République Algérienne Démocratique et Populaire
Ministère de l'enseignement supérieur et de la recherche scientifique
Université Larbi Tébessi - Tébessa
Faculté des Sciences Exactes et des Sciences de la Nature et de la Vie
Département: Mathématiques et Informatique

Mémoire de fin d'études
Pour l'obtention du diplôme de MASTER
Domaine: Mathématiques et Informatique
Filière: Mathématiques
Option: Equations aux dérivées partielles et applications

Thème

On some q-transformations

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Date de soutenance :08/06/2024

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Abstract

This dissertation deals with the most important definitions and mathematical properties of q-calculus, such as quantum derivative and Jackson integration, in addition to formulating quantum transformations such as q-Laplace transform and q-Mellin transform.

Introduction

The study and use of the special functions is a very old branch of mathematics. We mention for example the Bernoulli and Euler numbers and polynomials, gamma and hypergeometric functions, Jacobi's elliptic functions, Bessel functions, and the polynomials of Legendre, Laguerre and Hermite. Most of those functions were introduced to solve specific problems.

The study of basic hypergeometric series (also called q- hypergeometric series or q-series) essentially started in 1748 when Euler considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$ as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers. But it was not until about a hundred years later that the subject acquired an independent status when Heine converted a simple observation that $\lim_{q\to 1}[(1-q^a)/(1-q)] = a$ into a systematic theory of ${}_2\phi_1$ basic hypergeometric series parallel to the theory of Gauss' ${}_2F_1$ hypergeometric series.

Apart from some important work by J. Thomae and L. J. Rogers the subject remained somewhat dormant during the latter part of the nineteenth century until F. H. Jackson embarked on a lifelong program of developing the theory of basic hypergeometric series in a systematic manner, studying q-differentiation and q-integration and deriving q-analogues of the hypergeometric summation and transformation formulas that were discovered by A. C. Dixon, J. Dougall, L. Saalschiitz, F. J. W. Whipple, and others.

D. B. Sears, L. Carlitz, W. Hahn, and L. J. Slater were among the prominent contributors during the 1950's. Sears derived several transformation formulas for ${}_3\phi_2$ series, balanced ${}_4\phi_3$ series, and very-well-poised ${}_{n+1}\phi_n$ series.

During the 1960's R. P. Agarwal and Slater each published a book partially devoted to the theory of basic hypergeometric series, and G. E. Andrews initiated his work in number theory, where he showed how useful the summation and transformation formulas for basic hypergeometric series are in the theory of partitions. Andrews gave simpler proofs of many old results, wrote review articles pointing out many important applications and, during the mid 1970's, started a period of very fruitful collaboration with R. Askey. Thanks to these two mathematicians, basic hypergeometric series is an active field of research today. Since Askey's primary area of interest is orthogonal polynomials, q-series suddenly provided him and his co-workers with a very rich environment for deriving q-extensions of beta integrals and of the classical orthogonal polynomials of Jacobi, Gegenbauer, Legendre, Laguerre and Hermite. Askey and his students and collaborators who include W. A. AL-Salam, M. E. H. Ismail, T. H. Koornwinder, W. G. Morris, D. Stanton, and J. A. Wilson have produced a substantial amount of interesting work over the past sixteen years.

In summary, the theory of q-calculus presents a discrete analogue of the derivative's operator and the integral as well as of the factorial (i.e., which is referred to as shifted factorial!). In this theory, we shall take a fixed positive integer q ($0 < q < 1$ or $q > 1$), and then try to figure out some formulas that reduce to the classical one when the integer q goes to 1. It terms out, from this, that we can give the q-analogue (i.e., in terms of q), as far as we can, of all the definitions and problems. Note also that the transition of any classical expression to its q-analogue is not unique.

We plan in this work to give a simple overview of the q-calculus including the shift factorial, the definition of the q-derivation and q-integration. We shall also give a brief introduction to the constructions of the main basic (q-analogue) special functions, and to point out some q-integral transformations with special focus on the q-Laplace transform and q-Mellin transform.

Chapter 1

The quantum Number

The fundamental rules upon which the concept of quantitative calculus is built are as follows:

Definition 1.0.1. *The q-analogue of the integer number n is defined by this formula*

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \xrightarrow[q\to 1]{} n, \tag{1.1}$$

and the q-factorial of the integer n is defined by $[0]_q! = 1$ *and for* $n > 1$

$$[n]_q! = [n]_q \cdot [n-1]_q \ldots [2]_q \cdot [1]_q = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q)}{(1-q)^n} \tag{1.2}$$

Example 1.0.1. *From the definition we have*

$$[1]_q = 1, \quad [2]_q = 1 + q, \quad [3]_q = 1 + q + q^2, \quad [4]_q = 1 + q + q^2 + \mathrm{q}^3 ...$$

and

$$\begin{aligned}
[3]_q! &= \left(1 + q + q^2\right)(1 + q) = 1 + 2q + 2q^2 + q^3 \\
[4]_q! &= \frac{1-q^4}{1-q} \cdot \frac{1-q^3}{1-q} \cdot \frac{1-q^2}{1-q} \cdot \frac{1-q}{1-q} \\
&= \left(1 + q + q^2 + q^3\right)\left(1 + q + q^2\right)(1 + q) \\
&= 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6
\end{aligned}$$

Therefore, by analogy the quantum binomial theorem can be utilized in designing quantum algorithms and understanding the dynamics of complex quantum system, thereby facilitating applications and unraveling the mysteries of quantum phenomena

Definition 1.0.2. *The q-binomial coefficient is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q, \quad 0 \leqslant k \leqslant n. \tag{1.3}$$

Let us recall some elementary properties of the q-factorial needed in the sequel.

The q-binomial coefficient verifies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{1.4}$$

Proof 1.0.1. *Remark that we can write, for any $1 \leq k \leq n-1$,*

$$\begin{aligned} [n] &= \left(1 + q + \cdots + q^{k-1}\right) + q^k \left(1 + q + \cdots + q^{n-k-1}\right) \\ &= [k] + q^k [n-k]. \end{aligned}$$

Therefore,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n-1]!\,[n]}{[k]!\,[n-k]!} = \frac{[n-1]!\,[k]}{[k]!\,[n-k]!} + \frac{[n-1]!q^k\,[n-k]}{[k]!\,[n-k]!}$$

which proves the left equality. Now, for second equality at right most, it suffices to apply the left equality to the definition (1.3) of the q-binomial coefficient.

Example 1.0.2.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 1 + q + q^2 \neq 3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$$

1.1 The quantum Derivative

The importance of the difference operator, usually denoted Δ_w, lies in discrete models, i.e. the difference equations essential in particular in modeling (to describe the evolution of a population) as well as in the digital resolution and simulation. Another operator which is also of very great importance, in particular in quantum, is the Jackson operator denoted D_q which provides another generalization of the usual derivation operator. Let us now recall the Jackson operator as well as some of these elementary properties

Definition 1.1.1. *Let the q-difference operators Dq and σ_q defined, respectively, by*

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad d_q f(x) = f(qx) - f(x), \quad \sigma_q f(x) = f(qx). \tag{1.5}$$

For any function f, put $(q-1)x = h$ in the latter definition to obtain

$$\lim_{q \to 1} D_q f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

$$D_q f(\mathrm{x}) = \frac{d_q f(\mathrm{x})}{d_q \mathrm{x}} = \frac{f(\mathrm{qx}) - f(\mathrm{x})}{qx - x} \tag{1.6}$$

Example 1.1.1. 1 *Compute the q-derivative of $f(\mathrm{x}) = x^3$*

$$\begin{aligned} D_q x^3 &= \frac{(\mathrm{x}q)^3 - x^3}{xq - x} = \frac{(xq - x)\left(x^3 q^2 + x^2 q + x^2\right)}{xq - x} \\ &= \left(1 + q + q^2\right) x^2 = [3]_q x^2 \end{aligned}$$

2 *The q-derivativ of* $f(\mathrm{x}) = x^n, n \in \mathbb{N}$

$$\begin{aligned} D_q x^n &= \frac{(xq)^n - x^n}{xq - x} = \frac{q^n - 1}{q - 1} \cdot \frac{x^n}{x} \\ &= \frac{(q-1)\left(q^{n-1} + q^{n-2} + q + 1\right)}{q - 1} \cdot \frac{x^n}{x} \\ &= \left[q^{n-1} + q^{n-2} + \cdots + q + 1\right) x^{n-1} \\ &= [\mathrm{n}]_q x^{n-1} \end{aligned}$$

1.2 Properties of q-Derivatives

In this section we shall provide some algebraic operations on D_q defined by (1.5). For any two functions f and g, we have the following elementary operations

1

$$D_q(\alpha f(x) \pm \beta g(x)) = \alpha D_{\mathrm{q}} f(x) \pm \beta_{\mathrm{q}} D g(x) \tag{1.7}$$

2 *The q-product rule.*

$$D_q(f \cdot q)(x) = D_q f(x) g(x) + f(qx) D_q g(x) \tag{1.8}$$

3. *If $g(x) \neq g(qx) \neq 0$ then*

$$D_{\mathrm{q}}\left(\frac{f(x)}{g(x)}\right) = \frac{D_{\mathrm{q}}f(\mathrm{x})g(x) - f(x)D_{\mathrm{q}}g(t)}{g(x)g(qx)} \tag{1.9}$$

4. *when the function $f > 0$ So*

$$D_q\sqrt{f(x)} = \frac{D_q f(x)}{\sqrt{f(qx)} + \sqrt{f(x)}} \tag{1.10}$$

Proof 1.2.1. .

