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Classical Numbers defined by Generating functions

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بسم الله الرحمن الرحيم

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شكر وتقدير

الحمد لله الذي وهبني عقلا مفكرا, و لسانا ناطقا و أنار دربي, ويسر أمري لانهاء هذا العمل, و الصلاة و السلام على رسول الله صلى الله عليه وسلم.

فألهم لك الحمد و الشكر في الأولى و لك الحمد و الشكر في الاخرة و لك الحمد و الشكر من قبل و لك الحمد و الشكر من بعد و في كل حين و دائما و أبدا.

أتقدم بأسمى عبارات الشكر و التقدير الى كل من علمني حرفا و كل من أنار دربي الى كل منعلمني علما به انتفع و ادبا به ارتفع.

شكر خاص للأستاذ المشرف "صايب عبد الصادق" الذي أفادني بنصائحه و توجيهاته طيلة انجاز هذه المذكرة

كما أشكر أعضاء لجنة المناقشة التي شرفتني بقبولها مناقشة مذكرتي, أستاذنا القدير "بوعلي الطاهر" رئيسا و الأستاذ "دقايشية نوار" ممتحنا.

وفي الأخير أشكر كل من قدم لي يد العون و المساعدة سواء من قريب أو من بعيد و لو بكلمة طيبة او توجيه أو حتى بدعوة في ظهر الغيب لهم جزيل الشكر و العرفان و لكم مني فائق التقدير و الاحترام.

اهداء

أهدي ثمرة هذا العمل المتواضع الى أعذب كلمتين في الوجود وأسمى لفظين نطق بهما لساني الى من كانا سر وجودي و سبب بلوغي في هذه المرحلة والدي حفضهما الله و أطال في عمرهما. الى من قال فيهم الله عز وجل: **"و قل ربي ارحمهما كما ربياني صغيرا"** .

الى نور قلبي و ابتسامة حياتي, الى منبع الحنان التي سهرت لأجلي وضحت بالكثير حتى أبلغ هذه الدرجة, الى الشمس التي تنير حياتنا و ما كنا الا قمرا يعكس ضيائها الى" **أمي الغالية**" .

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الى اخوتي "**هيثم**" , "**وسيم**" و أختي الصغيرة

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الى كل من ساعدنا في اتمام هذه المذكرة :" **اكرام**" ," **ايمن**"

اسال الله لكم كل التوفيق و النجاح في مشواركم العلمي و المهني.

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Contents

Abstract

The main object of this dissertation is to investigate some classical numbers using the corresponding generating functions. For each sequence included in this dissertation, we shall provide an overview about its construction as well as some of its applications. Among classical numbers, we proposed Fibonacci, Lucas, Pell, Bell, Harmonic, Stirling both the first and second kind, ... among others.

Introduction

The solutions of problems in mathematics are sets of elements (finite or not), i.e. a collections of objects satisfying some specific properties. For instance, given a sequence of numbers, $\{a_n\}_n$, we can associate a formal power series $f(x)$ whose coefficients give the above sequence, i.e.

$$f(x) = \sum_{n\geqslant 0} a_n x^n.$$

The function f is called the generating function of the sequence $\{a_n\}_n$.
The generating function is a powerful mathematical tools used to represent and generate sequences, i.e. this function is represented as a power series expansion
Integer sequences appear in an amazingly wide range of subject areas besides discrete mathematics, including biology, engineering, chemistry, and physics, as well as in puzzles. An amazing database of different integer sequences can be found in the On-Line Encyclopedia of Integer Sequences.
Some counting problems can be solved by finding a closed form for the function that represents the problem and then manipulating the closed form to find the relevant coefficient. Among these functions, the most relevant are those satisfying recurrence relations in which the construction of the corresponding generating functions is more or less easy reached.
In probability, the most important use of generating functions is to understand moments of random variables and find explicitly either the random variables or their linear combinations, among others.
Each way to write a positive integer n as a sum of positive integers is called a partition of n. By introducing some enumerative combinatorics as a generalization of combinatorial notions using some special functions, they appeared a numbers of interesting sequence of numbers hidden inside such as the Stirling numbers both of the first and the second kind, Bell, Harmonic, Bernoulli ... and more. Moreover, the discrete version as well as q-analogue (q-numbers) provide many generalizations and give simplifications in counting objects mainly in partitions.
In this dissertation, we investigate properties of generation functions as well as the algebraic operations, i.e. addition, multiplication, differentiation, integration, shift and inversion.
In Chapter 2, we shall provide various type of classical numbers defined in terms of linear recurrence relations, then we construct either the ordinary generating functions or the

exponential generating functions. In case of higher order, we show that some specific sequences can be obtained as convolutions of some simple classical numbers.

In Chapter 3, we focus more or less in combinatorics of objects and discuss the construction of generating functions of the sequence of numbers. In particular, some illustrative graph are provided.

A conclusion is given at the end of the chapter 3, which explain further ideas and the relationship between the selection of objects and the coefficients of x^k in the generating functions.

Chapter 1

Preliminary

The first chapter provides different types of generating functions, the properties of the algebraic operations as well as the difference of their effect on different types. Illustrative example are provided.

$$a_r = \sum_s \binom{r}{s} b_s \quad \Longrightarrow \quad b_n = \sum_m \binom{n}{m} (-1)^{n-m} a_m.$$

1.1 Generating functions

A given infinite sequence $a_0, a_1, \ldots,$ can often be represented in a more compact form or in terms of itself, i.e. recursively. It can also be given with the help of other explicitly known sequences, among other. For instance, the generating function is a representation of an infinite sequence of numbers as the coefficients of a formal power series as

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

Generating functions are a powerful tool that allows us to encode an infinite sequence of numbers into a single function. They're used to study sequences of numbers in a systematic way, allowing us to perform operations on the sequences more easily.
Let us begin with the following sequence of positive integers

$$1,\ 1,\ 1,\ 1,\ 1, \ldots$$

Therefore, the corresponding generating function is

$$g(x) = 1 + x + x^2 + x^3 + \ldots$$

The explicit form of above generating function is well known because it's just a geometric series with a common ration x. However, the building method is as follows

$$\begin{array}{rll} g(x) & = 1+ & x + x^2 + x^3 + \ldots \\ xg(x) & = & x + x^2 + x^3 + x^4 + \ldots \\ \hline (1-x)g(x) & = 1 & \end{array}$$

Hence, we have the closed form of $g(x)$, i.e.

$$g(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{1.1}$$

Another interesting sequence is when $a_k = \binom{n}{k}$ for $0 \leqslant k \leqslant n$. It merely seen that

$$\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = (1+x)^n. \tag{1.2}$$

Once we obtained a closed expression of the generating function, we can use it to generate further sequences. Indeed, if we replace x by $-x$ in $g(x)$ we get the generating function of the sequence $1, -1, 1, -1, 1, \ldots$ as follows

$$g(-x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

This idea allows us to generate a plenty of sequences just by replacing x.
Unfortunately, we cannot, for example, give anything to x in $g(x)$ to generate the sequence $7,7,7,\ldots$. However, we remark that the latter sequence is just $7g(x)$. This leads to think about elementary operations on generating functions as well as on power series. Notice further that the sequence of numbers

$$7, \frac{7}{2}, \frac{7}{3!}, \frac{7}{4!}, \ldots \tag{1.3}$$

cannot be connected to $g(x)$ by any elementary operations. However, if we associate the latter sequence with the following generating function

$$\mathrm{G}(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} = a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots$$

In this case, we merely deduce that the generating function of the sequence of numbers (1.3) is $7\mathrm{G}(x) = 7e^x$. Now the function $\mathrm{G}(x)$ is referred to as the exponential generating function while $g(x)$ will be called the ordinary generating function. It is worthwhile to notice that there are various type of generating functions. Besides the exponential and the ordinary generating function, we quote for instance Poisson generating function, Dirichlet generating function, Bell series, Lambert series, among others.

