UNIVERSITY OF TEBESSA
L'ARBI TÉBESSI
MATH AND COMPUTER SCIENCE

The maximum time of existence of solutions for the evolution problem

Master

PDE AND THEIR APPLICATION

HANNI DRIDI

MOHAMED MOUDDEB

Supervisor: Abdarrazak Nabti
Dr

Tebessa, may 2017

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MATH AND COMPUTER SCIENCE

Some Statement

And another one ...

(Signature)

....

Hanni Dridi Mohamed Mouddeb

We dedicate this work to our parents
our Brother and sisters
our families and friends
for the sake of knowledge.........

Acknowledgements

We would like first to than **Allah** for giving us strength and capacity to complete this work.

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Hanni Dridi
Mohamed Mouddeb

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Abstract

The aim of this research is to study Maximum time of existence of solution (life span) to some evolution problems. In order to obtain a lower bound estimate of life span of solution, we invistigated the local existence of our problems. Furthermore, to give the upper bound estimate of life span of the blowing up solution, we studied the global non existence, the estimate of the life span of the blow-up of solutions, and both the local and global existence of solutions, in addition to both the lower and upper bound estimates of the life span all together.
This research is based on the test function and energy methods ...

Keywords

Nonlinear parabolic equation, blow-up, lifespan, critical exponent, Semilinear damped wave equation, lifespan, upper bound,Higher-order parabolic equation, critical exponent; life span.

Résumé

Dans ce travail, nous étudions le temps Maximal d'existence de solutions (la durée de vie) à quelques problèmes d'évolution. Pour obtenir une évaluation inférieure de la durée de vie de la solution, nous examinons l'existence (locale et globale) de nos problèmes. En outre, pour donner une borne supérieure de la durée de vie de solution, nous étudions l'explosion de solutions.
Ce travail est basé sur la méthode de la fonction test et la méthode d'énergie ...

Mots clés

Équation parabolique non linéaire, explosion, exposant critique, équation semi-linéaire d'ondes amorties, limite supérieure, équation parabolique de rang supérieur...

Notation

- p_F: Fujita critical exponent .
- p_c: Critical point .
- $L(\sigma)$, T_ε, τ: Life span of solutions.
- q: The Hölder conjugate of p satisfying $p^{-1} + q^{-1} = 1$
- Ω: Open set in $\mathbb{R}^N$.
- $\partial\Omega$: Topological boundary of Ω.
- Ω^c: complementary of Ω.
- Σ_L: The space defined by $\{x \in \mathbb{R}^N; \langle x\rangle^{2-a} \leqslant L(1+t)^{1+\beta}\}$.
- Σ_L^c: The complementary space defined by $\mathbb{R}^N \backslash \Sigma_L$.
- $E(t)$: The energy equation.
- $\rho(x)$, $h(x,t)$,: Continuous functions.
- $\varphi(x)$, $\psi(t)$, $\zeta(x,t)$, $\Phi(x)$, $\phi(t)$, $\eta(t)$, $\psi(t,x)$: Test functions.
- $C(\Omega)$: Continuous functions taking value in the reals defined on Ω.
- $C^k(\Omega)$: Functions f such that $\partial^a f \in C(\Omega)$ for every $|a| \leq k$.
- L^p: Lebesgue spaces.
- $W^{k,p}$: Sobolev spaces.
- $W^{1,2} = H^1$: Hilbert and banach spaces.
- $\|\cdot\|_p$:L^p norm.
- $div\, u = \nabla \cdot u = \sum_{i=1}^{i=N} \partial_{x_i} u$: Divergence of u.
- $\Delta u = \sum_{i=1}^{i=N} \partial_{xx_i} u$: Laplacian of u.
- $\Delta^m u = \sum_{i=1}^{i=N} \partial_{xx_i}^m u$: Laplacian of order m.
- $Du = (\partial_t, \nabla u)$.: Derivative operator.
- $\vec{n}$: Unit outer normal vector of $\partial\Omega$.

In Mathematics the partial differential equations are an important branch. They are used in the modeling of many phenomena of different natures.
Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. For example, the system of partial differential equations known as **Maxwell's** equations can be written on the back of a post card, yet from these equations one can derive the entire theory of electricity and magnetism, including light. Our goal here is to develop the most basic ideas from the theory of partial differential equations, and apply them to the simplest models arising from physics. In particular, we will present some of the elegant mathematics that can be used to describe the heat transfer that happens under specific conditions, we will see that the waves of all the phenomena of vibration are essentially a problem for the equation of **Bessel's**.

The solutions of initial value problems for partial differetial equations may not exist for all time, in other words, these solutions may blow up in some sense or other. Recently in connection with problems for some class of quasi-linear parabolic equations **Kaplan, Ito** and **Friedman** gave certain sufficient conditions under which the solutions blow up in a finite time. Although their results are not identiacl, we can say according to them that the solutiions are apt to blow up when the initial values are sufficiently large. On the other hand, it is commonly believed that the dimension of the $x-space$, x being the space variable, has a crucial influence on the conditins for the solutions of quasi-linear equations to exist for all time. As an example we can refer to the **Navier-Stokes** equation, for which the situation concerning global existence is quite different according as the dimension of the $x-space$ is 2 or 3.
The work presented in this thesis deals with some equations for Partial rings of the hyperbolic type and others of the parabolic type. In the theory of nonlinear equations of evolution, a solution is called global If it is defined for any positive time. In contrast to that, if a solution exists only on a Time interval $[0, T]$, it is said to be local. In the latter case and when the maximum time Of existence is connected to an alternative of explosion, it is also said that the solution explodes in time finished. However, to make sense of the notion of explosion in finite time, The space in which we work and with what standard we "measure" the solution.

In the **first chapter** we considered the following problem :

$$\rho(x)u_t - \Delta u^m = h(x,t)u^{1+p}, \quad x \in \mathbb{R}^N, \ t > 0,$$

with nonnegative, nontrivial, continuous initial condition,

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0, \ x \in \mathbb{R}^N.$$

An integral inequality is obtained that can be used to find an exponent p_c such that this problem has no nontrivial global solution when $p \leq p_c$. This integral inequality may also

be used to estimate the maximal time of existence $T > 0$ such that there is a solution for $0 \leq t < T$.
This is illustrated for the case $\rho \equiv 1$ and $h \equiv 1$ with initial condition $u(x,0) = \sigma u_0(x)$, $\sigma > 0$, by obtaining a bound of the form $T \leq C_0 \sigma^{-\vartheta}$.

In the **second chapter** we investigated the higher-order semilinear parabolic equation:

$$u_t + (-\Delta)^m u = |u|^p, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N,$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N.$$

We used the test function method to derive the blow-up critical exponent. And then based on integral inequalities, we estimated the life span of a blow-up solutions.

In the **third chapter** we considered the critical exponent problem for the semilinear wave equation with space–time dependent damping. When the damping is effective, it is expected that the critical exponent agrees only with the space dependent coefficient case. We proved that there exists a unique global solution for small data if the power of nonlinearity is larger than the expected exponent. Moreover, we did not assume that the data are compactly supported. However, it is still open whether there exists a blow-up solution if the power of nonlinearity is smaller than the expected exponent.
Furthermore our concerns estimates of the life span of solutions to the semilinear damped wave equation

$$u_{tt} - \Delta u + \phi(t,x) u_t = |u|^p, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N,$$

with the initial condition

$$(u, u_t)(0,x) = (u_0, u_1)(x); \quad x \in \mathbb{R}^N,$$

where the coefficient of the damping term is $\phi = \langle x \rangle^{-\alpha}(1+t)^{-\beta}$. Our novelty is to prove both the upper bound and the lower bound of the lifespan of solutions in subcritical cases $1 < p < 2/(N-\alpha)$.

Finally, in the **fourth chapter**, we have studied the life span of solution to the problem evoked in the third chapter, starting from the above we found results on the lower bound and the upper bound of the existence time of solution which is confirmed by the results of the study of the blow up of solution.

I

Chapter 1

Chapter 1: Life span of nonnegative solutions to certain quasilinear parabolic Cauchy problems

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1.1 Introduction

In this chapter, we investigate the maximal interval of existence of solutions for the problem

$$\rho(x)u_t - \Delta u^m = h(x,t)u^{1+p}, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

with nonnegative, nontrivial, continuous initial condition,

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0, \ x \in \mathbb{R}^N. \tag{1.2}$$

Fujita [1] studied this problem for the case where $m = 1$, $\rho(x) \equiv 1$ and $h(x,t) \equiv 1$ In 1966. He obtained the following, by now famous, results. When $0 < p < 2/N$ the problem fails to have a nontrivial global solution. That is to say that the maximal interval of existence of any solution is finite. When $p > 2/N$ there exists a global solution if $u_0(x) \leq Ae^{-k|x|^2}$ for some constant $k > 0$ provided that A is sufficiently small. The critical case, $p = p_c := 2/N$, was studied by Hayakawa [2], Kobayashi [3] and Weissler [4], they showed that there does not exist a nontrivial, nonnegative global solution in case $p = p_c$. Fujita's work has been extended and generalized by many others. In particular, we should mention that Qi [5] studied the problem

$$u_t - \Delta u^m = |x|^{\varsigma} t^r u^{1+p}.$$

He found that the critical exponent for this problem is $p_c = (m-1)(r+1)+(2+2r+\varsigma)/N > 0$. More references can be found, for example, in articles of [6] and [7] that motivated this work. In the first of these, Guedda and Kirane reconfigured the test function method of Pohozaev *et al.* [8, 9] and were able to find the critical exponent for equations of the form 1.1 as well as others. The basic idea of the test function methods can be found as far back as in articles of Baras and Pierre [10] and Baras and Kersner [11]. In this chapter we will take the test function method, but reconfigured once again, in order to study the relationship between the size of the initial condition and the length of the maximal interval of existence. In doing this we will extend some of the results of Tzong-Yow Lee and Wei-Ming Ni [7], who obtained such information for Fujita's problem, i.e. for the case $m = 1$, $h \equiv 1$ and $\rho \equiv 1$. For example, we will show that if u is a global solution with initial condition $u(x,0) = u_0(x)$, then an inequality of the form

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\, dx \leq C\hat{\eta}^{\kappa}$$

must be satisfied. Here Φ is a positive eigenfunction corresponding to the principal eigenvalue of the Dirichlet problem on the unit ball B_1, and normalized such that $\int_{B_1} \Phi(\xi)\, d\xi = 1$. The numbers C and κ depend on N, m, p, h, and ρ. When $m = 1$, $h \equiv 1$, and $\rho \equiv 1$, then $C = 1$ and $\kappa = 1/p$, a result obtained in [7]. We also obtain a bound for the maximal interval of existence. Suppose u_σ is a solution corresponding to a nontrivial, nonnegative initial condition $u(x,0) = \sigma u_0(x)$. Let $[0, T_\sigma)$ be its maximal interval of existence. We obtain a bound of the form $T_\sigma \leq C\sigma^{-\vartheta}$. When $m \geq 1$, $h \equiv 1$, and $\rho \equiv 1$ then $\vartheta = p + 1 - m$.