1.

$$\begin{aligned} D_q(\alpha f(x) \pm \beta g(x)) &= \frac{\alpha f(qx) \pm \beta g(qx) - \alpha f(x) \pm \beta g(x)}{xq - \mathrm{x}} \\ &= \frac{\alpha f(qx) - \alpha f(x)}{xq - x} \pm \frac{\beta g(qx) - \beta g(x)}{xq - x} \\ &= D_q f(x) \pm D_q g(x). \end{aligned}$$

2. *We have.*

$$\begin{aligned} D_q(fg)(x) =& \frac{f(qx)g(qx) - f(\mathrm{x})g(x)}{xq - x} \\ =& \frac{f(qx)g(qx) - f(qx)g(x) + f(qx)g(x) - f(x)g(x)}{xq - x} \\ =& \frac{f(qx)[g(qx) - g(\mathrm{x})]}{xq - x} + \frac{g(x)[f(qx) - f(x)]}{xq - x} \\ =& f(qx)D_q g(x) + g(x)D_q f(\mathrm{x}). \end{aligned}$$

By symmetry we car interchange f and g

$$D_q(fg)(x) = f(x)D_q g(x) + g(\mathrm{q}x)D_q f(\mathrm{x})$$

.

3. *we have*

$$g(\mathbf{x})\frac{f(x)}{g(x)} = f(\mathbf{x}) \tag{1.11}$$

We apply to (1,5) the rule (1) or we have:

$$g(x)D_q\left(\frac{f(x)}{g(x)}\right) + \frac{f(qx)}{g(qx)}D_q g(x) = D_q f(x)$$

if

$$D_q\left(\frac{f(x)}{g(x)}\right)=\frac{g(qx)D_qf(\mathrm{x})-f(qx)D_qg(\mathrm{x})}{g(\mathrm{x})g(qx)}. \tag{1.12}$$

the two formulas $(1,6)$ *and (3) are both valid.*

Example 1.2.1. *Let* $f,g:R\to R$ *be the functions* $f(x)=x^2 andg(x)=4x.$

$$D_q(fog)=4^2x(q+1)$$

and

$$D_qg(x)\cdot D_qf(g(x))=4^3x(q+1)$$

so

$$D_q(fog)\neq D_qg(x)\cdot D_qf(g(x))$$

4. *Provided that it is* $g(x)=ax^\beta$*, this is because the composition rule of two functions on the Jackson derivative is only valid under this condition*

We have

$$\begin{aligned}D_q(f\circ \mathrm{g})(x)&=\frac{f(\mathrm{qg}(x))-f(g(\mathrm{x}))}{qx-x}\\&=\frac{f\left(aqx^\beta\right)-f\left(ax^\beta\right)}{qx-x}\cdot\frac{aqx^\beta-ax^\beta}{aqx^\beta-ax^\beta}\\&=\frac{aqx^\beta-ax^\beta}{qx-x}\cdot\frac{f\left(aqx^\beta\right)-f\left(ax^\beta\right)}{aqx^B-ax^B}\\&=D_qg(x)\cdot D_qf(g(x))\end{aligned}$$

5. *we have*

$$\begin{aligned}D_q\sqrt{f(x)}=&\frac{\sqrt{f(qx)}-\sqrt{f(x)}}{qx-x}\\=&\frac{(\sqrt{f(qx)}-\sqrt{f(x)})(\sqrt{f(qx)}+\sqrt{f(x)})}{(qx-x)(\sqrt{f(qx)}+\sqrt{f(x)})}\\=&\frac{f(qx)-f(x)}{(qx-x)(\sqrt{f(qx)}+\sqrt{f(x)})}\\=&\frac{1}{(\sqrt{f(qx)}+\sqrt{f(x)}}\cdot\frac{f(qx)-f(x)}{(qx-x)}\\=&\frac{D_qf(x)}{\sqrt{f(qx)}+\sqrt{f(x)}}.\end{aligned}$$

Theorem 1.2.1. *Let's have the following binomial Gaussian formula be:*

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k \mathrm{x}^{n-k}$$

When we set $a = 1$, we find

$$(x+1)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \mathrm{x}^k \tag{1.13}$$

We obtain the q-analogue of $(1+x)_q^n, n \in \mathbb{N}$.

Proposition 1.2.1. *The q-analogue of $f(x) = \frac{1}{(1-x)_q^n}$ is given by.*

$$D_q f(\mathrm{x}) = \frac{D_\mathrm{q}}{(1-x)_\mathrm{q}^n} = \frac{[n]}{(1-\mathrm{x})_\mathrm{q}^{n+1}}$$

and

$$D_q^k f(x) = \frac{[n][n+1]\ldots[n+k-1]}{(1-\mathrm{x})_\mathrm{q}^{n+\mathrm{K}}}$$

when

$$\left(D_q^k f\right)(0) = [n][n+1] \cdot [n+k-1], \quad k \geqslant 1$$

and therefore

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{k=1}^{\infty} \frac{[n][n-1]\cdots[n+k-1]}{[k]!}. \tag{1.14}$$

The question arises, how can we apply the Jackson derivative (D_q) to e specific function successively.

Theorem 1.2.2. *For every $n \geqslant 0$, we have the flowing formula.*

$$D_q^n(f \cdot g)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} f\left(q^k x\right)\left(D_q^k g(\mathrm{x})\right). \tag{1.15}$$

Proof 1.2.2. *When $n = 1$, we simply obtain equation (1)*

Let's assume that equation $(1,24)$ *holds for* n*, and well prove it for* $n+1$.

$$\begin{aligned} D_q^{n+1}(fg)(\mathrm{x}) &= D_q\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} f(q^k x)\right)\left(D_q^k g(x)\right)\right). \\ &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^k D_q^{n+k+1} f\left(q^k x\right)\left(D_q^k g(\mathrm{x})\right) \\ &+ \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} f\left(q^{k+1} x\right)\right)\left(D_q^{k+1} g(x)\right). \end{aligned}$$

By Substituting $k+1 \to k$ *into the second sum we find:*

$$D_q^{n+1}(fg)(x) = \sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n+1-k} f(q^k x)\right)\left(D_q^k g(x)\right) + \sum_{k=1}^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q \left(D_q^{n+1-k} f(q^k x)\right)\left(D_q^k g(x)\right)$$

Then, using q-binomial coefficient formula (1.4) *we deduce.*

$$D_q^{n+1}(fg)(t) = \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left(D_q^{n+1-k} f\left(q^k \mathrm{x}\right)\right)\left(D_q^k g(\mathrm{x})\right) \tag{1.16}$$

and on the basis of (1.23), which is referred to as q-Leibniz. (1.22) is fulfilled.

Proposition 1.2.2. *The q-analogue of* $(x-a)^n$ *is the polynomial*

$$(x-a)_q^n = \begin{cases} 1 \\ (x-a)(x-qa) \ldots\ldots (x-q^{n-a}a) \end{cases} \tag{1.17}$$

for all $n \geq 1$

$$D_q (x-a)_q^n = [n](x-a)_q^{n-1} \tag{1.18}$$

to conclude for any integer, we have:

$$D_q \frac{1}{(x-a)_q^n} = [-n](x-q^n a)_q^{n-1} \tag{1.19}$$

$$D_q (x-a)_q^{-n} = -[n](x-q^n a)_q^{n-1} \tag{1.20}$$

$$D_q \frac{1}{(x-a)_q^n} = \frac{[n]}{(x-q^n a)_q^{n+1}} \tag{1.21}$$

Chapter 2

Jackson's q-integral

In this chapter, we shall investigate the concept of Jackson integration along with some important properties for q-integration, including q-analogue of the exponential functions and trigonometric functions such as cosine and sinus, as well as q-gamma and q-beta functions.

The Jackson q-integration can be defined as the inverse operation of the q-derivation. If $D_qF(x) = f(x)$*, then*

$$F(x) - F(qx) = (1-q)xf(x) \tag{2.1}$$

We deduce from above that

$$F(q^k x) - F(q^{k+1}x) = (1-q)xq^k f(q^k x), \quad k = 0, 1, 2, \ldots$$

Summing over $k = 0, 1, \ldots, n-1$ *we obtain*

$$F(x) - F(q^n x) = (1-q)x \sum_{k=0}^{n-1} q^k f(q^k x).$$

Suppose that $0 < q < 1$*, and then* $F(q^n x) \to F(0)$ *as* $n \to \infty$*. From which we deduce that*

$$F(x) - F(0) = (1-q)x \sum_{k=0}^{\infty} q^k f(q^k x).$$

Therefore, for $0 < q < 1$ *the q-integral of the function* f *on the interval* $[0, c]$ *is defined by*

$$\int_0^c f(x)d_qx = c(1-q)\sum_{k=0}^{\infty} q^k f(q^k x) = \sum_{r=0}^{\infty} (x_r - x_{r+1})\, f(x_r), \tag{2.2}$$

where $x_r = cq^r$*. For the interval* $[c, +\infty[$*, the q-integral of* f *is defined by*

$$\int_c^{\infty} f(x)d_qx = c(1-q)\sum_{k=1}^{\infty} q^{-k} f(q^{-k}x). \tag{2.3}$$

With $c = 1$ *in (2.2) and (2.3) and summing these two quantities we obtain*

Definition 2.0.1. *The Jackson q-integral of* f *over an infinite interval is given by the expression*

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n x). \tag{2.4}$$

Now we are able to set the following

Definition 2.0.2. *A function* f *is said to be absolutely q-integrable on* $[0, \infty[$*, if the series* $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ *converges absolutely.*

We write $L^1(\mathbb{R}_{q,+})$ *for the set of all functions that are absolutely q-integrable on* $[0, \infty[$*, where*

$\mathbb{R}_{q,+}$ *is the set*

$$\mathbb{R}_{q,+} = \{q^n : \ n \in \mathbb{Z}\}$$

By using geometric series in Riemann integral, Jackson was able to obtain the integral formula for the function $f(x)$

Theorem 2.0.1. *When* $0 < q < 1$ *and if* $|f(x)x^{\alpha}|$ *is bounded on the domain* $0 < a < 1$ *where* $0 \leqslant a < 1$*, then the Jackson integral* (2.3) *converges to* $F(x)$ *on the domain* $]0, a]$.

