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Option : PDE *and Applications*

Theme :

STUDY OF EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS OF SOME SYSTEMS AND PARTIAL DIFFERENTIAL EQUATIONS

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بسم الله الرحمن الرحيم

Thanks

At the beginning of my speech, I must first express my gratitude **to the Almighty God**, who has granted me the success to reach this high academic level

Also, I extend my gratitude and appreciation to **Dr. Mesloub Fatiha**, whose guidance and advice played a crucial role in the completion of my academic studies.

As well, I offer my profound thanks to all the esteemed committee members for agreeing to review my master's thesis.

Class of 2024

"Dedicated to"

With all my love, I dedicate the fruit of my success and graduation to the one who adorned my name with the most beautiful titles. The one who supported me boundlessly and gave me without expectation.

My Father

To the one whom God placed paradise beneath her feet. To the tender heart. To her whom I will never be able to repay, no matter what I do.

My Mother

To the one who brightens my life with her warm smile and infinite tenderness. Thank you for the unparalleled care and support.

My Grandmother

To the most wonderful brothers who form the heart of the family : "To my elder brother, **Mouhammed.**"

"To my sister **Zainab** and my brother **Islam.**"

Abstract

In this work we are concerned to prove the existence, uniqueness and the continuous dependance of a strong solution upon the data of some mixed problems with classical (Dirichlet, Neumann) and integral conditions for a hyperbolic class of equations and for a hyperbolic system. In a first case, we treate a mixed problem with purely classical conditions and in a second case we combine a classical and an integral condition.

In our proofs, we use functionnal analysis method.

Résumé

Dans ce mémoire, on démontre l'existence, l'unicité et la dépendance continue par rapport aux données de la solution forte de quelques problémes mixtes avec des conditions intégra- les et classiques (Dirichlet, Neumann), pour une classe d'équations hypérboliques de type Boussinesq, et un systéme hypérbolique; dans un premier cas, on étudie le probleme mixte avec des conditions pûrement classiques de type Dirichlet, et dans un autre avec des conditions classiques et intégrales. La méthode utilisée dans les démonstrations est celle de l'analyse fonctinnelle.

ملخص

الهدف من المذكرة انها تركز على دراسة وجود و وحدانية الحل وارتباطه المستمر بالنسبة لبعض المسائل المختلطة بشروط كلاسيكية وتكاملية للمعادلة التفاضلية الجزئية، بالإضافة إلى معادلتين تفاضليتين جزئيتين زائديتين , في الحالة الاولى بشروط كلاسيكية وتكاملية,وفي الحالة الثانية بشروط كلاسيكية بحتة. البراهين تعتمد على طريقة التحليل الدالي.

Contents

Introdution

The functional analysis method, also known as the method of energy inequalities or a priori estimates, has been developed to address a wide array of mixed problems across various classes of equations. It consistently emerges as one of the most effective techniques for tackling such diverse problem sets.

This present work endeavors to extend this method to a novel category of mixed problems, encompassing hyperbolic systems with purely classical or combined classical and integral conditions. Additionally, it aims to address a class of hyperbolic equations known as Boussinesq type, within the functional spaces of Sobolev.

This method was first formally presented by ***A.A.DEZIN*** . Its versatility has been demonstrated across a spectrum of equations, including elliptic, parabolic, hyperbolic, composite, mixed, non-classical, and operational equations. These have been tackled with diverse boundary conditions such as Cauchy, Dirichlet, Neumann, among others, as well as transmission problems. Integral conditions, representing averages, total energy, mass, and moments, have proven to be pivotal in practical applications.

Mathematical modeling of mixed problems with integral conditions finds application in heat transfer theory, population dynamics, plasma physics, metallurgy, thermoelasticity, and elasticity.

To effectively apply this method, certain steps must be adhered to:

- ❏ Obtaining a priori estimates.
- ❏ Assessing the density of the images of operators generated by the considered problems.
- ❏ Selecting the appropriate functional framework for each problem.

These steps constitute the general framework of the functional analysis method. This method aims to provide a systematic and effective approach to solving complex mathematical problems and serves as a valuable tool for researchers and engineers in various applied fields.

This method is based on the ideas of J. Leray [43], L. Garding [23]. It has been used for studying problems related to elliptic equations [37], [38], [44], parabolic equations [22], [36], [38], hyperbolic equations [5], [17], [22], [24], [25], [32], [36], [38], [45], composite equations [3], mixed equations [1], [29], [30], non-classical equations [2], [8], [9], [12], [33], and operational equations [14], [15], [20], [22], [30], [39], [40], [45], [46],with boundary conditions such as Cauchy, Dirichlet, Neumann,..., as well as for transmission problems [31], [34], [41].

In this work, we adhere to the outlined scheme with precision:

scheme:

The problem posed is written in operational form as:

$$Lu = \mathscr{F}, \qquad \forall u \in D(L) \tag{1}$$

where the operator L is considered from the Banach space B into the suitably chosen Hilbert space H. Then, the a priori estimation (known as the energy inequality) of the form :

$$\|u\|_B \leq c\|Lu\|_H \tag{2}$$

is established. It should be noted that for the problems posed in our work, this type of a priori estimations is obtained by multiplying the considered equation or system by a differential or integro-differential operator $\mathscr{M}u$ containing linear expressions in terms of the unknowns. This method was first proposed and used by K. O. Friedrichs and H. Lewy in [21], O.

A. Ladyzhenskaya in [36], and P. Lax in [42].
Next, it is shown that the operator L from B into H has a closure $\overline{L}$.
The solution to the operational equation:

$$\overline{L}u = \mathscr{F}, \qquad \forall u \in D(\overline{L}) \tag{3}$$

will be termed the strong solution of the given problem. Through a limit process, we extend the estimate (2) to strong solutions, resulting in:

$$\|u\|_B \leq c\|\overline{Lu}\|_H \tag{4}$$

From inequality (4), we deduce:

- ❏ The uniqueness of the strong solution of the posed problem and its continuous dependence on the $\mathscr{F} \in H$ data.
- ❏ The equality between the sets $R(\overline{L})$ and $\overline{R(L)}$.
- ❏ The invertibility of $\overline{L}$, where the inverse $\overline{L}^{-1}$ is defined on the range set $R(\overline{L})$ of the operator $\overline{L}$.

The final step involves establishing the density of the set $R(L)$ in H, thereby ensuring the existence of a strong solution to problem (1).
Our thesis consists of three chapters, which are:
The first chapter is devoted to some concepts in the theory of functional spaces and operator theory, along with the inequalities used in our study.
In the second chapter, we address a mixed problem for a class of hyperbolic equations of Boussinesq type with integral conditions.
The third chapter is dedicated to the study of a problem for a hyperbolic linear system with the Bessel operator under classic Dirichlet-type conditions.

Note:
I relied on this memory
To its owner : **Mansour Abdelouahab**

Theme:

Etude de l'existence et l'unicité des solutions fortes de quelques Problémes Mixtes Avec conditions classiques et intégrales Pour une classe d'équations aux dérivées partielles et systémes.
Date 2003.

Chapter 1

Preliminary Notations

1.1 *Normed Space*

Definition 1.1.1 (*The Norm*) *Let X be a real (or complex) vector space, a norm on the linear space X is a real-valued function $\|.\| : X \to [0,+\infty[$, where the value at x is denoted by $\|x\|$ and has the properties:*

- $\|x\| \geq 0$
- $\|\alpha x\| = |\alpha|\,\|x\|$ *for all scalars* α
- $\|x\| \neq 0$ *if* $x \neq 0$
- $\|x+y\| \leq \|x\| + \|y\|$

Definition 1.1.2 (*Normed Linear Space*) *A linear space $(X, \|.\|)$ on which a norm is defined is called a normed linear space.*

Definition 1.1.3 (*Complete Space*) *The normed space X is complete if every Cauchy sequence in X converges to an element in X (i.e) has a limit that is an element of X.*

1.2 *Banach Space*

Definition 1.2.1 (*Banach Space*:) *If a normed linear space $X = (X, \|.\|)$ is complete, it is called a Banach space.*

theorem 1.2.1 *Let X be a normed linear space, Then there exists a linear space Y such that Y is complete and X is a dense subset of Y.*
Up to isometry, the space Y is unique.

Definition 1.2.2 (*Completion*) *The space Y given by **Theorem 1.2.1** is called the completion (partition) of X.*

1.3 *Hilbert Space*

Definition 1.3.1 (Pre-Hilbert Space) *A complex linear space X is called a pre-Hilbert space if for each pair of elements x, y in X, there exists an associated complex number (x, y) called the pre-Hilbertian of x and y with the following properties:*

- $(x+y,z)=(x,z)+(y,z)$, for all $x,y,z\in X$
- $(x,y)=\overline{(y,x)}$, where the bar denotes complex conjugate.
- $(\alpha x,y)=\alpha(x,y)$, $\forall\alpha$
- $(x,x)\geq 0$, for all $x\in X$,
- $(x,x)=0\Leftrightarrow x=0$, for all $x\in X$

The pre-Hilbert space is a special case of a normed linear space, as expressed by the following lemma.

lemma 1.3.1 *: Let X be a linear space equipped with a scalar product. Then the expression $\|x\|=\sqrt{(x,x)}$, for all $x\in X$, defines a norm on X.*

Definition 1.3.2 (*Hilbert space*) *A Hilbert space is a complete pre-Hilbert space (a normed linear space).*
A distance on H is given by:

$$\|x-y\|=\sqrt{(x-y,x-y)}.$$

Therefore, pre-Hilbert spaces are normed spaces, and Hilbert spaces are Banach spaces.

1.4 *Space of Sobolev*

Definition 1.4.1 *Let $\Omega\subset\mathbb{R}^n$ be an open set, k a non-negative integer, and let $1\leq p\leq\infty$, We define $W_p^k(\Omega)$ to be the set of all distributions $u\in L^p(\Omega)$ such that $D^\alpha u\in L^p(\Omega)$ for $|\alpha|\leq k$, where $\alpha\in\mathbb{N}$, $\alpha=(\alpha_1,\alpha_2,....,\alpha_n)$, $\alpha_i\in\mathbb{N}$, $i=\overline{1,n}$ such that:*

$$W_p^k(\Omega)=\{u\in L^p(\Omega),\quad D^\alpha u\in L^p(\Omega);\quad\forall\alpha:|\alpha|\leq k\}$$

where

$$D^\alpha u=\frac{\partial^\alpha u}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2.......\partial^{\alpha_n}x_n},\quad |\alpha|=\sum_{i=1}^{n}\alpha_i$$

and

$$L^p(\Omega)=\{u:measurable\setminus\int|u(x)|^p dx<\infty\}$$

$$L^\infty(\Omega)=\{u:mesurable\setminus\exists c\quad such\ that\ |u(x)|<c\ on\quad\Omega\}.$$

$L^p(\Omega)$ is a complete space for the norm

$$\|u\|_{L^p(\Omega)}=(\int|u(x)|^p dx)^{\frac{1}{p}},\quad 1\leq p\leq\infty$$

And $L^\infty(\Omega)$ is a complete space for the norm:

$$\|u\|_{L^\infty(\Omega)}=\sup_{x\in\Omega}|u(x)|.$$

In $W_p^k(\Omega)$, a norm is defined by:

$$\|u\|_{W_p^k(\Omega)}=\sum_{|\alpha|\leq k}^{n}\left\|D^\alpha u\right\|_{L^p(\Omega)},\quad p<\infty$$

$$\|u\|_{W_p^\infty(\Omega)}=\max_{|\alpha|\leq k}\left\|D^\alpha u\right\|_{L^\infty(\Omega)}$$

For $p=2$, we define a scalar product as:

$$(u,v)_{W_2^k(\Omega)}=\sum_{|\alpha|\leq k}\left(D^\alpha u(x),D^\alpha v(x)\right)_{L^2(\Omega)}=\sum_{|\alpha|\leq k}\int_\Omega D^\alpha u(x)D^\alpha v(x)dx.$$

For $W_2^k(\Omega)$ we also use the notion of $H^k(\Omega)$.

1.5 *Orthogonality-Orthogonal Complement*

Definition 1.5.1 (*Orthogonality*) *Let H be a Hilbert space (more generally, a pre-Hilbert space). We say that two elements of H, x and y are orthogonal (or x is orthogonal to y) if:*

$$(x, y) = 0,$$

And we write $x \perp y$ Similarly, for the inclusion

$$A, B \subset H$$

We write:

$$x \perp y \quad if \quad x \perp a, \quad \forall a \in A$$

Definition 1.5.2 (*Orthogonal Complement*) *For each subspace M of H, we define the orthogonal complement as:*

$$M^{\perp} = \{x \in H \mid \quad \langle x, a \rangle = 0, \quad \forall a \in M\}$$

Which is the set of all vectors orthogonal to M.
It is clear that $M^{\perp}$ is a closed subspace. If M is also closed, then H is a direct sum of M and $M^{\perp}$:

$$H = M \oplus M^{\perp}$$

theorem 1.5.1 *Let H be a Hilbert space, A subspace M of H is dense if and only if $M^{\perp} = \{0\}$.*

$$M^{\perp} = \{0\} \Leftrightarrow \bar{M} = H$$

theorem 1.5.2 (*Completion*) *For any normed space (Pre-Hilbert space) X, there exists a Banach space (Hilbert space) X and an isometry A from X to a dense subspace W of X: the space X is unique, except for isometries.*
The following concepts are very useful for studying unbounded operators.

