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Study of existence and uniqueness of the solution of a class of linear and semi-linear equations

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شكر وعرفان

أود أن أبدأ بشكر الله عز وجل والذي منحني التوفيق ومهد لي الطريق
لأكون بينكم اليوم لأناقش مذكرتي هذه.
ونتقدم بجزيل الشكر والعرفان لأعضاء لجنة المناقشة الاستاذ "زارعي عبد
الرحمان" و "أ.بن زاهي مراد" و "أ.بوعلي الطاهر" لإطلاعهم على
هذه المذكرة وتحققهم من المعلومات الموجودة بها كما نشكرهم على
تدريسهم لنا قبل أن يكونوا أعضاء لجنة مناقشتنا و أتوجه بشكر خاص إلى
أستاذتي "مسلوب فتيحة" القديرة
على كل ما قدمته لي من توجيهات ومعلومات قيمة
ساهمت في إثراء موضوع دراستي من جوانب مختلفة
أطال الله في عمرها ورزقها الصحة والعافية

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Abstract

This memoir aims to study two cases of general mixed problems with integral conditions, both classical and non-local.

The memoir begins with an introduction providing background and interest in the addressed topic, along with mentioning some preliminary concepts useful later on.

In the second chapter, a problem involving a non-local boundary condition for a parabolic differential equation is addressed. We employ functional analysis method to prove the existence and uniqueness of the strong solution to the problem.

In the third chapter, our focus shifts to exploring the weak solution of the problem and proving its existence and uniqueness related to the specific issue, relying on the a priori estimation method used in the previous chapter for proving uniqueness. As for the existence of the solution, the Galerkin method is utilized. Finally, a fixed point argument is used to prove the existence of the local solution to the problem.

Résumé

Ce mémoire vise à étudier deux cas de problèmes mixtes généraux avec des conditions intégrales classiques et non locales. Le mémoire commence par une introduction qui expose le contexte et l'intérêt du sujet traité, ainsi que la mention de certains concepts préliminaires utiles ultérieurement.

Dans le premier chapitre, nous introduirons quelques concepts de base de la théorie des espaces fonctionnels et de la théorie des opérateurs, ainsi que quelques inégalités importantes que nous utiliserons dans les chapitres suivants.

Dans le deuxième chapitre, nous aborderons un problème mixte pour des equations hyperbolique de deuxieme degre', où nous combinerons une condition classique avec une condition intégrale. En utilisant la méthode d'estimation a priori, nous prouverons l'unicité de la solution pour le problème spécifique. En ce qui concerne l'existence de la solution, nous montrerons que l'ensemble de l'image de l'opérateur associé au problème étudié est dense dans l'ensemble d'arrivée.

Enfin, dans le dernier chapitre, nous étudierons une équation aux dérivées partielles parabolique avec des conditions classiques, où notre concentration sera sur l'exploration de la solution faible du problème et la preuve de son unicité associée au problème spécifique. Nous utiliserons la même méthode que dans le chapitre précédent, à savoir la méthode d'estimation a priori. En ce qui concerne l'existence de la solution, nous utiliserons la méthode de Galerkin. Enfin, nous utiliserons la théorie du point fixe pour prouver l'existence de la solution locale du problème.

الملخص

هذه المذكرة تهدف إلى دراسة حالتين من المسائل المختلطة العامة ذات شروط تكاملية كلاسيكية و غير محلية. حيث نبدأ بمقدمة تقدم الخلفية والاهتمام في الموضوع المعالج ثم بفصل أول، سنقدم فيه بعض المفاهيم الأساسية من نظرية الفضاءات الوظيفية ونظرية المؤثرات، إلى جانب بعض المتراجحات المهمة التي سنستخدمها في الفصول القادمة.

في الفصل الثاني، سنتناول مشكلة متنوعة تتعلق بمعادلات زائدية من الدرجة الثانية، حيث سنجمع بين شرط كلاسيكي وآخر تكاملي. و باستخدام طريقة التقدير المسبق، سنقوم بإثبات فرادة الحل للمشكلة المعينة. وبالنسبة لوجود الحل، سنوضح أن مجموعة صور المؤثر المتعلق بالمسألة المدروسة تكون كثيفة في مجموعة الوصول.

أما في الفصل الثالث، يكون تركيزنا إلى استكشاف الحل الضعيف للمسألة و اثبات فرادة ووجود هذا الحل المرتبط بالمسألة المحددة. و في الفصل الاخير، نقوم بدراسة معادلة تفاضل جزئي قطعية مع شروط كلاسيكية، حيث يكون تركيزنا على استكشاف الحل الضعيف للمسألة وإثبات فرادة هذا الحل المرتبط بالمشكلة المحددة. نعتمد في ذلك على نفس الطريقة المستخدمة في الفصل السابق، وهي طريقة التقدير المسبق. أما بالنسبة لوجود الحل، فنستعين بطريقة قالركن. وأخيرًا، نستخدم نظريةالنقطة الصامدة لإثبات وجود الحل المحلي للمسالة.

Introduction

The method of Priori estimate or energy inequalities, also known as the functional analysis method, originate from the work of I.G.Petrovsky [26] used in solving the cauchy problem related to (Hyperbolic, Elliptical, Parabolic) type equations. it has been applied and further developed in many works by A.A.Dezin [7], K.Friderichs [10] and N.I.Yurchuk [30]. The method subsequently underwent significant developments by J.Leray [17] and L.Garding [11].

The method of energgy inequalities or a priori estimates has proven to be an effective tool in the study of non-classical problems [9].

it has also been used to solve various problems in the fields of thermal conduction theory, plasma physics, electrochemistry, and others. During the application of this method, difficulties are encountered, among which we mention:

- The choice of the solution space.
- The choice of the multiplier.
- The choice of regularization operators.

The application of functional analysis, the energy method, and Priori estimate method are generally difficult to apply in solving problems of this type. It should be noted that the theoretical study of non-local problems is associated with significant difficulties.

This method has been used to solve mixed problems related to elliptic [16], parabolic [12], hyperbolic [23, 25, 18] and other types of differential equations.

In mixed problems models involving a non-local constraint with integral conditions, the fundamental reasons for their importance lie in the basic physical meaning of the integral condition, which includes average, flux, total energy, moment, and so on. These models are encountered in the theory of heat conduction, thermoelasticity, and semiconductors.

In this memoir, we applied the a priori estimation method (the energy inequality method) to certain linear mixed problems with non-local boundary values. We mainly proved the existence, uniqueness (you can refer to the theses from Bounama Fatiha [9]), and continuous dependence of the solution on the provided data (we showed that the posed problems are well-posed).

The memoir is organized as follows: In the first chapter, we will introduce some fundamental concepts from the theory of functional spaces and operator theory, along with some important inequalities that we will use in the subsequent chapters.

In the second chapter, we investigate a mixed problem for second-order hyperbolic equations by combining a classical condition with another integral one. Using a priori estimation, we establish the uniqueness of the solution to the given problem. Regarding the existence of the solution, we prove that the image of the operator generated by the studied problem is dense.

Therefore, we prove the existence and uniqueness of a generalized strong solution to the given problem.

In Chapter Three, we study a semi-linear problem for a parabolic equation with the Bessel operator and classical boundary conditions. Our focus is on finding the weak solution to the problem and proving its uniqueness related to the given issue. We rely on the same method used in the previous chapter, namely the a priori estimation method. As for the existence of the solution, we employ the Galerkin method. Finally, we use a fixed point argument to prove the local existence of the solution to the problem.

Chapter 1

Preliminary Notations

In this chapter, we revisit several fundamental theories in functional analysis and review how to classify second-order partial differential equations, while addressing some concepts and theories related to sequences. Additionally, we review some iportant functional inequalities and mention a set of additional theories.

1.1 Classification of second-order linear PDEs [22]

Consider the general form of a second order partial defferential equation (PDE) according to two independent variables (x and y):

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0$$

A fairly simple classification of this equation can be made on the basis of the coefficients associated with the high order derivatives A, B and C. we calculate the determinante defined by:

$$\Delta = B^2 - 4AC$$

The equation is said to be of type:

- Elliptical if $\Delta \succ 0$
- Parabolic if $\Delta = 0$
- Hyperbolic if $\Delta \prec 0$

1.2 Some properties of sequences [5]

Definition 1.2.1. ***(Convergent sequences and divergent sequences)***
$(u_n)_{n\in\mathbb{N}}$ a sequences value $u_n \in E$, we say about this sequence is convergent if exists $l \in E$ where $\lim_{n\to\infty} u_n = l \in E$ if:

$$\forall \varepsilon \succ 0, \exists n_\varepsilon \in, \forall n \geq n_\varepsilon \Rightarrow d(u_n, l) \prec \varepsilon.$$

A sequence which is not convergent is said to be divergent.

Theorem 1.2.1. *Any sequences extracted from a convergent sequences $(u_n)_{n\in\mathbb{N}}$ is convergent and has the same limit as $(u_n)_{n\in\mathbb{N}}$.*

Definition 1.2.2. ***(sequences of Cauchy)***
A sequence $(u_n)_{n\in\mathbb{N}}$ is called Cauchy if:

$$\forall \varepsilon \succ 0, \exists n_\varepsilon \in \mathrm{N}, \forall p, q \in \mathrm{N}, p, q \geq n_\varepsilon \Rightarrow d(u_q, u_q) \prec \varepsilon.$$

Theorem 1.2.2. *every convergent sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Theorem 1.2.3. *if (E, d) is a complete metric, then every Cauchy sequence is convergent in E.*

Theorem 1.2.4. ***(Bolzano Weierstrass)***
Every bounded sequence admits a convergent subsequence.

1.3 Characterization of injective and surjective linear application

Let $f : E \to F$ be a linear application from the vector K-space E to the vector K-space F.

- The mapping f is surjective if and only if its image is equal to the space F.

$$f \text{ Surjective} \Leftrightarrow \mathrm{Img}(f) = F$$

- The mapping f is injective if and only if its kernel contains only the zero vector.

$$f \text{ Injective} \Leftrightarrow Ker(f) = \{0_E\}$$

1.4 Linear Space

Let X be a set of elements denoted by: x, y...... ,We assume that each pair of elements (x, y) can be combined by a process called addition to give another element z denoted $z = x + y$.

We also assume that each real number α and each element x can be combined by a process denoted multiplication to give another element y denoted by $y = \alpha x$.

The set X with these two processes is called a linear space if the following axioms are satisfied:

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

In set X there exists a unique element, denoted by 0 called the zero element such that:

$$x + 0 = x \quad \text{for every } x \in X$$

For every $x \in X$ there exists a unique element denoted by $-x$ such that:

$$x + (-x) = 0$$

$$\alpha(x + y) = \alpha x + \alpha y$$

$$(\alpha + \beta)x = \alpha x + \beta x$$

$$(\alpha\beta)x = \alpha(\beta x)$$

$$1.x = x$$

$$0.x = 0$$

where α, β represents real numbers and x, y are elements in X.

Definition 1.4.1. ***(Linear Space):*** *Let X and Y be two linear spaces (both real or complex). Let A be a function from domain $D(A)$ in X and image $R(A)$ in Y.*
Then A is called a linear operator if $D(A)$ is a subspace of X and if:

- $A(x_1 + x_2) = Ax_1 + Ax_2$
- $A(\alpha x) = \alpha Ax$

Here α is a scalar and x_1, x_2, x_3 are vectors in $D(A)$.

If $D(A) = X$, we often say that A is a linear operator from X to Y.

An important subset of the domain of A is the null space of A, $N(A)$ where:

$$N(A) = \{x \in D(A) : Ax = 0\}$$

Theorem 1.4.1. *Let A be a linear operator on $D(A)$ in X and image $R(A)$ in Y, where X and Y are linear spaces Then:*

- *A^{-1} exists if and only if $N(A) = \{0\}$.*
- *If A^{-1} exists it is a linear operator.*

Definition 1.4.2. ***(Linear Functional):*** *Let X be a real (or complex) linear space and let Y be a scalar field associated with X. A linear operator on X to Y is then called a linear functional.*

Proposition 1.4.1. *Let $f \in BL(E; F)$, f is surjective if and only if $\mathrm{Img}(f) = f(E) = F$.*

Proposition 1.4.2. *Let $f \in BL(E; F)$, f is injective if and only if $Kerf = N(f) = \{0\}$.*

Remark 1.4.1. *$BL(E; F)$ is the set of linear application of E in F.*

1.5 Normed Space [6]

Definition 1.5.1. ***(the Norm):*** *Let X be a real (or complex) vector space, a norm on the linear space X is a real-valued function $\|.\| : X \to [0, +\infty[$, where the value at x is denoted by $\|x\|$ and has the properties:*

- $\|x\| \geq 0$
- $\|\alpha x\| = |\alpha| \, \|x\|$ *for all scalars α*
- $\|x\| \neq 0$ *if* $x \neq 0$
- $\|x + y\| \leq \|x\| + \|y\|$

Definition 1.5.2. ***(Normed Linear Space):*** *A linear space $(X, \|.\|)$ on which a norm is defined is called a normed linear space.*

Definition 1.5.3. ***(Complete Space):*** *The normed space X is complete if every Cauchy sequence in X converges to an element in X (i.e.) has a limit that is an element of X.*

1.6 Banach Space [21]

Definition 1.6.1. ***(Banach Space):*** *): If a normed linear space $X = (X, \|.\|)$ is complete, it is called a Banach space.*

Theorem 1.6.1. *Let X be a normed linear space, Then there exists a linear space Y such that Y is complete and X is a dense subset of Y.*
Up to isometry, the space Y is unique.