Proof 2.0.1. *It is readily seen that the finite series is convergent, therefore*

$$\begin{aligned} D_q F(x) =& \frac{1}{(q-1)x}\left((1-q)\mathrm{x}\sum_{n=0}^{\infty} q^n f\left(q^n \mathrm{x}\right) - (1-q)qx\sum_{n=0}^{\infty} q^n f\left(q^{n+1}x\right)\right) \\ =& \sum_{n=0}^{\infty} q^n f\left(q^n x\right) - \sum_{n=0}^{\infty} q^{n+1} f\left(q^{n+1}\mathrm{x}\right) \\ =& \sum_{n=0}^{\infty} q^n f\left(q^n x\right) - \sum_{n=1}^{\infty} q^n f\left(q^n \mathrm{x}\right) = f(\mathrm{x}). \end{aligned}$$

It is worthy to mention that from (2.2) we merely deduce that

Definition 2.0.3. *For* $0 < a < b$ *we have*

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.5}$$

It is worthy to mention that from definition (1.3), we can merely extract a more compact formulas

Proposition 2.0.1. *We have*

- $$\int_0^{\mathrm{b}} D_{\mathrm{q}} f(x)d_qx = f(b) - f(0) \tag{2.6}$$

- $$D_{\mathrm{q}}\int_0^x f(t)d_qt = f(x) \tag{2.7}$$

Proof 2.0.2. *Indeed, by definition we have*

$$\begin{aligned}\int_0^b D_q f(x) d_q x =& \int_0^b \frac{f(xq)-f(x)}{xq-x} d_q x.\\ =&(1-q)b\sum_{n=0}^{\infty} q^n \frac{f\left(bq^{n+1}\right)-f\left(bq^n\right)}{-bq^n(1-q)}\\ =&\sum_{n=0}^{\infty} f\left(bq^n\right)-f\left(bq^{n+1}\right)\\ =&\lim_{N\to\infty}\left(\sum_{n=0}^{N} f\left(bq^n\right)-\sum_{n=0}^{N} f\left(bq^{n+1}\right)\right)\\ =&f(\mathrm{~b})-f\left(aq^{N+1}\right) \quad |\mathrm{q}|<1\end{aligned}$$

Then

$$\int_0^b D_q f(\mathrm{x}) d_q x = f(b)-f(0)$$

and we have

$$\begin{aligned}D_q\int_0^x f(\mathrm{x}) d_q \mathrm{x} =& D_q\left((1-q)\mathrm{x}\sum_{n=0}^{\infty} q^n f\left(xq^n\right)\right)\\ =&\frac{1}{(1-q)x}\left((1-q)\mathrm{xq}\sum_{n=0}^{\infty} q^n f\left(xq^{n+1}\right)-(1-q)\mathrm{x}\sum_{n=0}^{\infty} q^n f\left(\mathrm{x}q^n\right)\right)\\ =&\sum_{n=0}^{\infty} q^n f\left(xq^n\right)-\sum_{n=0}^{\infty} q^{n+1} f\left(xq^{n+1}\right)=f(x)\end{aligned}$$

whence

$$D_q\int_0^x f(\mathrm{x}) d_q \mathrm{x} = f(x)$$

Theorem 2.0.2. *We have the following properties*

(a) For $u(x)=ax^\beta$ we have

$$\int_{u(a)}^{u(b)} f(u) d_q u = \int_0^b f(u(t)) d_{q^{1/\beta}} u(\mathrm{x}). \tag{2.8}$$

(b) The q-integration by parts

$$\int_0^b f(x) D_q g(\mathrm{x}) d_q x = f(b)g(b)-f(0)g(0)-\int_0^b g(qx) D_q f(x). \tag{2.9}$$

Proof 2.0.3. *using rule (2.6), we have*

$$\begin{aligned}
\int f(u(\mathrm{x}))d_{\mathrm{q}^{1/\beta}}u(\mathrm{x}) &= \sum_{n=0}^{\infty} f\left(u\left(q^{n/\beta}x\right)\right)\left(u\left(q^{n/\beta}\mathrm{x}\right)-u\left(q^{(n+1))/\beta}x\right)\right.\\
&= \sum_{n=0}^{\infty} f\left(aq^{n}x^{\beta}\right)\left(aq^{\beta}x^{\beta}-aq^{n+1}x^{\beta}\right)\\
&= \sum_{n=0}^{\infty} f\left(q^{n}u\right)\left(q^{n}u-q^{n+1}u\right)\\
&= \sum_{n=0}^{\infty} q^{n} f\left(q^{n}u\right)(1-q)u\\
&= (1-q)u\sum_{n=0}^{\infty} q^{n} f\left(q^{n}u\right) = \int f(u)d_{q}u
\end{aligned} \tag{2.10}$$

Second, using

$$D_q(fg)(x) = f(x)D_qg(x) + g(qx)D_qf(x)$$

we obtain the following

$$\begin{aligned}
\int_0^b D_q(fg)(x)d_qx &= \int_0^b f(x)D_qg(x)d_qx + \int_0^b g(qx)D_qf(t)d_q\\
f(b)g(b)-f(0)g(0) &= \int_0^b f(x)D_qg(x)d_qx + \int_0^b g(qx)D_qf(\mathrm{x})d_qx
\end{aligned}$$

thus

$$\int_0^b f(x)D_qg(x)d_qx = f(b)g(b)-f(0)g(0) - \int_0^b g(qx)D_qf(x)d_qx$$

from which we deduce

$$\int_a^b f(x)D_qg(\mathrm{x})d_qx = f(\mathrm{x})g(\mathrm{x})|_a^b - \int_a^b g(qx)D_pf(\mathrm{x})d_qx$$

Examples

$$\begin{aligned}
\int_0^x \mathrm{t}d_q\mathrm{t} &= (1-q)x\sum_{n=0}^{\infty} q^n x q^n\\
&= (1-q)x^2\sum_{n=0}^{\infty} q^{2n}\\
&= (1-q)x^2\frac{1}{1-q^2} = \frac{x^2}{1+q} = \frac{1}{[2]_q}x^2
\end{aligned}$$

Show that

$$\int_0^{\mathrm{x}} t^2 d_q \mathrm{t} = \frac{x^3}{1+q+q^2} = \frac{x^3}{[3]q}$$

and $h(t) = t^n$

$$\int_0^x t^n d_q \mathrm{t} = \frac{x^{n+1}}{[n+1]_q}$$

Proof

- $h(t) = t^n$

$$\begin{aligned}\int_0^x t^n d_q t &= (1-q) \times \sum_{n=0}^{\infty} q^n \left(xq^n\right)^n \\ &= (1-q)x^{n+1} \sum_{n=0}^{\infty} \left(q^{n+1}\right)^n \\ &= \frac{1-q}{1-q^{n+1}} x^{n+1} = \frac{x^{n+1}}{[n+1]_q}\end{aligned}$$

- $g(t) = \sqrt{t}$

$$\begin{aligned}\int_0^x g(t) d_q \mathrm{t} &= (1-q)\mathrm{x} \sum_{n=0}^{\infty} q^n \left(xq^n\right) \\ &= (1-q)\mathrm{x} \sum_{n=0}^{\infty} q^n \sqrt{\mathrm{x}q^n} \\ &= (1-q)\mathrm{x} \sum_{n=0}^{\infty} q^n x^{1/2} q^{n/2} \\ &= (1-\mathrm{q}) x^{3/2} \sum_{n=0}^{\infty} q^{3n/2} \\ &= x^{3/2}(1-\mathrm{q}) \cdot \frac{1}{1-q^{3/2}} \\ &= x^{3/2} \frac{(1-\sqrt{\mathrm{q}})(1+\sqrt{\mathrm{q}})}{(1-\sqrt{\mathrm{q}})(1+\sqrt{\mathrm{q}}+\mathrm{q})} = \frac{(1+\sqrt{\mathrm{q}})}{(1+\sqrt{\mathrm{q}}+\mathrm{q})} x^{3/2}\end{aligned}$$

- $k(t) = \log t$

$$\int_0^x k(t) d_q t = (1-q)\sum_{n=0}^{\infty} q^n k(xq^n) = (1-q)\sum_{n=0}^{\infty} q^n \log(xq^n)$$

$$= (1-q)\sum_{n=0}^{\infty} q^n \log(x) + (1-q)\sum_{n=0}^{\infty} q^n \log(q^n)$$

$$= x(1-q)\log(x)\sum_{n=0}^{\infty} q^n + x(1-q)\log(q)\sum_{n=1}^{\infty} nq^n$$

$$= (1-q)\log(\mathrm{x})\cdot\frac{1}{1-q} + (1-q)\mathrm{x}q\log(q)\sum_{n=0}^{\infty} nq^{n-1}$$

$$= x\log(t) + (1-q)xq\log(q)\left(D_q\sum_{n=0}^{\infty} q^n\right)$$

$$= x\log(x) + (1-q)xg\log(q)D_q\left(\frac{1}{1-q}\right)$$

$$= x\log(x) + (1-q)xq\log(q)\frac{1}{(1-q)^2} = x\log(x) + x\frac{q\log(q)}{1-q}$$

Remark 2.0.1.

