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Doctoral Thesis
Option: Evolution Problems

Theme

Existence and Asymptotic Behavior of Solutions for some Hyperbolic Systems

Presented by Mrs. ATMANIA Islah

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الإهداء

الحمد لله عدد ما خلق الحمد لله ملء ما خلق
الحمد لله عدد ما في السماوات والأرض الحمد لله ملء ما في السماوات والأرض
الحمد لله ما أحصى كتابه الحمد لله عدد كل شيء

إلى نفسي الطموحة
من قال أنا لها نالها وانا لها وان ابت اتيت بها صاغرة
لم تكن الرحلة قصيرة ولا ينبغي لها ان تكون ولم يكن الحلم قريبا ولا الطريق كان محفوفا بالتسهيلات لكنني فعلتها ونلتها

الى من كانت الداعمة الأولى الابدية، ملاكي الطاهر، من كان وجودها يمدني بالسعي دون ملل إلى التي ظلت دعواتها تضم اسمي دائما، معلمتي الاولى، دكتورتي الأولى
أمي ومحبوبتي وملهمتي

الى من كلله الله بالهيبة والوقار الى من أحمل اسمه بكل فخر ومن وضع كل فخره بي، ستبقى كلماتك نجوم اهتدي بها
إلى روح الروح والدي

الى من انتظر هذه اللحظات ليفتخر بي، الى رفيق الدرب وسندي وضلعي الثابت الذي لا يميل، الى من كان الأول في مساندتي وتشجيعي للوصول الى هدفي
إلى شريك حياتي

الى صاحبة الفضل، الى من كانت بجانبي كلما اوشكت ان أتعثر، الى من أزاحت عن طريقي كثيرا من متاعبي، الى من وجدتها سندا في أصعب اوقاتي
الى جدة ابنتي وصديقتها
الى الرجل المثقف المحب للعلم والداعي له، الى من تعلمت منه أن الدنيا كفاح وسلاحها العلم والمعرفة
الي جد ابنتي و حبيبها
الى التي صبرت علي شهور مديدة كنت حائرة فيها بين كتبي ومعتكفة بها في مكتبتي بين البحث والدراسة ، يشهد الله علي انني تحملت كل الصعوبات لأجلك ، ولأنني ارى فيك مستقبلي وما انا مقبلة عليه من رقي في درجات العلم، احدثك الآن واعلم انك لا تعي ما أقول لكن اخبئ لك هذه الرسالة ذكرى اعتزاز وامتنان لأعبر فيها عن مدى ما يجول في خاطري لتقرئيها حين تكبرين
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إلى رفيقات السنين و صديقات الرحلة و اصحاب الشدائد و الازمات اللائي شاركنني خطوات هذا الطريق وشجعنني على للمثابرة واكمال المسيرة وكن خير سند

دعاء مريم أحلام صحر سلمى رحمة

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البروفسور مسلوب فتيحة

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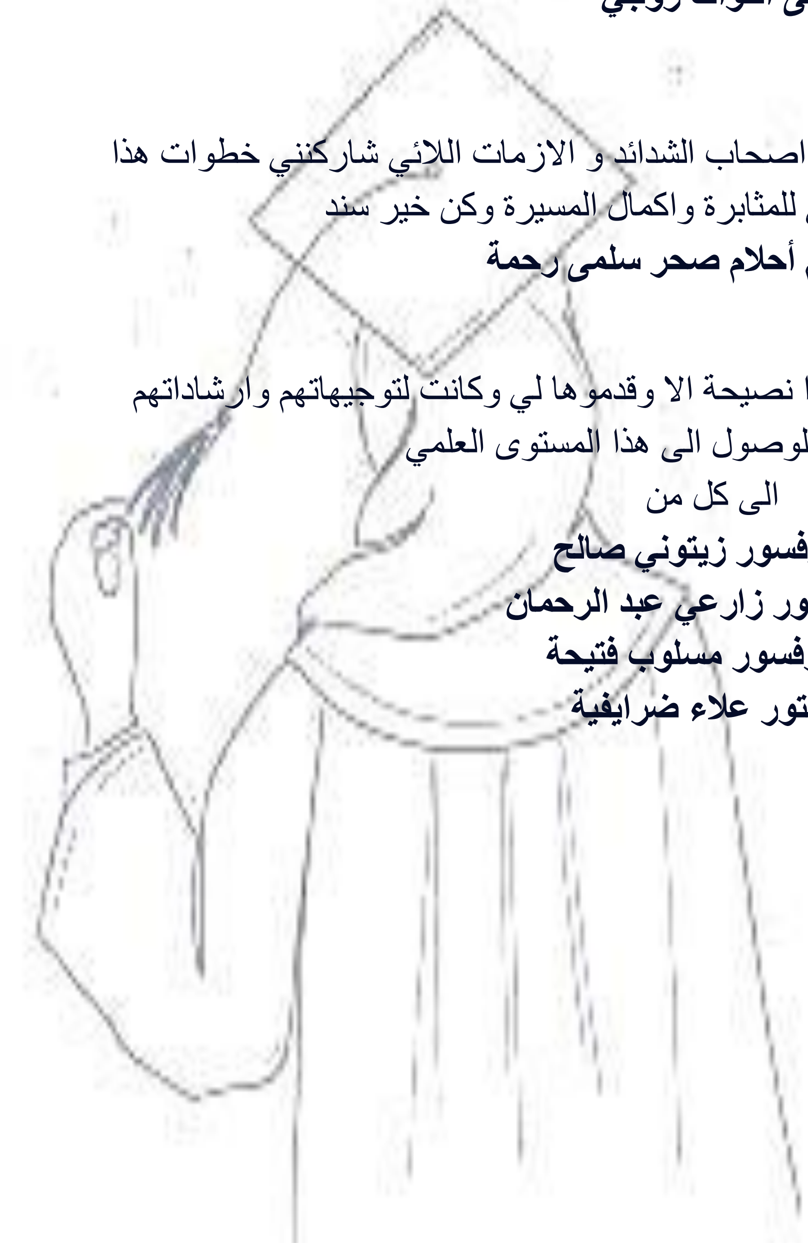

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Contents

Abstract

In this research, we examined the existence and asymptotic behavior of solutions for a Bresse-Timoshenko system, considering distributed delay and second sound effects.
In a section of this study,we demonstrated the global well-posedness of the initial and boundary value problem, given appropriate assumptions, utilizing Faedo-Galarkin approximations and several energy estimates.
In the final section, exponential stability was demonstrated through the utilization of Lyapunov functional and the multiplier technic.
Keywords : Bresse-Timoshenko-type systems, distributed delay term, second sound, well-posedness, exponential stability, Faedo-Galerkin approximations, Lyapunov functional, energy method.

Résumé

Dans cette recherche, nous avons examiné l'existence et le comportement asymptotique des solutions pour un système de Bresse-Timoshenko, en tenant compte des retards distribués et des effets du second son.

Dans une partie de cette étude, nous avons démontré que le problème des valeurs initiales et des valeurs limites était existe, unique et continue, compte tenu des hypothèses appropriées, en utilisant les approximations de Faedo-Galarkin et plusieurs estimations d'énergie.

Dans la dernière section, la stabilité exponentielle a été démontrée par l'utilisation de la fonction de Lyapunov et de la technique du multiplicateur.

Mots clés : Systèmes de type Bresse-Timochenko, terme de retard distribué, second son, stabilité exponentielle, approximations Faedo-Galerkin, fonctionnelle de Lyapunov, méthode d'énergie.

ملخص

درسنا في هذا البحث الوجود والسلوك التقاربي لحلول نظام براس-تيموشنكو (Bresse-Timoshenko) مع الأخذ بعين الاعتبار التأخير الموزع والتأثيرات الصوتية الثانية.

في قسم من هذه الدراسة، أثبتنا الوجود الكلي، الوحدانية و الاستمرارية للمشكلة ذات القيمة الابتدائية والحدودية، في ضوء الفرضيات المناسبة، وذلك باستخدام تقديرات فايدو-قلاركين (Faedo-Galarkin) التقريبية والعديد من تقديرات الطاقة.

في القسم الأخير، تم إثبات الاستقرار الأسي من خلال استخدام دالة ليابونوف (Lyapunov) وتقنية المضاعف.

الكلمات المفتاحية: نظام براس-تيموشنكو، حد التأخير الموزع، الوجود الكلي و الوحدانية، الصوت الثاني، تقريبات فايدو-قلاركين، الاستقرار الاسي، دالة ليابونوف، طريقة الطاقة.

Notations

$\Vert . \Vert$	Arbitrary norm of
$(.,.)$	The scalar product
Ω	Open set in $\mathbb{R}^n$
$\partial\Omega$	The boundary of domain
$\lim$	Limit
$L^p(\Omega)$	Lebesgue space with norm $\Vert . \Vert_p$
$L^2(\Omega)$	Space of integrable square functions
$C^m(\Omega)$	Space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$
$C^\infty(\Omega)$	$\bigcap_{m\in\mathbb{N}} C^m(\Omega)$
H	The Hilbert.space
$W^{m,p}(\Omega)$	Sobolev space with norm. $\Vert . \Vert_{m,p}$
$W_0^{m,p}(\Omega)$	is the closure of $C^\infty(\Omega)$ in $W^{m,p}(\Omega)$.
H^m	$W^{m,2}(\Omega)$
$\frac{\partial}{\partial x}$	Partial derivative.
$\frac{\partial^{\vert k \vert} v}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$	La dérivée généralisée.
Δu	Laplace operator.
∇u	Gradient of u.

General Introduction

In the middle of the seventeenth century, the art of prediction was revolutionized by a major event in the history of science. This led to the birth of differential equations, and in general, evolution equations. These equations seek to mathematically predict the evolution of phenomena by examining their "trends" or "infinitesimal differences". Remarkably, but not surprisingly, these equations emerged in the wake of the birth of calculus. The pioneers of this revolution were mathematicians, physicists, and even philosophers: Newton, Leibniz, Huygens... With them, we were able to master the trajectory of artillery shells, planets, and many other mechanical systems. Towards the conclusion of the 17th century, the famous Bernoulli brothers studied courses on differential equations in different cities in Europe. The second scientific revolution occurred a little less than a century after the invention of differential equations, with the emergence of evolutionary equations covering an entire domain, a completely unknown function. With Euler, D'Alembert, Lagrange, Laplace, and others, who dreamed of predicting the elusive movement of fluids. And this approach will be successful throughout the nineteenth century, including the Fourier equations that govern heat transfers, the Navier-Stokes equations that are now the basis of fluid simulations, and the Maxwell equations that govern electromagnetism. Transatlantic contacts would not have been possible without a deep understanding of partial differential equations... In 1890, in a visionary essay, Henri Poincaré began to talk about the classification of the great equations of mathematical physics, anticipating the extraordinary development of partial theory. Differential equations in the twentieth century.

With the development of computers in the 1950s, many equations were found that could previously only be studied qualitatively or in certain cases. Precision has continued to increase. Digital simulation became a major component of science and industry, with the development of numerical analysis and the interface between mathematical theory and arithmetic. This was seen as a third scientific revolution, which permeated the history of evolutionary equations.

Physically, Linear evolutionary equations are characterized by partial differential equations where time t serves as one of the independent variables. These equations originate not only from various mathematical disciplines but also find applications in other scientific domains such as physics, mechanics, and materials science.

Among these equations, we will focus on the Timoshenko system, Bresse system and Bresse-Timoshenko system.

The Timoshenko systems

The Timoshenko system is commonly regarded as a representation of the lateral vibration of a beam while disregarding any damping influences. Specifically, we refer to the model introduced by Timoshenko in 1921 (referenced in [1]), which is formulated as a set of two interrelated hyperbolic equations:

$$\begin{aligned} \rho\varphi_{tt} &= K\left(\varphi_x - \psi\right)_x, \text{ in } (0,L)\times\mathbb{R}_+, \\ I_\rho\psi_{tt} &= \left(EI\psi_x\right)_x + K\left(\varphi_x - \psi\right), \text{ in } (0,L)\times\mathbb{R}_+, \end{aligned} \tag{1}$$

the symbol φ denotes the transverse displacement of the beam, while ψ represents the rotation angle of the beam's filament. The parameters ρ, I_ρ, E, I and K correspondingly stand for the density (mass per unit length), the polar moment of inertia of a cross-section, Young's modulus of elasticity, the moment of inertia of a cross-section, and the shear modulus, respectively.
The system denoted by reference (1), along with boundary conditions as described in equation

$$\begin{aligned} EI\varphi_x\big|_{x=0}^{x=L} &= 0, \\ K\left(\varphi_x - \varphi\right)\big|_{x=0}^{x=L} &= 0, \end{aligned}$$

exhibits conservative behavior, ensuring the preservation of total energy as time progresses towards infinity. Various researchers have proposed distinct dissipative mechanisms to stabilize the aforementioned system, leading to the establishment of multiple findings regarding both uniform and asymptotic energy decay. Moving forward, we highlight several well-known results concerning the stabilization of the Timoshenko beam. A plethora of publications has addressed the stabilization of the Timoshenko system using diverse forms of damping. In a study referenced as [2], Kim and Renardy investigated the Timoshenko system referenced by (1) with two boundary controls specified in equation

$$\begin{aligned} K\varphi\left(L,t\right) - K\psi_x\left(L,t\right) &= \alpha\varphi_t\left(L,t\right), \text{on } \mathbb{R}_+, \\ EI\psi_x\left(L,t\right) &= -\beta\psi_t\left(L,t\right), \text{ on } \mathbb{R}_+, \end{aligned} \tag{2}$$

they succeeded in demonstrating an exponential decay phenomenon concerning the natural energy of the system refered to as (1). Additionally, they furnished numerical estimations for the eigenvalues associated with the same system. Feng et al. in [3] achieved a similar outcome while investigating the stabilization of vibrations within a Timoshenko system. Raposo et al., as documented in [4], delved into the study of Timoshenko (1) under homogeneous Dirichlet boundary conditions and two linear frictional dampings. In other words, they scrutinized the following system:

$$\rho\varphi_{tt} - k_1\left(\varphi_x - \psi\right)_x + \varphi_t = 0 \text{ in } (0,L)\times\mathbb{R}_+,$$
$$\rho\psi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x - \psi\right) + \psi_t = 0, \text{ in } (0,L)\times\mathbb{R}_+,$$
$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \text{ on } \mathbb{R}_+, \tag{3}$$

they demonstrated the exponential stability of the Timoshenko system (3). This finding echoes the result reported by Taylor [5], yet their contribution is distinct, as they emphasize the novelty of their approach grounded in the semigroup theory pioneered by Liu and Zheng [6].
Soufyane and Wehbe in [7] examined the Timoshenko system (1) incorporating a single internal distributed dissipation law. In other words, they investigated the following system:

$$\rho\psi_{tt} = \left(K\left(\psi_x - \varphi\right)\right)_x, \text{ in } (0,L)\times\mathbb{R}_+,$$
$$I_\rho\varphi_{tt} = \left(EI\varphi x\right)_x + K\left(\psi_x - \varphi\right) - b(x)\,\varphi t, \text{ in } (0,L)\times\mathbb{R}_+,$$
$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \text{ on } \mathbb{R}_+, \tag{4}$$

where $b(x)$ is a positive and continuous function, which satisfiies

$$b(x) \geq b_0 > 0, \ \forall x \in [a_0, a_1] \subset [0, L],$$

Their demonstration revealed that the Timoshenko system (4) achieves exponential stability solely under the condition of equal wave propagation rapidities, expressed as $\left(i.e\ \frac{K}{\rho} = \frac{EI}{I_\rho}\right)$. In cases where this condition isn't met, only asymptotic stability has been established. This advancement builds upon prior research by Soufyane [8] and Shi and Feng [9], who established exponential decay of the solution for (1) in conjunction with two locally distributed feedbacks.
Rivera and Racke [10] enhanced prior findings by demonstrating exponential decay of the solution within the system (4). This decay occurs under the condition where the coefficiient of the feedback allows for an indefinite sign. Additionally, in their work, Rivera and Racke [11] addressed a Timoshenko-type system formulated as:

$$\rho_1\varphi_{tt} - \sigma_1\left(\psi_x + \varphi\right)_x = 0,$$
$$\rho_2\psi_{tt} - \chi\left(\psi_x\right)_x + \sigma_2\left(\psi_x + \varphi\right) + d\psi_t = 0, \tag{5}$$

Within a one-dimensional bounded domain, frictional damping contributes to dissipation, operating exclusively within the equation governing rotation angle. The authors not only presented

an alternative proof establishing necessary and sufficient conditions for exponential stability in the linear case but also extended their analysis to demonstrate polynomial stability in a more general context. Additionally, their study delved into the examination of global existence for small smooth solutions and the investigation of exponential stability within the nonlinear framework.
Xu and Yung [12] demonstrated a system of Timoshenko beams incorporating pointwise feedback controls. They searched for information concerning the eigenvalues and eigenfunctions of the system, leveraging this data to analyze the systtem's stability.
Ammar-Khodja and colleagues [13] investigated a weaker form of dissipation, introducing a memory term $\int_0^t g(t-s)\psi_{xx}(s)\,ds$ into the equation governing rotation angle in (11). Utilizing multiplier techniques, they established that the system attains uniform stability if and only if condition (9) is met and the kernel g exhibits uniform decay. Specifically, they demonstrated that the decay rate is exponential (polynomial) if g decays exponentially (polynomially)

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\psi_x+\varphi)_x = 0, \text{ in } (0,L)\times\mathbb{R}_+, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)\,ds + K(\psi_x+\varphi) = 0, \text{ in } (0,L)\times\mathbb{R}_+, \end{aligned} \tag{6}$$

with homogeneous boundary conditions. They demonstrated that the system (6) achieves uniform stability only under the condition where the wave speeds are equal, and g exhibits uniform decay. Moreover, they established exponential decay if g decays exponentially, and polynomial decay if g decays polynomiall, They also required some technical conditions on both g' and g'' to acquire their findings, Ammar-Khodja and colleagues in [14] explored the decay rate of energy in a nonuniform Timoshenko beam with two boundary controlls affecting the rotation–angle equation. Santos [15] also investigated memory-type feedback within a Timoshenko system, showcasing its ability to uniformly stabilize the system with two memory–type feedbacks at a subset of the boundary, while also determining the energy decay rate, precisely reflecting the decay rate of relaxation functions. Particularly, under the condition of equal speed wave proppagation, they derived exponential decay results within an unknown finite-dimensional space of initial data and demonstrated the necessity of equal speed wave propagation for exponential stability. However, in cases of unequal speeds, no specifiic decay rate has been addrresed. Recently, Wehbe et al. in [16] enhanced the findings from [14], establishing nonuniform stability and optimal polynomial energy decay rates for the Timoshenko system with a single dissipation law applied solely at the boundary.
Shi and Feng [9] conducted a study on a nonuniform Timoshenko beam, where they demonstrated that the beam's vibration decays exponentially with the application of certain locally distributed controls. In their pursuit, the authors employed the frequency multiplier method to accomplish their objective.

For Timoshenko systems of classical thermoelasticity, Munoz Rivera and Racke in [17] showed the system of the form

$$\begin{aligned} &\rho_1\varphi_{tt}-\sigma\left(\psi_x+\varphi\right)_x \;=\;0,\;\text{in}\;\left(0,L\right)\times\mathbb{R}_+, \\ &\rho_2\psi_{tt}-b\left(\psi_{xx}\right)_x+k\left(\psi_x+\varphi\right)+\gamma\theta_x\;=\;0,\;\text{in}\;\left(0,L\right)\times\mathbb{R}_+, \\ &\rho_3\theta_t-k\theta_{xx}+\gamma\psi_{xt}\;=\;0,\;\text{in}\;\left(0,L\right)\times\mathbb{R}_+, \end{aligned} \tag{7}$$

where θ is the difference temperature, φ is the displacement and ψ is the rotation angle of filament of the beam and σ, ρ_1, ρ_2, b, k and γ are constituve constants. They demonstrated that, for the bouandary conditions

$$\varphi\left(x,t\right)=\psi\left(x,t\right)=\theta\left(x,t\right)=0,\;\text{for}\;x=0,L\;\text{and}\;t\geq 0, \tag{8}$$

the energy of the system described in (7) exhibits exponential decay if and onlly if

$$\frac{\rho_1}{k}=\frac{\rho_2}{b}, \tag{9}$$

and condition (9) is sufficient to exponentially stabilize system (7) under the given boundary conditions

$$\varphi\left(x,t\right)=\psi\left(x,t\right)=\theta\left(x,t\right)=0,\;\text{for}\;x=0,L\;\text{and}\;t\geq 0,$$

and non-exponential stability result for the case of different wave speeds of propagation. Munoz Rivera and Racke in [11] examined a Timoshenko system represented by the following form:

$$\begin{aligned} &\rho_1\varphi_{tt}-\sigma_1\left(\varphi_x+\psi\right)_x \;=\;0,\;\text{in}\;\left(0,L\right)\times\mathbb{R}_+, \\ &\rho_2\psi_{tt}-\chi\left(\psi_x\right)_x+\sigma_2\left(\varphi_x+\psi\right)+d\psi_t\;=\;0,\;\text{in}\;\left(0,L\right)\times\mathbb{R}_+, \end{aligned} \tag{10}$$

with homogeneous boundary conditions, they demonstrated that the Timoshenko system described in (10) achieves exponential stability when the wave propagation speeds are equal, conversely, in cases where the wave propagation speeds differ, the system exhibits only polynomial stability.
Within the aforementioned system, the heat flux is determined by Fourier's law. Consequently, a physical inconsistency arises, revealing an infinite heat propagation speed. This implies that any thermal disturbance occurring at a singular point instantaneously affects the entire medium. However, experimental observations have indicated that heat conduction in certain dielectric crystals at low temperatures does not conform to this paradox. Instead, disturbances, predominantly thermal in nature, proppagate at a finite speed. This intriguing phenomenon observed in dielectric crystals is referred to as second sound.

Numerous theories have been proposed to address this physical paradox. One such theory proposes the replacement of Fourier's law

$$q + k\theta_x = 0,$$

by what is known as Cattaneo's law

$$\tau q_t + q + k\theta_x = 0.$$

Alabau–Boussouira [18] expanded upon the findings of [11] to encompass the scenario involving nonlinear feedback $\alpha(\psi t)$ instead of $d\psi t$, where α is a globally Lipschitz function meeting certain growth conditions at the origin. Specifically, she examined the following system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \text{ in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \alpha\psi_x &= 0, \text{ in } (0, L) \times \mathbb{R}_+, \end{aligned} \tag{11}$$

with homogeneou boundary conditions. Indeed, if the wave propagation speeds are equal, she derived a comprehensive semi–explicit formula for the decay rate of energy at infinity. In contrast, she established polynomial decay in scenarios involving varying speeds of propagation for both linear and nonlinear globally Lipschitz feedbacks.

Regarding the Timoshenko system with delay, the inquiry commenced with the work of Houari and Laskri in their paper [19], where they examined the following problem

$$\begin{aligned} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) &= 0, \\ \rho_2\psi_{tt}(x,t) - \alpha\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) &= 0, \end{aligned} \tag{12}$$

under the assumption $\mu_1 \geq \mu_2$ regarding the weights of the two feedbacks, they demonstrated the well–posedness of the system. Furthermore, for $\mu_1 > \mu_2$, they demonstrated an exponential decay result in the scenario of equal–speed wave propagation.

Following this, the research presented in [19] was expanded to encompass the scenario of time–varying delay, expressed as $\psi_t(x, t - \tau(t))$, by Kirane, Said-Houari, and Anware [20]. Additionally, in [21], the casee where the damping terrm $\mu_1\psi_t$ is substituted by a history-type term $\int_0^\infty g(s)\psi_{xx}(x,t-s)\,ds$ (with either discrete delay $\mu_2\psi_t(x,t-\tau)$ or distributed delay $\int_0^\infty f(s)\psi_t(x,t-s)$ was addressed. In this study, various general decay estimates were established.

The Bresse systems

The issue concerning the arc, also referred to as the Bresse system, is widely recognized. Elastic structures of this kind find extensive applications in engineering, architecture, marine engineering, aeronautics, and various other fields. Specifically, the study of vibration in elastic structures holds significant importance in both engineering and mathematics. Within the realm of Mathematical Analysis, understanding the properties associated with energy behavior in the respective dynamic models becomes intriguing. For instance, when considering feedback laws, one may inquire about the conditions pertaining to the dynamic model necessary to achieve the decrease in energy from solutions in time t. Consequently, the concept of stabilization has garnered attention in the examination of dynamic issues concerning elastic structures, as expressed through partial differential equations.