Definition 1.6.2. ***(Completion):*** *The space Y given by* ***Theorem 1.3.1 is*** *called the completion (partition) of X.*

1.7 Hilbert Space [29]

Definition 1.7.1. ***(Pre-Hilbert Space):*** *A complex linear space X is called a pre-Hilbert space if for each pair of elements x, y in X, there exists an associated complex number (x, y) called the pre-Hilbertian of x and y with the following properties:*

- $(x + y, z) = (x, z) + (y, z)$, *for all* $x, y, z \in X$
- $(x, y) = \overline{(y, x)}$, *where the bar denotes complex conjugate.*
- $(\alpha x, y) = \alpha(x, y)$, $\forall \alpha$
- $(x, x) \geq 0$, *for all* $x \in X$,
- $(x, x) = 0 \Leftrightarrow x = 0$, *for all* $x \in X$

The pre-Hilbert space is a special case of a normed linear space, as expressed by the following lemma.

Lemma 1.7.1. *Let X be a linear space equipped with a scalar product. Then the expression $\|x\| = \sqrt{(x, x)}$, for all $x \in X$, defines a norm on X.*

Definition 1.7.2. ***(Hilbert space):*** *A Hilbert space is a complete pre-Hilbert space (a normed linear space).*
A distance on H is given by:

$$\|x - y\| = \sqrt{(x - y, x - y)}.$$

Therefore, pre-Hilbert spaces are normed spaces, and Hilbert spaces are Banach spaces.

1.8 Space of Sobolev [2, 20]

Definition 1.8.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, k a non-negative integer, and let $1 \leq p \leq \infty$, We define $W_p^k(\Omega)$ to be the set of all distributions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, where $\alpha \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in \mathbb{N}$, $i = 1, n$ such that:*

$$W_p^k(\Omega) = \{u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega);\ \forall \alpha : |\alpha| \leq k\}$$

where

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 .... \partial^{\alpha_n} x_n}, \quad |\alpha| = \sum_{i==1}^{n} \alpha_i$$

and

$$L^p(\Omega) = \{u : measurable \setminus \int |u(x)|^p dx \prec \infty\}$$

$$L^\infty(\Omega) = \{u : mesurable \setminus \exists c \ \ such\ that\ |u(x)| \prec c\ on\ \Omega\}.$$

$L^p(\Omega)$ is a complete space for the norm

$$\|u\|_{L^p(\Omega)} = (\int |u(x)|^p dx)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty$$

and $L^\infty(\Omega)$ is a complete space for the norm:

$$\|u\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |u(x)|.$$

in $W_p^k(\Omega)$, a norm is defined by:

$$\|u\|_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k}^{n} \|D^\alpha u\|_{L^p(\Omega)}, \quad p \prec \infty$$

$$\|u\|_{W_p^\infty(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

For $p = 2$, we define a scalar product as:

$$(u, v)_{W_2^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u(x), D^\alpha v(x))_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

for $W_2^k(\Omega)$ we also use the notion of $H^k(\Omega)$.

1.9 Orthogonality-Orthogonal Complement

Definition 1.9.1. ***(Orthogonality):*** *Let H be a Hilbert space (more generally, a pre-Hilbert space). We say that two elements of H, x and y are orthogonal (or x is orthogonal to y) if:*

$$(x, y) = 0,$$

and we write $x \perp y$ Similarly, for the inclusion

$$A, B \subset H$$

we write:

$$x \perp y \quad if \quad x \perp a, \quad \forall a \in A$$

Definition 1.9.2. ***(Orthogonal Complement):*** *For each subspace M of H, we define the orthogonal complement as:*

$$M^{\perp} = \{x \in H \backslash \quad \langle x, a \rangle = 0, \quad \forall a \in M\}$$

which is the set of all vectors orthogonal to M.
It is clear that $M^{\perp}$ is a closed subspace. If M is also closed, then H is a direct sum of M and $M^{\perp}$:

$$H = M \oplus M^{\perp}$$

Theorem 1.9.1. *Let H be a Hilbert space, A subspace M of H is dense if and only if $M^{\perp} = \{0\}$.*

$$M^{\perp} = \{0\} \Leftrightarrow \bar{M} = H$$

Theorem 1.9.2. ***(Completion):*** *For any normed space (Pre-Hilbert space) X, there exists a Banach space (Hilbert space) X and an isometry A from X to a dense subspace W of X: the space X is unique, except for isometries.*
The following concepts are very useful for studying unbounded operators.

1.10 Closed operators [15]

Definition 1.10.1. *The graph of a linear operator $A : X \to Y$ is the set of ordered pairs:*

$$\Gamma(A) = \{(x, Ax) \setminus x \in D(A)\} \subset X \times Y$$

Note that the graph is a subspace of $X \times Y$.

Lemma 1.10.1. *The operator $\tilde{A}$ is an extension of A if and only if:*

$$\Gamma(\tilde{A}) \supset \Gamma(A)$$

Definition 1.10.2. *An operator A is said to be closed if its graph is closed as a subset of $\subset X \times Y$*

Definition 1.10.3. *A is called closable if it has a closed extension, Every closable operator has a smallest closed extension called its closure and denoted by $\bar{A}$.*
The following lemma is a direct consequence of the definition of a closed operator.

Lemma 1.10.2. *An operator A is closed if and only if it has the following properties, Whenever there is a sequence $x_n \in D(A)$ such that:*

- $x_n \to x$
- $Ax_n \to f$

then

- $\forall x \in D(A)$ *then* $Ax = f$

A similar characterization holds for a closable operator.

Lemma 1.10.3. *An operator A is closed if for any sequence $x_n \in D(A)$ such that $x_n \to 0$, we have either:*

- $Ax_n \to 0$
- *or* $\lim_{n\to\infty} Ax_n$ *does not exist.*

Corollary 1.10.1. *If A is closed, then:*

$$\Gamma(\bar{A}) = \bar{\Gamma}(A)$$

1.11 Integration by part

Let $u, v \in H^1(\Omega)$, for all $1 \leq i \leq n$ we have

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i} v dx = \int\limits_{\Omega} \frac{\partial v}{\partial x_i} u dx + \int\limits_{\partial\Omega} u v \eta_i d\sigma,$$

Where $\eta_i(x) = \cos(\eta, x_i)$ is the direction cosine of the angle between the exterior normal at $\partial\Omega$ the point and the axis of x_i.

1.12 Green's Formula [8]

We suppose that Ω is a bounded domaine from $\mathbb{R}^n$, to boundary $\partial\Omega$ of class C^1, let u, v two class function C^2 in $\bar{\Omega}$, the following are called Green's formulas:

- **First green formula:**

$$\int\limits_{\Omega} \Delta u d\lambda = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma$$

  where $d\lambda$ is the lebesgue measure and $d\sigma$ the eculidean measure on $\partial\Omega$

- **Seconde green formula:**

$$\int\limits_{\Omega} (v\Delta u - u\Delta v) d\lambda = \int\limits_{\partial\Omega} \left(v\frac{\partial u}{dn} - u\frac{\partial v}{\partial n} \right) d\sigma$$

1.13 Regularization of Operators

Let W be a function of class C^∞, with the variables α such that:

$$W(\alpha) \succ 0; \ \ W = 0, \ \ |\alpha| \geq 1$$

and

$$\int\limits_{-\infty}^{\infty} W(\alpha) d\alpha = \int\limits_{-1}^{1} W(\alpha) d\alpha = 1$$

We denote by:

$$W_\varepsilon(x, y) = \frac{1}{\varepsilon} W(\frac{x-y}{\varepsilon})$$

For every $\varepsilon \succ 0$, we have:

$$\int_{-\infty}^{\infty} W_\varepsilon(x,y)dy = \int_{-1}^{1} W_\varepsilon(x,y)dy = 1,$$

and

$$W_\varepsilon(x,y) = 0, \ \ |x-y| \geq \varepsilon.$$

We define the smoothing operator $J_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ by the formula:

$$\begin{aligned} (J_\varepsilon v)(x) &= \int_{-\infty}^{\infty} W_\varepsilon(x,y)v(y)dy \\ &= \int_{|x-y|\prec\varepsilon} W_\varepsilon(x,y)v(y)dy, \end{aligned}$$

where $\Omega = (a,b) \subset \mathbb{R}$ and $v \in L^2(\Omega)$ This operator has the following properties:

- The function $J_\varepsilon v \in C^\infty$ if $v \in L^2(\Omega)$, and

$$\frac{\partial^m}{\partial x^m}(J_\varepsilon v) = J_\varepsilon\left(\frac{\partial^m}{\partial x^m} J_\varepsilon v\right), \quad \text{si} v \in C^m(\Omega)$$

- if $v \in L^2(\Omega)$, then

$$\|J_\varepsilon v - v\|_{L^2(\Omega)} \to 0, \ \text{ when } \ \varepsilon \to 0$$

and

$$\|J_\varepsilon v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}.$$

- if $\beta \in C(\Omega)$ and $v \in L^2(\Omega)$, then

$$\|\beta J_\varepsilon v - J_\varepsilon(\beta v)\|_{L^2(\Omega)} \to 0, \ \text{ when } \ \varepsilon \to 0$$

- if $\beta \in C^1(\Omega)$ and $v \in L^2(\Omega)$, then

$$\left\|\frac{\partial}{\partial x}(\beta J_\varepsilon v - J_\varepsilon(\beta v))\right\|_{L^2(\Omega)} \to 0, \ \text{ when } \ \varepsilon \to 0$$

1.14 Important Inequalities

1.14.1 Cauchy inquality

for all $a, b \in \mathbb{R}$, we have:

$$ab \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$$

1.14.2 Cauchy inequality with ε

The following inequality:

$$ab \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \quad a, b \in \mathbb{R}$$

remember for everything $\varepsilon \succ 0$.

1.14.3 Cauchy-Schwartz Integral Inequality [3, 1]

For all $u, v \in L^2(\Omega)$, we have the following inequality:

$$\int_{\Omega} u(x)v(x)dx \leq \left(\int_{\Omega} u^2(x)dx\right)^{\frac{1}{2}} \left(\int_{\Omega} v^2(x)dx\right)^{\frac{1}{2}}$$

is called the Cauchy-Schwartz integral inequality.

1.14.4 Holder inquality [13]

For all $u, v \in L^p(\Omega)$, we have the following inequality:

$$\int_{\Omega} u(x)v(x)dx \leq \left(\int_{\Omega} |u(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\Omega} |v(x)|^q dx\right)^{\frac{1}{q}},$$

where $p \succ 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, this inequality is the generalization of the inequality of the Cauchy-Schwartz Integral.

1.14.5 Young inquality

The generalization of the Cauchy-Schwartz inequality is denoted by Young's inequality which is given by:

$$ab \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^{\frac{p-1}{p}}; \quad a, b \in \mathbb{R}, \quad p \succ 1$$

1.14.6 Young inquality with ε

For all $\varepsilon \succ 0$, we have inquality:

$$ab \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p-1}{p}}; \quad a, b \in \mathbb{R}, \quad p \succ 1$$

1.14.7 Gronwall's Lemma [28, 27]

If $f_i(\tau)$ $i = 1, 2, 3$ are non-negative functions on $(0, T)$, $f_1(\tau)$, $f_2(\tau)$ are integrable functions, and $f_3(\tau)$ is a non-decreasing function on $(0, T)$, then if

$$\Im_\tau f_1 + f_2 \leq f_3 + c\Im_\tau f_2,$$

then

$$\Im_\tau f_1 + f_2 \leq \exp(c\tau) f_3(\tau),$$

where

$$\Im_\tau f_i = \int_0^\tau f_i(t)dt, \quad i = (1, 2).$$

1.14.8 Poincaré Inequality [14]

For all $u \in W_2^1(\Omega)$, we have inquality:

$$\int_\Omega u^2 dx \leq C_\Omega^2 \int_\Omega u_x^2 dx$$

where C_Ω^2 is a constant depending only on Ω.

1.14.9 Elementary Inequality [24]

$$\int_0^l (\Im_x(\xi u))^2 dx \leq \frac{l^3}{2} \|u(.,t)\|^2_{L^2(\Omega)},$$

$$\int_0^l \left(\Im_x^2(\xi u)\right)^2 dx \leq \frac{l^2}{2} \|\Im_x(\xi u)\|^2_{L^2(\Omega)},$$

$$\int_0^l \left(\Im_x^2(\xi u)\right)^2 dx \leq \frac{l^2}{2} \|\Im_x(\xi u)\|^2_{L^2(\Omega)}$$

1.15 Banach-Picard fixed point theorem [4]

Theorem 1.15.1. *Let (E, d) a complete metric space, and $f : E \to E$ a contracting application, that is to say that there exists $k \in [0; 1[$ such that, for all $(x, y) \in E^2$:*

$$d(f(x), f(y)) \leq kd(x, y).$$

Then f has a unique fixed point l where $f(l) = l$.

