stability of fractionarily discrete systems, the definition of fractionary sum operators, and Caputo's fractional difference.

Chapter 3: In this chapter, we presented the definitions of synchronization, and then we reviewed

the different types of
synchronization while presenting the active observer method, using a numerical example to prove the validity of this method.

Chapter 4: We studied two Sine-Term and sinusoidal fractal maps, studying their stability using the appropriate observer, and then we synchronized the two corresponding sinusoidal fractal systems, the master and the slave, using complete synchronization. To study stability, we used methd linear stability with appropriate observer control.

Notation

Γ : Gamma function.

$t^{(\alpha)}$: Falling function.

V:Lyapunov function.

x_f :Fixed point.

λ_i :The eigenvalues.

$\|.\|$:The norm.

$\lim$:The limit.

$\Delta^{-\upsilon}$:Fractional sum operators.

${}^{C}\Delta_{a}^{\upsilon}$:Caputo operator for difference.

$e(t)$:Error.

U :Controller

$\mathbb{R}^n$:The set of dimensional real n:

Chapter 1

Dynamics of Discrete Systems in a State of Chaos

1.1 Introduction

The aim of this chapter is to provide the main keys to studying behavior of a chaotics dynamics systems with a particular emphasis on discrete time, it is a system that will produce diferent behaviors in the long term when the initial conditions are disturbed very slightly. The sensibility to the initial conditions of chaotic systems made chaos an undesirable situation.We introduce, in particular, the notions of trajectory, flow, phase space, phase portrait, we also present an overview of the trajectories. Nowadays, chaos theory has invaded most sciences. Indeed, by name-many mathematical models of physical processes, biological phenomena, chemical reactions and economic systems were defined using systems chaotic dynamics in discrete time. In this chapter we give the notions of trajectory, flow, phase space, phase portrait . In com-starting by giving some characteristics of chaos, we focus on the attractors Strange as well as Chaos Detection. so we let give a famous example on chaotic systems in discrete time.

1.2 Discrete system

Discrete system discrete dynamic system is a set of recurrent algebraic equations described by:

$$X_{K+1} = F\left(X_K, \mu\right) \tag{1.1}$$

where: F is the recurrence matrix function,$X_K \in U \subseteq R^n$ the state vector at the moment t_k and $v \in V \subseteq R^p$ the parameter vector and $k \in \mathbb{N}$.

1.3 Notions of chaos theory

Definition 1.1 *"Trajectory"*

Either x_0 is a starting condition or $x(t, x_0)$ is the solution to the autonomous dynamic system. The collection of points $\forall t > 0$, $x(t, x_0)$ represents the trajectory in the state space that leads to point x_0 at the starting time.

Definition 1.2 *"float"*

Either $x(x_0, t)$, $x_0 \in D$, is a solution of the autonomous dynamic system with initial condition $x(0) = x_0$. The application $\phi_t : D \to R^n$, defined by$\phi_t(x_0) = x(x_0, t)$, is referred to as the system's float.

Definition 1.3 *"Phase Space"*

The set of possible states of a dynamic system can alternatively be defined as an abstract space, with each variable representing a dimension required to describe the system at a given moment, and the degree of freedom defining the phase space. It represents the order that corresponds to the dimension of the state space.

Definition 1.4 *"Phase Portrait"*

A phase portrait is a graph that gives the appearance of trajectories in the phase space.

Definition 1.5 *"Dissipative system"*

A dissipative system is one in which the volume of the phase space shrinks over time. Because the volume of the phase space decreases, a dissipative system is usually distinguished by the presence of an attractor. A non-dissipative system is categorized as conservative.

1.3.1 Attractors

Examining the asymptotic behavior of a dynamic system driven by many non-linear diarrheal equations frequently exposes the concept of an attractor, which is the compact set of the phase space that this entity maintains invariant and towards which all of the system's paths converge. A closed sub-region of the phase space that "attracted" all other orbits to it is commonly referred to as an attractors Some mathematical determinations of attraction are quoted here:

Definition 1.6 *:(Guckenheimer et Holmes)*[3]

Whether (X, f) is a discrete dynamic system, a subpart A of X is called attractor if and only if the following conditions are met:

1. A is closed,
2. A is positively invariant,
3. A is attractive, i.e. there is an open neighborhood U of A such that:

- U is positively invariant,
- U is attracted by $A; \forall u \in U, \lim_{t\to\infty} d(f^t(u), A) = 0.$

1.3.2 The Strange Attractors

The geometric shape of a weird attractor is complex and fractal, existing in finite space and having a dimension that is not a decimal integer. Its trajectory is complex, and practically none of the attractor's trajectories ever go through the same location twice, which means the aperiodic nature of every trajectory. Trajectories' apparent conflicting tendencies to diverge on the attractor and to be attracted to it give rise to this peculiar nature. Real systems are dissipative, meaning that frictional forces cause trajectories to converge towards the attractor, which is why attraction occurs. On the other hand, divergence results from initial condition sensitivity. In contrast to the local phenomena of exponential divergence between two trajectories, attractors with finite dimensions are unable to diverge infinitely and must fold back onto themselves. Consequently, the weird attractor is the outcome of three concurrent ,contraction, stretching, and folding processes result in a distinctive structure that is folded, stretched, and horseshoe-shaped. These attractors are known as weird attractors and are the hallmark of chaos because of their fractal geometry. The verification of chaotic behavior and its quantitative description inside the attraction basin are made possible by this signature. An example of a "definition" of an odd attractor would be as follows:

As long as there is a neighborhood U of A in the phase space, a bounded subset as A can be thought of as a peculiar attractor for a transformation T of the space. Stated otherwise, for any point in A, there exists a ballshaped region that, while abiding by the following set of conditions, includes that particular point and is confined within the real number system $\mathbb{R}$:

Attraction: Since U is an absorbing zone, all orbits with U as their beginning point are completely enclosed within U. Furthermore, each orbit of this kind approaches and stays as near to A as is desired.

Sensitivity: Initial conditions have a significant impact on orbits whose initial point is in U.
Mixing property: There are orbits starting in U that pass as close to any point in A as is desired.

1.4 Definition of chaos

Before we state the definition of chaos, some necessary concepts need to be defined
Let $f : (X, d) \to (X, d)$ where (X, d) is a metric space and

$$x(k+1) = f(x(k)); k = 0, 1, \ldots \tag{1.2}$$

Definition 1.7 *The map f is said to have sensitive dependence on the initial conditions, if there exists $\epsilon > 0$ such that for any $x(0) \in X$ and any open U containing $x(0)$, there exists $y(0) \in U$ and $k \in \mathbb{Z}^+$ satisfying*

$$d(f^k(x(0)), f^k(y(0))) > \epsilon \tag{1.3}$$

Definition 1.8 *Let f be a map on a metric space (X, d). Then f is said to be (topologically) transitive if for any nonempty open sets U and V , there exists a positive integer k such that*

$$f^k(U) \cap V \neq \theta \tag{1.4}$$

Definition 1.9 *Let Y be a subset of an arbitrary set X. Then, Y is said to be dense in X if for any $x \in X$, there exists a sequence $(y_n)_{n \in \mathbb{N}} \in Y$ such that $\lim_{n\to\infty}(y_n) = x$. i.e, the closure of Y is X.*

In this stage, we are ready to put the definition of chaos due to Devaney.

Theorem 1.1 *$f : X \to X$ is said to be chaotic if*

- f possesses sensitivity to initial values,
- f is topologically transitive,
- The set of periodic points is dense in X.

Although there isn't a single, widely accepted definition of chaos, this definition is still the most fascinating because it is based on ideas that are easily understood.

1.5 Characteristics of chaos

1.5.1 The non-linearity

The method used to forecast actual occurrences produced by dynamic systems is to build a mathematical model that demonstrates the connection between a number of objects and a set of causes. The phenomenon is linear if this relationship is an operation of proportionality. The outcome is not proportionate to the cause when the phenomenon is nonlinear. A linear system can not be chaotic; generally speaking, a chaotic system is a nonlinear dynamic system.

1.5.2 Determinism

Deterministic systems are those whose evolution can be predicted or calculated.with time. the exact understanding of the system's state at any one time,the first time, made it possible to calculate the system's condition precisely at any other time.It is utterly impossible to forecast the course of a random phenomenon.any particulate.Thus, the basic deterministic, non-probability rules of a chaotic system exist.Usually, it is controlled by recognized nonlinear diarrheal equations.hence by unbending, totally deterministic laws.

1.5.3 L'aspect arbitraire

The system's points fill the phase space if the movement is random.random: no discernible structure. In a chaotic movement, the points are random at first glance. However, upon observing the system as a whole, it has been noted for a while that the dots create a specific form. The figure(1.1) depicts the chaotic dynamic system's random emergence in. The Rössler system of currenc

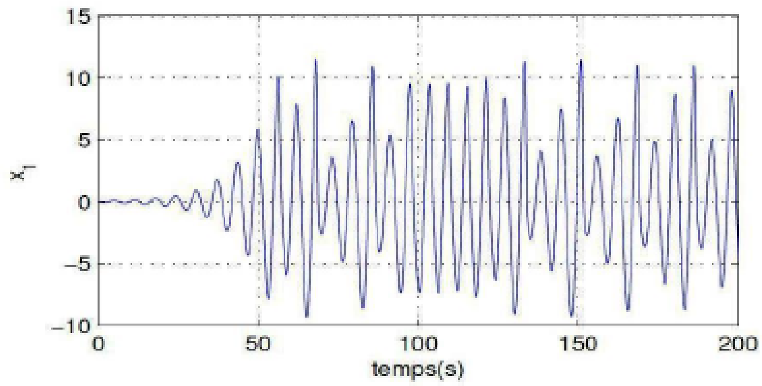


Figure 1.1: The random aspect of the Rössler system.

1.5.4 Sensitivity to initial conditions

Another property of chaotic phenomena is that they are very sensitive to disturbances. Eduard Lorenz had just discovered that in nonlinear systems, differences in the initial conditions would generate entirely different trajectories. Lorenz illustrated this fact by eating a butterfly. For two initially very close, arbitrary initial conditions, the two trajectories corresponding to these initial data diverge exponentially. Subsequently, the two trajectories are incomparable. One of the essential properties of chaos is, therefore, this sensitivity to initial conditions.which can be characterized by measuring the rates of divergence of trajectories from a mathematical point of view, it is said that f shows a sensitive dependence on initial conditions when:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D, \exists\, (y, p) \in D; \|x - y\| < \epsilon \Rightarrow \|f^p(x) - f^p(y)\| > \delta. \quad (1.5)$$

1.5.5 Power Spectrum

Therefore, in such a system, the variable's spectrum only comprises one assembly of strips that are situated at wi pulses, to their mwi harmonics with w_i, to combinations linear frequencies $mw_i + nw_j$ with a spectrum consisting of $m, n \in \mathbb{Z}$ Quasi-periodic frequencies are those with many frequencies but no one ratio. The presence of Broad spectrum is a necessary component of a system's chaotic dynamics.

1.6 Chaos Detection

There are several methods to determine whether nonlinear systems are chaotic or not. They are generally not very numerous, nor are they spread over a very long period of time across the system studied. Two of the most commonly used methods have been chosen: the fractal dimension and the Lyapunov exponents.