In the original, the Bresse system comprise three wave equations, with the primary variables delineating longitudinal, vertical, and shear angle displacements, as depicted (refer to [22]). The original formulation of the Bresse system is expressed through the following equations:

$$\begin{aligned}
\rho_1\varphi_{tt} &= Q_x + lN + F_1,\\
\rho_2\psi_{tt} &= M_x - Q + F_2,\\
\rho_1\omega_{tt} &= N_x - lQ + F_3,
\end{aligned} \tag{13}$$

These strengths encompass the deformation relationships (stress–strain) indicative of elastic behavior, and they are provided by

$$\begin{aligned}
N &= k_0\left(\omega_x - l\varphi\right),\\
Q &= k\left(\varphi_x + l\omega + \psi\right),\\
M &= b\psi_x,
\end{aligned} \tag{14}$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. By ω, φ and ψ we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A = \rho I$, $k_0 = EA$, $k = k^{'}GA$ and $l = R^{-1}$. To material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross–sectional area,I for the second moment of area of the cross–section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denoting external forces.

Taking into account the coupling of equations (13) and (14), we get

$$\begin{aligned}
\rho_1\varphi_{tt} - k\left(\varphi_x + l\omega + \psi\right)_x - k_0 l\left(\omega_x - l\varphi\right) &= F_1,\\
\rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + l\omega + \psi\right) &= F_2,\\
\rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + l\omega + \psi\right) &= F_3,
\end{aligned} \tag{15}$$

The system referenced as (15) is undamped, maintaining a constant energy as time progresses. Numerous authors have explored various damping mechanisms to stabilize this system (refer to [23] – [29]). Our discussion on stabilizing the elastic Bresse system begins with insights from

Wehbe and Youssef [30], who investigated the system under two locally internal dissipation laws. They established that exponential stability occurs only if the wave propagation speeds are identical, otherwise, polynomial stability prevails. Alabau–Boussouira et al. [23] scrutinized the same system with a globally distributed dissipation law and demonstrated that while exponential stability isn't guaranteed, polynomial decay exists, contingent upon specific coefficients' relations. Under Dirichlet-Dirichlet boundary conditions, they revealed energy decay rates of $t^{\frac{-1}{3}}$, and $t^{\frac{-2}{3}}$ if $k \equiv k_0$. These findings were complemented by Fatori and Montiero [31], who, under Dirichlet–Neumann-Neumann boundary conditions, demonstrated polynomial energy decay rates of $t^{\frac{-1}{2}}$ and if $k \equiv k_0$. Noun and Wehbe [32] extended the research of [23] and [31] by investigating the elastic Bresse system subject to locally distributed feedback, under either Dirichlet–Neumann–Neumann or Dirichlet–Dirichlet–Dirichlet boundary conditions. They demonstrated that by considering damping terms as infinite memories acting in the three equations.

In the study of the thermoelastic Bresse system addressed herein, two significant findings emerge. Initially demonstrated by Liu and Rao [33], they examined the Bresse system incorporating two distinct thermal dissipation laws. Their findings indicate an exponential energy decay when the wave speed of vertical displacement coincides with that of longitudinal or shear angle displacement. Conversely, polynomial decay rates dependent on boundary conditions are observed otherwise. Under Dirichlet–Neumann–Neumann boundary conditions, energy decay occurs at $t^{\frac{-1}{2}}$, while under fully Dirichlet boundary conditions, it diminishes at $t^{\frac{-1}{4}}$. Recent work by Fatori and Rivera [34] refines these findings, focusing on a globally dissipative mechanism controlled by a single temperature. They establish decay rates of $t^{\frac{-1}{3}}$ for both Dirichlet–Neumann–Neumann and Dirichlet–Dirichlet–Dirichlet boundary conditions. The primary contribution of this study lies in extending the results of [34], addressing scenarios where the therrmal dissipation law is locallly distributed within the angle displacement equation. In such cases, the damping coefficient becomes a variable function in $W^{2,\infty}(0, L)$, strictly positive within an open subinterval $]a, b[\subset]0, L[$ (where $a = 0$ or $b = L$ are not excluded), thereby enhancing the polynomial energy decay rate.
A new type of problem arises with the combination of the Timoshenko system [35] and Bresse system or the curved beam [36]. The coupled system from which we derive Bresse–Timoshenko is derived from Elishakov [37] and combines principle of D'Alembert of dynamic equilibrium with hypothesis of Timoshenko to produce the following coupled system

$$\begin{cases} \rho_1 \partial_{tt}\varphi - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2 \partial_{tt}\varphi_x - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{16}$$

The Cattaneo's law is one of the most well-known thermoelasticity laws, but it is unable to account for somee physical properties and cannot answer all questions, therefore, its applications are limited. This leads us to consider coupling the fiields of strain, temperature, and mass diffusion using the Gurtin–Pinkin model. Only a few authors have studied the stabilization of the Bresse–Timoshenko model.

Manevich and Kolakowski [38] obtained the first contribution in that direction where they analyzed the dynamics of a Timoshenko model in which, the damping mechanism is viscoelastic. More accurately, they deemed the dissipative system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x - \mu_1\left(\varphi_x + \psi\right)_{tx} = 0, \\ -\rho_2\psi_{tt} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) - \mu_2\psi_{ttx} + \mu_1\left(\varphi_x + \psi\right)_t = 0. \end{cases} \tag{17}$$

Second, based on Elishakoff's papers and collaborators and their studies on truncated versions for classical Timoshenko equations [39], Almeida Junior and Ramos [40] proved that the total energy for viscous damping acting on angle rotation of the simplified Timoshenko system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) + \mu_1\psi_t = 0, \end{cases} \tag{18}$$

There is a great difference in the model from a classical Timoshenko system, as it is consisted of three derivatives: two with respect to time and one with respect to space. This happened because the absence of the second spectrum, or nonphysical spectrum [39], [37], and its damage consequences for wave propagation speeds [40]. The historical and mathematical observations can be found in other works [39] and [37]. The similar results are accomplished for a dissipative truncated version, where the viscous damping acts on vertical displacement

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{19}$$

The study of the existence and stability of development problems has been the subject of many recent works so we indicate some related work, [41] in this work, Guesmia and Soufyane studied the well posedness and proved the stability for the following system

$$\begin{cases} \rho_1\varphi_{tt} - k_1\left(\varphi_x + \psi\right)_x + \lambda_1\varphi_t + \mu_1\varphi_t\left(x, t - \tau_1\right) = 0, \\ \rho_2\varphi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x + \psi\right) + \lambda_2\psi_t + \mu_2\psi_t\left(x, t - \tau_2\right) = 0. \end{cases} \tag{20}$$

In [42], the authors proved the well–posedness and establish uniform stability results for the following Timoshenko system with a linear frictional damping and an internal distributed delay acting on the transverse displacement

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \gamma_1\varphi_t + \int_{\tau_1}^{\tau_2}\gamma_2\varphi_t\left(x, t - s\right)ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{21}$$

In [43],the authors proved the well–posedness and established exponential stability results regardless of the speeds of wave propagation for the following thermoelastic system of Timoshenko type with a linear frictional damping and an internal distributed delay acting on the displacement equation

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2\left(s\right)\varphi_t\left(x, t-s\right)ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases} \tag{22}$$

In [44],they established the stability of the following Timoshenko-type-system

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2\left(s\right)\varphi_t\left(x, t-s\right)ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \int_0^t g\left(t-s\right)\left(a\left(x\right)\psi_x\right)_x ds \\ +\mu_3\left(t\right)b\left(x\right)f\left(\psi_t\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + kq_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + k\theta_x = 0. \end{cases} \tag{23}$$

In [45], Almeida Junior et al deemed two cases of dissipative systems for Bresse–Timoshenko–type-systems with constant delay cases. For the first one, the authors established the exponential decay for the system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau\right) = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{24}$$

For the second one, the authors also demonstrated the exponential decay for the system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) + \mu_1\varphi_t + \mu_2\psi_t\left(x, t-\tau\right) = 0, \end{cases} \tag{25}$$

The authors in [46] deemed two cases of dissipative systems for Bresse–Timoshenko–type systems with time-varying delay cases. For the fiirst one, the authors showed the exponential decay for the system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau\left(t\right)\right) = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) = 0, \end{cases} \tag{26}$$

for the second one, the authors also established the exponential decay result for the system presented by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) + \mu_1\varphi_t + \mu_2\psi_t\left(x, t-\tau\left(t\right)\right) = 0 \end{cases} \tag{27}$$

In [47], They used the Faedo-Galerkin approximations and some energy estimates to establish the global well–posedness of the initial and boundary value problem, and they proved the exponential decay of dissipative systems for the following Bresse–Timoshenko–type system with distributed delay, under appropriate assumptions

$$\begin{cases} \rho_1 \varpi_{tt} - K(\varpi_x + v)_x + \mu_1 \varpi_t + \int_{\tau_1}^{\tau_2} \mu_2(p)\, \varpi_t(x, t-p)\, dp = 0, \\ -\rho_2 \varpi_{ttx} - b v_{xx} + K(\varpi_x + v) = 0. \end{cases} \tag{28}$$

See other works in [48] and [49].
We escort the paper of [47] but in this present work, we deem the following Bresse–Timoshenko system of second sound with distrubted delay term

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_{\mathbf{x}} + \psi)_{\mathbf{x}} + \boldsymbol{\xi}_1 \varphi_t \\ + \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \varphi_t(\mathbf{x}, t-s)\, \mathbf{ds} = 0, \text{ in } (0,1)\times(0,\infty), \\ -\rho_2 \varphi_{tt\mathbf{x}} - b\psi_{\mathbf{xx}} + K(\varphi_{\mathbf{x}} + \psi) + \gamma\theta_{\mathbf{x}} = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3 \theta_t + \kappa q_{\mathbf{x}} + \gamma \psi_{t\mathbf{x}} = 0, \text{ in } (0,1)\times(0,\infty), \\ \tau_0 q_t + \delta q + \kappa \theta_{\mathbf{x}} = 0, \text{in } (0,1)\times(0,\infty), \\ \varphi(\mathbf{x},0) = \varphi_0(\mathbf{x}), \varphi_t(\mathbf{x},0) = \varphi_1(\mathbf{x}), \psi(\mathbf{x},0) = \psi_0(\mathbf{x}), \\ \psi_t(\mathbf{x},0) = \psi_1(\mathbf{x}), \text{ in } (0,1), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), \ q(\mathbf{x},0) = q_0(\mathbf{x}), \\ \varphi_t(\mathbf{x},-t) = f_0(\mathbf{x},t), \text{ in } (0,1)\times(0,\infty), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0, \ \forall t \geq 0, \end{cases} \tag{29}$$

where $t \in (0,\infty)$ denotes the time variable and $\mathbf{x} \in \mathcal{J} = (0,1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and t he rotation angle, the function θ is the temperature difference, $q = q(t,\mathbf{x}) \in \mathbb{R}$ is the heat flux, and ρ_1, ρ_2, ρ_3, γ, τ_0, δ, κ, $\boldsymbol{\xi}_1$, and K are positive constants, $\boldsymbol{\xi}_2 : \mathcal{L} = [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, \mathbf{ds} < \boldsymbol{\xi}_1, \tag{30}$$

where τ_1 and τ_2 two reall numbers satisfying $0 \leq \tau_1 \leq \tau_2$, and we study exponential stability results and the global well-posednes of a class of Bresse–Timoshenko system–type of second sound with distributed delay term.

Chapter 1

Preliminary Notations

In this chapter, we recollect some notions of the theory of functional spaces, and some important inequalities which we use in the next chapters

1.1 Some functional spaces

1.1.1 Banach Space

Definition 1.1 (Normed liinear space) *A normed vector space X is a vector space equipped with a norm $||.|| : X \to \mathbb{R}$ that satisfies the following properties:*

$$1.\ ||x+y|| \leq ||x|| + ||y||,$$

$$2.\ ||\alpha x|| = |\alpha|.||x||, \text{ for any scalar } \alpha,$$

$$3.\ ||x|| \geq 0, \text{ and } ||x|| \neq 0 \iff x = 0.$$

Recall that completeness of a normed vector space X means that all Cauchy sequences in X converge in X.

Definition 1.2 (Banach spaces) *A Banach space is a complete normed vector space.*

1.1.2 Hilbert Space

Definition 1.3 (Inner–Product Space) *A complex linear space X is called an Inner-Product space if to each pair of elements x, y of X, there is an associated complex number (x, y) (called the Inner Product of x and y) with the following properties:*

$$1.(x+y+z) = (x,z) + (y,z), \forall x,\ y,\ z \in X,$$

$$2.(x,y) = \overline{(y,x)},$$

where the bar denotes complex conjugate.

$$3.(\alpha x, y) = \alpha(x, y), \forall \alpha$$
$$4.\ (x, x) \geq 0, \forall x \in X,\ \textit{and}\ (x, x) = 0 \Leftrightarrow x = 0.$$

Inner–product spaces are special cases of Norrmed linear spaces. This is expressed by the following lemma.

Lemma 1.1 *Let X be a linear space with Inner Product $(.,.)$. Then the expression*

$$||x|| = \sqrt{(x, x)}, \forall x \in X,$$

defines a norm on X.

Definition 1.4 (Hilbert space) *A Hilbert space is an Inner Product space which (as a Normed linear space) is complete .*
A metric on H given by

$$||x - y|| = \sqrt{(x - y, x - y)},$$

Hence Inner Product spaces are Normed spaces, and Hilbert spaces are Banach spaces.

Definition 1.5 (Orthogonal complement) *For any Subspace $\mathcal{M}$ of H, we define the orthogonal complement by*

$$\mathcal{M}^{\perp} = \{x \in H \mid (x, y) = 0, \forall y \in \mathcal{M}\},$$

which is the set of all vectors orthogonal to $\mathcal{M}$.
It is clear that $\mathcal{M}^{\perp}$ is a closed subspace. If $\mathcal{M}$ is also closed, then H is a direct sum of $\mathcal{M}$ and $\mathcal{M}^{\perp}$: $H = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

Definition 1.6 *Let $(H, (.,.)_H)$ be an inner product space, we say that a sequencee $(x_i)_{i\in I}$ is orthogonali if*

$$\forall i; j \in I,\ i \neq j; (x_i, x_j)_H = 0,$$

and a sequence$(x_i)_{i\in I}$ is said to be orthonormal if

$$\forall i; j \in I : \quad (x_i, x_j)_H = \delta_{ij} = \begin{cases} 1;\ i \neq j \\ 0;\ i = j \end{cases}.$$

Definition 1.7 *A orthonormal sequence $(x_i)_{i\in I}$ in H is said to be an orthonormal basis of H or a Hilbert basis it satisfies $\{x_i;\ i \in I\}^{\perp} = 0$. In addition to strong convergence in H we also consider weak convergence or converges in the sense of inner product, $(u_n)_{n\in\mathbb{N}}$ of elements of H we say that it converges weakly to u if:*

$$(u_n - u, v)_H \longrightarrow 0 \text{ quand } n \longrightarrow \infty \text{ for any } v \in H,$$

and we denote by:

$$u_n \longrightarrow u$$

Proposition 1.1 *If $(u_n)_{n\in\mathbb{N}}$ converges in the norm to u, then it converges weakly to u. The reverse is not true in general. However if $u_n \longrightarrow u$and $\|u_n\| \longrightarrow \|u\|$, then in this case $u_n \xrightarrow{f} u$.*

1.1.3 The Sobolev spaces

The $L^p(\Omega)$ spaces

Definition 1.8 *Let $1 \leq p \leq \infty$, and let Ω be open set in $\mathbb{R}^n$, $n \in \mathbb{R}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ g : \Omega \longrightarrow \mathbb{R} : \ f \text{ is measurable and } \mid \int_{\Omega} |g(x)|^p dx < \infty \right\}.$$

Notation 1.1 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, denote by*

$$||g||_{L^p(\Omega)} = \left(\int_{\Omega} |g(x)|^p dx \right)^{\frac{1}{p}}$$

if $p = \infty$, we have

$$L^{\infty}(\Omega) = \left\{ \begin{array}{c} g : \Omega \longrightarrow \mathbb{R} : \ g \text{ is measurablee} \\ \text{and there exists a constant } C \text{ such that, } g(x)| \leq C \text{ a.e in } \Omega \end{array} \right\}.$$

Also, we denote byy

$$||g||_{\infty} = \inf \{C, \ |g(x)| \leq C \text{ a.e in } \Omega\}.$$

Notation 1.2 *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark 1.1 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle g, h \rangle_{L^2(\Omega)} = \int_{\Omega} g(x)\, h(x)\, dx,$$

is a Hilbert space.

The Sobolev spaces $\mathcal{W}^{k,p}(\Omega)$

Definition 1.9 *Let $k \in \mathbb{N}$ and $p \in [1,\infty]$. The $\mathcal{W}^{k,p}(\Omega)$ is the space of all $g \in L^p(\Omega)$ defined as*

$$\mathcal{W}^{k,p}(\Omega) = \left\{ \begin{array}{c} g \in L^p(\Omega), \text{ such that } \partial^\alpha g \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that} \\ |\alpha| = \sum_{j=1}^n \alpha_j \leq k, \text{ where } \partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n} \end{array} \right\}$$

Theorem 1.1 *$\mathcal{W}^{k,p}(\Omega)$ is a Banach space with their usual norm*

$$\|g\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha g\|_{L^p(\Omega)}, \ 1 \leq p \leq \infty, \text{ for all } g \in L^p(\Omega).$$

Definition 1.10 *When $p = 2$, we prefer to denote by $\mathcal{W}^{k,2}(\Omega) = H^k(\Omega)$ and $\mathcal{W}_0^{k,p}(\Omega) = H_0^k(\Omega)$ for $p \in [0,\infty[$ supplied with the norm*

$$\|f\|_{H^k(\Omega)} = \left(\sum_{|\alpha|\leq k}\left(\|\partial^\alpha f\|_{L^2(\Omega)}\right)^2\right)^{\frac{1}{2}},$$

which do at $H^k(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u,v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq k}\int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

The next result provides a basic characterization of functions in $\mathcal{W}_0^{1,p}(\Omega)$.

Theorem 1.2 *Let $u \in \mathcal{W}^{1,p}(\Omega)$. Thenn $u \in \mathcal{W}_0^{1,p}(\Omega)$ iif and only iif $u = 0$ on $\partial\Omega$.*

Remark 1.2 ***1.** The last theorem explains the central role played by the space $\mathcal{W}_0^{1,p}(\Omega)$. Differential equations (or partial differential equations) are often coupled with boundary conditions, i.e., the value of u is prescribed on $\partial\Omega$.*
***2.** We have the following characterization of $H_0^k(\Omega)$*

$$H_0^k(\Omega) = \left\{u \in H^k(\Omega), \ u = u' = \ldots = u^{(k-1)} = 0, \text{ on } \partial\Omega\right\}.$$

It is essential to notice the distinction between

$$H_0^2(\Omega) = \left\{u \in H^2(\Omega), \ u = u' = 0, \text{ on } \partial\Omega\right\},$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \left\{u \in H^2(\Omega), \ u = 0, \text{ on } \partial\Omega\right\}.$$

1.2 Important Inequalities

1.2.1 Gronwall's Lemma

If $f_i(\tau)(i = 1, 2, 3)$ are nonnegative functions on $(0, T)$, $f_1(\tau)$, $f_2(\tau)$ are integrable functions, and $f_3(\tau)$ is nondecreasing on $(0, T)$, then if

$$\Im_\tau f_1 + f_2 \leq f_3 + c\Im_\tau f_2,$$

then

$$\Im_\tau f_1 + f_2 \leq \exp(c\tau) f_3 (\tau),$$

where

$$\Im_\tau f_i = \int_0^\tau f_i(t)\, dt, \qquad (i = 1, 2).$$

1.2.2 Cauchy-Schwarz integral inequality

For any $u, v \in L^2(\Omega)$, we have the following inequality

$$\int_\Omega u(x)v(x)dx \leq \left(\int_\Omega u^2(x)dx\right)^{\frac{1}{2}} \left(\int_\Omega v^2(x)dx\right)^{\frac{1}{2}}.$$

which is called Cauchy -Schwarz integral inequality.

1.2.3 Cauchy's inequality

For any $a, b \in \mathbb{R}$, we have

$$ab \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2.$$

1.2.4 Cauchy's inequality with ε

The following inequality:

$$ab \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \quad a, b \in \mathbb{R},$$

holds for any $\varepsilon > 0$.

1.2.5 Young's inequality

The generalization of Cauchy inequality is called Young's inequality which is represented by

$$ab \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^{\frac{p-1}{p}}, \quad a, b \in \mathbb{R},\ p > 1.$$

1.2.6 Young's inequality with ε

For any $\varepsilon > 0$, we have the inequality

$$ab \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}|\frac{b}{\epsilon}|^{\frac{p-1}{p}}, \quad a, b \in \mathbb{R},\ p > 1.$$

which is the generalization of Cauchy inequality with ε.

1.2.7 Holder inequality

For any $u \in L^P(\Omega)$ and $v \in L^q(\Omega)$ we have the following inequality

$$\int_\Omega u(x)v(x)dx \leq \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^{\frac{p-1}{P}} dx\right)^{\frac{p-1}{p}}.$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. This inequality is the generalization of the Chauchy-Schwarz integral inequality.

1.2.8 Poincare' inequality

For all $u \in \overset{\circ}{W}{}_2^1(\Omega)$, we have the inequality

$$\int_\Omega u^2 dx \leq C_\Omega^2 \int_\Omega u_x^2 dx.$$

where C_Ω is a constant depending only on Ω.

1.3 Green's Formula

Suppose that Ω is a smooth bounded domain in $\mathbb{R}^n$, and $u, v \in C^2(\overline{\Omega})$ .The following are called Green's formulas

$$\int_{\partial\Omega} v\frac{\partial u}{\partial v} dS = \int_\Omega (v\triangle u + \nabla v \nabla u)\, dx, \tag{1.1}$$

$$\int_{\partial\Omega} (v\frac{\partial u}{\partial v} - u\frac{\partial v}{\partial v}) dS = \int_\Omega (v\triangle u - u\triangle v) dx, \tag{1.2}$$

where dS denotes the surface measure on $\partial\Omega$. in fact, Green's formulas (1.1) and (1.2) hold more generally for $u, v \in C^2(\overline{\Omega}) \cap C^1(\overline{\Omega})$, provided the integrals over Ω and $\partial\Omega$ converge.

Special Cases:

1. If we take $v = 1$ in (1.1), we obtain

$$\int_{\partial\Omega} \frac{\partial u}{\partial v} dS = \int_\Omega \triangle u\, dx,$$

2. If we take $u=v$ in (1.1) ,we obtain

$$\int_{\partial\Omega} u\frac{\partial u}{\partial v}dS=\int_{\Omega}\left(u\triangle u+|\nabla u|^2\right)dx.$$

1.4 Parametric integral

Let

$$f:\mathbb{R}^2\longrightarrow\mathbb{R}^2,$$

such that f and $\frac{\partial f}{\partial x}$ be continuous on $\mathbb{R}^2$ and let α and β two derivable functions from $\mathbb{R}^n$ in $\mathbb{R}^n$, then "parametric integral" F define on $\mathbb{R}^2$ by

$$F(x)=\int_{\alpha(x)}^{\beta(x)} f\left(x,y\right)dy,$$

is derivable

$$\frac{\partial}{\partial x}F(x)=\int_{\alpha(x)}^{\beta(x)}\frac{\partial}{\partial x}f\left(x,y\right)dy+\frac{\partial\beta\left(x\right)}{\partial x}f\left(x,\beta\left(x\right)\right)-\frac{\partial\alpha\left(x\right)}{\partial x}f\left(x,\alpha\left(x\right)\right)$$

Remark 1.3 *For a function f that depends only on the second variable, the fundamental theorem of analysis can be applied as follows*

$$\alpha\left(x\right)=\alpha,\beta\left(x\right)=\beta.$$

1.5 The Faedo-Galerkin method

In this section we give the Faedo-Galerkin scheme, and for this we give the following definition

1.5.1 The Faedo-Galerkin approximation

Definition 1.11 *Let V be a separable Hilbert space and $\{V_n\}_{n\in\mathbb{N}^*}$ a family of finite–dimensional vector spaces satisfying the axioms:*

$$V_n\subset V,\dim V_n<\infty, \tag{1.3}$$

$$V_n\rightarrow V \text{ quand } n\rightarrow\infty. \tag{1.4}$$

In the following sense: there exists $\bigcup\limits_{i=1}^{n}V_n$ a dense subspacee in V, such that for every $u\in V_n$ we can find a sequence $\{u_n\}_{\in\mathbb{N}^*}$ satisfying: for all n, $u_n\in V_n$ and $u_n\rightarrow u$ in V as $n\rightarrow\infty$. The space V_n is called a Galerkin approximation of orderr n.