1.2 Operations on generating functions

In this section we deal with ordinary generating functions, but the operations could be simply applied to other types of generating functions.
Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$ and $v(z) = \sum_{n=0}^{\infty} b_n z^n$ be two ordinary generating functions. Next, we shall define some algebraic operations on ordinary generating functions as follows

Addition

The sum of $u(z)$ and $v(z)$ is denoted by $u(z) + v(z)$ and is defined by

$$u(z) + v(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

Multiplication by a scalar

Multiplication of $u(z)$ by a scalar λ is denoted by $\lambda u(z)$ and is defined by multiplying its coefficient by this scalar, i.e. the multiplication of $u(z)$ by a scalar λ is the generating function of the sequence $\{\lambda a_n\}$

$$\lambda u(z) = \sum_{n=0}^{\infty} \lambda a_n z^n$$

Example 1.1 *From above we have*

$$\begin{array}{rlll} & \sum_{k=0}^{\infty} x^k & = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \\ + & \sum_{k=0}^{\infty} (-1)^k x^k & = 1 - x + x^2 - x^3 + x^4 + \cdots = \frac{1}{1+x} \\ \hline & \sum_{k=0}^{\infty} \left(1 + (-1)^k\right) x^k = \sum_{k=0}^{\infty} 2x^{2k} & = 2 + 2x^2 + 2x^4 + \cdots = \frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2} \end{array}$$

Convolution

The product of $u(z)$ and $v(z)$ denoted by $u(z)v(z)$, generates the sequence $\{c_n\}$ given by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, i.e.

$$\begin{aligned} u(z)v(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n = \sum_{n=0}^{\infty} c_n z^n \\ &= a_0 b_0 + \left(a_0 b_1 + a_1 b_0\right) z + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right) z^2 + \ldots \end{aligned}$$

While in case of the exponential generating function, the product $u(z)v(z)$ generates the sequence $d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, i.e.

$$\begin{aligned} u(z)v(z) &= \left(\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \right) \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} z^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) z^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} n! \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{z^n}{n!} \end{aligned}$$

The sequences $\{c_n\}$ and $\{d_n\}$ are called the Cauchy product and the binomial convolution of the sequences a_n and b_n, respectively.

We can evaluate the product $u(z)v(z)$ by using a table to identify all the cross-terms from the product of the sums

	$b_0 z^0$	$b_1 z^1$	$b_2 z^2$	$b_3 z^3$	...
$a_0 z^0$	$a_0 b_0 z^0$	$a_0 b_1 z^1$	$a_0 b_2 z^2$	$a_0 b_3 z^3$	...
$a_1 z^1$	$a_1 b_0 z^1$	$a_1 b_1 z^2$	$a_1 b_2 z^3$	...	
$a_2 z^2$	$a_2 b_0 z^2$	$a_2 b_1 z^3$	...		
$a_3 z^3$	$a_3 b_0 z^3$	...			

Example 1.2 (Vandermonde's Identity) *For all $m, n, r \in \mathbb{N}$*

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \ldots \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}. \tag{1.4}$$

Expanding the expressions on both sides from the identity

$$(1+x)^{n+m} = (1+x)^n (1+x)^m,$$

we have from (1.2) that

$$\sum_{k=0}^{n+m} \binom{n+m}{k} x^k = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{m} \binom{m}{j} x^j\right)$$
$$= \binom{n}{0}\binom{m}{0} x^0 + \left\{\binom{n}{0}\binom{m}{1} + \binom{n}{1}\binom{m}{0}\right\} x$$
$$+ \left\{\binom{n}{0}\binom{m}{2} + \binom{n}{1}\binom{m}{1} + \binom{n}{0}\binom{m}{2}\right\} x^2 + \cdots + \binom{n}{n}\binom{m}{m} x^{n+m}.$$

Now comparing the coefficients of x^r on both sides yields the result.

Remark 1.1 *If we take $m = n = r$ in identity (1.4) we obtain*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

Now by using the identity

$$\binom{n}{k} = \binom{n}{n-k}$$

we deduce the following identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Differentiation and Integration

The first derivative of the ordinary generating function $u(z)$ gives

$$u'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \quad \Longrightarrow \quad z u'(z) = \sum_{n=0}^{\infty} n a_n z^n$$

While the Integration on $[0, z]$ gives

$$\int_0^z u(t)\, dt = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n$$

Example 1.3 *By taking the derivative of both sides with respect to x and making a change* $n \to n+1$ *we obtain from (1.1) that*

$$\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}.$$

If you take the third power of (1.1) or take the second derivative of both sides you obtain

$$\sum_{n=0}^{\infty} \binom{n+2}{2} z^n = \frac{1}{(1-z)^3}.$$

Example 1.4 *Let us back to the generating function (1.1) and replace x by* x^2 *to get*

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{1+x}\right).$$

Integrating both sides from $[0,x]$ *we deduce that*

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = -\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh(x).$$

It is worthwhile to notice that the derivative of an exponential generating function $\mathrm{U}(z)$ would gives

$$\mathrm{U}'(z) = \sum_{n=0}^{\infty} a_{n+1} \frac{z^n}{n!} \quad \Longrightarrow \quad \mathrm{U}^{(r)}(z) = \sum_{n=0}^{\infty} a_{n+r} \frac{z^n}{n!}$$

Shifting

For $r \geqslant 1$, the generating function of the sequences $\{a_{n+r}\}_{n\geqslant 0}$ and $\{a_{n-r}\}_{n\geqslant r}$ are

$$\sum_{n=r}^{\infty} a_{n-r} z^n = z^r g(z),$$

$$\sum_{n=0}^{\infty} a_{n+r} z^n = \frac{g(z) - a_0 - z a_1 - \cdots - a_{r-1} z^{r-1}}{z^r}.$$

Therefore, if we combine the shift with the derivative, we merely obtain ($r = 0$ gives $g(z)$)

$$z^r g^{(r)}(z) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-r+1) a_n z^n. \tag{1.5}$$

Example 1.5 *From (1.1), we merely deduce that*

$$\sum_{n=0}^{\infty} \binom{n+r}{r} = \frac{1}{(1-x)^{r+1}}.$$

On the other hand, since

$$6\binom{n+3}{3} - 12\binom{n+2}{2} + 6\binom{n+1}{1} = n^3 - n$$

it then follows that

$$\sum_{n=0}^{\infty}(n^3-n)x^n = 6\sum_{n=0}^{\infty}\binom{n+3}{3}x^n - 12\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + 6\sum_{n=0}^{\infty}(n+1)x^n$$

Therefore, the generating function of the sequence $\{n(n^2-1)\}$ *is given by*

$$\sum_{n=0}^{\infty}(n^3-n)x^n = \frac{6}{(1-x)^4} - \frac{12}{(1-x)^3} + \frac{6}{(1-x)^2} = \frac{6x^2}{(1-x)^4}.$$

Inverse of a power series

The power series $\sum_{n=0}^{\infty} b_n z^n$ is said to be the inverse of the power series $\sum_{n=0}^{\infty} a_n z^n$ if:

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = 1$$

Proposition 1.1 *A power series* $\sum_{n=0}^{\infty} a_n z^n$ *is invertible if and only if* $a_0 \neq 0$.

Proof. Let $\sum_{n=0}^{\infty} b_n z^n$ be the inverse of the power series $\sum_{n=0}^{\infty} a_n z^n$ such that:

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = 1$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = 1$$

$$a_0 b_0 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = 1$$

By identification, we find:

$$a_0 b_0 = 1$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = 0$$

which gives the non-zero coefficient a_0.

Conversely, if a_0 is non-zero, then the triangular system of equations

$$\begin{aligned}
&a_0 b_0 = 1\\
&a_1 b_0 + a_0 b_1 = 0\\
&\vdots\\
&a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n = 0
\end{aligned}$$

has a unique solution. ■

Example 1.6 *We can merely check that*

1. *The series* $\sum_{n=0}^{\infty} z^n$ *is invertible and its inverse is* $1 - z$.
2. *We have the following*

$$a_r = \sum_s \binom{r}{s} b_s \quad \Longrightarrow \quad b_n = \sum_m \binom{n}{m} (-1)^{n-m} a_m.$$

Chapter 2

Generating functions from recurrence relations

In this chapter, we shall investigate some classical numbers defined in terms of recurrence relations of order two, three, and higher order. Some recurrence relations of higher are a convolution of other recurrence relations of less order. Furthermore, combinatorial interpretations in some cases are available.

$$
\begin{aligned}
\frac{1}{89} = \quad & 0.01 \\
& +0.001 \\
& +0.0002 \\
& +0.00003 \\
& +0.000005 \\
& +0.0000008 \\
& +0.00000013 \\
& +0.000000021 \\
& +0.0000000034 \\
& +0.00000000055 \\
& +0.000000000089 \\
& +0.0000000000144 \\
& +0.00000000000233 \\
& +0.000000000000377 \\
& +0.0000000000000610 \\
& +0.00000000000000987 \\
& + \quad \vdots \quad \text{Fibonacci numbers}
\end{aligned}
$$

2.1 Generating function of recurrence relation

In counting problems, it may be difficult to find the solution directly. However, it is often possible to express the n^{th} number in terms of the previous numbers in the sequence of solution. We call this interdependence a "recurrence relation" and the sequence may be expressed recursively using the previous numbers. For instance

1 The sequence of numbers $\{a_n\}$ defined recursively by

$$a_n = ca_{n-1}$$

involves a constant sequence, i.e. $a_n = c^n a_0$, $n \geqslant 1$. Therefore, if we want to find the corresponding ordinary generating function, we proceed as follows

$$a_n x^n = ca_{n-1}x^n \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n x^n = c\sum_{n=1}^{\infty} a_{n-1}x^n$$