1.6 *Closed operators*

Definition 1.6.1 *The graph of a linear operator $A : X \to Y$ is the set of ordered pairs:*

$$\Gamma(A) = \{(x, Ax) \ \backslash \ x \in D(A)\} \subset X \times Y$$

Note that the graph is a subspace of $X \times Y$.

lemma 1.6.1 *The operator $\tilde{A}$ is an extension of A if and only if:*

$$\Gamma(\tilde{A}) \supset \Gamma(A)$$

Definition 1.6.2 *An operator A is said to be closed if its graph is closed as a subset of $X \times Y$.*

Definition 1.6.3 *A is called closable if it has a closed extension, Every closable operator has a smallest closed extension called its closure and denoted by $\bar{A}$.*
The following lemma is a direct consequence of the definition of a closed operator.

lemma 1.6.2 *An operator A is closed if and only if it has the following properties, Whenever there is a sequence $x_n \in D(A)$ such that:*

- $x_n \to x$
- $Ax_n \to f$

then

- $\forall x \in D(A)$ *Then* $Ax = f$

A similar characterization holds for a closable operator.

lemma 1.6.3 *An operator A is closed if for any sequence $x_n \in D(A)$ such that $x_n \to 0$, we have either:*

- $Ax_n \to 0$ *or*
- $\lim_{n\to\infty} Ax_n$ *does not exist.*

Corollary 1.6.1 *If A is closed, then:*

$$\Gamma(\bar{A}) = \bar{\Gamma}(A)$$

1.7 *Regularization of Operators*

Let W be a function of class C^∞, with the variables α such that:

$$W(\alpha) > 0; \quad W = 0, \quad |\alpha| \geq 1$$

And

$$\int_{-\infty}^{\infty} W(\alpha)d\alpha = \int_{-1}^{1} W(\alpha)d\alpha = 1$$

We denote by:

$$W_\varepsilon(x,y) = \frac{1}{\varepsilon}W(\frac{x-y}{\varepsilon})$$

For every $\varepsilon > 0$, we have:

$$\int_{-\infty}^{\infty} W_\varepsilon(x,y)dy = \int_{-1}^{1} W_\varepsilon(x,y)dy = 1,$$

And

$$W_\varepsilon(x,y) = 0, \quad \left|x-y\right| \geq \varepsilon.$$

We define the smoothing operator $J_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ by the formula:

$$(J_\varepsilon v)(x) = \int_{-\infty}^{\infty} W_\varepsilon(x,y)v(y)dy = \int_{|x-y|<\varepsilon} W_\varepsilon(x,y)v(y)dy,$$

where $\Omega = (a,b) \subset \mathbb{R}$ and $v \in L^2(\Omega)$ This operator has the following properties:

1. The function $J_\varepsilon v \in C^\infty$ if $v \in L^2(\Omega)$, and

$$\frac{\partial^m}{\partial x^m}(J_\varepsilon v) = J_\varepsilon\left(\frac{\partial^m}{\partial x^m}J_\varepsilon v\right), \quad \text{si}\, v \in C^m(\Omega)$$

2. If $v \in L^2(\Omega)$, then
$$\|J_\varepsilon v - v\|_{L^2(\Omega)} \to 0, \quad \text{when} \quad \varepsilon \to 0$$
And
$$\|J_\varepsilon v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}.$$
3. If $\beta \in C(\Omega)$ and $v \in L^2(\Omega)$, then
$$\left\|\beta J_\varepsilon v - J_\varepsilon(\beta v)\right\|_{L^2(\Omega)} \to 0, \quad \text{when} \quad \varepsilon \to 0$$
4. If $\beta \in C^1(\Omega)$ and $v \in L^2(\Omega)$, then
$$\left\|\frac{\partial}{\partial x}\left(\beta J_\varepsilon v - J_\varepsilon(\beta v)\right)\right\|_{L^2(\Omega)} \to 0, \quad \text{when} \quad \varepsilon \to 0$$

1.8 *Important Inequalities*

1.8.1 *Gronwall's Lemma*

If $g_i(\tau)$ $i = 1,2,3$ are non-negative functions on $(0,T)$, $g_1(\tau)$, $g_2(\tau)$ are integrable functions, and $g_3(\tau)$ is a non-decreasing function on $(0,T)$, then if
$$\Im_\tau g_1 + g_2 \leq g_3 + c\Im_\tau g_2,$$
Then
$$\Im_\tau g_1 + g_2 \leq \exp(c\tau) g_3(\tau),$$
Where
$$\Im_\tau g_i = \int_0^\tau g_i(t)\,dt, \quad i = (1,2).$$

1.8.2 *Cauchy inqualty*

For all $a, b \in \mathbb{R}$, we have:
$$ab \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$$

1.8.3 *Cauchy inequality with ε*

The following inequality:
$$ab \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \quad a, b \in \mathbb{R}$$
Remember for everything $\varepsilon > 0$.

1.8.4 *Cauchy-Schwartz Integral Inequality*

For all $u, v \in L^2(\Omega)$, we have the following inequality:
$$\int_\Omega u(x)v(x)\,dx \leq \left(\int_\Omega u^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_\Omega v^2(x)\,dx\right)^{\frac{1}{2}}$$
Is called the Cauchy-Schwartz integral inequality.

1.8.5 *Young inquality*

The generalization of the Cauchy-Schwartz inequality is denoted by Young's inequality which is given by:

$$ab \le \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^{\frac{p-1}{p}}, \quad a,b \in \mathbb{R}, \quad p>1$$

1.8.6 *Young inquality with ε*

For all $\varepsilon > 0$, we have inquality:

$$ab \le \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p-1}{p}}, \quad a,b \in \mathbb{R}, \quad p>1$$

1.8.7 *Holder inquality*

For all $u, v \in L^p(\Omega)$, we have the following inequality:

$$\int_\Omega u(x)v(x)dx \le \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^q dx\right)^{\frac{1}{q}},$$

Where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, this inequality is the generalization of the inequality of the Cauchy-Schwartz Integral.

1.8.8 *Poincary Inequality*

For all $u \in W_2^1(\Omega)$, we have inquality:

$$\int_\Omega u^2 dx \le C_\Omega^2 \int_\Omega u_x^2 dx$$

where C_Ω^2 is a constant depending only on Ω.

1.8.9 *Elementary Inequality*

$$\int_0^l \left(\Im_x^2(\lambda u)\right)^2 dx \le \frac{l^2}{2} \|\Im_x(\lambda u)\|_{L^2(\Omega)}^2,$$

$$\int_0^l x(\Im_x(\lambda u))^2 dx \le l \|\Im_x(\lambda u)\|_{L^2(\Omega)}^2$$

$$\int_0^l (\Im_x(\lambda u))^2 dx \le \frac{l^3}{2} \|u(.,t)\|_{L^2(\Omega)}^2,$$

Chapter 2

Problems with an integral condition with weight for a class of hyperbolic equations of the Boussinesq type are considered.

In this chapter, we demonstrate the existence, uniqueness, and continuous dependence on the data of the strong solution of the posed problem (2.1)-(2.5). We establish an a priori estimate and demonstrate the density of the set of values of the operator generated by this problem.

2.1 *Problem setting:*

In the bounded domain $Q = (0, l) \times (0, T)$, where $l < \infty$ and $T < \infty$ The problem of determining the solution $u(x, t)$ of the differential equation is considered

$$\mathscr{L}u = u_{tt} - (b(x,t)u_x)_x - \beta \frac{\partial^4 u}{\partial t^2 \partial x^2} = f(x,t), \quad \forall (x,t) \in Q \tag{2.1}$$

where β is a strictly positive real constant, and the function $b(x, t)$ satisfies the following conditions:

1. $C_1 : b_0 \leq b(x,t) \leq b_1$, $b_t(x,t) \leq b_2$, $b_x(x,t) \leq b_3$,for all $(x,t) \in \overline{Q}$

2. $C_2 : b_{tt}(x,t) \leq b_4, b_{xt}(x,t) \leq b_5$, for all $(x,t) \in \overline{Q}$

Under the aforementioned conditions, as well as throughout, the b_i , $i = \overline{0,5}$ are strictly positive constants.
To equation (2.1), the following initial conditions are associated:

$$l_1 u = u(x,0) = \varphi_1(x), x \in (0,l) \tag{2.2}$$

$$l_2 u = u_t(x,0) = \varphi_2(x), x \in (0,l) \tag{2.3}$$

The Dirichlet boundary condition

$$u(0,t)=0, t\in(0,T) \tag{2.4}$$

And the integral condition

$$\int_0^l xu(x,t)dx=0, t\in(0,T) \tag{2.5}$$

Where f and φ_1,φ_2 are given functions satisfying compatibility conditions:

$$\varphi_1(0)=0, \int_0^l x\varphi_1(x)dx=0$$

$$\varphi_2(0)=0, \int_0^l x\varphi_2(x)dx=0$$

Firstly, it is observed that if $\beta=0$, problem (2.1)-(2.5) will be the problem addressed by Bouziani in [5].with integrable conditions and Neumann boundary conditions .

2.2 *Functional spaces*:

Are essential for studying the posed problem. Let $L^2(Q)$denote the Hilbert space consisting of (classes of) functions defined and square integrable on Q equipped with the scalar product given by:

$$(u,v)_{L^2(Q)}=\int_Q uvdxdt \tag{2.6}$$

And the associated norm

$$\parallel u\parallel_{L^2(Q)}=\left(\int_Q u^2dxdt\right)^{\frac{1}{2}} \tag{2.7}$$

Similarly, $L^2(0,l)$ .is defined with respect to the inner product given by:

$$(u,v)_{L^2(0,l)}=\int_0^l uvdx \tag{2.8}$$

And the norm:

$$\|u\|_{L^2(0,l)}=\left(\int_0^l u^2dx\right)^{\frac{1}{2}} \tag{2.9}$$

$L^2_\rho(Q)$ is the Hilbert space of square integrable functions with weight having finite norm.

$$\|u\|_{L^2_\rho(Q)}=\left(\int_Q xu^2dxdt\right)^{\frac{1}{2}} \tag{2.10}$$

The inner product in $L^2_\rho(Q)$ is defined as:

$$(u,v)_{L^2_\rho(Q)}=(xu,v)_{L^2(Q)} \tag{2.11}$$

$L^2_\rho(0,l)$ is defined in the same way by the scalar product

$$(u,v)_{L^2_\rho(0,l)}=\int_0^l xuvdx \tag{2.12}$$

And the norm

$$\|u\|_{L^2_\rho(0,l)} = \left(\int_0^l xu^2 dx\right)^{\frac{1}{2}} \tag{2.13}$$

The problem (2.1) - (2.5) can be written in the following operational form:

$$Lu = \mathscr{F}, \qquad u \in D(L) \tag{2.14}$$

Where

$$Lu = (\mathscr{L}u, l_1 u, l_2 u) \quad \text{and} \quad \mathscr{F} = (f, \varphi_1, \varphi_2) \tag{2.15}$$

The operator L is considered from the space B into the space H , where B is the Banach space consisting of functions $u \in L^2(Q)$, satisfying the conditions (2.3)-(2.5) and having finite norm:

$$\|u\|_B^2 = \sup_{0\le\tau\le T}\left[\|u(.,\tau)\|^2_{L^2(0,l)} + \|u_t(.,\tau)\|^2_{L^2(0,l)}\right] \tag{2.16}$$

And H is the Hilbert space $L^2(Q) \times L^2(0,l) \times L^2(0,l)$ .consisting of all elements $\mathscr{F} = (f, \varphi_1, \varphi_2)$.whose norm:

$$\|\mathscr{F}\|^2{}_H = \|f\|^2_{L^2_{(Q^\tau)}} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)} \tag{2.17}$$

Is finite.

The domain of definition $D(L)$ of the operator L is the set of all $u \in L^2(Q)$ functions for which

$$u_t, u_x, u_{tx}, u_{tt}, u_{ttx} \in L^2(Q)$$

And satisfying conditions (2.3) -(2.5).

2.3 *A priori estimation*

theorem 2.3.1 *If conditions $C_1 - C_2$ .are satisfied, then for any function .*
$u \in D(L)$, we have the a priori estimation :

$$\|u\|_B \le c\|Lu\|_H \tag{2.18}$$

Where c is a positive constant independent of the function u.