1.6.1 Lyapunov exponents

Chaotic evolution is difficult to understand, because the divergence of trajectories on the attractor is rapid. When this divergence grows exponentially with time for almost all initial conditions close to a given point, we have the phenomenon of sibility to initial conditions, an idea to which the Lyapounov exponents are attached, which give a quantitative measure of this local exponential divergence and measure in fact the degree of sensitivity of a dynamic system. Let us first recall this formula and let's see how Lyapunov was able to deduce such a formula.

$$\lambda = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left| f'\left(x_{i-1}\right) \right| \tag{1.6}$$

1.6.2 Fractal dimension

If, after calculating the dimension of the attractor of a system studied, we obtain a non-integer positive value, this means that the system has a strange attractor. we have several dimensions such as the Kolmogorov dimension, dimension correlation, Lyapunov dimension. There is a slight diference between each of these dimensions, but they all characterize,the strange attractor with its fractal dimension and satisfy the following three properties:

1. $A \subset B \Rightarrow d\left(A\right) \leq d(B)$.
2. $A = 0 \Rightarrow d\left(A\right) = 0$.
3. $d(A \times B) = d\left(A\right) \times d\left(B\right)$.

we will know Lyapunov dimention, the most famous and widely used dimension as follows:

Dimension of Lyapunov

This dimension is determined by Li and Yorke [14] and is given by :

$$d_l = m + \frac{\sum_{i=1}^{m} \lambda_i}{|\lambda_{m+1}|} \tag{1.7}$$

Ranking Lyapunov's exhibitors of the attractor of a dynamic system by :
$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ and m the largest integers such as :

$$\sum_{i=1}^{m} \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m+1} \lambda_i \leq 0. \tag{1.8}$$

1.7 Routes to chaos

It is yet unknown what circumstances lead a system to become chaotic. On the other hand, there are various ways in which a regular dynamic system can evolve toward chaos.Assume for the moment that a control parameter influences the dynamics under study [[4], [5], [6]].If this parameter is changed, the system may go from a stationary state to a periodic state, follow a transition scenario, and eventually enter a chaotic state if a particular threshold is reached. The change from a fixed point to chaos can be described by a number of situations.The transition from a fixed point to chaos is typically characterized by abrupt changes known as bifurcations rather than a steady process. The abrupt change from one dynamic regime to a different, qualitatively distinct one is indicated by a bifurcation. Each of these situations

1.7.1 Period doubling

The most well-known example of this shift toward chaos is presumably this one. The frequency of the periodic regime doubles when the experiment's control parameter is increased, and it is subsequently multiplied by 4, 8, 16, and so on. As the doublings get closer together, they trend to an accumulating point, at which an infinite frequency would presumably be obtained. This is the moment when chaos sets in with in the system.Robert May [7] has specifically examined it in population dynamics using the logistic map $X_{n+1} = rX_n(1 - X_n)$. The series either converges to a fixed point or does not, depending on the value of the parameter r. When an increases beyond 3, the system splits.Indicating that, around the fixed point, it oscillates between two values. This is referred to as a period-2 attractor cycle. These two attractors go away from the fixed point until a fresh bifurcation happens when r ieigenbaum came to understand the existence of a type of universality in this shift towards chaos a cascade of ncreases further. Every point splits, giving rise to a period-4 attractor cycle.This is referred to as period doubling. Fperiod doubling from this case.

1.7.2 Indeterminacy

In this situation, chaotic bursts arise erratically in a regularly oscillating system, which is characterized by indeterminacy. After a particular period of time, or "regularity," the system maintains a periodic or nearly periodic regime before destabilizing suddenly to create a chaotic explosion of sorts. After that, it stabilizes once more, paving the way for a subsequent burst. It has been noted that as one passes further away from the critical value of the constraint that caused the chaotic phases to occur, their frequency and length tend to grow. This transition phenomenon occurs when the limit cycle, which corresponds to the periodic state, bifurcates subcritically and there isn't an attractor in the vicinity. In the Rössler system, this is what is seen [8].

1.7.3 Quasi-periodicity

Ruelle and Takens' (1971) theoretical work [10] emphasized the quasi-periodicity scenario, which was exemplified by the Lorenz model (1963) [9]. Numerous studies, including the well-known thermohydrodynamic ones, have confirmed cette situation.Among other things, the Belousov-Zhabotinsky reaction in chemistry and the Rayleigh-Bénard convection in a tiny box. The dynamical system's "competition" between various frequences is what leads to chaos. A seconde frequency emerge quand a parameter is changed in a system that exhibits periodic behavior at a single frequency. The behavior is periodic if the ratio between the two frequences is reasonable. Nevertheless, if the ratio is The behavior is quasi-periodic, irrational, and, in this instance, the trajectories span the surface of a torus. When we adjust the setting once again, a third frequency starts to show up, and so on, till chaos. Additionally, some systems go straight from two frequencies to chaos.

1.8 0–1 Test

The benefit of this approach is that the test may be done right away to the series data $x(n)$ and does not require phase space reconstruction: The outcome is either 0 or 1, depending on how regular or chaotic the system is. The actual digits 1 and 0 are not here,therefore the test is still good as long as K is close enough to them. In practice, we applied the $0-1$ test method directly to the state $x(n)$. For an arbitrary constant c in $(0, \pi)$,we calculate the translation variables

$$p_c(n) = \sum_{j=1}^{n} x(j)\cos(jc), \; q_c(n) = \sum_{j=1}^{n} x(j)\sin(jc), \; n = 1, 2, \ldots, N. \tag{1.9}$$

The dynamics of the translation components (p_c, q_c) provide a visual test. Basically,if the dynamic is regular then the behavior of trajectories in the $(p_c - q_c)$ plane is bounded, whereas if the dynamic is chaotic then the $(p_c - q_c)$ trajectories depict Brownian like behavior.In order to examine the diffusive behavior of p_c and q_c, we define the meawn square displacement as:

$$M_c(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left((p_c(j+n) - p_c(j))^2 + (q_c(j+n) - q_c(j))^2 \right), \; n \leq \frac{N}{10} \tag{1.10}$$

with the asymptotic growth rate of

$$K_c = \lim_{n\to\infty} \frac{\log M_c(n)}{\log c} \tag{1.11}$$

In practice, the final result K can be determined numerically by computing the median of K_c When $K \simeq 1$, the behavior is classified as chaotic and when K is close to 0 the behavior is regular. In order to further confirm the influence of the fractional order $\upsilon = (\upsilon_1, \upsilon_2)$ on the properties of the fractional-order maps, we apply a new approach proposed by Gottwald and Melbourne called the $0 - 1$ test method.

1.9 Applications for chaos

- **Physics:** In order to understand the behavior of complex systems including fluid dynamics, online dynamics, and celestial mechanics, chaos theory has been employed in physics.
- **Engineering**: To improve the design and administration of complex systems, such as power plants, chemical reactors, and communication networks, chaos theory has been used to engineering.
- **Biology:** The workings of biological systems, such as brain networks, heart cycles, and ecological systems, are understood through the application of chaos theory in biology.
- **Finance**: To explain market behavior and create models that can forecast market volatility, chaos theory has been applied to finance.
- **computer science:** Algorithms for data analysis and optimization have been developed using chaos theory.Chaos theory has been utilized to music and art to examine the relationship between creativity and unpredictability and to generate new forms of expression.

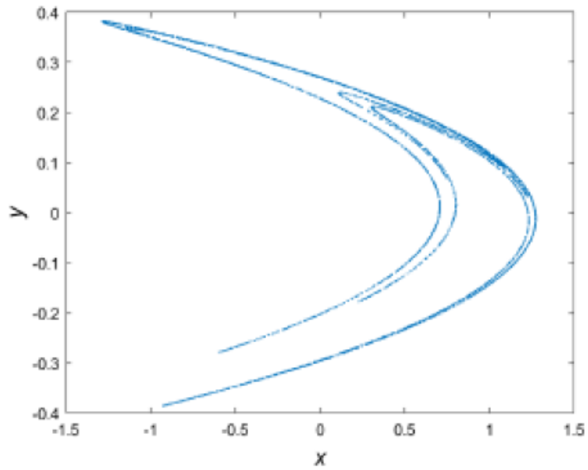


Figure 1.2: The Hénon attractor for $a = 1.4$ and $b = 0.3$.

1.10 Examples for discrete systems

Example 1.1 *01* ***The Hénon system[22]***

$$\begin{cases} X_{k+1} = 1 - aX_k^2 + Y_k \\ Y_{k+1} = bX_k \end{cases}$$

while a and b are parameters.

Example 1.2 *02* ***The Lozi system[23]***

$$\begin{cases} X_{k+1} = 1 - a|X_k| + bY_k \\ Y_{k+1} = X_k \end{cases}$$

where the system's nonlinear behavior is controlled by the parameters a and b.

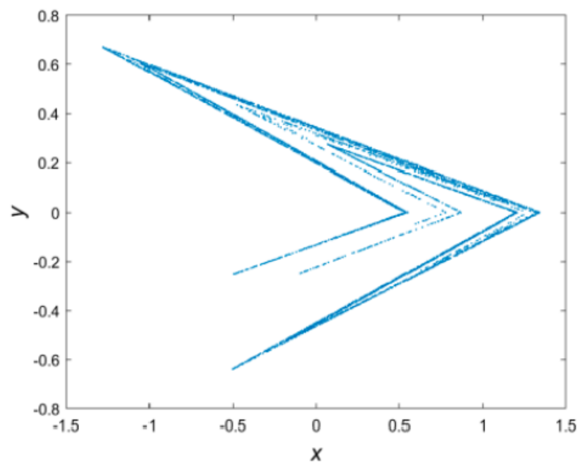


Figure 1.3: The Lozi attractor for $(x_0, y_0) = (0.0)\,, \nu = 0.98, a = 1.7$ and $b = 0.5.$

1.11 Conclusion

The interest of this chapter is to provide the basic notions of chaotic systems,in discrete time such as: phase space, trajectories, phase space, fixed point etc...

Then, we presented the characteristics of chaos, the transition and route of chaos, ending with an example of a discrete system.

Chapter 2

Fundamentals of Discrete Fractional Calculation

2.1 Introduction

We identify the most widely used formulation in the discrete fractional, the Caputo-operator difference, and provide several essential features based on the difference operator and Gamma function. which is useful for the remaining tasks, will be mentioned. Here we start by providing some essential fundamental definitions. such as the fractionl sum and differenceoperator, like caputo operator for différence, We recall some concepts relating to the stability of discrete chaotic systems of fractional order and their types of stability, in particular the Lyapunov function stability.

2.2 Basic notion

2.2.1 Gamma Function

he factorial for all real numbers is simply extended to represent the Gamma function in its most basic form[3]. It is defined by:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \ x \in \mathbb{R}^+$$

Definition 2.1 :

We define the decreasing factorial power for $n \in \mathbb{N}$ as:

$$t^{(n)} = t(t+1)(t+2)\dots(t+1-n) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}$$

2.2.2 Falling function

The definition of the falling function of real order is

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} \tag{2.1}$$

Keep in mind that $t^{(0)} = 1$, and that $t^{(\alpha)} = 0$ for any nonpositive $t - \alpha + 1$.

2.3 Fractional sum and difference operators

2.3.1 Fractional sum operators

Allow $X : \mathbb{N}_a \to \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, where $a \in \mathbb{R}$. Suppose that a function X defined on $\mathbb{N}_a$ has a difference operator stated in the following form:

$$\Delta X(t) = X(t+1) - X(t),\, t \in \mathbb{N}_a \tag{2.2}$$

Next, it is possible to define the fractional sum of positive fractional order υ.
The following defines the fractional sum of positive fractional order υ:

$$\Delta_a^{-\upsilon} X(t) = \frac{1}{\Gamma(\upsilon)} \sum_{s=a}^{t-\upsilon} (t - \sigma(s))^{(\upsilon-1)} X(s)), \tag{2.3}$$

$a \in \mathbb{R}$, $t \in \mathbb{N}_{a+\upsilon}$, and $\sigma(s) = s + 1$.
In the following definition, we are now prepared to precisely state the Caputo fractional difference operator.

2.3.2 Like Caputo operator for difference

The Caputo-like difference, denoted by an ${}^C\Delta_a^{\upsilon} X(t)$, is defined as follows:

$${}^C\Delta_a^{\upsilon} X(t) = \Delta^{-(n-\upsilon)} \Delta^n X(t) = \frac{1}{\Gamma(n-\upsilon)} \sum_{s=a}^{t-(n-\upsilon)} (t - \sigma(s))^{(n-\upsilon-1)} \Delta^n X(s)). \tag{2.4}$$

where

$$\Delta^n X(s) = \sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k)!} (-1)^{n-k} X(s+k). \tag{2.5}$$

In the event when n is a positive integer and the fractional order υ.

$$^{C}\Delta_a^{\upsilon}X(t) = \Delta^n X(t),\ t \in \mathbb{N}_a. \tag{2.6}$$

We list some of the different operators's properties below.
Here, we provide a few attributes associated with the definition given above.