If we denote by $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=0}^{\infty} a_n x^n - a_0 = cx\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = cx\sum_{n=0}^{\infty} a_n x^n \quad \Longleftrightarrow \quad f(x) - a_0 = cxf(x)$$

from which we obtain the expression

$$f(x) = \frac{a_0}{1-cx}.$$

2 Let us consider the sequence of numbers $\{b_n\}$ defined recursively by

$$b_{n+1} - 4b_n = 5^n, \quad n \geqslant 0, \qquad b_0 = 1.$$

We shall obtain its generating function $h(x)$ using the above recurrence relation as follows

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_{n+1}x^{n+1} - 4\sum_{n=0}^{\infty} b_n x^{n+1} = \sum_{n=0}^{\infty} 5^n x^{n+1}\\
\Longleftrightarrow \quad &\sum_{n=0}^{\infty} b_{n+1}x^{n+1} - 4x\sum_{n=0}^{\infty} b_n x^n = x\sum_{n=0}^{\infty} (5x)^n\\
\Longleftrightarrow \quad &\big(h(x) - b_0\big) - 4xh(x) = \frac{x}{1-5x}\\
\Longleftrightarrow \quad &h(x)\big(1-4x\big) = \frac{x}{1-5x} + 1 = \frac{1-4x}{1-5x}\\
\Longrightarrow \quad &h(x) = \frac{1}{1-5x} \quad \Longrightarrow \quad b_n = 5^n, \quad n \geqslant 0.
\end{aligned}$$

2.2 Recurrence relations of order 2

As shown above, if the sequence of numbers satisfying some kind of recurrence relations, then the generating function could be, more or less given in a closed form. Among classical numbers, there are a numerous families satisfying recurrence relations such as Fibonacci, Lucas, Pell, Tribonacci, Padovan, among others. We shall next select some classical numbers satisfying a second order recurrence relation. To begin with, let us first provide an overview of each chosen sequence.

Fibonacci sequence

Fibonacci numbers are the following positive integers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

in which each number is the sum of the two previous numbers. Therefore, if we denote these numbers F_n, we obtain the following recurrence

$$F_{n+1} = F_n + F_{n-1}, \quad F_1 = 1, \quad F_0 = 0.$$

Let $F(x)$ be the generating function for this sequence of numbers, then

$$\begin{aligned} F(x) - F_0 - F_1 x = F(x) - x &= \sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})\, x^n \\ &= x \sum_{n=0}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \end{aligned}$$

accordingly,

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left[\frac{1}{1-\phi x} - \frac{1}{1-\overline{\phi} x} \right] = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\phi^n - \overline{\phi}^n \right] x^n \tag{2.1}$$

In particular, F_n is given explicitly by

$$F_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{2.2}$$

p-fibonacci numbers

For every integer $p > 0$, the p-Fibonacci sequence denoted throughout $\{F_n(p)\}_{n\in\mathbb{N}}$, is defined in terms of the following recurrence relation

$$F_{n+1}(p) = pF_n(p) + F_{n-1}(p), \quad \text{with} \quad F_0(p) = 0,\ F_1(p) = 1. \tag{2.3}$$

The first few p-Fibonacci numbers are $\{0, 1, p, p^2+1, p^3+2p, \dots\}$. Notice that for a different given value of p will reduce the above sequence to other known families. In particular, for p = 1 the p-Fibonacci reduces to the standard Fibonacci numbers. While the case p=2 involves the recurrence relation of Pell numbers. Since the initial condition coincide with those of standard Fibonacci, then following the same process we merely obtain the ordinary generating function of p-Fibonacci numbers

$$\tilde{\mathrm{F}}(x) = \sum_{n=0}^{\infty} \mathrm{F}_n(p)x^n = \frac{x}{1-px-x^2}. \tag{2.4}$$

p-Lucas numbers

p-Lucas numbers are defined by the recurrence relation (2.3) with the initial conditions $\mathrm{L}_0(p) = 2$ and $\mathrm{L}_1(p) = p$. As customary, for $p = 1$ the latter numbers gives the classical Lucas numbers while the case $p = 2$ is called in the literature the Lucas-Pell numbers. Now, taking in account the initial conditions, their generating function will be

$$\mathrm{L}(x) = \sum_{n=0}^{\infty} \mathrm{L}_n(p)x^n = \frac{2-px}{1-px-x^2}.$$

General second order recurrence relations

In order to unify the study of every sequence of numbers satisfying a second order recurrence relation, we shall take the following general definition. Let the sequence of numbers E_n defined by the following recurrence relation

$$\mathrm{E}_{n+1} = p\mathrm{E}_n + q\mathrm{E}_{n-1}, \quad \mathrm{E}_1 = b, \quad \mathrm{E}_0 = a. \tag{2.5}$$

The Table 2.1 below provides recurrence relations for certain classical numbers such as p-Fibonacci, p-Lucas, p-Pell, p-Jacobsthal, p-Mersenne, ... among others.

Sequence of numbers $\{\mathrm{E}_n\}$	E_0	E_1	p	q
p-Fibobacci sequence $\mathrm{F}_n(p)$	0	1	p	1
p-Lucas sequence $\mathrm{L}_n(p)$	2	p	p	1
p-Pell sequence $\mathrm{P}_n(p)$	0	1	2	p
p-Jacobsthal sequence $\mathrm{J}_n(p)$	0	1	p	2
p-Mersenne sequence $\mathrm{M}_n(p)$	0	1	$3p$	-2

Table 2.1: Recurrence relations for some specific numbers

2.2.1 The odd and the even p-numbers

From the above definition (2.5), we have

$$\begin{aligned}E_{2n+1} &= pE_{2n} + qE_{2n-1} = p\left(pE_{2n-1} + qE_{2n-2}\right) + qE_{2n-1}\\ &= \left(p^2+q\right)E_{2n-1} + q\left(pE_{2n-2}\right) = \left(p^2+q\right)E_{2n-1} + q\left(E_{2n-1} - qE_{2n-3}\right)\end{aligned}$$

hence, by induction we can prove the following formulas

$$E_{2n+1} = (p^2+2q)E_{2n-1} - q^2E_{2n-3}, \tag{2.6}$$

$$E_{2n} = (p^2+2q)E_{2n-2} - q^2E_{2n-4}. \tag{2.7}$$

Therefore, if $\tilde{E}_o(x)$ and $\tilde{E}_e(x)$ are the generating function of the odd p-number and even p-numbers, respectively, defined by (2.5), then we have from (2.6) and (2.7) the following

$$\begin{aligned}\left\{1-\left(p^2+2q\right)x+(qx)^2\right\}\tilde{E}_o(x) &= E_1 + E_3x - \left(p^2+2q\right)E_1x\\ &= E_1 + \left(\left(p^2+q\right)E_1 + pqE_0\right)x - \left(p^2+2q\right)E_1x\\ &= E_1 + q\left(pE_0 - E_1\right)x\\ \left\{1-\left(p^2+2q\right)x+(qx)^2\right\}\tilde{E}_e(x) &= E_0 + E_2x - \left(p^2+2q\right)E_0x\\ &= E_0 + \left(pE_1 - \left(p^2+q\right)E_0\right)x,\end{aligned}$$

from which we obtain, taking into account the initial conditions, explicitly the generating functions

$$\tilde{E}_o(x) = \frac{b+q\left(ap-b\right)x}{1-(p^2+2)x+(qx)^2}, \qquad \tilde{E}_e(x) = \frac{a+\left(bp-a\left(p^2+q\right)\right)x}{1-(p^2+2)x+(qx)^2}. \tag{2.8}$$

2.2.2 Generating function of some convolution p-numbers

In order to use the properties of the convolution, we shall prove first some identities. For the sake of simplicity we shall denote the p-Fibonacci and the p-Lucas by $F_n := F_n(p)$ and $L_n := L_n(p)$, respectively.