Proof. Consider the scalar product in $L^2(Q^\tau)$ of equation (2.1) and the integrated-differential operator

$$Mu = x\Im_x^* u_t - \Im_x^*(\rho u_t)$$

Where

$$Q^\tau = (0,l) \times (0,\tau) \ \text{ with } \ 0 \le \tau \le T$$

And

$$\Im_x^* v = \int_x^l v(\xi, t) d\xi,$$

We obtain

$$(\mathscr{L}u, Mu)_{L^2(Q^\tau)} = (\mathscr{L}u, x\Im_x^* u_t)_{L^2(Q^\tau)} - (\mathscr{L}u, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \tag{2.19}$$

We have:

$$\begin{aligned}(\mathscr{L}u, x\Im_x^* u_t)_{L^2(Q^\tau)} &= (u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} - ((b(x,t)u_x)_x, x\Im_x^* u_t)_{L^2(Q^\tau)} \\ &\quad -\beta(u_{ttxx}, x\Im_x^* u_t)_{L^2(Q^\tau)}\end{aligned} \tag{2.20}$$

By integrating by parts each term of the right-hand side of equality (2.20) and taking into account conditions (2.2) - (2.5), we obtain:

$$\begin{aligned}(u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} &= \int_{Q^\tau} x u_{tt} \Im_x^* u_t dxdt \\ &= \int_0^\tau \Im_x^* u_{tt} x \Im_x^* u_t|_{x=0}^{x=l} dt + \int_{Q^\tau} \Im_x^* u_{tt} \Im_x^* u_t dxdt - \int_{Q^\tau} \Im_x^* u_{tt} x u_t dxdt \\ &= \frac{1}{2}\left\|\Im_x^* u_t(.,\tau)\right\|_{L^2(0,l)}^2 - \frac{1}{2}\left\|\Im_x^* \varphi_2\right\|_{L^2(0,l)}^2 - (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\end{aligned} \tag{2.21}$$

$$\begin{aligned}&-((b(x,t)u_x)_x, x\Im_x^* u_t)_{L^2(Q^\tau)} \\ =& -\int_0^\tau b(x,t)u_x x\Im_x^* u_t|^{x=l}{}_{x=0} dt + \int_{Q^\tau} b(x,t)u_x x \Im_x^* u_t dxdt - \int_{Q^\tau} b(x,t) u_x x u_t dxdt \\ =& \int_0^\tau b(x,t) u \Im_x^* u_t|_{x=0}^{x=l} dt + \int_{Q^\tau} b(x,t) u u_t dxdt - \int_{Q^\tau} b(x,t) u \Im_x^* u_t dxdt - (b(x,t)u_x, u_t)_{L^2(Q^\tau)} \\ =& \frac{1}{2}\left\|\sqrt{b(.,\tau)}u(.,\tau)\right\|_{L^2(0,l)}^2 - \frac{1}{2}\left\|\sqrt{b(.,\tau)}\varphi_1\right\|_{L^2(0,l)}^2 - \frac{1}{2}\left\|\sqrt{b_t(.,t)}u\right\|_{L^2(Q^\tau)}^2 \\ &-(b_x(x,t)u, \Im_x^* u_t)_{L^2(Q^\tau)} - (b_x(x,t)u_x, u_t)_{L^2_\rho(Q^\tau)}\end{aligned} \tag{2.22}$$

$$\begin{aligned}&-\beta(u_{ttxx}, x\Im_x^* u_t)_{L^2(Q^\tau)} = \\ =& -\beta\int_0^\tau u_{ttx} x \Im_x^* u_t|_{x=0}^{x=l} dt + \beta\int_{Q^\tau} u_{ttx} \Im_x^* u_t dxdt - \beta \int_{Q^\tau} u_{ttx} x u_t dxdt \\ =& \beta \int_0^\tau u_{tt} \Im_x^* u_t|_{x=0}^{x=l} dt + \beta \int_{Q^\tau} u_{ttx} u_t dxdt - \beta\int_{Q^\tau} u_{ttx} x u_t dxdt \\ =& \frac{\beta}{2}\left\|u_t(.,\tau)\right\|_{L^2(0,l)}^2 - \frac{\beta}{2}\left\|\varphi_2\right\|_{L^2(0,l)}^2 - \beta(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}\end{aligned} \tag{2.23}$$

We also have :

$$\begin{aligned}&-(\mathscr{L}u, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\ =& -(u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} + ((b(x,t)u_x)_x, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} + \beta(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)}\end{aligned} \tag{2.24}$$

By integrating by parts each term of the right-hand side of equality (2.24) and taking into account conditions (2.2) - (2.5) we obtain,

$$\begin{aligned}&-(u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\ =& \int_0^\tau \Im_x^* u_{tt} \Im_x^*(\rho u_t)|_{x=0}^{x=l} dt + \int_{Q^\tau} \Im_x^* u_{tt} x u_t dxdt \\ =& (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\end{aligned} \tag{2.25}$$

$$\begin{aligned} &((b(x,t)u_x)_x, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\ = &\int_0^\tau b(x,t)u_x \Im_x^*(\rho u_t)|_{x=0}^{x=l} dt + \int_{Q^\tau} b(x,t)u_x x u_t dx dt \\ = &(b(x,t)u_x, u_t)_{L^2_\rho(Q^\tau)} \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\beta(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\ = &\beta\int_0^\tau u_{ttx} \Im_x^*(\rho u_t)|_{x=0}^{x=l} dt + \beta\int_{Q^\tau} u_{ttx} x u_t dx dt \\ = &\beta(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)} \end{aligned} \tag{2.27}$$

Let's substitute equalities (2.21) - (2.23) and (2.25) - (2.27) into equality (2.19),
It follows

$$\begin{aligned} &\frac{1}{2}\left\|\Im_x^* u_t(.,\tau)\right\|^2_{L^2(0,l)} + \frac{1}{2}\left\|\sqrt{b(.,t)}u(.,\tau)\right\|^2_{L^2(0,l)} + \frac{\beta}{2}\left\|\Im_x^* u_t(.,\tau)\right\|^2_{L^2(0,l)} \\ = &\frac{1}{2}\left\|\Im_x^*\varphi_2\right\|^2_{L^2(0,l)} + \frac{1}{2}\left\|\sqrt{b(.,t)}\varphi_1\right\|^2_{L^2(0,l)} + \frac{\beta}{2}\left\|\varphi_2\right\|^2_{L^2(0,l)} + \frac{1}{2}\left\|\sqrt{b_t}u\right\|^2_{L^2(Q_2)} \\ &+(b_x(x,t)u, \Im_x^* u_t)_{L^2(Q^\tau)} + (\mathscr{L}u, x\Im_x^* u_t)_{L^2(Q^\tau)} - (\mathscr{L}u, \Im_x^*(\rho u_t))_{L^2(Q^\tau)}. \end{aligned} \tag{2.28}$$

Using the Cauchy inequality and using conditions C_1 , we estimate the last three terms of the right-hand side of (2.28). We find

$$(b_x(x,t)u, \Im_x^* u_t)_{L^2(Q^\tau)} \leq \frac{b_3^2}{2}\left\|u\right\|^2{}_{L^2(Q^\tau)} + \frac{1}{2}\left\|\Im_x^* u_t\right\|^2_{L^2(Q^\tau)}, \tag{2.29}$$

$$(\mathscr{L}u, x\Im_x^* u_t)_{L^2(Q^\tau)} \leq \frac{1}{2}\left\|\mathscr{L}u\right\|^2_{L^2(Q^\tau)} + \frac{l^4}{4}\left\|u_t\right\|^2_{L^2(Q^\tau)}, \tag{2.30}$$

$$-(\mathscr{L}u, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \leq \frac{1}{2}\left\|\mathscr{L}u\right\|^2_{L^2(Q^\tau)} + \frac{l^4}{4}\left\|u_t\right\|^2_{L^2(Q^\tau)}, \tag{2.31}$$

Let's substitute inequalities (2.29) - (2.31)into (2.28) and using conditions C_1 we obtain:

$$\begin{aligned} &\frac{b_0}{2}\left\|u(.,\tau)\right\|^2_{L^2(0,l)} + \frac{\beta}{2}\left\|u(.,\tau)\right\|^2_{L^2(0,l)} + \frac{1}{2}\left\|\Im_x^* u_t(.,\tau)\right\|^2_{L^2(0,l)} \\ \leq &\left\|\mathscr{L}u\right\|^2_{L^2(Q^\tau)} + \frac{b_1}{2}\left\|\varphi_1\right\|^2_{L^2(0,l)} + \frac{\beta_1 + l^2}{2}\left\|\varphi_2\right\|^2_{L^2(0,l)} + \frac{b_3^2}{2}\left\|u\right\|^2_{L^2(Q^\tau)} \\ &+\frac{l^4}{2}\left\|u_t\right\|^2_{L^2(Q^\tau)}, + \frac{1}{2}\left\|\Im_x^* u_t\right\|^2_{L^2(Q^\tau)} + \frac{b_2}{2}\left\|u\right\|^2_{L^2(Q^\tau)} \end{aligned} \tag{2.32}$$

So,

$$\begin{aligned} &\left\|u(.,\tau)\right\|^2_{L^2(0,l)} + \left\|u_t(.,\tau)\right\|^2_{L^2(0,l)} + \left\|\Im_x^* u_t(.,\tau)\right\|^2_{L^2(0,l)} \\ \leq &k[\left\|f\right\|^2_{L^2(Q^\tau)} + \left\|\varphi_1\right\|^2_{L^2(0,l)} + \left\|\varphi_2\right\|^2_{L^2(0,l)}\left\|u\right\|^2_{L^2(Q^\tau)} \\ &+\left\|u_t\right\|^2_{L^2(Q^\tau)} + \left\|\Im_x^* u_t\right\|^2_{L^2(Q^\tau)}] \end{aligned} \tag{2.33}$$

Where

$$k = \frac{\max(2, b_1, \beta + l^2, b_3^2 + b_2, l^4)}{\min(1, b_0, \beta)} \tag{2.34}$$

Applying the term, the Gronwall lemma to (2.33) and neglecting the term $\left\|\Im_x^* u_t(.,\tau)\right\|^2_{L^2(0,l)}$ on the lefthand side of the inequality that results, We obtain

$$\begin{aligned} & \|u(.,\tau)\|^2_{L^2(0,l)} + \|u_t(.,\tau)\|^2_{L^2(0,l)} \\ \leq\ & \exp(k\tau)\left(\|f\|^2_{L^2(Q^\tau)} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)}\right) \\ \leq\ & \exp(kT)\left(\|f\|^2_{L^2(Q^\tau)} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)}\right) \end{aligned} \tag{2.35}$$

As the right-hand side of the inequality (2.35) is independent of τ, we pass to the lefthand side or supremum with respect to τ from 0 to T. We obtain inequality (2.18) with $c = \sqrt{k}\exp(k\frac{T}{2})$

■

Proposition 2.3.1 *The operator L defined from B into H has a closure $\overline{L}$.*

Proof. (According to [10]) We must check that, if $u_n \in D(L)$ such that in

$$u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } B$$

And

$$Lu_n = (\mathscr{L}u_n, l_1 u_n, l_2 u_n) \underset{n\longrightarrow\infty}{\longrightarrow} \mathscr{F} = (f, \varphi_1, \varphi_2) \qquad \text{in } H \tag{2.36}$$

So

$$f \equiv 0, \varphi_1 \equiv 0, \varphi_2 \equiv 0$$

The convergence of (u_n) in B implies that:

$$u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } \mathscr{D}'(Q)$$

Where $\mathscr{D}'(Q)$ is the space of distributions on Q.
According to the continuity of the derivation of $\mathscr{D}'(Q)$ in $\mathscr{D}'(Q)$. we have

$$\mathscr{L}u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } \mathscr{D}'(Q)$$

And since the convergence of $\mathscr{L}u_n$ to f in $L^2(Q)$ implies,

$$\mathscr{L}u_n \underset{n\longrightarrow\infty}{\longrightarrow} f \qquad \text{in } \mathscr{D}'(Q)$$

According to the uniqueness of the limit in $\mathscr{D}'(Q)$ we conclude that: $f \equiv 0$ Furthermore, it follows from (2.36) that

$$l_1 u_n \underset{n\longrightarrow\infty}{\longrightarrow} \varphi_1 \qquad \text{in } L^2(0,l) \tag{2.37}$$

As the canonical injection of $L^2(0,l)$ into $D'(0,l)$ is continuous
we deduce from (2.37) that

$$l_1 u_n \underset{n\longrightarrow\infty}{\longrightarrow} \varphi_1 \qquad \text{in } \mathscr{D}'(0,l) \tag{2.38}$$

On the other hand, according to the definitions of $\|.\|_{L^2(0,l)}$ and $\|.\|_B$ we have:

$$\|l_1 u_n\|_{L^2(0,l)} \leq \|u_n\|_B , \quad \forall n$$

And as in:

$$u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } B$$

Then, we have:

$$l_1 u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } L^2(0,l)$$

Consequently:

$$l_1 u_n \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in } \mathscr{D}'(0,l) \tag{2.39}$$

By virtue of the uniqueness of the limit in $\mathscr{D}'(0,l)$ we conclude from (2.38) and (2.39) that $\varphi_1 \equiv 0$.
Similarly, we prove that $\varphi_2 \equiv 0$.
Let $\overline{L}$ be the closure of the operator L and $D(\overline{L})$its domain of definition. ■

Definition 2.3.1 *The solution of the equation* $\overline{L}u = \mathscr{F}$ *is* $u \in D(\overline{L})$ *called the strong solution of the problem (2.1)-(2.5). By passing to the limit, we extend inequality (2.18) to strong solutions, then we have*

$$\|u\|_B \leq c \left\|\overline{L}u\right\|_H , \quad \forall u \in D(\overline{L}), \tag{2.40}$$

From inequality (2.40), we have the following results:

Corollary 2.3.1 *The strong solution of the problem (2.1)-(2.5), if it exists, is unique and continuously depends on the data* f, φ_1, φ_2.

Corollary 2.3.2 *The set of values* $R(\overline{L})$ *of the operator* $\overline{L}$ *is equal to the closure* $\overline{R(L)}$ *of* $R(L)$.