- $\lim_{q\to 1}\left(\frac{x^2}{[2]_q}\right) = \frac{x^2}{2}$
- $\lim_{\mathrm{q}\to 1}\left(\frac{x^{n+1}}{[n+1]_q}\right) = \frac{x^{n+1}}{n+1}.$
- $\lim_{q\to 1}\left(\frac{1-\sqrt{q}}{1+\sqrt{q}+q}x^{3/2}\right) = \frac{2}{3}x^{3/2}.$
- $\lim_{\mathrm{q}\to 1}\left(x\log(x) + x\frac{\mathrm{q}\log(\mathrm{q})}{1-\mathrm{q}}\right) = x\log(x) - x$

2.1 The q-analogue of the exponential function

Several different methods have been proposed for constructing a q-exponential function and in this chapter, we will present the approach that relies on the q-differential equation.

$$\begin{cases} D_q y(\mathrm{x}) = y(\mathrm{x}) & 0 < q < 1, x \in \mathbb{R}^+ \\ y(0) = 1 \end{cases} \tag{2.11}$$

from $(2,11)$ *we have*

$$y(x) = \frac{y(\mathrm{q}x) - y(x)}{(\mathrm{q}-1)x}$$

and from it

$$y(qx) = [1 + (q-1)x]y(\mathrm{x})$$

and

$$y(\mathrm{x}) = \frac{y(qx)}{[1+(q-1)x]}$$

if

$$y(qx) = \frac{y\left(q^2 x\right)}{1+(q-1)\mathrm{q}x}$$

then we conclude the following.

$$\begin{aligned} y(x) &= \frac{y\left(q^2 x\right)}{(1+(q-1)x)(1+(q-1)qx)} \\ y(x) &= \frac{y\left(q^n x\right)}{(1+(q-1)x)(1+(q-1)qx)\left(1+(q-1)q^2 x\right)\cdots\left(1+(q-1)q^n \mathrm{x}\right.} \\ &= \frac{y\left(q^n \mathrm{x}\right)}{\prod_{k=1}^n\left[1+\left(q^k-1\right)x\right]} \end{aligned}$$

and we have $\lim_{n\to\infty} \mathrm{q}^n = 0$

$$y(\mathrm{x}) = \frac{y(0)}{\prod_{k=1}^n\left[1+\left(q^k-1\right)x\right]}. \tag{2.12}$$

Now, let us find solutions for equation (2.14) in the following form

$$y(\mathrm{x}) = \sum_{n=0}^{\infty} c_n x^n. \tag{2.13}$$

where c_n *is a real number for every natural integer* n*, and from* (2.14) *and* (2.15) *we find*

$$y(x) = \sum_{n=0}^{\infty} c_n D_q x^n = \sum_{n=0}^{\infty} c_n x^n.$$

and we have $\quad D_q x^n = [n]_q x^{n-1} \quad \forall n \subset N.$

so we get

$$c_{n+1} = \frac{1}{[n+1]_{\mathrm{q}}} c_n$$

if

$$\begin{aligned} c_n &= c_0 \prod_{j=0}^{\mathrm{n}} \frac{1}{[k]_q!} \\ c_n &= \frac{c_0}{[\mathrm{n}]_q!}. \end{aligned} \tag{2.14}$$

Substituting (2.14) *into* (2.13), *we find*

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$
$$y(x) = c_0 e_q \tag{2.15}$$

and

$$D_q\left(\frac{y(\mathrm{x})}{e_q(\mathrm{x})}\right) = \frac{D_q\mathrm{y}(x)e_q(x) - y(\mathrm{x}))\,D_q e_q(x)}{e_q(x)e_q(qx)} = \frac{xy(x)e_q(x) - xy(x)e_q(\mathrm{x})}{e_q(x)e_q(qx)} = 0$$

This shows that the function $\frac{y(x)}{e_q(x)}$ *is a constant function, so we conclude*

$$\frac{y(x)}{e_q(x)} = \frac{y(0)}{e_q(0)} = 1$$

So

$$y(x) = e_q(x)$$

from (2.15) *and* (2.14) *we find*

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{\prod_{k=1}^{n}\left[1 + \left(q^k - 1\right)x\right]}$$

Definition 2.1.1. *for all* $x \in \mathbb{R}, 0 < q < 1$ *we define the q-exponential function* $e_q(\mathrm{x})$ *is by following:*

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

Notice that $x \in \mathbb{R} \quad 0 < \mathrm{q} < 1$

$$\lim_{q\to 1} e_q(\mathrm{x}) = \lim_{q\to 1} \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e(x)$$

2.2 The function big q-exponential E_q

In the same way as before, we can extract the function q-exponential $E_q(x)$ *as solution of the following*

$$\begin{cases} D_{\mathrm{q}}y(\mathrm{x}) = y(qx) \\ y(0) = 1 \end{cases} \tag{2.16}$$

For $0 < q < 1.$ *and* $x > 0$, *we have:*

$$y(qx) = \frac{y(qx) - y(\mathrm{x})}{(q-1)x}$$
$$\Rightarrow \quad y(x) = [1 + (1-q)x]y(qx)$$

As above, we merely have

$$y(qx) = \prod_{k=1}^{2}\left[1 + \left(1-q^{k}\right)x\right] y\left(q^{2}\mathrm{x}\right)$$
$$y(qx) = \prod_{k=1}^{n}\left[1 + \left(1-q^{k}\right)x\right] y\left(q^{n}\mathrm{x}\right)$$

and we have $\lim_{n\to\infty} q^n = 0$.

Therefore,

$$y(\mathrm{x}) = y(0)\prod_{k=1}^{\infty}\left[1 + \left(1-q^{k}\right)\mathrm{x}\right]. \tag{2.17}$$

assuming the following formula to find solution to equation (2.19).

$$y(\mathrm{x}) = \sum_{n=0}^{\infty} d_n x^n \tag{2.18}$$

Where d_n *is a real number for any natural number* n, *and from* $(2, 19)$ *and* (2.20) *we have*

$$\sum_{n=0}^{\infty} d_n \mathrm{D}_q x^n = \sum_{n=0}^{\infty} d_{\mathrm{n}}\left(q\mathrm{x}^{n}\right)$$

and since $\mathrm{D}_q x^n = [n]_q x^{n-1}$, *then we obtain*

$$d_{n+1} = \frac{\mathrm{q}^{n}}{[n+1]_q!} d_n$$

and we have.

$$\forall n \in \mathbb{N}^n \quad d_n = d_0 \prod_{n=1}^{n}\frac{q^{n-1}}{[n]_q} d_n$$
$$d_n = \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!} d_0$$

by substituting (2.21) *for* (2.20), *we find:*

$$y(\mathrm{x}) = d_0 \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_{\mathrm{q}!}}$$
$$\Rightarrow y(\mathrm{x}) = d_0 E_q(\mathrm{x})$$

and

$$D_{\mathrm{q}}\left(\frac{y(\mathrm{x})}{E_q(\mathrm{x})}\right) = \frac{D_q y(\mathrm{x}) E_q(\mathrm{x}) - y(\mathrm{x})\mathrm{D}_q E(\mathrm{x})}{E_q(\mathrm{x}) E_q(qx)}$$
$$= \frac{xy(\mathrm{x})E_q(x) - xy(\mathrm{x})E_q(\mathrm{x})}{E_q(\mathrm{x})E_q(qx)} = 0$$

This shows that the function $\frac{y(t)}{E_q(t)}$ *is a constant function.*

So we conclude

$$\frac{y(\mathrm{x})}{E_q(\mathrm{x})} = \frac{y(0)}{E_q(0)} = 1$$

whence

$$y(\mathrm{x}) = E_q(\mathrm{x})$$

from the foregoing, we conclude the following formula:

$$E_q(\mathrm{x}) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{\mathrm{x}^n}{[n]_q!} d_n = \prod_{k=1}^{n} \left[1 + \left(1 - q^k\right) x\right] y\left(q^n \mathrm{x}\right)$$

Definition 2.2.1. *For all* $x \in \mathbb{R}, 0 < q < 1$ *we define the* q*-exponential function* $E_q(x)$ *as follows*

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!} \tag{2.19}$$

Proposition 2.2.1. *For the* q*-exponential functions we have the following properties*

1.
$$D_q e_q(\mathrm{x}) = e_q(\mathrm{x}) \tag{2.20}$$

2.
$$D_q E_q(\mathrm{x}) = E_q(\mathrm{x}) \tag{2.21}$$

3.
$$D_q E_q(-x) = -E_q(-qx) \tag{2.22}$$

4

$$e_q(\mathrm{x})e_q(y) = e_q(\mathrm{x}+y) \tag{2.23}$$

5

$$e_q(-\mathrm{x})E_q(\mathrm{x}) = e_q(\mathrm{x})E_q(-\mathrm{x}) = 1 \tag{2.24}$$