1.2 Prelimineries

In this section, we present some preliminaries that will be used in the next sections.

Definition 1.1. *Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R}; \quad f \textit{ is measurable and } \quad |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L_p} = \|f\|_p = \left[\int_\Omega |f|^p \, d\upsilon\mu \right]^{1/p}.$$

We shall check later on that

$\|\cdot\|_p$ is a norm.

Definition 1.2. *We set*

$$L^\infty(\Omega) = \{ f : \Omega \longrightarrow \mathbb{R} \textit{ suchthat } \quad |f(x)| \leq C \textit{ on } \Omega \}.$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C; |f(x)| \leq C \textit{ on } \Omega \}.$$

The following remark implies that

$\|.\|_\infty$ is a norm:

Remark 1. *If $f \in L^\infty$ then we have*

$$|f(x)| \leq \|f\|_\infty \textit{ a.e.on } \Omega.$$

Indeed, there exists a sequence C_n such that $C_n \longrightarrow \|f\|_\infty$ and for each n, $|f(x)| \leq C_n$ a.e. on Ω. Therefore $|f(x)| \leq C_n$ for all $x \in \Omega_n$, with $|E_N| = 0$. We set $E = \cup_{n=1}^{\infty} E_n$, so that $|E| = 0$ and

$$|f(x)| \leq C_n \qquad \forall n, \quad \forall x \in \Omega;$$

it follows that $|f(x)| \leq \|f\|_\infty \qquad \forall x \in \Omega$.

Definition 1.3. *A function $f \in L^1_{loc}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f\partial_i\varphi \, dx = -\int_\Omega g_i\varphi \, dx for \, all \varphi \in C_c^\infty(\Omega).$$

The function g_i is called the weak it's partial derivative of f, and is denoted by $\partial_i f$. Thus, for weak derivatives, the integration by parts formula

$$\int_\Omega f\partial_i\varphi \, dx = -\int_\Omega \partial_i f\varphi \, dx$$

holds by definition for all $\varphi \in C_c^\infty(\Omega)$. Since C_c^∞ is dense in $L^1_{loc}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. The existence of a weak derivative is, however, not equivalent to the existence of a point wise derivative almost every where.

Definition 1.4. *Suppose that Ω is an open set in $\mathbb{R}^n$, $k \in N$, and $1 \leqslant p \leqslant \infty$. The Sobolev space $w^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \longrightarrow \mathbb{R}^n$ such that*

$$\partial^a f \in L^p(\Omega) for 0 \leqslant |a| \leqslant k.$$

We write $w^{k,2}(\Omega) = H^k(\Omega)$.
The Sobolev space $w^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{w^{k,p}(\Omega)} = \left(\sum_{|a|\leqslant k} \int_\Omega |\partial^a f|^p \, dx \right)^{1/p}$$

for $1 \leqslant p < \infty$ and

$$\|f\|_{w^{k,p}(\Omega)} = \max_{|a|\leqslant k} sup_\Omega |\partial^a f|.$$

Proposition 1.1. *If $f \in L^1_{loc}(\Omega)$ has weak partial derivative $\partial_i f \in L^1_{loc}$ and $\psi \in C^\infty$, then ψf is weakly differentiable with respect to x_i and*

$$\partial_i(\psi f) = (\partial_i\psi)f + \psi(\partial_i f).$$

Proof. Let $\varphi \in C_c^\infty(\Omega)$ be any test function .Then $\psi\varphi \in C_c^\infty$ and the weak differentiability of f implies that

$$\int_\Omega f\partial_i(\psi\varphi) \, dx = -\int_\Omega (\partial_i f)\psi\varphi \, dx.$$

Expanding $\partial_i(\psi\varphi) = \psi(\partial_i\varphi) + (\partial_i\psi)\varphi$ in this equation and rearranging the result,we get

$$\int_\Omega \psi f(\partial_i\varphi) \, dx = -\int_\Omega [(\partial_i\psi)f + \psi(\partial_i f)]\varphi \, dx$$

Thus, ψf is weakly differentiable and its weak derivative. □

Lemma 1.1. *(Young inequality)*
Let $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{p}, \quad a, b > 0.$$

Proof. The mapping $x \longmapsto e^x$ is convex ,and consequently,

$$ab = e^{loga+logb} = e^{\frac{1}{p}loga^p + \frac{1}{q}logb^q}$$
$$\leqslant \frac{1}{p}e^{loga^p} + \frac{1}{q}e^{logb^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Lemma 1.2. *(Young inequality with ε)*
Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leqslant \varepsilon\frac{a^p}{p} + \frac{1}{\varepsilon^{q/p}}\frac{b^q}{q}, \quad a, b > 0.$$

Lemma 1.3. *(Hölder inequality)*
Let $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then if $u \in L^p(\Omega)$, we have

$$\int_\Omega |uv|\, dx \leqslant \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

Proof.
- In the cases where $p = \infty$ or $q = \infty$, it is easy, because there exists a subset $\Omega' \subset \Omega$, with $|\Omega'| = |\Omega|$, such that $\sup_{\Omega'} |u| = \|u\|_{L^\infty(\Omega)}$ or $\sup_{\Omega'} |v| = \|v\|_{L^\infty(\Omega)}$.
- In the cases where $1 < p, q < \infty$. By the homogeneity of the inequality, we may assume that $\|u\|_{L^p(\Omega)} = \|v\|_{L^q(\Omega)} = 1$. Then the Young inequality implies that

$$\int_\Omega |uv|dx \leqslant \frac{1}{p}\int_\Omega |u|^p dx + \frac{1}{q}\int_\Omega |u|^q dx = 1 = \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

□

Definition 1.5. *The hypothesis (HC), which appears in all the applications of the flux-divergence theorem, is that at least one of the objects considered is compact. This will not detailed every time, we give only two examples:*

In the identity $\int_\Omega divFd\mathcal{H}_n = \int_{\partial\Omega} v.Fd\mathbb{H}^{n-1}$, We suppose either F with compact support, or Ω Relatively compact.

In the identity $\int_\Omega(\partial_j g)d\mathcal{H}_n = \int_{\partial\Omega} v_j(fg)d\mathbb{H}^{n-1} - \int_\Omega(\partial_j f)gd\mathcal{H}_n$, We assume: one of the u and v with compact support, or Ω relatively compact.

Theorem 1.1. *(Théorème du flux-divergence)*
Ω is an open Lipschitz.$F \in C^1(\overline{\Omega}; \mathbb{R}^N)$. We have (HC). Conclusion. We have

$$\int_\Omega divFd\mathcal{H}_n = \int_{\partial\Omega} v.Fd\mathbb{H}^{n-1}.$$

Corollary 1.1. *(Integration by parts)*

- *Hypotheses.* $f \in C^1(\Omega)$. $g \in C^1_c(\Omega)$.
  Conclusion.
$$\int_\Omega f(\partial_j g) = -\int_\Omega (\partial_j f) g.$$
- *Hypotheses.* $f, g \in C^1(\overline{\Omega})$, Ω *Lipschitz. We have (HC).*
  Conclusion.
$$\int_\Omega f(\partial_j g) d\mathcal{I}_n = \int_{\partial\Omega} (\nu_j f g) d\mathbb{H}^{n-1} - \int_\Omega (\partial_j f) g d\mathcal{I}_n.$$

Theorem 1.2. *(Green Formulas)*

Hypotheses. $\Omega \subset \mathbb{R}^N$ *is Lipschitz.* $u, v \in C^2(\overline{\Omega})$. *At least one of the sets* Ω, *suppu and suppv is relatively compact.*
Conclusions. We have

- *Green's first formula*
$$\int_\Omega u\Delta v = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - \int_\Omega \nabla u . \nabla v.$$
- *Green's second formula*
$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} \right).$$

Proof. By integrating by parts colrollary 1.1 ,we have
$$\int_\Omega u\partial_{ii} v = \int_{\partial\Omega} u\nu_i \partial_i v - \int_\Omega \partial_i u \partial_i v.$$
By summing on i, is obtained by subtracting from first formula the identity obtained by exchanging u and v in first formula . □

Corollary 1.2. *Hypotheses.* $\Omega \subset \mathbb{R}^N$, $\omega \subset \Omega$ *is an open Lipschitz.* u *is harmonic in* Ω.
Conclusion. We have
$$\int_{\partial\omega} \frac{\partial u}{\partial \nu} = 0.$$
Proof. Take $v = 1$ in the first formula of Green. □

Theorem 1.3. *(Maximum principles)*

Hypotheses. $\Omega \subset \mathbb{R}^N$ *field.* u *subharmonic in* Ω. u *has a maximum point.*
Conclusion.u *constant .*

Proof. Let $M = \max u$ and $F = \{x \in \Omega; u(x) = M\}$.$\Omega$ Being connected and F being closed in Ω not empty, it suffices to show that F is open.Let $x_0 \in F$. Let $0 < R < dist(x_0, \partial\Omega)$.
So
$$M = u(x_0) \leq \int_{B(x_1,R)} u(x)\, dx \leq M.$$
We find $u = M$ in $B(x_1, R)$, and therefore $B(x_1, R) \subset F$.

□

Theorem 1.4. *(lemma of hopf)*

Hypotheses. *B a ball.* $x_0 \in \partial B$. *u over-harmonic in B.* $u \in C^1(\overline{B}).u > u(x_0)$ *in B.* ν *the Normal outside B at* x_0.
Conclusion. $\frac{\partial u}{\partial \nu}(x_0) < 0$.

1.3 The test function method

In this section, we will use the test function methode to show the non existence result of global solution. Suppose that u is a solution of 1.1-1.2 on $\mathbb{R}^N \times [0, t_*)$.
Let $B_R := \{x \in \mathbb{R}^N \ : \ |x| < R\}$. We assume that

$$0 < m < p + 1,$$

and that there exists a continuous function h_0 defined on $B_1 \times [0, \infty)$, and real constants β and $\mu \geq 0$ such that for each $T > 0$ and $R > R_0$ we have

$$h(R\xi, R^\beta \tau) \geq R^\mu h_0(\xi, \tau) \quad \forall \xi \in B_1, \ \forall \tau \in [0, T], \tag{1.3}$$

where

$$\int_0^T \int_{B_1} h_0(\xi, \tau)^{-a} \, d\xi \, d\tau < \infty,$$

for $a = 1/p$ and for $a = m/(p + 1 - m)$. The simplest examples of functions satisfying these hypotheses are those of the form $h(x, t) = A|x|^\varsigma t^r$ where A is a positive constant and ς and r are sufficiently small: $\varsigma < Np$, $\varsigma < N(p + 1 - m)/m$, $r < p$, and $r < (p + 1 - m)/m$.