2.4 *Existence of the solution :*

theorem 2.4.1 *If conditions* $C_1 - C_2$ *are satisfied, then for each:*

$$\mathscr{F} = (f, \varphi_1, \varphi_2) \in H$$

There exists a unique strong solution

$$u = \overline{L}^{-1}\mathscr{F} = \overline{L^{-1}}\mathscr{F}$$

Of the problem (2.1)-(2.5) satisfying the estimate:

$$\|u\|_B \leq c\,\|\mathscr{F}\|_H$$

Where c *is a positive constant independent of the solution* u.

Proof. From inequality (2.40). We only deduce that the operator $\overline{L}$ of $D(\overline{L})$ whose $R(\overline{L})$ admits the inverse $\overline{L}^{-1}$ And from corollary 2.3.2 , we deduce that the image of $R(\overline{L})$ of the operator $\overline{L}$ is closed. So it suffices to show the density of the set $R(L)$ in the space H (i.e) $\overline{R(L)} = H$
For this, we need the following proposition ■

Proposition 2.4.1 *If the conditions of theorem 2.4.1 are satisfied, and it is for $\omega \in L^2(Q)$ we have*

$$(\mathcal{L}u, \omega)_{L^2(Q)} = 0, \tag{2.41}$$

For any function $u \in D_0(L) = \{u/u \in D(L) : l_1 u = l_2 u = 0\}$ then ω almost vanishes everywhere, in Q.

Proof. (of proposition 2.4.1) The relation (2.41) is given for any $u \in D_0(L)$ So we can express it in a particular form. Let u_{tt} be a solution of the equation:

$$b(\sigma, t)\left[x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt})\right] = h(x, t) \tag{2.42}$$

Where σ is a constant in $(0, l)$. And

$$h(x, t) = \int_t^T \omega(x, \tau)d\tau.$$

And let the function u be defined by:

$$u = \begin{cases} 0 & \text{si } 0 \le t \le s, \\ \int_s^t (t-\tau)u_{\tau\tau}d\tau & \text{si } s \le t \le T. \end{cases} \tag{2.43}$$

Relations (2.42) and (2.43) imply that u is in $D_0(L)$ and that:

$$\begin{aligned} \omega(x, \tau) &= \Im_x^{*^{-1}} h \\ &= -\left[b(\sigma, t)(x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt}))\right]_t \\ &= \left[b(\sigma, t)\Im_x^*(\rho - x)u_{tt}\right]_t \end{aligned} \tag{2.44}$$

To continue the proof, we need the following lemma. ∎

lemma 2.4.1 *If the conditions of theorem 2.4.1 are satisfied, then the function ω defined by (2.44) is in $L^2(Q)$*

Proof. (of lemma 2.4.1) First, we will prove the inequality:

$$\left\|\Im_x^*(\rho - x)u_{tt}\right\|_{L^2(0,l)}^2 \le \frac{l^4}{12}\|u_{tt}\|_{L^2(0,l)}^2$$

We have:

$$\begin{aligned} \left\|\Im_x^*(\rho - x)u_{tt}\right\|^2 &= \left(\int_x^l (\rho - x)u_{tt}d\rho\right)^2 \\ &\le \int_x^l (\rho - x)d\rho \int_x^l {u_{tt}}^2 d\rho \\ &= \frac{(l-x)^3}{3}\int_x^l {u_{tt}}^2 d\rho \end{aligned}$$

If we integrate over $(0, l)$ with respect to x we obtain:

$$\int_0^l (\Im_x^*(\rho - x)u_{tt})^2 dx \le \frac{1}{3}\left|\int_0^l (l-x)^3 dx\right|\left|\int_0^l u_{tt}^2 dx\right| = \frac{l^4}{12}\|u_{tt}\|_{L^2(0,l)}^2$$

From this inequality, and given that conditions C_1 are satisfied, we deduce that:

$$b_t(\sigma,t)(\Im_x^*(\rho-x)u_{tt} \quad \text{and in} \quad L^2(Q)$$

And as

$$\begin{aligned}\omega(x,t) &= \left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t \\ &= b_t(\sigma,t)\Im_x^*(\rho-x)u_{tt}+b(\sigma,t)\Im_x^*(\rho-x)u_{ttt}\end{aligned}$$

So, it remains to demonstrate that:

$$b(\sigma,t)\Im_x^*(\rho-x)u_{ttt}\in L^2(Q).$$

For this, we must introduce the regularization T-operations ρ_ε of the form:

$$(\rho_\varepsilon f)(x,t)=\frac{1}{\varepsilon}\int_0^T\omega(\frac{t-s}{\varepsilon})f(x,s)ds.$$

where

$$\omega\in C_0^\infty(0,T),\quad \omega\geq 0 \qquad \int_{-\infty}^{+\infty}\omega(s)ds=1$$

$$\omega\equiv 0 \quad \text{in the vicinity of} \quad t\leq 0 \quad \text{and} \quad t\geq T$$

Let's apply operators ρ_ε and $\frac{\partial}{\partial t}$ to the equation:

$$b(\sigma,t)\left[b(\sigma,t)x\Im_x^*u_{tt}-\Im_x^*(\rho u_{tt})\right]=h(x,t)$$

To the equation:

$$-b(\sigma,t)\Im_x^*(\rho-x)u_{tt}=h(x,t)$$

We obtain

$$\begin{aligned}&\frac{\partial}{\partial t}(-b(\sigma,t)\Im_x^*(\rho-x)u_{tt})\\ =\ &\frac{\partial}{\partial t}\left[-b(\sigma,t)\Im_x^*(\rho-x)u_{tt}+\rho_\varepsilon(b(\sigma,t)\Im_x^*(\rho-x)u_{tt})\right]-\frac{\partial}{\partial t}\rho_\varepsilon h.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right\|^2_{L^2(Q)}\\ \leq\ &2\left\|\frac{\partial}{\partial t}\left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}+\rho_\varepsilon(b(\sigma,t)\Im_x^*(\rho-x)u_{tt})\right]\right\|^2_{L^2(Q)}\\ &+2\left\|\frac{\partial}{\partial t}\rho\varepsilon h\right\|^2_{L^2(Q)}\end{aligned}$$

As

$$\rho_\varepsilon f \underset{\varepsilon \longrightarrow 0}{\longrightarrow} f$$

And $\frac{\partial}{\partial t}(b(\sigma,t)\Im_x^*(\rho-x)u_{tt})$is bounded in $L^2(Q)$, then $\omega \in L^2(Q)$.
Let's now return to the proof of proposition 2.4.1, replacing ω in (2.41) by its representation (2.44), we obtain:

$$\begin{aligned} & (u_{tt}, \left[(b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t)_{L^2(Q)} \\ = \; & ((b(x,t)u_x)_x, \left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t)_{L^2(Q)} + \beta(u_{ttxx}, \left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_l)_{L^2(Q)} \end{aligned} \tag{2.45}$$

After integration by parts, taking into account conditions (2.3)-(2.5) and the particular form given by (2.42) and (2.43), equality (2.45) can be expressed in a simpler form. To do this, let's integrate each term of this equality separately over the sub-domain

$$Q_s = (0,l) \times (s,T) \quad \text{where} \quad 0 \le s \le T$$

$$(u_{tt}\left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt})\right]) = \tag{2.46}$$

$$\begin{aligned} = \; & \int_Q b(\sigma,t)u_{tt}\Im_x^*(\rho-x)u_{ttt}dxdt + \int_Q b_t(\sigma,t)u_{tt}\Im_x^*(\rho-x)u_{ttt}dxdt \\ = \; & -\int_s^T b(\sigma,t)\Im_x^* u_{tt}\Im_x^*(\rho-x)u_{ttt} \mid_{x=0}^{x=1} dt - \int_{Q_s} b(\sigma,t)\Im_x^* u_{tt}\Im_x^* u_{ttt}dxdt \\ & -\int_s^T b_t(\sigma,t)\Im_x^* u_{tt}\Im_x^*(\rho-x)u_{ttt} \mid_{x=0}^{x=1} - \int_{Q_s} b(\sigma,t)(\Im_x^* u_{tt})^2 dxdt \\ = \; & -\frac{1}{2}\int_0^l b(\sigma,t)(\Im_x^* u_{tt})^2 \mid_{t=s}^{t=T} dx + \frac{1}{2}\int_{Q_s} b(\sigma,t)(\Im_x^* u_{tt})^2 dxdt - \int_{Q_s} b(\sigma,t)(\Im_x^* u_{tt})^2 dxdt \\ = \; & \frac{1}{2}\int_0^l b(\sigma,s)(\Im_x^* u_{tt}(x,s))^2 dx - \frac{1}{2}\int_{Q_s} b_t(\sigma,t)(\Im_x^* u_{tt})^2 dxdt \\ = \; & \frac{1}{2}\left\|\sqrt{b(\sigma,s)}\Im_x^* u_{tt}(.,s)\right\|_{L^2(Q_s)}^2 - \frac{1}{2}\left\|\sqrt{b_t(\sigma,.)}\Im_x^* u_{tt}\right\|_{L^2(Q_s)}^2 \end{aligned}$$

$$((b(x,t)u_x)_x, \left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t)_{L^2(Q)} = \tag{2.47}$$

$$\begin{aligned}
&= \int_s^T b(x,t)u_x\left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t|_{x=0}^{x=l}\,dt+\int_{Q_s}(b(x,t)u_x\left[b(\sigma,t)\Im_x^*u_{tt}\right]_t dxdt\\
&= b(x,t)u\left[b(\sigma,t)\Im_x^*u_{tt}\right]_t|_{x=0}^{x=l}\,dt+\int_{Q_s}(b(x,t)u\left[b(\sigma,t)u_{tt}\right]_t dxdt\\
&\quad -\int_{Q_s} b_x(x,t)u\left[b(\sigma,t)\Im_x^*u_{tt}\right]_t dxdt\\
&= \int_0^l b(x,t)ub(\sigma,t)u_{tt}|_{t=s}^{t=T}\,dx-\int_{Q_s} b(x,t)u_t\left[b(\sigma,t)u_{tt}\right]dxdt-\\
&\quad -\int_{Q_s} b_t(x,t)u\left[b(\sigma,t)u_{tt}\right]dxdt-\int_0^l b_x(x,t)ub(\sigma,t)\Im_x^*u_{tt}\,|dx\ +\\
&\quad \int_{Q_s} b_x(x,t)u_t b(\sigma,t)\Im_x^*u_{tt}dxdt+\int_{Q_s} b_{xt}(x,t)ub(\sigma,t)\Im_x^*u_{tt}dxdt\\
&= -\frac{1}{2}\int_{Q_s} b(x,t)b(\sigma,t)\left[u_t\right]^2|_{t=s}^{t=T}\,dx+\frac{1}{2}\int_{Q_s}\left[b_t(x,t)b(\sigma,t)+\right.\\
&\quad \left.b(x,t)b_t(\sigma,t)\right](u_t)^2dxdt-\int_0^l b_t(x,T)b(\sigma,T)u(x,T)u_t(x,T)dx\\
&\quad +\int_{Q_s}\left[b_{tt}(x,t)b(\sigma,t)+b_t(x,t)b_t(\sigma,t)\right]uu_tdxdt+\int_{Q_s} b_t(x,t)b(\sigma,t)(u_t)^2dxdt\\
&\quad +\int_{Q_s}\left[b_x(x,t)u_t+b_{xt}(x,t)u\right]b(\sigma,t)\Im_x^*u_{tt}dxdt\\
&= -\frac{1}{2}\left\|\sqrt{b(.,T)b(\sigma,T)}u_t(.,T)\right\|_{L^2(0,l)}^2+\frac{1}{2}\int_{Q_s}\left[3b_t(x,t)b(\sigma,t)+b(x,t)b_t(\sigma,t)\right]\\
&\quad (u_t)^2dxdt-\int_0^l b_t(x,T)b(\sigma,T)u(x,T)u_t(x,T)dx+\int_{Q_s}\left[b_{tt}(x,t)b(\sigma,t)+\right.\\
&\quad \left.+b_t(x,t)b_t(\sigma,t)\right]uu_tdxdt+\int_{Q_s}\left[b_x(x,t)u_t+b_{xt}(x,t)u\right]b(\sigma,t)\Im_x^*u_{tt}dxdt
\end{aligned}$$

$$\beta(u_{ttxx,}\left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t)_{L^2(Q)}= \tag{2.48}$$

$$\begin{aligned}
&= \beta\int_s^T u_{ttx,}\left[b(\sigma,t)\Im_x^*(\rho-x)u_{tt}\right]_t|_{x=0}^{x=l}\,dt+\beta\int_{Q_s} u_{ttx}\left[b(\sigma,t)\Im_x^*u_{tt}\right]_t dtdx\\
&= \beta\int_s^T u_{tt}\left[b(\sigma,t)\Im_x^*u_{tt}\right]_t|_{x=0}^{x=l}\,dt+\beta\int_{Q_s} u_{tt}\left[b(\sigma,t)u_{tt}\right]_t dtdx\\
&= \beta\int_{Q_s} b_t(\sigma,t)(u_{tt})^2dxdt+\beta\int_{Q_s} b(\sigma,t)u_{tt}u_{tt}dtdx\\
&= \beta\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|_{L^2(Q_s)}^2+\frac{\beta}{2}\int_0^l b(\sigma,t)(u_{tt})^2|_{t=s}^{t=T}\,dx-\frac{\beta}{2}\int_{Q_s} b_t(\sigma,t)(u_{tt})^2dxdt\\
&= \frac{\beta}{2}\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|_{L^2(Q_s)}^2-\frac{\beta}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|_{L^2(0,l)}^2
\end{aligned}$$