1.16 Galerkin methods [19]

In mathematics, in the field of numerical analysis, Galerkin methods are a class of methods for transforming a continuous problem (for example differential equation) into a discrete problem. This approach is attributed to Russian engineers Ivan Bubnov (1911) and Boris Galerkin (1913).

This method is commonly used in the finite element method.

we start from the weak formulation of the problem. The solution belongs to a functional space satisfying well defined regularity properties. the Galerkin method consists of using a mesh of the study domain and considering the restriction of the solution function on each of the meshes.

from a more formal point of view, we write the weak formulation in the form:

$$\{\text{find } u \in V; \ \forall v \in V, \ \text{where } a(u, v) = L(v)\}$$

Where a is a bilinear form, and L a linear form.

The set V being generally of infinite dimension, we construct a space $V_h \subset V$ with $\dim V_h \prec \infty$ and we rewrite the problem as follows:

$$\{\text{find } \ u_h \in V_h; \ \forall v_h \in V_h, \ \text{ where } \ a\left(u_h, v_h\right) = L\left(v_h\right)\}$$

Typically, the space V_h considered is the set of continuous functions such that the restriction of the function on a mesh is a polynomial.

Chapter 2

Mixed problem with a weighted integral condition for a hyperbolic equation with dissipation

2.1 Problèm Statement

In the bounded domain $Q = (0, l) \times (0, T)$, Where $l < \infty$ et $T < \infty$ we consider the problem of determining the solution $u(x, t)$ of the differential equation.

$$\mathcal{L}u = u_{xx} + \frac{1}{x}u_x - (a(t)u_{tt} + b(t)u_t) = f(x, t), \quad \forall (x, t) \in Q \tag{2.1}$$

Where the functions $a(t)$ and $b(t)$ satisfy the following conditions:

$\mathbf{C}_1 \cdot c_0 \leq a(t) \leq c_1, c_2 \leq a_t(t) \leq c_3$ for all $t \in [0, T]$,

$\mathbf{C}_2 \cdot c_4 \leq b(t) \leq c_5, c_6 \leq b_t(t) \leq c_7, b_{tt}(t) \leq c_8$ for all $t \in [0, T]$.

In the conditions above, as well as throughout, the $c_i, i = \overline{0, 11}$ are strictly positive constants. To equation (1,1), we associate the initial conditions:

$$l_1 u = u(x, 0) = \varphi_1(x), \tag{2.2}$$

$$l_2 u = u_t(x, 0) = \varphi_2(x), \tag{2.3}$$

The Dirichlet type boundary condition

$$u(l,t)=0, \tag{2.4}$$

And the integral condition

$$\int_0^l xu(x,t)dx=0, \tag{2.5}$$

Where f, φ_1 and φ_2 are given functions satisfying compatibility conditions:

$$\begin{aligned} \varphi_1(l)=0, &\quad \int_0^l x\varphi_1(x)dx=0, \\ \varphi_2(l)=0, &\quad \int_0^l x\varphi_2(x)dx=0. \end{aligned}$$

In this chapter, we demonstrate the existence and uniqueness of the strong solution to the posed problems (2,1)-(2,5). We establish a priori estimation and demonstrate the density of the set of values of the operator generated by the considered problem.

2.2 Functional Spaces

For the study of the posed problem, we need some functional spaces, namely $L^2(Q)$ the Hilbert space, consisting of (classes of) square integrable functions defined in Q equipped with a given scalar product:

$$(u,w)_{L^2(Q)}=\int_Q uwdxdt \tag{2.6}$$

And the associated norm:

$$\|u\|_{L^2(Q)}=\left(\int_Q u^2dxdt\right)^{\frac{1}{2}} \tag{2.7}$$

$L^2_\rho(Q)$ is the Hilbert space of square integrable functions with weighted norms:

$$\|u\|_{L^2_\rho(Q)}=\left(\int_Q xu^2dxdt\right)^{\frac{1}{2}} \tag{2.8}$$

The scalar product in $L^2_\rho(Q)$ is defined by:

$$(u,w)_{L^2_\rho(Q)}=(xu,w)_{L^2(Q)} \tag{2.9}$$

$V_\rho^{1,0}(Q)$ is the Hilbert space with the scalar product.

$$(u, w)_{V_\rho^{1,0}(Q)} = (u, w)_{L_\rho^2(Q)} + (u_x, w_x)_{L_\rho^2(Q)}, \tag{2.10}$$

And the norm

$$\|u\|^2_{V_\rho^{1,0}(Q)} = \|u\|^2_{L_\rho^2(Q)} + \|u_x\|^2_{L_\rho^2(Q)}. \tag{2.11}$$

$V_\rho^{1,1}(Q)$ is the Hilbert space having the scalar product

$$(u, w)_{V_\rho^{1,1}(Q)} = (u, w)_{L_\rho^2(Q)} + (u_x, w_x)_{L_\rho^2(Q)} + (u_t, w_t)_{L_\rho^2(Q)}, \tag{2.12}$$

And equipped with the norm.

$$\|u\|^2_{V_\rho^{1,1}(Q)} = \|u\|^2_{L_\rho^2(Q)} + \|u_x\|^2_{L_\rho^2(Q)} + \|u_t\|^2_{L_\rho^2(Q)}. \tag{2.13}$$

Weighted spaces on the interval $(0, l)$ are also used, such as $L_\rho^2(0, l)$ and $V_\rho^{1,0}(0, l)$. Their definitions are analogous to those of the spaces on Q ,For example $V_\rho^{1,0}(0, l)$ is the subspace of $L_\rho^2(0, l)$ with the associated norm

$$\|u\|^2_{V_\rho^{1,0}(0,l)} = \|u\|^2_{L_\rho^2(0,l)} + \|u_x\|^2_{L_\rho^2(0,l)}. \tag{2.14}$$

The problem (2.3)-(2.5) can be written in the following operational form:

$$Lu = \mathcal{F}, \quad u \in D(L). \tag{2.15}$$

Where

$$Lu = (\mathcal{L}u, l_1u, l_2u), \quad \text{et} \quad \mathcal{F} = (f, \varphi_1, \varphi_2).$$

The operator L is considered from the space E to the space F, where E is the Banach space, Consisting of functions $u \in L_\rho^2(Q)$, satisfying the conditions (2.3)-(2.5) and in the finite norm:

$$\|u\|^2_E = \sup_{0 \le \tau \le T} \left[\|u(x, \tau)\|^2_{V_\rho^{1,1}(0,l)} + \|\Im_x^* (\rho u_t(., \tau))\|^2_{L^2(0,l)} \right] \tag{2.16}$$

And F is the Hilbert space $L_\rho^2(Q) \times V_\rho^{1,0}(0, l) \times L_\rho^2(0, l)$, Consisting of all elements, $\mathcal{F} = (f, \varphi_1, \varphi_2)$ including the norm:

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2_\rho(Q)} + \|\varphi_1\|^2_{V^{1,0}_\rho(0,l)} + \|\varphi_2\|^2_{L^2_\rho(0,l)} \tag{2.17}$$

is finite

The domain of definition $D(L)$ of the operator L is the set of all functions $u \in L^2_\rho(Q)$ for which $u_t, u_x, u_{tx}, u_{xx}, u_{tt} \in L^2_\rho(Q)$ and satisfying the conditions (2.3)-(2.5).

2.3 Priori estimate

Theorem 2.3.1. *If conditions $\mathbf{C}_1 - \mathbf{C}_2$ are satisfied, then for all functions $u \in D(L)$, we have the a priori estimate:*

$$\|u\|_E \leq c\|Lu\|_F, \tag{2.18}$$

Where c is a positive constant independent of the solution u.

Proof 2.3.1. *Consider the inner product in $L^2(Q^\tau)$ of equation (2.1) and the integrated-differential operator*

$$Mu = xu_t - x\Im_x^{*^2}(\rho u_t), \tag{2.19}$$

Where $Q^\tau = (0,l) \times (0,\tau)$ with $0 \leq \tau \leq T$, and $\Im_x^{*^2}(\rho u_t) = \int_x^l \int_\xi^l \rho u_t(\rho, t) d\rho d\xi$,

We obtain

$$\begin{aligned}
&(\mathcal{L}u, Mu)_{L^2(Q^\tau)} \\
&= \left((xu_x)_x, u_t\right)_{L^2(Q^\tau)} - \left((xu_x)_x, \Im_x^{*^2}(\rho u_t)\right)_{L^2(Q^\tau)} \\
&\quad - (au_{tt}, u_t)_{L^2_\rho(Q^\tau)} + \left(au_{tt}, \Im_x^{r^2}(\rho u_t)\right)_{L^2_\rho(Q^\tau)} \\
&\quad - (bu_t, u_t)_{L^2_\rho(Q^\tau)} + \left(bu_t, \Im_x^{*^2}(\rho u_t)\right)_{L^2_\rho(Q^\tau)},
\end{aligned} \tag{2.20}$$

By integrating each term of the right-hand side of (2.20) and taking into account conditions (2.2)-(2.5), we obtain:

$$\begin{aligned}
&\left((xu_x)_x, u_t\right)_{L^2(Q^\tau)} \\
&= \int_0^\tau xu_x u_t \Big|_{x=0}^{x=1} dt - \int_{Q^\tau} xu_x u_{tx} dx dt \\
&= -\frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|\varphi_{1x}\|^2_{L^2_\rho(0,l)},
\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
& -\left((xu_x)_x\,, \Im_x^{*2}(\rho u_t)\right)_{L^2(Q^\tau)} \\
&= -\int_0^\tau xu_x \Im_x^{*2}(\rho u_t)\Big|_{x=0}^{x=l} dt - \int_{Q^\tau} xu_x \Im_x^{*}(\rho u_t)\, dxdt \\
&= -\int_{Q^\tau} xu_x \Im_x^{*}(\rho u_t)\, dxdt,
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
& -(au_{tt}, u_t)_{L^2_\rho(Q^\tau)} \\
&= -\frac{1}{2}\int_0^l axu_t^2\Big|_{t=0}^{t=\tau} dx + \frac{1}{2}\int_{Q^\tau} a_t xu_t^2 dxdt \\
&= -\frac{1}{2}\int_0^l axu_t^2(.,\tau)dx + \frac{1}{2}\int_0^l ax\varphi_2^2(x)dx \\
&\quad + \frac{1}{2}\int_{Q^\tau} a_t xu_t^2 dxdt
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
& \left(au_{tt}, \Im_x^{*^2}(\rho u_t)\right)_{L^2_\rho(Q^\tau)} \\
&= -\int_0^\tau a\Im_x^{*}(\rho u_{tt})\, \Im_x^{2^2}(\rho u_t)\Big|_{x=0}^{x=l} dt - \int_{Q^\tau} a\Im_x^{*}(\rho u_{tt})\, \Im_x^{*}(\rho u_t)\, dxdt \\
&= -\frac{1}{2}\int_0^l a\left(\Im_x^{*}(\rho u_t)\right)^2\Big|_{t=0}^{t=\tau} dx + \frac{1}{2}\int_{Q^\tau} a_t\left(\Im_x^{*}(\rho u_t)\right)^2 dxdt \\
&= -\frac{1}{2}\int_0^l a\left(\Im_x^{*}(\rho u_t(.,\tau))\right)^2 dx + \frac{1}{2}\int_0^l a\left(\Im_x^{*}(\rho\varphi_2)\right)^2 dx \\
&\quad + \frac{1}{2}\int_{Q^\tau} a_t\left(\Im_x^{*}(\rho u_t)\right)^2 dxdt,
\end{aligned} \tag{2.24}
$$

$$
\begin{aligned}
& \left(bu_t, \Im_x^{*2}(\rho u_t)\right)_{L^2_\rho(Q^\tau)} \\
&= -\int_0^\tau b\Im_x^{*}(\rho u_t)\, \Im_x^{*}(\rho u_t)\Big|_{x=0}^{x=l} dt - \int_{Q^\tau} b\left(\Im_x^{*}(\rho u_t)\right)^2 dxdt \\
&= -\int_{Q^\tau} b\left(\Im_x^{*}(\rho u_t)\right)^2 dxdt,
\end{aligned} \tag{2.25}
$$