Proposition 2.1 *The p-Fibonacci and p-Lucas numbers defined above satisfying the following identities*

$$F_nF_m = \frac{1}{p^2+4}\left(L_{n+m} - (-1)^mL_{n-m}\right), \tag{2.9}$$

$$L_nL_m = L_{n+m} + (-1)^mL_{n-m}. \tag{2.10}$$

Proof. A simple proof of these identities can be done using the characteristic equation, i.e. the representation of the numbers in terms of the solution of the characteristic equation as in standard case (2.2). For the p-numbers, the characteristic equation is the following

$$0 = x^2 - px + 1 = (x - \sigma_1)(x - \sigma_2), \qquad \sigma_{1,2} = \frac{p \pm \sqrt{p^2+4}}{2}.$$

Taking into account the initial conditions, we have the following representations

$$\mathrm{F}_n = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}, \qquad \mathrm{L}_n = \sigma_1^n + \sigma_2^n. \tag{2.11}$$

Let us remark, by taking $\sigma_1 > 0$, $\sigma_2 < 0$, that we have

$$\sigma_1\sigma_2 = -1, \quad \sigma_1 + \sigma_2 = p, \quad \sigma_1 - \sigma_2 = \sqrt{p^2+4}, \quad \sigma^2 = p\sigma + 1.$$

Therefore, the left hand side of (2.9) will be

$$\begin{aligned}
\mathrm{F}_n\mathrm{F}_m &= \frac{1}{p^2+4}\left(\sigma_1^n - \sigma_2^n\right)\left(\sigma_1^m - \sigma_2^m\right) = \frac{1}{p^2+4}\left(\sigma_1^{n+m} + \sigma_2^{n+m} - \left(\sigma_1^n\sigma_2^m + \sigma_1^m\sigma_2^n\right)\right) \\
&= \frac{1}{p^2+4}\left(\mathrm{L}_{n+m} - \left(\sigma_1^{n-m+m}\sigma_2^m + \sigma_1^m\sigma_2^{n-m+m}\right)\right) \\
&= \frac{1}{p^2+4}\left(\mathrm{L}_{n+m} - (\sigma_1\sigma_2)^n\left(\sigma_1^{n-m} + \sigma_2^{n-m}\right)\right) = \frac{1}{p^2+4}\left(\mathrm{L}_{n+m} - (-1)^n\mathrm{L}_{n-m}\right).
\end{aligned}$$

Now the second identity for p-Lucas numbers is clear. ∎

Some interesting identities could be easily obtained from (2.9)-(2.10). For instance, since $\mathrm{F}_1 = 1$, remark then that for $m = 1$ the identity (2.9) reduces to the following

$$\mathrm{F}_n = \frac{1}{p^2+4}\left(\mathrm{L}_{n+1} + \mathrm{L}_{n-1}\right). \tag{2.12}$$

While the choice $n = k+1$ and $m = k$ transforms the identities to the following

$$\mathrm{F}_k\mathrm{F}_{k+1} = \frac{1}{p^2+4}\left(\mathrm{L}_{2k+1} - p(-1)^k\right), \tag{2.13}$$

$$\mathrm{L}_k\mathrm{L}_{k+1} = \mathrm{L}_{2k+1} + p(-1)^k. \tag{2.14}$$

From another hand, we merely obtain that

$$\mathrm{F}_k^2 = \frac{1}{p^2+4}\left(\mathrm{L}_{2k} - 2(-1)^k\right), \qquad \mathrm{L}_k^2 = \mathrm{L}_{2k} + 2(-1)^k. \tag{2.15}$$

$$\mathrm{F}_k\mathrm{L}_k = \mathrm{F}_{2k}. \tag{2.16}$$

Let us now denote by $\mathrm{E}_1(x)$ and $\mathrm{E}_2(x)$ the ordinary generating functions of the sequence $\{\mathrm{F}_n(p)\mathrm{F}_{n+1}(p)\}_{n\in\mathbb{N}}$ and $\{\mathrm{L}_n(p)\mathrm{L}_{n+1}(p)\}_{n\in\mathbb{N}}$, respectively. Then

Theorem 2.1 *The generating functions* $E_1(x)$ *and* $E_2(x)$ *are given explicitly as follows*

$$E_1(x) = \frac{px}{1-(p^2+1)(x+x^2)+x^3}, \qquad E_2(x) = \frac{p\left(2-p^2x+2x^2\right)}{1-(p^2+1)(x+x^2)+x^3}.$$

Proof. For $E_1(x)$ we shall use (2.13) together with (2.8). From one hand, we know that the ordinary generating function of the sequence $\left\{(-1)^n\right\}$ is $\frac{1}{1+x}$. From another hand, taking into account the initial conditions, the generating function of the odd p-Lucas numbers together with (2.13) involve

$$\begin{aligned} E_1(x) &= \frac{1}{p^2+4}\left(\frac{p(1+x)}{1-(p^2+2)x+x^2} - \frac{p}{1+x}\right) \\ &= \frac{1}{p^2+4}\left(\frac{p(1+x)^2 - p(1-(p^2+2)x+x^2)}{(1-(p^2+2)x+x^2)(1+x)}\right) \\ &= \frac{px}{1-(p^2+1)(x+x^2)+x^3}. \end{aligned}$$

In the same way, the ordinary generating function $E_2(x)$ could be checked. ∎

Next, we shall denote by $E_3(x)$, $E_4(x)$ and $E_5(x)$ the ordinary generating functions of the sequence of numbers $\left\{F_n^2(p)\right\}_{n\in\mathbb{N}}$, $\left\{L_n^2(p)\right\}_{n\in\mathbb{N}}$ and $\left\{F_n(p)L_n(p)\right\}_{n\in\mathbb{N}}$, respectively. Then

Theorem 2.2 *The generating functions* $E_3(x)$, $E_4(x)$ *and* $E_5(x)$ *are given explicitly as follows*

$$\begin{aligned} E_3(x) &= \frac{x-x^2}{1-(p^2+1)(x+x^2)+x^3}, \\ E_4(x) &= \frac{4-\left(4+3p^2\right)x-p^2x^2}{1-(p^2+1)(x+x^2)+x^3}, \\ E_5(x) &= \frac{px}{1-(p^2+1)(x+x^2)+x^3}. \end{aligned}$$

Proof. The generating functions are a direct calculations from (2.15)-(2.16) together with (2.8) after replacing the initial conditions at the latter identity. ∎

2.2.3 Inhomogeneous recurrence relations

Some interesting inhomogeneous recurrence relation of order 2 worth to be mentioned. The first one we want to invoke is the so called Leonardo sequence $\{D_n\}$ defined in terms of the following recurrence [2]

$$D_n = D_{n-1} + D_{n-2} + 1, \quad n \geqslant 2. \tag{2.17}$$

with initial conditions $D_0 = D_1 = 1$.

As each sequence, Leonardo numbers have their own properties. We shall provide some of their amazing properties. First, by induction we can prove in few lines that

Proposition 2.2 *For $n \geqslant 0$, the D_n is an odd number.*

Unexpected properties is that Leonardo and Fibonacci numbers are expressed in each other as follows

Proposition 2.3 *For $n \geqslant 0$, the Leonardo numbers D_n are given in terms of Fibonacci numbers F_n via*

$$\mathrm{D}_n = 2\mathrm{F}_{n+1} - 1. \tag{2.18}$$

Proof. For $n = 0$ and $n = 1$ the identity (2.18) is true according to the initial conditions of (2.17) . By induction, assume that (2.18) is true up to n. Therefore, from (2.17)

$$\begin{aligned}\mathrm{D}_{n+1} &= \mathrm{D}_n + \mathrm{D}_{n-1} + 1 = \left(2\mathrm{F}_{n+1} - 1\right) + \left(2\mathrm{F}_n - 1\right) + 1\\ &= 2\left(\mathrm{F}_{n+1} + \mathrm{F}_n\right) - 1 = 2\mathrm{F}_{n+2} - 1.\end{aligned}$$

Whence the result. ■

The ordinary generating function of Leonardo numbers can be easily calculated using the initial conditions. Indeed, let us denote the latter by $\mathrm{GL}(x)$, then we have

$$\mathrm{GL}(x) = \sum_{n=0}^{\infty} \mathrm{D}_n x^n = \frac{1 - x + x^2}{1 - 2x + x^3}.$$

Since the recurrence relation (2.17) is inhomogeneous, then substituting n by $n+1$ and substracting the resulting equality from (2.17) we infer that

$$\mathrm{D}_{n+1} = 2\mathrm{D}_n - \mathrm{D}_{n-2}, \quad n \geqslant 2.$$

The latter identity shows that Leonardo numbers satisfy a recurrence relation of order 3, this will be the main objective of the next section. Before moving to the next section, we shall mention further generalizations of Leonardo numbers. In [6] the authors propose the following generalization of Leonardo numbers, which we call p-Leonardo and denoted by $\{\mathscr{D}_n\}$. Again, these numbers are defined in terms of the following inhomegeneous second order recurrence relation

$$\mathscr{D}_n = \mathscr{D}_{n-1} + \mathscr{D}_{n-2} + p, \quad n \geqslant 2. \tag{2.19}$$

with initial conditions $\mathscr{D}_0 = \mathscr{D}_1 = 1$.