We assume that there exists a continuous function ρ_0 defined on $B_1 \times [0, \infty)$, and a positive constant ω such that for each $R > R_0$, we have

$$\rho(R\xi) \leq R^\omega \rho_0(\xi), \quad \forall \xi \in B_1, \tag{1.4}$$

where

$$\int_{B_1} \rho_0(\xi)^{(p+1)/p} \, d\xi < \infty.$$

Let λ_R be the principal eigenvalue for the Dirichlet problem on the ball of radius R:

$$-\Delta\Phi(x) = \lambda\Phi(x), \quad x \in B_R,$$
$$\Phi(x) = 0 \quad x \in \partial B_R.$$

We note that $\lambda_R = \lambda_1/R^2$. Let Φ denote the unique nonnegative eigenfunction corresponding to the principal eigenvalue λ_1 such that

$$\int_{B_1} \Phi(x) \, dx = 1.$$

Of course Φ is radially symmetric: $\Phi(x) = \Phi_0(|x|)$.

For $0 \leq S < T$, we define

$$\psi(t) := \begin{cases} 1 & \text{if } t < S, \\ (1-(t-S)/(T-S))^{\partial} & \text{if } S \leq t \leq T, \\ 0 & \text{if } t > T. \end{cases}$$

We also define

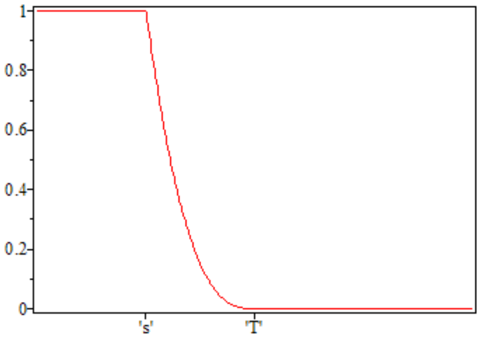

Figure 1.1: *Test function.*

$$\zeta(x,t) := \psi(t/R^{\beta})\Phi(x/R),$$

and, for $TR^{\beta} < t_*$,

$$J_R(S,T) := \int_{SR^{\beta}}^{TR^{\beta}} \int_{B_R} h(x,t)u^{(1+p)}\zeta(x,t)\,dx\,dt.$$

Using 1.1 and 1.2 and integration by parts and we have

$$\begin{aligned} J_R(0,T) &= \int_0^{TR^{\beta}} \int_{B_R} [\rho(x)u_t - \Delta u^m]\psi(t/R^{\beta})\Phi(x/R)\,dx\,dt \\ &= -\int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx - \int_0^{TR^{\beta}} \int_{B_R} R^{-\beta}u\rho\psi'(t/R^{\beta})\Phi(x/R)\,dx\,dt \\ &\quad + \int_0^{TR^{\beta}} \int_{\partial B_R} [-\frac{\partial u^m}{\partial \upsilon}\psi(t/R^{\beta})\Phi(x/R) + u^m\psi(t/R^{\beta})R^{-1}\Phi_0'(|x|/R)]\,dS\,dt \\ &\quad + \int_0^{TR^{\beta}} \int_{B_R} u^m\psi(t/R^{\beta})R^{-2}\hbar_1\Phi(x/R)\,dx\,dt\,. \end{aligned}$$

Note that by the Maximum Principle 1.3, u cannot attain the value zero in $\mathbb{R}^N \times (0,\infty)$ and consequently the surface integral must be negative. Using the notations

$$V_R := \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx,$$

by Hopf lemma 1.4, we have

$$\int_0^{TR^{\beta}} \int_{\partial B_R} [-\frac{\partial u^m}{\partial \upsilon}\psi(t/R^{\beta})\Phi(x/R) + u^m\psi(t/R^{\beta})R^{-1}\Phi_0'(|x|/R)]\,dS\,dt < 0.$$

Since $\psi'(t) = 0$ except on (S, T),using the Hölder inequality 1.3, we have

$$\begin{aligned}
&J_R(0,T) + V_R \\
&< + \int_{SR^\beta}^{TR^\beta} \int_{B_R} u[h\psi(t/R^\beta)\Phi(x/R)]^{\frac{1}{p+1}} \rho R^{-\beta} \\
&\quad \times [-\psi'(t/R^\beta)\psi(x/R^\beta)^{-\frac{1}{p+1}}] h^{-\frac{1}{p+1}} \Phi(x/R)^{\frac{p}{p+1}}\, dx\, dt \\
&\quad + \int_0^{TR^\beta} \int_{B_R} u^m [h\psi(t/R^\beta)\Phi(x/R)]^{\frac{m}{p+1}} R^{-2} \hbar_1 \\
&\quad \times h^{-\frac{m}{p+1}} \psi(t/R^\beta)^{\frac{p+1-m}{p+1}} \Phi(x/R)^{\frac{p+1-m}{p+1}}\, dx\, dt \\
&\le + J_R(S,T)^{\frac{1}{p+1}} R^{-\beta} \Big[\int_{SR^\beta}^{TR^\beta} \int_{B_R} \rho^{\frac{p+1}{p}} \\
&\quad \times \Big[[-\psi'(t/R^\beta)]^{\frac{p+1}{p}} \psi(x/R^\beta)^{-1/p} \Big] h^{-\frac{1}{p}} \Phi(x/R)\, dx\, dt \Big]^{p/(p+1)} \\
&\quad + J_R(0,T)^{\frac{m}{p+1}} \hbar R^{-2} \Big[\int_0^{TR^\beta} \int_{B_R} h^{-\frac{m}{p+1-m}} \psi(t/R^\beta)\Phi(x/R)\, dx\, dt \Big]^{\frac{p+1-m}{p+1}}.
\end{aligned}$$

Making the change of variables $\xi = x/R$ and $\tau = t/R^\beta$, and using 1.3 and 1.4, we have

$$\begin{aligned}
&J_R(0,T) + V_R \\
&< J_R(S,T)^{\frac{1}{p+1}} R^{s_1} \\
&\quad \times \Big[\int_S^T \int_{B_1} \rho_0(\xi)^{\frac{p+1}{p}} (-\psi'(\tau))^{\frac{p+1}{p}} \psi(\tau)^{-1/p} h_0(\xi,\tau)^{-1/p} \Phi(\xi)\, d\xi\, d\tau \Big]^{p/(p+1)} \\
&\quad + J_R(0,T)^{\frac{m}{p+1}} \hbar R^{s_2} \Big[\int_0^T \int_{B_1} h_0(\xi,\tau)^{-\frac{m}{p+1-m}} \psi(\tau)\Phi(\xi)\, d\xi\, d\tau \Big]^{\frac{p+1-m}{p+1}},
\end{aligned}$$

where

$$s_1 := \omega + \frac{Np - \mu - \beta}{p+1}, \quad s_2 := -2 + N + \beta - \frac{(N+\beta+\mu)m}{p+1}.$$

Defining

$$A(S,T) := \int_S^T \int_{B_1} \rho_0(\xi)^{\frac{p+1}{p}} (-\psi'(\tau))^{\frac{p+1}{p}} \psi(\tau)^{-1/p} h_0(\xi,\tau)^{-1/p} \Phi(\xi)\, d\xi\, d\tau,$$

$$B(T) := \hbar \int_0^T \int_{B_1} h_0(\xi,\tau)^{-\frac{m}{p+1-m}} \psi(\tau)\Phi(\xi)\, d\xi\, d\tau,$$

for $R > R_0$, we have

$$J_R(0,T) + V_R < J_R(S,T)^{\frac{1}{p+1}} R^{s_1} A(S,T)^{\frac{p}{p+1}} + J_R(0,T)^{\frac{m}{p+1}} \hbar R^{s_2} B(T)^{\frac{p+1-m}{p+1}}. \tag{1.5}$$

Next we choose β such that $s_1 = s_2$:

$$\beta := \frac{(p+1)(\omega+2) + (m-1)(\mu+N)}{p+2-m}, \tag{1.6}$$

so that $s_1 = s_2 = s$ where

$$s := \frac{(N+\omega)(p+1-m)-\mu-2}{p+2-m}. \tag{1.7}$$

It is our objective to use 1.5 to obtain information on the relationship between the initial condition $u_0(x)$ and the length of the maximum interval of existence. However, it does also provide a proof to the following result:

Theorem 1.5. *If $s \leq 0$, that is to say*

$$p \leq p_c := m - 1 + \frac{2+\mu}{N+\omega},$$

then the problems 1.1-1.2 has no global solution except for $u \equiv 0$.

Proof. When $s < 0$ we take the limit as R tends to infinity on both sides of 1.5 and obtain

$$\int_0^\infty \int_{\mathbb{R}^N} h(x,t)u^{(1+p)}\zeta(x,t)\,dx\,dt + \int_{\mathbb{R}^N} \rho(x)u_0(x)\Phi(0)\,dx = 0, \tag{1.8}$$

so that $u \equiv 0$ is the only global solution.
If $s = 0$ we first note that $J_R(0,T)$ is uniformly bounded for all R. This means that we can make $J_R(S,T)$ arbitrarily small by choosing S large enough and hence we can make the first term on the right hand side of 1.5 arbitrarily small, provided we keep $T-S$ bounded. Next we can make the second term arbitrarily small by making $|T-S|$ sufficiently small. Once again we have 1.8. □

It should be noted that the choice of β depends on the value of μ and that these quantities are already related by hypothesis 1.3. This means, that in order to apply this result one needs to compute μ and β simultaneously. We illustrate this with the following example.

Example

Suppose that $h(x,t) = |x|^{\varsigma}t^r$, where we assume that $p \neq p_* := (r+1)*(m-1)-1$. Then $\mu = \varsigma + r\beta$. Solving this equation and equation 1.6 simultaneously for β and μ we obtain

$$\mu = \frac{(p+1)(\omega r + 2r + \varsigma) + (m-1)(Nr - \varsigma)}{p+1+(r+1)(1-m)},$$
$$\beta = \frac{(\omega+2)(p+1) + (m-1)(N+\varsigma)}{p+1+(r+1)(1-m)}.$$

We also compute

$$s = \frac{(N+\omega)(p - rm + 1 - m) + rN - 2r - 2 - \varsigma}{p+1+(r+1)(1-m)}.$$

We may solve the above equation for p when $s = 0$ in order to see that the critical exponent is

$$p_c = (m + rm - 1) + \frac{-rN + 2 + \varsigma + 2r}{N+\omega},$$

which agrees with the result in [5] when $\omega = 0$. Since $p_c > p_*$, the restriction $p \neq p_*$ does not affect the determination of the critical exponent.

1.4 Life span of blowing-up solution

For the rest of this chapter, we assume that $S = 0$ and that the value of β is given by 1.6. Suppressing arguments and subscripts 1.5 becomes

$$J + V < J^{\frac{1}{p+1}} R^s A^{\frac{p}{p+1}} + J^{\frac{m}{p+1}} \hat{\jmath} R^s B^{\frac{p+1-m}{p+1}}. \tag{1.9}$$

We will use this to obtain an estimate for V. First we give the following two lemmas.