6

$$e_{1/q}(\mathrm{x}) = E_q(\mathrm{x}) \tag{2.25}$$

Proof 2.2.1. 1

$$\begin{aligned} D_q e_q(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{D}_q x^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[n]_q x^{n-1}}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[n]_q x^{n-1}}{[n]_q[n-1]_\mathrm{q}!} \\ &= \sum_{n=0}^{\infty} \frac{x^{n-1}}{[n-1]_\mathrm{q}!} = \sum_{m=0}^{\infty} \frac{x^m}{[m]_q!} = e_q(x) \end{aligned}$$

2

$$\begin{aligned} D_q E_q(\mathrm{x}) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{D_q x^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{[n]_q \mathrm{x}^{n-1}}{[n]_q!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{[n]_q x^{n-1}}{[n]_\mathrm{q}[n-1]_q} \\ &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^{n-1}}{[n-1]_q!} = \sum_{\mathrm{m}=0}^{\infty} q^{\binom{n}{2}} \frac{x^m}{[m]_q!} = E_q(\mathrm{x}) \end{aligned}$$

3

$$\begin{aligned} D_q E_q(-x) &= D_q\left(1+\sum_{n=1}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(-x)^n}{[n]_q!}\right) = D_q\left(1+\sum_{n=1}^{\infty} \frac{(-1)^n \cdot q^{\frac{n(n-1)}{2}}}{[n]_q!} x^n\right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{(n-1)}{2}}}{[n]_q!} D_q x^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n-1]_q!} x^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}}{[n]_q!} x^n \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n-1)}{2}}}{[n]_q!} (qx)^n = -E_q(qx) \end{aligned}$$

4

$$\begin{aligned} e_q(\mathrm{x})e_q(y) &= \left(\sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{[n]_q!}\right) \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^k y^n}{[k]_q![n]_q!} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{[n+k]_q!}{[k]_q![n]_q!} \frac{x^k y^n}{[n+k]_q!} \end{aligned}$$

if we change the variable from $m = n + k$, then we have

$$\begin{aligned} e_q(\mathrm{x})e_q(y) &= \sum_{m=0}^{\infty} \frac{1}{[m]_q!} \left(\sum_{k=0}^{m} \frac{[m]_q!}{[k]_q![m-k]q'!} x^k y^{m-k} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \right) \frac{1}{[m]_q!} = \sum_{m=0}^{\infty} \frac{(\mathrm{x}+y)^m}{[m]_q!} \\ &= e_q(x+y) \end{aligned}$$

5

$$\begin{aligned} e_q(\mathrm{x})E_q(-x) &= \frac{1}{\prod_{k=1}^{n} \left[1 + \left(q^k - 1\right)\mathrm{x}\right]} \cdot \prod_{k=1}^{n} \left[1 + \left(1 - q^k\right)(-\mathrm{x})\right] \\ &= \frac{\prod_{k=1}^{n} \left[1 + \left(q^k - 1\right)(x)\right]}{\prod_{k=1}^{n} \left[1 + \left(q^k - 1\right)x\right]} = 1 \end{aligned}$$

$$\begin{aligned} e_{1/q} &= \sum_{n=0}^{\infty} \frac{(1-1/q)^n x^n}{(1-1/q)\left(1-1/q^2\right)\ldots\left(1-1/q^n\right)} \\ &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(1-1/q)^n x^n}{(1-q)\left(1-q^2\right)\ldots\left(1-q^n\right)} \\ &= E_q(x) \end{aligned}$$

2.3 The q-trigonometric functions

The q-trigonometric functions are q-analogue the classical trigonometric functions. These functions are defined as follows:

$$\cos_q x = \frac{e_q(ix) + e_q(-\mathrm{i}x)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{[2n]_q!} \quad \forall x \in \mathbb{C}, |x| < 1 \tag{2.26}$$

$$\mathrm{Cos}_q x = \frac{E_q(ix) + E_q(-\mathrm{i}x)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\mathrm{n}(2n-1)} x^{2n}}{[2n]_q!} \quad \forall x \in \mathbb{C} \tag{2.27}$$

$$\sin_q x = \frac{e_q(ix) - e_q(-\mathrm{i}x)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{[2n+1]_q!} \quad \forall x \in \mathbb{C}, |x| < 1 \tag{2.28}$$

$$\mathrm{Sin}\, x = \frac{E_q(ix) - E_q(-\mathrm{i}x)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\mathrm{n}(2n+1)} x^{2n+1}}{[2n+1]_q!} \quad \forall x \in \mathbb{C} \tag{2.29}$$

Proposition 2.3.1. 1 *According to* (2.25), *we have*

$$\mathrm{Sin}_q(x) = \sin_{1/q}(t) \tag{2.30}$$

$$\cos_q(x) = \cos_{1/q}(t) \tag{2.31}$$

2 *From* (2.20) *and* (2.21) *together with* $u(x) = ix$, *we find*

$$\begin{aligned} D_q \sin_q x &= \cos_q(t) & .D_q \sin_q x &= \cos(qx) \\ D_q \cos_q x &= -\sin q(t) & .D_q \cos_q x &= -\sin_q(qx) \end{aligned} \tag{2.32}$$

3 *We have*

$$\cos_q x \cdot \cos_q x = \frac{e_q(ix)E_q(ix) + e_q(-ix)E_q(-ix) + 2}{4}$$

and

$$\sin_q \mathrm{x} \cdot \sin_q x = -\frac{e_q(ix)E_q(ix) + e_q(-\mathrm{i}\mathrm{x})E_q(-\mathrm{i}\mathrm{x}) - 2}{4}$$

and thus we have

$$\cos_q x \cos_q \mathrm{x} + \sin_q \mathrm{x} \sin_q x = 1$$

It is the q-analogue of the equation

$$\sin^2 x + \cos^2 x = 1$$

4 *Further properties easily verified using only the definition are the following*

$$\cos_q(x) + i\sin_q(x) = e_q(x)$$

$$\cos_q^2(x) + \sin_q^2(x) = e_q(ix)e(-ix)$$

$$\sin_q(x)Cos_q(x) = \cos_q(x)Sin_q(x)$$

$$\cos_q^2(x) + \sin_q^2(x) = E_q(ix)E_q(-ix)$$

2.4 The Function q-Gamma and q-Beta

The q-Gamma and q-Beta functions are generalizations of the classical Gamma and Beta functions, respectively. They are used in mathematics, particularly in the context of special number theory and special functions and often appear in the context of q-series theory.

The two formulas are introduced by Euler as they are related to solutions of certain special *differential equation.*

$$\Gamma\left(t\right)=\int\limits_{0}^{\infty}x^{t-1}e^{-x}dx,\quad t>0 \tag{2.33}$$

$$\beta\left(t,s\right)=\int\limits_{0}^{\infty}\frac{x^{t-1}}{(1+x)^{s+t}}dx,\quad t>0 \tag{2.34}$$

they are gamma and beta functions, respectively. Some of their properties are the following

$$\Gamma(t+1)=t\Gamma(t) \tag{2.35}$$

$$\Gamma(n)=(n-1)!.\quad \textit{if } n \textit{ is a positive integer} \tag{2.36}$$

$$\beta(t,s)=\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)} \tag{2.37}$$

Formula (2.38) shows that the gamma function can be considered as a generalization of the factorials. Next, we study the q-analogues of these two functions, where $0<q<1$ is involved.
We have $q\in\mathbb{C}$, and for each $0<|q|<1$, the q-Gama $\Gamma_q(x)$ is given as follows:

$$\Gamma_q(x)=\frac{(q,q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \tag{2.38}$$

Where $(q,q)=\prod_{k=0}^{\infty}\left(1-aq^k\right)$. It is a meromorphic function with poles at $x=-n\pm 2\pi ik/\log q$ where k and n are non-negative integers
When $q>1$, using the inverse of observation (1.2) we obtain

$$\Gamma_q(x)=q^{\binom{n}{2}}\Gamma_{1/q}(x)=\frac{(q^{-1},q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}} \tag{2.39}$$

Definition 2.4.1. *Let the formula for q-gamma be the following, $t>0$*

$$\Gamma_q(t)=\int_0^\infty x^{t-1}E_q^{-qx}d_qx \tag{2.40}$$

and

$$\hat{\Gamma}_q(t)=\int_0^\infty x^{t-1}e_q^{-x}d_qx\quad x\in\mathbb{C},\mathrm{Re}(x)>0 \tag{2.41}$$

Proposition 2.4.1. *for $t\in\mathbb{N}^*$, we have.*

(1)

$$\Gamma_q[t+1]=t\Gamma[t],\qquad \Gamma_q(1)=1,\quad \hat{\Gamma}_q(1)=1 \tag{2.42}$$

(2)

$$\hat{\Gamma}_q(t+1) = q^{-t}[t]_q\hat{\Gamma}(t) \tag{2.43}$$

Proof 2.4.1. *(1) According to the property* (2.22)*, we have*

$$\Gamma_q(t+1) = \int_0^\infty x^t \cdot E_q^{-\mathrm{qx}} d_q x$$
$$\Gamma_q(t+1) = -\int_0^\infty x^t D_q E_q^{-\mathrm{x}} d_q x$$

Using q-integral by parts (2.11) we have

$$\Gamma_q[t+1] = -\left.\frac{qx^t E_1^{-\mathrm{q}t} + (q\mathrm{x})^t E_{\mathrm{q}}^{-\mathrm{q}^2 x}}{2q}\right|_0^\infty + \int_0^\infty E_q^{-\mathrm{qx}} \mathrm{D}_q X^t d_q x$$
$$= \left.\frac{q\lim_{x\to\infty}\left(x^t E_1^{-\mathrm{q}t}\right) + \lim_{x\to\infty}\left((\mathrm{qx})^t E_{\mathrm{q}}^{-\mathrm{q}^2 x}\right)}{2q}\right|_0^\infty + \int_0^\infty E_q^{-\mathrm{qx}} \mathrm{D}_q X^t d_q x$$
$$= \int_0^r (t) f^{t-1} E_q^{-qx} d_q t = [t]_q r_q(t).$$

$$\lim_{T\to\infty}\left(x^t E_q^{-qt}\right) = \lim_{x\to\infty}\left((qx)^t E_q^{-q^2 x}\right) = 0.$$