Substituting equalities (2.21)-(2.25) into èquality (1.20), it follows:

$$\begin{aligned}
&\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \int_0^l axu_t^2(.,\tau)dx \\
&+ \int_0^l a\left(\Im_x^*\left(\rho u_t(.,\tau)\right)\right)^2 dx \\
=&\|\varphi_{1x}\|^2_{L^2_\rho(0,l)} + \int_0^l ax\varphi_2^2(x)dx \\
&+ \int_0^l a\left(\Im_x^*\left(\rho\varphi_2\right)\right)^2 dx + \int_{Q^\tau} a_t xu_t^2 dxdt \\
&+ \int_{Q^\tau} a_t\left(\Im_x^*\left(\rho u_t\right)\right)^2 dxdt - 2\int_{Q^\tau} xu_x\Im_x^*\left(\rho u_t\right)dxdt \\
&- 2\int_{Q^\tau} b\left(\Im_x^*\left(\rho u_t\right)\right)^2 dxdt - 2\int_{Q^\tau} bxu_t^2 dxdt \\
&+ \int_{Q^\tau} xfu_t dxdt - \int_{Q^\tau} xf{\Im_x}^{*2}(pu_t)dxdt
\end{aligned} \tag{2.26}$$

and By elementary inequality credit

$$\begin{aligned}
\leq&\|\varphi_{1x}\|^2_{L^2_\rho(0,l)} + c_1\|\varphi_2\|^2_{L^2_\rho(0,l)} \\
&+ c_3\|u_t\|^2_{L^2_\rho(Q^\tau)} + c_1\|\Im_x^*\left(\rho\varphi_2\right)\|^2_{L^2(0,l)} + c_3\|\Im_x^*\left(\rho u_t\right)\|^2_{L^2(Q^\tau)} \\
&+ \frac{l}{2c_4}\|u_x\|^2_{L^2_\rho(Q^\tau)} + 2c_4\|\Im_x^*\left(\rho u_t\right)\|^2_{L^2(Q^\tau)} + 2c_4\|u_t\|^2_{L^2_\rho(Q^\tau)} \\
&+ \frac{1}{2c_4}\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + \frac{l^3}{2}\|\Im_x^*\left(\rho u_t\right)\|^2_{L^2(Q^\tau)}
\end{aligned} \tag{2.27}$$

Hence

$$\begin{aligned}
&\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + c_0\|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} \\
&+ c_0\|\Im_x^*\left(\rho u_t(.,\tau)\right)\|^2_{L^2(0,l)} \\
\leq&\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \int_0^l axu_t^2(.,\tau)dx \\
&+ \int_0^l a\left(\Im_x^*\left(\rho u_t(.,\tau)\right)\right)^2 dx \\
\leq&\|\varphi_{1x}\|^2_{L^2_\rho(0,l)} + c_1\left(1+\frac{l^4}{2}\right)\|\varphi_2\|^2_{L^2_\rho(0,l)} \\
&+ \left(\frac{1}{2c_4}+1\right)\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + c_3\|u_t\|^2_{L^2_\rho(Q^\tau)} \\
&+ \frac{l}{2c_4}\|u_x\|^2_{L^2_\rho(Q^\tau)} + \left(c_3+\frac{l^3}{2}\right)\|\Im_x^*\left(\rho u_t\right)\|^2_{L^2(Q^\tau)}
\end{aligned} \tag{2.28}$$

Let the inequality:

$$\|u(.,\tau)\|^2_{L^2_\rho(0,l)} \leq \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_t\|^2_{L^2_\rho(Q^\tau)} + \|\varphi_1\|^2_{L^2_\rho(0,l)} \tag{2.29}$$

Indeed:

On one hand:

$$\begin{aligned}\int_{Q^\tau} \frac{\partial}{\partial t}\left(xu^2\right) dxdt &= \int_0^l xu^2\Big|_{t=0}^{t=\tau} dx \\ &= \int_0^l xu^2(.,\tau)dx - \int_0^l x\varphi_1^2(x)dx\end{aligned}$$

And on the other hand:

$$\int_{Q^r} \frac{\partial}{\partial t}\left(xu^2\right) dxdt = \int_{Q^r} 2xuu_t dxdt$$

So

$$\int_0^1 xu^2(.,\tau)dx - \int_0^1 x\varphi_1^2(x)dx = \int_{Q^r} 2xuu_t dxdt$$

After applying the Cauchy inequality to the right term, we obtain:

$$\int_0^l xu^2(.,\tau)dx \leq \int_0^l x\varphi_1^2(x)dx + \int_{Q^\tau} xu^2dxdt + \int_{Q^\tau} xu_t^2dxdt.$$

Now summing up (2.28) and (2.29) term by term, we get:

$$\begin{aligned}&\|u(.,\tau)\|^2_{V_\rho^{1,0}(0,l)} + c_0\|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} \\ &+ c_0\|\Im_x^*(\rho u_t(.,\tau))\|^2_{L^2(0,l)} \\ \leq &\left(1+\frac{1}{2c_4}\right)\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + c_1\left(1+\frac{l^4}{2}\right)\|\varphi_2\|^2_{L^2_\rho(0,l)} \\ &+ \|\varphi_1\|^2_{V_\rho^{1,0}(0,l)} + (1+c_3)\|u_t\|^2_{L^2_\rho(Q^\tau)} + \|u\|^2_{L^2_\rho(Q^\tau)} \\ &+ \frac{l}{2c_4}\|u_x\|^2_{L^2_\rho(Q^\tau)} + \left(c_3+\frac{l^3}{2}\right)\|\Im_x^*(\rho u_t)\|^2_{L^2(Q^\tau)}.\end{aligned}$$

And consequently:

$$\begin{aligned}&\min(1,c_0)\left[\|u(.,\tau)\|^2_{V_\rho^{1,1}(0,l)} + \|\Im_x^*(\rho u_t(.,\tau))\|^2_{L^2(0,l)}\right] \\ \leq &\max\left(\left(1+\frac{1}{2c_4}\right), c_1\left(1+\frac{l^4}{2}\right), (1+c_3), \left(c_3+\frac{l^3}{2}\right), \frac{l}{2c_4}\right)\Big[\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} \\ &+ \|\varphi_1\|^2_{V_\rho^{1,0}(0,l)} + \|\varphi_2\|^2_{L^2_\rho(0,l)} + \|u\|^2_{V_\rho^{1,1}(Q^\tau)} + \|\Im_x^*(\rho u_t)\|^2_{L^2(Q^\tau)}\Big]\end{aligned}$$

From this last inequality, it follows that:

$$\begin{aligned}&\|u(.,\tau)\|^2_{V^{1,1}_\rho(0,l)}+\|\Im^*_x(\rho u_t(.,\tau))\|^2_{L^2(0,l)}\\ \leq & c_9\left(\|\mathcal{C}u\|^2_{L^2_\rho(Q^\tau)}+\|\varphi_1\|^2_{V^{1,0}_\rho(0,l)}+\|\varphi_2\|^2_{L^2_\rho(0,l)}\right)\\ &+c_9\left(\|u\|^2_{V^{1,1}_\rho(Q^\tau)}+\|\Im^*_x(\rho u_t)\|^2_{L^2(Q^\tau)}\right),\end{aligned} \tag{2.30}$$

Where:

$$c_9=\frac{\max\left(1+\frac{1}{2c_4},c_1\left(1+\frac{l^4}{2}\right),1+c_3,c_3+\frac{l^3}{2},\frac{l}{2c_4}\right)}{\min(1,c_0)}.$$

Applying the Gronwall's lemma to (2.30), denoting by:

$$f_1(\tau)=0$$

$$f_2(\tau)=\|u(.,\tau)\|^2_{V^{1,1}_\rho(0,l)}+\|\Im^*_x(\rho u_t(.,\tau))\|^2_{L^2(0,l)}$$

$$f_3(\tau)=c_9(\|f\|^2_{L^2_\rho(Q^\tau)}+\|\varphi_1\|^2_{V^{1,0}_\rho(0,l)}+\|\varphi_2\|^2_{L^2_\rho(0,l)})$$

We have:

$$\begin{aligned}&\|u(.,\tau)\|^2_{V^{1,1}_\rho(0,l)}+\|\Im^*_x(\rho u_t(.,\tau))\|^2_{L^2(0,l)}\\ \leq & c_9\exp(c_9\tau)\left(\|f\|^2_{L^2_\rho(Q^r)}+\|\varphi_1\|^2_{V^{1,0}_\rho(0,l)}+\|\varphi_2\|^2_{L^2_\rho(0,l)}\right)\\ \leq & c_9\exp(c_9T)\left(\|f\|^2_{L^2_\rho(Q^r)}+\|\varphi_1\|^2_{V^{1,0}_\rho(0,l)}+\|\varphi_2\|^2_{L^2_\rho(0,l)}\right).\end{aligned} \tag{2.31}$$

Since the right-hand side of inequality (2.31) is independent of τ, we pass it to the left-hand side or supremum. With respect to τ from 0 to T, we obtain inequality (2.18) with $c=c_9^{\frac{1}{2}}\exp\left(c_9\frac{T}{2}\right)$.

$$\|u\|_E\leq c_9^{\frac{1}{2}}\exp\left(c_9\frac{T}{2}\right)\|Lu\|_F,$$

Proposition 2.3.1. *The operator L defined from E to F has a closure.*

Proof 2.3.2. *We must verify that: if $u_n\in D(L)$ such that:*

$$u_n\underset{n\to\infty}{\longrightarrow}0 \ dans\ E \tag{2.32}$$

and

$$Lu_n=(\mathcal{L}u_n,l_1u_n,l_2u_n)\underset{n\to\infty}{\longrightarrow}\mathcal{F}=(f,\varphi_1,\varphi_2)\qquad in\ F, \tag{2.33}$$

Then

$$f \equiv 0, \varphi_1 \equiv 0, \;\; et \;\varphi_2 \equiv 0.$$

The relation (2.32) implies that:

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad in\ \mathcal{D}'(Q), \tag{2.34}$$

Where $\mathcal{D}'(Q)$ *is the space of distributions over* Q*.*

Due to the continuity of the derivation of $\mathcal{D}'(Q)$ *in* $\mathcal{D}'(Q)$*, (2.34) implies:*

$$\mathcal{L}u_n \underset{n\to\infty}{\longrightarrow} 0 \ dans\ \mathcal{D}'(Q), \tag{2.35}$$

And, since the convergence of $\mathcal{L}u_n$ *towards* f *in* $L^2_\rho(Q)$ *implies:*

$$\mathcal{L}u_n \underset{n\to\infty}{\longrightarrow} f \ dans\ \mathcal{D}'(Q), \tag{2.36}$$

According to the uniqueness of the limit in $\mathcal{D}'(Q)$*, we conclude from (2.34) and (2.35) that* $f \equiv 0$.

Moreover, it follows from (2.33), that:

$$l_1 u_n \underset{n\to\infty}{\longrightarrow} \varphi_1 \qquad in\ V^{1,0}_\rho(0,l), \tag{2.37}$$

As the canonical injection of $V^{1,0}_\rho(0,l)$ *into* $\mathcal{D}'(0,l)$ *is continuous.*

We deduce from (2.37) that:

$$l_1 u_n \underset{n\to\infty}{\longrightarrow} \varphi_1 \qquad in\ \mathcal{D}'(0,l), \tag{2.38}$$

On the other hand, as in

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad in\ E$$

and

$$\|l_1 u_n\|_{V^{1,0}_\rho(0,l)} \leq \|u_n\|_E \quad \forall n$$

Then, we have

$$l_1 u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad dans\ V^{1,0}_\rho(0,l). \ , \tag{2.39}$$

As a result

$$l_1 u_n \underset{n\to\infty}{\longrightarrow} 0 \quad dans\ \mathcal{D}'(0,l), \tag{2.40}$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(0,l)$, *we conclude from (2.38) and (2.40) that* $\varphi_1 \equiv 0$, *Similarly it can be shown that* $\varphi_2 \equiv 0$.

Let $\bar{L}$ *be the closure of the operator* L, *and* $D(\bar{L})$ *it's domain of definition.*

Definition 2.3.1. *The solution of the equation* $\bar{L}u = \mathcal{F}$ *for* $u \in D(\bar{L})$ *is called the strong solution of the problem (2.1)-(2.5).*

By passing to the limit, we extend inequality (1.18) to strong solutions, and then:

$$\|u\|_E \leq c\|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}) \tag{2.41}$$

From inequality (1.41), we have the following results:

Corollary 2.3.1. *The strong solution of the problem (2.1)-(2.5) if it exists, is unique and continuously depends on* $\mathcal{F} = (f, \varphi_1, \varphi_2) \in F$.

The proof is trivial.