Lemma 1.4. *Suppose that a, b, r, and q are positive constants.Define the functions $F(x) := ax^q - bx^r$, $G(x) := ax^{-q} + bx^r$ on $0 < x < \infty$. Then*

$$\max_{x>0} F(x) = (1 - q/r)a^{\frac{r}{r-q}}(\frac{q}{br})^{\frac{q}{r-q}},$$
$$\min_{x>0} G(x) = (1 + q/r)a^{\frac{r}{r+q}}(\frac{br}{q})^{\frac{q}{r+q}}.$$

Lemma 1.5. *Let $0 < \omega_1, \omega_2 < 1$, $\omega_1 \neq \omega_2$. On $[0, \infty)$ define*

$$\Upsilon(x) := \max(x^{\omega_1}, x^{\omega_2}).$$

Let η be an arbitrary positive number, then

$$\Psi(\omega_1, \omega_2; \eta) := \max_x(\eta\Upsilon(x) - x) = \max_i \left((1 - \omega_i)\omega_i^{\frac{\omega_i}{1-\omega_i}} \eta^{\frac{1}{1-\omega_i}}\right).$$

For η sufficiently large

$$\Psi(\omega_1, \omega_2; \eta) = (1 - \overline{\omega})\overline{\omega}^{\frac{\overline{\omega}}{1-\overline{\omega}}} \eta^{\frac{1}{1-\overline{\omega}}}, \tag{1.10}$$

where $\overline{\omega} = \max(\omega_1, \omega_2)$.

Proof. The function $\eta\Upsilon(x) - x$ has at most three critical points: the cusp at $x = 1$ and the points where the functions $\eta x^{\omega_1} - x$ and $\eta x^{\omega_2} - x$ attain their maxima. It is easy to see that $\eta\Upsilon(x) - x$ cannot attain its maximum at the cusp. Applying the previous lemma, we see that the maximum value of $\eta\Upsilon(x) - x$ must be the larger of the two values

$$(1 - \omega_i)\omega_i^{\frac{\omega_i}{1-\omega_i}} \eta^{\frac{1}{1-\omega_i}}.$$

The last assertion is obvious. □

We will use the notation $\overline{m} := \max(1, m)$ and

$$J_{\overline{m}} := (1 - \overline{m}/(p + 1))(\frac{\overline{m}}{p + 1})^{\frac{\overline{m}}{p+1-\overline{m}}}.$$

Then, for η sufficiently large

$$\Psi(\frac{1}{p+1},\frac{m}{p+1},\eta) = J_{\overline{m}}\,\eta^{\frac{p+1}{p+1-\overline{m}}}.$$

Theorem 1.6. *If u is a nonnegative solution of 1.1-1.2 on $B_{R_*}\times[0,t_*)$, s is given by 1.7. Let*

$$A(T) := \int_0^T\int_{B_1}\rho_0(\xi)^{\frac{p+1}{p}}(-\psi'(\tau))^{\frac{p+1}{p}}\psi(\tau)^{-1/p}h_0(\xi,\tau)^{-1/p}\Phi(\xi)\,d\xi\,d\tau,$$
$$B(T) := \int_0^T\int_{B_1}h_0(\xi,\tau)^{-\frac{m}{p+1-m}}\psi(\tau)\Phi(\xi)\,d\xi\,d\tau.$$

Then for all $(R,T)\in\{(\rho,\tau)\ :\ R_0\le\rho\le R_*,\ 0\le\tau\le t_*\rho^{-\beta}\}$, we have

$$\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx < \Psi(\frac{1}{p+1},\frac{m}{p+1};([A(T)^{\frac{p}{p+1}}+\mathcal{J}B(T)^{\frac{p+1-m}{p+1}}]R^s). \tag{1.11}$$

In particular, if u is a global nonnegative solution then

$$\limsup_{R\to\infty} R^{-S}\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx \le J_{\overline{m}}\inf_T\left[A(T)^{\frac{p}{p+1}}+\mathcal{J}B(T)^{\frac{p+1-m}{p+1}}\right]^{\frac{p+1}{p+1-\overline{m}}}, \tag{1.12}$$

where

$$S := \frac{s(p+1)}{p+1-\overline{m}} = \frac{(p+1)[(N+\omega)(p+1-m)-\mu-2]}{(p+1-\overline{m})(p+2-m)}.$$

Proof. For the sake of convenience we define

$$\Theta(T) = A(T)^{\frac{p}{p+1}}+\mathcal{J}B(T)^{\frac{p+1-m}{p+1}}.$$

From 1.9, we see that $V\le\Upsilon(J)\Theta(T)R^s-J$, where

$$\Upsilon(\sigma) := \max\{\sigma^{\frac{1}{p+1}},\sigma^{\frac{m}{p+1}}\}.$$

Then by Lemma 1.5, we have 1.11. For R sufficiently large we can use equation 1.10 to conclude the validity of 1.12. □

Corollary 1.3. *Suppose that there exist positive constants ρ_c and h_c such that for $R>R_0$,*

$$h(R\xi,R^\beta\tau)\ge h_cR^\mu,\quad and\quad \rho(R\xi)\le\rho_cR^\omega,$$

where β is given by 1.6. Suppose that u is a nonnegative global solution. Then

$$\limsup_{R\to\infty}R^{-S}\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx\le J_{\overline{m}}\,K_m^{\frac{p+1}{p+1-\overline{m}}}\,\mathcal{J}^{\frac{p+1}{(p+2-m)(p+1-\overline{m})}}$$

where

$$K_m := (p+2-m)\Big(\frac{\rho_c^{(p+1-m)}}{(p+1-m)^{(p+1-m)}h_c}\Big)^{1/(p+2-m)}. \tag{1.13}$$

Proof. We easily obtain

$$A(T) \le A_0 \equiv \frac{\rho_c^{\frac{p+1}{p}} h_c^{-\frac{1}{p}} \partial^{\frac{p+1}{p}}}{(\partial - 1/p) T^{\frac{1}{p}}}, \quad \text{and} \quad B(T) \le B_0 \equiv \frac{h_c^{-\frac{m}{p+1-m}} T}{\partial + 1}.$$

Then

$$V < R^s (J^{\frac{1}{p+1}} A_0^{\frac{p}{p+1}} + J^{\frac{m}{p+1}} \mathcal{J} B_0^{\frac{p+1-m}{p+1}}) - J \le R^s \Theta_0(T) \Upsilon(J) - J,$$

where

$$\Theta(T) \le \Theta_0(T) := a_0 T^{-\frac{1}{p+1}} + \beta_0 T^{\frac{p+1-m}{p+1}},$$

with

$$a_0 := \frac{\rho_c h_c^{-1/(p+1)} \partial}{(\partial - 1/p)^{p/(p+1)}}, \quad \beta_0 = \frac{\mathcal{J} h_c^{-m/(p+1)}}{(\partial + 1)^{(p+1-m)/(p+1)}}.$$

By lemma 1.4

$$\begin{aligned}
\Theta_{00} &:= \min(\Theta_0(T)) \\
&= \left[(p+1-m)^{-1} a_0\right]^{(p+1-m)/(p+2-m)} \beta_0^{1/(p+2-m)} [p+2-m] \\
&= \frac{(p+2-m)(p+1-m)^{-\frac{p+1-m}{p+2-m}} \rho_c^{\frac{p+1-m}{p+2-m}} h_c^{-\frac{1}{(p+2-m)}} \mathcal{J}^{\frac{1}{p+2-m}} \partial^{\frac{p+1-m}{p+2-m}}}{(\partial - 1/p)^{p(p+1-m)/[(p+1)(p+2-m)]} [\partial + 1]^{(p+1-m)/[(p+1)(p+2-m)]}}.
\end{aligned}$$

Taking the limit as $\partial \to \infty$ we have $\lim_{\partial \to \infty} \Theta_{00} = K_m \mathcal{J}^{1/(p+2-m)}$. Then after substituting this into equation 1.12, the proof is complete. □

When we are dealing with the problem originally considered by Fujita ($\rho \equiv \rho_0 \equiv \rho_c \equiv 1$, $h \equiv h_0 \equiv h_c \equiv 1$, and $m = 1$), then $J_{\overline{m}} = p(p+1)^{-(p+1)/p}$ and $K_m = p^{-1}(p+1)^{(p+1)/p}$ and we see that the above inequality reduces to

$$\limsup_{R \to \infty} R^{-N+2/p} \int_{B_R} \rho(x) u_0(x) \Phi(x/R)\, dx \le \mathcal{J}^{1/p}. \tag{1.14}$$

This is precisely the result found in [7]. As done in that article we can deduce the following result.

Corollary 1.4. *When* $N \ge S$, *1.5 and 1.3 remain valid if we replace*

$$\limsup_{R \to \infty} R^{-S} \int_{B_R} \rho(x) u_0(x) \Phi(x/R)\, dx$$

by $\liminf_{|x| \to \infty} |x|^{N-S} \rho(x) u_0(x)$.

Proof. The statement of this corollary follows from the inequalities:

$$\begin{aligned}
&\lim_{R\to\infty} R^{-S}\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx\\
&\geq \lim_{R\to\infty} R^{-S}\int_{B_R\setminus B_k}\inf_{R\geq|x|\geq k}(|x|^{N-S}\rho(x)u_0(x))R^{S-N}\Phi(x/R)\,dx\\
&\geq \lim_{R\to\infty}\inf_{R\geq|x|\geq k}(|x|^{N-S}\rho(x)u_0(x))\int_{B_R\setminus B_k} R^{-N}\Phi(x/R)\,dx\\
&= \lim_{R\to\infty}\inf_{R\geq|x|\geq k}(|x|^{N-S}\rho(x)u_0(x))\int_{B_1\setminus B_{k/R}}\Phi(\xi)\,d\xi\\
&= \inf_{|x|\geq k}(|x|^{N-S}\rho(x)u_0(x)).
\end{aligned}$$

The proof is complete by letting k tend to infinity. □

Inequality 1.11 can also be used to obtain an upper bound for the length of the maximal interval of existence. Consider problem 1.1-1.2. By the *life span* for initial condition u_0, we mean the least upper bound of all values T such that $[0,T)$ is a maximal interval of existence of a solution to 1.1-1.2. Let us fix u_0, $u_0\not\equiv 0$ and $u_0(x)\geq 0$ for all $x\in\mathbb{R}^N$. We denote by $L(\sigma)$, $\sigma>0$, the life span corresponding to initial condition σu_0. Assume the hypotheses of 2.8 are satisfied, then there exists a value Λ such that

$$\Lambda V_R = \Psi(R^s\Theta(T_M)),$$

where T_M is the value of T at which $\Theta(T)$ attains its minimum value. Let Θ_L denote the restriction of Θ to the interval $[0,T_M)$. If we take $\sigma>\Lambda$, then $L(\sigma)<\infty$ and we see from 1.12 that

$$L(\sigma)\leq R^\beta\Theta_L^{-1}\left(R^{-s}\Psi^{-1}(\sigma V_R)\right). \tag{1.15}$$

In the next result we use this inequality to obtain an explicit upper bound for the life span of a solution.