The substitution of equalities (2.46)-(2.48) into the identity (2.45) yields

$$\begin{aligned}
&\frac{1}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|^2_{L^2(0,l)}+\frac{1}{2}\left\|\sqrt{b(.,T)b(\sigma,T)}u_t(.,T)\right\|^2_{L^2(0,l)}\\
&+\frac{\beta}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|^2_{L^2(0,l)}\\
=\ &\frac{1}{2}\left\|\sqrt{b_t(\sigma,s)}\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{\beta}{2}\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|^2_{L^2(Q_s)}\\
&+\frac{1}{2}\int_{Q_s}\left[3b_t(x,t)b(\sigma,t)+b(x,t)b_t(\sigma,t)\right](u_t)^2dxdt\\
&-\int_0^l b_t(x,T)b(\sigma,T)u(x,T)u_t(x,T)dx+\int_{Q_s}\left[b_{tt}(x,t)b(\sigma,t)+b_t(x,t)b_t(\sigma,t)\right]uu_tdxdt\\
&+\int_{Q_s}\left[b_x(x,t)u_t+b_{xt}(x,t)u\right]b(\sigma,t)\Im_x^* u_{tt}dxdt
\end{aligned}\tag{2.49}$$

Let's estimate the last three terms on the right-hand side of (2.49) using Cauchy's inequalities ε and taking into account conditions and $C_1 - C_2$, we find:

$$\begin{aligned}
&-\int_0^l b_t(x,T)b(\sigma,T)u(x,T)u_t(x,T)dx\\
\leq\ &\frac{\varepsilon b_1^2}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)}+\frac{b_2^2}{2\varepsilon}\left\|u(.,T)\right\|^2_{L^2(0,l)}
\end{aligned}\tag{2.50}$$

$$\begin{aligned}
&\int_{Q_s}\left[b_{tt}(x,t)b(\sigma,t)+b_t(x,t)b_t(\sigma,t)\right]uu_tdxdt\\
\leq\ &\frac{b_2^2+b_4^2}{2}\left\|u\right\|^2_{L^2(Q_s)}+\frac{b_2^2+b_1^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}
\end{aligned}\tag{2.51}$$

$$\begin{aligned}
&\int_{Q_s}\left[b_x(x,t)u_t+b_{xt}(x,t)u\right]b(\sigma,t)\Im_x^* u_{tt}dxdt\\
\leq\ &b_1^2\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{b_3^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}+\frac{b_5^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}
\end{aligned}\tag{2.52}$$

By combining (2.50)-(2.52) with (2.49) and setting $\varepsilon=\dfrac{b_0^2}{2b_1^2}$, we get:

$$\begin{aligned}
&\frac{b_0}{2}\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)}+\frac{b_0^2}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)}\frac{\beta b_0}{2}\left\|u_t(.,s)\right\|^2_{L^2(0,l)}\\
\leq\ &\frac{b_2}{2}\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{\beta b_2}{2}\left\|u_{tt}\right\|^2_{L^2(Q_s)}+2b_1b_2\left\|u_t\right\|^2_{L^2(Q_s)}\\
&+\frac{b_1^2b_2^2}{b_0^2}\left\|u(.,T)\right\|^2_{L^2(0,l)}+\frac{b_2^2+b_4^2}{2}\left\|u\right\|^2_{L^2(Q_s)}+\frac{b_2^2+b_1^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}\\
&+b_1^2\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{b_3^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}+\frac{b_5^2}{2}\left\|u\right\|^2_{L^2(Q_s)}
\end{aligned}\tag{2.53}$$

$$\begin{aligned}
&\frac{b_0}{2}\left[\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)}+\frac{b_0}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)}+\beta\left\|u_{tt}(.,s)\right\|^2_{L^2(0,l)}\right]\\
\leq\ &\left(b_1^2+\frac{b_2}{2}\right)\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\frac{\beta b_2}{2}\left\|u_{tt}\right\|^2_{L^2(Q_s)}+\\
&\frac{b_2^2+b_1^2+4b_1b_2+b_3^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}+\frac{b_2^2+b_4^2+b_5^2}{2}\left\|u\right\|^2_{L^2(Q_s)}\\
&+\frac{b_1^2b_2^2}{b_0^2}\left\|u(.,T)\right\|^2_{L^2(0,l)}
\end{aligned}\tag{2.54}$$

Let's estimate the last term on the right-hand side of the inequality (2.54), according to the elementary equality:

$$\frac{b_1^2 b_2^2}{b_0^2}\|u(.,T)\|_{L^2(0,l)}^2 \leq \frac{b_1^2 b_2^2}{b_0^2}\|u\|_{L^2(Q_s)}^2 + \frac{b_1^2 b_2^2}{b_0^2}\|u_t\|_{L^2(Q_s)}^2 \tag{2.55}$$

we will have

$$\begin{aligned} & \left\|\Im_x^* u_{tt}(.,s)\right\|_{L^2(0,l)}^2 + \frac{b_0}{2}\|u_t(.,T)\|_{L^2(0,l)}^2 + \beta\|u_{tt}(.,s)\|_{L^2(0,l)}^2 \\ \leq\quad & \frac{\left(2b_1^2+b_2\right)}{b_0}\left\|\Im_x^* u_{tt}\right\|_{L^2(Q_s)}^2 + \frac{\beta b_2}{2}\|u_{tt}\|_{L^2(Q_s)}^2 + \\ & \frac{\frac{2b_1^2 b_2^2}{b^2} + 2b_1 b_2 + (b_1 b_2)^2 + b_3^2}{b_0}\|u_t\|_{L^2(Q_s)}^2 + \frac{b_0^2(b_2^2+b_4^2+b_5^2)+b_1^2 b_2^2}{b_0^3}\|u\|_{L^2(Q_s)}^2 \end{aligned} \tag{2.56}$$

To estimate the last term on the right-hand side of (2.56), we must demonstrate that:

$$\|u\|_{L^2(Q_s)}^2 \leq 24T^2\|u_t\|_{L^2(Q_s)}^2$$

Indeed:

As

$$\begin{aligned} \int_s^T u^2 dt &= tu^2 \mid_{t=s}^{t=T} - 2\int_s^T tuu_t dt \\ &= Tu^2(x,T) - 2\int_s^T tuu_t dt \end{aligned}$$

We then have, according to Cauchy's inequality with ε:

$$\begin{aligned} \int_s^T u^2 dt &= Tu^2(x,T) - 2\int_s^T tuu_t dt \\ &\leq Tu^2(x,T) + T\varepsilon_1\int_s^T u^2 dt + \int_s^T \frac{T}{\varepsilon_1}u_t^2 dt \end{aligned}$$

Let's set $\varepsilon_1 = \frac{1}{2T}$ then we have

$$\int_s^T u^2 dt \leq Tu^2(x,T) + \frac{1}{2}\int_s^T u^2 dt + 2T^2\int_s^T u_t^2 dt$$

$$\frac{1}{2}\int_s^T u^2 dt \leq Tu^2(x,T) + 2T^2\int_s^T u_t^2 dt \tag{2.57}$$

On the other hand, we have

$$\frac{\partial}{\partial t}u^2 = 2uu_t$$

Hence

$$\int_s^T \frac{\partial}{\partial t}u^2 dt = 2\int_s^T uu_t dt$$

Therefore

$$u^2(x,t) = 2\int_s^T uu_t dt$$

Also, according to Cauchy's inequality with ε, we have

$$u^2(x,t) \leq \varepsilon_2 \int_s^T u^2 dt + \frac{1}{\varepsilon_2}\int_s^T u_t^2 dt$$

So

$$Tu^2(x,t) \leq T\varepsilon_2 \int_s^T u^2 dt + \frac{T}{\varepsilon_2}\int_s^T u_t^2 dt$$

Let's set $\varepsilon_2 = \dfrac{1}{4T}$ then we obtain

$$Tu^2(x,t) \leq \frac{1}{4}\int_s^T u^2 dt + 4T^2\int_s^T u_t^2 dt \tag{2.58}$$

Inequalities (2.57) and (2.58) give:

$$\frac{1}{2}\int_s^T u^2 dt \leq \int_s^T u^2 dt + 4T^2\int_s^T u_t^2 dt + 2T^2\int_s^T u_t^2 dt$$

Hence :

$$\frac{1}{4}\int_s^T u^2 dt \leq 6T^2\int_s^T {u_t}^2 dt$$

(i.e):

$$\int_s^T u^2 dt \leq 24T^2\int_s^T u_t^2 dt$$

Let's integrate both sides over $(0, l)$ with respect to x we get

$$\|u\|^2_{L^2(Q_s)} \leq 24T^2\,\|u_t\|^2_{L^2(Q_s)} \tag{2.59}$$

By combining (2.59) and (2.56) we have:

$$\begin{aligned} & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \beta \left\| u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \frac{b_0}{2}\left\| u_t(.,T)\right\|^2_{L^2(0,l)} \\ \leq\ & \frac{2b_1^2+b_2}{b_0}\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \frac{\beta b_2}{2}\left\| u_{tt}\right\|^2_{L^2(Q_s)} + k(b_i,T)\left\| u_t\right\|^2_{L^2(Q_s)} \end{aligned}$$

with

$$k(b_i,T) = \frac{2b_1^2b_2^2 + b_0^2\left[2b_1b_2 + (b_2b_1)^2 + b_3^2\right] + 24T^2 b(b_2^2+b_4^2+b_5^2) + b_1^2b_2^2}{b_0^3}$$

which gives us

$$\begin{aligned} & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\| u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\| u_t(.,T)\right\|^2_{L^2(0,l)} \\ \leq\ & k\left[\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\| u_{tt}\right\|^2_{L^2(Q_s)} + \left\| u_t\right\|^2_{L^2(Q_s)}\right] \end{aligned} \tag{2.60}$$

with

$$k = \frac{\max(\beta b_2, (2b_1^2+b_2), b_0 k(b_i T))}{b_0 \min(1,\beta,\frac{b_0}{2})}.$$

To continue, we will make a change of variables: we introduce the function $v(x,t)$such that

$$v(x,t) = \int_t^T u_{\tau\tau}\, d\tau$$

Then

$$u_t(x,t) = v(x,s) - v(x,t), \quad \text{and} \quad u_t(x,T) = v(x,s)$$

Inequality (2.60) becomes

$$\begin{aligned} & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\| u_{tt}(.,s)\right\|^2_{L^2(0,l)} + (1-2k(T-s))\left\| v(.,s)\right\|^2_{L^2(0,l)} \\ \leq\ & 2k\left(\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\| u_{tt}\right\|^2_{L^2(Q_s)} + \left\| v\right\|^2_{L^2(Q_s)}\right) \end{aligned} \tag{2.61}$$

$s_0 > 0$ satisfies

$$(1-2k(T-s_0)) = \frac{1}{2}.$$

Then inequality (2.61) implies

$$\begin{aligned} & \left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\| u_{tt}(.,s)\right\|^2_{L^2(0,l)} + \left\| v(.,s)\right\|^2_{L^2(0,l)} \\ \leq\ & 4k\left(\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)} + \left\| u_{tt}\right\|^2_{L^2(Q_s)} + \left\| v\right\|^2_{L^2(Q_s)}\right) \end{aligned} \tag{2.62}$$

For all $s \in [T-s_0, T]$

Let's set

$$Y(s) = \left\| \Im_x^* u_{tt} \right\|^2_{L^2(Q_s)} + \| u_{tt} \|^2_{L^2(Q_s)} + \| v \|^2_{L^2(Q_s)}$$

Then we have

$$Y^{'}(s) = -\left(\left\| \Im_x^* u_{tt}(.,s) \right\|^2_{L^2(0,l)} + \| u_{tt}(.,s) \|^2_{L^2(0,l)} + \| v(.,s) \|^2_{L^2(0,l)} \right)$$

Hence and according to (2.62), we will have

$$-Y^{'}(s) \leq 4kY(s)$$

Consequently

$$-\frac{\partial}{\partial s}(Y(s)\exp(4ks)) \leq 0$$

By integrating this last inequality over (s,T) and taking into account that $Y(T) = 0$, we obtain

$$Y(s)\exp(4ks) \leq 0$$

Hence $Y(s) = 0$ for all $s \in [T - s_0, T]$.

hence $\omega = 0$ almost everywhere in Q_{T-s_0} and since the length s. is independent of the choice of the origin, proceeding with the same reasoning a finite number of times, we show that $\omega = 0$ in Q.

The proposition being established, let's now return to the proof of the theorem

We must show the validity of the equality $\overline{R(L)} = H$.

As H is a Hilbert space, the equality $\overline{R(L)} = H$ is true if, from equality

$$(Lu, W)_H = (\mathscr{L}u, \omega)_{L^2(Q_s)} + (l_1 u, \omega_1)_{L^2(0,l)} + (l_2 u, \omega_2)_{L^2(0,l)} = 0. \tag{2.63}$$

where $W = (\omega, \omega_1, \omega_2) \in R(L)^{\perp}$ it follows that $\omega \equiv 0, \omega_1 \equiv 0, \omega_2 \equiv 0$ almost everywhere in Q.