1. **Definition 1.12 (The scheme of the Faedo-Galerkiin method)**

The scheme of the Faedo-Galerkiin method Let P be the exact problem for which we seek to demonstrate the existence of the solution in a function space constructed on a separable Hilbert space V, and let u be the solution of the problem P. After choosing a Galerkin approximation V_n of V, it is appropriate to define an approximate problem P_n in the finite–dimensional space V_n having a unique solution u_n. The course of the study is then as follows:

(a) *We defiine the solution u_n of problem P_n.*

(b) *We establish estimates on u_n ("a priori estimation") to show that u_n is uniformly bounded.*

(c) *By using the results that u_n is uniformly bounnded, it is possible to extract from $\{u_n\}_{n\in\mathbb{N}^*}$ a subsequencee $\{u'_n\}_{n\in\mathbb{N}^*}$ that has a limit in the weak topologyy of the spaces involved in the estimates of step b. Let u be the obtained limit.*
Our objective is to construct an approximation procedure that provides us with a demonstration of the existence of the solution at the limit. This procedure amounts to approximating $u_n(x,t)$ as a linear combination of basis functions $Z_i(t)$, such that

$$u_n(x,t) = \sum_{i=1}^{n} C_i(t) Z_i(x) \quad (x,t) \in \Omega \times [0,T], \tag{1.5}$$

where the $C_i(t)$ are then solutions to a system of n linear differential equations.

1.6 Stabilization method (Lyapunov functional)

To establish the desired stability results of the systems, we use the multiplier method. The multiplier method is mainly based on the construction of an appropriate Lyapunov function L which is equivalent to the solution's energy. By equivalence $L \sim E$, we mean

$$\alpha E(t) \leq L(t) \leq \gamma E(t), \ \forall t > 0, \ (\alpha, \gamma > 0). \tag{1.6}$$

To prove the exponential stability, we show that L satisfies

$$L'(t) \leq \beta L(t), \ \forall t > 0, \ \beta > 0. \tag{1.7}$$

A simple integration of (1.7) over $(0,t)$ together with (1.6) gives the desired exponential stability result.

1.7 Stabilization types

There are several types of stabilization, of which

- **Strong stabilization:** It consists of analyzing simply the decay of energy solutions to 0, i.e

$$E(t) \to 0 \text{ as } t \to \infty.$$

- **Exponential (uniform) stabilization:** we're interested in the fastest energy decay, i.e. when it tends exponentially towards 0, i.e

$$E(t) \leq \alpha \exp(-\gamma t), \forall t > 0, (\alpha, \gamma > 0).$$

- **Polynomial stabilization:**

$$E(t) \leq \alpha t^{-\gamma}, \ \forall t > 0, (\alpha, \gamma > 0).$$

1.8 Time delays

Time delays are a common occurrence in various applications where phenomena are governed by partial differential equations. These delays introduce a modulation where the evolution of a system is influenced not just by its current state but also by its past states. Delay differential equations (DDEs) represent a class of differential equations where the current state of unknown functions is determined by their values at previous time instance. Mathematically, a simple delay differential equation for $y(t) \in \mathbb{R}^n$ takes the form

$$\frac{d}{dt} y(t) = g(t, y_t),$$

where $y_t = \{y(\tau), \ \tau \leq t\}$ represents the trajectory of the solution in the past. The functional operator g takes a time input and a continuous function y_t and genratees a real number $\frac{d}{dt} y(t)$ as its output. Examples of such equation include:

- Discrete / constant delay: $\frac{d}{dt} y(t) = g(t, y(t-\tau))$.
- Time–varying delay: $\frac{d}{dt} y(t) = g(t, y(t-\tau(t)))$.
- Distributed delay: $\frac{d}{dt} y(t) = g\left(t, \int_0^\tau \mu(s) y(t-s) \, ds\right)$,
  where τ is the delay in time.

Chapter 2

Well-posedness of problem

In this chapter, we will utilize the classical Faedo-Galerkin method to establish the global existence and we will establish the uniqueness and the continuous dependence on the initial data of the generalized solution to problem (29).

2.1 Introduction

Introducing the following new variable

$$\boldsymbol{\varpi}(\mathbf{x}, \boldsymbol{\rho}, \mathrm{s}, t) = \varphi_t(\mathbf{x}, t - \boldsymbol{\rho}\mathbf{s}) \quad \text{in } (0,1) \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty), \tag{2.1}$$

then, we obtain

$$\mathbf{s}\boldsymbol{\varpi}_t(\mathbf{x}, \boldsymbol{\rho}, \mathrm{s}, t) + \boldsymbol{\varpi}_\rho(\mathbf{x}, \boldsymbol{\rho}, \mathrm{s}, t) = 0. \tag{2.2}$$

Consequently, the problem (29) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_{\mathbf{x}}+\psi\right)_{\mathbf{x}} + \boldsymbol{\xi}_1\varphi_t \\ +\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{ds} = 0, \text{ in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{tt\mathbf{x}} - b\psi_{\mathbf{xx}} + K\left(\varphi_{\mathbf{x}}+\psi\right) + \gamma\theta_{\mathbf{x}} = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + \kappa q_{\mathbf{x}} + \gamma\psi_{t\mathbf{x}} = 0, \text{ in } (0,1)\times(0,\infty), \\ \tau_0 q_t + \delta q + \kappa\theta_{\mathbf{x}} = 0, \text{ in } (0,1)\times(0,\infty), \\ \mathbf{s}\boldsymbol{\varpi}_t\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right) + \boldsymbol{\varpi}_{\boldsymbol{\rho}}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right) = 0, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ \mathbf{s}\boldsymbol{\varpi}_{tt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right) + \boldsymbol{\varpi}_{\boldsymbol{\rho}t}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right) = 0, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ \varphi(\mathbf{x},0) = \varphi_0\left(\mathbf{x}\right), \varphi_t(\mathbf{x},0) = \varphi_1\left(\mathbf{x}\right), \ \psi(\mathbf{x},0) = \psi_0\left(\mathbf{x}\right), \\ \psi_t(\mathbf{x},0) = \psi_1\left(\mathbf{x}\right), \text{ in } (0,1), \\ \theta\left(\mathbf{x},0\right) = \theta_0\left(\mathbf{x}\right), \ q\left(\mathbf{x},0\right) = q_0\left(\mathbf{x}\right), \text{ in } (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0, \ \forall t \geq 0, \\ \boldsymbol{\varpi}\left(x,\rho,s,0\right) = f_0\left(\mathbf{x},-\rho s\right), \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.3}$$

Let $\mathcal{V}\left(Q := (0,1)\times(0,\infty)\right)$ and $\mathcal{W}\left(Q\right)$ be the set spaces defined respectively by

$$\mathcal{V}\left(Q\right) := \left\{ \begin{array}{c} \left(\varphi,\psi,\theta,q,\boldsymbol{\varpi}\right) : \varphi \in L^2\left(\mathbb{R}_+, H^2\cap H_0^1\right), \ \varphi_t \in L^2\left(\mathbb{R}_+, H^1\right), \ \varphi_{tt} \in L^2\left(\mathbb{R}_+, L^2\right), \\ \psi \in L^2\left(\mathbb{R}_+, H_0^1\cap H^2\right), \ \psi_t \in L^2\left(\mathbb{R}_+, H^1\right), \ \theta, \ q \in L^2\left(\mathbb{R}_+, H_0^1\right), \\ \theta_t, \ q_t \in L^2\left(\mathbb{R}_+, L^2\right), \ \boldsymbol{\varpi} \in L^2\left(\mathbb{R}_+, H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)\right), \\ \boldsymbol{\varpi}_t \in L^2\left(\mathbb{R}_+, H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)\right), \end{array} \right\},$$

and

$$\mathcal{W}\left(Q\right) := \left\{ \left(\varphi,\psi,\theta,q,\boldsymbol{\varpi}\right) \in \mathcal{V}\left(Q\right) : \lim_{T\to\infty} w_l\left(T\right) = \lim_{T\to\infty} s_l\left(T\right) = \lim_{T\to\infty} v_l\left(T\right) = \lim_{T\to\infty} r_l\left(T\right) = \lim_{T\to\infty} p_l\left(T\right) = 0 \right\}.$$

Consider the system

$$\begin{aligned} &\rho_1\left(\varphi_{tt},u\right) + K\left(\left(\varphi_{\mathbf{x}}+\psi\right),u_x\right) + \boldsymbol{\xi}_1\left(\psi_t,u\right) \\ &+ \left(\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{ds}, u\right) = 0, \\ &\rho_2\left(\varphi_{tt},v_{\mathbf{x}}\right) + b\left(\psi_{\mathbf{x}},v_x\right) + K\left(\left(\varphi_{\mathbf{x}}+\psi\right),v\right) \\ &+\gamma\left(\theta_{\mathbf{x}},v\right) = 0, \\ &\rho_3\left(\theta_t,w\right) + \left(q_{\mathbf{x}},w\right) + \gamma\left(\psi_{t\mathbf{x}},w\right) = 0, \\ &\tau_0\left(q_t,z\right) + \delta\left(q,z\right) + \kappa\left(\theta_{\mathbf{x}},z\right) = 0, \\ &s\left(\boldsymbol{\varpi}_t\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\phi\right) + \left(\boldsymbol{\varpi}_{\boldsymbol{\rho}}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\phi\right) = 0, \\ &s\left(\boldsymbol{\varpi}_{tt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\phi\right) + \left(\varpi_{\boldsymbol{\rho}t}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\phi\right) = 0, \end{aligned} \tag{2.4}$$

where $(.,.)_{L^2(Q)}$ stands for the inner product in $L^2(Q)$, $(\varphi,\psi,\theta,q,\varpi)$ is supposed to be a solution of the problem (2.3) and $(u,v,w,z,\phi)\in\mathcal{W}(Q)$. Evaluation of the inner product in (2.4) and use of the Dirichlet conditions $(2.3)_8$ leads to

$$\begin{aligned}
&-\rho_1(\varphi_t,u_t)_{L^2(Q)}-\rho_1(\varphi_t(\mathbf{x},0),u(\mathbf{x},0))_{L^2(0,1)}+K((\varphi_{\mathbf{x}}+\psi),u_{\mathbf{x}})_{L^2(Q)}\\
&-\mu_1(\psi,u_t)_{L^2(Q)}-\boldsymbol{\xi}_1(\psi(\mathbf{x},0),u(\mathbf{x},0))_{L^2(0,1)}\\
&+\left(\int_{\mathcal{L}}\boldsymbol{\xi}_2(s)\,\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{ds},u\right)_{L^2(Q)}=0,\\
&-\rho_2(\varphi_t,v_{\mathbf{x}t})_{L^2(Q)}-\rho_2(\varphi_t(\mathbf{x},0),v_{\mathbf{x}}(\mathbf{x},0))_{L^2(0,1)}+b(\psi_{\mathbf{x}},v_{\mathbf{x}})_{L^2(Q)}\\
&+K((\varphi_{\mathbf{x}}+\psi),v)_{L^2(Q)}+\gamma(\theta_{\mathbf{x}},v)_{L^2(Q)}=0,\\
&-\rho_3(\theta,w_t)_{L^2(Q)}-\rho_3(\theta(\mathbf{x},0),w(\mathbf{x},0))_{L^2(0,1)}\\
&+(q_x,w)_{L^2(Q)}-\gamma(\psi_{\mathbf{x}},w_t)_{L^2(Q)}\\
&-\gamma(\psi_{\mathbf{x}}(x,0),w(\mathbf{x},0))_{L^2(0,1)}=0,\\
&-\tau_0(q,z_t)_{L^2(Q)}-\tau_0(q,z)_{L^2(0,1)}+\delta(q,z)_{L^2(Q)}\\
&+\kappa(\theta_{\mathbf{x}},z)_{L^2(Q)}=0,\\
&-\mathbf{s}(\boldsymbol{\varpi}(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t),\phi_t)_{L^2(Q)}-\mathbf{s}(\boldsymbol{\varpi}(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0),\phi(x,\rho,s,0))_{L^2(0,1)}\\
&+(\boldsymbol{\varpi}_{\boldsymbol{\rho}}(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t),\phi)_{L^2(Q)}=0,\\
&-\mathbf{s}(\boldsymbol{\varpi}_t(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t),\phi_t)_{L^2(Q)}-\mathbf{s}(\boldsymbol{\varpi}_t(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0),\phi(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0))_{L^2(0,1)}\\
&+(\boldsymbol{\varpi}_{\boldsymbol{\rho}t}(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t),\phi)=0.
\end{aligned}\tag{2.5}$$

Definition 2.1 *Functions $(\varphi,\psi,\theta,q,\varpi)\in\mathcal{V}(Q)$ are called a generalized solution of system (2.3) if it satisfies (2.5) for each $(u,v,w,z,\phi)\in\mathcal{W}(Q)$.*

Theorem 2.1 *If $\varphi_0,\ \psi_0\in H_0^1(0,1)\cap H^2(0,1)$, $\varphi_1,\ \psi_1\in H^1(0,1)$, $q_0,\ \theta_0\in H_0^1(0,1)$,, $f_0\in H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)$, and $f_1\in H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)$, then there is at least one generalized solution in $\mathcal{V}(Q)$ to system.(2.3).*

By using Faedo-Galerkin approximations, we prove the global existence of the generalized solution of system (2.3). for more detail, we refer to reader to see [46], [50], and [51].

2.2 Approximate problem

let $\{u_j\}, \{v_j\}, \{w_j\}, \{z_j\}$ be the Galerkin basis, For $m \geq 1$, let

$$\begin{aligned} L_m &= span\{u_1, u_2, ..., u_n\}, \\ \Gamma_m &= span\{v_1, v_2, ..., v_n\}, \\ W_m &= span\{w_1, w_2, ..., w_n\}, \\ K_m &= span\{z_1, z_2, ..., z_n\}, \end{aligned} \tag{2.6}$$

we define for $1 \leq j \leq n$ the sequence $\phi_j(x, \rho, s)$ by

$$\phi_j(x, 0, s) = u_j(x), \tag{2.7}$$

then, we can extend $\phi_j(x, 0, s)$ by $\phi_j(x, \rho, s)$ over $L^2\left((0,1)^2 \times (\tau_1, \tau_2)\right)$ and denote $Z^m = span\{\phi_1, \phi_2, ..., \phi$
Given initial data $\varphi_0,\ \psi_0 \in H_0^1(0,1) \cap H^2(0,1),\ \varphi_1, \psi_1 \in H^1(0,1),\ q_0,\ \theta_0 \in H_0^1(0,1),\ f_0,\ f_1 \in H^1\left((0,1)^2 \times (\tau_1, \tau_2)\right)$, define the approximations

$$\begin{aligned} \varphi_m &= \sum_{j=1}^{n} \xi_{jm}(t)\, u_j(\mathbf{x}), \\ \psi_m &= \sum_{j=1}^{n} k_{jm}(t)\, v_j(\mathbf{x}), \\ \theta_m &= \sum_{j=1}^{n} l_{jm}(t)\, w_j(\mathbf{x}), \\ q_m &= \sum_{j=1}^{n} f_{jm}(t)\, z_j(\mathbf{x}), \\ \boldsymbol{\varpi}_m &= \sum_{j=1}^{n} h_{jm}(t)\, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}), \end{aligned} \tag{2.8}$$

where the constants $\xi_{jm}(t), k_{jm}(t), l_{jm}(t), f_{jm}(t)$, and $h_{jm}(t)$ are defined by the conditions

$$\begin{aligned} \xi_{jm}(t) &= (\varphi_m, u_j(\mathbf{x}))_{L^2(0,1)}, \\ k_{jm}(t) &= (\psi_m, v_j(\mathbf{x}))_{L^2(0,1)}, \\ l_{jm}(t) &= (\theta_m, w_j(\mathbf{x}))_{L^2(0,1)}, \\ f_{jm}(t) &= (q_m, z_j(\mathbf{x}))_{L^2(0,1)}, \\ h_{jm}(t) &= \left(\varpi_m, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\right)_{L^2(0,1)}, \end{aligned} \tag{2.9}$$

and can be determined from the relations

$$
\begin{aligned}
&\rho_1\left(\varphi_{mtt}, u_l\right) + K\left(\left(\varphi_{m\mathbf{x}} + \psi_m\right), u_{l\mathbf{x}}\right) + \boldsymbol{\xi}_1\left(\psi_{mt}, u_l\right) \\
&+ \left(\int_{\mathcal{L}} \boldsymbol{\xi}_2\left(s\right) \boldsymbol{\varpi}_m\left(\mathbf{x}, \mathbf{1}, \mathbf{s}\right) \mathbf{ds}, u_l\right) \;=\; 0, \\
&\rho_2\left(\varphi_{mtt}, v_{l\mathbf{x}}\right) + b\left(\psi_{m\mathbf{x}}, v_{l\mathbf{x}}\right) + K\left(\left(\varphi_{m\mathbf{x}} + \psi_m\right), v_l\right) \\
&+\gamma\left(\theta_{m\mathbf{x}}, v_l\right) \;=\; 0, \\
&\rho_3\left(\theta_{mt}, w_l\right) + \left(q_{m\mathbf{x}}, w_l\right) + \gamma\left(\psi_{mt\mathbf{x}}, w_l\right) \;=\; 0, \\
&\tau_0\left(q_{mt}, z_l\right) + \delta\left(q_m, z_l\right) + \kappa\left(\theta_{m\mathbf{x}}, z_l\right) \;=\; 0, \\
&s\left(\boldsymbol{\varpi}_{mt}\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right), \phi_l\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right)\right) + \left(\boldsymbol{\varpi}_{m\rho}\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right), \phi_l\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right)\right) \;=\; 0, \\
&s\left(\boldsymbol{\varpi}_{mtt}\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right), \phi_l\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right)\right) + \left(\varpi_{m\boldsymbol{\rho}t}\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right), \phi_l\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}\right)\right) \;=\; 0,
\end{aligned}
\tag{2.10}
$$

substitution of (2.8) into (2.10) gives for $l = 1, .., n$

$$
\begin{aligned}
&\int_{\mathcal{J}} \sum_{j=1}^{n} \Big\{ \rho_1 \xi''_{jm}(t)\, u_j(\mathbf{x})\, u_l(\mathbf{x}) + K \xi_{jm}(t)\, u_{j\mathbf{x}}(\mathbf{x})\, u_{l\mathbf{x}}(\mathbf{x}) \\
&+ K k_{jm}(t)\, v_j(\mathbf{x})\, u_{l\mathbf{x}}(\mathbf{x}) + \boldsymbol{\xi}_1 k'_{jm}(t)\, v_j(\mathbf{x})\, u_l(\mathbf{x}) \\
&+ u_l(\mathbf{x}) \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, h_{jm}(t)\, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\, \mathbf{ds} \Bigg\} \mathbf{dx} = 0, \\
&\int_{\mathcal{J}} \sum_{j=1}^{n} \Big\{ \rho_2 \xi''_{jm}(t)\, u_j(\mathbf{x})\, v_{lx}(\mathbf{x}) + b k_{jm}(t)\, v_{j\mathbf{x}}(\mathbf{x})\, v_{l\mathbf{x}}(\mathbf{x}) \\
&+ K \xi_{jm}(t)\, u_{j\mathbf{x}}(\mathbf{x})\, v_l(\mathbf{x}) + K k_{jm}(t)\, v_j(\mathbf{x})\, v_l(\mathbf{x}) \\
&+ \gamma l_{jm}(t)\, w_{j\mathbf{x}}(\mathbf{x})\, v_l(\mathbf{x}) \} \mathbf{dx} = 0, \\
&\int_{\mathcal{J}} \sum_{j=1}^{n} \Big\{ \rho_3 l'_{jm}(t)\, w_j(\mathbf{x})\, w_l(\mathbf{x}) + \kappa f_{jm}(t)\, z_{j\mathbf{x}}(\mathbf{x})\, w_l(\mathbf{x}) \\
&+ \gamma k'_{jm}(t)\, v_{j\mathbf{x}}(\mathbf{x})\, w_l(\mathbf{x}) \Big\} \mathbf{dx} = 0, \\
&\int_{\mathcal{J}} \sum_{j=1}^{n} \Big\{ \tau_0 f'_{jm}(t)\, z_j(\mathbf{x})\, z_l(\mathbf{x}) + \delta f_{jm}(t)\, z_j(\mathbf{x})\, z_l(\mathbf{x}) \\
&+ \kappa l_{jm}(t)\, w_{j\mathbf{x}}(\mathbf{x})\, z_l(\mathbf{x}) \} \mathbf{dx} = 0, \\
&\int_{\mathcal{J}} \sum_{j=1}^{n} \Big\{ s h'_{jm}(t)\, \phi_j(x, \rho, s)\, \phi_l(x, \rho, s) \\
&+ h_{jm\rho}(t)\, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\, \phi_l(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \Big\} \mathbf{dx} = 0 \\
&\int_{\mathcal{J}} \sum_{j=1}^{n} s h''_{jm}(t)\, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\, \phi_l(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \\
&+ h_{jm\rho}(t)\, \phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\, \phi_l(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \Big\} \mathbf{dx} = 0.
\end{aligned} \tag{2.11}
$$

From (2.11), it follows that

$$\begin{aligned}
&\sum_{j=1}^{n}\Big\{\rho_1\xi''_{jm}(t)\left(u_j(\mathbf{x}),u_l(\mathbf{x})\right)_{L^2(0,1)}+K\xi_{jm}(t)\left(u_{j\mathbf{x}}(\mathbf{x}),u_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)}\\
&+Kk_{jm}(t)\left(v_j(\mathbf{x}),u_{lx}(\mathbf{x})\right)_{L^2(0,1)}+\boldsymbol{\xi}_1k'_{jm}(t)\left(v_j(\mathbf{x}),u_l(\mathbf{x})\right)_{L^2(0,1)}\\
&+\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,h_{jm}(t)\left(\phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s}),u_l(\mathbf{x})\right)_{L^2(0,1)}ds\Bigg\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_2\xi''_{jm}(t)\left(u_j(\mathbf{x}),v_{lx}(\mathbf{x})\right)_{L^2(0,1)}+bk_{jm}(t)\left(v_{j\mathbf{x}}(\mathbf{x}),v_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)}\\
&+K\xi_{jm}(t)\left(u_{j\mathbf{x}}(\mathbf{x}),v_l(\mathbf{x})\right)_{L^2(0,1)}+Kk_{jm}(t)\left(v_j(\mathbf{x})\,v_l(\mathbf{x})\right)_{L^2(0,1)}\\
&+\gamma l_{jm}(t)\left(w_{j\mathbf{x}}(\mathbf{x}),v_l(\mathbf{x})\right)_{L^2(0,1)}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_3l'_{jm}(t)\left(w_j(\mathbf{x}),w_l(\mathbf{x})\right)_{L^2(0,1)}+\kappa f_{jm}(t)\left(z_{j\mathbf{x}}(\mathbf{x}),w_l(\mathbf{x})\right)_{L^2(0,1)}\\
&+\gamma k'_{jm}(t)\left(v_{jx}(\mathbf{x}),w_l(\mathbf{x})\right)_{L^2(0,1)}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\tau_0f'_{jm}(t)\left(z_j(\mathbf{x}),z_l(\mathbf{x})\right)_{L^2(0,1)}+\delta f_{jm}(t)\left(z_j(\mathbf{x}),z_l(\mathbf{x})\right)_{L^2(0,1)}\\
&+\kappa l_{jm}(t)\left(w_{j\mathbf{x}}(\mathbf{x}),z_l(\mathbf{x})\right)_{L^2(0,1)}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh'_{jm}(t)\left(\phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s}),\phi_l(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}\\
&+h_{jm\boldsymbol{\rho}}(t)\left(\phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s}),\phi_l(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh''_{jm}(t)\left(\phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s}),\phi_l(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}\\
&+h'_{jm\boldsymbol{\rho}}(t)\left(\phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s}),\phi_l(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}\Big\}\ =0
\end{aligned}\tag{2.12}$$