Theorem 1.7. *Assume the hypotheses of Corollary 1.3 Let u_0 be a nonnegative nontrivial continuous function on $\mathbb{R}^N$. There exist positive numbers Λ_m, C_1 and σ_1 so that the life span $L(\sigma)$ corresponding to the initial condition σu_0 with $\sigma>\Lambda_m$ satisfies*

$$L(\sigma)\leq C_1\sigma^{-(p+1-\overline{m})}. \tag{1.16}$$

Proof. Decreasing the value of T_M to a value T_m if needed, we may assume that the function Θ_0, introduced above, is decreasing on $(0,T_m)$. We can choose Λ_m such that $\Lambda_m V_R\geq\Psi(R^s\Theta(T_m))$ and also so that $\Lambda_m V_R\geq C_3$ where C_3 is a sufficiently large constant so that whenever $\sigma>\Lambda_m$ then

$$\Psi^{-1}(\sigma V_R) = [(1-\overline{\omega})^{-1}\overline{\omega}^{-\frac{\overline{\omega}}{1-\overline{\omega}}}]^{1-\overline{\omega}}(\sigma V_R)^{\frac{p+1-\overline{m}}{p+1}},$$

with $\overline{\omega}=\overline{m}/(p+1)$. We write

$$\Psi^{-1}(\sigma V_R) = \gamma_0 V_R^{\frac{p+1-\overline{m}}{p+1}}\,\sigma^{\frac{p+1-\overline{m}}{p+1}},$$

where $\gamma_0 := (p+1)(p+1-\overline{m})^{-\frac{p+1-\overline{m}}{p+1}}\overline{m}^{-\frac{\overline{m}}{p+1}}$. Since

$$\Theta(T) \le \Theta_0(T) \le \alpha_0 T^{-\frac{1}{p+1}} + \beta_0 T_m^{\frac{p+1-m}{p+1}}$$

on $[0, T_m)$, it follows that

$$\Theta_L^{-1}(\eta) \le \left[\frac{\eta - \beta_0 T_m^{\frac{p+1-m}{p+1}}}{\alpha_0}\right]^{-(p+1)}, \quad \text{for } \eta > \beta_0 T_m^{\frac{p+1-m}{p+1}}.$$

Let $[0, T_\infty)$ be the maximal interval of existence of u and let $T = \tau R^{-\beta}$ where $0 < \tau < T_\infty$). We define

$$G(R,\sigma) := R^\beta \alpha_0^{p+1}\left[\gamma_0 R^{-s} V_R^{\frac{p+1-\overline{m}}{p+1}} \sigma^{\frac{p+1-\overline{m}}{p+1}} - \delta_0\right]^{-(p+1)},$$

where $\delta_0 := \beta_0 T_m^{\frac{p+1-m}{p+1}}$. Whenever $\tau < L(\sigma)$ we have $\tau \le G(R,\sigma)$. Therefore

$$L(\sigma) \le G(R,\sigma). \tag{1.17}$$

It is easily seen that this implies inequality 1.16. □

Inequality 1.17 must be satisfied for all $R > R_0$, However, because the domains depend on R we cannot improve our bound by merely taking the infimum over all $R \ge R_0$. Nevertheless, it is sometimes possible to do so by finding the envelope of the curves $\tau = G(R,\sigma)$. We illustrate this in the next section.

1.5 Application of results to the problem $u_t = \Delta u^m + u^{p+1}$

Suppose that $m \ge 1$, $\rho \equiv 1$, $h \equiv 1$, and for some nonnegative constant δ, $|x|^{-\delta}u_0$ is bounded from below by a positive constant. Let u_σ be a solution of 1.1 with initial condition $u_\sigma(x,0) = \sigma u_0(x)$. In this case

$$\beta = \frac{2(p+1) + N(m-1)}{p+2-m}, \quad s = \frac{N(p+1-m)-2}{p+2-m}.$$

We could substitute these values into 1.17, obtain $G(R,\sigma)$, and then find an envelope for the R-parameterized curves $y = G(R,\sigma)$. However, the R-dependence of the domains and the fact that Ψ is piecewise defined complicate matters. So it is easier to use inequality 1.11 directly. The left side of this inequality is greater than

$$\sigma \int_{B_R} K|x|^\delta \Phi(x/R)\,dx = \sigma K R^{N+\delta}\int_{B_1} |\xi|^\delta \Phi(\xi)\,d\xi = K_1\sigma R^{N+\delta}.$$

Let $[0, T_\sigma)$ be the maximal interval of existence of u_σ. We assume that σ is sufficiently large to ensure that $T_\sigma < \infty$. We may replace Θ in right hand side of 1.11 by Θ_0 and obtain

$$K_1\sigma R^{N+\delta} \le \Psi(\Theta_0(\tau R^{-\beta})R^s)$$

whenever $0 < \tau < T_\sigma$. Therefore, $\sigma \le \max(F_1(R;\tau), F_2(R;\tau))$, where

$$F_i(R;\tau) := C_i R^{-\delta - N}\left[a_0 \tau^{-\frac{1}{p+1}} R^{\frac{\beta}{p+1}+s} + \beta_0 \tau^{\frac{p+1-m}{p+1}} R^{-\beta\left(\frac{p+1-m}{p+1}\right)+s}\right]^{q_i},$$

where C_1 and C_2 are certain positive constants and $q_1 := (p+1)/p$ and $q_2 := (p+1)/(p+1-m)$. Now, we define

$$\Omega_1^{(i)} := \beta/(p+1) + s - (N+\delta)/q_i, \quad \Omega_2^{(i)} := \beta(p+1-m)/(p+1) - s + (N+\delta)/q_i,$$

$\omega_1 := 1/(p+1)$, and $\omega_2 := (p+1-m)/(p+1)$. Then we may write simply

$$F_i(R;\tau) = C_i[a_0 \tau^{-\omega_1} R^{\Omega_1^{(i)}} + \beta_0 \tau^{\omega_2} R^{-\Omega_2^{(i)}}]^{q_i}.$$

If we can find functions $y = F_i(\tau)$ such that

$$F_i(R,\tau) \ge F_i(\tau), \quad \forall \tau > 0,$$

and such that for each value of τ there exists a value $R_\tau^{(i)}$ where

$$F_i(R_\tau^{(i)}, \tau) = F_i(\tau),$$

then $\sigma \le F_i(R,\tau)$ for all R if and only if $\sigma \le F_i(\tau)$. We make our mission somewhat easier by making a change of variables: let $z_i := R^{\Omega_1^{(i)}+\Omega_2^{(i)}}$ and $\eta := \tau^{\omega_1+\omega_2}$, so that $F_i(R;\tau) = C_i \tau^{-\omega_1 q_i} h_i(z_i;\eta)^{q_i}$, where

$$h_i(z_i;\eta) = a_0 z_i^{1-\gamma_i} + \beta_0 z_i^{-\gamma_i}\eta,$$

and $\gamma_i := \Omega_2^{(i)}/(\Omega_1^{(i)} + \Omega_2^{(i)})$. For the rest of this article, we suppress the index i. We easily find the envelope

$$y = h(\eta) := a_0^\gamma \beta_0^{1-\gamma}\Big[\Big(\frac{\gamma}{1-\gamma}\Big)^{1-\gamma} + \Big(\frac{1-\gamma}{\gamma}\Big)^{\gamma}\Big]\eta^{1-\gamma},$$

which leads us to define $F(\tau) := C\tau^{-\omega_1 q} h(\eta)^q$. If we define $\eta_z := a_0 \beta_0^{-1}(1-\gamma)\gamma^{-1} z$, then we may write

$$h(\eta) = \left[a_0 z^{1-\gamma_i} \eta_z^{\gamma-1} + \beta_0 z^{-\gamma}\eta_z^{\gamma}\right]\eta^{1-\gamma},$$

which immediately shows that the parameterized family of lines $y = h(z,\eta)$ are tangent to the concave curve $y = h(\eta)$ at the respective points $(\eta_z, h(\eta_z))$. Consequently $h(z,\eta) \ge h(\eta)$ for all $z > 0$ and $\eta > 0$, which implies that $F(R;\tau) \ge F(\tau)$. Tracing back through the change of variables we find that $F(R_\tau,\tau) = F(\tau)$ provided we pick $R_\tau = z^{1/(\Omega_1+\Omega_2)}$ where z is the solution of $\eta_z = \tau^{\omega_1+\omega_2}$. Going back to the use of the index i, we see that $\sigma \le \max(F_1(\tau), F_2(\tau))$ where

$$F_i(\tau) := C_i \tau^{-\omega_1 q_i}[h_i(\tau^{\omega_1+\omega_2})]^{q_i} = M_i \tau^{\vartheta_i},$$

for some positive constants M_1 and M_2 and with

$$\vartheta_i = [(1-\gamma_i)\omega_2 - \gamma_i\omega_1]\, q_i. \tag{1.18}$$

Therefore, $\sigma \leq \max(M_1\tau^{\partial_1}, M_2\tau^{\partial_2})$. Suppose that the exponents ∂_i are negative and let $\vartheta_i := -1/\partial_i$. Then it is clear that $\tau \leq C_0\sigma^{-\vartheta}$ for some constant C_0, provided we take $\vartheta := \min(\vartheta_1, \vartheta_2)$ and provided σ is restricted to sufficiently large values. Using equation (18) we can compute the values of ϑ_i, and then obtain the following result.

Corollary 1.5. *For each $\sigma > 0$, let u_σ be a solution of the problem*

$$u_t = \Delta u^m + u^{p+1},$$
$$u(x,0) = \sigma u_0(x),$$

on $\mathbb{R}^N \times [0, T_\sigma)$ where $[0, T_\sigma)$ is its maximum interval of existence. Assume that $0 < m < p+1$ and $u_0(x) \geq K|x|^\delta$ for some constants δ and $K > 0$, and that the numbers ϑ_1 and ϑ_2 given below are positive:

$$\vartheta_1 = \frac{[2(p+1) + N(m-1)]p}{2(p+1) + N(m-1) + \delta p(p+2-m)},$$
$$\vartheta_2 = \frac{(2p+2+Nm-N)(p+1-m)}{2(p+1) - N(m-1)(p+1-m) + \delta(p+1-m)(p+2-m)}.$$

Then there exist positive constants C_0 and σ_0 such that

$$T_\sigma \leq C_0\sigma^{-\vartheta},$$

for all $\sigma > \sigma_0$, where $\vartheta = \min(\vartheta_1, \vartheta_2)$.

Remark 2. *Note that in case $m = 1$ and $\delta = 0$, ϑ is simply equal to p, agreeing with the asymptotic result in [7].*

II

Chapter 2

Chapter 2: Life span of blow-up solutions for higher-order semilineare parabolic equations

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2.1 Introduction

In this chapter, we concerns the following cauchy problem for the higher-order semi-linear parabolic equation

$$\begin{aligned} u_t + (-\Delta)^m u = |u|^p, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{aligned} \tag{2.1}$$

where $m, p > 1$. Higher-order semilinear and quasilinear heat equations appear in numerous applications such as thin film theory, flame propagation, bi-stable phase transition and higher-order diffusion. For examples of these mathematical models, we refer the reader to the monograph [12]. For studies of higher-order heat equations we refer also to [13, 14, 15, 16, 17, 18] and the references therein.