- For a given constant C, the Caputo difference is

$$^{C}\Delta_a^{\upsilon}C = 0,\ 0 < \upsilon < 1$$

- For $\upsilon > 0$,and $f : \mathbb{N}_a \to \mathbb{R}$

$$^{C}\Delta_a^{-\upsilon}\Delta_{a+n-\upsilon}^{\upsilon} f\,(t) = f\,(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!}\Delta^k f(a).$$

Specifically, if $0 < \upsilon < 1$ then

$$^{C}\Delta_a^{-\upsilon}\Delta_{a+n-\upsilon}^{\upsilon} f\,(t) = f\,(t) - f(a).$$

We are now prepared to define an initial value problem with discrete fractions.

$$\begin{cases} {}^{C}\Delta_a^{\upsilon}X\,(t-\upsilon) = f(t+\upsilon-1), X\,(t+\upsilon-1) \\ \Delta^k X(a) = X_k,\ n = [\upsilon]+1,\ k = 0,\ldots,n-1 \end{cases} \tag{2.7}$$

For the discrete fractional issue (1.28), the corresponding discrete fractional equation is as follows:

$$X\,(t) = X_0(t) + \frac{1}{\Gamma\,(\upsilon)} \sum_{s=a+n-\upsilon}^{t-\upsilon} (t-\sigma\,(s))^{(\upsilon-1)} f(s+\upsilon-1, X(s+\upsilon-1)),\ t \in \mathbb{N}_{a+n}, \tag{2.8}$$

where:

$$X_0(t) = \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma(k+1)}\Delta^{(k)}X(k). \tag{2.9}$$

We can now define the equivalent numerical formula, when $a = 0$, thanks to Theorem 1.2.1.
For the discrete fractional problem (1.28), the discrete fractional equation is equal to the following numerical formula:

$$X\,(t) = X_0\,(t) + \frac{1}{\Gamma\,(\upsilon)} \sum_{j=0}^{t-\upsilon} \frac{\Gamma\,(t+\upsilon-1-j)}{\Gamma\,(t+\upsilon-j)} f\,(j, X\,(j))\,, \tag{2.10}$$

where the initial point is $X(0)$.

2.4 Fixed point

The study of the behavior of a discrete dynamic system is equivalent to the study of point stability. It's the dynamic system

$$x(k+1) = F(x(k)), \quad k = 1, 2, \ldots \tag{2.11}$$

where $x(k) = (x_1(k), \ldots, x_n(k))^T$ and $F : \mathbb{R}^n \to \mathbb{R}^n$
If there is a fixed point in the system, one can:

$$x_f = F(x_f) \tag{2.12}$$

In graphs, an equilibrium point is the intersection point between the two curves $y = f(x)$and $y = x$.

2.5 Stability

The notion of stability of a dynamic system characterizes the behavior of its trajectories around equilibrium points. Analysis of the stability of a dynamic system therefore allows us to study the evolution of its state trajectory when the initial state is close from a point of equilibrium. There are some concepts for the stability of dynamic systems. mics such as stability in the sense of Lyapunov.types of stability
Examining the vector difference equation will help.

$$\begin{cases} x(k+1) = F(k, x(k), \\ x(k_0) = x_0 \end{cases} \tag{2.13}$$

2.5.1 Stability type

Definition 2.2 *(Stability)*

The origin is a stable equilibrium point x in the Lyapunov sense of the system (1.14).

$$\forall \epsilon > 0, \exists \delta = \delta\left(\epsilon\right) > 0 : ||x(k_0) - x_f|| < \delta \Rightarrow ||x(k, x(k_0)) - x_f|| < \epsilon, \forall k \geq k_0$$

Definition 2.3 *(uniform stability)*

The origin is a point of equilibrium x uniformly stable in Lyapunov's sense of the system (1.14), which is:

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 : ||x(k_0) - x_f|| < \delta \Rightarrow ||x(k, x(k_0)) - x_f|| < \epsilon, \forall k \geq k_0$$

Definition 2.4 *(attractive)*

The fixed point x_f is attractive when there is convergence between the initial state x and the final state x_f over an infinite amount of time, provided that the initial criteria $x(k_0)$ are met, that is:

$$\forall k_0 \in \mathbb{N}; \exists \delta_0(k_0), \text{Such as: } \|x(k_0) - x_f\| < \delta_0(k_0) \Rightarrow \lim x(k, k_0, x(k_0)) = x_f, \tag{2.14}$$

When $\delta_0(t_0)$ it is said that the fixed point x_f is globally attractive.

Definition 2.5 *(asymptotic stability)*

When a fixed point x_f is both attractif (respectively globally asymptotically) and stable in the Lyapunov sense, it is said to be asymptotically (respectively asymptotically globally)

$$\exists \delta > 0 : ||x(k_0) - x_f|| < \delta \Rightarrow \lim_{t \to +\infty} ||x(k, k_0, x(k_0)) - x_f|| = 0$$

Thus, asymptotic stability means that one can determine a neighborhood of the equilibrium point, such as any trajectory derived from a point $x(0)$ belonging to a neighborhood. of x_f, tends to $x \rightarrow +\infty$

Definition 2.6 *(Exponential stability)*

The origin is a locally exponentially stable equilibrium point x_f of the system (1.14) if there are two constants strictly positive a ,b such as

$$\forall \epsilon > 0, \exists \delta > 0 : ||x(k_0) - x_f|| < \delta \Rightarrow ||x(k, x(k_0)) - x_f|| < a||x(k_0) - x_f|| \exp(-bk), \forall k \geq k_0, \forall \in B_r.$$

When $B_r = \mathbb{R}^n$, the origin is said to be globally exponentially stable.

Definition 2.7 *(Instability)*

The equilibrium point x_f is said to be unstable if it is not stable,.in the sense of Lyapunov.

2.5.2 Fractional order difference system stability

Stability of linear systems

[11]An equation of a linear system at zero:

$$^{C}\Delta_a^{\nu}X(k) = AX(k+\nu-1) \tag{2.15}$$

assumes asymptotic stability if:

$$\lambda \in \left\{ z \in \mathbb{C} : |z| < (2\cos\frac{|\arg z| - \pi}{2-\upsilon})^{\upsilon}, \text{ and } |\arg z| > \frac{\upsilon\pi}{2} \right\},$$

when $X(k) = (x_1(k), \ldots, x_n(k))$, and $0 < \upsilon \leq 1$ for each of A eigenvalues λ.

Example 2.1 :

Considre the following linear fractional discrete system:

$$\begin{cases} ^{C}\Delta_a^{\nu}X(t) = Y(k+\nu-1) - X(k+\nu-1), \\ ^{C}\Delta_a^{\nu}Y(t) = -X(k+\nu-1), \end{cases} \tag{2.16}$$

where $0 < \nu < 1$, $k \in \mathbb{N}_{a-\nu+1}$ and the matrix A is given by

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

The eigen values of the matrix A are $\lambda_1 = -1$, $\lambda_2 = -1$. Hence

According to Theorem 2.1, the trivial solution of the system (2.16) is asymptotically stable. The time evolution of the states of the system (2.16) is shown in Figure 2.1.

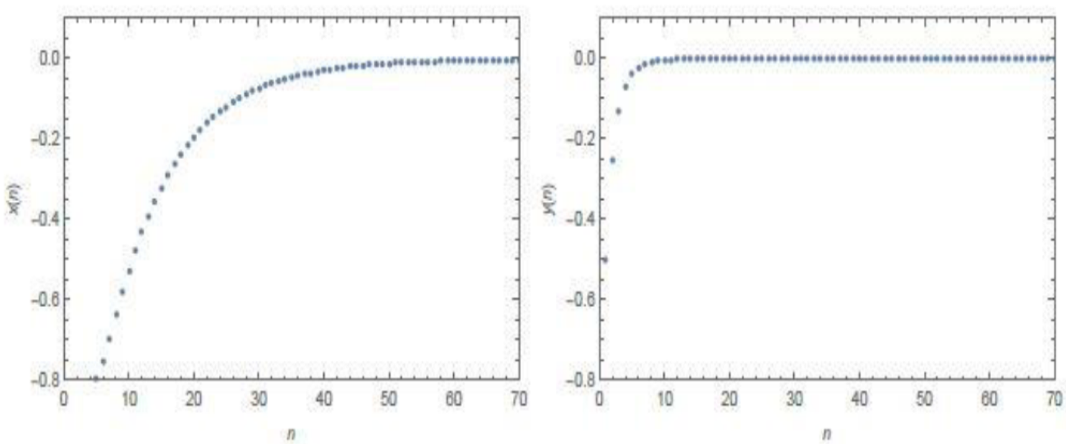


Figure 2.1: Time evolution of the system 2.16

Stability of non-linear systems

The discrete fractional nonlinear system:

$$ {}_{q}^{C}\Delta_{a}^{\upsilon} = F(k+\upsilon-1, X(k+\upsilon-1)), \text{ for every } k \in \mathbb{N}_{a+n-\upsilon}, \tag{2.17}$$

about the presence and stability of asymptotic outcomes. The stabilization of the origin equilibrium point and Lyapunov stability are the two most frequently applied theorems. The following theorems declare them, accordingly.

Asymptotically stable fractional nonlinear discrete system (3.1) exists if, given the equilibrium point $x = 0$, there is a positive definite and decreasing scalar function $V(t; X(t))$ such that ${}^{C}\Delta_{a}^{\upsilon}V(t, X(t)) \leq 0$.

The following inequality is true for each $t \in \mathbb{N}_{a+n-\upsilon}$:

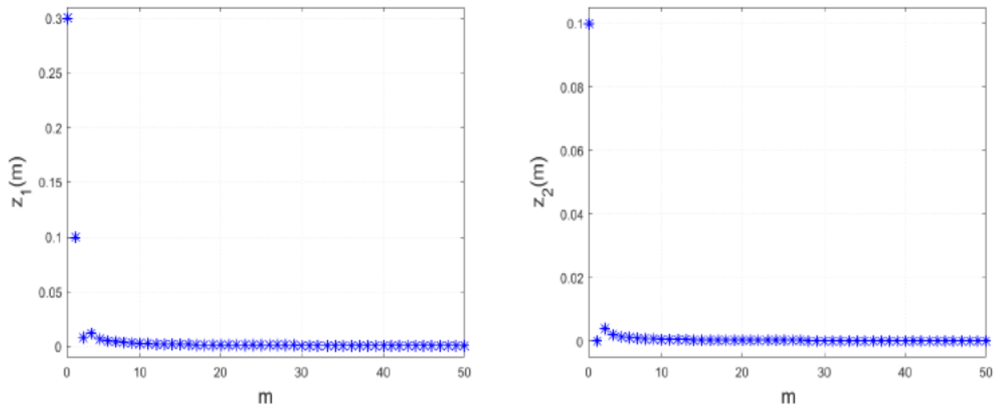
$$\frac{1}{2}{}^{C}\Delta_{a}^{\upsilon}X^{T}(t)\,X(t) \leq X^{T}(t+\upsilon-1){}^{C}\Delta_{a}^{\upsilon}X(t)\,, \quad 0<\upsilon\leq 1. \tag{2.18}$$

Examine the non-linear fractional system that follows.

$$\begin{cases} {}^{C}\Delta_{a}^{\upsilon}X(t) = -0.1X(t+\upsilon-1) \\ {}^{C}\Delta_{a}^{\upsilon}Y(t) = -0.5Y(t+\upsilon-1) \end{cases} \tag{2.19}$$

where $0<\upsilon<1$

In accordance with Lemma and the Lyapunov function $V = \frac{1}{2}(x^2+y^2)$, we generate



(a) (b)

Figure 2.2: Time evolution of system

$$\begin{aligned}{}^{C}\Delta_a^{\upsilon}V &= X(t+\upsilon-1)^{C}\Delta_a^{\upsilon}X(t)+Y(t+\upsilon-1)^{C}\Delta_a^{\upsilon}Y(t)\\ &\leq -0.1X^2(t+\upsilon-1)-0.5Y^2(t+\upsilon-1)\\ &\leq 0\end{aligned}$$

Therefore, the system (2.19) is asymptotically stable based on Theorem 2.1.Figure 2.2 displays the condition of the system's time progression (2.19).