If we consider any element of $D_0(L)$

$$D_0(L) = \{u \in D(L) : \ l_1 u = l_2 u = 0\}$$

From equality (2.63), we obtain:

$$\forall u \in D_0(L), \ \ (\mathscr{L}u, \omega)_{L^2(Q_s)} = 0.$$

Hence, by virtue of proposition 2.4.1, we deduce only that $\omega = 0$

Therefore, from equality (2.63) we obtain

$$(l_1 u, \omega_1)_{L^2(0,l)} + (l_2 u, \omega_2)_{L^2(0,l)} = 0.$$

Since l_1 and l_2 are independent and the sets of values of operators l_1 and l_2 are everywhere in the Hilbert space $L^2(0,l)$ Then $\omega_1 \equiv 0$ and $\omega_2 \equiv 0$ and consequently $W = 0$, which completes the proof of theorem2.4.1 ■

Chapter 3

On a hyperbolic linear system with Dirichlet boundary conditions

In this chapter, we will study a problem for a coupled system of two hyperbolic equations with the Bessel operator, with Dirichlet boundary conditions. We demonstrate the existence of transition limits using the method of energy inequalities.

3.1 *Problem statement:*

In the rectangle Q defined by

$$Q = (0,l) \times (0,T), 0 < T < +\infty, l \in I\mathbb{R}_+^*$$

we consider the hyperbolic system

$$\mathcal{L}_1 u = u_{tt} - \frac{1}{x}(xu_x)_x + \alpha(x)(u - v) = f(x,t) \tag{3.1}$$

$$\mathcal{L}_2 v = v_{tt} - \frac{1}{x}(xv_x)_x + \alpha(x)(v - u) = g(x,t) \tag{3.2}$$

where the function $\alpha(x)$ satisfies conditions

$$\mathbf{C_1} \qquad a_0 \leq \alpha(x) \leq a_1$$

For all $(x,t) \in Q$, where a_0 and a_1 are positive constants.
First, we notice that if $\alpha(x) = 0$ the system (3.1)-(3.2) will decouple and be a double problem that has been addressed by Masloub in [47].(Page 25.37) with integral conditions with weights and Neumann boundary conditions.
To problem (3.1)-(3.2), we associate the initial conditions

$$\ell_1 u = u(x,0) = u_0(x), \qquad 0 < x < l \tag{3.3}$$

$$\ell_2 u = u_t(x,0) = u_1(x), \qquad 0 < x < l \tag{3.4}$$

$$\ell_3 v = v(x,0) = v_0(x), \qquad 0 < x < l \tag{3.5}$$

$$\ell_4 v = v_t(x,0) = v_1(x), \qquad 0 < x < l \tag{3.6}$$

And Dirichlet boundary conditions

$$u(l,t) = 0, \quad 0 < t < T \tag{3.7}$$

$$v(l,t) = 0, \quad 0 < t < T \tag{3.8}$$

Where f, g, u_0, u_1, v_0, v_1 are given functions and such that

$$u_0(l) = u_1(l) = v_0(l) = v_1(l) = 0 \tag{3.9}$$

3.2 *Associated functional spaces:*

For the study of the posed problems, we need some functional spaces, namely $L^2_\rho(Q)$ the weighted Hilbert space of (class) of functions defined and square integrable on Q
whose inner product is defined by

$$(u,v)_{L^2_\rho(Q)} = \int_Q xuvdxdt$$

And the associated norm is defined by

$$\|u\|^2_{L^2_\rho(Q)} = \int_Q xu^2dxdt$$

$V^1_\rho(0,l)$ The Hilbert space equipped with the scalar product

$$(u,v)_{V^1_\rho(0,l)} = (u,v)_{L^2_\rho(0,l)} + (u_x,v_x)_{L^2_\rho(0,l)}$$

And the norm

$$\|u\|^2_{V^1_\rho(0,l)} = \|u\|^2_{L^2_\rho(0,l)} + \|u_x\|^2_{L^2_\rho(0,l)}$$

The problem (3.1) – (3.8) is equivalent to the operational equation

$$AZ = F \tag{3.10}$$

where Z, AZ and F are respectively the pairs

$$Z = (u,v) \tag{3.11}$$

$$AZ = (L_1u, L_2v) \tag{3.12}$$

And

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \tag{3.13}$$

The right-hand sides of (3.12) and (3.13) are respectively defined

$$L_1 u = \{\mathcal{L}_1 u, \ell_1 u, \ell_2 u\}, \qquad L_2 v = \{\mathcal{L}_2 v, \ell_3 v, \ell_4 v\} \tag{3.14}$$

And

$$\mathcal{F}_1 = \{f, u_0, u_1\} \quad , \quad \mathcal{F}_2 = \{g, v_0, v_1\} \tag{3.15}$$

The operator A is considered from the Banach space :

$$B = B_1 \times B_2 = \{(u, v) \in (L^2_\rho(Q))^2, \text{checking the conditions (3.3)-(3.8)}\}$$

And having finite norm

$$\begin{aligned} \|Z\|^2_B &= \sup_{0\le\tau\le T} (\|u(.,\tau)\|^2_{V^1_\rho(0,l)} + \|v(.,\tau)\|^2_{V^1_\rho(0,l)} + \\ &\quad \|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} + \|v_t(.,\tau)\|^2_{L^2_\rho(0,l)}) \end{aligned} \tag{3.16}$$

In the Hilbert space $H = H_1 \times H_2$ which is the completion of $\left(\{L^2_\rho(Q) \times V^1_\rho(0,l) \times L^2_\rho(0,l)\}\right)^2$ with respect to the finite final norm of

$$\begin{aligned} \|\mathcal{F}\|^2_H &= (\|f\|^2_{L^2_\rho(Q)} + \|g\|^2_{L^2_\rho(Q)} + \|u_0\|^2_{V^1_\rho(0,l)} + \\ &\quad \|v_0\|^2_{V^1_\rho(0,l)} + \|u_1\|^2_{L^2_\rho(0,l)} + \|v_1\|^2_{L^2_\rho(0,l)}) \end{aligned} \tag{3.17}$$

The domain of definition $D(A)$ of the operator A is defined by

$$\begin{aligned} D(A) &= \left\{Z = (u, v) \in (L^2_\rho(Q))^2\right\} / u_t, v_t, u_x, v_x, u_{tt}, v_{tt} \in L^2_\rho(Q) \\ &\quad \text{and checking the conditions (3.3)-(3.8)}\} \end{aligned}$$

3.3 *A priori estimation and its applications:*

theorem 3.3.1 *If the function $\alpha(x)$ satisfies conditions C_1 , then for any function $Z = (u, v) \in D(A)$, there exists a positive constant C independent of Z such that ,*

$$\|Z\|_B \le C\|AZ\|_H \tag{3.18}$$

Proof. Let's consider the differential operator $\mathcal{M}$ defined by

$$\mathcal{M}Z = \{\mathcal{M}_1 u, \mathcal{M}_2 v\}$$

At

$$\mathcal{M}_1 u = u_t \quad , \qquad \mathcal{M}_2 v = v_t \tag{3.19}$$

And then $\mathcal{L}Z = \{\mathcal{L}_1 u, \mathcal{L}_2 v\}$ We consider the inner product

$$(\mathcal{L}Z, \mathcal{M}Z)_{L^2_\rho(Q^\tau)} = (\mathcal{L}_1 u, \mathcal{M}_1 u)_{L^2_\rho(Q^\tau)} + (\mathcal{L}_2 v, \mathcal{M}_2 v)_{L^2_\rho(Q^\tau)} \tag{3.20}$$

Where

$$Q^\tau = (0,\tau)\times(0,l) \quad \text{and} \quad 0 \le \tau \le T$$

Let's calculate

$$\begin{aligned}(\mathcal{L}_1 u, \mathcal{M}_1 u)_{L^2_\rho(Q^\tau)} &= (\mathcal{L}_1 u, u_t)_{L^2_\rho(Q^\tau)} \\ &= (u_t, u_{tt})_{L^2_\rho(Q^\tau)} - (u_t, \frac{1}{x}(xu_x)_x)_{L^2_\rho(Q^\tau)} + (u_t, \alpha(x)(u-v))_{L^2_\rho(Q^\tau)}\end{aligned} \tag{3.21}$$

Using conditions (3.3)-(3.9) and integrating by parts each term of the right-hand side of equality (3.21), we obtain

$$(u_t, u_{tt})_{L^2_\rho(Q^\tau)} = \frac{1}{2}\|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|u_1\|^2_{L^2_\rho(0,l)} \tag{3.22}$$

$$\begin{aligned}(u_t, \frac{1}{x}(x\,u_x)_x)_{L^2_\rho(Q^\tau)} &= \int_{Q^\tau} u_t(x\,u_x)_x dxdt \\ &= \int_0^\tau x u_t u_x|_{x=0}^{x=l} = dt - \int_{Q^\tau} x u_{tx}\, u_x dxdt \\ &= \frac{1}{2}\|u_x(.,0)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)}\end{aligned} \tag{3.23}$$

$$(u_t, \alpha(x)(u-v))_{L^2_\rho(Q^\tau)} = (u_t, \alpha(x)u)_{L^2_\rho(Q^\tau)} - (u_t, \alpha(x)v)_{L^2_\rho(Q^\tau)} \tag{3.24}$$

$$\begin{aligned}(u_t, \alpha(x)u)_{L^2_\rho(Q^\tau)} &= \frac{1}{2}\int_0^l x\alpha(x)(u(x,t))^2|_{x=0}^{x=\tau} dx \\ &= \frac{1}{2}\|\sqrt{\alpha}u(.,\tau)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|\sqrt{\alpha}u_0\|^2_{L^2_\rho(0,l)}\end{aligned}$$

Hence (3.24) becomes

$$\begin{aligned}&(u_t, \alpha(x)(u-v))_{L^2_\rho(Q^\tau)} \\ =\; &\frac{1}{2}\|\sqrt{\alpha}u(.,\tau)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|\sqrt{\alpha}u_0\|^2_{L^2_\rho(0,l)} - (u_t, \alpha(x)v)_{L^2_\rho(Q^\tau)}\end{aligned} \tag{3.25}$$

Now let's calculate

$$\begin{aligned}(\mathcal{L}_2 v, \mathcal{M}_2 v)_{L^2_\rho(Q^\tau)} &= (\mathcal{L}_2 v, v_t)_{L^2_\rho(Q^\tau)} \\ &= (v_2, v_{tt})_{L^2_\rho(Q^\tau)} - (v_t, \frac{1}{x}(xv_x)_x)_{L^2_\rho(Q^\tau)} + (u_t, \alpha(x)(u-v))_{L^2_\rho(Q^\tau)}\end{aligned} \tag{3.26}$$

By a similar calculation to that which we did fo $(\mathcal{L}_1 u, \mathcal{M}_1 u)_{L^2_\rho(Q^\tau)}$;
We find

$$(v_t, v_{tt})_{L^2_\rho(Q^\tau)} = \frac{1}{2}\|v_t(.,\tau)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|v_1\|^2_{L^2_\rho(0,l)} \tag{3.27}$$

$$(v_t, \frac{1}{x}(xv_x)_x)_{L^2_\rho(Q^\tau)} = \frac{1}{2}\|v_x(.,0)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} \tag{3.28}$$

$$\begin{aligned} & (v_t, \alpha(x)(u-v))_{L^2_\rho(Q^\tau)} \\ = & \frac{1}{2}\left\|\sqrt{\alpha}v(.,\tau)\right\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\left\|\sqrt{\alpha}v_0\right\|^2_{L^2_\rho(0,l)} - (u_t, \alpha(x)u)_{L^2_\rho(Q^\tau)} \end{aligned} \tag{3.29}$$

Let's substitute equalities (3.22), (3.23), (3.25) into (3.21) and (3.27)- (3.29) into (3.26) and sum member to member, we have:

$$\begin{aligned} & \frac{1}{2}\|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} - \frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\left\|\sqrt{\alpha}u(.,\tau)\right\|^2_{L^2_\rho(0,l)} \\ & +\frac{1}{2}\|v_t(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\left\|\sqrt{\alpha}v(.,\tau)\right\|^2_{L^2_\rho(0,l)} \\ = & \frac{1}{2}\|u_1\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\left\|\sqrt{\alpha}u_0\right\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|v_1\|^2_{L^2_\rho(0,l)} \\ & \frac{1}{2}\|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\left\|\sqrt{\alpha}v_0\right\|^2_{L^2_\rho(0,l)} + (u_t, \alpha(x)v)_{L^2_\rho(Q^\tau)} + (v_t, \alpha(x)u)_{L^2_\rho(Q^\tau)} \\ & +(u_t, \mathcal{L}_1 u)_{L^2_\rho(Q^\tau)} + (v_t, \mathcal{L}_2 v)_{L^2_\rho(Q^\tau)} \end{aligned} \tag{3.30}$$