In [17], under the assumption that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \not\equiv 0$ and

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \geq 0, \tag{2.2}$$

Galaktionov and Pohozaev studied the Fujita critical exponent of problem 2.1 and showed that $p_F = 1 + 2m/N$. The critical exponents p_F is calculated from both sides:

(i) blow-up of any solutions with 2.2 for $1 < p \leq p_F$;

(ii) global existence of small solutions for $p > p_F$.

Egorov et al [16] studied the asymptotic behavior of global solutions with suitable initial data in the supercritical Fujita range $p > p_F$ by constructing self-similar solutions of higher-order parabolic operators and through a stability analysis of the autonomous dynamical system. For other studies of the problem, we refer to [15] where global non-existence was proved for $p \in (1, p_F]$ by using the test function approach, and [13] where a general situation was discussed with nonlinear function $h(u)$ in place of $|u|^p$.

In [18],they have discussed the following system

$$\begin{aligned} u_t + (-\Delta)^m u = |v|^p, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\ v_t + (-\Delta)^m v = |u|^q, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}^N. \end{aligned} \tag{2.3}$$

It is proved that if $N/(2m > \max\{\frac{1+p}{pq-1}, \frac{1+q}{pq-1}\}$ then solutions of 2.3 with small initial data exist globally in time. Moreover the decay estimates $\|u(t)\|_\infty \le C(1+t)^{-\sigma_1}$ and $\|v(t)\|_\infty \le C(1+t)^{-\sigma_2}$ with $\sigma_1 > 0$ and $\sigma_2 > 0$ are also satisfied. On the other hand, under the assumption that

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x > 0, \quad \int_{\mathbb{R}^N} v_0(x)\mathrm{d}x > 0,$$

if $N/(2m) \le \max\{\frac{1+p}{pq-1}, \frac{1+q}{pq-1}\}$ then every solution of 2.3 blows up in finite time.

Exploiting the test function method, we shall give the life span of blow-up solution for some special initial data. The main idea comes from [19] for discussing cauchy problem of the second order equation

$$\begin{aligned} \rho(x)u_t - \Delta u^m &= h(x,t)u^{1+p}, \quad (t,x) \in \mathbb{R}_+^1 \times \mathbb{R}^N, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^N. \end{aligned} \tag{2.4}$$

Using the test function method, the author gave the blow-up type critical exponent and the estimates for life span $[0, T)$ like that in [20]. For the construction of a test function, the author mainly based on the eigenfunction Φ corresponding to the principle eigenvalue λ_1 of the Dirichlet problem on unit ball B_1,

$$\begin{aligned} -\Delta w(x) &= \lambda_1 w(x), \quad x \in B_1, \\ w(x) &= 0, \quad x \in \partial B_1. \end{aligned}$$

However, for the operator $(-\Delta)^m$, the eigenfunction Φ corresponding to the principal eigenvalue λ_1 of the Dirichlet problem may change sign (see [21]). We will use a non-negative smooth function Φ constructed in [13] and [17].

2.2 Fujita critical exponent

In this section, we shall use the test function method to derive the Fujita critical exponent and some useful inequalities. From the reference [17], we know that if $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the solution $u(t,\cdot) \in C^1([0,T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ for some $T > 0$. Therefore, without loss of generality, we may consider $u_0(x)$ concentrated around the origin and bounded below by a positive constant in some neighborhood of origin. Further, $u_0(x) \to 0$ as $|x| \to \infty$. With these choices, the solution u and its spatial derivatives vanish as $|x| \to \infty$ for $t > 0$.

First we construct a test function. For this aim, we shall use a non-negative smooth function Φ which was constructed in the papers [13] and [17].

Let

$$\Phi(x) = \Phi(|x|) > 0, \quad \Phi(0) = 1; \quad 0 < \Phi(r) \le 1 \quad \text{for } r > 0,$$

where $\Phi(r)$ is decreasing and $\Phi(r) \to 0$ as $r \to \infty$ sufficiently fast. Moreover, there exists a constant $\lambda_1 > 0$ such that

$$|\Delta^m \Phi| \le \lambda_1 \Phi, \quad x \in \mathbb{R}^N, \tag{2.5}$$

and such that

$$\|\Phi\|_1 = \int_{\mathbb{R}^N} \Phi(x)\mathrm{d}x = 1.$$

This can be done by letting $\Phi(r) = e^{-r^\nu}$ for $r \gg 1$ with $\nu \in (0, 1]$, and then extending Φ to $[0, \infty)$ by a smooth approximation.
Take $\partial > p/(p-1)$, and define

$$\varphi(t) = \begin{cases} 0, & t > T, \\ (1-(t-S)/(T-S))^{\partial}, & 0 \le t \le T, \\ 1, & t < S, \end{cases}$$

where $0 \le S < T$. Now set

$$\xi(t, x) = \varphi(t/R^{2m})\Phi(x/R), \quad R > 0.$$

Suppose that u exists in $[0, t_*) \times \mathbb{R}^N$. For $TR^{2m} < t_*$, multiply both sides of equation 2.1 by ξ and integrate over $[0, TR^{2m}) \times \mathbb{R}^N$ by parts to obtain

$$\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|^p\xi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\xi(0, x)\mathrm{d}x \le \int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\{|\xi_t| + |\Delta^m\xi|\}\mathrm{d}x\mathrm{d}t. \tag{2.6}$$

Denote

$$I(S, T) = \int_{SR^{2m}}^{TR^{2m}}\int_{\mathbb{R}^N} |u|^p\varphi(t/R^{2m})\Phi(x/R)\mathrm{d}x\mathrm{d}t, \quad J = \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x.$$

We now estimate $I(0, T) + J$. Using the Hölder inequality, since $\varphi'(t) = 0$ except on (S, T), we obtain

$$\begin{aligned}\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u||\xi_t|\mathrm{d}x\mathrm{d}t &= \int_{SR^{2m}}^{TR^{2m}}\int_{\mathbb{R}^N} |u|\varphi(t/R^{2m})^{1/p}|\varphi'(t/R^{2m})| \\ &\quad \times \varphi(t/R^{2m})^{-1/p}\,\Phi(x/R)R^{-2m}\mathrm{d}x\mathrm{d}t \\ &\le I(S, T)^{1/p}R^{-2m}\Big(\int_{SR^{2m}}^{TR^{2m}}\int_{\mathbb{R}^N} |\varphi'(t/R^{2m})|^{p/(p-1)} \\ &\quad \times \varphi(t/R^{2m})^{-1/(p-1)}\Phi(x/R)\mathrm{d}x\mathrm{d}t\Big)^{(p-1)/p}.\end{aligned} \tag{2.7}$$

Since $\Delta_x^m \Phi(x/R) = R^{-2m}\Delta_y^m \Phi(y)$ for $y = x/R$, using the Hölder inequality and 2.5 we have

$$\begin{aligned}
&\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u||\Delta^m \xi| \mathrm{d}x\mathrm{d}t \\
&= \int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\varphi(t/R^{2m})|\Delta^m \Phi(x/R)|\mathrm{d}x\mathrm{d}t \\
&= R^{-2m}\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\varphi(t/R^{2m})|\Delta^m_{x/R} \Phi(x/R)|\mathrm{d}x\mathrm{d}t \\
&\leq \mathcal{A}_1 R^{-2m}\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\varphi(t/R^{2m})\Phi(x/R)\mathrm{d}x\mathrm{d}t \\
&\leq I(0,T)^{1/p}\mathcal{A}_1 R^{-2m}\Big(\int_0^{TR^{2m}}\int_{\mathbb{R}^N} \varphi(t/R^{2m})\Phi(x/R)\mathrm{d}x\mathrm{d}t\Big)^{(p-1)/p}.
\end{aligned} \tag{2.8}$$

Making the change of variables $\tau = t/R^{2m}$ and $\eta = x/R$, from 2.6, 2.7 and 2.8 we deduce that

$$\begin{aligned}
&I(0,T) + J \\
&\leq I(S,T)^{1/p}R^s\Big(\int_{SR^{2m}}^{TR^{2m}}\int_{\mathbb{R}^N} |\varphi'(\tau)|^{p/(p-1)}\varphi(\tau)^{-1/(p-1)}\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p} \\
&+ I(0,T)^{1/p}\mathcal{A}_1 R^s\Big(\int_0^T\int_{\mathbb{R}^N}\varphi(\tau)\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p},
\end{aligned} \tag{2.9}$$

where $s = -2m + (2m+N)(p-1)/p$. Set

$$A(S,T) = \Big(\int_S^T\int_{\mathbb{R}^N} |\varphi'(\tau)|^{p/(p-1)}\varphi(\tau)^{-1/(p-1)}\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p},$$
$$B(T) = \Big(\int_0^T\int_{\mathbb{R}^N}\varphi(\tau)\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p}.$$

Thus 2.9 can be simply written as

$$I(0,T) + J \leq R^s[I(S,T)^{1/p}A(S,T) + \mathcal{A}_1 I(0,T)^{1/p}B(T)]. \tag{2.10}$$

We have the following result:

Theorem 2.8 (Fujita critical exponent)**.** *If*

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \geq 0,\ \ u_0(x) \not\equiv 0$$

and $s \leq 0$, that is to say $p \leq p_c = 1 + 2m/N$, then 2.1 *has no global solution.*

Proof. By slightly shifting the origin in time, we may assume

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x > 0. \tag{2.11}$$

Let u be a global solution with u_0 satisfying 2.11, then

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t > 0.$$

Suppose $s < 0$. Letting R tend to infinity in 2.10 to obtain

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x = 0.$$

Hence $u \equiv 0$, a contradiction.