2.6 Conclusion

In this chapter is to study the stability of the solution of a system of differential equations linear and non-linear fractional. The derivatives considered are in Caputo, and order between 0 and 1. The theoretical results given clearly show that the stability condition for systems fractional order differs from the well-known state for integer order systems. Especially, the left half-plane (stable region) for integer order systems applies in the angular sector $|\arg\lambda| > \left(\frac{(\upsilon\pi)}{2}\right)$ in the case of fractional order systems, indicating that the stability region becomes larger and larger when the value of the fractional order is diminished.

Chapter 3

Theory of synchronization

3.1 Introduction:

The phenomenon of synchronization is manifested when two dynamic systems evolve read in the same way over time. One of the synchronization configurations. The most popular configuration is the master-slave configuration for which a system dynamic called slave system follows the rhythm and trajectory imposed by another dynamic system called master system. The prospects for using chaos in various applications have motivated researchers to study the question of the possible posability to synchronize chaos. The prospects of using chaos in various applications have motivated researchers to study the question of the possible possibility of ability to synchronize chaos. This synchronization seems difficult to achieve, because the diference of synchronization where we seek to reproduce only a period oscillation, chaotic synchronization presents more constraints.

3.2 Synchronization definition

3.2.1 General Definition

One of the first things humans have noticed is the synchronization of human relationships. For example, a baby that responds to its mother's smile simply synchronizes its own facial expressions with hers.

Definition 3.1 *(Larousse)*

Synchronization is a Greek word divided by two parts: Syn wants to say together, and Chrono wants to say time. Phase-setting is the process of arranging many operations to occur simultaneously, depending on the time.

Definition 3.2 *(general)*

One way to manage a periodic movement (or chaotic movement) is through synchronization. When two dynamic systems synchronize, it means that one system evolves by imitating the other's behavior.

3.2.2 Mathematical definitions of synchronization

We have the two systems master-slave as folloing.

$$\begin{cases} \Delta X(k+1-\alpha) = f(X(k)) \\ \Delta Y(k+1-\alpha) = g(Y(k)) + U \end{cases} \tag{3.1}$$

where $X(k) = (x_1(k), x_2(k), ..., x_n(k))^T \in \mathbb{R}^n$ and $Y(k) = (y_1(k), y_2(k), ..., y_n(k))^T \in \mathbb{R}^n$,with $X, Y \in \mathbb{R}^n$, f and g of defined nonlinear functions of $\mathbb{R}^n \to \mathbb{R}^n$. Both systems are called synchronized if

$$\lim_{t\to+\infty} \|e(k)\| = 0$$

where $e(k) = X(k) - Y(k)$ represents the synchronization error.

.

Definition 3.3 *Brown and Kocarev [24] The subsystems in equation (3.1) are synchronized on the path of $\varphi(w_0)$, relative to the g_x and g_y properties, if there is a instant independent of application h such that $||h(g_x, g_y)|| = 0$.With the choice of g_x, g_y, and h, we can determine the type of synchronization. This approach leads to the idea that there are different types of synchronization that could be They are engaged in the same formalism.*

Theorem 3.1 *The master system and the slave system are synchronized only if all of the slave system's Lyapunov exposants, also known as conditional Lyapunov exposants, are negative.*

3.3 Master-slave system

let consider a master system as follow:

$$\begin{cases} {}^{C}\Delta_{0}^{\alpha_1} x_1 \left(k+1-\alpha_1\right) = f_1\left(X\left(k\right)\right), \\ {}^{C}\Delta_{0}^{\alpha_2} x_2 \left(k+1-\alpha_2\right) = f_2\left(X\left(k\right)\right), \\ \\ {}^{C}\Delta_{0}^{\alpha_n} x_n \left(k+1-\alpha_n\right) = f_n\left(X\left(k\right)\right), \end{cases} k = 0, 1, ..., \tag{3.2}$$

the Caputo fractional difference of order α_i denoted by ${}^{C}\Delta_{0}^{\alpha_i}$ where $0 < \alpha_i \leq 1$, for $i = 1, 2, .., n$, $X(k) = (x_1(k), x_2(k), ..., x_n(k))^T \in \mathbb{R}^n$ is the state of the system (1) and $(f_1, f_2, ..., f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$. A slave system is given by:

$$\begin{cases} {}^{C}\Delta_{0}^{\alpha_1} y_1 \left(k+1-\alpha_1\right) = g_1\left(Y\left(k\right)\right) + U_1, \\ {}^{C}\Delta_{0}^{\alpha_2} y_2 \left(k+1-\alpha_2\right) = g_2\left(Y\left(k\right)\right) + U_2, \\ \\ {}^{C}\Delta_{0}^{\alpha_n} y_n \left(k+1-\alpha_n\right) = g_n\left(Y\left(k\right)\right) + U_n, \end{cases} k = 0, 1, ..., \tag{3.3}$$

where $U = (U_1, U_2, ..., U_n)^T \in \mathbb{R}^n$ is a control vector to be determined, and $Y(k) = (y_1(k), y_2(k), ..., y_n(k))^T$ $\mathbb{R}^n$ is the state of the system (2), $(g_1; g_2, ..., g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$.

3.4 Synchronization types

We have an extensive bibliography, we have been advisedd ifferent types and schemes of the synchronization tell you that the complete synchronization (CS), anti-synchronization (AS), the generalized synchronization (GS), the projective synchronization (PS), full state hybrid projective synchronization Synchronization Q- S ,etc.

Complete synchronization (C.S)

Definition 3.4 *[12] The problem of complete synchronization is to determine the control U so that*

$$\lim_{k\to\infty} \|Y(k) - X(k)\| = 0, \tag{3.4}$$

where $\|.\|$is the euclidean norm.

Remark 3.1 *If $(f_1, f_2, ..., f_n) = (g_1, g_2, ..., g_n)$, the relationship becomes identical complete synchronization.*

If $(f_1, f_2, ..., f_n) \neq (g_1, g_2, ..., g_n)$, it is a non-identical complete synchronization.

Anti-synchronization

Definition 3.5 *[13] The problem of anti-synchronization is to determine the control U so that*

$$\lim_{k\to\infty} \|Y(k) + X(k)\| = 0, \tag{3.5}$$

Projective synchronization

Definition 3.6 *[14]*

The master system $X(k) = (x_i(k))$ and the drive system $Y(k) = (y_i(k))$are said to be projective synchronized, if there exists non zero vector $\alpha = (\alpha_i)_{1 \leq i \leq n}$ such that

$$\lim_{k \to \infty} |y_i(k)| - \alpha_i x_i(k)| = 0, \quad \forall (x(0), y(0)),\ 1 \leq i \leq n$$

FSHP synchronization (FSHPS)

Definition 3.7 *[15] We say that we have a FSHP synchronization (full state hybrid projective synchronization) between the master system (3.2) and the slave system (3.3), if there exists a controls $Ui, 1 \leq i \leq n$, and a constants $\left(\gamma_{ij}\right)_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$, such as:*

$$\lim_{k \to \infty} \left| y_i(k) - \sum_{j=1}^{n} \gamma_{ij} x_j(k) \right| = 0, \quad i = 1, ..., n. \tag{3.6}$$

FSHPI synchronization (IFSHPS)

Definition 3.8 *[16] We say that we have a FSHP inverse synchronization between the master system(3.2) and the slave system (3.3), if there exists a controls$Ui, 1 \leq i \leq n$, and a constants $\left(\beta_{ij}\right)_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$, so that synchronization errors*

$$\lim_{k \to \infty} \left| x_i(k) - \sum_{j=1}^{n} \beta_{ij} y_j(k) \right| = 0, \quad i = 1, ..., n. \tag{3.7}$$

Generalized synchronization (G.S)

Definition 3.9 *[17] If there exists a controller U and a function $\phi : \mathbb{R}^n \to \mathbb{R}^n$, check*

$$\lim_{k \to \infty} \|Y(k) - \phi(X(k))\| = 0, \tag{3.8}$$

then, systems (3.2) and (3.3) synchronize in the generalized sense with respect to the function ϕ.

Remark 3.2 *Generalized synchronization is considered to be a generalization of complete synchronization, anti-synchronization, projective synchronization and FSHP synchronization.*

Inverse generalized synchronization (IGS)

Definition 3.10 *[18] If there exists a controller U and a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, check*

$$\lim_{k\to\infty} \|X(k) - \varphi(Y(k))\| = 0, \tag{3.9}$$

then, systems (3.2) and (3.3) synchronize in the generalized inverse sense with respect to the function φ.

Remark 3.3 *If the function ϕ is defined by $\phi(Y(k)) = DY(k)$ where $B = (\beta_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$, we say that we have a inverse full-state hybrid projective synchronization.*

Synchronization $\mathbf{Q-S}$

Definition 3.11 *[19] We say that systems (3.2) and (3.3) are in $Q-S$ synchronization in dimension d, if there is a controller U and two functions $Q : \mathbb{R}^n \to \mathbb{R}^d, S : \mathbb{R}^n \to \mathbb{R}^d$ such that*

$$\lim_{k\to\infty} \|Q(X(k)) - S(Y(k))\| = 0. \tag{3.10}$$

Remark 3.4 *$Q-S$ synchronization is considered to be a generalization of all types of previous synchronizations.*

3.5 Method of an active controller:

The active controller method is a powerful methodology that has demonstrated its ability to synchronize not only identical systems but also non-identical systems with different dimensions. There are two synchronization systems, master and slave, which are defined as follows:

$${}^{c}D_t^q X(k) = F(X(k)) \tag{3.11}$$

and

$${}^{c}D_t^q Y(k) = G(Y(k)) + U \tag{3.12}$$

Where: $X(k), Y(k) \in \mathbb{R}^n$ are the state vectors of the master and slave systems, respectively One control vector to determine is $F : \mathbb{R}^n \to \mathbb{R}^n, G : \mathbb{R}^m \to \mathbb{R}^m$, $0 < q \le 1, U \in \mathbb{R}^m$.
When time tends to infinity, the error between the two systems' trajectories must converge towards zero for the two systems to synchronize. This error is obtained as follows:

$$e(k) = Y(k) - X(k) \tag{3.13}$$

so

$$\begin{aligned} {}^{c}D_{t}^{q}e(k) &= {}^{c}D_{t}^{q}Y(k) - {}^{c}D_{t}^{q}X(k) \\ &= F(X(k)) - G(Y(k)) + U \end{aligned}$$

If the quantity $F(X(k)) - G(Y(k))$ may be expressed in the following way:

$$-F(X(k)) + G(Y(k)) = Ae(k) + H(X(k), Y(k)) \tag{3.14}$$

The error can be expressed as follows:

$$ {}^{c}D_{t}^{q}e(k) = Ae(k) + H(X(k), Y(k)) + U \tag{3.15}$$

In which $A \in \mathbb{R}^{n}$is a constant matrix and H is a non-linear function. This is how the controller U is suggested:

$$U = V - H(X(k), Y(k)) \tag{3.16}$$

where V is the active controller, as defined by:

$$V = -Le(k) \tag{3.17}$$

in which L is an unknown control matrix. As a result, the error's final formula becomes:

$$ {}^{c}D_{t}^{q}e(k) = (A - L)\, e(k) \tag{3.18}$$

Thus, the synchronization issue between the master system (3.1) and the slave system (3.2) becomes the zero-stabilized system issue (8). The following theory is an immediate consequence of the theory of fractional-order system stability.