1. $\Gamma_q(1) = \int_0^p E_q^{-qx} d_q t = E_q^0 - E_q^\infty = 1$

2. $\hat{\Gamma}_q(1) = \int_0^\infty e_q^{-x} dx$

3. $\hat{\Gamma}_q(t+1) = \int_0^\infty x^t e_q^{-x} d_q x$

Using q-integral by parts (2.9)*, we have*

$$u = x^t, D_q v(e) = e_q(-x) dqx$$

from which we obtain

$$\hat{\Gamma}_q(t+1) = -\int_0^\alpha [t] x_1^{t-1}\left(e_q(-qx)\right) dq_q x$$
$$= [t]_q \int_0^\infty x^{t-1} e_q(-qt) dq_q x$$

and setting $\alpha = qx$, it follows that

$$\hat{\Gamma}_q \quad (t+1) = \frac{[t]_q}{q^t}\int_0^\infty \alpha^{t-1} e_q(-\alpha) d_q\alpha$$
$$= q^{-t}[t]_q \hat{\Gamma}_q(t).$$

Definition 2.4.2. *The q-Beta function is defined for $t,\ s > 0$ by the following*

$$\beta_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x \tag{2.44}$$

From the definition of the q-integral we have

$$\begin{aligned}\beta(t,\infty) &= (1-q)\sum_{j=0}^{\infty} q^j \left(q^j a\right)^{t-1}\left(1-q^{j+1}\right)_q^{\infty} \\ &= (1-q)\sum_{j=1}^{\infty} q^j \left(q^j a\right)^{t-1}\left(1-q^{j+1}\right)_{-\infty}^{\infty} \\ &= \int_0^{\infty} x^{t-1}(1-qx)_q^{\infty} d_q x\end{aligned}$$

Using the following relation $(1-q^{j+1})_q^{\infty} = 0$ for any non negative integer j, we have from the above $E_q^x = (1+(1-q)x)_q^{\infty}$, and thus we obtain

$$\mathbf{B_q(t,\infty)} = \int_0^{\infty} \mathbf{x^{t-1} E_q^{-\frac{qx}{(1-q)}} d_q x}$$

by the change the variable $x = (1-q)y$

$$\beta_q(t,\infty) = (1-q)^t \int_0^{\infty} y(t-1) E_q^{-qy} d_q y$$

or

$$\Gamma_q(t) = \frac{\beta_q(t,\infty)}{(1-q)^t} \tag{2.45}$$

Introducing another variable might seem like a step backward at first glance, but in reality, it increases our freedom in handling functions and simplifying the problem.

Proposition 2.4.2. *If $t > 0$, and $n \in \mathbb{Z}^+$, we have.*

$$\beta_q(t,n) = \frac{(1-q)(1-q)_q^{n-1}}{(1-q^t)_q^n} \tag{2.46}$$

For $t, s > 0$, we have:

$$\beta_q(t,s) = \frac{(1-q)(1-q)_q^{\infty}\left(1-1^{t+s}\right)_q^{\infty}}{(1-q^t)_q^{\infty}\left(1-q^s\right)_q^{\infty}} \tag{2.47}$$

Chapter 3

The q-Laplace transform

The Laplace transform is a mathematical tool used to convert time domain functions into frequency domain functions. It's useful in solving differential equations and analyzing linear time invariant systems, as it transforms different: operations into algebraic operations that are easier to handle.

The Laplace transform of the function f, is defined in terms of integral as follows

$$\mathcal{L}(f(t))(s) = F(s) = \int_0^1 f(t) \cdot e(-st)dt$$

For instance, the Laplace transform of the constant function, i.e. $f(t) = 1$, is $\frac{1}{s}$

$$\mathcal{L}(1)(s) = \int_0^\infty e(-st)dt = -\frac{1}{s}[e(-st)]_0^\infty = \frac{1}{s}$$

3.1 The q-analogue of Laplace transform

The q-Laplace transform is a q-version of the standard Laplace transform. Since there are two version of the q-exponential functions, it's obvious that we should have at least two version of q-Laplace transform. From another hand, because the q-exponentials are the inverse of each other, we shall only consider one of them.

Definition 3.1.1. *The q-Laplace transform of a function f is given by*

$$\mathcal{L}_q(f(t))(p) = \int_0^{\infty} e_q(-pt)f(t)dqt$$

where f is defined over the positive real axis and $\operatorname{Re}(p) > 0$.

Example 3.1.1. .

1. $f(t) = 1$

$$L_q(1)(s) = \int_0^{\infty} e_q(-st)d_qt = -\frac{1}{s}\int_0^{1} D_qe_q(-st)d_qt = \frac{1}{s}\left[e_q(-st)\right]_0^{\infty} = \frac{1}{s}$$

2. $f(t) = t$

$$\begin{aligned}\mathcal{L}_q(t)(s) &= \int_0^{\infty} +e_q(-st)dqt = -\frac{1}{s}\int_0^{1} tD_qe_q(-st)dqt \\ &= -\frac{1}{s}\left\{t\, e_q(st)|_0^{\infty} - \int\limits_0^{\infty} e_q(-st)d_qt\right\} = \frac{1}{s}\left\{\mathcal{L}_q(1)(s)\right\} = \frac{1}{s^2}\end{aligned}$$

3. $f(t) = t^2$

$$\begin{aligned}L_q\left(t^2\right)(s) &= \int\limits_0^{\infty} t^2e_q(-st)dqt = -\frac{1}{s}\int_0^{\infty} t^2D_qe_q(-st)d_qt \\ &= -\frac{1}{s}\left\{t^2e_q(-st)|_0^{\infty} - 2 - \int\limits_0^{\infty} te_q(-st)dqt\right\} = \frac{2}{s}\left\{f_q \mid t)(s)\right\} = \frac{2}{s^3}\end{aligned}$$

4. $f(t) = t^{\alpha}$ *and* $\alpha > -1$*, with the change of variable* $\rho t = x$

$$\mathcal{L}_q\left(t^2\right)(s) = \int_0^{\infty} t^{\alpha}e_q(-st)dqt = \frac{1}{s^{\alpha+1}}\int_0^{\infty} e_q(-x)x^2dqx = \frac{1}{s^{\alpha+1}}\hat{\Gamma}_q(\alpha+1)$$

Proposition 3.1.1. *we have the q-Laplace transform of some elementary functions*

1.
$$\mathcal{L}_q\left(e_q(at)\right)(s) = \sum_{n=0} \frac{a^n}{s^{n-1}} q^{-\binom{x^{n+1}}{}} \tag{3.1}$$

2.
$$\mathcal{L}_q\left(E_q(at)\right)(s) = \frac{q}{qs-a} \tag{3.2}$$

3.
$$\mathcal{L}_q(\cos_q(at))(s) = \frac{1}{s}\sum_{n=0}(-1)^n q^{-\binom{2n+1}{2}}\left(\frac{a}{s}\right)^{2n} \tag{3.3}$$

4.
$$\mathcal{L}_q\left(\cos_q(at)\right)(s) = \frac{q^2 s}{(qs)^2+a^2} \tag{3.4}$$

5.
$$\mathcal{L}_q\left(\sin_q(at)\right)(s) = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n q^{\binom{2n+1}{2}}\left(\frac{a}{s}\right)^{2n+1} \tag{3.5}$$

6.
$$\mathcal{L}_q\left(\sin_q(at)\right)(s) = \frac{aq}{(qs)^2+a^2} \tag{3.6}$$

Proof 3.1.1. *1.*

$$\begin{aligned}
\mathcal{L}_q\left(e_q(at)\right)(s) &= \int_0^\infty e_q(-st)e_q(at)dt = \int_0^\infty e_q(-st)\sum_{n=0}^{\infty}\frac{a^n t^n}{[n]!}dqt \\
&= \sum_{n=0}^{i}\frac{i}{[n]!}\int_0^\infty e_q(-st)t^n dqt = \sum_{n=0}\frac{a^n}{(n)}\mathcal{I}_q\left(t^n\right)(p) \\
&= \sum_{n=0}^{\infty}\frac{a^n}{[n-1]}\frac{1}{q}\frac{\hat{q}(n+1)}{s^{n+1}} = \sum_{n=0}^{\infty}\frac{a^n}{s^{n+1}}q^{(n)-1}
\end{aligned}$$

2.

$$\begin{aligned}
\mathcal{L}_q\left(E_q'(at)\right)(s) &= \int_0^\infty e_q(-st)E_q(qt)dqt = \int_0^\infty e_q(-st)\sum_{n=0}^{2}\frac{a^n t_n^n}{[n!}q^n\Bigg)dt \\
&= \sum_{n=0}^{\infty}\frac{\hat{a}q^{(n)}}{\ln)_q}\int_0^\infty e_q(-st)tdt = \sum_{n=0}^{\infty}\frac{aq^{(i)}}{[n]!}\mathcal{L}_q\left(t^n\right)(s) \\
&= \sum_{n=0}^{n}\frac{a^n q^{(n)}}{(n)_q^{q!}}\cdot\frac{\bar{q}^{n+2})}{s^{n+1}}[n]_q! = \frac{1}{s}\sum_{n=0}^{2}\left(\frac{a}{sq}\right)^n = \frac{1}{s}\cdot\frac{1}{1-\frac{a}{sq}} = \frac{q}{sq-a}
\end{aligned}$$