Let's estimate the last 4 terms of the right-hand side of (3.30) using Inequality (**see chapter (1)**) and applying conditions $\mathbf{C}_1$ we find:

$$\begin{aligned} (u_t, \alpha(x)v)_{L^2_\rho(Q^\tau)} &\leq \frac{a_1}{2}\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{a_1}{2}\|v\|^2_{L^2_\rho(Q^\tau)} \\ (v_t, \alpha(x)u)_{L^2_\rho(Q^\tau)} &\leq \frac{a_1}{2}\|v_t\|^2_{L^2_\rho(Q^\tau)} + \frac{a_1}{2}\|u\|^2_{L^2_\rho(Q^\tau)} \\ (u_t, \mathcal{L}_1 u)_{L^2_\rho(Q^\tau)} &\leq \frac{1}{2}\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|\mathcal{L}_1 u\|^2_{L^2_\rho(Q^\tau)} \\ (v_t, \mathcal{L}_2 v)_{L^2_\rho(Q^\tau)} &\leq \frac{1}{2}\|v_t\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|\mathcal{L}_2 v\|^2_{L^2_\rho(Q^\tau)} \end{aligned}$$

By substituting into(3.30) and applying conditions C_1 it follows that:

$$\begin{aligned} & \frac{1}{2}\|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{a_0}{2}\|u(.,\tau)\|^2_{L(0,l)} \\ & +\frac{1}{2}\|v_t(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{a_0}{2}\|v(.,\tau)\|^2_{L^2_\rho(0,l)} \\ \leq & \frac{1}{2}\|u_1\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{a_0}{2}\|u_0\|^2_{L^2_\rho(0,l)} + \frac{1}{2}\|v_1\|^2_{L^2_\rho(0,l)} \\ & +\frac{1}{2}\|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \frac{a_0}{2}\|v_0\|^2_{L^2_\rho(0,l)} + \frac{a_1}{2}\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{a_1}{2}\|v\|^2_{L^2_\rho(Q^\tau)} \\ & +\frac{a_1}{2}\|v_t\|^2_{L^2_\rho(Q^\tau)} + \frac{a_1}{2}\|u\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|v_t\|^2_{L^2_\rho(Q^\tau)} \\ & +\frac{1}{2}\|\mathcal{L}_1 u\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|\mathcal{L}_2 v\|^2_{L^2_\rho(Q^\tau)} \end{aligned} \tag{3.31}$$

i.e:

$$\begin{aligned}
& \|u(.,\tau)\|^2_{L^2_\rho(0,l)} + \|u_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} \\
& + \|v(.,\tau)\|^2_{L^2_\rho(0,l)} + \|v_x(.,\tau)\|^2_{L^2_\rho(0,l)} + \|v_t(.,\tau)\|^2_{L^2_\rho(0,l)} \\
\leq \ & k[\|u_0\|^2_{V^1_\rho(0,l)} + \|v_0\|^2_{V^1_\rho(0,l)} + \|u_1\|^2_{L^2_\rho(0,l)} + \|v_1\|^2_{L^2_\rho(0,l)} \\
& + \|f\|^2_{L^2_\rho(Q^\tau)} + \|g\|^2_{L^2_\rho(Q^\tau)} + \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_t\|^2_{L^2_\rho(Q^\tau)} \\
& + \|v\|^2_{L^2_\rho(Q^\tau)} + \|v_t\|^2_{L^2_\rho(Q^\tau)}]
\end{aligned} \tag{3.32}$$

Hence

$$k = \frac{a_1+1}{\min(a_0,1)}$$

In virtue of the same lemme (**see chapter(1) 1.8.1**), we have

$$\begin{aligned}
& \|u(.,\tau)\|^2_{V^1_\rho(0,l)} + \|v(.,\tau)\|^2_{V^1_\rho(0,l)} + \|u_t(.,\tau)\|^2_{L^2_\rho(0,l)} + \|v_t(.,\tau)\|^2_{L^2_\rho(0,l)} \\
\leq \ & ke^{k\tau}[\|u_0\|^2_{V^1_\rho(0,l)} + \|v_0\|^2_{V^1_\rho(0,l)} + \|u_1\|^2_{L^2_\rho(0,l)} + \|v_1\|^2_{L^2_\rho(0,l)} + \|f\|^2_{L^2_\rho(Q^\tau)} + \|g\|^2_{L^2_\rho(Q^\tau)}] \\
\leq \ & ke^{kT}[\|u_0\|^2_{V^1_\rho(0,l)} + \|v_0\|^2_{V^1_\rho(0,l)} + \|u_1\|^2_{L^2_\rho(0,l)} + \|v_1\|^2_{L^2_\rho(0,l)} + \|f\|^2_{L^2_\rho(Q^\tau)} + \|g\|^2_{L^2_\rho(Q^\tau)}]
\end{aligned} \tag{3.33}$$

Passing to the supremum with respect to τ on the interval $[0,T]$ we obtain the desired inequality (3.18)with $C = \sqrt{k}\exp(\frac{kT}{2})$

■

Proposition 3.3.1 *the operator $A : B \longrightarrow H$ has a closure $\overline{A}$*

Proof. if $Z_n = (u_n, v_n) \in D(A)$, a sequence such that

$$Z_n = (u_n, v_n) \underset{n\longrightarrow\infty}{\longrightarrow} (0,0) \qquad \text{in } B \tag{3.34}$$

and

$$AZ_n = (L_1 u_n, L_2 v_n) \underset{n\longrightarrow\infty}{\longrightarrow} \mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \qquad \text{in } H \tag{3.35}$$

We aim to show that $(\mathscr{F}_1, \mathscr{F}_2) = (0,0)$ that is:

$$f = g = u_0 = u_1 = v_0 = v_1 = 0$$

The convergence of (u_n, v_n) to $(0,0)$ in B implies that

$$(u_n, v_n) \underset{n\longrightarrow\infty}{\longrightarrow} (0,0) \qquad \text{in } \mathscr{D}'(Q) \times \mathscr{D}'(Q) \tag{3.36}$$

in $\mathscr{D}'(Q)$ is the distribution space over Q

According to the continuity of the derivation from $\mathscr{D}'(Q) \times \mathscr{D}'(Q)$ to $\mathscr{D}'(Q) \times \mathscr{D}'(Q)$, (3.36) implies

$$(\mathscr{L}_1 u_n, \mathscr{L}_2 v_n) \underset{n\longrightarrow\infty}{\longrightarrow} (0,0) \qquad \text{in } \mathscr{D}'(Q) \times \mathscr{D}'(Q) \tag{3.37}$$

But as

$$(\mathscr{L}_1 u_n, \mathscr{L}_2 v_n) \underset{n\longrightarrow\infty}{\longrightarrow} (f,g) \qquad \text{in } L^2_\rho(Q) \times L^2_\rho(Q) \tag{3.38}$$

Then

$$(\mathcal{L}_1 u_n, \mathcal{L}_2 v_n) \underset{n \longrightarrow \infty}{\longrightarrow} (f, g) \qquad \text{in } \mathcal{D}'(Q) \times \mathcal{D}'(Q) \tag{3.39}$$

By the uniqueness of the limit in $\mathcal{D}'(Q) \times \mathcal{D}'(Q)$ we conclude from (3.37) and (3.39) that:

$$(f, g) = (0, 0)$$

From (3.35) we have:

$$\ell_1 u_n \underset{n \longrightarrow \infty}{\longrightarrow} u_0 \quad \text{in } V_\rho^1(0, l) \tag{3.40}$$

And from the fact that the canonical injection from $V_\rho^1(0,l)$ into $\mathcal{D}'(0,l)$ is deduced from (3.40), we deduce that

$$\ell_1 u_n \underset{n \longrightarrow \infty}{\longrightarrow} u_0 \quad \text{in } \mathcal{D}'(0, l) \tag{3.41}$$

On the other hand, as

$$Z_n \underset{n \longrightarrow \infty}{\longrightarrow} 0 \qquad \text{in } B$$

And

$$\|\ell_1 u_n\|_{V_\rho^1(0,l)} \le \|Z_n\|_B \qquad \forall n$$

Then, we have

$$\ell_1 u_n \underset{n \longrightarrow \infty}{\longrightarrow} 0 \quad \text{in } V_\rho^1(0, l) \tag{3.42}$$

Consequently

$$\ell_1 u_n \underset{n \longrightarrow \infty}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(0, l) \tag{3.43}$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(0,l)$, we conclude from ((3.41)) and ((3.43)) that $u_0 = 0$.
In the same way, it is demonstrated that $u_1 = 0$, $v_0 = 0$ and $v_1 = 0$, therefore A is closable and we note by $\overline{A}$ its closure in the domain of definition is denoted by $D(\overline{A})$ ■

Definition 3.3.1 : *The solution of the equation:*

$$\overline{A} Z = \mathcal{F}$$

Is called a strong solution of problem (3.1)-(3.8). *By passing to the limit, we extend inequality* (3.18) *to strong solutions. Then, there exists a positive constant* **C** *such that*

$$\|Z\|_B \le C \left\| \overline{A} Z \right\|_H, \quad \forall Z \in D(\overline{A}) \tag{3.44}$$

Hence the following two corollaries.

Corollary 3.3.1 *The strong solution of problem* (3.1)-(3.8), *if it exists, is unique and depends continuously on:* $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in H$

$$\mathcal{F}_1 = \{f, u_0, u_1\} \quad \text{and} \quad \mathcal{F}_2 = \{g, v_0, v_1\}$$

Corollary 3.3.2 : *The set of values* $R(\overline{A})$ *of the operator* $\overline{A}$ *is closed and satisfies* $R(\overline{A}) = \overline{R(A)}$

3.4 *Existence of the solution:*

theorem 3.4.1 *for all:*

$$(f,g) \in (L^2_\rho(Q))^2 \ , u_0 \in V^1_\rho(0,l) \ , u_1 \in L^2_\rho(0,l) \text{ and } v_1 \in L^2_\rho(0,l)$$

There exists one and only one strong solution $Z = \overline{A}^{-1}\mathscr{F} = \overline{A^{-1}}\mathscr{F}$ *of problem* (3.1)-(3.8) *with*

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \in H \ , \mathscr{F}_1 = \{f, u_0, u_1\} \ , \mathscr{F}_2 = \{g, v_0, v_1\} \ , Z = (u, v)$$

And

$$\|Z\|_B \leq C\,\|AZ\|_H$$

Where C *is a positive constant independent of* Z

Proof. : It suffices to demonstrate that $R(A)$, the image of A, is dense, where

$$H = H_1 \times H_2$$

be $\Phi = (\Phi_1, \Phi_2) = (\{w_1, w_3, w_4\}, \{w_2, w_5, w_6\}) \in R(A)^\perp$ such that :

$$\begin{aligned}
(AZ,\Phi)_H &= (\{L_1 u, L_2 v\}, \{\Phi_1, \Phi_2\})_H \\
&= (\{\mathscr{L}_1 u, \ell_1 u, \ell_2 u\}, \{\mathscr{L}_2 v, \ell_3 v, \ell_4 v\}, \{w_1, w_3, w_4\}, \{w_2, w_5, w_6\})_H \\
&= (\mathscr{L}_1 u, w_1)_{L^2_\rho(Q)} + (\ell_1 u, w_3)_{V^1_\rho(0,l)} + (\ell_2 u, w_4)_{L^2_\rho(0,l)} \\
&= (\mathscr{L}_2 v, w_2)_{L^2_\rho(Q)} + (\ell_3 v, w_5)_{V^1_\rho(0,l)} + (\ell_4 v, w_6)_{L^2_\rho(0,l)} \\
&= 0
\end{aligned}$$

For $Z \in D_0(A)$, we have

$$(\mathscr{L}_1 u, w_1)_{L^2_\rho(Q)} + (\mathscr{L}_2 v, w_2)_{L^2_\rho(Q)} = 0$$

We will demonstrate in the following proposition that, necessarily

$$(w_1, w_2) = (0,0)$$

■

Proposition 3.4.1 *If the conditions and theorems **3.3.1** are satisfied, then for all functions*

$$W = (w_1, w_2) \in (L^2_\rho(Q))^2$$

and for all

$$Z \in D_0(A) = \{Z \in D(A) : \ell_1 u = \ell_2 u = \ell_3 v = \ell_4 v = 0\}$$

such that

$$(\mathscr{L}_1 u, w_1)_{L^2_\rho(Q)} + (\mathscr{L}_2 v, w_2)_{L^2_\rho(Q)} = 0 \tag{3.45}$$

Then W *vanishes almost everywhere in* Q

Proof. Now, the relation (3.45) holds for any $Z \in D_0(A)$,We can express it in a particular form. First, let's define the functions h_i , $(i = \overline{1,2})$ by

$$\begin{cases} h_1(x,t) = \int_t^T w_1(x,\tau)d\tau \\ h_2(x,t) = \int_t^T w_2(x,\tau)d\tau \end{cases} \tag{3.46}$$

let u_{tt} , v_{tt} solution of the equations

$$u_{tt} = h_1(x,t) \text{ and } v_{tt} = h_2(x,t) \tag{3.47}$$

successively and let $Z = (u, v)$ given by

$$Z = \begin{cases} (0,0) & 0 \le t \le s \\ (\int_s^t (\tau - t)u_{\tau\tau}d\tau, \int_s^t (\tau - t)v_{\tau\tau}d\tau) & s \le t \le T \end{cases} \tag{3.48}$$

Now we have

$$\begin{cases} w_1 = -u_{tt} \\ w_2 = -v_{tt} \end{cases} \tag{3.49}$$

and that

$$Z \in D_0(A)$$

■

lemma 3.4.1 *: The function $W = (w_1, w_2)$ defined by (3.48) is in $(L^2_\rho(Q))^2$.*