Example 3.1 *Let us consider master the 2D fractional lorenz map [8] as follow*

$$\begin{cases} {}^{C}\Delta_{a}^{\alpha_1}x_1(k+1-\alpha_1) = \sum_{j=1}^{2} a_{1j}x_j(k) + f_1(x_1(k), x_2(k)), \\ {}^{C}\Delta_{a}^{\alpha_2}x_2(k+1-\alpha_2) = \sum_{j=1}^{2} a_{2j}x_j(k) + f_2(x_1(k), x_2(k)), \end{cases} \quad k = 0, 1, ..., \tag{3.19}$$

whene

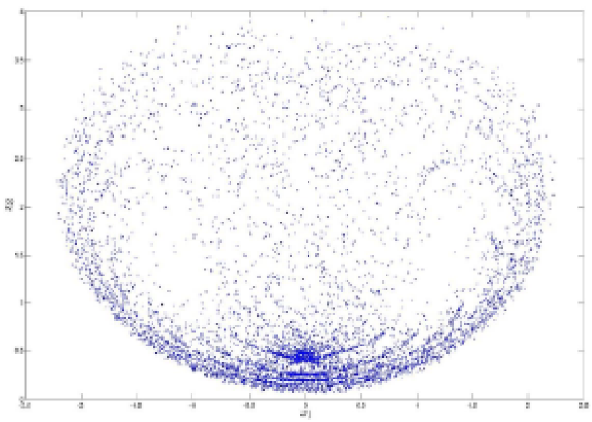


Figure 3.1: Chaotic attractor of the fractional order Lorenz map for$\alpha_1 = 0.98, \alpha_2 = 0.8$.

$$\begin{cases} (a_{11}, a_{12}, a_{21}, a_{22}) = (0.93751, 0, 0, -0, 75)\,, \\ f_1(x_1(k), x_2(k)) = -0.75x_1(k)x_2(k), \\ f_2(x_1(k), x_2(k)) = 0.75x_1^2(k), \\ \alpha_1 \neq \alpha_2, a = 0, \end{cases}$$

and and initial conditions are $x_1\left(0\right) = 0.1, x_2\left(0\right) = 0.$

Figure 3.1 showing the resulting chaotic attractor nd its general shape is similar to that of the integer order one. We choose the fractional flow map suggested in for the slave. We defined of the slave system as follow:

$$\begin{cases} {}^{C}\Delta_a^{\alpha_1} y_1\left(k+1-\alpha_1\right) = \sum_{j=1}^{2} b_{1j} y_j\left(k\right) + g_1\left(y_1\left(k\right), y_2\left(k\right)\right) + U_1, \\ {}^{C}\Delta_a^{\alpha_2} y_2\left(k+1-\alpha_2\right) = \sum_{j=1}^{2} b_{2j} y_j\left(k\right) + g_2\left(y_1\left(k\right), y_2\left(k\right)\right) + U_2, \end{cases} \quad k = 0, 1, ..., \tag{3.20}$$

where y_1, y_2 are states of the slave systems respectively, and

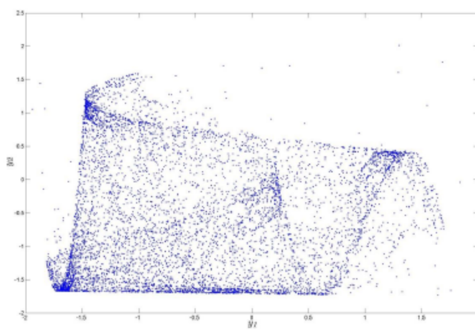


Figure 3.2: Fractional order chaotic attractor Flow map for $\alpha_1 = 0.98, \alpha_2 = 0.8.$

$$\begin{cases} (b_{11}, b_{12}, b_{21}, b_{22}) = (-1.1, 1, 0, -1), \\ g_1\left(y_1\left(k\right), y_2\left(k\right)\right) = 0 \\ g_2\left(y_1\left(k\right), y_2\left(k\right)\right) = y_1^2\left(k\right) - 1.7, \\ a = 0, \alpha_1 \neq \alpha_2, \end{cases}$$

and $U_i\left(k\right), i = 1, 2,$ are controllers, the uncontrolled map (12) with $U_1 = U_2 = 0$ is chaotic as presented in Figure3.2.

Theorem 4.1 states that condition (3.14) can be satisfied by B-C if there is a control matrix C. One may select the case, for example

$$C = \begin{pmatrix} -0.1 & 0 \\ 0 & 0 \end{pmatrix},$$

It manifestly meets the requirement, and hence, systems (11) and (12) are in 2D synchronization. The control law may now be constructed quite easily using Theorem . The error system that results looks like this:

$$\begin{cases} {}^{C}\Delta_a^{\alpha_1} e_1\left(k + 1 - \alpha_1\right) = -e_1\left(k\right) + e_2\left(k\right), \\ {}^{C}\Delta_a^{\alpha_2} e_2\left(k + 1 - \alpha_2\right) = -e_2\left(k\right), \end{cases} \quad k = 0, 1,$$

Figure3.3 shows the errors time evolution. Since the errors converge to zero in a reasonable amount of time, synchronization is evidently achieved.

The numerical results show how accurate the suggested scheme

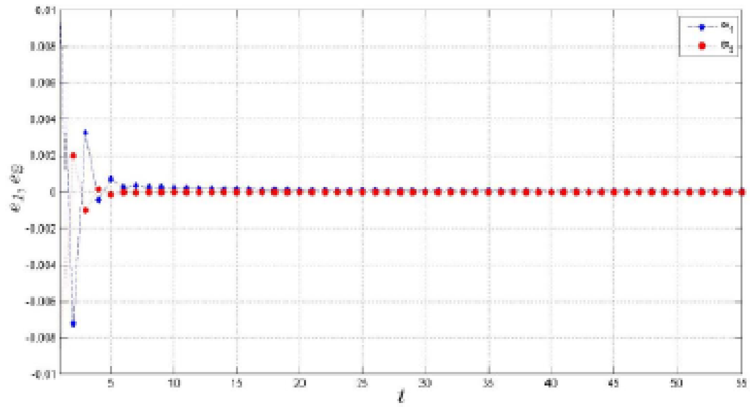


Figure 3.3: Evolution of states of the error system for $\alpha_1 = 0.98, \alpha_2 = 0.8$.

3.6 Conclusion

The main objective of this chapter was to present the different types of synchronization and the most efficient active synchronization control method. we have presented the definitions of chaotic synchronization. We then defined several synchronization types of chaotic systems of fractional order. The method of active control most used to achieve synchronization of chaotic dynamic systems.

Chapter 4

On the Dynamics and Control of Fractional Chaotic Maps with Sine Terms

This study examines the behavior of two chaotic maps with fractional orders, which are derived from standard chaotic maps with sine terms. The analysis of this map's dynamics is conducted using numerical methods like phase plots, bifurcation diagrams, Lyapunov exponents,and the 0-1 test. By varying the fractional order, it is demonstrated that the fractional maps proposed in this study display various dynamic behaviors, including the presence of coexisting attractors. Furthermore, three control schemes are presented. The first two controllers work to stabilize the states of the proposed maps and guarantee their asymptotic convergence to zero. The third controller is responsible for synchronizing a pair of non-identical fractional maps. Numerical results are utilized to validate the conclusions.

4.1 Map of fractional sines

The two-dimensional iterated map presented by Zeraoulia and Sprott in 2008 [4] and provided by: is of relevance to us in this part.

$$\begin{cases} x(n+1) = 1 - \alpha \sin x(n) + \beta y(n), \\ y(n+1) = x(n), \end{cases} \tag{4.1}$$

where x and y represent the discrete-time system's states, α and β stand for some bifurcation parameters. All it took to create this map was to swap out the x^2 word in the typical Hénon map using the $\sin x$ trigonometric term. By use of a period–doubling bifurcation path to chaos, this was found to generate a C^∞ mapping that may be regarded as a generalization of the chaotic attractor with "multifolds".

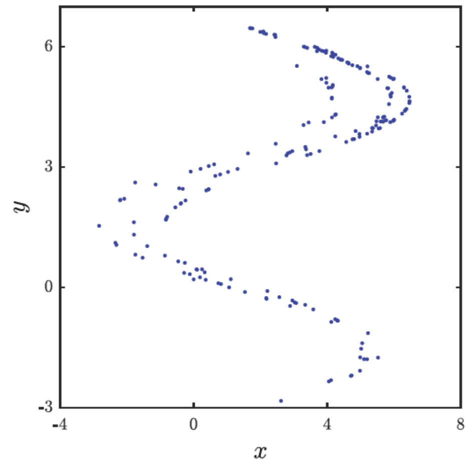


Figure 4.1: Phase portrait of the sine map for parameter values $(\alpha, \beta) = (3.8, 0.3)$

[4] states that when α falls between $[-150, 200]$ and $\beta = 0.3$, the map displays chaotic behavior. The map for $\alpha = 0.3$ is examined in the following. Phase space is what mapped out in Figure 1, confirming the existence of chaos.

The fractional Caputo-type form of (1), which we derive first, begins with the delta-differences taken as follows:

$$\begin{cases} \Delta x(n) = x(n+1) - x(n) \\ \Delta y(n) = y(n+1) - y(n) \end{cases}$$

$$\begin{cases} \Delta x(n) = 1 - \alpha \sin x(n) + \beta y(n) - x(n), \\ \\ \Delta y(n) = x(n) - y(n). \end{cases} \tag{4.2}$$

Next, using the definition of the Caputo-like delta difference to substitute the first-order difference, we obtain:-

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = 1 - \alpha \sin x(t+\upsilon-1) + \beta y(t+\upsilon-1) - x(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases} \tag{4.3}$$

where an is the beginning point, $0 < \upsilon < 1$, and $t \in \mathbb{N}_{a+1-\upsilon}$.We will refer to (4.2) as the fractional sine map from now on.

We use Theorem 1 to derive a numerical formula for system (4.2) in order to do numerical analysis. To begin, we take the equivalent discrete integral for $t \in \mathbb{N}_{a+1-\upsilon}$ and $0 < \upsilon < 1$. This yields:

$$\begin{cases} x(t) = x(a) + \frac{1}{\Gamma(\upsilon)} \sum_{s=t+1-\upsilon}^{t-\upsilon} \left(t - \sigma\left(s\right)\right)^{(\upsilon-1)} \left(1 - \alpha \sin x(t+\upsilon-1) + \beta y(t+\upsilon-1) - x(t+\upsilon-1)\right), \\ \\ y(t) = y(a) + \frac{1}{\Gamma(\upsilon)} \sum_{s=t+1-\upsilon}^{t-\upsilon} \left(t - \sigma\left(s\right)\right)^{(\upsilon-1)} \left(x(t+\upsilon-1) - y(t+\upsilon-1)\right), \end{cases} \tag{4.4}$$

The reciprocal $\frac{(t-\sigma(s))^{(\upsilon-1)}}{\Gamma(\upsilon)}$, which stands for a discrete kernel function, can be understood as follows:

$$\frac{\left(t - \sigma\left(s\right)\right)^{(\upsilon-1)}}{\Gamma\left(\upsilon\right)} = \frac{\Gamma(t-s)}{\Gamma\left(\upsilon\right)\Gamma\left(t-s-\upsilon+1\right)}. \tag{4.5}$$

Therefore, by selecting a zero initial value, that is, $a = 0$, the numerical formula is defined as follows:

$$\begin{cases} x(n) = x(0) + \frac{1}{\Gamma(\upsilon)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\upsilon)}{\Gamma(n-j+1)} \left(1 - \alpha \sin x(j-1) + \beta y(j-1) - x(j-1)\right). \\ \\ y(n) = y(0) + \frac{1}{\Gamma(\upsilon)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\upsilon)}{\Gamma(n-j+1)} (x(j-1) - y(j-1)), \end{cases} \tag{4.6}$$

We take into consideration the same bifurcation parameter values as previously employed for numerical simulation. Calculus formula (4.6) is used to examine the impact of the fractional order υ regarding the fractional sine map's dynamics (4.2). The bifurcation diagram that results when $\upsilon = 1$ is first shown in Figure 2, where α is varied in steps of $\Delta\alpha = 0.003$ and maintained within the interval $[-1, 4]$. Our fractional map should reduce to the conventional one since $\upsilon = 1$, with the answer $x(n)$ based on all historical data $x(n-1), x(n-2), ..., x(0)$. The map's dynamics change when the fractional order υ is changed from 1 to smaller values. For $\upsilon = 0.976, \upsilon = 0.78$, and $\upsilon = 0.65$, the phase portraits are Figure 4 shows the equivalent bifurcation diagrams for