Corollary 2.3.2. *The set of values* $R(\bar{L})$ *of the operator* $\bar{L}$ *is equal to the closure* $\overline{R(L)}$ *of* $R(L)$, *where* $R(\bar{L}) = \overline{R(L)}$

Proof 2.3.3. ***of Corollary 2.3.2**: From the definition of the set* $R(\bar{L})$, *it follows that,* $R(\bar{L}) \subset \overline{R(L)}$, *it remains to show the reverse inclusion.*

Let $z \in \overline{R(L)}$ *Then, there exists a Cauchy sequence* (z_n) *in* F *consisting of elements of the set* $R(L)$ *such that*

$$\lim_{n\longrightarrow\infty} z_n = z.$$

There exists then a corresponding sequence $u_n \in D(L)$ *such that*

$$Lu_n = z_n$$

From estimation (2.18), we have:

$$\|u_n - u_m\|_E \leq c\,\|Lu_n - Lu_m\|_F \longrightarrow 0$$

As n *and* m *tend to infinity, we deduce that* (u_n) *is a Cauchy sequence in* E*.*

Therefore, there exists a function $u \in E$ *such that* $u_n \underset{n\to\infty}{\longrightarrow} u$ *in* E*.*

By virtue of the definition of $\bar{L}$*, the function* u *satisfies* $u \in D(\bar{L})$ *and* $\bar{L}u = z$*, thus* $z \in R(\bar{L})$*.*

2.4 Existence of the solution:

Theorem 2.4.1. *If conditions* $\mathbf{C}_1 - \mathbf{C}_2$ *are satisfied, then for each* $f \in L^2_\rho(Q)$*,*$\varphi_1 \in V^{1,0}_\rho(0,l)$ *and* $\varphi_2 \in L^2_\rho(0,l)$*, there exists a unique strong solution* $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ *of the problem (2.1)-(2.5) satisfying the estimation*

$$\|u\|_E \leq c\|\mathcal{F}\|_F \tag{2.42}$$

Where c *is a positive constant independent of the solution* u*.*

Proof 2.4.1. *From inequality (2.41), we deduce that the operator* $\bar{L}$*.from* E *To* E *has an inverse* $\bar{L}^{-1}$*, and from the* ***Corollary 2.3.2****, we deduce that the image* $R(\bar{L})$ *of the operator* $\bar{L}$ *is closed, Therefore, it suffices to show the density of the set* $R(L)$ *in the space* F *(i.e)*

$$\overline{R(L)} = F,$$

. For this, we need the following proposition:

Proposition 2.4.1. *If the conditions of* ***theorem 2.4.1*** *are satisfied, if for* $\omega \in L^2(Q)$*, we have*

$$(\mathcal{L}u, \omega)_{L^2_\rho(Q)} = 0 \tag{2.43}$$

For any function $u \in D_0(L) = \{u/u \in D(L) : l_1u = l_2u = 0\}$*.*

Then ω *vanishes almost everywhere in* Q*.*

Proof 2.4.2. ***of proposition 2.4.1****: The relation (1.43) holds for all* $u \in D_0(L)$*, It can be expressed in a special form.*

Let u_{tt} *be a solution of the equation:*

$$u_{tt} - \Im_x^{*^2}(\rho u_{tt}) = \Psi(x,t) \tag{2.44}$$

Where

$$\Psi(x,t) = \int_t^T \omega(x,\tau)d\tau \tag{2.45}$$

And let the function u be given by :

$$u = \begin{cases} 0, & si\ 0 \leq t \leq s \\ \int_s^t (t-\tau) u_{\tau\tau} d\tau, & si\ s \leq t \leq T \end{cases} \tag{2.46}$$

The relations (2.44) and (2.46) imply that u is in $D_0(L)$ and from (1.44) and (1.45) we have:

$$\begin{aligned} \omega(x,t) &= \left(-u_{tt} + \Im_x^{*^2}(\rho u_{tt})\right)_t \\ &= -u_{ttt} + \Im_x^{*^2}(\rho u_{ttt}), \end{aligned} \tag{2.47}$$

Lemma 2.4.1. *If the conditions of **theorem 2.4.1** are satisfied, then the function ω defined by (1.47) is in $L^2_\rho(Q)$.*

Proof 2.4.3. ***of Lemma 2.4.1*** *:To demonstrate that $\omega \in L^2_\rho(Q)$, we must introduce regularization t-operators ρ_ϵ of the form:*

$$(\rho_\epsilon h)(x,t) = \frac{1}{\varepsilon}\int_0^T \omega\left(\frac{s-t}{\varepsilon}\right) h(x,s)ds,$$

Where

$$\omega \in C_0^\infty(0,T), \quad \omega \geq 0, \quad \int_{-\infty}^{+\infty} \omega(t)dt = 1$$

Applying the operators ρ_ϵ and $\frac{\partial}{\partial t}$ to equation (2,44), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(u_{tt} - \Im_x^{*^2}(\rho u_{tt})\right) \\ =&\frac{\partial}{\partial t}\left[\left(u_{tt} - \Im_x^{*^2}(\rho u_{tt})\right) - \rho_\varepsilon\left(u_{tt} - \Im_x^{*^2}(\rho u_{tt})\right)\right] + \frac{\partial}{\partial t}\rho_\varepsilon \Psi \end{aligned}$$

Where

$$\begin{aligned} &\left\| \frac{\partial}{\partial t}\left(u_{tt} - \Im_x^{*2}(\rho u_{tt})\right)\right\|^2_{L^2_\rho(Q)} \\ \leq& 2\left\| \frac{\partial}{\partial t}\left[\left(u_{tt} - \Im_x^{*^2}(\rho u_{tt})\right) - \rho_\varepsilon\left(u_{tt} - \Im_x^{*^2}(\rho u_{tt})\right)\right]\right\|^2_{L^2_\rho(Q)} \\ &+ 2\left\|\frac{\partial}{\partial t}\rho_\varepsilon \Psi\right\|^2_{L^2_\rho(Q)} \end{aligned}$$

Hence

$$\rho_\varepsilon g \underset{\varepsilon \to 0}{\longrightarrow} g$$

And $\frac{\partial}{\partial t}\left(u_{tt}-\Im_x^{*^2}(\rho u_{tt})\right)$ *is bounded in* $L^2_\rho(Q)$*, then* $\omega\in L^2_\rho(Q)$*.*

Now, returning to the proof of the proposition, by substituting ω *in (2,43) with its representation (2.47) we obtain:*

$$\begin{aligned}&-(u_{xx},u_{ttt})_{L^2_\rho(Q)}-(u_x,u_{ttt})_{L^2(Q)}\\&+(au_{tt},u_{ttt})_{L^2_\rho(Q)}+(bu_t,u_{ttt})_{L^2_\rho(Q)}\\&+\left(u_{xx},\Im_x^{*^2}(\rho u_{ttt})\right)_{L^2_\rho(Q)}+\left(u_x,\Im_x^{*^2}(\rho u_{ttt})\right)_{L^2(Q)}\\&-\left(au_{tt},\Im_x^{*^2}(\rho u_{ttt})\right)_{L^2_\rho(Q)}-\left(bu_t,\Im_x^{*^2}(\rho u_{ttt})\right)_{L^2_\rho(Q)}\\&=0\end{aligned}\tag{2.48}$$

After integration by parts, taking into account the conditions (2.3)-(2.5), $\mathbf{C}_1-\mathbf{C}_2$ *and the special form given by (2.44) and (2.46), the equality (2.48) can be written in a simpler form. For this, let's consider each term of the equality (2.48):*

$$\begin{aligned}&-(u_{xx},u_{ttt})_{L^2_\rho(Q)}\\&=-\int_Q xu_{xx}u_{ttt}dxdt\\&=-\int_s^T xu_xu_{ttt}\Big|_{x=0}^{x=l}dt+\int_{Q_s}u_xu_{ttt}dxdt+\int_{Q_s}xu_xu_{tttx}dxdt\\&=(u_x,u_{ttt})_{L^2(Q_s)}+\int_0^l xu_xu_{ttx}\Big|_{t=s}^{t=T}dx-\int_{Q_s}xu_{tx}u_{ttx}dxdt\\&=(u_x,u_{ttt})_{L^2(Q_s)}-\frac{1}{2}\int_0^l xu_{tx}^2\Big|_{t=s}^{t=T}dx\\&=(u_x,u_{ttt})_{L^2(Q_s)}-\frac{1}{2}\int_0^l xu_{tx}^2(.,T)dx,\end{aligned}\tag{2.49}$$

$$\begin{aligned}&(au_{tt},u_{tt})_{L^2_\rho(Q)}\\&=\frac{1}{2}\int_0^l axu_{tt}^2\Big|_{t=s}^{t=T}dx-\frac{1}{2}\int_{Q_s}a_txu_{tt}^2dxdt\\&=-\frac{1}{2}\int_0^l axu_{tt}^2(.,s)dx-\frac{1}{2}\int_{Q_s}a_txu_{tt}^2dxdt\end{aligned}\tag{2.50}$$

$$
\begin{aligned}
&(bu_t, u_{tt})^l_{L^2_\rho(Q)}\\
&= \int_0^l bxu_t u_{tt}\Big|_{t=s}^{t=T} dx - \int_{Q_s} b_t x u_t u_{tt} dxdt\\
&- \int_{Q_s} bxu_{tt}^2 dxdt\\
&= -\frac{1}{2}\int_0^l b_t x u_t^2\Big|_{t=s}^{t=T} dx + \frac{1}{2}\int_{Q_s} b_{tt} x u_t^2 dxdt\\
&- \int_{Q_s} bxu_{tt}^2 dxdt\\
&= -\frac{1}{2}\int_0^l b_t x u_t^2(.,T)dx + \frac{1}{2}\int_{Q_s} b_{tt} x u_t^2 dxdt\\
&\quad - \int_{Q_s} bxu_{tt}^2 dxdt,
\end{aligned}
\tag{2.51}
$$

$$
\begin{aligned}
&\left(u_{xx}, \Im_x^{*^2}(\rho u_{ttt})\right)_{L^2(Q_s)}\\
&= \int_s^T xu_x \Im_x^{*^2}(\rho u_{ttt})\Big|_{x=0}^{x=l} dt + \int_{Q_s} xu_x \Im_x^{*}(\rho u_{ttt})\, dxdt\\
&- \int_{Q_s} u_x \Im_x^{*^2}(\rho u_{ttt})\, dxdt\\
&= \int_0^l xu_x \Im_x^{*}(\rho u_{tt})\Big|_{t=s}^{t=T} dx - \int_{Q_s} xu_{xt} \Im_x^{*}(\rho u_{tt})\, dxdt\\
&- \left(u_x, \Im_x^{*^2}(\rho u_{ttt})\right)_{L^2(Q_s)}\\
&= -\left(u_{xt}, \Im_x^{*}(\rho u_{tt})\right)_{L^2_\rho(Q_s)} - \left(u_x, \Im_x^{*2}(\rho u_{ttt})\right)_{L^2(Q_s)},
\end{aligned}
\tag{2.52}
$$

$$
\begin{aligned}
&-\left(au_{tt}, \Im_x^{*^2}(\rho u_{ttt})\right)_{L^2_\rho(Q)}\\
&= \int_s^T a\Im_x^{*}(\rho u_{tt})\, \Im_x^{*^2}(\rho u_{ttt})\Big|_{x=0}^{x=l} dt + \int_{Q_s} a\Im_x^{*}(\rho u_{tt})\, \Im_x^{*}(\rho u_{ttt})\, dxdt\\
&= \frac{1}{2}\int_0^l a\left(\Im_x^{*}(\rho u_{tt})\right)^2\Big|_{t=s}^{t=T} dx - \frac{1}{2}\int_{Q_s} a_t\left(\Im_x^{*}(\rho u_{tt})\right)^2 dxdt\\
&= -\frac{1}{2}\int_0^l a\left(\Im_x^{*}(\rho u_{tt}(.,s))\right)^2 dx - \frac{1}{2}\int_{Q_s} a_t\left(\Im_x^{*}(\rho u_{tt})\right)^2 dxdt,
\end{aligned}
\tag{2.53}
$$

$$
\begin{aligned}
&-\left(bu_t, \Im_x^{*^2}\left(\rho u_{ttt}\right)\right)_{L_\rho^2(Q)} \\
&=\int_s^T b\Im_x^*\left(\rho u_t\right)\Im_x^{*^2}\left(\rho u_{tt}\right)\Big|_{x=0}^{x=l} dt+\int_{Q^s} b\Im_x^*\left(\rho u_t\right)\Im_x^*\left(\rho u_{ttt}\right)dxdt \\
&=\int_0^l b\Im_x^*\left(\rho u_t\right)\Im_x^*\left(\rho u_{tt}\right)\Big|_{t=s}^{t=T} dx-\int_{Q_s} b\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt \\
&\quad-\int_{Q_s} b_t\Im_x^*\left(\rho u_t\right)\Im_x^*\left(\rho u_{tt}\right)dxdt \\
&=-\int_{Q_s} b\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt-\frac{1}{2}\int_0^l b_t\left(\Im_x^*\left(\rho u_t\right)\right)^2\Big|_{t=s}^{t=T} dx \\
&\quad+\frac{1}{2}\int_{Q_s} b_{tt}\left(\Im_x^*\left(\rho u_t\right)\right)^2 dxdt \\
&=-\int_{Q_s} b\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt-\frac{1}{2}\int_0^l b_t\left(\Im_x^*\left(\rho u_t(.,T)\right)\right)^2 dx \\
&\quad+\frac{1}{2}\int_{Q_s} b_{tt}\left(\Im_x^*\left(\rho u_t\right)\right)^2 dxdt.
\end{aligned}
\tag{2.54}
$$