At first sight, the connection between p-Leonardo and Fibonacci constitutes a generalization of (2.18) and can be again proved by induction in few lines

Proposition 2.4 *For $n \geqslant 0$, the p-Leonardo numbers $\mathscr{D}_n$ are given in terms of Fibonacci numbers F_n via*

$$\mathscr{D}_n = \left(p+1\right)\mathrm{F}_{n+1} - p. \tag{2.20}$$

2.3 Recurrence relations of higher order

A linear homogeneous recurrence relation of order k with constant coefficients is a recurrence of the form

$$u_n + d_1 u_{n-1} + d_2 u_{n-2} + \ldots + d_k u_{n-k} = 0, \qquad d_k \neq 0.$$

Remark that if $u_n = z^n$ is a solution of the equation with $u_n \neq 0$ and verifying

$$z^k + d_1 z^{k-1} + d_2 z^{k-2} + \ldots + d_k z^{n-k} = 0.$$

In particular, if $n = k$, we find the latter equation is the characteristic equation, i.e.

$$z^k + d_1 z^{k-1} + d_2 z^{k-2} + \ldots + d_k = 0.$$

2.3.1 Recurrence relation of order 3

Next we shall give illustrative examples for third order recurrence relation.

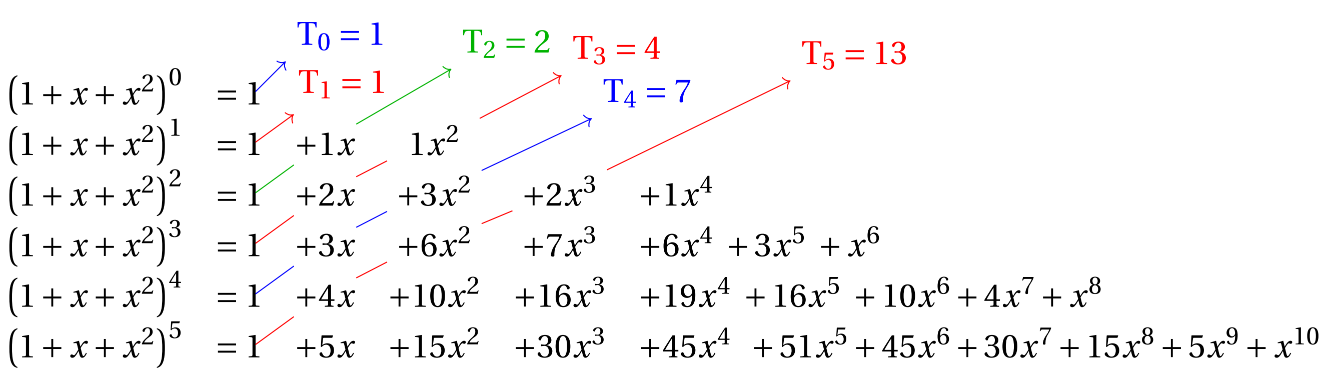
1 Let us begin with the so-called the Tribonacci sequence $\{T_n\}$ and Tribonacci-Lucas sequence $\{K_n\}$ are defined by following third order recurrence relation

$$Y_n = Y_{n-1} + Y_{n-2} + Y_{n-3}, \qquad n \geqslant 3$$

with the initial condition $T_0 = 0$, $T_1 = T_2 = 1$ and $K_0 = K_2 = 3$ and $K_1 = 1$, respectively. By taking into account the initial conditions, it's not difficult to see that

$$\sum_{n=0}^{\infty} T_n x^n = \frac{x}{1 - x - x^2 - x^3}, \qquad \sum_{n=0}^{\infty} K_n x^n = \frac{3 - 2x - x^3}{1 - x - x^2 - x^3}.$$

It has been shown that Tribonacci numbers could be obtained by computing sums of the elements in a specific direction from triangular numbers. The trinomial numbers obtained in the expansion of the polynomial $\left(1 + x + x^2\right)^n$, $n \geqslant 0$, explains such idea [4]



Tribonacci numbers $\quad T_n = T_{n-1} + T_{n-2} + T_{n-3}$

2 A second type of examples are special ones. We shall consider the so-called the Padovan sequence, denoted $\{C_n\}$ and the Perrin sequence $\{R_n\}$ defined by the following recurrence relation

$$Z_n = Z_{n-2} + Z_{n-3}, \qquad n \geqslant 3$$

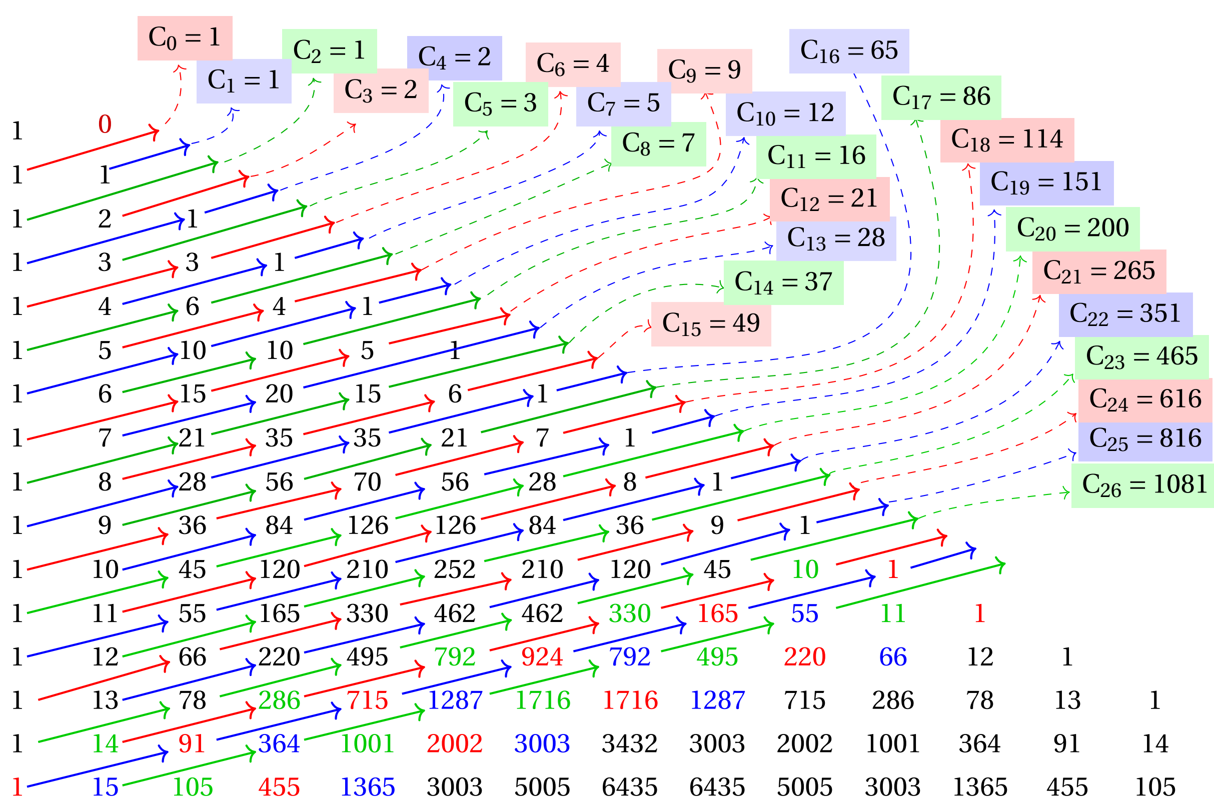
with the initial conditions $C_0 = C_1 = C_2 = 1$ and $R_0 = 3,\ R_1 = 0,\ R_2 = 2$, respectively.
In Literature, it has been remarked that Padovan and Perrin numbers satisfying further recurrence relations such as

$$Z_n \ = \ Z_{n-1} + Z_{n-5} \ = \ Z_{n-3} + Z_{n-4} + Z_{n-5} \dots$$

Their ordinary generating functions are

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1+x}{1-x^2-x^3}, \qquad \sum_{n=0}^{\infty} R_n x^n = \frac{3-x^2}{1-x^2-x^3},$$

respectively. Moreover, there are some combinatorial interpretations of Padovan numbers mainly their appearance in partitions. It is also worthy to mention that Padovan numbers could be obtained from Pascal triangle trough some specific directions.



Padovan numbers obtained through the above direction $\qquad C_n = C_{n-2} + C_{n-3}$

2.3.2 Recurrence relation of order greater than 3

We can also provide generating functions of the convolution of some of the above sequences. For instance, if we consider the following fifth order recurrence relation

$$\mathrm{I}_{n+5} = \mathrm{I}_{n+4} + 2\mathrm{I}_{n+3} - 2\mathrm{I}_{n+1} - \mathrm{I}_n, \qquad n \geqslant 3 \tag{2.21}$$

with the initial conditions $\mathrm{I}_0 = \mathrm{I}_1 = 1$, $\mathrm{I}_2 = 2$, $\mathrm{I}_3 = 4$ and $\mathrm{I}_4 = 8$, then its ordinary generating function will be

$$\sum_{n=0}^{\infty} \mathrm{I}_n x^n = \frac{x + x^2}{1 - x - 2x^2 + 2x^4 + x^5}. \tag{2.22}$$

Next, we shall discuss some convolutions of classical numbers as well as decompositions of some sequences of numbers defined by higher order recurrence relations in terms of the classical number's sequences.