Let

$$
\begin{aligned}
\left(u_j(\mathbf{x}), u_l(\mathbf{x})\right)_{L^2(0,1)} &= \delta_{jl} = \begin{cases} 1, & j = l \\ 0, & j \neq l \end{cases}, \\
\left(u_{j\mathbf{x}}(\mathbf{x}), u_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)} &= \gamma_{jl}, \\
\left(v_j(\mathbf{x}), u_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)} &= \sigma_{jl}, \\
\left(u_j(\mathbf{x}), v_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)} &= a_{jl} \\
\left(v_{j\mathbf{x}}(\mathbf{x}), v_{l\mathbf{x}}(\mathbf{x})\right)_{L^2(0,1)} &= \vartheta_{jl}, \\
\left(u_{j\mathbf{x}}(\mathbf{x}), v_l(\mathbf{x})\right)_{L^2(0,1)} &= \nu_{jl}, \\
\left(v_j(\mathbf{x}), u_l(\mathbf{x})\right)_{L^2(0,1)} &= \varsigma_{jl}, \\
\left(v_j(\mathbf{x}), v_l(\mathbf{x})\right)_{L^2(0,1)} &= \delta_{jl} = \begin{cases} 1, & j = l \\ 0, & j \neq l \end{cases}, \\
\left(w_{j\mathbf{x}}(\mathbf{x}), v_l(\mathbf{x})\right)_{L^2(0,1)} &= \chi_{jl}, \\
\left(\phi_j(\mathbf{x}, \mathbf{1}, \mathbf{s}), u_l(\mathbf{x})\right)_{L^2(0,1)} &= \omega_{jl}, \\
\left(w_j(\mathbf{x}), w_l(\mathbf{x})\right)_{L^2(0,1)} &= \delta_{jl} = \begin{cases} 1, & j = l \\ 0, & j \neq l \end{cases}, \\
\left(z_{j\mathbf{x}}(\mathbf{x}), w_l(\mathbf{x})\right)_{L^2(0,1)} &= \varrho_{jl}, \\
\left(v_{j\mathbf{x}}(\mathbf{x}), w_l(\mathbf{x})\right)_{L^2(0,1)} &= \alpha_{jl}, \\
\left(z_j(\mathbf{x}), z_l(\mathbf{x})\right)_{L^2(0,1)} &= \delta_{jl} = \begin{cases} 1, j = l \\ 0, j \neq l \end{cases}, \\
\left(w_{j\mathbf{x}}(\mathbf{x}), z_l(\mathbf{x})\right)_{L^2(0,1)} &= \beta_{jl}, \\
\left(\phi_j(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}), \phi_l(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\right)_{L^2(0,1)} &= \delta_{jl} = \begin{cases} 1, & j = l \\ 0, & j \neq l \end{cases}.
\end{aligned}
$$

Then (2.12) can be written as

$$
\begin{aligned}
&\sum_{j=1}^{n}\Big\{\rho_1\xi_{jm}^{''}(t)\,\delta_{jl}+K\xi_{jm}(t)\,\gamma_{jl,}+Kk_{jm}(t)\,\sigma_{jl}+\boldsymbol{\xi}_1k_{jm}^{'}(t)\,\varsigma_{jl}\\
&+\int_{\tau_1}^{\tau_2}\boldsymbol{\xi}_2(s)\,h_{jm}(t)\,\omega_{jl}ds\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_2\xi_{jm}^{''}(t)\,a_{jl}+bk_{jm}(t)\,\vartheta_{jl}+K\xi_{jm}(t)\,\nu_{jl}+Kk_{jm}(t)\,\delta_{jl}\\
&+\gamma l_{jm}(t)\,\chi_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_3l_{jm}^{'}(t)\,\delta_{jl}+\kappa f_{jm}(t)\,\varrho_{jl}+\gamma k_{jm}^{'}(t)\,\alpha_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\tau_0f_{jm}^{'}(t)\,\delta_{jl}+\delta f_{jm}(t)\,\delta_{jl}+\kappa l_{jm}(t)\,\beta_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh_{jm}^{'}(t)\,\delta_{jl}+h_{jm\rho}(t)\,\delta_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh_{jm}^{''}(t)\,\delta_{jl}+h_{jm\rho}^{'}(t)\,\delta_{jl}\Big\}\ =0.
\end{aligned}
\tag{2.13}
$$

We put $\displaystyle\int_{\mathcal{L}}\boldsymbol{\xi}_2\,(\mathbf{s})\,\mathbf{ds}=c$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n}\Big\{\rho_1\xi_{jm}^{''}(t)\,\delta_{jl}+K\xi_{jm}(t)\,\gamma_{jl,}+Kk_{jm}(t)\,\sigma_{jl}+\boldsymbol{\xi}_1k_{jm}^{'}(t)\,\varsigma_{jl}\\
&+ch_{jm}(t)\,\omega_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_2\xi_{jm}^{''}(t)\,a_{jl}+bk_{jm}(t)\,\vartheta_{jl}+K\xi_{jm}(t)\,\nu_{jl}+Kk_{jm}(t)\,\delta_{jl}\\
&+\gamma l_{jm}(t)\,\chi_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\rho_3l_{jm}^{'}(t)\,\delta_{jl}+\kappa f_{jm}(t)\,\varrho_{jl}+\gamma k_{jm}^{'}(t)\,\alpha_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{\tau_0f_{jm}^{'}(t)\,\delta_{jl}+\delta f_{jm}(t)\,\delta_{jl}+\kappa l_{jm}(t)\,\beta_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh_{jm}^{'}(t)\,\delta_{jl}+h_{jm\boldsymbol{\rho}}(t)\,\delta_{jl}\Big\}\ =0,\\
&\sum_{j=1}^{n}\Big\{sh_{jm}^{''}(t)\,\delta_{jl}+h_{jm\rho}^{'}(t)\,\delta_{jl}\Big\}\ =0,
\end{aligned}
\tag{2.14}
$$

with

$$\begin{aligned}
\xi_{jm}(0) &= (\varphi_m(\mathbf{x},0), u_j(\mathbf{x}))_{L^2(0,1)}, \\
\xi'_{jm}(0) &= (\varphi_{mt}(\mathbf{x},0), u_j(\mathbf{x}))_{L^2(0,1)}, \\
k_{jm}(0) &= (\psi_m(\mathbf{x},0), v_j(\mathbf{x}))_{L^2(0,1)}, \\
l_{jm}(0) &= (\theta_m(\mathbf{x},0), w_j(\mathbf{x}))_{L^2(0,1)}, \\
f_{jm}(0) &= (q_m(\mathbf{x},0), z_j(\mathbf{x}))_{L^2(0,1)}, \\
h_{jm}(0) &= \left(\boldsymbol{\varpi}_m(\mathbf{x},\boldsymbol{\rho},0), \phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}, \\
h'_{jm}(0) &= \left(\boldsymbol{\varpi}_{mt}(\mathbf{x},\boldsymbol{\rho},0), \phi_j(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right)_{L^2(0,1)}.
\end{aligned} \tag{2.15}$$

We obtain a system of differential equations of two orders with respect to the variable t with constant coefficients and the initial conditions (2.15), consequently, we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. Thus for every m there exists a function $(\varphi_m, \psi_m, \theta_m, q_m, \varpi_m)$ satisfying (2.10).

2.3 A priori estimate I

Firstly, multiplying the first equation of (2.10) by ξ'_{lm} and integrating over $\mathcal{J} = (0,1)$, we get

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} + K\int_{\mathcal{J}} (\varphi_{m\mathbf{x}} + \psi_m)\varphi_{mt\mathbf{x}} \mathbf{dx} \\
&+\boldsymbol{\xi}_1 \int_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} + \int_{\mathcal{J}} \varphi_{mt} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \boldsymbol{\varpi}_m(\mathbf{x},1,\mathbf{s})\, \mathbf{dsdx} = 0.
\end{aligned} \tag{2.16}$$

Then, multiplyiing the seccond equatiion of (2.10) by k'_{lm} and integratiing overr $\mathcal{J} = (0,1)$, we get

$$\begin{aligned}
&\rho_2 \int_{\mathcal{J}} \varphi_{mtt}\psi_{m\mathbf{x}t} \mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int_{\mathcal{J}} \psi_{m\mathbf{x}}^2 \mathbf{dx} + K\int_{\mathcal{J}} (\varphi_{m\mathbf{x}} + \psi_m)\psi_{mt} \mathbf{dx} \\
&+\gamma \int_{\mathcal{J}} \theta_{m\mathbf{x}} \psi_{mt} \mathbf{dx} = 0,
\end{aligned} \tag{2.17}$$

now, substituting: $\psi_{m\mathbf{x}t} = \frac{\rho_1}{K}\varphi_{mttt} - \varphi_{m\mathbf{xx}t} + \frac{\boldsymbol{\xi}_1}{K}\varphi_{mtt} + \frac{1}{K}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mt}(\mathbf{x},1,\mathbf{s})\,\mathbf{ds}$, obtained from the first equation of (2.10) ,we get

$$\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int\limits_{\mathcal{J}} \varphi_{mtt}^2\mathbf{dx} + \frac{\rho_2}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2\mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \psi_{m\mathbf{x}}^2\mathbf{dx}$$
$$+K\int\limits_{\mathcal{J}}\int_0^1 (\varphi_{m\mathbf{x}} + \psi_m)\,\psi_{mt}\mathbf{dx} + \frac{\rho_2\boldsymbol{\xi}_1}{K}\int\limits_{\mathcal{J}} \varphi_{mtt}^2\mathbf{dx}$$
$$+\gamma\int\limits_{\mathcal{J}} \theta_{m\mathbf{x}}\psi_{mt}\mathbf{dx} + \frac{\rho_2}{K}\int\limits_{\mathcal{J}} \varphi_{mtt}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mt}(\mathbf{x},1,\mathbf{s})\,\mathbf{dsdx} = 0. \tag{2.18}$$

Next, multiplying the third equation of (2.10) by l_{lm} and integrating over $\mathcal{J} = (0,1)$, we get

$$\frac{\rho_3}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \theta_m^2\mathbf{dx} + \kappa\int\limits_{\mathcal{J}} q_{m\mathbf{x}}\theta_m\mathbf{dx} - \gamma\int\limits_{\mathcal{J}} \psi_{mt}\theta_{m\mathbf{x}}\mathbf{dx} = 0. \tag{2.19}$$

Finally, multiplying the fourth equation of (2.10) by f_{lm} and integrating over $\mathcal{J} = (0,1)$, we get

$$\frac{\tau_0}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} q_m^2\mathbf{dx} + \delta\int\limits_{\mathcal{J}} q_m^2\mathbf{dx} - \kappa\int\limits_{\mathcal{J}} \theta_m q_{m\mathbf{x}}\mathbf{dx} = 0. \tag{2.20}$$

By combining (2.16) , (2.18) , (2.19) and (2.20), we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \Big[\rho_1\varphi_{mt}^2 + K(\varphi_{m\mathbf{x}} + \psi_m)^2 + \frac{\rho_1\rho_2}{K}\varphi_{mtt}^2 + \rho_2\varphi_{mt\mathbf{x}}^2 + b\psi_{m\mathbf{x}}^2$$
$$+\rho_3\theta_m^2 + \tau_0 q_m^2\Big]\mathbf{dx} + \boldsymbol{\xi}_1\int\limits_{\mathcal{J}} \varphi_{mt}^2\mathbf{dx} + \delta\int\limits_{\mathcal{J}} q_m^2\mathbf{dx} + \frac{\rho_2\boldsymbol{\xi}_1}{K}\int\limits_{\mathcal{J}} \varphi_{mtt}^2\mathbf{dx}$$
$$+\int\limits_{\mathcal{J}} \varphi_{mt}\int_{\tau_1}^{\tau_2} \boldsymbol{\xi}_2(s)\,\boldsymbol{\varpi}_m(\mathbf{x},1,\mathbf{s})\,\mathbf{dsdx}$$
$$+\frac{\rho_2}{K}\int_0^1 \varphi_{mtt}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mt}(\mathbf{x},1,\mathbf{s})\,\mathbf{dsdx} = 0. \tag{2.21}$$

Now, multiplyiing the fifth equatiion of (2.10) by $\boldsymbol{\xi}_2\left(\mathbf{s}\right)h_{lm}$ and integratiing overr $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\boldsymbol{\xi_2}\left(\mathbf{s}\right)\boldsymbol{\varpi}_m^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\boldsymbol{\xi_2}\left(\mathbf{s}\right)\boldsymbol{\varpi}_m^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\
= \; &-\frac{1}{2}\iint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\int \boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}_m^2\left(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau}\right)\mathbf{dsdxd}\boldsymbol{\tau}\\
&+\frac{1}{2}\left(\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\boldsymbol{\tau}}^2\mathbf{dxd}\boldsymbol{\tau}. \qquad (2.22)
\end{aligned}$$

Then, multiplying the last equation of (2.10) by $\boldsymbol{\xi}_2\left(\mathbf{s}\right)h_{lm}^{'}$ and integrating over $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\boldsymbol{\xi_2}\left(\mathbf{s}\right)\boldsymbol{\varpi}_{mt}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\boldsymbol{\xi_2}\left(\mathbf{s}\right)\boldsymbol{\varpi}_{mt}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\
= \; &-\frac{\rho_2}{2K}\iint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\int \boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}_{m\tau}^2\left(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau}\right)\mathbf{dsdxd}\boldsymbol{\tau}\\
&+\frac{\rho_2}{2K}\left(\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\boldsymbol{\tau\tau}}^2\mathbf{dxd}\tau. \qquad (2.23)
\end{aligned}$$

Next, integrating (2.21) over $\mathcal{I}=(0,t)$ and using (2.22) and (2.23), we obtain

$$\begin{aligned}
& E_m(t)+\left(\boldsymbol{\xi}_1-\frac{1}{2}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau}^2\mathbf{dxd}\boldsymbol{\tau} \\
& +\frac{\rho_2}{K}\left(\xi_1-\frac{1}{2}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau}^2\mathbf{dxd}\boldsymbol{\tau} \\
& +\delta\iint\limits_{\mathcal{I}\times\mathcal{J}}q_m^2\mathbf{dxd}\boldsymbol{\tau}+\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\varpi_m(x,1,s)\,\mathbf{dsdxd}\boldsymbol{\tau} \\
& +\frac{\rho_2}{K}\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
& +\frac{1}{2}\iiint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_m^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau}- \\
& +\frac{\rho_2}{2K}\iiint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau}^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
= \; & E_m(0),
\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}
E_m(t) = \; & \frac{1}{2}\int\limits_{\mathcal{J}}\Big[\rho_1\varphi_{mt}^2+K(\varphi_{m\mathbf{x}}+\psi_m)^2+\frac{\rho_2\rho_1}{K}\varphi_{mtt}^2 \\
& +\rho_2\varphi_{m\mathbf{xt}}^2+b\psi_{m\mathbf{x}}^2+\rho_3\theta_m^2+\tau_0 q_m^2\Big]\,\mathbf{dx} \\
& +\frac{1}{2}\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_m^2(x,\rho,s)\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\
& +\frac{\rho_2}{2K}\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx},
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
& \iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{mt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_m(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
\geq \; & -\left(\frac{1}{2}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\boldsymbol{\tau}}^2\mathbf{dxd}\boldsymbol{\tau} \\
& -\frac{1}{2}\iiint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_m^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau},
\end{aligned} \tag{2.25}$$

and

$$
\begin{aligned}
&\frac{\rho_2}{K}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi_{mtt}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \boldsymbol{\varpi}_{m\tau}(\mathbf{x},\mathbf{1},\mathbf{s})\, \mathbf{dsdxd}\boldsymbol{\tau} \\
\geq\ & -\frac{\rho_2}{2K}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \mathbf{ds}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
& -\frac{\rho_2}{2K}\iint\int\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \boldsymbol{\varpi}^2_{m\boldsymbol{\tau}}(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\, \mathbf{dsdxd}\boldsymbol{\tau}.
\end{aligned}
\tag{2.26}
$$

Which, together with (2.24), yields

$$
\begin{aligned}
& E_m(t) + \left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
& +\frac{\rho_2}{K}\left(\xi_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
& +\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q_m^2\mathbf{dxd}\boldsymbol{\tau} \\
\leq\ & E_m(0),
\end{aligned}
$$

implies

$$
\begin{aligned}
& E_m(t) + \eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
& +\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q_m^2\mathbf{dxd}\boldsymbol{\tau} + \frac{\rho_2}{K}\eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
\leq\ & E_m(0),
\end{aligned}
\tag{2.27}
$$

where $\eta_0 = \boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \mathbf{ds} > 0.$

So, we have

$$
E_m(t) \leq E_m(0),
\tag{2.28}
$$

and make use of the following inequality

$$
\begin{aligned}
\rho_1\int\limits_{\mathcal{J}} \varphi_m^2\mathbf{dx}\ \leq\ & \rho_1\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi_m^2(\mathbf{x},\boldsymbol{\tau})\, \mathbf{dxd}\boldsymbol{\tau} \\
& +\rho_1\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau}}(\mathbf{x},\boldsymbol{\tau})\, \mathbf{dxd}\boldsymbol{\tau} + \rho_1\int\limits_{\mathcal{J}} \varphi_m^2(\mathbf{x},\mathbf{0})\, \mathbf{dx},
\end{aligned}
\tag{2.29}
$$

combining inequalities (2.28) and (2.29), we get

$$\begin{aligned} E_m(t) + \rho_1 \int_{\mathcal{J}} \varphi_m^2 \mathbf{dx} &\leq E_m(0) + \rho_1 \iint_{\mathcal{I}\times\mathcal{J}} \varphi_m^2(\mathbf{x}, \boldsymbol{\tau})\, \mathbf{dxd}\boldsymbol{\tau} \\ &+ \rho_1 \iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\tau}^2(\mathbf{x}, \boldsymbol{\tau})\, \mathbf{dxd}\boldsymbol{\tau} + \rho_1 \int_{\mathcal{J}} \varphi_m^2(x, 0)\, \mathbf{dx}, \end{aligned}$$

we put

$$\mathcal{P}_m(t) = E_m(t) + \rho_1 \int_{\mathcal{J}} \varphi_m^2 \mathbf{dx}, \tag{2.30}$$

we get

$$\mathcal{P}_m(t) \leq \mathcal{P}_m(0) + \int_{\mathcal{I}} \mathcal{P}_m(\boldsymbol{\tau})\, \mathbf{d}\boldsymbol{\tau}. \tag{2.31}$$

Applying the Gronwall inequality to (2.31), we obtain

$$\mathcal{P}_m(t) \leq \mathcal{P}_m(0) \exp(T),$$

thus, there exist a positive constant C independent on m such that

$$\mathcal{P}_m(t) \leq C,\ t \geq 0, \tag{2.32}$$

it follows from (30) and (2.32) that

$$\begin{aligned} &\rho_1 \int_{\mathcal{J}} \varphi_m^2 \mathbf{dx} + \rho_1 \int_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} + K \int_{\mathcal{J}} (\varphi_{m\mathbf{x}} + \psi_m)^2\, \mathbf{dx} \\ &+ \frac{\rho_1 \rho_2}{K} \int_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} + \rho_2 \int_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2 \mathbf{dx} + b \int_{\mathcal{J}} \psi_{m\mathbf{x}}^2 \mathbf{dx} \\ &+ \rho_3 \int_{\mathcal{J}} \theta_m^2 \mathbf{dx} + \tau_0 \int_0^1 q_m^2 \mathbf{dx} \\ &+ \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\, |\boldsymbol{\xi}_2(\mathrm{s})|\, \boldsymbol{\varpi}_m^2(\mathbf{x}, \boldsymbol{\rho}, \mathrm{s})\, \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\ &+ \frac{\rho_2}{K} \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\, |\boldsymbol{\xi}_2(\mathrm{s})|\, \boldsymbol{\varpi}_{mt}^2(\mathbf{x}, \boldsymbol{\rho}, \mathrm{s})\, \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\ \leq\ & C. \end{aligned} \tag{2.33}$$

2.4 A priori estimate II

Firstly, differentiating the first equation of (2.10) and multiplying by ξ''_{lm}, and then integrating the result over $\mathcal{J} = (0,1)$,we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} + K\int_{\mathcal{J}} (\varphi_{m\mathbf{x}t} + \psi_{mt})\,\varphi_{mtt\mathbf{x}}\mathbf{dx}$$

$$+\boldsymbol{\xi}_1\int_{\mathcal{J}} \varphi_{mtt}^2\mathbf{dx} + \int_{\mathcal{J}} \varphi_{mtt}\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mt}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = . \tag{2.34}$$

Next, differentiating the second equation of (2.10) and multiplying by k''_{lm},and integrating over $\mathcal{J} = (0,1)$, we obtaiin

$$\rho_2\int_{\mathcal{J}} \varphi_{mttt}\psi_{m\mathbf{x}tt}\mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int_{\mathcal{J}} \psi_{m\mathbf{x}t}^2 dx + K\int_{\mathcal{J}} (\varphi_{m\mathbf{x}t} + \psi_{mt})\,\psi_{mtt}\mathbf{dx}$$

$$+\gamma\int_{\mathcal{J}} \theta_{m\mathbf{x}t}\psi_{mtt}\mathbf{dx} = 0,$$

now, substituting: $\psi_{m\mathbf{x}tt} = \frac{\rho_1}{K}\varphi_{mtttt} - \varphi_{m\mathbf{xx}tt} + \frac{\boldsymbol{\xi}_1}{K}\varphi_{mttt} + \frac{1}{K}\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mtt}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{ds}$, obtained from the first equation of (2.10) ,we obtain

$$\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_{\mathcal{J}} \varphi_{mttt}^2\mathbf{dx} + \frac{\rho_2}{2}\frac{d}{dt}\int_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2\mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int_{\mathcal{J}} \psi_{m\mathbf{x}t}^2\mathbf{dx}$$

$$+K\int_{\mathcal{J}} (\varphi_{m\mathbf{x}t} + \psi_{mt})\,\psi_{mtt}\mathbf{dx} + \frac{\rho_2\boldsymbol{\xi}_1}{K}\int_{\mathcal{J}} \varphi_{mttt}^2\mathbf{dx}$$

$$+\gamma\int_{\mathcal{J}} \theta_{m\mathbf{x}t}\psi_{mtt}\mathbf{dx} + \frac{\rho_2}{K}\int_{\mathcal{J}} \varphi_{mttt}\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mtt}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0. \tag{2.35}$$

Then, differentiating the third equation of (2.10) and multiplying by l'_{lm}, and integrating over $\mathcal{J} = (0,1)$, we obtain

$$\frac{\rho_3}{2}\frac{d}{dt}\int_{\mathcal{J}} \theta_{mt}^2\mathbf{dx} + \kappa\int_{\mathcal{J}} q_{mt\mathbf{x}}\theta_{mt}\mathbf{dx} - \gamma\int_{\mathcal{J}} \psi_{mtt}\theta_{mt\mathbf{x}}\mathbf{dx} = 0. \tag{2.36}$$

Finally, differentiating the fourth equation of (2.10) and multiplying by f'_{lm}, and integrating over $\mathcal{J} = (0,1)$, we obtain

$$\frac{\tau_0}{2}\frac{d}{dt}\int_{\mathcal{J}} q_{mt}^2 dx + \delta\int_{\mathcal{J}} q_{mt}^2 dx - \kappa\int_{\mathcal{J}} \theta_{mt}q_{mxt}dx = 0. \tag{2.37}$$

By combining (2.34), (2.35), (2.36) and (2.37), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{J}}\left[\rho_1\varphi_{mtt}^2+K\left(\varphi_{m\mathbf{x}t}+\psi_{mt}\right)^2+\frac{\rho_1\rho_2}{K}\varphi_{mttt}^2+\rho_2\varphi_{mtt\mathbf{x}}^2+b\psi_{m\mathbf{x}t}^2\right.$$
$$\left.+\rho_3\theta_{mt}^2+\tau_0 q_{mt}^2\right]\mathbf{dx}+\xi_1\int_{\mathcal{J}}\varphi_{mtt}^2\mathbf{dx}+\delta\int_{\mathcal{J}}q_{mt}^2\mathbf{dx}+\frac{\rho_2\boldsymbol{\xi}_1}{K}\int_{\mathcal{J}}\varphi_{mttt}^2\mathbf{dx}$$
$$+\int_{\mathcal{J}}\varphi_{mtt}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mt}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$
$$+\frac{\rho_2}{K}\int_{\mathcal{J}}\varphi_{mttt}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{mtt}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}\;=\;0. \tag{2.38}$$