The substitutions of equalities (2.49)-(2.54) into the identity (2.48), yield

$$
\begin{aligned}
&\int_0^l xu_{tx}^2(.,T)dx+\int_0^l axu_{tt}^2(.,s)dx+\int_0^l b_txu_t^2(.,T)dx \\
&+\int_0^l a\left(\Im_x^*\left(\rho u_{tt}(.,s)\right)\right)^2 dx \\
&+\int_0^l b_t\left(\Im_x^*\left(\rho u_t(.,T)\right)\right)^2 dx+\int_{Q_s} a_txu_{tt}^2dxdt \\
&+2\int_{Q_s} bxu_{tt}^2dxdt+\int_{Q_s} a_t\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt \\
&+2\int_{Q_s} b\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt \\
&=\int_{Q_s} b_{tt}xu_t^2dxdt+\int_{Q_s} b_{tt}\left(\Im_x^*\left(\rho u_t\right)\right)^2 dxdt-2\left(u_{xt}, \Im_x^*\left(\rho u_{tt}\right)\right)_{L_\rho^2(Q_s)}
\end{aligned}
\tag{2.55}
$$

Using the Cauchy inequality with ε, we estimate the last term of the right-hand side of the equality (2.55) as follows:

$$
\begin{aligned}
&-2\left(u_{xt}, \Im_x^*\left(\rho u_{tt}\right)\right)_{L_\rho^2(Q_s)} \\
&\leq\int_{Q_s} xu_{xt}^2dxdt+\frac{l}{\varepsilon}\int_{Q_s}\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt \\
&\quad \textit{By putting}\left(\varepsilon=\frac{l}{c_2+2c_4}\right) \\
&\leq\left(\frac{l}{c_2+2c_4}\right)\int_{Q_s} xu_{xt}^2dxdt+\left(c_2+2c_4\right)\int_{Q_s}\left(\Im_x^*\left(\rho u_{tt}\right)\right)^2 dxdt,
\end{aligned}
\tag{2.56}
$$

Taking into account the conditions $\mathbf{C}_1 - \mathbf{C}_2$, and combining the inequality (2.56) and the equality (2.55):

$$\begin{aligned}&\|u_{tx}(.,T)\|^2_{L^2_\rho(0,l)} + c_0\|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + c_6\|u_t(.,T)\|^2_{L^2_\rho(0,l)}\\&+ c_0\|\Im^*_x\left(\rho u_{tt}(.,s)\right\|^2_{L^2(0,l)} + c_6\|\Im^*_x\left(\rho u_t(.,T)\right\|^2_{L^2(0,l)}\\&+ c_2\|u_{tt}\|^2_{L^2_\rho(Q_s)} + 2c_4\|u_{tt}\|^2_{L^2_\rho(Q_s)} + c_2\|\Im^*_x(\rho u_{tt})\|^2_{L^2(Q_s)}\\&+ 2c_4\|\Im^*_x(\rho u_{tt})\|^2_{L^2(Q_s)}\\&\leq c_8\|u_t\|^2_{L^2_\rho(Q_s)} + c_8\|\Im^*_x(\rho u_t)\|^2_{L^2(Q_s)}\\&+\left(\frac{l}{c_2+2c_4}\right)\|u_{xt}\|^2_{L^2_\rho(Q_s)} + (c_2+2c_4)\|\Im^*_x(\rho u_{tt})\|^2_{L^2(Q_s)}\end{aligned}$$

neglecting the sixth and seventh terms of the left-hand side of the previous inequality, it follows that:

$$\begin{aligned}&\|u_{tx}(.,T)\|^2_{L^2_\rho(0,l)} + \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)}\\&+\|u_t(.,T)\|^2_{L^2_\rho(0,l)} + \|\Im^*_x\left(\rho u_{tt}(.,s)\right\|^2_{L^2(0,l)}\\&+\|\Im^*_x\left(\rho u_t(.,T)\right\|^2_{L^2(0,l)}\\&\leq c_{11}\left(\|u_t\|^2_{L^2_\rho(Q_s)} + \|u_{xt}\|^2_{L^2_\rho(Q_s)} + \|\Im^*_x(\rho u_t)\|^2_{L^2(Q_s)}\right),\end{aligned} \tag{2.57}$$

Where

$$c_{11} = \frac{\max\left(c_8, \frac{1}{c_2+2c_4}\right)}{\min(1, c_0, c_6)}.$$

To proceed, we introduce a new function $v(x,t)$, such that:

$$v(x,t) = \int_t^T u_{\tau\tau}d\tau \tag{2.58}$$

Then

$$u_t(x,t) = v(x,s) - v(x,t), \quad \text{and} \quad u_t(x,T) = v(x,s). \tag{2.59}$$

The inequality (2.57) becomes:

$$\begin{aligned}&\|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|\Im^*_x\left(\rho u_{tt}(.,s)\right\|^2_{L^2(0,l)}\\&+(1-2c_{11}(T-s))\left(\|v_x(.,s)\|^2_{L^2_\rho(0,l)}\right.\\&\left.+\|v(.,s)\|^2_{L^2_\rho(0,l)} + \|\Im^*_x\left(\rho v(.,s)\right\|^2_{L^2(0,l)}\right)\\&\leq 2c_{11}\left(\|v_x\|^2_{L^2_\rho(Q_s)} + \|v\|^2_{L^2_\rho(Q_s)} + \|\Im^*_x(\rho v)\|^2_{L^2(Q_s)}\right).\end{aligned} \tag{2.60}$$

If $s_0 > 0$ satisfies:

$$(1 - 2c_{11}(T - s)) = \frac{1}{2},$$

Then the inequality (2.60) implies

$$\begin{aligned}
&\|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|\Im^*_x \left(\rho u_{tt}(.,s)\|^2_{L^2(0,l)}\right.\\
&+ \|v_x(.,s)\|^2_{L^2_\rho(0,l)} + \|v(.,s)\|^2_{L^2_\rho(0,l)}\\
&+ \|\Im^*_x \left(\rho v(.,s)\|^2_{L^2(0,l)}\right.\\
\leq & 4c_{11}\left(\|v_x\|^2_{L^2_\rho(Q_S)} + \|v\|^2_{L^2_\rho(Q_S)} + \|\Im^*_x(\rho v)\|^2_{L^2(Q_s)}\right)
\end{aligned} \tag{2.61}$$

For all $s \in [T - s_0, T]$.

We set

$$h(s) = \|v_x\|^2_{L^2_\rho(Q_S)} + \|v\|^2_{L^2_\rho(Q_s)} + \|\Im^*_x(\rho v)\|^2_{L^2(Q_s)}. \tag{2.62}$$

Then (2.61) and (2.62) yield.

$$\begin{aligned}
&\|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|\Im^*_x \left(\rho u_{tt}(.,s)\|^2_{L^2(0,l)} - \frac{\partial h}{\partial s}\right.\\
\leq & 4c_{11}h(s).
\end{aligned} \tag{2.63}$$

Consequently

$$-\frac{\partial}{\partial s}\left(h(s)\exp\left(4c_{11}s\right)\right) \leq 0 \tag{2.64}$$

By integrating (2.64) over (s, T) and taking into account that $h(T) = 0$, we obtain:

$$h(s)\exp\left(4c_{11}s\right) \leq 0 \tag{2.65}$$

Where $\omega = 0$ Almost everywhere in Q_{T-s_0}, and since the length s is independent of the choice of the origin, proceeding with the same reasoning, After a finite number of times, it is shown that $\omega = 0$ in Q.

With the proposition established, let us now return to the proof of the theorem. We must show the validity of the equality $\overline{R(L)} = F$.

As F is a Hilbert space, the equality $\overline{R(L)} = F$ is true if the equality:

$$(Lu, W)_F = (\mathcal{L}u, \omega)_{L^2_\rho(Q)} + (l_1u, \omega_1)_{V^{1,0}_\rho(0,l)} + (l_2u, \omega_2)_{L^2_\rho(0,l)} = 0, \tag{2.66}$$

Where $W = (\omega, \omega_1, \omega_2) \in R(L)^\perp$, *it follows that* $\omega \equiv 0, \omega_1 \equiv 0$ *and* $\omega_2 \equiv 0$, *almost everywhere in* Q.

Considering any element of $D_0(L)$, *from equality (2.66), we obtain:*

$$\forall u \in D_0(L), \quad (\mathcal{L}u, \omega)_{L^2_\rho(Q)} = 0 \tag{2.67}$$

Hence, by virtue of ***proposition 2.4.1***, *we deduce that* $\omega \equiv 0$.

So, from equality (2.66), we obtain:

$$(l_1 u, \omega_1)_{V^{1,0}_\rho(0,l)} + (l_2 u, \omega_2)_{L^2_\rho(0,l)} = 0 \tag{2.68}$$

Since $l_1 u$ *and* $l_2 u$ *are independent and the set of values of operators* $l_1 u$ *and* $l_2 u$ *is everywhere dense in the Hilbert spaces* $V^{1,0}_\rho(0,l)$ *and* $L^2_\rho(0,l)$ *respectively.*

Hence $\omega_1 \equiv 0$ *And* $\omega_2 \equiv 0$ *And consequently* $W = 0$, *which completes the proof of* ***theorem 2.4.1***.

Chapter 3

On a semi-linear problem for a parabolic equation with the Bessel operator

3.1 Problèm Statement

In the rectangle $Q = (0,l) \times (0,T)$ where $l < \infty$ and $T < \infty$, we consider the semi-linear heat equation with the Bessel operator:

$$u_t - \frac{1}{x}(xu_x)_x = |u|^{p-2}u, \quad (x,t) \in (0,l) \times [0,T] \tag{3.1}$$

Where p will be chosen later.

To equation (3.1) we associate the initial condition

$$lu = u(x,0) = \varphi(x), \quad x \in (0,l), \tag{3.2}$$

And the boundary condition:

$$u(l,t) = 0, \quad t \in [0,T] \tag{3.3}$$

Where φ is a given function and satisfies the compatibility condition:

$$\varphi(l) = 0.$$

In this chapter, we demonstrate the local existence and uniqueness of the weak solution to problem (3.1)-(3.3) The proofs are essentially based on a priori estimates as well as fixed point arguments.

Let's start by studying the corresponding linear problem.

In the bounded domain $Q = (0, l) \times (0, T)$ where $l < \infty$ and $T < \infty$, Consider the second-order parabolic equation:

$$\mathcal{L}u = u_t - \frac{1}{x}(xu_x)_x = f(x,t), \quad x \in (0,l), t \in (0,T) \tag{3.4}$$

To equation (3.4), we associate the initial condition:

$$lu = u(x,0) = \varphi(x), \quad x \in (0,l), \tag{3.5}$$

And the boundary condition:

$$u(l,t) = 0, \quad t \in (0,T) \tag{3.6}$$

Where f is a given function belonging to $L^2\left((0,T), L^2_\rho(0,l)\right)$, (i.e) $\int_0^T \|f\|^2_{L^2_\rho(0,l)} dt < \infty$.

In this linear case, we demonstrate the existence and uniqueness of the weak solution to problem (3.4)-(3.6). The uniqueness is demonstrated using the method of the previous chapter. For existence, we use the Galerkin method.

3.2 Functional space

For the study of the posed problem (3.4)-(3.6), we need some functional spaces.
Let $L^2_\rho(Q)$ be the Hilbert space, of square integrable functions with weights having finite norm:

$$\|u\|_{L^2_\rho(Q)} = \left(\int_Q xu^2 dxdt\right)^{\frac{1}{2}},$$

The inner product in $L^2_\rho(Q)$ is defined by:

$$(u,w)_{L^2_\rho(Q)} = (xu, w)_{L^2(Q)}$$

$H^1_\rho(0,l)$ is the Hilbert space with finite norm:

$$\|u\|^2_{H^1_\rho(0,l)} = \|u\|^2_{L^2_\rho(0,l)} + \|u_x\|^2_{L^2_\rho(0,l)},$$

And the inner product:

$$(u,w)_{H^1_\rho(0,l)} = (u,w)_{L^2_\rho(0,l)} + (u_x, w_x)_{L^2(0,l)},$$

$L^\infty\left((0,T), H^1_\rho(0,l)\right)$ is the space of measurable functions $u : [0,T] \longrightarrow H^1_\rho(0,l)$ having finite norm:

$$\|u\|_{L^\infty\left((0,T),H^1_\rho(0,l)\right)} = \underset{0\le t\le T}{\text{supess}}\|u(.,t)\|^2_{H^1_\rho(0,l)},$$

Where $H^1_\rho(0,l)$ is the space with weights on the interval $(0,l)$.