Convolution of Fibonacci and Padovan

To begin with, let us back to the sequence of numbers defined by the linear recurrence relation (2.21). In order to understand the behavior of the sequence, we shall simplify or decompose its generating function. Indeed, the latter can be considered as a convolution of two sequences when we decompose the generating function as a product of two functions, i.e.

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{I}_n x^n &= \frac{x + x^2}{1 - x - 2x^2 + 2x^4 + x^5} = \left(\frac{x}{1 - x - x^2}\right)\left(\frac{1 + x}{1 - x^2 - x^3}\right)\\ &= \left(\sum_{n=0}^{\infty} \mathrm{F}_n x^n\right)\left(\sum_{n=0}^{\infty} \mathrm{C}_n x^n\right)\end{aligned}$$

which suggests that the numbers I_n are the product of Fibonacci and Padovan, that is to say, $\mathrm{I}_n = \mathrm{F}_n \mathrm{C}_n$. Therefore, we have obtained, by multiplying numbers of sequences satisfying recurrence relations of order two and three, a new sequence of numbers satisfying a fifth order recurrence relation.

Convolution of Fibonacci and Perrin

Now, under the initial conditions $\mathrm{I}_0 = 0$, $\mathrm{I}_1 = \mathrm{I}_2 = 3$, $\mathrm{I}_3 = 8$ and $\mathrm{I}_4 = 14$, the generating function of the recurrence relation (2.21) takes the following form

$$\sum_{n=0}^{\infty} \mathrm{I}_n x^n = \frac{3x - x^3}{1 - x - 2x^2 + 2x^4 + x^5}. \tag{2.23}$$

as above, using the decomposition of the denominator, in this case the generating function is a convolution of Fibonacci and Perrin sequences, i.e.

$$\sum_{n=0}^{\infty} \mathrm{I}_n x^n = \frac{3x - x^3}{1 - x - 2x^2 + 2x^4 + x^5} = \left(\frac{x}{1 - x - x^2}\right)\left(\frac{3 - x^2}{1 - x^2 - x^3}\right)$$
$$= \left(\sum_{n=0}^{\infty} \mathrm{F}_n x^n\right)\left(\sum_{n=0}^{\infty} \mathrm{R}_n x^n\right).$$

2.4 Binomial transformation

For a sequence of numbers $\{a_n\}$, its binomial transform is a new sequence $\{\hat{a}_n\}$ defined by the rule

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} a_k \qquad \text{with inversion} \qquad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \hat{a}_k. \tag{2.24}$$

It could be also defined in the symmetric version as follows

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} a_k \qquad \text{with inversion} \qquad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} \hat{a}_k.$$

It has been proved that the binomial transform of many classical numbers also satisfy recurrence relations. For instance, we can prove by induction the following results

Theorem 2.3 *Under the notations above, the binomial transforms of p-Fibonacci, Tribonacci, Padovan and Perrin satisfy the following recurrence relations*

$$\hat{\mathrm{F}}_{n+2} = (p+2)\hat{\mathrm{F}}_{n+1} - p\hat{\mathrm{F}}_n, \qquad \hat{\mathrm{T}}_{n+3} = 4\hat{\mathrm{T}}_{n+2} - 4\hat{\mathrm{T}}_{n+1} + 2\hat{\mathrm{T}}_n,$$
$$\hat{\mathrm{C}}_{n+3} = 3\hat{\mathrm{C}}_{n+2} - 2\hat{\mathrm{C}}_{n+1} + \hat{\mathrm{C}}_n, \qquad \hat{\mathrm{E}}_{n+3} = 3\hat{\mathrm{E}}_{n+2} - 2\hat{\mathrm{E}}_{n+1} + \hat{\mathrm{E}}_n,$$

respectively. Moreover, their ordinary generating functions could be computed explicitly. Indeed, the generating function of binomial transform of p-Fibonacci, Tribonacci, Padovan and Perrin numbers are

$$\sum_{n=0}^{\infty} \hat{\mathrm{F}}_n x^n = \frac{x}{1 - (p+2)x + px^2}, \qquad \sum_{n=0}^{\infty} \hat{\mathrm{T}}_n x^n = \frac{x - x^2}{1 - 4x + 4x^2 - 2x^3},$$
$$\sum_{n=0}^{\infty} \hat{\mathrm{E}}_n x^n = \frac{3 - 6x + 2x^2}{1 - 3x + 2x^2 - x^3}, \qquad \sum_{n=0}^{\infty} \hat{\mathrm{C}}_n x^n = \frac{1 - x}{1 - 3x + 2x^2 - x^3},$$

respectively.

Chapter 3

Generating functions beyond recurrence relations

In this chapter we introduce the Pochhammer symbol, i.e. the ascending factorial and descending factorial. The coefficients of their expansions appear in many situations and referred to as the Stirling numbers. In fact, many classical numbers are somehow connected.

$$\sum_{n\geq 0}(x)_n z^n = \cfrac{1}{1-xz-\cfrac{1\cdot xz^2}{1-(x+2)z-\cfrac{2(x+1)z^2}{1-(x+4)z-\cfrac{3(x+2)z^2}{\cdots}}}}.$$

$$\sum_{n\geq 0} n!\cdot z^n = \cfrac{1}{1-z-\cfrac{1^2\cdot z^2}{1-3z-\cfrac{2^2z^2}{\cdots}}}$$

3.1 Enumerative Combinatorics

The rising factorial, sometimes called the ascending factorial, is defined by $(x|w)_0 = 1$ and

$$(x|w)_n = x(x+w)(x+2w)...(x+(n-1)w), \quad n \geq 1,$$

and generalized falling factorial, also called descending factorial, is defined by $\langle x|w\rangle_0 = 1$ and

$$\langle x|w\rangle_n = x(x-w)(x-2w)...(x-(n-1)w), \quad n \geq 1.$$

When $w = 1$, the rising factorial gives the Pochhammer symbol, i.e.

$$(x)_n := (x|1)_n = x(x+1)(x+2)...(x+n-1), \quad n \geq 1, \tag{3.1}$$

while the falling factorial becomes

$$\langle x\rangle_n := \langle x|1\rangle_n = x(x-1)(x-2)...(x-n+1), \quad n \geq 1, \tag{3.2}$$

By expanding the rising and the falling factorial, we obtain polynomials in x, i.e. power series. Indeed, the following are few terms

$$\begin{array}{l|l}
(x)_1 = 1 & \langle x\rangle_1 = 1 \\
(x)_2 = x(x+1) = x^2 + x & \langle x\rangle_2 = x(x-1) = x^2 - x \\
(x)_3 = x(x+1)(x+2) = x^3 + 3x^2 + 2x & \langle x\rangle_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x \\
(x)_4 = x^4 + 6x^3 + 11x^2 + 6x & \langle x\rangle_4 = x^4 - 6x^3 + 11x^2 - 6x \\
(x)_5 = x^5 + 10x^4 + 35x^3 + 50x^2 + 24x & \langle x\rangle_5 = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x.
\end{array}$$

It is worthwhile to notice some of the properties of the falling and the rising factorial. We have

$$(x)_n = \langle x+n-1\rangle_n = (-1)^n \langle -x\rangle_n, \qquad \langle x\rangle_n = (x-n+1)_n = (-1)^n (-x)_n$$

$$\langle x\rangle_n = n!\binom{x}{n}, \qquad (x)_n = n!\binom{x+n-1}{n}, \qquad \langle n\rangle_n = (1)_n = n!$$

The rising as well as falling factorial can be extended to real x with help of gamma function as follows

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \qquad \langle x\rangle_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

3.2 Stirling numbers

The coefficients of the power series (3.1) and (3.2) constitute a sequence of numbers called the Stirling numbers of the first kind. That is to say, the sequence of numbers obtained from the following expansion

$$\langle x\rangle_n = \sum_{k=0}^{n} s(n,k)x^k. \tag{3.3}$$

The sequence numbers obtained from the inverse of the expansion are called the Stirling numbers of the second kind. In other words, Stirling numbers of the first and second kind can be considered inverses of one another. Roughly speaking, we have from one hand

$$x^n = \sum_{k=0}^{n} \mathrm{S}(n,k)\,\langle x\rangle_k. \tag{3.4}$$

and from the other hand, they constitute matrix inverses of one another. That is, if we denote by $s = (s_{nk})$ the lower triangular matrix of Stirling numbers of the first kind, i.e. $s_{nk} = s(n,k)$. Then the inverse of this matrix is the lower triangular matrix $s^{-1} = \mathrm{S} = (\mathrm{S}_{nk})$ with $\mathrm{S}_{nk} = \mathrm{S}(n,k)$ the Stirling numbers of the second kind.
Another place where you can encounter Stirling numbers is the following: Let $\mathrm{D} = d/dx$, then the differential operators $x^n\mathrm{D}^n$ and $(x\mathrm{D})^n$ are connected through the following relations [9]

$$(x\mathrm{D})^n = \sum_{k=0}^{n} \mathrm{S}(n,k)x^k\mathrm{D}^k, \qquad x^n\mathrm{D}^n = \sum_{k=0}^{n} s(n,k)(x\mathrm{D})^k = \langle x\mathrm{D}\rangle_n$$