3. *according to equation (2.26) we have*

$$L_q\left(cosat\right)(s) = L_q\left(\frac{e_q\left(iat\right)+e_q\left(-iat\right)}{2}\right)(s)$$
$$= \frac{1}{2}\left[L_q\left(e_q\left(iat\right)(s)\right)+L_q\left(e_q\left(-iat\right)\right)(s)\right]$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} q^{\frac{-n(n+1)}{2}}\frac{(ia)^n}{s^{n+1}}+\sum_{n=0}^{\infty} q^{\frac{-n(n+1)}{2}}\frac{(-ia)^n}{s^{n+1}}\right]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{-n(2n+1)}}{s^{2n+1}}\left[(ia)^n+(-ia)^n\right]$$
$$= \sum_{n=0}^{\infty}\frac{q^{-n(2n+1)}}{s^{2n+1}}(-1)^n a^{2n} = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n q^{-\binom{2n+1}{2}}\left(\frac{a}{s}\right)^{2n}$$

4. *according to equation (2.27), we have*

$$\mathcal{L}_q(\cos(at))(s) = \mathcal{L}_q\left(\frac{E_q(iat)+E_q(-iat)}{2}\right) = \frac{1}{2}\left[\mathcal{L}_q\left(E_q(iat)\right)(s)+\mathcal{L}_q\left(E_q(-iat)\right)(s)\right]$$
$$= \frac{1}{2}\left[\frac{q}{ps-ia}+\frac{q}{ps+ia}\right] = \frac{q^2 s}{q^2s^2+a^2}$$

5. *according to equation (2.28), we have*

$$\mathcal{L}_q(\text{sing}(at))(s) = \mathcal{L}_q\left(\frac{e_q(iat)-e_q(-iat)}{2}\right)(s) = \frac{1}{2}\left[\mathcal{L}_q\left(e_q(iat)\right)(s)-\mathcal{L}_q\left(e_q(iat)\right)(s)\right]$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} q^{-\frac{n(n+1)}{2}}\frac{(ia)^n}{s^{n-1}}-\sum_{n=0}^{2} q^{-\frac{n(n+1)}{2}}\frac{(i-ia)^n}{s^{n+1}}\right]$$
$$= \sum_{n=0}^{2}(-1)^n\cdot q^{-\binom{2n+2}{2}}\frac{a^{2n+1}}{q^{2n+2}} = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n q^{-\left(\frac{2n+2}{2}\right)}\left(\frac{a}{s}\right)^{2n+1}$$

6. *according to equation (2.29), we have*

$$\mathcal{L}_q\left(\sin_q(at)\right)(s) = \mathcal{L}_q\left(\frac{E_q(iat)-E_q(-iat)}{2i}\right)(s)$$
$$= \frac{1}{2i}\left[\mathcal{L}_q\left(E_q(iat)(s)-\mathcal{L}_q\left(E_q\right)\right)-iat\right)(s)\right]$$
$$= \frac{1}{2i}\left[\frac{q}{sq-ia}-\frac{q}{sq+ia}\right] = \frac{aq}{q^2s^2+a^2}$$

In terms of the big q-exponential function, there is another definition of the second kind q-

Laplace transform denoted (L_q)*, and defined as follows*

$$L_q\left\{f(t)\right\}(s) = \int\limits_0^{\infty} f(t)E_q\left(-qst\right)d_qt$$

Example 3.1.2. *In terms of the second kind we have*

1. $f(t) = 1$

$$L_q\left(1\right)\left(s\right) = \int_0^{\sigma} E_q(-qst)dt = -\frac{1}{qs}\int_0^{\infty} D_qE_q(-qst)dt = -\frac{1}{qs}\left[E_q(-qst)\right]_0^{\infty} = \frac{1}{qs}$$

2. $f(t) = t$

$$\begin{aligned} L_q(t)(s) &= \int_0^{\infty} E_q(-qst)tdq^t = -\frac{1}{qs}\int_0^{\infty} tD_qE(-qst)dqt \\ &= -\frac{1}{qs}\left\{\left[tE_q(\cdot qst)|_0^{\infty} - \int_0^{\infty} E_q(-qst)dqt\right.\right. = \frac{1}{qs}\left\{\mathcal{L}_q(1)(s)\right\} = \frac{1}{(qs)^2} \end{aligned}$$

3. $f(t) = t^{\alpha},\ \alpha > -1$*, with the change of variable* $st = x$ *we obtain*

$$L_q\left(t^{\alpha}\right)\left(s\right) = \int_0^{\infty} E_q(-qst)t^{\alpha}dqt = \frac{1}{s^{\alpha+1}}\int_0^{\infty} E_q(-qx)x^{\alpha}dd_qx = \frac{1}{s^{\alpha+1}}\Gamma_q(t+1)$$

Proposition 3.1.2. *The q-Laplace transform* (L_q) *of elementary functions, we have for instance the following*

1.

$$L_q\left(e_q(t)\right)\left(s\right) = \sum_{n=0} \frac{1}{s^{n+1}}q^{\frac{n(n-1)}{2}} \tag{3.7}$$

2.

$$\mathrm{Lg}\left(E_q(t)\right)\left(s\right) = \frac{1}{s-1} \tag{3.8}$$

3.

$$L_q\left(\cos_q(t)\right)\left(s\right) = \sum_{n=0}^{n}(-1)^n q^{-(2n-1)}\frac{1}{s^{n+1}} \tag{3.9}$$

4.

$$\mathrm{L}_q\left(\cos_q(t)\right)\left(s\right) = \frac{s}{s^2+1} \tag{3.10}$$

5.

$$\mathrm{L}_q \sin(t))\,(s) = \sum_{n=0}(1)^n q^{(m-1)(n-1)} \frac{1}{s^{2n+2}} \tag{3.11}$$

6.

$$\mathrm{L}_q \left\{\sin_q(t)\right\}(s) = \frac{1}{s^2+1} \tag{3.12}$$

Proof 3.1.2. *1.*

$$\begin{aligned} L_q\left(e_q(t)\right)(s) &= \int_0^\infty E_q(-qst)e_q(t)dqt = \int_0^\infty E_q(-qst)\sum_{n=0}^{\infty}\frac{t}{[n]_q!}dqt \\ &= \sum_{n=0}^{\infty}\frac{1}{[n]_q!}\int_0^\infty E_q(-qst)t^n dqt = \sum_{n=0}^{\infty}\frac{1}{[n]_q!}\int_q (t^n)(s) \\ &= \sum_{n=0}^{\infty}\frac{1}{[n]}\cdot\frac{r_q(n+1)}{s^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{s^{n+1}}\cdot q^{\frac{n(n-1)}{2}} \end{aligned}$$

2.

$$\begin{aligned} L_q\left(E_q(t)\right)(s) &= \int_0^\infty E_q(-qst)E_q(t)dqt = \int_0^\infty E_q(-qst)\sum_{n=0}^{\infty}\frac{t^n}{[n]q!}q\binom{n}{2} \\ &= \sum_{n=0}^{\infty}\frac{q^{(n)}}{[n]}\int_q^\infty E_0(-qst)t^n d_q t = \sum_{n=0}^{\infty}\frac{q^{(n)}}{[n]}L_q\left(t^n\right)(s) \\ &= \sum_{n=0}^{\infty}\frac{q^{(n)}}{[n]}\frac{r_q(n+1)}{s^{\alpha+1}} = \frac{1}{s-1} \end{aligned}$$

in the same manner as before, we deduce(3).

3. $L_q\left(\cos_q(t)\right)(s) = \sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)}\frac{1}{s^{n+1}}$

4. $L_q\left(\mathrm{Cos}_q(t)\right)(s) = \frac{s}{s^2+1}$

5. $L_q\left(\sin_q(t)\right)(s) = \sum_{n=0}^{\infty}(-1)^n q^{(2n-1)(n-2)}\frac{1}{p^{2n+1}}$

6. $\mathrm{L}_q(\mathrm{Sin_q}(t))(s) = \frac{1}{s^2+1}$.