3.3 Uniqueness of the solution

Definition 3.3.1. *we say that the function $u(x,t) \in L^\infty\left((0,T), H^1_\rho(0,l)\right)$ is a weak solution of the problem (3.4)-(3.6) if it satisfies:*

- $\int_0^l x u_t \eta dx + \int_0^l x u_x \eta_x dx = \int_0^l x f \eta(x) dx$ *For all $\eta \in H^1_\rho(0,l)$, and for almost all $t \in [0,T]$.*
- $u(x.0) = \varphi(x), \quad x \in (0,l).$

Theorem 3.3.1. *The problem (3.4)-(3.6) cannot have more than one weak solution.*

Proof 3.3.1. *let u_1 and u_2 be two solutions of (3.4)-(3.6). Then $w = u_1 - u_2$ is a solution of:*

$$w_t - \frac{1}{x}(xw_x)_x = 0, \quad x \in (0,l), t \in [0,T], \tag{3.7}$$

$$w(l,t) = 0, t \in [0,T], \tag{3.8}$$

$$w(x,0) = 0, x \in (0,l), \tag{3.9}$$

If we scalar multiply equation (3.7) by xw, we obtain:

$$(w_t, xw)_{L^2(0,l)} - ((xw_x)_x, w)_{L^2(0,l)} = 0 \tag{3.10}$$

Hence:

$$\int_0^l xww_t dx - \int_0^l (xw_x)_x w dx = 0 \tag{3.11}$$

By integrating by parts each term of equation (3.11) and taking into account conditions (3.8)-(3.9), *we obtain:*

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^l xw^2dx + \int_0^l xw_x^2dx = 0 \tag{3.12}$$

So

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^l xw^2dx = -\int_0^l xw_x^2dx \tag{3.13}$$

From (3.13), we conclude that:

$$\frac{\partial}{\partial t}\int_0^l xw^2dx \leq 0 \tag{3.14}$$

Integrating (3.14) over $[0, t]$*, we get:*

$$\begin{aligned}\int_0^l xw^2(x,t)dx &\leq \int_0^l xw^2(x,0)dx \\ &= 0\end{aligned}$$

Therefore

$$w(x,t) = 0, \quad \text{and from him} \quad u_1 = u_2.$$

3.4 Existence de la solution faible

Theorem 3.4.1. *Let* $f \in L^2\left((0,T), L^2_\rho(0,l)\right)$ *and* $\varphi \in H^1_\rho(0,l)$*, then (2.4)-(2.6) has a unique weak solution in* $L^\infty\left((0,T), H^1_\rho(0,l)\right)$*,* $u_t \in L^2\left((0,T), L^2_\rho(0,l)\right)$*.*

Proof 3.4.1. *we use the Galerkin method where the solution can be obtained as a limit of approximate solution.*

Let $(\varphi_k(x))_{k\in N}$ *be a fundamental system in* $H^1_\rho(0,l)$ *with:*

$$(\varphi_k, \varphi_j)_{L^2(0,l)} = \delta_k^j = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k \end{cases}.$$

we are lookig for an approximate solution $u^N(x,t)$ *under the form* $u^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x)$ *and verify:*

$$\left(u_t^N, \varphi_j\right)_{L^2_\rho(0,l)} + \left(u_x^N, \varphi_{jx}\right)_{L^2_\rho(0,l)} = (f, \varphi_j)_{L^2_\rho(0,l)}, \quad j = \overline{1,N} \tag{3.15}$$

And the relation:

$$C_j^N(0) = (\varphi, \varphi_j) \tag{3.16}$$

If we multiply each equation of (3.15) suitably by par $\frac{d}{dt}C_j^N(t)$*, and sum from* $j=1$ *to* $j=1$ *as follows:*

$$\left(u_t^N, \frac{d}{dt}C_1^N(t)\varphi_1\right)_{L_\rho^2(0,l)} + \left(u_x^N, \frac{d}{dt}C_1^N(t)\varphi_{1x}\right)_{L_\rho^2(0,l)} = \left(f, \frac{d}{dt}C_1^N(t)\varphi_1\right)_{L_\rho^2(0,l)},$$
$$\left(u_t^N, \frac{d}{dt}C_2^N(t)\varphi_2\right)_{L_\rho^2(0,l)} + \left(u_x^N, \frac{d}{dt}C_2^N(t)\varphi_{2x}\right)_{L_\rho^2(0,l)} = \left(f, \frac{d}{dt}C_2^N(t)\varphi_2\right)_{L_\rho^2(0,l)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left(u_t^N, \frac{d}{dt}C_N^N(t)\varphi_N\right)_{L_\rho^2(0,l)} + \left(u_x^N, \frac{d}{dt}C_N^N(t)\varphi_{Nx}\right)_{L_\rho^2(0,l)} = \left(f, \frac{d}{dt}C_N^N(t)\varphi_N\right)_{L_\rho^2(0,l)}, 0<t<T$$

from where after summation, we have:

$$\left(u_t^N, u_t^N\right)_{L_\rho^2(0,l)} + \left(u_x^N, u_{tx}^N\right)_{L_\rho^2(0,l)} = \left(f, u_t^N\right)_{L_\rho^2(0,l)}, 0<t<T \tag{3.17}$$

Integrating both sides of (3.17) over $(0,t)$*, we have:*

$$\begin{aligned}&\int_0^t \int_0^t x\left(u_t^N\right)^2 dxds + \frac{1}{2}\int_0^t x\left(u_x^N(x,t)\right)^2 dx \\ =&\frac{1}{2}\int_0^t x\left(u_x^N(x,0)\right)^2 dx + \int_0^t \int_0^l x f u_t^N dxds, \forall t\in[0,T].\end{aligned} \tag{3.18}$$

Applying Cauchy's inequality to the second term of the right side of (3.18), we get:

$$\begin{aligned}&\int_0^t \int_0^t x\left(u_t^N\right)^2 dxds + \int_0^t x\left(u_x^N(x,t)\right)^2 dx \\ \leq&\int_0^t x\left(u_x^N(x,0)\right)^2 dx + \int_0^t \int_0^l x f^2 dxds, \quad \forall t\in[0,T]\end{aligned} \tag{3.19}$$

But as:

$$\int_0^1 x\left(u_x^N(x,0)\right)^2 dx \leq \int_0^1 x\varphi_x^2(x)dx$$

We then have:

$$\begin{aligned}\int_0^l x\left(u_x^N(x,t)\right)^2 dx + \int_0^t \int_0^l x\left(u_t^N\right)^2 dxds &\leq \int_0^l x\varphi_x^2(x)dx + \int_0^t \int_0^l x f^2 dxds \\ &\leq \left(\int_0^l x\varphi_x^2(x)dx + \int_0^l x\varphi^2(x)dx\right) + \int_0^t \int_0^l x f^2 dxds \\ &\leq \|\varphi\|^2_{H_\rho^1(0,l)} + \|f\|^2_{L^2([0,T],L_\rho^2(0,l))}\end{aligned}$$

Hence, we have:

$$\int_0^t \int_0^l x\left(u_t^N\right)^2 dxds \leq c_1$$

With c_1, a positive constant independent of N.

Consequently

$$\left\|u_t^N\right\|^2_{L^2\left((0,T),L^2_\rho(0,l)\right)} \leq c_1. \tag{3.20}$$

So the sequence $\left(u_t^N\right)_{N\in N}$ is bounded in $L^2\left((0,T),L^2_\rho(0,l)\right)$.

Now let's consider the inequality:

$$\begin{aligned}
&\int_0^l x\left(u^N(x,t)\right)^2 dx \\
\leq &\int_0^l x\left(u^N(x,0)\right)^2 dx + \int_0^t \int_0^l x\left(u^N\right)^2 dxds \\
&+ \int_0^t \int_0^l x\left(u_t^N\right)^2 dxds.
\end{aligned} \tag{3.21}$$

$\forall t \in [0.T]$.

Summing (3.19) and (3.21) term by term, we obtain:

$$\begin{aligned}
&\int_0^l x\left(u^N(x,t)\right)^2 dx + \int_0^l x\left(u_x^N(x,t)\right)^2 dx \\
\leq &\int_0^l x\left(u^N(x,0)\right)^2 dx + \int_0^l x\left(u_x^N(x,0)\right)^2 dx \\
&\int_0^t \int_0^l x\left(u^N\right)^2 dxds + \int_0^l \int_0^l xf^2 dxds
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|u^N(.,t)\right\|^2_{H^1_\rho(0,l)} &\leq \|\varphi\|^2_{H^1_\rho(0,l)} + \|f\|^2_{L^2\left([0,T],L^2_\rho(0,l)\right)} + \left\|u^N\right\|^2_{L^2_\rho(Q^t)} \\
&\leq \|\varphi\|^2_{H^1_\rho(0,l)} + \|f\|^2_{L^2\left([0,T],L^2_\rho(0,l)\right)} + \left\|u^N\right\|^2_{H^1_\rho(Q^t)},
\end{aligned} \tag{3.22}$$

For all $t \in [0,T]$.

Applying Gronwall's lemma to inequality (3.22), we get:

Where

$$f_1 = 0 \;\; ; \;\; f_2 = \left\|u^N(.,t)\right\|^2_{H^1_p(0,l)} \;\; ; \;\; f_3 = \|\varphi\|^2_{H^1_p(0,l)} + \|f\|^2_{L^2([0,T],L^2(0,l))}$$

We have

$$\left\|u^N(.,t)\right\|^2_{H^1_\rho(0,l)} \leq \exp(t)\left(\|\varphi\|^2_{H^1_\rho(0,l)} + \|f\|^2_{L^2\left([0,T],L^2_\rho(0,l)\right)}\right) \\ \leq \exp(T)\left(\|\varphi\|^2_{H^1_\rho(0,l)} + \|f\|^2_{L^2\left([0,T],L^2_\rho(0,l)\right)}\right) \tag{3.23}$$

And since the right-hand side of inequality (3.23) is independent of t, we pass to the left-hand side or supremum with respect to t from 0 to T we obtain:

$$\sup_{0\leq t\leq T}\left\|u^N(.,t)\right\|^2_{H^1_\rho(0,l)} \leq \exp(T)\left(\|\varphi\|^2_{H^1_\rho(0,l)} + \|f\|^2_{L^2\left([0,T],L^2_\rho(0,l)\right)}\right) \tag{3.24}$$

And consequently

$$\left\|u^N\right\|^2_{L^\infty\left((0,T),H^1_\rho(0,l)\right)} \leq c_2, \tag{3.25}$$

Where c_2 is a constant independent of N.

From (3.20), we deduce that $\left(u^N_t\right)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2\left([0,T],L^2_\rho(0,l)\right)$, Then we can extract from this sequence a subsequence $\left(u^{N_k}_t\right)_{N_k\in\mathbb{N}}$ that converges weakly to $u_t \in L^2\left((0,T),L^2_\rho(0,l)\right)$. From (4.11), we deduce that $\left(u^N\right)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty\left((0,T),H^1_\rho(0,l)\right)$ **(Bolzano Weierstrass theorem)**, Thus we can extract a subsequence $\left(u^{N_k}\right)_{N_k\in\mathbb{N}}$ that converges weakly to $u \in L^\infty\left((0,T),H^1_\rho(0,l)\right)$.

Let's show that this limit u is a weak solution of (3.4)-(3.6).

Multiply equation (3.4) by x, we have:

$$xu_t - (xu_x)_x = xf(x,t) \tag{3.26}$$

As

$$u^N(x,t) = \sum_{k=1}^{N} C^N_k(t)\varphi_k(x), \quad u^N_t(x,t) = \sum_{k=1}^{N}\frac{d}{dt}C^N_k(t)\varphi_k(x)$$

Then u^N satisfies:

$$\begin{cases} \sum_{k=1}^{N}\frac{d}{dt}C^N_k(t)\int_0^l x\varphi_k\cdot\varphi_j dx + \sum_{k=1}^{N}C^N_k(t)\int_0^l x\frac{\partial\varphi_k}{\partial x}\cdot\frac{\partial\varphi_j}{\partial x}dx \\ = \int_0^l xf(x,t)\varphi_j dx, 0<t<T, j=\overline{1,N} \\ \sum_{k=1}^{N}C^N_k(0)\int_0^l x\varphi_k(x)\varphi_j(x)dx = \int_0^l x\varphi(x)\varphi_j(x)dx, j=\overline{1,N} \end{cases} \tag{3.27}$$

The relation (3.27) is a system of N ordinary differential equations of unknowns $C_k^N(t), (k = \overline{1,N})$ as $f \in L^2\left((0,T), H_\rho^1(0,l)\right)$, then the standard theory of ordinary differential equations implies the existence of functions $C_k^N(t)$ such that $u^N(x,t)$ defined by $\sum_{k=1}^N C_k^N(t)\varphi_k(x)$ satisfies (3.27).

From (3.27), we have:

$$\frac{d}{dt}\int_0^l xu^N(x,t)\varphi_j dx + \int_0^l x\frac{\partial}{\partial x}u^N(x,t)\frac{\partial \varphi_j}{\partial x}dx = \int_0^l xf\varphi_j dx, \text{ pour } j = \overline{1,N}$$

From this last inequality, for each $N \geq j$, for each $j \geq 1$, the function u satisfies:

$$\frac{d}{dt}\int_0^l xu\varphi_j dx + \int_0^l x\frac{\partial}{\partial x}u\frac{\partial \varphi_j}{\partial x}dx = \int_0^l xf\varphi_j dx, \text{ pour presque tout } t \in [0,T]$$

Consequently, for each $\eta(x) \in H_\rho^1(0,l)$:

$$\frac{d}{dt}\int_0^l xu\eta dx + \int_0^l xu_x\eta_x dx = \int_0^l xf\eta dx, \text{ pour presque tout } t \in [0,T]$$

That is

$$\int_0^l xu_t\eta dx + \int_0^l xu_x\eta_x dx = \int_0^l xf\eta dx, \text{ pour tout } t \in [0,T]$$

Therefore u is a weak solution of the posed problem.