In combinatorics Stirling numbers of the first kind $s(n,k)$ count the number of permutations of n elements with k disjoint cycles (circular permutations). While Stirling numbers of the second kind denoted $\mathrm{S}(n,k)$ count the number of ways to partition a set of n elements into k nonempty subsets.
Now, to get the generating function of the Stirling numbers, there are many ways using either the definition or some of their properties. We shall here use some of these techniques alternatively. Let us start with the first kind numbers. From the above properties we have from one hand

$$\begin{aligned}(1+z)^x &= \sum_{n=0}^{\infty}\binom{x}{n}z^n = \sum_{n=0}^{\infty} n!\binom{x}{n}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\langle x\rangle_n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n} s(n,k)x^k = \sum_{k=0}^{\infty} x^k \sum_{k=n}^{\infty} s(n,k)\frac{z^n}{n!}.\end{aligned}$$

From the other hand, since

$$(1+z)^x = e^{x\ln(1+z)} = \sum_{k=0}^{\infty} \big(\ln(1+z)\big)^k \frac{x^k}{k!}$$

it follows then that

Proposition 3.1 *The exponential generating function of the first kind Stirling numbers is*

$$\sum_{k=n}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{(\ln(1+z))^k}{k!}. \tag{3.5}$$

Although James Stirling had discovered the Stirling numbers of the second kind in a purely algebraic context in 1730, Masanobu Saka was the first person to realize the combinatorial significance of the latter numbers in 1782. [5]. Indeed, Saka studied the number $S(n,k)$ of ways that a set of n elements can be partitioned into k subsets where he discovered the following recurrence relation which can be proved by induction

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \qquad 1 \leqslant k < n. \tag{3.6}$$

In order to use the latter recurrence, let us now consider the following sums

$$\text{a}\ A_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n, \qquad \text{b}\ T_n(x) = \sum_{k=0}^{\infty} S(n,k)x^k. \tag{3.7}$$

Theorem 3.1 *We have the following*

$$A_k(x) = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)} \qquad T_n(x) = x(1+D)B_{n-1}(x).$$

Proof. Since $A_0(x) = B_0(x) = 1$, then by using the above recurrence relation (3.6) we obtain for $k \geqslant 1$ that

$$\begin{aligned} A_k(x) = \sum_{n=1}^{\infty} S(n,k)x^n &= \sum_{n=1}^{\infty} S(n-1,k-1)x^n + k\sum_{n=1}^{\infty} S(n-1,k)x^n \\ &= x\sum_{n=1}^{\infty} S(n-1,k-1)x^{n-1} + kx\sum_{n=1}^{\infty} S(n-1,k)x^{n-1} \end{aligned}$$

Therefore,

$$A_k(x) = xA_{k-1}(x) + kxA_k(x) \quad \Longrightarrow \quad A_k(x) = \frac{x}{1-kx}A_{k-1}(x) = \frac{x^k}{(1-kx)\dots(1-2x)(1-x)}$$

The same thing for $T_n(x)$. ■

It is worthy to mention that $T_n(x)$ is a polynomial in x referred to as Touchard polynomial. Moreover, the Stirling numbers of the first kind satisfy the following analogue recurrence relation which can be used, as above, to extract the ordinary generating function

$$s(n+1,k) = s(n,k-1) - ns(n,k).$$

The following tables provide the first few values of these numbers

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	1								
2	-1	1							
3	2	-3	1						
4	-6	11	-6	1					
5	24	-50	35	-10	1				
6	-120	274	-225	85	-15	1			
7	720	-1764	1624	-735	175	-21	1		
8	-5040	13068	-13132	6769	-1960	322	-28	1	
9	40320	-109584	118124	-67284	22449	-4536	546	-36	1

Table 3.1: The first values of Stirling numbers of the first kind

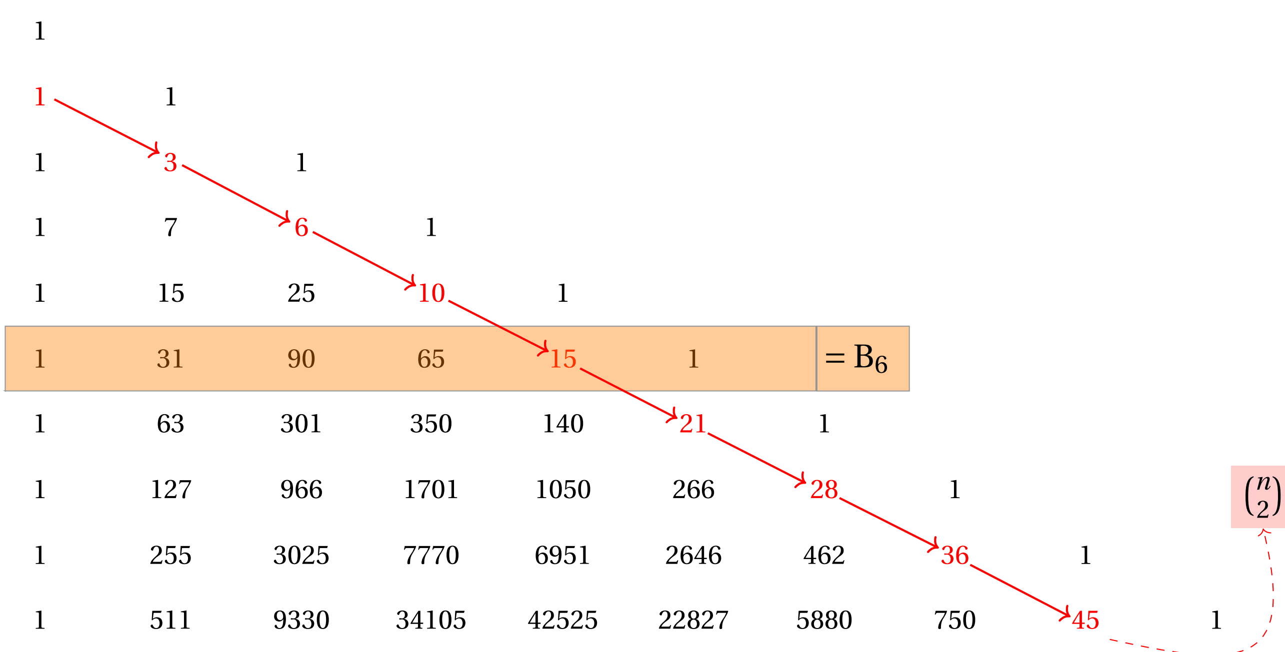


Figure 3.1: The first values of Stirling numbers of the second Kind

3.3 Bell numbers

If we look at the table of Stirling numbers of the second kind, we can extract many properties regarding these numbers. At the first sight, we can see that the diagonal entries is always 1. The second remark is that the line right below the main diagonal constitutes the binomial coefficient $\binom{n}{2}$ (as explained in red at the table of these numbers).
Now, since the Stirling numbers of the second kind count the number of partition of n-set into k-nonempty parts, then the total number of partitions of the integer n is the sum of the corresponding row. This interesting sequence of numbers is called the n^{th} Bell numbers and denoted by B_n. That is to say, the Bell numbers are the total number of ways of partitioning a et of n elements, i.e.

$$\mathrm{B}_n = \sum_{k=0}^{n} \mathrm{S}(n,k).$$

It is worthy to mention that the Bell numbers had been studied by many mathematicians. One of the earliest appearances of theme is in Japan around the year 1500. Moreover, in order to simplify the computation of thee numbers, we shall provide some of their properties mainly recurrence relations. For this end we have [8]

Proposition 3.2 *Let B_n be the number of set partitions of $[n]$. Then B_n satisfies the following recurrence relation*

$$\mathrm{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k, \quad \mathrm{B}_0 = 1. \tag{3.8}$$

Moreover, their exponential generating function can be easily computed. First, we have

$$\sum_{n=k}^{\infty} \frac{x^n}{(n-k)!k!} = \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!k!} = \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^k e^x}{k!}. \tag{3.9}$$