Now, differentiating the fifth equation of (2.10) and multiplying by $\boldsymbol{\xi}_2(\mathbf{s})\,h_{lm}^{'}$, and integrating over $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{1}{2}\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\boldsymbol{\xi}_{\mathbf{2}}(\mathbf{s})\,\boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s},\mathbf{t})\,\mathbf{ds}d\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{1}{2}\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\boldsymbol{\xi}_{\mathbf{2}}(\mathbf{s})\,\boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s},\mathbf{0})\,\mathbf{ds}d\boldsymbol{\rho}\mathbf{dx}\\
=\;&-\frac{1}{2}\iint\int_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau}^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdx}d\boldsymbol{\tau}\\
&+\frac{1}{2}\left(\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau}^2\mathbf{dx}d\boldsymbol{\tau},
\end{aligned}\tag{2.39}$$

then, differentiating the last equation of (2.10) and multiplying by $\boldsymbol{\xi}_2(\mathbf{s})\,h_{lm}^{''}$, and integrating over $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\boldsymbol{\xi}_{\mathbf{2}}(\mathbf{s})\,\boldsymbol{\varpi}_{mtt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s},\mathbf{t})\,\mathbf{ds}d\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{\rho_2}{2K}\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\boldsymbol{\xi}_{\mathbf{2}}(\mathbf{s})\,\boldsymbol{\varpi}_{mtt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s},\mathbf{0})\,\mathbf{ds}d\boldsymbol{\rho}\mathbf{dx}\\
=\;&-\frac{\rho_2}{2K}\iint\int_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\boldsymbol{\tau}\boldsymbol{\tau}}^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdx}d\boldsymbol{\tau}\\
&+\frac{\rho_2}{2K}\left(\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\boldsymbol{\tau}\boldsymbol{\tau}\boldsymbol{\tau}}^2\mathbf{dx}d\boldsymbol{\tau}.
\end{aligned}\tag{2.40}$$

next, integrating (2.38) over $\mathcal{I}=(0,t)$ and using (2.39) and (2.40), we obtain

$$\begin{aligned}
&\mathcal{M}_m(t)+\left(\boldsymbol{\xi}_1-\frac{1}{2}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau}^2\mathbf{dxd}\boldsymbol{\tau}\\
&+\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1-\frac{1}{2}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau\tau}^2\mathbf{dxd}\boldsymbol{\tau}\\
&+\delta\iint_{\mathcal{I}\times\mathcal{J}}q_{m\tau}^2\mathbf{dxd}\boldsymbol{\tau}\\
&+\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd}\boldsymbol{\tau}\\
&+\frac{\rho_2}{K}\iint_{\mathcal{I}\times\mathcal{J}}\varphi_{m\tau\tau\tau}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau\tau}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd}\boldsymbol{\tau}\\
&+\frac{1}{2}\iiint_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau}^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau}-\\
&+\frac{\rho_2}{2K}\iiint_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\tau\tau}^2(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau}\\
=\;&\mathcal{M}_m(0),
\end{aligned}\tag{2.41}$$

where

$$\begin{aligned}
\mathcal{M}_m(t)=\;&\frac{1}{2}\int_{\mathcal{J}}\Big[\rho_1\varphi_{mtt}^2+K(\varphi_{m\mathbf{x}t}+\psi_{mt})^2+\frac{\rho_2\rho_1}{K}\varphi_{mttt}^2\\
&+\rho_2\varphi_{m\mathbf{x}tt}^2+b\psi_{m\mathbf{x}t}^2+\rho_3\theta_{mt}^2+\tau_0q_{mt}^2\Big]\,\mathbf{dx}\\
&+\frac{1}{2}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\
&+\frac{\rho_2}{2K}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_{mtt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx},
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
&\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi_{m\boldsymbol{\tau\tau}} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\varpi_{m\boldsymbol{\tau}}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd\boldsymbol{\tau}} \\
\geq\;& -\left(\frac{1}{2}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,d\mathbf{s}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
&-\frac{1}{2}\iint\int\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\varpi^2_{m\boldsymbol{\tau}}(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd\boldsymbol{\tau}},
\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}
&\frac{\rho_2}{K}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi_{m\boldsymbol{\tau\tau\tau}} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\varpi_{m\boldsymbol{\tau\tau}}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd\boldsymbol{\tau}} \\
\geq\;& -\frac{\rho_2}{2K}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,d\mathbf{s}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
&-\frac{\rho_2}{2K}\iint\int\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\varpi^2_{m\boldsymbol{\tau\tau}}(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd\boldsymbol{\tau}}.
\end{aligned} \tag{2.43}$$

Which, together with (2.41), yields

$$\begin{aligned}
&\mathcal{M}_m(t) + \left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,d\mathbf{s}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
&+\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
&+\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q^2_{m\tau}\mathbf{dxd\boldsymbol{\tau}} \\
\leq\;& \mathcal{M}_m(0),
\end{aligned} \tag{2.44}$$

implies

$$\begin{aligned}
&\mathcal{M}_m(t) + \eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
&+\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q^2_{m\boldsymbol{\tau}}\mathbf{dxd\boldsymbol{\tau}} + \frac{\rho_2}{K}\eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\boldsymbol{\tau\tau\tau}}\mathbf{dxd\boldsymbol{\tau}} \\
\leq\;& \mathcal{M}_m(0),
\end{aligned} \tag{2.45}$$

where $\eta_0 = \boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\mathbf{ds} > 0.$

Then

$$\mathcal{M}_m(t) \leq \mathcal{M}_m(0), \tag{2.46}$$

thus, there exist a positive constant C independent on m such that

$$\mathcal{M}_m(t) \leq C,\ t \geq 0, \tag{2.47}$$

it follows from (30) and (2.47) that

$$\begin{aligned}
&\rho_1 \int\limits_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} + K \int\limits_{\mathcal{J}} (\varphi_{m\mathbf{x}t} + \psi_{mt})^2 \mathbf{dx} \\
&+ \frac{\rho_1 \rho_2}{K} \int\limits_{\mathcal{J}} \varphi_{mttt}^2 \mathbf{dx} \\
&+ \rho_2 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{x}t}^2 \mathbf{dx} \\
&+ \rho_3 \int\limits_{\mathcal{J}} \theta_{mt}^2 \mathbf{dx} + \tau_0 \int\limits_{\mathcal{J}} q_{mt}^2 \mathbf{dx} \\
&+ \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&+ \frac{\rho_2}{K} \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_{mtt}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
\leq\ & C.
\end{aligned} \tag{2.48}$$

2.5 A priori estimate III

Firstly, let $u_l = -\varphi_{mt\mathbf{xx}}$ in the first equation of (2.10), we get

$$\begin{aligned}
&\frac{\rho_1}{2} \frac{d}{dt} \int\limits_{\mathcal{J}} \varphi_{mt\mathbf{x}}^2 \mathbf{dx} + K \int\limits_{\mathcal{J}} (\varphi_{m\mathbf{xx}} + \psi_{m\mathbf{x}})\, \varphi_{mt\mathbf{xx}} \mathbf{dx} \\
&+ \boldsymbol{\xi}_1 \int\limits_{\mathcal{J}} \varphi_{mt\mathbf{x}}^2 \mathbf{dx} + \int\limits_{\mathcal{J}} \varphi_{mt\mathbf{x}} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \boldsymbol{\varpi}_{m\mathbf{x}}(\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{ds}\mathbf{dx} = 0.
\end{aligned} \tag{2.49}$$

Then, let $v_l = -\psi_{mt\mathbf{xx}}$ in the second equation of (2.10) , we get

$$\rho_2 \int_{\mathcal{J}} \varphi_{m\mathbf{x}tt} \psi_{m\mathbf{x}\mathbf{x}t} \mathbf{dx} + \frac{b}{2} \frac{d}{dt} \int_{\mathcal{J}} \psi_{m\mathbf{xx}}^2 \mathbf{dx}$$
$$+K \int_{\mathcal{J}} \left(\varphi_{m\mathbf{xx}} + \psi_{m\mathbf{x}}\right) \psi_{m\mathbf{x}t} \mathbf{dx}$$
$$+\gamma \int_{\mathcal{J}} \theta_{m\mathbf{xx}} \psi_{m\mathbf{x}t} \mathbf{dx} = 0, \tag{2.50}$$

now, substituting: $\psi_{m\mathbf{xx}t} = \frac{\rho_1}{K} \varphi_{m\mathbf{x}ttt} - \varphi_{m\mathbf{xxx}t} + \frac{\boldsymbol{\xi}_1}{K} \varphi_{m\mathbf{x}tt} + \frac{1}{K} \int_{\mathcal{L}} \boldsymbol{\xi}_2 (\mathbf{s})\, \boldsymbol{\varpi}_{mt\mathbf{x}} (\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{ds}$, obtained from the first equation of (2.10) ,we get

$$\frac{\rho_2 \rho_1}{2K} \frac{d}{dt} \int_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} + \frac{\rho_2}{2} \frac{d}{dt} \int_{\mathcal{J}} \varphi_{m\mathbf{xx}t}^2 \mathbf{dx}$$
$$+\frac{b}{2} \frac{d}{dt} \int_{\mathcal{J}} \psi_{m\mathbf{xx}}^2 \mathbf{dx} + K \int_{\mathcal{J}} \left(\varphi_{m\mathbf{xx}} + \psi_{m\mathbf{x}}\right) \psi_{m\mathbf{x}t} \mathbf{dx}$$
$$+\frac{\rho_2 \boldsymbol{\xi}_1}{K} \int_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} + \gamma \int_{\mathcal{J}} \theta_{m\mathbf{xx}} \psi_{m\mathbf{x}t} \mathbf{dx}$$
$$+\frac{\rho_2}{K} \int_{\mathcal{J}} \varphi_{m\mathbf{x}tt} \int_{\mathcal{L}} \boldsymbol{\xi}_2 (\mathbf{s})\, \boldsymbol{\varpi}_{m\mathbf{x}t} (\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsdx} = 0. \tag{2.51}$$

Next, let $w_l = -\theta_{m\mathbf{xx}}$ in the third equation of (2.10) , we get

$$\frac{\rho_3}{2} \frac{d}{dt} \int_{\mathcal{J}} \theta_{m\mathbf{x}}^2 \mathbf{dx} - \kappa \int_{\mathcal{J}} q_{m\mathbf{x}} \theta_{m\mathbf{xx}} \mathbf{dx} + \gamma \int_{\mathcal{J}} \psi_{mt\mathbf{xx}} \theta_{m\mathbf{x}} \mathbf{dx} = 0. \tag{2.52}$$

Finally, let $z_l = -q_{m\mathbf{xx}}$ in the fourth equation of (2.10) , we get

$$\frac{\tau_0}{2} \frac{d}{dt} \int_{\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dx} + \delta \int_{\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dx} + \kappa \int_{\mathcal{J}} \theta_{m\mathbf{xx}} q_{m\mathbf{x}} \mathbf{dx} = 0. \tag{2.53}$$

By combining (2.49) , (2.51) , (2.52) and (2.53) , we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\left[\rho_1\varphi_{m\mathbf{xt}}^2+K\left(\varphi_{m\mathbf{xx}}+\psi_{m\mathbf{x}}\right)^2+\frac{\rho_1\rho_2}{K}\varphi_{m\mathbf{x}tt}^2\right.\\
&\left.+\rho_2\varphi_{mt\mathbf{xx}}^2+b\psi_{m\mathbf{xx}}^2+\rho_3\theta_{m\mathbf{x}}^2+\tau_0 q_{m\mathbf{x}}^2\right]\mathbf{dx}\\
&+\boldsymbol{\xi}_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}t}^2\mathbf{dx}+\delta\int\limits_{\mathcal{J}}q_{m\mathbf{x}}^2\mathbf{dx}+\frac{\rho_2\boldsymbol{\xi}_1}{K}\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}tt}^2\mathbf{dx}\\
&+\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}t}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\boldsymbol{\varpi}_{mx}\left(\mathbf{x},\mathbf{1},\mathrm{s}\right)\mathbf{dsdx}\\
&+\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}tt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{x}t}\left(\mathbf{x},\mathbf{1},\mathrm{s}\right)\mathbf{dsdx}\;=\;0. \qquad (2.54)
\end{aligned}$$

Now, let $\phi_l=-\boldsymbol{\xi}_2\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{xx}}$ in 5^{th} equation of (2.10) , and integrating over $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{1}{2}\iint\int\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathrm{s}\boldsymbol{\xi}_{\mathbf{2}}\left(\mathrm{s}\right)\boldsymbol{\varpi}_{mx}^2\left(\mathbf{x},\boldsymbol{\rho},\mathrm{s},\mathbf{t}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{1}{2}\iint\int\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathrm{s}\boldsymbol{\xi}_{\mathbf{2}}\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{x}}^2\left(\mathbf{x},\boldsymbol{\rho},\mathrm{s},\mathbf{0}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
=\;&-\frac{1}{2}\iint\int\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{x}}^2\left(\mathbf{x},\mathbf{1},\mathrm{s},\boldsymbol{\tau}\right)\mathbf{dsdx}\mathbf{d}\boldsymbol{\tau}\\
&+\frac{1}{2}\left(\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\boldsymbol{\tau}\mathbf{x}}^2\mathbf{dx}\mathbf{d}\boldsymbol{\tau}, \qquad (2.55)
\end{aligned}$$

and let $\phi_l=-\boldsymbol{\xi}_2\left(\mathrm{s}\right)\varpi_{m\mathbf{xxt}}$ in the last equation of (2.10) , and integrating over $\mathcal{I}\times\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\iint\int\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathrm{s}\boldsymbol{\xi}_{\mathbf{2}}\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{x}t}^2\left(\mathbf{x},\boldsymbol{\rho},\mathrm{s},\mathbf{t}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{\rho_2}{2K}\iint\int\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathrm{s}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\varpi_{m\mathbf{x}t}^2\left(\mathbf{x},\boldsymbol{\rho},\mathrm{s},\mathbf{0}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
=\;&-\frac{\rho_2}{2K}\iint\int\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\boldsymbol{\varpi}_{m\mathbf{x}\boldsymbol{\tau}}^2\left(\mathbf{x},\mathbf{1},\mathrm{s},\boldsymbol{\tau}\right)\mathbf{dsdx}\mathbf{d}\boldsymbol{\tau}\\
&+\frac{\rho_2}{2K}\left(\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathrm{s}\right)\mathbf{ds}\right)\int_0^t\int_0^1\varphi_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}}^2\mathbf{dx}\mathbf{d}\boldsymbol{\tau}. \qquad (2.56)
\end{aligned}$$

Next, integrating (2.54) over $(0,t)$ and using (2.55) and (2.56), we obtain

$$\begin{aligned}
&\mathcal{K}_m(t) + \left(\boldsymbol{\xi}_1 - \frac{1}{2}\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\mathbf{ds}\right) \iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}}^2 \mathbf{dxd}\boldsymbol{\tau} \\
&+\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \frac{1}{2}\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\mathbf{ds}\right) \iint_{\mathcal{I}\times\mathcal{J}} \varphi_{mx\tau\tau}^2 \mathbf{dxd}\boldsymbol{\tau} \\
&+\delta \iint_{\mathcal{I}\times\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dxd}\boldsymbol{\tau} + \iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\varpi_{m\mathbf{x}}(\mathbf{x},\mathbf{1},\mathrm{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
&+\frac{\rho_2}{K}\iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\boldsymbol{\varpi}_{m\mathbf{xt}}(\mathbf{x},\mathbf{1},\mathrm{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
&+\frac{1}{2}\iiint_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\boldsymbol{\varpi}_{m\mathbf{x}}^2(\mathbf{x},\mathbf{1},\mathrm{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau} - \\
&+\frac{\rho_2}{2K}\iiint_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\varpi_{m\mathbf{x}\boldsymbol{\tau}}^2(\mathbf{x},\mathbf{1},\mathrm{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
=\ &\mathcal{K}_m(0), \qquad (2.57)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_m(t) =\ & \frac{1}{2}\int_{\mathcal{J}} \Big[\rho_1\varphi_{m\mathbf{xt}}^2 + K(\varphi_{m\mathbf{xx}} + \psi_{m\mathbf{x}})^2 + \frac{\rho_2\rho_1}{K}\varphi_{m\mathbf{xtt}}^2 \\
&+\rho_2\varphi_{m\mathbf{xxt}}^2 + b\psi_{m\mathbf{xx}}^2 + \rho_3\theta_{m\mathbf{x}}^2 + \tau_0 q_{m\mathbf{x}}^2\Big]\,\mathbf{dx} \\
&+\frac{1}{2}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\,|\boldsymbol{\xi}_2(\mathrm{s})|\,\boldsymbol{\varpi}_{m\mathbf{x}}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\
&+\frac{\rho_2}{2K}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\,|\boldsymbol{\xi}_2(\mathrm{s})|\,\boldsymbol{\varpi}_{m\mathbf{xt}}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx},
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
&\iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathrm{s})\,\boldsymbol{\varpi}_{m\mathbf{x}}(\mathbf{x},\mathbf{1},\mathrm{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
\geq\ & -\left(\frac{1}{2}\int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathrm{s})|\,\mathbf{ds}\right)\iint_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}}^2 \mathbf{dxd}\boldsymbol{\tau} \\
&-\frac{1}{2}\iiint_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathrm{s})|\,\boldsymbol{\varpi}_{m\mathbf{x}}^2(\mathbf{x},\mathbf{1},\mathrm{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau}, \qquad (2.58)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\rho_2}{K}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_{m\mathbf{x}\boldsymbol{\tau}}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdxd}\boldsymbol{\tau} \\
\geq\ & -\frac{\rho_2}{2K}\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
&-\frac{\rho_2}{2K}\iiint\limits_{\mathcal{I}\times\mathcal{J}\times\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}^2_{m\mathbf{x}\boldsymbol{\tau}}(\mathbf{x},\mathbf{1},\mathbf{s},\boldsymbol{\tau})\,\mathbf{dsdxd}\boldsymbol{\tau}. \qquad (2.59)
\end{aligned}$$

Which, together with (2.57), yields

$$\begin{aligned}
&\mathcal{K}_m(t) + \left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\mathbf{x}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
&+\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
&+\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q^2_{m\mathbf{x}}\mathbf{dxd}\boldsymbol{\tau} \\
\leq\ & \mathcal{K}_m(0),
\end{aligned}$$

implies

$$\begin{aligned}
&\mathcal{K}_m(t) + \eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\mathbf{x}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
&+\delta\iint\limits_{\mathcal{I}\times\mathcal{J}} q^2_{m\mathbf{x}}\mathbf{dxd}\boldsymbol{\tau} + \frac{\rho_2}{K}\eta_0\iint\limits_{\mathcal{I}\times\mathcal{J}} \varphi^2_{m\mathbf{x}\boldsymbol{\tau}\boldsymbol{\tau}}\mathbf{dxd}\boldsymbol{\tau} \\
\leq\ & \mathcal{K}_m(0), \qquad (2.60)
\end{aligned}$$

where $\eta_0 = \boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds} > 0.$

Then

$$\mathcal{K}_m(t) \leq \mathcal{K}_m(0), \qquad (2.61)$$

and make use of the following inequality

$$\begin{aligned}
&\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2\mathbf{dx}\\
\le\ &\beta_1\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{x}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}+\beta_2\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{xx}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}\\
&+\beta_1\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{x}\boldsymbol{\tau}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}+\beta_2\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{xx}\boldsymbol{\tau}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}\\
&+\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2(0)\,\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2(0)\,\mathbf{dx}.
\end{aligned}\tag{2.62}$$

Combining inequalities (2.61) and (2.62), we get

$$\begin{aligned}
&\mathcal{K}_m(t)+\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2\mathbf{dx}\\
\le\ &\mathcal{K}_m(0)+\beta_1\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{x}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}+\beta_2\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{xx}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}\\
&+\beta_1\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{x}\boldsymbol{\tau}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}+\beta_2\iint\limits_{\mathcal{I}\times\mathcal{J}}\varphi_{m\mathbf{xx}\boldsymbol{\tau}}^2(\mathbf{x},\boldsymbol{\tau})\,\mathbf{dxd}\boldsymbol{\tau}\\
&+\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2(0)\,\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2(0)\,\mathbf{dx},
\end{aligned}$$

we put

$$\mathcal{S}_m(t)=\mathcal{K}_m(t)+\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2\mathbf{dx},\tag{2.63}$$

we get

$$\mathcal{S}_m(t)\le\mathcal{S}_m(0)+\int\limits_{\mathcal{I}}\mathcal{S}_m(\boldsymbol{\tau})\,\mathbf{d}\boldsymbol{\tau}.\tag{2.64}$$

Applying the Gronwall inequality to (2.64), we obtain

$$\mathcal{S}_m(t)\le\mathcal{S}_m(0)\exp(T),\tag{2.65}$$

thus, there exists a positive constant C independent on m such that

$$\mathcal{S}_m(t)\le C,t\ge 0,\tag{2.66}$$

it follows from (30) and (2.66) that

$$\begin{aligned}
&\rho_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xt}}^2\mathbf{dx}+K\int\limits_{\mathcal{J}}\left(\varphi_{m\mathbf{xx}}+\psi_{m\mathbf{x}}\right)^2\mathbf{dx}+\tau_0\int\limits_{\mathcal{J}}q_{m\mathbf{x}}^2\mathbf{dx}\\
&+\beta_1\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}}^2\mathbf{dx}+\beta_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}}^2\mathbf{dx}+\frac{\rho_2\rho_1}{K}\int\limits_{\mathcal{J}}\varphi_{m\mathbf{x}tt}^2\mathbf{dx}\\
&+\rho_2\int\limits_{\mathcal{J}}\varphi_{m\mathbf{xx}t}^2\mathbf{dx}+b\int\limits_{\mathcal{J}}\psi_{m\mathbf{xx}}^2\mathbf{dx}+\rho_3\int\limits_{\mathcal{J}}\theta_{m\mathbf{x}}^2\mathbf{dx}\\
&+\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}_{m\mathbf{x}}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
&+\frac{\rho_2}{K}\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\varpi_{m\mathbf{x}t}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
\leq\ & C. \qquad (2.67)
\end{aligned}$$