Proof. We demonstrate that $u_{ttt} \in L^2_\rho(Q)$ For this, let's use the t-regularization operators : ρ_ε of the form

$$(\rho_\varepsilon f)(x,t) = \frac{1}{\varepsilon}\int_0^T w\left(\tfrac{t-s}{\varepsilon} \right) f(x,s)ds$$

Where $w \in C^\infty(0,T)$, $w \ge 0$, and $w \equiv 0$ in the vicinity of $t = 0$, $t = T$, outside the interval $(0,T)$ and $\int_{IR} w(s)ds = 1$ let's apply for this purpose the operators ρ_ε and $\frac{\partial}{\partial t}$ to the first equation of (3.47), we obtain

$$u_{ttt} = \frac{\partial}{\partial t}(u_{tt} - \rho_\varepsilon u_{tt}) + \frac{\partial}{\partial t}(\rho_\varepsilon h_1)$$

Hence

$$\|u_{ttt}\|^2_{L^2_\rho(Q)} \le 2\left\|\frac{\partial}{\partial t}(u_{tt} - \rho_\varepsilon u_{tt})\right\|^2_{L^2_\rho(Q)} + 2\left\|\frac{\partial}{\partial t}(\rho_\varepsilon h_1)\right\|^2_{L^2_\rho(Q)}$$

Or

$$\left\|\frac{\partial}{\partial t}(u_{tt} - \rho_\varepsilon u_{tt})\right\|^2_{L^2_\rho(Q)} \longrightarrow 0 \text{ qd } \varepsilon \longrightarrow 0$$

And consequently

$$\|u_{ttt}\|^2_{L^2_\rho(Q)} \le 2\left\|\frac{\partial}{\partial t}(\rho_\varepsilon h_1)\right\|^2_{L^2_\rho(Q)}$$

With the same method, we demonstrate that $u_{ttt} \in L^2_\rho(Q)$
Hence $W \in (L^2_\rho(Q))^2$

To continue the demonstration of proposition **3.4.1**, we replace W in (3.45) by its representation (3.49) we will have:

$$(-u_{ttt},\mathscr{L}_1 u)_{L^2_\rho(Q)}+(-v_{ttt},\mathscr{L}_2 v)_{L^2_\rho(Q)}=0$$

(i,e):

$$\begin{aligned}&(u_{tt},u_{ttt})_{L^2_\rho(Q)}-(u_{ttt},(xu_x)_x)_{L^2_\rho(Q)}+(u_{ttt},\alpha(x)(u-v))_{L^2_\rho(Q)}\\&+(v_{tt},v_{ttt})_{L^2_\rho(Q)}-(v_{ttt},(xv_x)_x)_{L^2_\rho(Q)}+(v_{ttt},\alpha(x)(v-u))_{L^2_\rho(Q)}\\&=0\end{aligned} \tag{3.50}$$

Taking into account the conditions (3.3)-(3.8), and the particular form of Z, defined by relations (3.47) and (3.49), identity (3.50) reduces to a simpler one. For this purpose, we integrate by parts each term of equality (3.50) on the subdomain

$$Q_s=(0,l)\times(s,T) \quad \text{where} \quad 0\le s\le T$$

We obtain

$$(u_{tt},u_{ttt})_{L^2_\rho(Q)}=-\frac{1}{2}\|u_{tt}\|^2_{L^2_\rho(Q)} \tag{3.51}$$

$$\begin{aligned}&-(u_{ttt},(xu_x)_x)_{L^2_\rho(Q)}\\=&-\int_0^l u_{tt}(xu_x)_x|_{t=s}^{t=T}dx+\int_{Q_s}u_{tt}(xu_x)_{xt}dxdt\\=&\int_s^T u_{tt}(xu_x)_t|_{x=0}^{x=l}dt-\int_{Q_s}u_{ttx}(xu_x)_t dxdt\\=&-\frac{1}{2}\|u_{xt}(.,T)\|^2_{L^2_\rho(0,l)}\end{aligned} \tag{3.52}$$

$$\begin{aligned}&(u_{ttt},\alpha(x)(u-v))_{L^2_\rho(Q)}\\=&\int_0^l x\alpha(x)\int_s^T u_{ttt}u\,dtdx-\int_{Q_s}x\alpha(x)vu_{ttt}dtdx\\=&\int_0^l x\alpha(x)[u_{tt}u]_{t=s}^{t=T}dx-\int_{Q_s}x\alpha(x)u_{tt}u_t dtdx-\int_{Q_s}x\alpha(x)vu_{ttt}dtdx\\=&-\frac{1}{2}\left\|\sqrt{\alpha}u_t(.,T)\right\|^2_{L^2_\rho(0,l)}-\int_0^l x\alpha(x)[u_{tt}v]_{t=s}^{t=T}dx+\int_{Q_s}x\alpha(x)u_{tt}v_t dtdx\\=&-\frac{1}{2}\left\|\sqrt{\alpha}u_t(.,T)\right\|^2_{L^2_\rho(0,l)}+(v_t,\alpha(x)(u_{tt})_{L^2_\rho(Q)}\end{aligned} \tag{3.53}$$

A similar calculation gives us

$$(v_{tt},v_{ttt})_{L^2_\rho(Q)}=-\frac{1}{2}\|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} \tag{3.54}$$

$$-(v_{ttt},(xv_x)_x)_{L^2_\rho(Q)}=-\frac{1}{2}\|v_{xt}(.,T)\|^2_{L^2_\rho(0,l)} \tag{3.55}$$

$$
\begin{aligned}
& (v_{ttt}, \alpha(x)(v-u))_{L^2_\rho(Q)} \\
= \quad & -\frac{1}{2}\left\|\sqrt{\alpha}v_t(.,T)\right\|^2_{L^2_\rho(0,l)} + (u_t, \alpha(x)v_{tt})_{L^2_\rho(Q_s)}
\end{aligned}
\tag{3.56}
$$

By substituting equality (3.51)-(3.56) into (3.50), it follows

$$
\begin{aligned}
& \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|u_{xt}(.,T)\|^2_{L^2_\rho(0,l)} \\
& + \left\|\sqrt{\alpha}u_t(.,T)\right\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} \\
& + \|v_{xt}(.,T)\|^2_{L^2_\rho(0,l)} + \left\|\sqrt{\alpha}v_t(.,T)\right\|^2_{L^2_\rho(0,l)} \\
= \quad & 2(v_t, \alpha(x)u_{tt})_{L^2_\rho(Q_s)} + 2(u_t, \alpha(x)v_{tt})_{L^2_\rho(Q_s)}
\end{aligned}
\tag{3.57}
$$

Now let's estimate the two terms of the right side of (3.57), using the Cauchy inequality, taking into account conditions $\mathbf{C}_1$, we obtain

$$
2(u_t, \alpha(x)v_{tt})_{L^2_\rho(Q_s)} \le \|u_t\|^2_{L^2_\rho(Q_s)} + a_1^2\|v_{tt}\|^2_{L^2_\rho(Q_s)} \tag{3.58}
$$

$$
2(v_t, \alpha(x)u_{tt})_{L^2_\rho(Q_s)} \le \|v_t\|^2_{L^2_\rho(Q_s)} + a_1^2\|u_{tt}\|^2_{L^2_\rho(Q_s)} \tag{3.59}
$$

By combining (3.57), (3.58) and (3.59), we obtain

$$
\begin{aligned}
& \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \left\|\sqrt{\alpha}u_t(.,T)\right\|^2_{L^2_\rho(0,l)} \\
& + \left\|\sqrt{\alpha}v_t(.,T)\right\|^2_{L^2_\rho(0,l)} + \|u_{xt}(.,T)\|^2_{L^2_\rho(0,l)} + \|v_{xt}(.,T)\|^2_{L^2_\rho(0,l)} \\
\le \quad & a_1^2\|u_{tt}\|^2_{L^2_\rho(Q_s)} + a_1^2\|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|u_t\|^2_{L^2_\rho(Q_s)} + \|v_t\|^2_{L^2_\rho(Q_s)}
\end{aligned}
\tag{3.60}
$$

Neglecting the last two terms on the left side of (3.60) and using conditions $\mathbf{C}_1$
we find

$$
\begin{aligned}
& \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} \\
& + a_0\|u_t(.,T)\|^2_{L^2_\rho(0,l)} + a_0\|v_t(.,T)\|^2_{L^2_\rho(0,l)} \\
\le \quad & a_1^2\|u_{tt}\|^2_{L^2_\rho(Q_s)} + a_1^2\|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|u_t\|^2_{L^2_\rho(Q_s)} + \|v_t\|^2_{L^2_\rho(Q_s)}
\end{aligned}
\tag{3.61}
$$

Hence

$$
\begin{aligned}
& \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|u_t(.,T)\|^2_{L^2_\rho(0,l)} + \|v_t(.,T)\|^2_{L^2_\rho(0,l)} \\
\le \quad & k\left[\|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|u_t\|^2_{L^2_\rho(Q_s)} + \|v_t\|^2_{L^2_\rho(Q_s)}\right]
\end{aligned}
\tag{3.62}
$$

With

$$
k = \frac{\max(1, a_1^2)}{\min(1, a_0)}
$$

To continue, let's introduce the function V defined by

$$
V = (\alpha(x,t), \beta(x,t))
$$

where

$$\alpha(x,t) = \int_t^T u_{\tau\tau} d\tau \quad \text{and} \quad \beta(x,t) = \int_t^T v_{\tau\tau} d\tau$$

We then have

$$\begin{cases} u_t(x,t) = \alpha(x,s) - \alpha(x,t) & u_t(x,T) = \alpha(x,s) \\ v_t(x,t) = \beta(x,s) - \beta(x,t) & v_t(x,T) = \beta(x,s) \end{cases} \tag{3.63}$$

By combining equality (3.62) and relations (3.63), we obtain

$$\begin{aligned} & \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} \\ & +(1-2k(T-s))\left(\|\alpha(.,s)\|^2_{L^2_\rho(0,l)} + \left\|\beta(.,s)\right\|^2_{L^2_\rho(0,l)}\right) \\ \leq \quad & 2k\left[\|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|\alpha\|^2_{L^2_\rho(Q_s)} + \left\|\beta\right\|^2_{L^2_\rho(Q_s)}\right] \end{aligned} \tag{3.64}$$

If for $s_0 > 0$, we have $1-2k(T-s) = \frac{1}{2}$, it follows from (3.64) that

$$\begin{aligned} & \|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|\alpha(.,s)\|^2_{L^2_\rho(0,l)} + \left\|\beta(.,s)\right\|^2_{L^2_\rho(0,l)} \\ \leq \quad & 4k\left[\|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|\alpha\|^2_{L^2_\rho(Q_s)} + \left\|\beta\right\|^2_{L^2_\rho(Q_s)}\right] \end{aligned} \tag{3.65}$$

For all $s \in [T-s_0, T]$
let's now put

$$h(s) = \|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|v_{tt}\|^2_{L^2_\rho(Q_s)} + \|\alpha\|^2_{L^2_\rho(Q_s)} + \left\|\beta\right\|^2_{L^2_\rho(Q_s)}$$

we have

$$h'(s) = -\left(\|u_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|v_{tt}(.,s)\|^2_{L^2_\rho(0,l)} + \|\alpha(.,s)\|^2_{L^2_\rho(0,l)} + \left\|\beta(.,s)\right\|^2_{L^2_\rho(0,l)}\right)$$

Hence (3.65) becomes

$$h'(s) \leq 4kh(s)$$

and consequently, we have

$$h(s)' exp(4ks) \leq 0$$

from this last inequality, we deduce that

$$h(s) = 0 \qquad \forall s \in [T-s_0, T].$$

Hence, it follows that $W = (w_1, w_2) = (0,0)$ almost everywhere Q_{T-s_0}Therefore, by proceeding in the same way, a finite number of times, we prove that

$$W = (w_1, w_2) = (0,0) \qquad \text{in} \quad Q$$

in the cube. Let's now return to theorem **3.4.1**,
the relation

$$(AZ, \Phi)_H = 0$$

implies that

$$(\ell_1 u, w_3)_{V^1_\rho(0,l)} + (\ell_2 u, w_4)_{L^2_\rho(0,l)} + (\ell_3 v, w_5)_{V^1_\rho(0,l)} + (\ell_4 v, w_6)_{L^2_\rho(0,l)} = 0$$

for all $Z \in D(A)$
since $\ell_1 u$, $\ell_2 u$, $\ell_3 v$ and $\ell_4 v$ are independent, and the sets of values of the trace operators $\ell_1, \ell_2, \ell_3$and ℓ_4 are everywhere dense, respectively, in the hyper stage

$$V^1_\rho(0,l), L^2_\rho(0,l), V^1_\rho(0,l) \quad \text{and} \quad L^2_\rho(0,l)$$

therefore, and consequently $\Phi = 0$, which completes the proof of theorem **3.4.1**. ■

Conclusion

In this thesis, we applied the a priori estimation method to demonstrate the uniqueness of two problems:

- On a hyperbolic linear system with Dirichlet boundary conditions
- Problems with an integral condition with weight for a class of hyperbolic equations of the Boussinesq type are considered

The goal it is to :Expanding the functional analysis method to encompass a new category of mixed problems.
Demonstrating the versatility and effectiveness of the proposed method in solving a variety of equations and problems, including those with different boundary and integral conditions.
And also this method has been developed to study the existence and uniqueness of strong solutions for fractional problem.

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