Therefore, using (3.8) we obtain

$$\begin{aligned} \mathrm{B}(t) = \sum_{n=0}^{\infty} \mathrm{B}_n \frac{t^n}{n!} &= 1 + \sum_{n=0}^{\infty} \mathrm{B}_n \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{B}_k \frac{t^n}{n!} \\ &= 1 + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k \frac{t^{n+1}}{(n+1)!} = 1 + \sum_{k=0}^{\infty} \mathrm{B}_k \sum_{n=k}^{\infty} \binom{n}{k} \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

The first derivative with respect to t together with the use of (3.9) give

$$\mathrm{B}'(t) = \sum_{k=0}^{\infty} \mathrm{B}_k \sum_{n=k}^{\infty} \binom{n}{k} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \mathrm{B}_k \sum_{n=k}^{\infty} \frac{t^n}{(n-k)!k!} = e^t \sum_{k=0}^{\infty} \mathrm{B}_k \frac{t^k}{k!} = e^t \mathrm{B}(t).$$

Solving the latter differential equation with the initial condition $\mathrm{B}(0) = 1$, we deduce that

Theorem 3.2 *The exponential generating function of Bell numbers is*

$$\sum_{n=0}^{\infty} \mathrm{B}_n \frac{t^n}{n!} = e^{e^x-1}. \tag{3.10}$$

For people who don't like combinatorics, we refer to [7] for comprehensive information on the exponential polynomials.

Now, since $\mathrm{B}_0 = 1$, then the reciprocal of B_n can be defined (in literature the reciprocal of Bell numbers are called Uppuluri-Carpenter numbers and denoted C_n). Hence, by definition we have $\mathrm{B}_n\mathrm{C}_n = 1$ from which we deduce the exponential generating function

$$\sum_{k=0}^{n} \mathrm{C}_n \frac{x^n}{n!} = e^{1-e^x}, \qquad \mathrm{C}_n = \sum_{k=0}^{n} (-1)^k \mathrm{S}(n,k). \tag{3.11}$$

Furthermore, if we add, and then subtract, the expressions of B_n and C_n, we obtain [7]

$$\mathrm{E}_n := \frac{1}{2}(\mathrm{B}_n + \mathrm{C}_n) = \mathrm{S}(n,2) + \mathrm{S}(n,4) + \ldots + \mathrm{S}(n,k) = \sum_{k=0}^{[\frac{n}{2}]} 2\mathrm{S}(n,2i)$$

$$\mathrm{O}_n := \frac{1}{2}(\mathrm{B}_n - \mathrm{C}_n) = \mathrm{S}(n,1) + \mathrm{S}(n,3) + \ldots + \mathrm{S}(n,l) = \sum_{k=0}^{[\frac{n-1}{2}]} \mathrm{S}(n,2i+1),$$

where k and l is the largest even integer and the largest odd integer that is less than or equal to n, respectively. From these, the exponential generating functions are given explicitly

$$\sum_{k=0}^{n} \mathrm{E}_n \frac{x^n}{n!} = \cosh\left(e^x - 1\right), \qquad \sum_{k=0}^{n} \mathrm{O}_n \frac{x^n}{n!} = \sinh\left(e^x - 1\right).$$

Notice further that the Touchard polynomials (3.7) evaluated at 1 is nothing else but Bell numbers, i.e. $\mathrm{T}_n(1) = \mathrm{B}_n$. Therefore, we can invoke further properties. For instance, the geometric polynomials (also known as Fubini polynomials) are slight modification of Touchard polynomials, obtained from the latter by multiplying the coefficient of x^k by $k!$. Roughly peaking, by setting $x = 1$ the Fubini polynomials give the so-called geometric numbers (or preferential arrangement numbers or Fubini numbers) G_n as

$$\mathrm{G}_n = \sum_{k=0}^{n} \mathrm{S}(n,k)k!. \tag{3.12}$$

It is easy to see that the exponential and the ordinary generating function are explicitly given

$$\sum_{n=0}^{\infty} \mathrm{G}_n \frac{x^n}{n!} = \frac{1}{2-e^x}, \qquad \sum_{n=0}^{\infty} \mathrm{G}_n x^n = \frac{n!x^n}{(1-x)(1-2x)\ldots(1-nx)}.$$

3.4 Harmonic numbers

The Harmonic numbers denoted $\{H_n\}_{n\geqslant 1}$ are defined by the general term

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Therefore, we shall look for a closed formula for the generating function given by

$$H(x) = \sum_{n=1}^{\infty} H_n x^n = \sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) x^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right) x^n$$

which is clearly the generating function of the sequence $\{1, 1, 1, \ldots\}$ times the generating function of the sequence $\{H_n\}_{n\geqslant 1}$.
Let us remark first, that the derivative of the generating function of the latter sequence gives

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} x^n\right)' = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Accordingly, by integrating both sides we get that the generating function of the sequence $\{H_n\}_{n\geqslant 1}$ is

$$-\ln(1-x) = \ln\left(\frac{1}{1-x}\right).$$

Finally, the convolution shows that the generating function of the Harmonic numbers is

$$H(x) = \sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \ln\left(\frac{1}{1-x}\right).$$

Next we shall invoke very amazing properties by performing diagonal sums in the Harmonic triangle. Let us begin with explanations of this triangle. In fact, the Harmonic triangle is related to reciprocals of the elements in Pascal's triangle and is formed by taking successive differences of terms of the harmonic series.

Therefore, after the first row, each entry is the difference of the two elements immediately above it. It is worth to mention that each element is the sum of the element to its right and the element below it in the array. For example, for $\frac{1}{20}$ we see that at its right is $\frac{1}{12}$ and bellow this number we find $\frac{1}{30}$, by checking we find that $\frac{1}{12} - \frac{1}{30} = \frac{1}{20}$.

$$
\begin{array}{ccccccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} & & \frac{1}{8} & & \cdots \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & & \frac{1}{56} & & \cdots & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{105} & & \frac{1}{168} & & \cdots & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \frac{1}{140} & & \frac{1}{280} & & \cdots & & & \\
& & & & \frac{1}{5} & & \frac{1}{30} & & \frac{1}{105} & & \frac{1}{280} & & \cdots & & & & \\
& & & & & \frac{1}{6} & & \frac{1}{42} & & \frac{1}{168} & & \cdots & & & & & \\
& & & & & & \frac{1}{7} & & \frac{1}{56} & & \cdots & & & & & & \\
& & & & & & & \frac{1}{8} & & \cdots & & & & & & &
\end{array}
$$

From another hand, each entry is the sum of the infinite series formed by the entries in the row below and to the right, in other words, each row has the first element in the row above it as its sum. Further remark is that each rising diagonal contains elements which are $\frac{1}{n}$ times the reciprocal of the similarly placed elements in Pascal's triangle.

In contrast to the harmonic triangle, each element in any row after the first is the sum of all terms in the row above it and to the left, while it is also the difference of the two terms in the row beneath it, and the sum of the element to its left and the element above it. Since the nth row in the harmonic triangle has sum $\frac{1}{n-1}$, if we multiply the row by n, we can immediately write the sum of the reciprocals of elements found in the columns of Pascal's triangle written in left-justified form as

$$\frac{n}{n-1} = \sum_{k=n}^{\infty} \binom{k}{n}^{-1}, \qquad n > 1.$$

3.5 Conclusion

There are plenty of interesting sequences of numbers that we meet in our daily life. Solving recurrence relation can be given in term of generating functions which help in tern to provide some connections between coefficients and and object. Indeed, if we consider various ways of selecting objects from a set S = $\{a, b, c\}$, then

1 Select one object from S we have

$$\{a} \quad \text{or} \quad \{b\} \quad \text{or} \quad \{c\} \quad \big(\text{denoted by} \quad a+b+c\big)$$

2 Select two objects from S we have

$$\{a,b\} \quad \text{or} \quad \{a,c\} \quad \text{or} \quad \{b,c\} \quad \big(\text{denoted by} \quad ab+ac+bc\big)$$

3 Select three objects from S we have

$$\{a,b,c\} \quad \big(\text{denoted by} \quad abc\big)$$

Remark that these symbols can be found in the following expression

$$(1+ax)(1+bx)(1+cx) = 1x^0 + (a+b+c)\,x^1 + (ab+ac+bc)\,x^2 + (abc)\,x^3.$$

We my write $1+ax = x^0 + ax^1$ which could be interpreted as

"a is not selected or a is selected once".

The latter technique of modeling is very practical mainly in combinatorics to choose objects in different ways, arrangements, configurations, looking for the shortest routes in rectangular grid, ... etc.

This simple interpretation shows how one can associate objects to coefficients in generating functions. There are many techniques for modeling problem in daily life and across different fields such as calculus, biology, physics, electronics, random variables, ... among others.

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