Suppose $s = 0$. We first show $J \geq 0$ for all $R > 0$. In fact, from the assumptions on initial datum, there exists $\varepsilon_0 > 0$ such that $u_0(x) \geq \delta > 0$ for $|x| \leq \varepsilon_0$. Set

$$\begin{aligned}
J &= \int_{|x|\leq\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\
&> \delta \int_{|x|\leq\varepsilon_0} \Phi(x/R)\mathrm{d}x + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\
&= \delta R^N \int_{|\eta|\leq\varepsilon_0/R} \Phi(\eta)\mathrm{d}\eta + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\
&\geq \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x.
\end{aligned}$$

By the choice of Φ, we have

$$\lim_{R\to 0} \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x = 0.$$

And so there exists $R_0 > 0$ such that $J \geq 0$ for all $0 < R < R_0$. On the other hand, there exists $M > 0$ such that

$$\int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x > \int_{|x|>R_0 M} |u_0(x)|\mathrm{d}x.$$

In addition, by a slight modification of Φ, we may set $\Phi(x) \equiv 1$ in $\{x : |x| \leq M\}$. Note that since $0 \leq \Phi \leq 1$ we have, for $R \geq R_0$,

$$\begin{aligned}
J &= \int_{|x|\leq R_0 M} u_0(x)\Phi(x/R)\mathrm{d}x + \int_{|x|>R_0 M} u_0(x)\Phi(x/R)\mathrm{d}x \\
&\geq \int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x - \int_{|x|>R_0 M} |u_0(x)|\Phi(x/R)\mathrm{d}x \\
&\geq \int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x - \int_{|x|>R_0 M} |u_0(x)|\mathrm{d}x > 0.
\end{aligned}$$

Now we are in the position to complete the proof of case $s = 0$. Since

$$A(S,T) = \frac{\partial(T-S)^{-1/p}}{[\partial - 1/(p-1)]^{(p-1)/p}}, \quad B(T) = \Big[S + \frac{T-S}{\partial + 1}\Big]^{(p-1)/p},$$

we may choose S small and ∂ large, $T - S$ bounded, such that

$$B(T) \leq \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \Big/ \Big[2\mathcal{A}_1\Big(\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t\Big)^{1/p}\Big]. \tag{2.12}$$

Moreover, note that $J \geq 0$, from 2.10 we get that $I(0,T)$ is uniformly bounded for all $R > 0$. Then, keeping $T - S$ bounded,

$$\lim_{R\to\infty} I(S,T)^{1/p} A(S,T) = 0. \tag{2.13}$$

Letting $R \to \infty$, 2.10–2.13 give

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x = 0,$$

which also implies $u \equiv 0$. □

Let σ be an arbitrary positive number. For $x \in [0,\infty)$ and $0 < \omega < 1$, define

$$\Psi(\omega;\sigma) := \max_x (\sigma x^\omega - x).$$

It is easy to check that $\Psi(\omega;\sigma) = (1-\omega)\omega^{\frac{\omega}{1-\omega}}\sigma^{\frac{1}{1-\omega}}$. Set

$$A(T) = A(0,T), \quad S(T) = A(T) + \mathcal{A}_1 B(T).$$

We have the following result.

Theorem 2.9. *If u is a solution of 2.1 defined on $[0,t_*) \times \mathbb{R}^N$. Then, for $R > 0$ and $0 \leq \tau \leq t_* R^{-2m}$, we have*

$$\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \leq \Psi\Big(\frac{1}{p};\ S(T)R^s\Big). \tag{2.14}$$

Moreover, if u is a global solution of 2.1, then

$$\lim_{R\to\infty} \sup R^{-\hat{s}} \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \leq \mathcal{A}_1^{1/(p-1)}, \tag{2.15}$$

where $\hat{s} = sp/(p-1)$.

Proof. Denote $I(T) = I(0,T)$. Firstly, by the definition of Ψ, from 2.10 we know that

$$J \leq I(T)^{1/p} S(T) R^s - I(T) \leq \Psi\Big(\frac{1}{p};\ S(T)R^s\Big).$$

This is exactly 2.14. By means of 2.14, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x &\leq \Psi\Big(\frac{1}{p};\ S(T)R^s\Big) \\ &= (1-1/p)(1/p)^{\frac{1/p}{1-1/p}}[S(T)R^s]^{\frac{1}{1-1/p}} \\ &= (p-1)p^{p/(1-p)}R^{sp/(p-1)}S(T)^{\frac{p}{p-1}},\end{aligned} \tag{2.16}$$

which leads to

$$\lim_{R\to\infty} \sup R^{-\hat{s}} \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \leq (p-1)p^{p/(1-p)}[\inf_T S(T)]^{\frac{p}{p-1}}. \tag{2.17}$$

To estimate $S(T)$, we need estimate $A(T)$ and $B(T)$ respectively. Denote

$$a_p = \frac{\partial}{[\partial - 1/(p-1)]^{(p-1)/p}}, \quad b_p = \frac{\lambda_1}{(\partial + 1)^{(p-1)/p}}.$$

We obtain

$$S(T) = a_p T^{-1/p} + b_p T^{(p-1)/p}.$$

Since

$$\begin{aligned}\min_T S(T) &= p[a_p/(p-1)]^{(p-1)/p} b_p^{1/p} \\ &= \frac{p(p-1)^{-(p-1)/p} \lambda_1^{1/p} \partial^{(p-1)/p}}{[\partial - 1/(p-1)]^{(p-1)^2/p^2} (1+\partial)^{(p-1)/p^2}},\end{aligned}$$

we have

$$\lim_{\partial \to \infty} \min_T S_p(T) = p(p-1)^{-(p-1)/p} \lambda_1^{1/p}. \tag{2.18}$$

Combining 2.17 and 2.18, we obtain 2.15. The proof is complete. □

2.3 Life span of blow-up solutions

In this section, we shall estimate the life span of the blow-up solution with some special initial datum. To this aim, we assume that u_0 satisfies

(H) There exist positive constants C_0, L such that

$$u_0(x) \geq \begin{cases} \delta, & |x| \leq \varepsilon_0, \\ C_0 |x|^{-\kappa}, & |x| > \varepsilon_0, \end{cases}$$

where δ and ε_0 are as in the proof of Theorem 2.8, and $N < \kappa < 2m/(p-1)$ if $p < 1 + 2m/N$; $0 < \kappa < N$ if $p = 1 + 2m/N$.

Now we state the main result.

Theorem 2.10. *Let* (H) *be fulfilled and u_ε be the solution of* 2.1 *with initial data $u_\varepsilon(0,x) = \varepsilon u_0(x)$, where $\varepsilon > 0$. Denote $[0, T_\varepsilon)$ be the life span of u_ε. Then there exists a positive constant C such that $T_\varepsilon \leq C\varepsilon^{1/\hat{\beta}}$, where*

$$\hat{\beta} = \frac{\kappa}{2m} - \frac{1}{p-1} < 0.$$

Remark 3. When $p = 1 + 2m/N$, note that $\hat{\beta} = (\kappa - N)/(2m)$.

Proof. Choose R such that $R \geq R^0 > 0$. By the definition of J and the assumptions of initial data, we have

$$\begin{aligned} J &= \varepsilon \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \\ &\geq \varepsilon \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\ &= \varepsilon R^N \int_{|\eta|>\varepsilon_0/R} u_0(R\eta)\Phi(\eta)\mathrm{d}\eta \\ &\geq \varepsilon C_0 R^{N-\kappa} \int_{|\eta|>\varepsilon_0/R} |\eta|^{-\kappa}\Phi(\eta)\mathrm{d}\eta \\ &\geq \varepsilon C_0 R^{N-\kappa} \int_{|\eta|>\varepsilon_0/R^0} |\eta|^{-\kappa}\Phi(\eta)\mathrm{d}\eta \\ &= \widetilde{C}R^{N-\kappa}. \end{aligned} \tag{2.19}$$

Using 2.16, we know from 2.19 that, for $0 < \tau < T_\varepsilon$,

$$\begin{aligned} \varepsilon &\leq R^{\kappa-N}\widetilde{C}^{-1}(p-1)p^{p/(1-p)}[R^s S(T)]^{p/(p-1)} \\ &= \widetilde{C}^{-1}(p-1)p^{p/(1-p)}H(\tau,R), \end{aligned} \tag{2.20}$$

where $H(\tau,R) = R^{\kappa-N}[S(\tau R^{-2m})R^s]^{p/(p-1)}$. We write

$$H(\tau,R) = [a_p\tau^{-1/p}R^{a_1} + b_p\tau^{(p-1)/p}R^{-a_2}]^{p/(p-1)},$$

where $a_1 = (p-1)\kappa/p$, $a_2 = 2m - (p-1)\kappa/p$. The choice of κ implies $a_1, a_2 > 0$. Now we derive some estimates on $H(\tau,R)$. If we can find a function $G(\tau)$ such that

$$H(\tau,R) \geq G(\tau), \quad \forall\ \tau > 0,$$

and for each value of $R \geq R^0$ there exists a value of τ_R such that $H(\tau_R,R) = G(\tau_R)$, then 2.20 holds for all $R \geq R^0$ if and only if

$$\varepsilon \leq \widetilde{C}^{-1}(p-1)p^{p/(1-p)}G(\tau). \tag{2.21}$$

Set

$$y = R^{a_1+a_2} = R^{2m}, \quad \beta_1 = a_2/(a_1+a_2) = a_2/(2m).$$

Then

$$H(\tau,R) = \tau^{-1/(p-1)}h(\tau,y)^{p/(p-1)},$$

with $h(\tau,y) = a_p y^{1-\beta_1} + b_p y^{-\beta_1}\tau$. Denote

$$\sigma = a_p b_p^{-1}(1-\beta_1)\beta_1^{-1}y, \quad G(\tau) = \tau^{-1/(p-1)}g(\tau)^{p/(p-1)},$$

where

$$g(\tau) = [a_p y^{1-\beta_1}\sigma^{\beta_1-1} + b_p y^{-\beta_1}\sigma^{\beta_1}]\tau^{1-\beta_1}.$$

It is easy to check that $0 < \beta_1 < 1$. Then, $\zeta = g(\tau)$ ia a concave curve. Furthermore, $\zeta = h(\tau,y)$ is a tangent line of $\zeta = g(\tau)$ at the point of $(\sigma, g(\sigma))$. Therefore, we get that

$h(\tau, y) \geq g(\tau)$, for all $\tau > 0$. Hence $H(\tau, R) \geq G(\tau)$, for all $\tau > 0$. Moreover, $H(\tau, R_\tau) = G(\tau)$ with

$$\tau_R = a_p b_p^{-1}(1-\beta_1)\beta_1^{-1}R^{2m}.$$

By computations,

$$G(\tau) = \tau^{-1/(p-1)}g(\tau)^{p/(p-1)} = C_1\tau^{\hat{\beta}}. \quad (2.22)$$

for some positive constant C, where

$$\hat{\beta} = \frac{\kappa}{2m} - \frac{1}{p-1}.$$

The choice of κ implies that $\hat{\beta} < 0$. Combining 2.21 and 2.22, we find that

$$\varepsilon \leq K\tau^{\hat{\beta}}, \quad (2.23)$$

for some $K > 0$. From 2.23, it follows that

$$\tau \leq C\varepsilon^{1/\hat{\beta}},$$

for some $C > 0$. The proof is complete. □

III

Chapter 3

Chapter 3: Results of global and local existence for the semilinear wave equation with space–time dependent damping

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3.1 Introduction

In this chapter, we shall prove the existence of local and global solutions with small data,and after that we give an estimate of the life span of solutions.
We consider the Cauchy problem for the semilinear damped wave equation

$$u_{tt} - \Delta u + \varphi(t,x)u_t = f(u), \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{3.1}$$

with the initial condition

$$(u, u_t)(0,x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N. \tag{3.2}$$

The nonlinear term $f(u)$ is given by $f(u) = |u|^p$, where $u = u(t,x)$ is a real-valued unknown function of (t,x), $p > 1$, $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. The coefficient of the damping term is given by

$$\varphi(t,x) = \langle x\rangle^{-\alpha}(1+t)^{-\beta}.$$

With $\alpha \in [0,1)$,$\beta \in (-1,1)$ and $\alpha\beta = 0$. Here $\langle x\rangle$ denotes $\sqrt{1+|x|^2}$.
The power p satisfies

$$1 < p \leqslant \frac{N}{N-2} \quad (N \geqslant 3), 1 < p < \infty \quad (N = 1,2).$$

Our aim is to determine the critical exponent p_c, which is a number defined by the following property:

- If $p_c < p$, for all small data,the solutions of 3.1 are global,
- if $1 < p \leqslant pc$, the time-local solution cannot be extended time globally for some data.