3.2 The q-Laplace transform and the q-derivative

Together with Jackson operator (D_q), the q-Laplace transform involve some perturbations, i.e. depends on the initial conditions of functions

Proposition 3.2.1. *Let f be a function defined on $\mathbb{R}_+$ with q-Laplace transform of $D_d f$ exist. Then*

$$\mathcal{L}_q\left(D_q f(t)\right)(s) = -f(0) + \frac{s}{q}\mathcal{L}_q\left(f(t)\right)\left(\frac{s}{q}\right) \tag{3.13}$$

Proof 3.2.1. *By definition, using q-integral by part, we find:*

$$\begin{aligned}\mathcal{L}_q\left(D_q f(t)\right)(s) &= \left[f(t)e_q(-st)|_0^\infty - \int_0^\infty f(qt)D_q e_q\left(-std_d\right]_d^t\right.\\ &= -f(0) - \int_0^\infty f(qt)D_q e_q(-st)d_q t\\ &= -f(0) + s\int_0^\infty f(qt)e_q(-st)dqt\\ &= -f(0) + \frac{s}{q}\int_0^\sigma f(x)e_q\left(-\frac{s}{q}x\right)dq^x\\ &= -f(0) + \frac{s}{q}\mathcal{L}_q(f(c))\left(\frac{s}{q}\right)\end{aligned}$$

More generally, we can apply the q-Laplace transform of $D_q^n f$ to obtain the $D_q^{n+1} f$

Proposition 3.2.2. *Let f be a function defined on $\mathbb{R}^+$ and assume that its q-Laplace transform of $D_q^n f$ exist. Then*

$$L_q\left(D_q^n f(t)\right)(s) = s^n q^{-\binom{n+1}{2}} L_q\left(f(t)\right)\left(\frac{s}{q^n}\right) - \sum_{i=0}^{n-1} s^{n-i-1} q^{-\binom{n-i}{2}} D_q^i f(0). \tag{3.14}$$

Proof 3.2.2. *We assume that equation (3.13) holds true for n and we proceed it for $n+1$.*

$$\begin{aligned}L_q\left(D_q^{n+1} f(t)\right)(s) &= L_q\left(D_q^n\left(D_q f(t)\right)\right)(s)\\ &= s^n q^{-\binom{n+1}{2}} L_q\left(D_q f(t)\right)\left(\frac{s}{q^n}\right) - \sum_{i=0}^{n-1} s^{n-i-1} q^{-\binom{n-i}{2}} D_q^i\left(D_q f\right)(0)\\ &= s^n q^{-\binom{n+1}{2}} L_q\left(D_q f(t)\right)\left(\frac{s}{q^n}\right) - \sum_{i=1}^{n} s^{n-i} q^{-\binom{n+1-i}{2}} D_q^i\left(f\right)(0).\end{aligned}$$

Now by using (3.13), we deduce that

$$\begin{aligned}\mathcal{L}_q\left(D_q^{n+1} f(t)\right)(p) &= p^n q^{-\binom{n+1}{2}}\left\{-f(0) + \frac{p}{q^{n+1}}\mathcal{L}_q\left(f(t)\right)\left(\frac{p}{q^{n+1}}\right)\right\} - \sum_{i=1}^{n} p^{n-i} q^{-\binom{n+1-i}{2}} D_q^i\left(f\right)(0)\\ &= p^{n+1} q^{-\binom{n+2}{2}}\mathcal{L}_q\left(f(t)\right)\left(\frac{p}{q^{n+1}}\right) - \sum_{i=0}^{n} p^{n-i} q^{-\binom{n+1-i}{2}} D_q^i\left(f\right)(0).\end{aligned}$$

where we have used the identity $\binom{n+1}{2} = \binom{n}{2} + n$.

Chapter 4

The q-Mellin transform

Initiated by Hjalmar Mellin (1854-1933), the Mellin transform of a suitable function f over $]0,\infty[$ *is given by*

$$M(f)(s) = \int_0^\infty f(x)x^{s-1}dx$$

The inversion formula for the Mellin transform is given by the following line integral,

$$f(x) = \frac{1}{2\pi in}\int_{c-i\infty}^{c+i\infty} M(f)(s)x^{-s}ds.$$

The definition of the Mellin convolution product of suitable functions f and g is

$$f *_M g(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}.$$

In this chapter we are interested with the q-analogue of Mellin transform in terms of q-Jackson integral.

Definition 4.0.1. *Let $f \in \mathbb{R}_{q,+}$, the q-Mellin transform of f is given by*

$$M_q(t)(s) = M_q[f(t)](s) = \int_0^\infty t^{s-1} f(t) d_q t. \tag{4.1}$$

There exists a (possibly empty) maximal open vertical strip denoted (x_0, x_1) in which the q-integral (4.1) is well defined. Such strip will be called a fundamental strip.

As the classical Mellin transform, we have interesting properties of q-Mellin transform which coincide with the classical Mellin transform.

Proposition 4.0.1. *Let $f, (a_i)_i \in \mathbb{R}_{q,+}$, we have the following properties*

$M_q[f(at)](s) = s^{-s} M_q[f(t)](s)$	$\frac{d}{ds} M_q[f(t)](s) = M_q[\log(f(t))](s)$
$M_q\left[f\left(\frac{1}{t}\right)\right](s) = M_q[f(t)](-s)$	$M_q\left[\frac{1}{t} f\left(\frac{1}{t}\right)\right](s) = M_q[f(t)](1-s)$
$M_q[tD_q f(t)](s) = [-s]_q M_q[f(t)](s)$	$M_q[D_q f(t)](s) = [1-s]_q M_q[f(t)](s-1)$
$M_q\left[\int_0^t f(x)\, d_q x\right](s) = \frac{1}{[-s]_q} M_q[f(t)](s+1)$	$M_q\left[f\left(t^\theta\right)\right](s) = \left[\frac{1}{\theta}\right]_{q^\theta} M_{q^\theta}[f(t)]\left(\frac{s}{\theta}\right)$
$M_q\left[D_q^n f(t)\right](s) = [1-s]_q[2-s]_q...[n-s]_q M_q[f(t)](s-n)$	
$M_q\left[\sum_{i=0}^{\infty} b_i f(a_i t)\right](s) = \left(\sum_{i=0}^{\infty} \frac{b_i}{a_i^s}\right) M_q[f(t)](s), \quad (b_i)_i \in \mathbb{C}$	

Example 4.0.1. *We summarize some q-transformation of elementary q-functions*

$M_q\left[E_q^{-qt}\right](s) = \Gamma_q(s)$	$M_q\left[E_q^{-t}\right](s) = q^s \Gamma_q(s)$
$M_q\left[E_{q^\alpha}^{-q^\alpha t^\alpha}\right](s) = \left[\frac{1}{\alpha}\right]_{q^\alpha} \Gamma_{q^\alpha}\left(\frac{s}{\alpha}\right)$	
$M_q\left[\cos_{q^2}(t)\right](s) = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}} q^{s-1/2} ((1+q))^{s-1/2} \frac{\Gamma_{q^2}\left(\frac{s}{2}\right)}{\Gamma_{q^2}\left(\frac{1-s}{2}\right)}$	
$M_q\left[\sin_{q^2}(t)\right](s) = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}} q^{s+1/2} (1+q)^{s-1/2} \frac{\Gamma_{q^2}\left(\frac{s+1}{2}\right)}{\Gamma_{q^2}\left(\frac{2-s}{2}\right)}$	

4.1 The q-Mellin inversion formula

The most interesting question of integral transformation is whether the inverse exist or not. In this section we shall discuss the inverse of the q-Mellin transform. The inversion formula is given by the next Theorem

Theorem 4.1.1. *Let $f \in \mathbb{R}_{q,+}$ and c in the fundamental strip, then we can inverse and obtain the expression of f from its q-Mellin transform by the formula*

$$f(x) = \frac{\log(q)}{2\pi i(1-q)} \int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(f)(s)x^{-s}ds. \tag{4.2}$$

Proof 4.1.1. *Let $x = q^n \in \mathbb{R}_{q,+}$, we have using the definition of q-Jackson integral*

$$\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(f)(s)x^{-s}ds = (1-q)\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} \sum_{-\infty}^{\infty} q^{s(k-n)} f(q^k)ds.$$

The above series converges uniformly with respect to s, therefore

$$\begin{aligned}\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(f)(s)x^{-s}ds &= i(1-q)\sum_{k=-\infty}^{\infty} q^{c(k-n)} f(q^k) \int_{-\frac{i\pi}{\log(q)}}^{\frac{i\pi}{\log(q)}} q^{i(k-n)t}dt \\ &= \frac{2i\pi(1-q)}{\log(q)} \sum_{k=-\infty}^{\infty} q^{c(k-n)} f(q^k)\delta_{k,n} \\ &= \frac{2i\pi(1-q)}{\log(q)} f(q^n) = \frac{2i\pi(1-q)}{\log(q)} f(x).\end{aligned}$$

Whence the desired formula.

For the convolution there are further properties q-analogue to the classical ones. We mention some of them here.

Definition 4.1.1. *The q-Mellin convolution product of the functions f and g is the function as denoted above by $f *_M g$ defined by*

$$f *_M g(x) = \int_0^{\infty} f(y)g\left(\frac{x}{y}\right)\frac{d_q y}{y}, \quad x \in \mathbb{R}_{q,+}. \tag{4.3}$$

provided the q-integral exists.

Theorem 4.1.2. *If the q-Mellin convolution product of f and g exists, then*

1. $f *_M g = g *_M f$
2. $M_q[f *_M g] = M_q(f)M_q(g)$

Theorem 4.1.3. *For the suitable functions f and g, we have the following relations*

1. $$\frac{\log(q)}{2\pi i(1-q)}\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(f)(s)M_q(g)(1-s)ds = \int_0^\infty f(x)g(x)d_qx$$
2. $$\frac{\log(q)}{2\pi i(1-q)}\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(f)(s)M_q(g)(s)ds = \int_0^\infty f(y)g\left(\frac{1}{y}\right)\frac{d_qy}{y},$$

4.2 Conclusion

It is worthy to mention that Mellin and its q-analogue is quite different of Laplace and Fourier transforms as well as their q-analogues. Indeed, we can combine the Fourier cosine and Fourier sine all together to obtain the Laplace transform with a slate modification and this true either for classical transformations or their q-analogues.

We remarked at the construction stage of some q-analogues of elementary functions that we always have two choices: one for $0 < q < 1$ and one when $q > 1$, and this corresponds somehow to the right and left fractional integral calculus. That wealth has been approved with their applications mainly at recent discover of the quantum theory and application in q-information, q-bit, ... etc.

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