Now let's return to the semilinear problem where $f \equiv |u|^{p-2}u$, and demonstrate the local existence of problem (3.1)-(3.3). Let's start with Lemma:

Lemma 3.4.1. *If $v \in H_\rho^1(0,l)$, and $2 < p < 3$ then $|v|^{p-2}v \in L_\rho^2(0,l)$.*

Proof 3.4.2. ***of Lemma 3.4.1*** *We have*

$$\begin{aligned}\int_0^l x\left(|v|^{p-2}v\right)^2 dx &= \int_0^l xv^{2p-2}dx \\ &= \int_0^l \left(xv^2\right)^{p-1}x^{2-p}dx \\ &\leq \int_0^l \left(\sup_{0\leq x\leq l} xv^2\right)^{p-1}x^{2-p}dx \\ &\leq \left(\sup_{0\leq x\leq l} xv^2\right)^{p-1}\int_0^l x^{2-p}dx\end{aligned} \tag{3.28}$$

*Applying **Lemma 2.4.1** from **Chapter 2**, (3.28) becomes:*

$$\begin{aligned}\int_0^l x\left(|v|^{p-2}v\right)^2 dx &\leq \left(c\int_0^l xv_x^2dx\right)^{p-1} \frac{1}{3-p}x^{3-p}\Big|_{x=0}^{x=l} \\ &= \frac{l^{3-p}}{3-p}\left(c\int_0^l xv_x^2dx\right)^{p-1} \\ &= \frac{c^{p-1}l^{3-p}}{3-p}\left(\int_0^l xv_x^2dx\right)^{p-1} \\ &\leq \frac{c^{p-1}l^{3-p}}{3-p}\left(\int_0^l xv^2dx+\int_0^l xv_x^2dx\right)^{p-1} \\ &= \frac{c^{p-1}l^{3-p}}{3-p}\left(\|v\|^2_{H^1_\rho(0,1)}\right)^{p-1}.\end{aligned} \tag{3.29}$$

Therefore $|v|^{p-2}v \in L^2_\rho(0,l)$.

3.5 Existence and uniqueness of the solution

Let's use a fixed point argument to demonstrate the local existence of the solution to problem (3.1)-(3.3).

Theorem 3.5.1. *Let* $\varphi \in H^1_\rho(0,l), 2 < p < 3$*. then (1.1)-(1.3), admits a unique weak solution in* $L^\infty\left((0,T^*), H^1_\rho(0,l)\right)$ *and* $u_t \in L^2\left((0,T^*), L^2_\rho(0,l)\right)$*, for all* $T^* < T$*.*

Proof 3.5.1. *Let* $B = L^\infty\left((0,T), H^1_\rho(0,l)\right)$ *and* $H = L^2\left((0,T), L^2_\rho(0,l)\right)$*, define for* $M > 0$*,* $T > 0$*, the set of functions* $K(M,T)$ *defined by:*

$$\begin{aligned}K &= K(M,T) \\ &= \left\{u \in B : N(u,u_t,T) = \|u_t\|^2_H + \|u\|^2_B \leq M^2, u(x,0) = \varphi(x)\right\}\end{aligned} \tag{3.30}$$

For all $v \in K(M,T)$*, we can solve the problem:*

$$\begin{cases} u_t - \frac{1}{x}(xu_x)_x = |v|^{p-2}v \quad \text{for} \quad x\in(0,l), \quad t \succ 0 \\ u(l,t) = 0, \quad t \geq 0 \\ u(x,0) = \varphi(x), \quad x \in (0,l). \end{cases} \tag{3.31}$$

*according to **Theorem 2.4.1** , the problem (2.31), admits a unique solution* $u \in B$ *for all* $v \in K(M,T)$*.*

We can define an operator h *where* $u = hv$ *which associates each* $v \in K(M,T)$*, with the solution*

of the linear problem (3.31). If we can demonstrate that h *admits a fixed point* u *in* $K(M,T)$*, then* u *is the solution to problem (3.1)-(3.3), demonstrating that* h *is a strictly contracting application. First, let's prove that* h *transforms* $K(M,T)$ *into itself,* $h : K \longrightarrow K$*.*
if we multiply equation (3.31) by xu_t*, we have:*

$$\int_0^t \int_0^l xu_t^2 dxds - \int_0^t \int_0^t (xu_x)_x u_t dxds = \int_0^l \int_0 xu_t |v|^{p-2} v dxds,$$

By integrating by parts, we get:

$$\int_0^t \int_0^l xu_t^2 dxds + \frac{1}{2} \int_0^l xu_x^2 \Big|_{s=0}^{s=\ell} dx = \int_0^t \int_0^l xu_t |v|^{p-2} v dxds,$$

Hence

$$2\int_0^t \int_0^l xu_t^2 dxds + \int_0^l xu_x^2(x,t)dx = \int_0^t x\varphi_x^2 dx + 2\int_0^t \int_0^l xu_t |v|^{p-2} v dxds$$

Applying Cauchy's inequality, it follows:

$$\begin{aligned} &2\int_0^t \int_0^l xu_t^2 dxds + \int_0^l xu_x^2(x,t)dx \\ &\leq \int_0^l x\varphi_x^2 dx + \int_0^t \int_0^l xu_t^2 dxds + \int_0^t \int_0^l x|v|^{2p-2} dxds \end{aligned}$$

And consequently

$$\begin{aligned} &\int_0^l xu_x^2(x,t)dx + \int_0^t \int_0^l xu_t^2 dxds \\ &\leq \int_0^l x\varphi_x^2 dx + \int_0^t \int_0^l x|v|^{2p-2} dxds \\ &\leq d\left(\|\varphi\|^2_{H^1_\rho(0,l)} + \int_0^t \int_0^l x|v|^{2p-2} dxds \right), \forall t \in [0,T]. \end{aligned} \tag{3.32}$$

As

$$\int_0^l x|v|^{2p-2} dx \leq \frac{C}{3-p} \|v\|^{2p-2}_{H^1_p(0,l)}, \tag{3.33}$$

Where $C = c^{p-1} l^{3-p}$*.*

Integrating (3.33) with respect to t from 0 to T, we have:

$$\begin{aligned}\int_0^T\int_0^l x|v|^{2p-2}dxdt &\leq \frac{C}{3-p}\int_0^T \|v\|^{2p-2}_{H^1_\rho(0,l)}dt\\ &\leq \frac{C}{3-p}\int_0^T \sup_{0<t<T}\|v\|^{2p-2}_{H^1_\rho(0,l)}dt\\ &= \frac{CT}{3-p}\sup_{0<t<T}\|v\|^{2p-2}_{H^1_\rho(0,l)}\\ &\leq \frac{CT}{3-p}\left(\sup_{0<t<T}\|v\|^{2p-2}_{H^1_\rho(0,l)}+\|v_t\|^{2p-2}_{L^2\left([0,T],L^2_\rho(0,l)\right)}\right)\\ &\leq \frac{CT}{3-p}M^{2p-2}.\end{aligned}$$

If we set $\frac{C}{3-p}=K_1$ then we have:

$$\int_0^T\int_0^l x|v|^{2p-2}dxdt \leq K_1TM^{2p-2}$$

And consequently (3.32) becomes:

$$\begin{aligned}&\sup_{0<t<T}\int_0^l xu_x^2(.,t)dx+\int_0^T\int_0^l xu_t^2dxdt\\ \leq&d\|\varphi\|^2_{H^1_\rho(0,l)}+\frac{dCT}{3-p}M^{2p-2},\\ =&A+dK_1TM^{2p-2},\end{aligned}\tag{3.34}$$

Where A is a constant dependent on the norm of φ in $H^1_\rho(0,l)$.

Let M be large enough, and choose T^* to be small enough, then from (3.34), it follows that:

$$\sup_{0<t<T^*}\int_0^l xu_x^2(.,t)dx+\int_0^{T*}\int_0^l xu_t^2dxdt \leq M^2$$

And consequently h transforms $K(M,T)$ into itself.

Now let's demonstrate that h is a contracting application for T^* small enough.

let $v_1, v_2 \in K(M,T)$ such that $u_1 = hv_1$ and $u_2 = hv_2$ are their images, then $W = u_1 - u_2$ Verifies:

$$\begin{cases} W_t-\frac{1}{x}\left(xW_x\right)_x=|v_1|^{p-2}v_1-|v_2|^{p-2}v_2, & x\in(0,l), \quad t>0\\ W(l,t)=0, & t\geq 0\\ W(x,0)=0, & x\in(0,l)\end{cases}\tag{3.35}$$

Multiplying (3.35) by $2xW_t$ *and integrating over* $(0,l)\times(0,t)$*, We have:*

$$2\int_0^t\int_0^l xW_t^2dxds - 2\int_0^t\int_0^l (xW_x)_x W_tdxds = 2\int_0^t\int_0^l xW_t\left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right)dxds$$

By integrating by parts, using Cauchy's inequality with ε*, we obtain:*

$$\begin{aligned}&2\int_0^t\int_0^l xW_t^2dxds + \int_0^l xW_x^2(.,t)dx\\ \leq&\int_0^t\int_0^l xW_t^2dxds + \int_0^l\int_0^l x\left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right)^2dxds\end{aligned}\tag{3.36}$$

Hence

$$\begin{aligned}&\int_0^l xW_x^2(.,t)dx + \int_0^t\int_0^l xW_t^2dxds\\ \leq&\int_0^t\int_0^l x\left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right)^2dxds.\end{aligned}\tag{3.37}$$

By estimating the right member of (3.37), it follows:

$$\int_0^l xW_x^2(.,t)dx + \int_0^t\int_0^l xW_t^2dxds \leq d\int_0^t\int_0^l x(v_1-v_2)^2\left(|v_1|^{2p-4} + |v_2|^{2p-4}\right)dxds$$

Where d *is a positive constant independent of* v_1, v_2 *and* t*.*

If we set $V = v_1 - v_2$ *it follows:*

$$\begin{aligned}&\int_0^l xW_x^2(.,t)dx + \int_0^t\int_0^l xW_t^2dxds\\ \leq& d\int_0^t\int_0^l xV^2\left(|v_1|^{2p-4} + |v_2|^{2p-4}\right)dxds\\ \leq& d\int_0^t\int_0^l\left(\sup_{0<x<l} xV^2\right)\left(|v_1|^{2p-4} + |v_2|^{2p-4}\right)dxds\\ \leq& d\int_0^t\left(\sup_{0<x<l} xV^2\right)\int_0^l\left(|v_1|^{2p-4} + |v_2|^{2p-4}\right)dxds\\ \leq& C\int_0^t\left(\int_0^l xV_x^2dx\right)\int_0^l\left(|v_1|^{2p-4} + |v_2|^{2p-4}\right)dxds.\end{aligned}\tag{3.38}$$

Now let's estimate $\int_0^l |v_1|^{2p-4}dx$ *et* $\int_0^l |v_2|^{2p-4}dx$*.*

We have

$$\begin{aligned}\int_0^l v_1^{2p-4}dx &= \int_0^l x^{p-2}v_1^{2p-4}x^{2-p}dx\\ &= \int_0^l \left(xv_1^2\right)^{p-2}x^{2-p}dx\\ &\leq \int_0^l \left(\sup_{0<x<l} xv_1^2\right)^{p-2}x^{2-p}dx\\ &\leq \left(c\int_0^l xv_{1x}^2dx\right)^{p-2}\int_0^l x^{2-p}dx\\ &\leq \left(c\int_0^l xv_{1x}^2dx\right)^{p-2}\cdot\frac{1}{3-p}l^{3-p}\\ &\leq CM^{2p-4}.\end{aligned}$$

Hence from (3.38), we have:

$$\int_0^l xW_x^2(.,t)dx + \int_0^t\int_0^l xW_t^2dxds \leq CM^{2p-4}\int_0^t\int_0^l xV_x^2dxds$$

Passing to the supremum with respect to t *From* 0 *To* T^**, we obtain:*

$$\begin{aligned}&\sup_{0\leq t\leq T^*}\int_0^l xW_x^2(.,t)dx + \int_0^{T^*}\int_0^l xW_t^2dxdt\\ \leq &CM^{2p-4}\int_0^{T*}\int_0^l xV_x^2(.,t)dxdt\\ \leq &CT^*M^{2p-4}\sup_{0\leq t\leq T^*}\int_0^l xV_x^2(.,t)dx\\ \leq &CT^*M^{2(p-2)}\left(\sup_{0\leq t\leq T^*}\int_0^l xV_x^2(.,t)dx + \int_0^{T*}\int_0^l xV_t^2dxdt\right),\end{aligned}$$

Therefore

$$N\left(W, W_t, T^*\right) \leq CT^*M^{2(p-2)}N\left(V, V_t, T^*\right).$$

If we choose T^* *to be small enough such that* $CT^*M^{2(p-2)} < 1$*, then* h *is a contracting application from* $K(M,T)$ *into itself. And consequently, the theorem of the contracting application, (Banach fixed point theorem), asserts the existence of a unique fixed point* u *which is the desired solution of problem (3,1)-(3.3).*

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