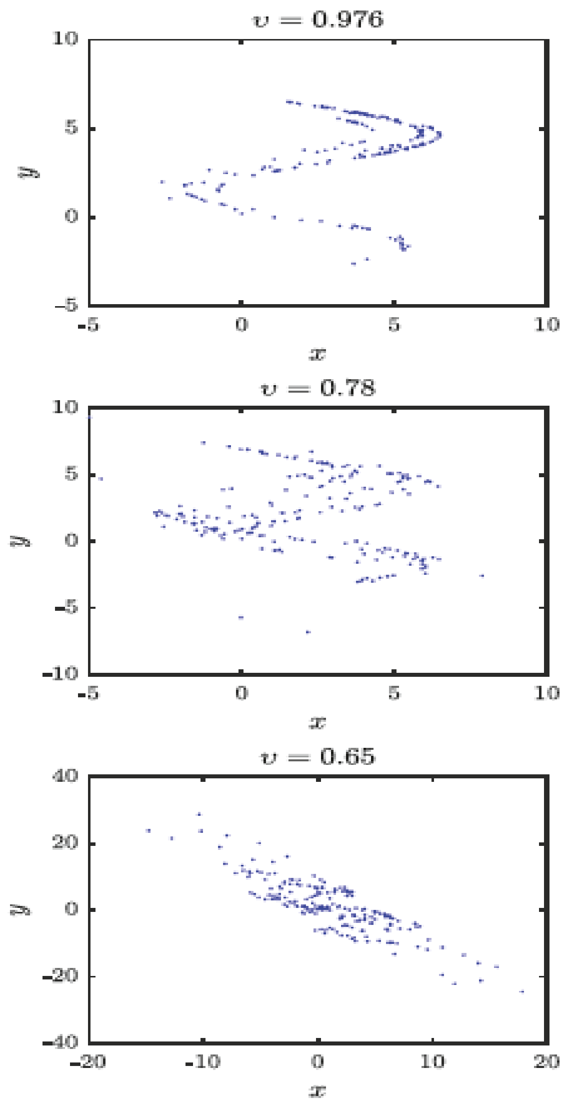


Figure 4.2: Attractors of the fractional sine map for different fractional orders υ.

$\alpha \in [-1, 4]$. It has been noted that when the bifurcation parameters are fixed α and β and alter the fractional order υ value, the bounded attractor of the fractional map disperses over wider areas. Figure 5 illustrates the transitory condition that is seen when $\upsilon = 0.63$. At first, the solution converges to a confined attractor, but it then steadily diverges to infinity in a different direction.

To demonstrate the presence of chaotic dynamics in the fractional-order sine map under specific system parameters $(\alpha, \beta) = (3.8, 0.3)$, the $0-1$ test technique is utilized. The $0-1$ test involves generating a random walk process from time series data for binary testing purposes. A straight-forward method for assessment involves plotting the motion paths of the translation component within the $p-q$ plane. Typically, unbounded trajectories in the $p-q$ plane indicate chaotic behavior, while bounded trajectories suggest regular behavior. In this study, the test is directly applied to the solution $x(n)$ with outcomes presented in Figures 4.3 and 4.4. Figure 4.3 displays the evolution of the translation components (p, q), where unbounded trajectories signify chaotic behavior. Figure 4.4 illustrates the asymptotic growth rate K approaching 1 with increasing n, indicating chaotic dynamics. Additionally, Figure 4.5 exhibits the temporal progression of chaotic states for $\upsilon = 0.78$. To further investigate the impact of the fractional order υ on the system dynamics of the new map (4.2), bifurcation and largest Lyapunov exponent (LLE) diagrams are employed with υ as a critical parameter. Figure 9 showcases bifurcation and LLE diagrams for the parameter set $(\alpha, \beta) = (3, 0.3)$ across the range $\upsilon \in [0.65, 1]$. Decreasing below 1 leads to a transition of the fractional υ order sine map (4.2) from chaotic to periodic states, returning to chaos at 0.7124. Further reduction in υ results in the system converging towards an unbounded attractor. Overall, these findings support the notion that the fractional order υ can serve as a bifurcation parameter

Now, we switch to a new two-dimensional chaotic map with two sine terms. The definition of the so-called sine-sine map is:

$$\begin{cases} x(n+1) = \sin x(n) - \sin 2y(n), \\ y(n+1) = x(n), \end{cases} \tag{4.7}$$

In this case, the dependent state variables are x and y. A globally attractive map with a class-1 basin, System (4.7), was suggested in [7]. The initial circumstances' phase space
Figure 10 displays $(x(0), y(0)) = (1, 1)$. The attractor's shape is consistent with the findings that are stated in [7].
The fractional sine-sine map for $t \in \mathbb{N}_{a+1-\upsilon}$ can be written as follows, using the fractional discrete calculus notation outlined in Section 2 as in the previous section:

$$\begin{cases} \Delta x(n) = x(n+1) - x(n), \\ \Delta y(n) = y(n+1) - y(n), \end{cases}$$

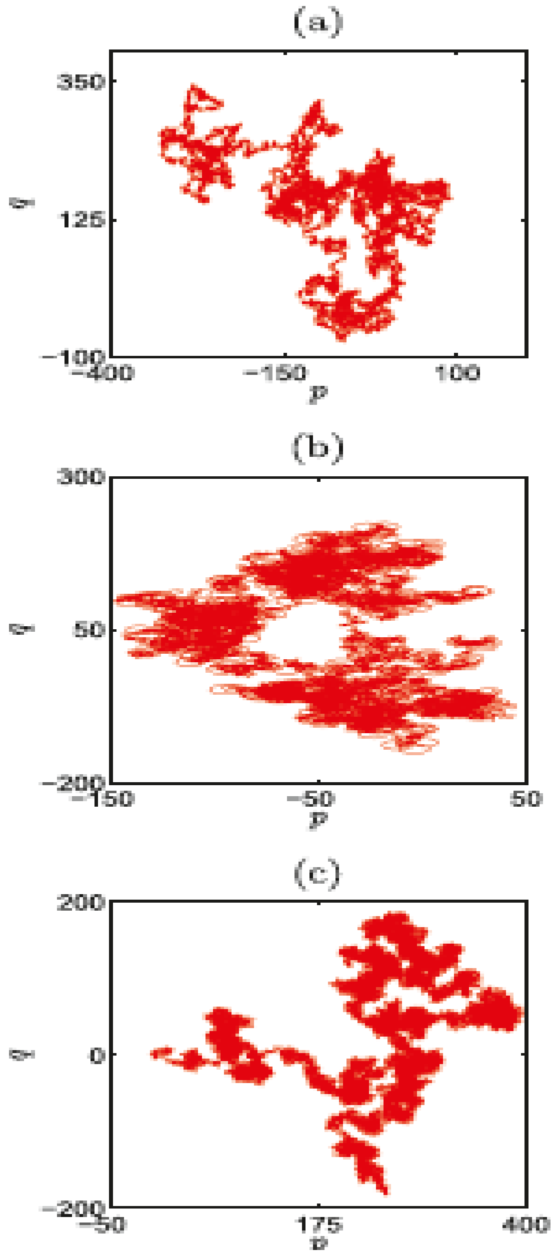


Figure 4.3: The 0–1 test (dynamics of translation components$p$ and q)of the fractional sine map for different fractional orders: $(a)\upsilon = 0.976, (b)\upsilon = 0.78, (c)\nu = 0.65$.

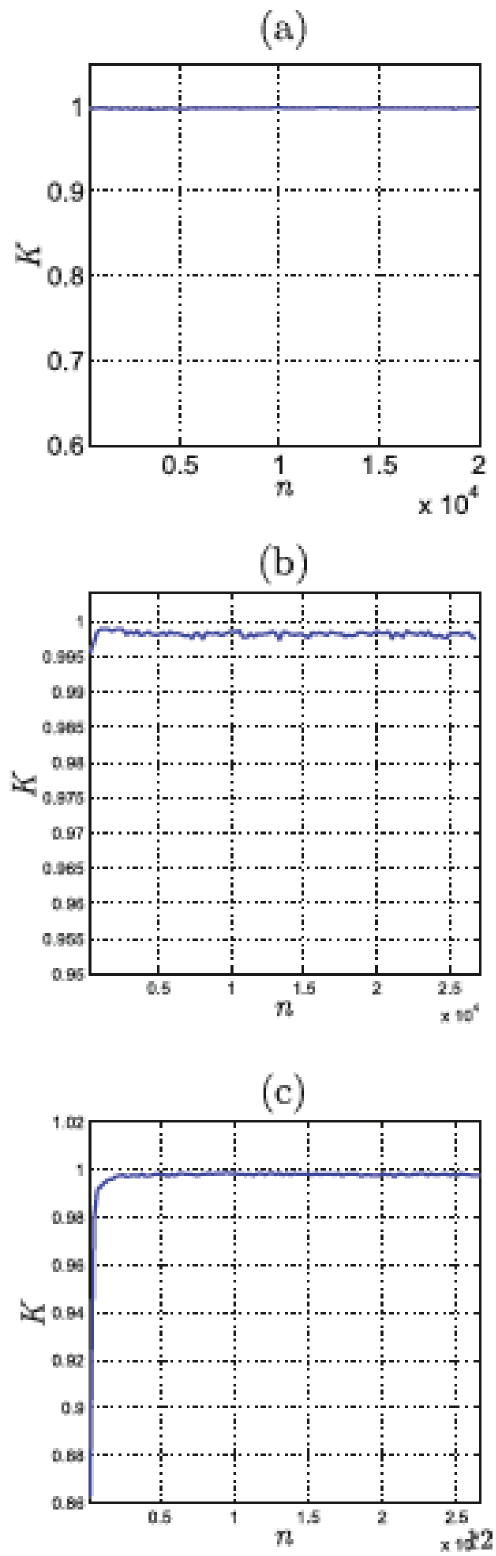


Figure 4.4: The 0–1 test (asymptotic growth rate versus n) of thefractional sine map for different fractional orders:$(a)\upsilon = 0.976, (b)\upsilon = 0.78, (c)\upsilon = 0.65.$

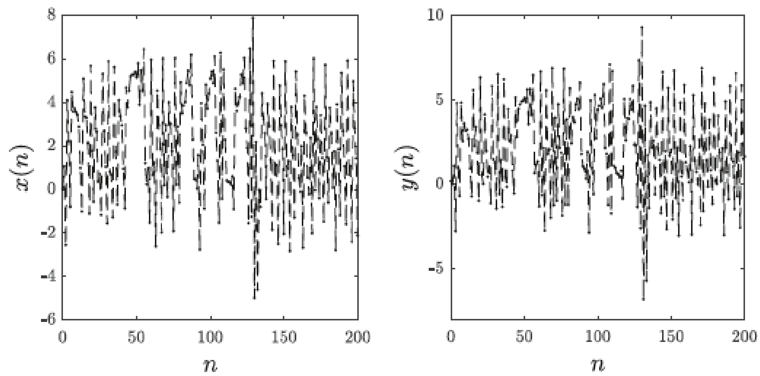


Figure 4.5: Time evolution of states for the fractional sine map with $\upsilon = 0.78$.