Now, combining inequalities (2.67), (2.48), and (2.33) , we obtain

$$
\begin{aligned}
&\rho_1 \int\limits_{\mathcal{J}} \varphi_m^2 \mathbf{dx} + \beta_1 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}}^2 \mathbf{dx} + \rho_1 \int\limits_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} \\
&+ (\rho_1+\rho_2) \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2 \mathbf{dx} + \left(\rho_1 + \frac{\rho_1\rho_2}{K}\right) \int\limits_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} \\
&+\beta_2 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{xx}}^2 \mathbf{dx} + \left(\rho_2 + \frac{\rho_2\rho_1}{K}\right) \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} \\
&+\frac{\rho_1\rho_2}{K} \int\limits_{\mathcal{J}} \varphi_{mttt}^2 \mathbf{dx} + \rho_2 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{xx}t}^2 \mathbf{dx} \\
&+b \int\limits_{\mathcal{J}} \psi_{m\mathbf{x}}^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{x}t}^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{xx}}^2 \mathbf{dx} \\
&+K \int\limits_{\mathcal{J}} \left(\varphi_{m\mathbf{x}} + \psi_m\right)^2 \mathbf{dx} + K \int\limits_{\mathcal{J}} \left(\varphi_{m\mathbf{xx}} + \psi_{m\mathbf{x}}\right)^2 \mathbf{dx} \\
&+K \int\limits_{\mathcal{J}} \left(\varphi_{m\mathbf{x}t} + \psi_{mt}\right)^2 \mathbf{dx} + \rho_3 \int\limits_{\mathcal{J}} \theta_m^2 \mathbf{dx} \\
&+\rho_3 \int\limits_{\mathcal{J}} \theta_{m\mathbf{x}}^2 \mathbf{dx} + \rho_3 \int\limits_{\mathcal{J}} \theta_{mt}^2 \mathbf{dx} \\
&+\tau_0 \int\limits_{\mathcal{J}} q_m^2 \mathbf{dx} + \tau_0 \int\limits_{\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dx} + \tau_0 \int\limits_{\mathcal{J}} q_{mt}^2 \mathbf{dx} \\
&+ \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\left|\boldsymbol{\xi}_2(\mathrm{s})\right| \boldsymbol{\varpi}_m^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\, \mathbf{ds d\boldsymbol{\rho} dx} \\
&+ \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\left|\boldsymbol{\xi}_2(\mathrm{s})\right| \boldsymbol{\varpi}_{m\mathbf{x}}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\, \mathbf{ds d\boldsymbol{\rho} dx} \\
&+ \left(1+\frac{\rho_2}{K}\right) \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\left|\boldsymbol{\xi}_2(\mathrm{s})\right| \boldsymbol{\varpi}_{mt}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\, \mathbf{ds d\boldsymbol{\rho} dx} \\
&+\frac{\rho_2}{K} \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\left|\boldsymbol{\xi}_2(\mathrm{s})\right| \varpi_{m\mathbf{x}t}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\, \mathbf{ds d\boldsymbol{\rho} dx} \\
&+ \iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\left|\boldsymbol{\xi}_2(\mathrm{s})\right| \boldsymbol{\varpi}_{mtt}^2(\mathbf{x},\boldsymbol{\rho},\mathrm{s})\, \mathbf{ds d\boldsymbol{\rho} dx} \\
\leq\ & C, \qquad (2.68)
\end{aligned}
$$

using Young's inequality with ε, we have

$$\begin{aligned}
& \rho_1 \int_{\mathcal{J}} \varphi_m^2 \mathbf{dx} + \beta_1 \int_{\mathcal{J}} \varphi_{m\mathbf{x}}^2 \mathbf{dx} + \rho_1 \int_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} \\
& + (\rho_1 + \rho_2) \int_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2 \mathbf{dx} + \left(\rho_1 + \frac{\rho_1 \rho_2}{K}\right) \int_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} \\
& + \beta_2 \int_{\mathcal{J}} \varphi_{m\mathbf{xx}}^2 \mathbf{dx} + \left(\rho_2 + \frac{\rho_2 \rho_1}{K}\right) \int_{\mathcal{J}} \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} \\
& + \frac{\rho_1 \rho_2}{K} \int_{\mathcal{J}} \varphi_{mttt}^2 \mathbf{dx} + \rho_2 \int_{\mathcal{J}} \varphi_{m\mathbf{xx}t}^2 \mathbf{dx} \\
& + b \int_{\mathcal{J}} \psi_{m\mathbf{x}}^2 \mathbf{dx} + b \int_{\mathcal{J}} \psi_{m\mathbf{x}t}^2 \mathbf{dx} + b \int_{\mathcal{J}} \psi_{m\mathbf{xx}}^2 \mathbf{dx} \\
& + K \left(1 - \frac{1}{\varepsilon}\right) \int_{\mathcal{J}} \varphi_{m\mathbf{x}}^2 \mathbf{dx} + K (1 - \varepsilon) \int_{\mathcal{J}} \psi_m^2 \mathbf{dx} \\
& + K \left(1 - \frac{1}{\varepsilon}\right) \int_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2 \mathbf{dx} + K (1 - \varepsilon) \int_{\mathcal{J}} \psi_{mt}^2 \mathbf{dx} + \rho_3 \int_{\mathcal{J}} \theta_m^2 \mathbf{dx} \\
& + \rho_3 \int_{\mathcal{J}} \theta_{m\mathbf{x}}^2 \mathbf{dx} + \rho_3 \int_{\mathcal{J}} \theta_{mt}^2 \mathbf{dx} \\
& + \tau_0 \int_{\mathcal{J}} q_m^2 \mathbf{dx} + \tau_0 \int_{\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dx} + + \tau_0 \int_{\mathcal{J}} q_{mt}^2 \mathbf{dx} \\
& + \iiint_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left|\boldsymbol{\lambda}_2 (\mathbf{s})\right| \boldsymbol{\varpi}_m^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd\boldsymbol{\rho}dx} \\
& + \iiint_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left|\boldsymbol{\lambda}_2 (\mathbf{s})\right| \boldsymbol{\varpi}_{m\mathbf{x}}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd\boldsymbol{\rho}dx} \\
& + \left(1 + \frac{\rho_2}{K}\right) \iiint_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left|\boldsymbol{\xi}_2 (\mathbf{s})\right| \boldsymbol{\varpi}_{mt}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd\boldsymbol{\rho}dx} \\
& + \frac{\rho_2}{K} \iiint_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left|\boldsymbol{\xi}_2 (\mathbf{s})\right| \varpi_{m\mathbf{x}t}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd\boldsymbol{\rho}dx} \\
& + \iiint_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left|\boldsymbol{\xi}_2 (\mathbf{s})\right| \boldsymbol{\varpi}_{mtt}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd\boldsymbol{\rho}dx} \\
\leq \ & C, \qquad (2.69)
\end{aligned}$$

we choose $\varepsilon = \frac{1}{2}$, and $\lambda = \beta_1 - K > 0$,such that $\beta_1 = \rho_1 + \rho_2 > K$, we obtain

$$
\begin{aligned}
& \mathcal{N}_m(t) = \rho_1 \int\limits_{\mathcal{J}} \varphi_m^2 \mathbf{dx} + \lambda \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}}^2 \mathbf{dx} + \rho_1 \int\limits_{\mathcal{J}} \varphi_{mt}^2 \mathbf{dx} \\
& + \lambda \int\limits_{\mathcal{J}} \varphi_{m\mathbf{x}t}^2 \mathbf{dx} + \left(\rho_1 + \frac{\rho_1 \rho_2}{K} \right) \int\limits_{\mathcal{J}} \varphi_{mtt}^2 \mathbf{dx} \\
& + \beta_2 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{xx}}^2 \mathbf{dx} + \left(\rho_2 + \frac{\rho_2 \rho_1}{K} \right) \int_0^1 \varphi_{m\mathbf{x}tt}^2 \mathbf{dx} \\
& + \frac{\rho_1 \rho_2}{K} \int\limits_{\mathcal{J}} \varphi_{mttt}^2 \mathbf{dx} + \rho_2 \int\limits_{\mathcal{J}} \varphi_{m\mathbf{xx}t}^2 \mathbf{dx} \\
& + \frac{K}{2} \int\limits_{\mathcal{J}} \psi_m^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{x}}^2 \mathbf{dx} \\
& + \frac{K}{2} \int\limits_{\mathcal{J}} \psi_{mt}^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{x}t}^2 \mathbf{dx} + b \int\limits_{\mathcal{J}} \psi_{m\mathbf{xx}}^2 \mathbf{dx} \\
& + \rho_3 \int\limits_{\mathcal{J}} \theta_m^2 \mathbf{dx} + \rho_3 \int\limits_{\mathcal{J}} \theta_{m\mathbf{x}}^2 \mathbf{dx} + \rho_3 \int\limits_{\mathcal{J}} \theta_{mt}^2 \mathbf{dx} \\
& + \tau_0 \int\limits_{\mathcal{J}} q_m^2 \mathbf{dx} + \tau_0 \int\limits_{\mathcal{J}} q_{m\mathbf{x}}^2 \mathbf{dx} + + \tau_0 \int\limits_{\mathcal{J}} q_{mt}^2 \mathbf{dx} \\
& + \iiint\limits_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} s \left| \boldsymbol{\xi}_2(\mathbf{s}) \right| \boldsymbol{\varpi}_m^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \, \mathbf{ds} \mathbf{d}\boldsymbol{\rho} \mathbf{dx} \\
& + \iiint\limits_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left| \boldsymbol{\xi}_2(\mathbf{s}) \right| \boldsymbol{\varpi}_{m\mathbf{x}}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \, \mathbf{ds} \mathbf{d}\boldsymbol{\rho} \mathbf{dx} \\
& + \left(1 + \frac{\rho_2}{K} \right) \iiint\limits_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left| \boldsymbol{\xi}_2(\mathbf{s}) \right| \boldsymbol{\varpi}_{mt}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \, \mathbf{ds} \mathbf{d}\boldsymbol{\rho} \mathbf{dx} \\
& + \frac{\rho_2}{K} \iiint\limits_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left| \boldsymbol{\xi}_2(\mathbf{s}) \right| \boldsymbol{\varpi}_{m\mathbf{x}t}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \, \mathbf{ds} \mathbf{d}\boldsymbol{\rho} \mathbf{dx} \\
& + \iiint\limits_{\mathcal{J} \times \mathcal{K} \times \mathcal{L}} \mathbf{s} \left| \boldsymbol{\xi}_2(\mathbf{s}) \right| \boldsymbol{\varpi}_{mtt}^2 (\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \, \mathbf{ds} \mathbf{d}\boldsymbol{\rho} \mathbf{dx} \\
\leq \; & C. \qquad (2.70)
\end{aligned}
$$

2.6 Passage to limit

Now, to prove that (2.10) holds, we multiply each of the equation (2.10) by a functions $w_l(t)$, $s_l(t)$, $v_l(t)$, $r_l(t)$, and $p_l(t)$ respectivly, we obtain

$$\begin{aligned}
&\rho_1(\varphi_{mtt}, u_l) w_l(t) + K((\varphi_{m\mathbf{x}} + \psi_m), u_{lx}) w_l(t) + \xi_1(\psi_{mt}, u_l) w_l(t) \\
&+ \left(\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varpi}_m(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{ds}, u_l\right) w_l(t) = 0, \\
&\rho_2(\varphi_{mtt}, v_{lx}) s_l(t) + b(\psi_{m\mathbf{x}}, v_{l\mathbf{x}}) s_l(t) + K((\varphi_{m\mathbf{x}} + \psi_m), v_l) s_l(t) \\
&+\gamma(\theta_{m\mathbf{x}}, v_l) s_l(t) = 0, \\
&\rho_3(\theta_{mt}, w_l) v_l(t) + (q_{m\mathbf{x}}, w_l) v_l(t) + \gamma(\psi_{mt\mathbf{x}}, w_l) v_l(t) = 0, \\
&\tau_0(q_{mt}, z_l) r_l(t) + \delta(q_m, z_l) r_l(t) + \kappa(\theta_{m\mathbf{x}}, z_l) r_l(t) = 0, \\
&s(\varpi_{mt}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \phi_l) p_l(t) + (\varpi_{m\boldsymbol{\rho}}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \phi_l) p_l(t) = 0, \\
&s(\varpi_{mtt}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \phi_l) p_l(t) + (\varpi_{m\boldsymbol{\rho}t}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \phi_l) p_l(t) = 0. \qquad (2.71)
\end{aligned}$$

Then, summing over l from 1 to m and if we let

$$\begin{aligned}
\lambda_m &= \sum_{l=1}^{l=m} u_l(\mathbf{x}) w_l(t), \quad \gamma_m = \sum_{l=1}^{l=m} v_l(\mathbf{x}) s_l(t), \\
\mu_m &= \sum_{l=1}^{l=m} w_l(\mathbf{x}) v_l(t), \quad \eta_m = \sum_{l=1}^{l=m} z_l(\mathbf{x}) r_l(t), \\
\sigma_m &= \sum_{l=1}^{l=m} \phi_l(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) p_l(t), \qquad (2.72)
\end{aligned}$$

then, we have

$$\begin{aligned}
&\rho_1(\varphi_{mtt}, \lambda_m) + K((\varphi_{m\mathbf{x}} + \psi_m), \lambda_{m\mathbf{x}}) + \xi_1(\psi_{mt}, \lambda_m) \\
&+ \left(\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varpi}_m(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{ds}, \lambda_m\right) = 0, \\
&\rho_2(\varphi_{mtt}, \gamma_{m\mathbf{x}}) + b(\psi_{m\mathbf{x}}, \gamma_{m\mathbf{x}}) + K((\varphi_{m\mathbf{x}} + \psi_m), \gamma_m) \\
&+\gamma(\theta_{m\mathbf{x}}, \gamma_m) = 0, \\
&\rho_3(\theta_{mt}, \mu_m) + (q_{m\mathbf{x}}, \mu_m) + \gamma(\psi_{mt\mathbf{x}}, \mu_m) = 0, \\
&\tau_0(q_{mt}, \eta_m) + \delta(q_m, \eta_m) + \kappa(\theta_{m\mathbf{x}}, \eta_m) = 0, \\
&s(\boldsymbol{\varpi}_{mt}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \sigma_m(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t)) + (\varpi_{m\boldsymbol{\rho}}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \sigma_m(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t)) = 0, \\
&s(\boldsymbol{\varpi}_{mtt}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \sigma_m(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t)) + (\varpi_{m\boldsymbol{\rho}t}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t), \sigma_m(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}, t)) = 0. \qquad (2.73)
\end{aligned}$$

Now, we integrate over t on $(0,\infty)$, we obtain

$$
\begin{aligned}
&-\rho_1\left(\varphi_{mt},\lambda_{mt}\right)_{L^2(Q)}-\rho_1\left(\varphi_{mt}\left(\mathbf{x},0\right),\lambda_m\left(\mathbf{x},0\right)\right)_{L^2(0,1)}\\
&+K\left(\left(\varphi_{m\mathbf{x}}+\psi_m\right),\lambda_{m\mathbf{x}}\right)_{L^2(Q)}-\xi_1\left(\psi_m,\lambda_{mt}\right)_{L^2(Q)}\\
&-\xi_1\left(\psi_m\left(\mathbf{x},0\right),\lambda_m\left(\mathbf{x},0\right)\right)_{L^2(0,1)}\\
&+\left(\int_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}_m\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{ds},\lambda_m\right)_{L^2(Q)}=0,\\
&-\rho_2\left(\varphi_{mt},\gamma_{m\mathbf{x}t}\right)_{L^2(Q)}-\rho_2\left(\varphi_{mt}\left(\mathbf{x},0\right),\gamma_{m\mathbf{x}}\left(\mathbf{x},0\right)\right)_{L^2(0,1)}\\
&+b\left(\psi_{m\mathbf{x}},\gamma_{m\mathbf{x}}\right)_{L^2(Q)}+K\left(\left(\varphi_{m\mathbf{x}}+\psi_m\right),\gamma_m\right)_{L^2(Q)}\\
&+\gamma\left(\theta_{m\mathbf{x}},\gamma_m\right)_{L^2(Q)}=0,\\
&-\rho_3\left(\theta_m,\mu_{mt}\right)_{L^2(Q)}-\rho_3\left(\theta_m\left(\mathbf{x},0\right),\mu_m\left(\mathbf{x},0\right)\right)_{L^2(0,1)}+\left(q_{m\mathbf{x}},\mu_m\right)_{L^2(Q)}\\
&-\gamma\left(\psi_{m\mathbf{x}},\mu_{mt}\right)_{L^2(Q)}-\gamma\left(\psi_{m\mathbf{x}}\left(\mathbf{x},0\right),\mu_m\left(\mathbf{x},0\right)\right)_{L^2(0,1)}=0,\\
&-\tau_0\left(q_m,\eta_{mt}\right)_{L^2(Q)}-\tau_0\left(q_m\left(\mathbf{x},0\right),\eta_m\left(\mathbf{x},0\right)\right)_{L^2(0,1)}\\
&+\delta\left(q_m,\eta_m\right)_{L^2(Q)}+\kappa\left(\theta_{m\mathbf{x}},\eta_m\right)_{L^2(Q)}=0,\\
&-s\left(\varpi_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\sigma_{mt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\right)_{L^2(Q)}\\
&-s\left(\varpi_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right),\sigma_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right)\right)_{L^2(0,1)}\\
&+\left(\varpi_{m\boldsymbol{\rho}}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\sigma_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\right)_{L^2(Q)}=0,\\
&-s\left(\varpi_{mt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\sigma_{mt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\right)_{L^2(Q)}\\
&-s\left(\varpi_{mt}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right),\sigma_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},0\right)\right)_{L^2(0,1)}\\
&+\left(\varpi_{m\boldsymbol{\rho}}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right),\sigma_m\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s},t\right)\right)_{L^2(Q)}=0.
\end{aligned}
\tag{2.74}
$$

From (2.32), (2.47) and (2.66), we conclude that for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\varphi_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^2 \cap H_0^1\right),\\
&\varphi_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\right),\\
&\varphi_{mtt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&\psi_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^2 \cap H_0^1\right),\\
&\psi_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\right),\\
&\theta_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H_0^1\right),\\
&\theta_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&q_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H_0^1\right),\\
&q_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&\boldsymbol{\varpi}_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right),\\
&\boldsymbol{\varpi}_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right).
\end{aligned} \tag{2.75}$$

Thus, we get

$$\begin{aligned}
&\varphi_m \text{ weakly in } L^2\left(\mathbb{R}_+, H^2 \cap H_0^1\right),\\
&\varphi_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\right),\\
&\varphi_{mtt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&\psi_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&\psi_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\right),\\
&\theta_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&\theta_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&q_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&q_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&\boldsymbol{\varpi}_m \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right),\\
&\boldsymbol{\varpi}_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right).
\end{aligned} \tag{2.76}$$

Thus, the limit function $(\varphi, \psi, \theta, q, \boldsymbol{\varpi})$ satisfies (2.10) for every (2.72). We denote by Q_m the totality of all functions of the forme (2.72) with $\lim\limits_{T\to\infty} w_l(T) = \lim\limits_{T\to\infty} s_l(T) = \lim\limits_{T\to\infty} v_l(T) = \lim\limits_{T\to\infty} r_l(T) = \lim\limits_{T\to\infty} p_l(T) = 0$. But $\cup_{m=1}^{\infty} Q_m$ is dense in $\mathcal{W}(Q)$, then the relation (2.5) holds for all $(\varphi, \psi, \theta, q, \boldsymbol{\varpi}) \in \mathcal{W}(Q)$. Thus, we have shown that the limit function $(\varphi, \psi, \theta, q, \boldsymbol{\varpi})$ is a generalized solution of problem (2.3) in $\mathcal{V}(Q)$.

2.7 Continuous dependence on the initial data and uniqueness

First, we prove the continuous dependence and uniqueness for weak solutions of problem (2.3) .Let $(\varphi, \varphi_t, \varphi_{tt}, \psi, \theta, q, \boldsymbol{\varpi}, \boldsymbol{\varpi}_t)$ and $(\Gamma, \Gamma_t, \Gamma_{tt}, \Xi, \Omega, \xi, \Pi, \Pi_t)$ be two global solutions of problem (2.3) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, q_0, \Theta_0, \Theta_1)$ and $(\Gamma_0, \Gamma_1, \Gamma_2, \Xi_0, \Omega_0, \xi_0, \Phi_0, \Phi_1)$, respectively. Let

$$\begin{aligned} \Lambda(t) &= \varphi - \Gamma, \\ \Sigma(t) &= \psi - \Xi, \\ \mathcal{M}(t) &= \theta - \Omega, \\ \mathcal{R}(t) &= q - \xi, \\ \varrho(t) &= \Pi - \Phi. \end{aligned} \tag{2.77}$$

Then, $(\Lambda, \lambda, \mathcal{M}, \mathcal{R}, \boldsymbol{\varrho})$ verifies (2.3) , and we have

$$\begin{aligned} &\rho_1 \Lambda_{tt} - K(\Lambda_{\mathbf{x}} + \Sigma)_{\mathbf{x}} + \boldsymbol{\xi}_1 \Lambda_t + \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varrho}(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{ds} = 0, \\ &-\rho_2 \Lambda_{tt\mathbf{x}} - b\lambda_{\mathbf{xx}} + K(\Lambda_{\mathbf{x}} + \lambda) + \gamma \mathcal{M}_{\mathbf{x}} = 0, \\ &\rho_3 \mathcal{M}_t + \kappa \mathcal{R}_{\mathbf{x}} + \gamma \lambda_{t\mathbf{x}} = 0, \\ &\tau_0 \mathcal{R}_t + \delta \mathcal{R} + \kappa \mathcal{M}_{\mathbf{x}} = 0, \\ &s\boldsymbol{\varrho}_t + \boldsymbol{\varrho}_\rho = 0, \\ &s\boldsymbol{\varrho}_{tt} + \boldsymbol{\varrho}_{\rho t} = 0. \end{aligned} \tag{2.78}$$

Now, multiplying $(2.78)_1$, $(2.78)_2$, $(2.78)_3$ and $(2.78)_4$ by Λ_t, λ_t, $\mathcal{M}_t$, $\mathcal{R}_t$, respectively, and integrating over $\mathcal{J} = (0, 1)$ (the same arguments as in energy method), we get

$$\begin{aligned} &\frac{\rho_1}{2} \frac{d}{dt} \int_{\mathcal{J}} \Lambda_t^2 \mathbf{dx} + K \int_{\mathcal{J}} (\Lambda_x + \Sigma) \Lambda_{t\mathbf{x}} \mathbf{dx} + \boldsymbol{\xi}_1 \int_{\mathcal{J}} \Lambda_t^2 \mathbf{dx} \\ &+ \int_{\mathcal{J}} \Lambda_t \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varrho}(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} = 0, \end{aligned} \tag{2.79}$$

then,

$$\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_{\mathcal{J}}\Lambda_{tt}^2\mathbf{dx}+\frac{\rho_2}{2}\frac{d}{dt}\int_{\mathcal{J}}\Lambda_{\mathbf{x}t}^2\mathbf{dx}+\frac{b}{2}\frac{d}{dt}\int_{\mathcal{J}}\Sigma_{\mathbf{x}}^2\mathbf{dx}$$
$$+K\int_{\mathcal{J}}\Sigma_t\left(\Lambda_x+\Sigma\right)\mathbf{dx}+\gamma\int_{\mathcal{J}}\Sigma_t\mathcal{M}_x\mathbf{dx}$$
$$\frac{\rho_2\boldsymbol{\xi}_1}{K}\int_{\mathcal{J}}\Lambda_{tt}^2\mathbf{dx}+\frac{\rho_2}{K}\int_{\mathcal{J}}\Lambda_{tt}\int_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varrho}_t\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx}=0, \tag{2.80}$$

after that,

$$\frac{\rho_3}{2}\frac{d}{dt}\int_{\mathcal{J}}\mathcal{M}^2\mathbf{dx}+\kappa\int_{\mathcal{J}}\mathcal{M}\mathcal{R}_{\mathbf{x}}\mathbf{dx}-\gamma\int_{\mathcal{J}}\Sigma_t\mathcal{M}_{\mathbf{x}}\mathbf{dx}=0, \tag{2.81}$$

finally,

$$\frac{\tau_0}{2}\frac{d}{dt}\int_{\mathcal{J}}\mathcal{R}^2\boldsymbol{dx}+\delta\int_{\mathcal{J}}\mathcal{R}^2\mathbf{dx}-\kappa\int_{\mathcal{J}}\mathcal{M}\mathcal{R}_{\mathbf{x}}\mathbf{dx}=0. \tag{2.82}$$

By combining (2.79), (2.80), (2.81) and (2.82), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{J}}\Big[\rho_1\Lambda_t^2+K\left(\Lambda_{\mathbf{x}}+\Sigma\right)^2+\frac{\rho_2\rho_1}{K}\Lambda_{tt}^2+b\Sigma_{\mathbf{x}}^2+\rho_2\Lambda_{\mathbf{x}t}^2 \tag{2.83}$$
$$\rho_3\mathcal{M}^2+\tau_0\mathcal{R}^2\Big]\mathbf{dx}+\xi_1\int_{\mathcal{J}}\Lambda_t^2\mathbf{dx}+\delta\int_{\mathcal{J}}\mathcal{R}^2\mathbf{dx}$$
$$+\int_{\mathcal{J}}\Lambda_t\int_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\varrho\left(\mathbf{x},1,s\right)\mathbf{dsdx}+\frac{\rho_2\lambda_1}{K}\int_{\mathcal{J}}\Lambda_{tt}^2\mathbf{dx}$$
$$+\frac{\rho_2}{K}\int_{\mathcal{J}}\Lambda_{tt}\int_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\varrho_t\left(\mathbf{x},1,s\right)\mathbf{dsdx}=0.$$

Now, multiplying $(2.78)_5$ by $|\boldsymbol{\xi}_2\left(\mathbf{s}\right)|\,\boldsymbol{\varrho}\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)$ and integrating over $\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varrho}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}$$
$$-\frac{1}{2}\left(\int_{\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\mathbf{ds}\right)\int_{\mathcal{J}}\Lambda_t^2\mathbf{dx}$$
$$+\frac{1}{2}\iint_{\mathcal{J}\times\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varrho}^2\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx}=0. \tag{2.84}$$

Then, multiplying $(2.78)_6$ by $|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}_t(\mathbf{x},\boldsymbol{\rho},\mathbf{s})$ and integrating over $\mathcal{J}\times\mathcal{K}\times\mathcal{L}=(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned}
&\frac{\rho_2}{2K}\frac{d}{dt}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
&-\frac{\rho_2}{2K}\left(\int\limits_{\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int\limits_{\mathcal{J}}\Lambda_{tt}^2\mathbf{dx}\\
&+\frac{\rho_2}{2K}\int\limits_{\mathcal{J}\times\mathcal{L}}\int |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}=0.
\end{aligned}\tag{2.85}$$

By combining (2.83), (2.84) and (2.85), we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &= -\left(\boldsymbol{\xi}_1-\frac{1}{2}\int\limits_{\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int\limits_{\mathcal{J}}\Lambda_t^2\mathbf{dx}\\
&\quad-\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1-\frac{1}{2}\int\limits_{\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int_0^1\Lambda_{tt}^2\mathbf{dx}\\
&\quad-\int\limits_{\mathcal{J}}\Lambda_t\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varrho}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}-\delta\int\limits_{\mathcal{J}}\mathcal{R}^2\mathbf{dx}\\
&\quad-\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\Lambda_{tt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varrho}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}\\
&\quad-\frac{1}{2}\int\limits_{\mathcal{J}\times\mathcal{L}}\int|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}\\
&\quad-\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}\\
&\le -\eta_0\int\limits_{\mathcal{J}}\Lambda_t^2\mathbf{dx}-\eta_0\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\Lambda_{tt}^2\mathbf{dx}-\delta\int\limits_{\mathcal{J}}\mathcal{R}^2\mathbf{dx}\le 0\\
&\le c\left(\int\limits_{\mathcal{J}}\left[\Lambda_t^2+(\Lambda_{\mathbf{x}}+\Sigma)^2+\Lambda_{tt}^2+\Lambda_{\mathbf{x}t}^2+\Sigma_{\mathbf{x}}^2+\mathcal{M}^2+\mathcal{R}^2\right]\mathbf{dx}\right.\\
&\quad+\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\\
&\quad\left.+\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varrho}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}\right).
\end{aligned}\tag{2.86}$$

By integrating (2.86) over $(0,t)$, we obtain

$$E(t)-E(0) \leq c\left(\int_{\mathcal{I}}\left[\|\Lambda_t\|^2+\|(\Lambda_{\mathbf{x}}+\Sigma)\|^2+\|\Lambda_{tt}\|^2+\|\Lambda_{\mathbf{x}t}\|^2+\|\Sigma_{\mathbf{x}}\|^2+\|\mathcal{M}\|^2+\|\mathcal{R}\|^2\right]\mathbf{d}\boldsymbol{\tau}\right.$$
$$\left.+\iint_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\|^2\mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau}+\iint_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}_t(x,\rho,s)\|^2\mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau}\right),$$

implies,

$$\begin{aligned} E(t) \leq\ & E(0)+c\int_{\mathcal{I}}\left[\|\Lambda_t\|^2+\|(\Lambda_{\mathbf{x}}+\Sigma)\|^2+\|\Lambda_{tt}\|^2\right.\\ &\left.+\|\Lambda_{\mathbf{x}t}\|^2+\|\mathcal{M}\|^2+\|\Sigma_{\mathbf{x}}\|^2+\|\mathcal{R}\|^2\right]\mathbf{d}\boldsymbol{\tau}\\ &+c\iint_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\|^2\mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau}\\ &+c\iint_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}_t(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\|^2\mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau}. \end{aligned} \tag{2.87}$$

On the other hand, we have

$$\begin{aligned} E(t) =\ & \frac{1}{2}\int_{\mathcal{J}}\left[\rho_1\Lambda_t^2+K(\Lambda_{\mathbf{x}}+\Sigma)^2+\frac{\rho_2\rho_1}{K}\Lambda_{tt}^2+\rho_2\Lambda_{xt}^2+b\Sigma_x^2+\rho_3\mathcal{M}^2+\tau_0\mathcal{R}^2\right]\mathbf{dx}\\ &+\frac{1}{2}\iint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\boldsymbol{\varrho}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\ &+\frac{\rho_2}{2K}\iint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\boldsymbol{\varrho}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}\\ \geq\ & \frac{1}{2}\min\left(\rho_1,K,\frac{\rho_2\rho_1}{K},\rho_2,b,\rho_3,\tau_0,\frac{\rho_2}{K},1\right)\\ &\times\left(\|\Lambda_t\|^2+\|(\Lambda_{\mathbf{x}}+\lambda)\|^2+\|\Lambda_{tt}\|^2+\|\Lambda_{\mathbf{x}t}\|^2+\|\Sigma_{\mathbf{x}}\|^2+\|\mathcal{M}\|^2+\|\mathcal{R}\|^2\right.\\ &+\iint_{\mathcal{K}\times\mathcal{L}} \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\|^2\mathbf{dsd}\boldsymbol{\rho}\\ &\left.+\iint_{\mathcal{K}\times\mathcal{L}} \mathbf{s}\left|\boldsymbol{\xi}_2(\mathbf{s})\right|\|\boldsymbol{\varrho}_t(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\|^2\mathbf{dsd}\boldsymbol{\rho}\right), \end{aligned}$$

implies,

$$\begin{aligned}
E(t) \geq\ & m_0 \Big(\|\Lambda_t\|^2 + \|(\Lambda_{\mathbf{x}} + \lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{\mathbf{x}t}\|^2 + \|\Sigma_{\mathbf{x}}\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 \\
& + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \\
& + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}_t(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \Big). \qquad (2.88)
\end{aligned}$$

So, we have

$$\begin{aligned}
& m_0 \Big(\|\Lambda_t\|^2 + \|(\Lambda_{\mathbf{x}} + \lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{\mathbf{x}t}\|^2 \\
& + \|\Sigma_{\mathbf{x}}\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \\
& + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}_t(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \Big) \\
\leq\ & E(0) + c \int\limits_{\mathcal{I}} \big[\|\Lambda_t\|^2 + \|(\Lambda_{\mathbf{x}} + \lambda)\|^2 + \|\Lambda_{tt}\|^2 \\
& + \|\Lambda_{\mathbf{x}t}\|^2 + \|\mathcal{M}\|^2 + \|\Sigma_{\mathbf{x}}\|^2 + \|\mathcal{R}\|^2 \big] \, \mathbf{d}\boldsymbol{\tau} \\
& + c \iiint\limits_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau} \\
& + c \iiint\limits_{\mathcal{I}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}_t(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho}\mathbf{d}\boldsymbol{\tau}. \qquad (2.89)
\end{aligned}$$

Applying Gronwall's inequality to (2.89), we get

$$\begin{aligned}
& \|\Lambda_t\|^2 + \|(\Lambda_{\mathbf{x}} + \lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{\mathbf{x}t}\|^2 + \|\lambda_x\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 \\
& + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \\
& + \iint\limits_{\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \, \|\boldsymbol{\varrho}_t(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s})\|^2 \, \mathbf{dsd}\boldsymbol{\rho} \\
\leq\ & \frac{1}{m_0} E(0) \exp(Mt), \qquad (2.90)
\end{aligned}$$

where $M = \frac{c}{m_0}$.

This shows that solution of system (2.3) ,depends continuously on the initial data and unique..

Chapter 3

Exponential stability results

In this chapter, we will prove exponential stability results for problem(2.3) under the assumption (30). To achieve our goal, we utilize the energy method.to construct an appropriate Lyapunov functional, resulting in a proof of exponential stability.

3.1 The energy functional

In this section, we introduce the following lemma needed for the proof of our main result

Lemma 3.1 *Define the energy of solution as*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\mathcal{J}} \left[\rho_1\varphi_t^2 + K(\varphi_{\mathbf{x}} + \psi)^2 + \frac{\rho_2\rho_1}{K}\varphi_{tt}^2 + \rho_2\varphi_{\mathbf{x}t}^2 + b\psi_{\mathbf{x}}^2 + \rho_3\theta^2 + \tau_0 q^2\right]\mathbf{dx} \\ &+\frac{1}{2}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\ &+\frac{\rho_2}{2K}\iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}, \end{aligned}$$

satisfies

$$
\begin{aligned}
E'(t) &\leq -\left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, ds\right) \int\limits_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \delta \int\limits_{\mathcal{J}} q^2 \mathbf{dx} \\
&\quad -\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, ds\right) \int\limits_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} \\
&\leq -\eta_0 \int\limits_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \delta \int\limits_{\mathcal{J}} q^2 \mathbf{dx} \\
&\quad -\frac{\rho_2}{K}\eta_0 \int\limits_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} \\
&\leq 0,
\end{aligned}
$$

where $\eta_0 = \left(\boldsymbol{\xi}_1 - \int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, ds\right) > 0.$

Proof. First, multiplying $(2.3)_1$, $(2.3)_2$, $(2.3)_3$ and $(2.3)_4$ by φ_t, ψ_t, θ and q,respectively, and integrating over $\mathcal{J} = (0,1)$, using integration by parts and the boundary conditions, we obtain

$$
\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \varphi_t^2 \mathbf{dx} + K\int\limits_{\mathcal{J}} \varphi_{t\mathbf{x}}(\varphi_{\mathbf{x}} + \psi)\,\mathbf{dx} + \boldsymbol{\xi}_1 \int\limits_{\mathcal{J}} \varphi_t^2 \mathbf{dx} \\
&+\int\limits_{\mathcal{J}} \varphi_t \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\, \boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0,
\end{aligned} \tag{3.1}
$$

then,

$$
\begin{aligned}
&\rho_2 \int\limits_{\mathcal{J}} \psi_{t\mathbf{x}}\varphi_{tt}\mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} + K\int\limits_{\mathcal{J}} \psi_t(\varphi_{\mathbf{x}} + \psi)\,\mathbf{dx} \\
&+\gamma \int\limits_{\mathcal{J}} \psi_t \theta_{\mathbf{x}} \mathbf{dx} = 0,
\end{aligned} \tag{3.2}
$$

next,

$$
\frac{\rho_3}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} \theta^2 \mathbf{dx} - \kappa \int\limits_{\mathcal{J}} \theta_{\mathbf{x}} q \mathbf{dx} - \gamma \int\limits_{\mathcal{J}} \theta_{\mathbf{x}}\psi_t \mathbf{dx} = 0. \tag{3.3}
$$

finally,

$$
\frac{\tau_0}{2}\frac{d}{dt}\int\limits_{\mathcal{J}} q^2 \mathbf{dx} + \delta \int\limits_{\mathcal{J}} q^2 \mathbf{dx} + \kappa \int\limits_{\mathcal{J}} q\theta_{\mathbf{x}} \mathbf{dx} = 0. \tag{3.4}
$$

now, substituting $\psi_{t\mathbf{x}} = \frac{\rho_1}{K}\varphi_{ttt} - \varphi_{\mathbf{xx}t} + \frac{\boldsymbol{\xi}_1}{K}\varphi_{tt} + \frac{1}{K}\int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{ds}$ into first integral of (3.2) and using the integral by parts, we get

$$\begin{aligned}
&\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int\limits_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} + \frac{\rho_2}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\varphi_{\mathbf{x}t}^2\mathbf{dx} + \frac{b}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx}\\
&+K\int\limits_{\mathcal{J}}\psi_t(\varphi_{\mathbf{x}}+\psi)\,\mathbf{dx} + \gamma\int\limits_{\mathcal{J}}\psi_t\theta_{\mathbf{x}}\mathbf{dx} + \frac{\boldsymbol{\xi}_1\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx}\\
&+\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{tt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0, \qquad (3.5)
\end{aligned}$$

summing $(3.1), (3.3), (3.4)$ and (3.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\Big[\rho_1\varphi_t^2 + K(\varphi_{\mathbf{x}}+\psi)^2 + \frac{\rho_2\rho_1}{K}\varphi_{tt}^2\\
&+\rho_2\varphi_{\mathbf{x}t}^2 + b\psi_{\mathbf{x}}^2 + \rho_3\theta^2 + \tau_0 q^2\Big]\,\mathbf{dx}\\
&+\boldsymbol{\xi}_1\int\limits_{\mathcal{J}}\varphi_t^2\mathbf{dx} + \delta\int\limits_{\mathcal{J}}q^2\mathbf{dx} + \int\limits_{\mathcal{J}}\varphi_t\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}\\
&+\xi_1\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} + \frac{\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{tt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0. \qquad (3.6)
\end{aligned}$$

Second, multiplying $(2.3)_5$ by $(\boldsymbol{\varpi}\,|\boldsymbol{\xi}_2(\mathbf{s})|)$, integrating the product overr $\mathcal{J}\times\mathcal{K}\times\mathcal{L} = (0,1)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $\boldsymbol{\varpi}(\mathbf{x},\mathbf{0},\mathbf{s}) = \varphi_t$, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} - \frac{1}{2}\left(\int\limits_{\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int\limits_{\mathcal{J}}\varphi_t^2dx\\
&+\frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0, \qquad (3.7)
\end{aligned}$$

now, differentiating and multiplying $(2.3)_5$ by $(\boldsymbol{\varpi}_t\,|\boldsymbol{\xi}_2(\mathbf{s})|)$, integrating the product over $\mathcal{J}\times\mathcal{K}\times\mathcal{L} = (0,1$ $(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\frac{d}{dt}\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} - \frac{\rho_2}{2K}\left(\int\limits_{\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int\limits_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx}\\
&+\frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\lambda}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} = 0, \qquad (3.8)
\end{aligned}$$

A combination of (3.6), (3.7) and (3.8), gives

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\left[\rho_1\varphi_t^2+K\left(\varphi_{\mathbf{x}}+\psi\right)^2+\frac{\rho_2\rho_1}{K}\varphi_{tt}^2+\rho_2\varphi_{\mathbf{xt}}^2+b\psi_{\mathbf{x}}^2+\rho_3\theta^2+\tau_0 q^2\right]\mathbf{dx}$$
$$+\frac{1}{2}\frac{d}{dt}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}$$
$$+\frac{\rho_2}{2K}\frac{d}{dt}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}_t^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}$$
$$= -\left(\boldsymbol{\xi}_1-\frac{1}{2}\left(\int\limits_{\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\mathbf{ds}\right)\right)\int\limits_{\mathcal{J}}\varphi_t^2\mathbf{dx}-\delta\int\limits_{\mathcal{J}}q^2\mathbf{dx}$$
$$-\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1-\frac{1}{2}\left(\int\limits_{\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\mathbf{ds}\right)\right)\int\limits_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx}$$
$$-\int\limits_{\mathcal{J}}\varphi_t\int\limits_{\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx}-\frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}^2\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx}$$
$$-\frac{\rho_2}{K}\int\limits_{\mathcal{J}}\varphi_{tt}\int\limits_{\mathcal{L}}\boldsymbol{\xi}_2\left(\mathbf{s}\right)\boldsymbol{\varpi}_t\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx}-\frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{L}}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}_t^2\left(\mathbf{x},\mathbf{1},\mathbf{s}\right)\mathbf{dsdx},$$

where

$$\begin{aligned} E\left(t\right) &= \frac{1}{2}\frac{d}{dt}\int\limits_{\mathcal{J}}\left[\rho_1\varphi_t^2+K\left(\varphi_{\mathbf{x}}+\psi\right)^2+\frac{\rho_2\rho_1}{K}\varphi_{tt}^2\right. \\ &\left.+\rho_2\varphi_{\mathbf{xt}}^2+b\psi_{\mathbf{x}}^2+\rho_3\theta^2+\tau_0 q^2\right]\mathbf{dx} \\ &+\frac{1}{2}\frac{d}{dt}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\ &+\frac{\rho_2}{2K}\frac{d}{dt}\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\left|\boldsymbol{\xi}_2\left(\mathbf{s}\right)\right|\boldsymbol{\varpi}_t^2\left(\mathbf{x},\boldsymbol{\rho},\mathbf{s}\right)\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}, \end{aligned} \tag{3.9}$$

and

$$\frac{d}{dt}E(t) = -\left(\boldsymbol{\xi}_1 - \frac{1}{2}\left(\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\right)\int\limits_{\mathcal{J}} \varphi_t^2\mathbf{dx} - \delta\int\limits_{\mathcal{J}} q^2\mathbf{dx}$$

$$-\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \frac{1}{2}\left(\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\right)\int\limits_{\mathcal{J}} \varphi_{tt}^2\mathbf{dx}$$

$$-\int\limits_{\mathcal{J}} \varphi_t \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$

$$-\frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$

$$-\frac{\rho_2}{K}\int\limits_{\mathcal{J}} \varphi_{tt} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$

$$-\frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}. \tag{3.10}$$

Meanwhile, using Young's and Cauchy Shwarz's inequalities, we have

$$-\int\limits_{\mathcal{J}} \varphi_t \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$

$$\leq \frac{1}{2}\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \frac{1}{2}\left(\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int_0^1 \varphi_t^2 dx, \tag{3.11}$$

and

$$-\frac{\rho_2}{K}\int\limits_{\mathcal{J}} \varphi_{tt} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s})\,\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}$$

$$\leq \frac{\rho_2}{2K}\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \frac{\rho_2}{2K}\left(\int\limits_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\mathbf{ds}\right)\int\limits_{\mathcal{J}} \varphi_{tt}^2\mathbf{dx}. \tag{3.12}$$

Now, substituting (3.11) and (3.12) into (3.10), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) \leq & -\left(\boldsymbol{\xi}_1 - \left(\int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \, \mathbf{ds}\right)\right) \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \delta \int_{\mathcal{J}} q^2 \mathbf{dx} \\ & -\frac{\rho_2}{K}\left(\boldsymbol{\xi}_1 - \left(\int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \, \mathbf{ds}\right)\right) \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} \\ \leq & -\eta_0 \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \delta \int_{\mathcal{J}} q^2 \mathbf{dx} - \frac{\rho_2}{K}\eta_0 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx}, \end{aligned} \tag{3.13}$$

where $\eta_0 = \left(\boldsymbol{\xi}_1 - \int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \, \mathbf{ds}\right) > 0$.then we obtain that E is decreasing. ■

3.2 Lyapunov functional and main results

In this section, we construct a Lyapunov functional $\mathcal{L}$ equivalent to E. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.2 *The functional*

$$F_1(t) = -\frac{\xi_1}{2} \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - K \int_{\mathcal{J}} \varphi_{t\mathbf{x}} \varphi_{\mathbf{x}} \mathbf{dx}, \tag{3.14}$$

satisfies

$$\begin{aligned} F_1'(t) \leq & -K \int_{\mathcal{J}} \varphi_{t\mathbf{x}}^2 \mathbf{dx} + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \varepsilon_1 \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\ & + c \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \, \varpi^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \, \mathbf{dsdx}. \end{aligned} \tag{3.15}$$

Proof. A simple differentiation of $F_1(t)$, using parametric integral, $(2.3)_1$, integration by parts, Young's and Poincaré inequalities, we gett

$$\begin{aligned}
F_1'(t) &= \rho_1 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} - K \int_{\mathcal{J}} \varphi_{tt}\psi_{\mathbf{x}} \mathbf{dx} - K \int_{\mathcal{J}} \varphi_{t\mathbf{x}}^2 \mathbf{dx} \\
&\quad + \int_{\mathcal{J}} \varphi_{tt} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varpi}(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\leq -K \int_{\mathcal{J}} \varphi_{t\mathbf{x}}^2 \mathbf{dx} + \rho_1 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \frac{K^2}{\varepsilon_1} \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \varepsilon_1 \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\
&\quad + \rho_1 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \frac{\boldsymbol{\xi}_1}{\rho_1} \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\leq -K \int_{\mathcal{J}} \varphi_{t\mathbf{x}}^2 \mathbf{dx} + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \varepsilon_1 \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\
&\quad + \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx}.
\end{aligned}$$

where $c = \max\left(\frac{\boldsymbol{\xi}_1}{\rho_1}, 2\rho_1\right)$ and

$$\begin{aligned}
&\int_{\mathcal{J}} \varphi_{tt} \int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \boldsymbol{\varpi}(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&= \int_{\mathcal{J}} \varphi_{tt} \int_{\mathcal{L}} \sqrt{|\boldsymbol{\xi}_2(\mathbf{s})|} \sqrt{|\boldsymbol{\xi}_2(\mathbf{s})|} \boldsymbol{\varpi}(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\leq \int_{\mathcal{J}} \varphi_{tt} \left[\left(\int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \mathbf{ds} \right)^{\frac{1}{2}} \left(\int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{ds} \right)^{\frac{1}{2}} \right] \mathbf{dx} \\
&\leq \frac{\varepsilon}{2} \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \frac{1}{2\varepsilon} \int_{\mathcal{J}} \left[\int_{\mathcal{L}} \boldsymbol{\xi}_2(\mathbf{s}) \mathbf{ds} \int_{\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{ds} \right] \mathbf{dx} \\
&\leq \rho_1 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \frac{\boldsymbol{\lambda}_1}{\rho_1} \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx}.
\end{aligned}$$

■

Lemma 3.3 *The functional*

$$F_2(t) = \rho_1 \int_{\mathcal{J}} \varphi\varphi_t \mathbf{dx} + \frac{\boldsymbol{\xi}_1}{2} \int_{\mathcal{J}} \varphi^2 \mathbf{dx} + \frac{\xi_1 \rho_2}{2K} \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} + \rho_2 \int_{\mathcal{J}} \varphi_{t\mathbf{x}} \varphi_{\mathbf{x}} \mathbf{dx}, \tag{3.16}$$

satisfies

$$F_2'(t) \leq -\frac{b}{2} \int_{\mathcal{J}} \psi_x^2 \mathbf{dx} - \frac{K}{2} \int_{\mathcal{J}} (\varphi_x + \psi)^2 \mathbf{dx} - \frac{\rho_1 \rho_2}{2K} \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \rho_2 \int_{\mathcal{J}} \varphi_{tx}^2 \mathbf{dx} + \frac{\rho_3 \kappa}{4} \int_{\mathcal{J}} \theta^2 \mathbf{dx} + \rho_1 \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} + c \int\int_{\mathcal{J} \times \mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx}. \tag{3.17}$$

Proof. *A simple differentiation of* $F_2(t)$*, using parametric integral,* $(2.3)_1$*,* $(2.3)_2$ *integration by* *parts, Young's, Cauchy Schwarz and Poincaré inequalities, we get*

$$\begin{aligned}
F_2'(t) &= \rho_1\int_{\mathcal{J}}\varphi_t^2dx - K\int_{\mathcal{J}}(\varphi_x+\psi)^2dx - \int_{\mathcal{J}}\varphi\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\varpi(x,1,s)\,dsdx \\
&\quad -\frac{\rho_1\rho_2}{K}\int_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} - b\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} + \gamma\int_{\mathcal{J}}\theta\psi_{\mathbf{x}}\mathbf{dx} + \rho_2\int_{\mathcal{J}}\varphi_{t\mathbf{x}}^2\mathbf{dx} \\
&\quad -\frac{\rho_2}{K}\int_{\mathcal{J}}\varphi_{tt}\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} \\
&\le -K\int_{\mathcal{J}}(\varphi_{\mathbf{x}}+\psi)^2\mathbf{dx} - b\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} - \frac{\rho_1\rho_2}{K}\int_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} + \rho_1\int_{\mathcal{J}}\varphi_t^2\mathbf{dx} \\
&\quad +\frac{\gamma^2}{2b}\int_{\mathcal{J}}\theta^2\mathbf{dx} + \frac{b}{2}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} + \rho_2\int_{\mathcal{J}}\varphi_{t\mathbf{x}}^2\mathbf{dx} \\
&\quad +\frac{K}{2}\int_{\mathcal{J}}(\varphi_{\mathbf{x}}+\psi)^2\mathbf{dx} + \frac{\xi_1}{2}\iint_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} \\
&\quad +\frac{\rho_1\rho_2}{2K}\int_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} + \frac{\xi_1\rho_2}{2\rho_1K}\iint_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} \\
&\le -\frac{K}{2}\int_{\mathcal{J}}(\varphi_{\mathbf{x}}+\psi)^2\mathbf{dx} - \frac{b}{2}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} - \frac{\rho_1\rho_2}{2K}\int_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} \\
&\quad +\rho_2\int_{\mathcal{J}}\varphi_{t\mathbf{x}}^2\mathbf{dx} + \rho_1\int_{\mathcal{J}}\varphi_t^2\mathbf{dx} + \frac{\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&\quad +c\iint_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx},
\end{aligned}$$

where $c = \frac{\boldsymbol{\xi}_1}{2}\left(1+\frac{K}{\rho_1\rho_2}\right)$,

$$\begin{aligned}
&-\int_{\mathcal{J}}\varphi\int_{\mathcal{L}}\boldsymbol{\xi}_2(\mathbf{s})\boldsymbol{\varpi}(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} \\
&\le \frac{K}{2}\int_{\mathcal{J}}\varphi_{\mathbf{x}}^2\mathbf{dx} + \frac{\boldsymbol{\xi}_1}{2}\iint_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} \\
&\le \frac{K}{2}\int_{\mathcal{J}}(\varphi_{\mathbf{x}}+\psi)^2\mathbf{dx} + \frac{\boldsymbol{\lambda}_1}{2}\iint_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx},
\end{aligned}$$

and

$$
\begin{aligned}
& -\int\limits_{\mathcal{J}} \varphi_{tt} \int\limits_{\mathcal{L}} \boldsymbol{\xi}_2 (\mathbf{s})\, \boldsymbol{\varpi} (\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsdx} \\
\leq\ & \frac{\rho_1}{2} \int\limits_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} + \frac{\boldsymbol{\xi}_1}{2\rho_1} \iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2 (\mathbf{s})|\, \boldsymbol{\varpi}^2 (\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsdx}.
\end{aligned}
$$