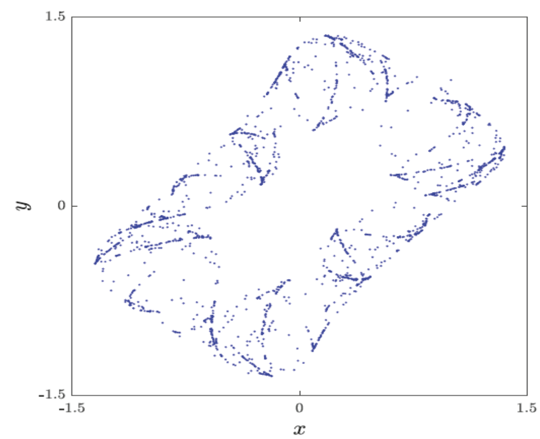


Figure 4.6: Initial conditions (x (0), y (0)) = (1, 1) for the classic sine-sine map's phase portrait.

$$\begin{cases} \Delta x(n) = \sin x(n) - \sin 2y(n) - x(n), \\ \Delta y(n) = x(n) - y(n), \end{cases} \tag{4.8}$$

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = \sin x(t+\upsilon-1) - \sin 2y(t+\upsilon-1) - x(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases} \tag{4.9}$$

in which $0 < \upsilon < 1$. The numerical formula can also be expressed as follows:

$$\begin{cases} x(n) = x(0) + \frac{1}{\Gamma(\upsilon)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\upsilon)}{\Gamma(n-j+1)} \left(\sin x(j-1) - \sin 2y(j-1) - x(j-1)\right), \\ \\ y(n) = y(0) + \frac{1}{\Gamma(\upsilon)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\upsilon)}{\Gamma(n-j+1)} \left(x(j-1) - y(j-1)\right), \end{cases}$$

Using numerical formula (17) with two different fractional orders $\upsilon = 0.989$ and $\upsilon = 0.976$ yields the phase space portraits shown in Figure 4.7. The largest Lyapunov exponents for these two

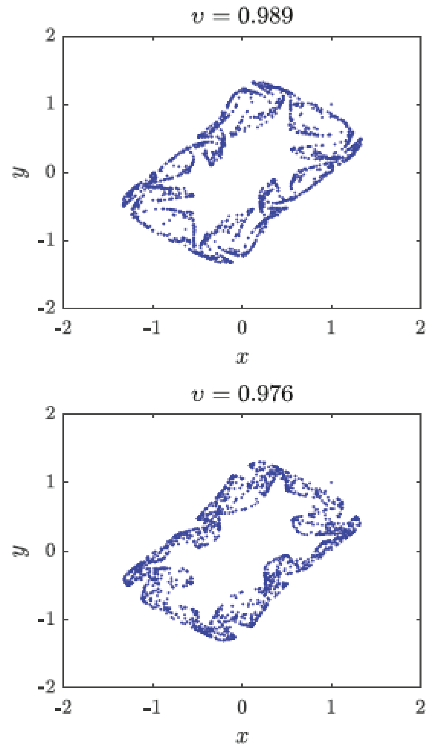


Figure 4.7: Attractors of the fractional sine–sine map for two different fractional orders $\upsilon = 0.989$ and $\upsilon = 0.976$.

orbits are illustrated in Figure 4.8.Since the fractional order map has a positive largest Lyapunov exponent, the phase portrait in Figure 4.7 are therefore a chaotic attractor. The chaotic states for 5 $\upsilon = 0.976$ isdepicted in Figure 4.9. The calculated values K of the $0-1$ test for these two orbits are also presented in Figure 4.10. We observe that K approaches 1 for $\upsilon = 0.989$ and $\upsilon = 0.976$.Therefore, the 0–1 test confirms the existence of chaos.

To analyze further the effect of the fractional-order υ on the dynamic behavior of the fractional-order sine-sine map (4.7), we chose to vary υ from 0 to 1 in steps of 0.001 and observe the behavior of the map.

from (x, y) to $(-x, -y)$. From Figure 16, we find that even with a fractional order, the map still exhibits a chaotic behavior. The chaotic properties of the map disappear in the interval $\upsilon \in [0.952, 0.967]$ and reappear for $\upsilon \in [0.939, 0.951]$ as seen in Figure 16. Finally, when $\upsilon \leq 0.03$, chaos disappears once more and the system becomes stable.

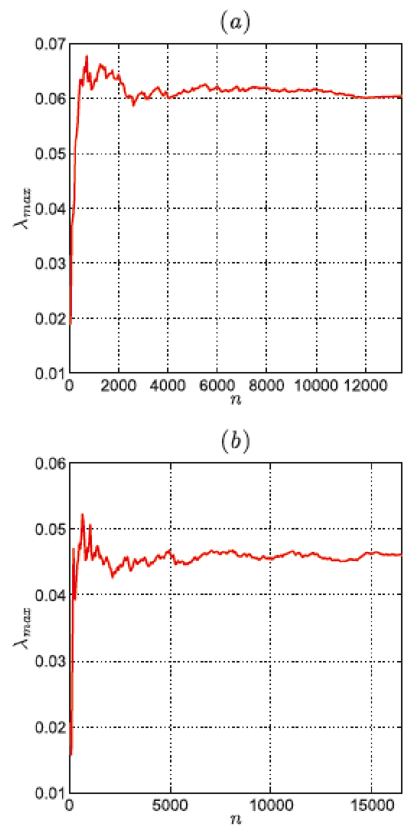


Figure 4.8: LLE of the fractional sine–sine map for different fractional orders: $(a)\upsilon = 0.989$ and $(b)\upsilon = 0.976$.

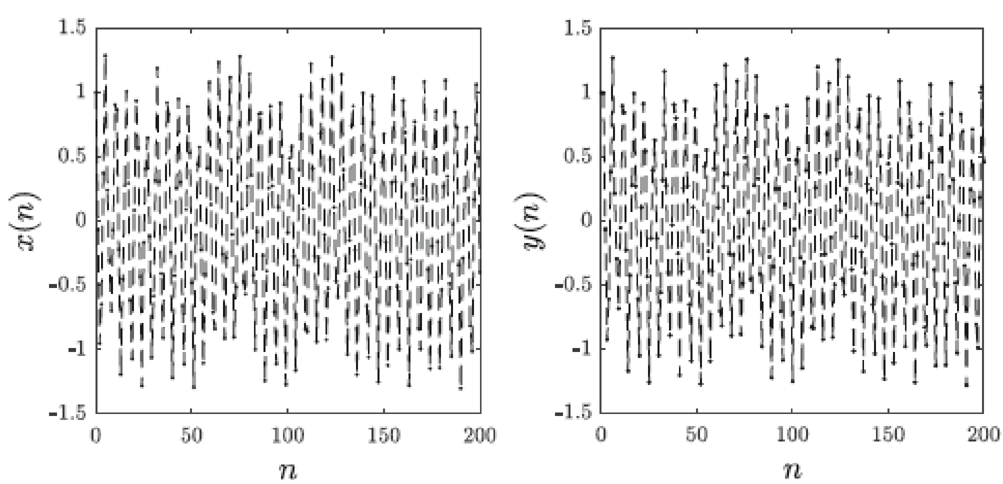


Figure 4.9: Time evolution of states for the fractional sine–sine map with $\upsilon = 0.976$.

4.2 Control of the fractional sine maps

4.2.1 Stabilization

In order to stabilize both the fractional sine map and the fractional sine-sine map, we developed an active controller in this section. Typically, the stabilization mechanism.The challenge lies in determining an appropriate adaptive control law that will cause the system's states to asymptotically converge towards zero.

Theorem 4.1 *The* $1D$ *control law can be used to govern the* $2D$ *fractional sine map:*

$$U(t) = 2x(t) + \alpha \sin x(t) - 1 - \left(\frac{3}{4} + \beta\right) y(t) \tag{4.10}$$

Proof. The fractional sine map that is controlled looks like this:

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = 1 - \alpha \sin x(t+\upsilon-1) + \beta y(t+\upsilon-1) - x(t+\upsilon-1) + U(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases}$$

The intended controller is denoted by u (t). To achieve the new dynamics, we first replace our controlled system (4.12) with the proposed control law (4.11).

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = 1 - \alpha \sin x(t+\upsilon-1) + \beta y(t+\upsilon-1) - x(t+\upsilon-1) + 2x(t+\upsilon-1) \\ \qquad +\alpha \sin x(t+\upsilon-1) - 1 - \frac{3}{4} y(t+\upsilon-1) - \beta y(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases}$$

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = x(t+\upsilon-1) - \frac{3}{4} y(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases} \tag{4.11}$$

We then demonstrate the globally asymptotically stable nature of (4.13) zero equilibrium. To do this, the error system can be expressed in the concise form shown below:

$${}^{C}\Delta_a^{\upsilon} \left(x(t), y(t)\right)^T = A(x(t), y(t))T \tag{4.12}$$

Where ■

$$A = \begin{pmatrix} 1 & -\frac{3}{4} \\ 1 & -1 \end{pmatrix}$$

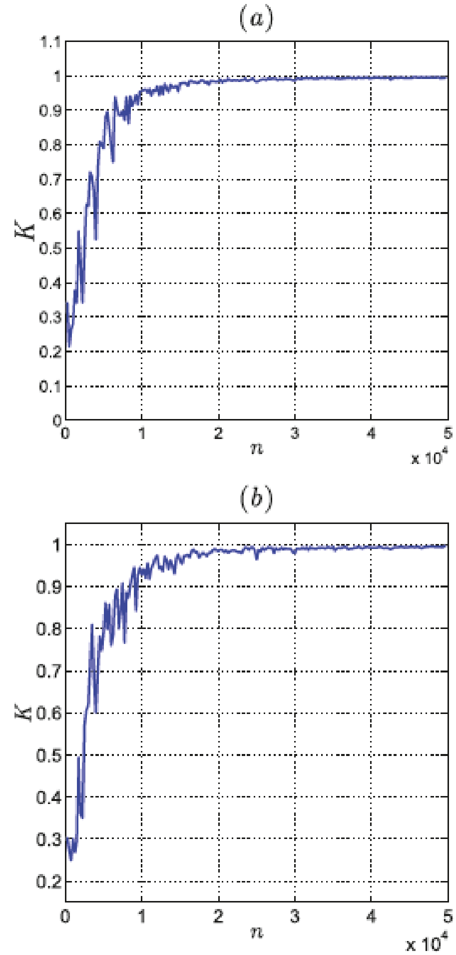


Figure 4.10: The $0-1$ test of the fractional sine–sine map for different fractional orders: (a)$\upsilon = 0.989$ and (b)$\upsilon = 0.976$.

The eigen values of the matrix A are $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{1}{2}$.meet the requirements listed below:

$$|\lambda_i| < \left(2\cos\frac{|\arg\lambda_i| - \pi}{2-\upsilon}\right)^{\upsilon} \text{ and } |\arg\lambda_i| > \frac{\upsilon\pi}{2}, i = 1, 2.$$

Since this satisfies Theorem requirement, the zero equilibrium of equation (4.13) is globally asymptotically stable. The states will therefore undoubtedly gravitate to asymptotically to zero.The control law (4.11) proposed for the fractional Hénon-like map was numerically tested for $\alpha = 0.976$ using the same parameters and initial conditions as before. Figure 4.11 illustrates the convergence of the states towards zero through time-evolution and phase space plots.

By following the identical process, we can also announce the following outcome concerning the stabilization of the fractional
sine-sine map.

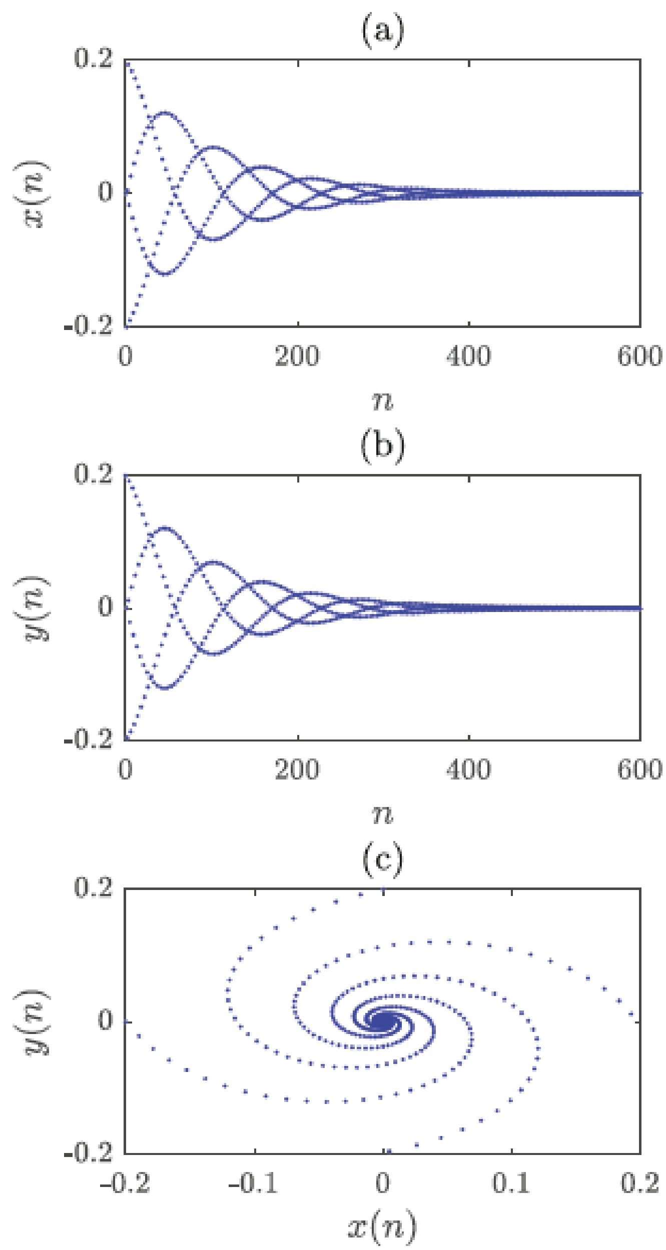


Figure 4.11: (a) State $x(n)$, (b) state $y(n)$, (c) attractor of the fractional sine-map after being stabilized for $\upsilon = 0.976$.

The 2D controlled fractional sine–sine map

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = \sin x(t+\upsilon-1) - \sin 2y(t+\upsilon-1) - x(t+\upsilon-1) + U(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases} \tag{4.13}$$

is kept stable by the 1D control law:

$$u(t) = \sin 2y(t) - \sin x(t) + 2(x(t) - y(t)). \tag{4.14}$$

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = \sin x(t+\upsilon-1) - \sin 2y(t+\upsilon-1) - x(t+\upsilon-1) \\ + \sin 2y(t+\upsilon-1) - \sin x(t+\upsilon-1) + 2x(t+\upsilon-1) - 2y(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases}$$

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x(t) = x(t+\upsilon-1) - 2y(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y(t) = x(t+\upsilon-1) - y(t+\upsilon-1), \end{cases} \tag{4.15}$$

where

$$A' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

The eigen values of the matrix A are $\lambda_1 = -i$, $\lambda_2 = i$.
meet the requirements listed below:

$$|\lambda_i| < \left(2\cos\frac{|\arg\lambda_i| - \pi}{2-\upsilon}\right)^{\upsilon} \text{ and } |\arg\lambda_i| > \frac{\upsilon\pi}{2}, i = 1, 2.$$

Assuming a fractional order of $\upsilon = 0.976$, and using the same parameters and initial conditions as discussed in Section 4, Figure4.12 illustrates the resulting states and phase plot. The outcomes validate the effectiveness of the suggested law in asymptotically stabilizing the systems states.

4.3 Synchronization

In this section, we will explore an alternative method of controlling fractional sine maps and proposed sine maps. Our main objective here is to achieve chaos synchronization, which involves arranging the states of the fractal sine map in such a way that they mirror the precise trajectories of the primary fractal sine map. In this context,we denote the master states and the slave

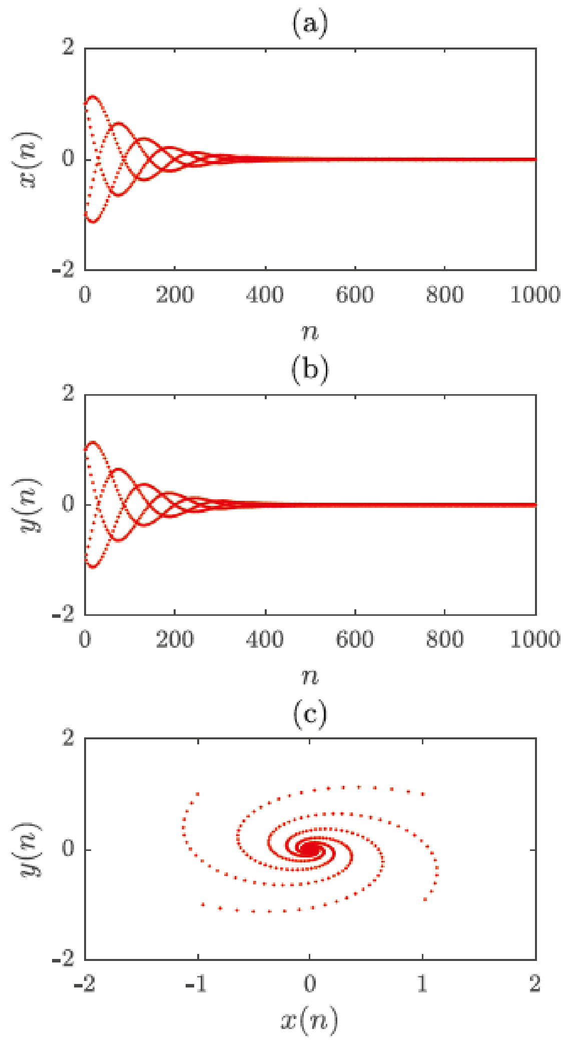


Figure 4.12: (a) State $x(n)$, (b) state $y(n)$,(c) attractor of the fractional sine–sine map after being stabilized for $\upsilon = 0.976$.

states Therefore, we can express the master-slave pair as:Now, let's look at an alternative control scheme for the fractional sine and sine-sine maps that have been suggested. In this case, the goal of chaotic synchronization is to match the precise trajectories of a fractional sine master map, regulate the states of the fractional sine-sine map.The subscripts m and s stand for the master and slave states, respectively. Thus, we can declare the master-slave pair as:

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x_m(t) = 1 - \alpha \sin x_m(t+\upsilon-1) + \beta y_m(t+\upsilon-1) - x_m(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y_m(t) = x_m(t+\upsilon-1) - y_m(t+\upsilon-1), \end{cases} \tag{4.16}$$

and

$$\begin{cases} {}^{C}\Delta_a^{\upsilon} x_s(t) = \sin x_s(t+\upsilon-1) - \sin 2y_s(t+\upsilon-1) - x_s(t+\upsilon-1) + U_1(t+\upsilon-1), \\ \\ {}^{C}\Delta_a^{\upsilon} y_s(t) = x_s(t+\upsilon-1) - y_s(t+\upsilon-1) + U_2(t+\upsilon-1), \end{cases} \tag{4.17}$$

Let's define what synchronization means before we provide our findings. The following are the synchronization errors:

$$\begin{cases} e_1(t) = x_s(t) - x_m(t), \\ \\ e_2(t) = y_s(t) - y_s(t) \end{cases} \tag{4.18}$$

A master-slave pair (4.18)–(4.19) is considered synchronized if and only if:

$$\lim_{t\to\infty} ||e_i(t)|| = 0 \text{ for } i = 1, 2. \tag{4.19}$$

It suggests that the states of the slaves converge on those of the master. Our result is given in the following theorem.

Theorem 4.2 *The synchronization of the master-slave pair system (4.18)–(4.19) is contingent upon:*

$$\begin{cases} u_1 = -\sin x_s(t) + \sin 2y_s(t) + 1 - \alpha \sin x_m(t) + \beta y_m(t), \\ u_2 = 0. \end{cases} \tag{4.20}$$

$$\begin{cases} e_1(t) = (\sin x_s(t+\upsilon-1) - \sin 2y_s(t+\upsilon-1) - x_s(t+\upsilon-1) + u_1(t+\upsilon-1)) \\ \quad -(1 - \alpha \sin x_m(t+\upsilon-1) + \beta y_m(t+\upsilon-1) - x_m(t+\upsilon-1)), \\ \\ e_2(t) = (x_s(t) - y_s(t) + U_2(t)) - (x_m(t) - y_m(t)). \end{cases}$$

The error system (4.20) fractional Caputo-type differences can be expressed as follows:

$$\begin{cases} {}^C\Delta_a^v e_1(t) = \sin x_s(t+v-1) - \sin 2y_s(t+v-1) - x_s(t+v-1) + u_x(t+v-1) \\ \qquad -1 + \alpha \sin x_m(t+v-1) - \beta y_m(t+v-1) + x_m(t+v-1), \\ \\ {}^C\Delta_a^v e_2(t) = x_s(t+v-1) - y_s(t+v-1) + U_y(t+v-1) - x_m(t+v-1) + y_m(t+v-1). \end{cases} \tag{4.21}$$

where

$$\begin{cases} {}^C\Delta_a^v e_1(t) = -x_s(t+v-1) + x_m(t+v-1), \\ {}^C\Delta_a^v e_2(t) = x_s(t+v-1) - x_m(t+v-1) - y_s(t+v-1) + y_m(t+v-1). \end{cases}$$

Changing the control law results in the following extremely basic dynamics:

$$\begin{cases} {}^C\Delta_a^v e_1(t) = -e_1(t+v-1), \\ \\ {}^C\Delta_a^v e_2(t) = e_1(t+v-1) - e_2(t+v-1). \end{cases} \tag{4.22}$$

It is simple to demonstrate that zero is an equilibrium for (4.24) and that it is globally asymptotically stable. The error system can then be written in a concise form as in the equation (4.25) that follows:

$${}^C\Delta_a^v e(t) = Ae(t), \tag{4.23}$$

Where $e(t) = (e_1(t), e_2(t))^T$ and

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

The eigenvalues $(\lambda_i = 1,\ i = 1,2)$ of A clearly meet the requirements listed below:

$$|\lambda_i| < \left(2\cos\frac{|\arg\lambda_i| - \pi}{2-v}\right)^v \text{ and } |\arg\lambda_i| > \frac{v\pi}{2}, i = 1,2.$$

Because of Theorem 2.1, the master-slave pair is synchronized since the zero equilibrium is globally asymptotically stable.

We use numerical results to guarantee the effectiveness of the suggested one-dimensional synchronization control. The pair of master and slave (4.18)–(4.19) was assessed.using the necessary numerical formulas in accordance with Theorem , while keeping the control law (4.22) in mind. The outcomes, which are displayed in Figure 4.13, demonstrate that the pair's synchronization was effective because the errors distinctly converge towards zero.

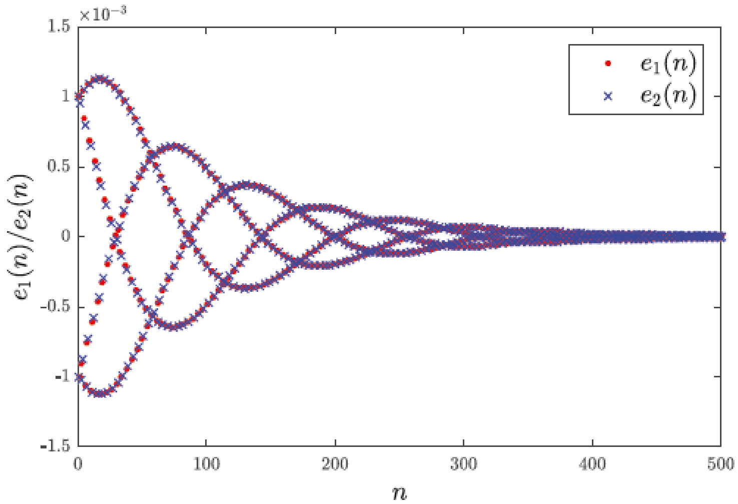


Figure 4.13: Time–evolution of the synchronization errors.

4.4 Conclusions

This chapter began with an introduction to the topic of the dynamics of discrete systems in a state of chaos in fractional calculus. It then moved on to discuss discrete fractional-order systems' stability. Finally, it explored synchronization theory, covering different types of synchronization and the control mechanisms that go along with them. In the last part of the chapter, two fractional maps were created using the traditional chaotic maps combined with trigonometric sine functions.

4.5 General conclusion

This memo began with an introduction to the topic of the dynamics of discrete systems in a state of chaos in fractional calculus. It then moved on to discuss discrete fractional-order systems' stability. Finally, it explored synchronization theory, covering different types of synchronization and the control mechanisms that go along with them. In the last part of the research, two fractional maps were created using the traditional chaotic maps combined with trigonometric sine functions.

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