■

Lemma 3.4 *The functional*

$$
\begin{aligned}
F_3 (t) &= -\rho_3 \tau_0 \int\limits_{\mathcal{J}} q \int_0^x \theta (y, t)\, dy \mathbf{dx} \\
&\quad -\tau_0 \gamma \int\limits_{\mathcal{J}} q \psi \mathbf{dx},
\end{aligned}
\tag{3.18}
$$

satisfies

$$
\begin{aligned}
F_3' (t) &\leq -\frac{\kappa \rho_3}{2} \int\limits_{\mathcal{J}} \theta^2 \mathbf{dx} + \frac{b}{4} \int\limits_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\
&\quad + c \int\limits_{\mathcal{J}} q^2 \mathbf{dx}.
\end{aligned}
\tag{3.19}
$$

Proof. A simple differentiation of $F_3(t)$, using parametric integral, $(2.3)_3$, $(2.3)_4$, integration by parts, Young's, Poincaré and Cauchy Schwarz inequalities, we get

$$\begin{aligned}
F_3'(t) &= \rho_3\delta\int_{\mathcal{J}} q\int_0^x \theta(y,t)\,dy\mathbf{dx} - \rho_3\kappa\int_{\mathcal{J}}\theta^2\mathbf{dx} + \tau_0\kappa\int_{\mathcal{J}} q^2\mathbf{dx} \\
&\quad + \gamma\delta\int_{\mathcal{J}}\psi q\mathbf{dx} - \gamma\kappa\int_{\mathcal{J}}\psi_{\mathbf{x}}\theta\mathbf{dx} \\
&\leq \rho_3\delta\left(\int_{\mathcal{J}} q^2\mathbf{dx}\right)^{\frac{1}{2}}\left(\int_{\mathcal{J}}\left(\int_0^x \theta(y,t)\,dy\right)^2\mathbf{dx}\right)^{\frac{1}{2}} - \rho_3\kappa\int_{\mathcal{J}}\theta^2\mathbf{dx} + \tau_0\kappa\int_{\mathcal{J}} q^2\mathbf{dx} \\
&\quad + \frac{\gamma\delta C}{2}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} + \frac{\gamma\delta}{2}\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{(\gamma\kappa)^2}{2\varepsilon_4}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} + \frac{\varepsilon_4}{2}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&\leq \frac{(\rho_3\delta)^2}{2}\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{1}{2}\int_{\mathcal{J}}\left(\int_0^x \theta(y,t)\,dy\right)^2\mathbf{dx} - \rho_3\kappa\int_{\mathcal{J}}\theta^2\mathbf{dx} + \tau_0\kappa\int_{\mathcal{J}} q^2\mathbf{dx} \\
&\quad + \frac{\gamma\delta}{2}\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{b}{4}\int_{\mathcal{J}}\psi_x^2\mathbf{dx} + \frac{\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&\leq c\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{C}{2}\int_{\mathcal{J}}\left(\int_0^x \theta_y(y,t)\,dy\right)^2\mathbf{dx} - \rho_3\kappa\int_{\mathcal{J}}\theta^2\mathbf{dx} + \frac{b}{4}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} \\
&\quad + \frac{\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&\leq c\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{C}{2}\int_{\mathcal{J}}\left(\theta(x,t)^2 - \theta(0,t)^2\right)\mathbf{dx} - \rho_3\kappa\int_{\mathcal{J}}\theta^2\mathbf{dx} + \frac{b}{4}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} \\
&\quad + \frac{\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&\leq c\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} - \frac{3\rho_3\kappa}{4}\int_{\mathcal{J}}\theta^2\mathbf{dx} + \frac{b}{4}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx} \\
&\leq -\frac{\rho_3\kappa}{2}\int_{\mathcal{J}}\theta^2\mathbf{dx} + c\int_{\mathcal{J}} q^2\mathbf{dx} + \frac{b}{4}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx},
\end{aligned}$$

where $\varepsilon_4 = C = \frac{\rho_3\kappa}{2}$, $\frac{\gamma\delta C}{2} + \frac{(\gamma\kappa)^2}{2\varepsilon_4} = \frac{b}{4}$. ■

Lemma 3.5 *The functional*

$$F_4(t) = \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathrm{s}\exp(-\mathbf{s}\boldsymbol{\rho})\left|\boldsymbol{\xi}_2(\mathrm{s})\right|\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx}, \tag{3.20}$$

satisfies

$$\begin{aligned} F_4'(t) \leq & -m_1 \int\int_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \varpi^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \boldsymbol{\xi}_1 \int_{\mathcal{J}} \varphi_t^2(t)\,\mathbf{dx} \\ & -m_1 \iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds d\boldsymbol{\rho} dx}. \end{aligned} \tag{3.21}$$

Proof. A simple differentiation of $F_4(t)$, using parametric integral and $(2.3)_5$, we get

$$\begin{aligned} F_4'(t) = & -\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \frac{\partial}{\partial\boldsymbol{\rho}}\left[\exp(-\mathbf{s}\boldsymbol{\rho})\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right]\mathbf{ds d\boldsymbol{\rho} dx} \\ & -\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\exp(-s\rho)\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds d\boldsymbol{\rho} dx} \\ = & -\int\int_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \int_{\mathcal{K}} \frac{\partial}{\partial\boldsymbol{\rho}}\left[\exp(-s\rho)\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right]\mathbf{d\boldsymbol{\rho} dsdx} \\ & -\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds d\boldsymbol{\rho} dx} \\ = & -\int\int_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \left[\exp(-\mathbf{s})\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s}) - \boldsymbol{\varpi}^2(\mathbf{x},\mathbf{0},\mathbf{s})\right]\mathbf{dsdx} \\ & -\iint\int_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds d\boldsymbol{\rho} dx}, \end{aligned}$$

UsIing the equaliIty $\boldsymbol{\varpi}(\mathbf{x},\mathbf{0},\mathbf{s})=\varphi_t(t)$ and $-\exp(-\mathbf{s}\boldsymbol{\rho})\leq-\exp(-\mathbf{s})\leq-\exp(-\boldsymbol{\tau}_2)$ for alll $0\leq\boldsymbol{\rho}\leq 1$ and $\boldsymbol{\tau}_1\leq\mathbf{s}\leq\boldsymbol{\tau}_2$, we gett

$$\begin{aligned}
F_4'(t) &= -\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\exp(-\mathbf{s})\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}+\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\varphi_t^2(t)\,\mathbf{dsdx}\\
&\quad-\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx}\\
&\leq -\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\exp(-\boldsymbol{\tau}_2)\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}+\boldsymbol{\xi}_1\int\limits_{\mathcal{J}}\varphi_t^2(t)\,\mathbf{dx}\\
&\quad-\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}s\exp(-\boldsymbol{\tau}_2)\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx}\\
&\leq -\exp(-\boldsymbol{\tau}_2)\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}+\xi_1\int\limits_{\mathcal{J}}\varphi_t^2(t)\,\mathbf{dx}\\
&\quad-\exp(-\boldsymbol{\tau}_2)\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}s\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx}\\
&\leq -m_1\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\varpi^2(\mathbf{x},\mathbf{1},\mathbf{s})\,d\mathbf{sdx}+\boldsymbol{\xi}_1\int\limits_{\mathcal{J}}\varphi_t^2(t)\,\mathbf{dx}\\
&\quad-m_1\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}s\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx},
\end{aligned}$$

where $m_1=\exp(-\boldsymbol{\tau}_2)$. ■

Lemma 3.6 *The functional*

$$F_5(t)=\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\exp(-s\rho)\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx},\tag{3.22}$$

satisfies

$$\begin{aligned}
F_5'(t) &\leq -m_1\iint\limits_{\mathcal{J}\times\mathcal{L}}|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx}+\boldsymbol{\xi}_1\int\limits_{\mathcal{J}}\varphi_{tt}^2(t)\,dx\\
&\quad-m_1\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}s\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx}.
\end{aligned}\tag{3.23}$$

Proof. A simple differentiation of $F_5(t)$, using parametric integral and $(2.3)_6$, we get

$$\begin{aligned}
F_5'(t) &= -\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int |\boldsymbol{\xi}_2(\mathbf{s})| \frac{\partial}{\partial\boldsymbol{\rho}}\left[\exp(-\mathbf{s}\boldsymbol{\rho})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right]\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx} \\
&\quad -\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int s\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx} \\
&= -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \int\limits_{\mathcal{K}} \frac{\partial}{\partial\rho}\left[\exp(-\mathbf{s}\boldsymbol{\rho})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\right]\mathbf{d}\boldsymbol{\rho}\,\mathbf{ds}\,\mathbf{dx} \\
&\quad -\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int s\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx} \\
&= -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \left[\exp(-\mathbf{s})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s}) - \boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{0},\mathbf{s})\right] dsdx \\
&\quad -\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int s\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\,\mathbf{d}\boldsymbol{\rho}\,\mathbf{dx},
\end{aligned}$$

Using the equality $\boldsymbol{\varpi}_t(\mathbf{x},\mathbf{0},\mathbf{s}) = \varphi_{tt}(t)$ and $-\exp(-\mathbf{s}\boldsymbol{\rho}) \leq -\exp(-\mathbf{s}) \leq -\exp(-\boldsymbol{\tau}_2)$ for all $0 \leq \rho \leq 1$ and $\tau_1 \leq s \leq \tau_2$, we get

$$\begin{aligned}
F_5'(t) &= -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \left[\exp(-\mathbf{s})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s}) - \varphi_{tt}^2(t)\right] \mathbf{dsdx} \\
&\quad -\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&= -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\exp(-\mathbf{s})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\varphi_{tt}^2(t)\,\mathbf{dsdx} \\
&\quad -\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\exp(-\mathbf{s}\boldsymbol{\rho})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\exp(-\mathbf{s})\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \boldsymbol{\xi}_1\int\limits_{\mathcal{J}} \varphi_{tt}^2(t)\,\mathbf{dx} \\
&\quad -\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\exp(-\mathbf{s})\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq -\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\exp(-\boldsymbol{\tau}_2)\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \boldsymbol{\xi}_1\int\limits_{\mathcal{J}} \varphi_{tt}^2(t)\,\mathbf{dx} \\
&\quad -\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\exp(-\boldsymbol{\tau}_2)\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq -\exp(-\boldsymbol{\tau}_2)\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \boldsymbol{\xi}_1\int\limits_{\mathcal{J}} \varphi_{tt}^2(t)\,\mathbf{dx} \\
&\quad -\exp(-\tau_2)\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} s\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\varpi_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq -m_1\iint\limits_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\mathbf{1},\mathbf{s})\,\mathbf{dsdx} + \boldsymbol{\xi}_1\int\limits_{\mathcal{J}} \varphi_{tt}^2(t)\,\mathbf{dx} \\
&\quad -m_1\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx},
\end{aligned}$$

where $m_1 = \exp(-\tau_2)$. ■

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E to prove the next theorem.

Theorem 3.1 *Assume that* (30) *holds, then there exist positive constants* ℓ_1 *and* ℓ_2 *such that the* *energy functional* (3.9) *satisfies*

$$E(t) \leq \ell_2 \exp(-\ell_1 t), \ \forall t \geq 0. \tag{3.24}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2(F_2(t) + F_3(t)) + N_4(F_4(t) + F_5(t)), \tag{3.25}$$

where N, N_1, N_2, $N_4 > 0$, by differentiating (3.25) and using (3.13), (3.15), (3.17), (3.19), (3.21) and (3.23), we have

$$\begin{aligned}
\mathcal{L}'(t) &= NE'(t) + N_1 F_1'(t) + N_2(F_2'(t) + F_3'(t)) + N_4(F_4'(t) + F_5'(t)) \\
&\leq -[N\eta_0 - N_2\rho_1 - N_4\xi_1] \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} \\
&\quad - \left[N\frac{\rho_2}{K}\eta_0 + N_2\frac{\rho_1\rho_2}{2K} - N_1 c\left(1 + \frac{1}{\varepsilon_1}\right) - N_4\xi_1\right] \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} \\
&\quad - [N\delta - cN_2] \int_{\mathcal{J}} q^2 \mathbf{dx} \\
&\quad - [N_1 K - N_2\rho_2] \int_{\mathcal{J}} \varphi_{\mathbf{x}t}^2 \mathbf{dx} \\
&\quad - \left[N_2\frac{b}{2} - N_1\varepsilon_1 - N_2\frac{b}{4}\right] \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\
&\quad - N_2\frac{K}{2} \int_{\mathcal{J}} (\varphi_{\mathbf{x}} + \psi)^2 \mathbf{dx} \\
&\quad - N_2\frac{\rho_3\kappa}{4} \int_{\mathcal{J}} \theta^2 \mathbf{dx} \\
&\quad - [N_4 m_1 - N_1 c - N_2 c] \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \varpi^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\quad - N_4 m_1 \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})| \varpi_t^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\quad - N_4 m_1 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \varpi^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\
&\quad - N_4 m_1 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \varpi_t^2(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx},
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{bN_2}{8N_1}.$$

Once N_2 is fixed, we then choose N_1 large enough such that

$$\gamma_4 = N_1 K - N_2 \rho_2 > 0.$$

Then we choose N_4 large enough so that

$$\begin{aligned}\gamma_5 &= m_1 N_4 > 0, \\ \gamma_6 &= N_4 m_1 - N_1 c - N_2 c > 0.\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq\ & -(N\eta_0 - c)\int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \left(N\frac{\rho_2}{K}\eta_0 + \gamma_0 - c\right)\int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} \\ & -\gamma_1 \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} - \gamma_2 \int_{\mathcal{J}} (\varphi_{\mathbf{x}} + \psi)^2 \mathbf{dx} - \gamma_3 \int_{\mathcal{J}} \theta^2 \mathbf{dx} \\ & -(N\eta_0 - c)\int_{\mathcal{J}} q^2 \mathbf{dx} \\ & -\gamma_4 \int_{\mathcal{J}} \varphi_{\mathbf{x}t}^2 \mathbf{dx} - \gamma_6 \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsdx} \\ & -\gamma_5 \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\xi}_2(\mathbf{s})|\, \boldsymbol{\varpi}_t^2(\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsdx} \\ & -\gamma_5 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\, |\boldsymbol{\xi}_2(\mathbf{s})|\, \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\ & -\gamma_5 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s}\, |\boldsymbol{\xi}_2(\mathbf{s})|\, \boldsymbol{\varpi}_t^2(\mathbf{x}, \mathbf{1}, \mathbf{s})\, \mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}, \end{aligned} \tag{3.26}$$

where $\gamma_0 = N_2 \frac{\rho_1 \rho_2}{2K}, \gamma_1 = N_2 \frac{b}{8}, \gamma_2 = N_2 \frac{K}{2}$ and $\gamma_3 = N_2 \frac{\rho_3 \kappa}{4}$ .On the other hand, if we let

$$\mathcal{Z}(t) = N_1 F_1(t) + N_2 (F_2(t) + F_3(t)) + N_4 (F_4(t) + F_5(t)),$$

then

$$
\begin{aligned}
|\mathcal{Z}(t)| &= |N_1 F_1(t) + N_2(F_2(t) + F_3(t)) + N_4(F_4(t) + F_5(t))| \\
&\leq N_1|F_1(t)| + N_2|F_2(t)| + N_2|F_3(t)| + N_4|F_4(t)| + N_4|F_5(t)| \\
&\leq N_1\frac{\xi_1}{2}\int\limits_{\mathcal{J}}\varphi_t^2\mathbf{dx} + N_1K\int\limits_{\mathcal{J}}|\varphi_{t\mathbf{x}}\varphi_{\mathbf{x}}|\,\mathbf{dx} \\
&\quad + N_2\rho_1\int\limits_{\mathcal{J}}|\varphi\varphi_t|\,\mathbf{dx} + N_2\frac{\xi_1}{2}\int\limits_{\mathcal{J}}\varphi^2\mathbf{dx} \\
&\quad + N_2\frac{\xi_1\rho_2}{2K}\int\limits_{\mathcal{J}}\varphi_t^2\mathbf{dx} + N_2\rho_2\int\limits_{\mathcal{J}}|\varphi_{\mathbf{x}}\varphi_{t\mathbf{x}}|\,\mathbf{dx} \\
&\quad + N_2\rho_3\tau_0\int\limits_{\mathcal{J}}\left|q\int_0^x\theta(y,t)\,dy\right|\mathbf{dx} + N_2\int\limits_{\mathcal{J}}|q\psi|\,\mathbf{dx} \\
&\quad + N_4\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\exp(-\mathbf{s}\boldsymbol{\rho})|\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx} \\
&\quad + N_4\iiint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\mathbf{s}\,|\exp(-\mathbf{s}\boldsymbol{\rho})|\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{dsd}\boldsymbol{\rho}\mathbf{dx}.
\end{aligned}
$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, we get

$$\begin{aligned}
|\mathcal{Z}(t)| &\leq \left[N_1\frac{\lambda_1}{2}+\frac{N_2\rho_1}{2}+N_2\frac{\lambda_1\rho_2}{2K}\right]\int_{\mathcal{J}}\varphi_t^2\mathbf{dx} \\
&+\left[\frac{N_1K}{2}+\frac{N_2\rho_2}{2}\right]\int_{\mathcal{J}}\varphi_{t\mathbf{x}}^2\mathbf{dx} \\
&+\frac{N_2c}{2}\int_{\mathcal{J}}\psi_{\mathbf{x}}^2\mathbf{dx}+\int_{\mathcal{J}}\varphi_{tt}^2\mathbf{dx} \\
&+\left[\frac{N_1K}{2}+\frac{N_2\rho_1c}{2}+N_2\frac{\lambda_1c}{2}+\frac{N_2\rho_2}{2}\right]\int_{\mathcal{J}}(\varphi_{\mathbf{x}}+\psi)^2\mathbf{dx} \\
&+\left(\frac{N_2\rho_3\tau_0}{2}+\frac{N_2}{2}\right)\int_{\mathcal{J}}q^2\mathbf{dx}+\frac{N_2\rho_3\tau_0C}{2}\int_{\mathcal{J}}\theta^2\mathbf{dx} \\
&+N_4\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&+N_4\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq c\int_{\mathcal{J}}\left[\varphi_t^2+\varphi_{tt}^2+\varphi_{tx}^2+\psi_x^2+(\varphi_x+\psi)^2+q^2+\theta^2\right]dx \\
&+c\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&+c\iint\limits_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}}\int \mathbf{s}\,|\boldsymbol{\xi}_2(\mathbf{s})|\,\boldsymbol{\varpi}_t^2(\mathbf{x},\boldsymbol{\rho},\mathbf{s})\,\mathbf{ds}\mathbf{d}\boldsymbol{\rho}\mathbf{dx} \\
&\leq cE(t).
\end{aligned}$$

Consequently

$$|\mathcal{Z}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t),$$

implies

$$-cE(t) \leq \mathcal{L}(t) - NE(t) \leq cE(t),$$

which yields

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (c+N)E(t).$$

Now, we choose N large enough so that

$$N\frac{\rho_2}{K}\eta_0 - c > 0, N\eta_0 - c > 0, N\delta - c > 0, N - c > 0,$$

we obtain

$$\eta_1 E(t) \leq \mathcal{L}(t) \leq \eta_2 E(t), \forall t \geq 0, \tag{3.27}$$

where η_1 and η_2 are positive constants.
Hence

$$\begin{aligned}
\mathcal{L}'(t) &\leq -\gamma_7 \int_{\mathcal{J}} \varphi_t^2 \mathbf{dx} - \gamma_8 \int_{\mathcal{J}} \varphi_{tt}^2 \mathbf{dx} - \gamma_1 \int_{\mathcal{J}} \psi_{\mathbf{x}}^2 \mathbf{dx} \\
&\quad -\gamma_2 \int_{\mathcal{J}} (\varphi_{\mathbf{x}} + \psi)^2 \mathbf{dx} - \gamma_4 \int_{\mathcal{J}} \varphi_{\mathbf{x}t}^2 \mathbf{dx} \\
&\quad -\gamma_9 \int_{\mathcal{J}} q^2 \mathbf{dx} - \gamma_3 \int_{\mathcal{J}} \theta^2 \mathbf{dx} \\
&\quad -\gamma_6 \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\lambda}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\quad -\gamma_5 \iint_{\mathcal{J}\times\mathcal{L}} |\boldsymbol{\lambda}_2(\mathbf{s})| \boldsymbol{\varpi}_t^2(\mathbf{x}, \mathbf{1}, \mathbf{s}) \mathbf{dsdx} \\
&\quad -\gamma_5 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\lambda}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{ds} \boldsymbol{d\rho} \mathbf{dx} \\
&\quad -\gamma_5 \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\lambda}_2(\mathbf{s})| \boldsymbol{\varpi}_t^2(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{ds} \boldsymbol{d\rho} \mathbf{dx} \\
&\leq -\eta \int_0^1 \left[\varphi_t^2 + \varphi_{tt}^2 + \varphi_{tx}^2 + \psi_x^2 + (\varphi_x + \psi)^2 + q^2 + \theta^2\right] dx \\
&\quad -\eta \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{ds} \boldsymbol{d\rho} \mathbf{dx} \\
&\quad -\eta \iiint_{\mathcal{J}\times\mathcal{K}\times\mathcal{L}} \mathbf{s} |\boldsymbol{\xi}_2(\mathbf{s})| \boldsymbol{\varpi}^2(\mathbf{x}, \boldsymbol{\rho}, \mathbf{s}) \mathbf{ds} \boldsymbol{d\rho} \mathbf{dx} \\
&\leq -\eta E(t), \ \forall t \geq 0,
\end{aligned} \tag{3.28}$$

where $\eta = \max(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9) > 0$.
A combination (3.27) with (3.28) gives,

$$\mathcal{L}'(t) \leq -\eta E(t) \leq -\frac{\eta}{\eta_2} \mathcal{L}(t).$$

We choose $h_1 = \frac{m}{\alpha_2}$, we get

$$\mathcal{L}'(t) \leq -\ell_1 \mathcal{L}(t), \ \forall t \geq 0. \tag{3.29}$$

A simple integration of (3.29) over $(0,t)$ and using (3.28), we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)\exp(-\ell_1 t), \ \forall t \geq 0,$$

implies

$$E(t) \leq \ell_2 \exp(-\ell_1 t), \ \forall t \geq 0,$$

because

$$\eta_1 E(t) \leq \mathcal{L}(t) \leq \mathcal{L}(0)\exp(-\ell_1 t), \ \forall t \geq 0,$$

where $\ell_2 = \frac{\mathcal{L}(0)}{\eta_1} \leq \frac{\eta_2}{\eta_1} E(0)$. ■

General Conclusion and Perspectives

In conclusion, this thesis focused on investigating the global well-posedness (the existence, uniqueness, Continuous dependence on the initial data) and exponential stability of solutions of a Bresse-Timoshenko type system with distributed delay and second sound. This system models a vibratory motion of a beam named the Bresse-Timoshenko beam under the effect of temperature (we demonstrate the existence of the solution and the type of stability of this beam).

To achieve the desired outcomes, the study commenced by examining the global well-posedness of the initial and boundary value problem under specific assumptions using Faedo-Galarkin approximations and energy estimates. Subsequently, the research explored the exponential stability, established through the Lyapunov functional and multiplier technique.

Looking ahead, thereis a keen interest in further exploring practical applications, including the utilization of numerical methods for approximating solutions and conducting numerical simulations for this particular type of problem.

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