It is expected that the critical exponent of 3.1 is given by

$$p_c = 1 + \frac{2}{N-\alpha}.$$

In this chapter we shall prove the existence of global solutions with small data when $p > 1 + 2/(N-\alpha)$. However, it is still open whether there exists a blow-up solution when

$1 < p \leq 1 + 2/(Na)$. When the damping term is missing and $f(u) = |u|^p$, that is

$$\begin{cases} u_{tt} - \Delta v = |u|^p & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & u_t(0, x) = u_1(x), \ x \in \mathbb{R}^N. \end{cases}$$

There are few results about solution to the linear part of 3.1 is expressed asymptotically by:

$$u(t, x) \sim v(t, x) + \exp^{-t/2} w(t, x),$$

where $v(t, x)$ is the solution of the corresponding heat equation :

$$\begin{cases} v_t - \Delta v = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ v(0, x) = u_0(x) + u_1(x) & x \in \mathbb{R}^N. \end{cases}$$

And $w(t, x)$ is the solution of the free wave equation

$$\begin{cases} w_{tt} - \Delta w = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ w(0, x) = u_0(x) & x \in \mathbb{R}^N. \end{cases}$$

By using a refined multiplier method. Their method also depends on the finite propagation speed property. Recently, Nishihara [22] and Lin et al.[23] considered the semilinear wave equation with time-dependent damping.

3.2 Prelimineries

In this section, we present some preliminaries that will be used in the next sections.

Theorem 3.11. *(Cauchy-Schwarz inequality)*

Let $f, g \in C([0, 1], \mathbb{R})$. So: $\displaystyle\int_0^1 |fg| \leq \left(\int_0^1 |f|^2\right)^{1/2} \left(\int_0^1 |g|^2\right)^{1/2}$.

Theorem 3.12. *(Poincare inequality, first version)*

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $p \in [1, \infty)$. Then there exists a constant $C(\Omega, p)$, depending only on Ω and p, such that

$$||u||_{L^p} \leq C(\Omega, p)||\nabla u||_{W^{1,p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$

In addition $C(\Omega) \leq C(n, p) diam(\Omega)$.

The proof of this result can be simplified by means of these properties:

- $H_0^{1,p}(\Omega) \subset H_0^{1,p}(\Omega')$ if $\Omega \subset \Omega'$(monotonicity).
- if $C(\Omega, p)$ denotes the best constant, then $C(\hbar\Omega, p) = (\Omega, p)$ (scaling invariance)and $C(\Omega + h, p) = C(\Omega, p)$(translation invariance).

The first fact is a consequence of the definition of the spaces $H_0^{1,p}$ in terms of regular functions, while the second one (translation invariance is obvious) follows by:

$$u_\rho(x) = u(\rho x) \in H_0^{1,p}(\Omega), \quad \forall u \in H_0^{1,p}(\rho\Omega).$$

Proof. By the monotonicity and scaling properties, it is enough to prove the inequality for $\Omega = Q \subset \mathbb{R}^N$ where Q is the cube centered at the origin, with sides parallel to the coordinate axis and length 2. We write $x = (x_1, x')$ with $x' = (x_2, \cdots, x_n)$. By density, we may also assume $u \in C_c^1(\Omega)$ and hence use the following representation formula:

$$u(x_1, x') = \int_{-1}^{x_1} \frac{\partial u}{\partial x_1}(t, x')\, dt.$$

Hölder's inequality gives

$$|u|^p(x_1, x') \leq 2^{p-1} \int_{-1}^{1} |\frac{\partial u}{\partial x_1}|^p(t, x')\, dt.$$

And hence we just need to integrate w.r.t. x_1 to get

$$\int_{-1}^{1} \frac{\partial u}{\partial t}(x_1, x')\, dx_1 \leq 2^p \int_{-1}^{1} |\frac{\partial u}{\partial x_1}|^p(t, x')\, dt.$$

Now, integrating w.r.t.x', repeating the previous argument for all the variables

$$x_j; \quad j = 1, \cdots, n$$

and summing all such inequalities we obtain the thesis with $C(Q, p) \leq 2/n^{1/p}$. □

Theorem 3.13. *(Poincare inequality, second version)*

Let us consider a bounded, regular and connected domain $\Omega \in \mathbb{R}^n$ and an exponent $1 \leq p < \infty$, so that by Rellich's theorem we have the compact immersion $w^{1,P}(\Omega) \hookrightarrow L^p$. Then, there exists a constant $C(\Omega, p)$ such that

$$\int_\Omega |u - u_\Omega|^p\, dx \leq C \int_\Omega |\nabla u|^p\, dx, \quad \forall u \in W^{1,p}(\Omega),$$

where

$$u_\Omega = \int_\Omega u\, dx.$$

Proof. By contradiction, if the desired inequality were not true, exploiting its homogeneity and translation invariance we could find a sequence $(u_n) \subset W^{1,p}(\Omega)$ such that

- $(u_n)_\Omega = 0$ for all $n \in \mathbb{N}$.
- $\int_\Omega |\nabla u_n|^p\, dx \longrightarrow 0$ for $n \longrightarrow \infty$.

By Rellich's theorem there exists (up to a subsequence) a limit point $u \in L^p$, that is $u_n \longrightarrow u$ in L^p. It is now a general fact that if ∇u_n has some weak limit point g then necessarily $g = \nabla u$ Therefore, in this case we have by comparison $\nabla u = 0$ in $L^p(\Omega)$ and hence, by connectivity of the domain and the constancy theorem, we deduce that u must be equivalent to a constant. By taking limits we see that u satisfies at the same time

$$\int_\Omega u\,dx = 0 \quad \text{and} \quad \int_\Omega |u|^p\,dx = 1,$$

which is clearly impossible. □

Lemma 3.6. *(Gagliardo-Nirenberg) Let $p, q, r (1 \leqslant p, q, r \leqslant \infty)$ and $\sigma \in [0,1]$ satisfy*

$$\frac{1}{p} = \sigma\left(\frac{1}{r} - \frac{1}{n}\right) + (1-\sigma)\frac{1}{q},$$

except for $p = \infty$ or $r = n$ when $n \geqslant 2$. Then for some constant $C = C(p,q,r,n) > 0$, the inequality

$$\|u\|_{L^p} \leqslant C\|u\|_{L^q}^{1-\sigma}\|\nabla u\|_{L^r}^{\sigma},$$

for any $u \in C_0^1(\mathbb{R}^n)$

Proof. Let $u \in D^p$ satisfy the constraint

$$J[u] := \frac{1}{2p}\int_{\mathbb{R}^d} |u(x)|^{2p}dx = J_\infty.$$

For $\Lambda > 0$, we consider the scaled function

$$u_\Lambda(x) = \Lambda^{\frac{d}{2p}} u(\Lambda, x).$$

which still satisfies $J[u_\Lambda] = J[\infty]$. Then for each $\Lambda > 0$,

$$G(u_\Lambda) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx \Lambda^{d/p-(d-2)} + \frac{1}{p+1}\int_{\mathbb{R}^d} u^{p+1} dx \Lambda^{-d(p-1)/2p} \geqslant I_\infty.$$

Minimizing the left hand side of the above expression in $\Lambda > 0$ yields

$$C_*[\|\nabla u\|_2^{\vartheta}\|u\|_{p+1}^{1-\vartheta}]^{\sigma} \geqslant I_\infty,$$

where

$$C_* = \frac{1}{2}\Lambda_*^{d/p-(d-2)} + \frac{1}{p+1}\Lambda_*^{-d(p-1)/2p}, \quad \Lambda_* = \frac{d}{d-p(d-2)}\frac{p-1}{p+1},$$
$$\sigma = 2p\frac{d+2-(d-2)p}{4p-d(p-1)}, \quad \vartheta = \frac{d(p-1)}{p(d+2-p(d-2))}.$$

Since $\|u\|_{2p} = 2pJ_\infty$, we may write:

$$\|\nabla u\|_2^{\vartheta}\|u\|_{p+1}^{1-\vartheta} \geqslant \left(\frac{I_\infty}{C_*}\right)^{1/\sigma}\frac{\|u\|_{2p}}{(2pJ_\infty)^{1/(2p)}}.$$

By homogeneity, the above inequality actually holds for any $u \in D^p$, with optimal constant

$$C(2pJ_\infty)^{1/(2p)}\left(\frac{C_*}{I_\infty}\right)^{1/\sigma}.$$

□

Theorem 3.14. (Gronwall) *Let x, Ψ and χ be real continuous functions defined in $[a, b]$, $\chi(t) \geq 0$ for $t \in [a, b]$. We suppose that on $[a, b]$ we have the inequality*

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)x(s)\, ds.$$

Then

$$x(t) \leq \left\{\Psi(t) + \int_a^t \chi(s)\Psi(s) \exp\left[\int_s^t \chi(u)\, du\right] ds\right\} \in [a, b].$$

Proof. Let us consider the function $y(t) := \int_a^t \chi(u)x(u)du \in [a, b]$.
Then we have $y(a) = 0$ and

$$\begin{aligned} y'(t) = \chi(t)x(t) &\leq \chi(t)\Psi(t) + \chi(t) \int_a^b \chi(s)x(s)ds \\ &= \chi(t)\Psi(t) + \chi(t)y(t), \quad t \in (a, b). \end{aligned}$$

By multiplication with $\exp\left(-\int_a^t \chi(s)ds\right) > 0$, we obtain

$$\frac{d}{dt}\left(y(t) \exp\left(-\int_a^t \chi(s)ds\right)\right) \leq \Psi(t)\chi(t) \exp\left(-\int_a^t \chi(s)ds\right).$$

By integration on $[a, t]$, one gets

$$y(t) \exp\left(-\int_a^t \chi(s)ds\right) \leq \int_a^t \Psi(u)\chi(u) \exp\left(-\int_a^u \chi(s)ds\right) du.$$

From where results

$$y(t) \leq \int_a^t \Psi(u)\chi(u) \exp\left(\int_u^t \chi(s)ds\right) du, \quad t \in [a, b].$$

Since $x(t) \leq \Psi(t) + y(t)$, the theorem is thus proved. □

Definition 3.6. *Let X be a topological space and let $T : X \longrightarrow X$ be a map. A point $x \in X$ is a fixed point if $T(x) = x$.*

Definition 3.7. *Let (X, d) be a metric space.A mapping $T : X \longrightarrow X$ is a contraction mapping ,or contraction, if there exists a constant c with $0 \leq c < 1$, such that*

$$d(T(x), T(y)) \leq cd(x, y) \quad \text{for all} \quad x, y \in X. \tag{3.3}$$

Thus, a contraction maps points closer together.In particular,for every $x \in X$, and any $r > 0$,all points y in the ball $B_r(x)$, are mapped into a ball $B_s(Tx)$, with $s < r$.
If $T : X \longrightarrow X$, a fixed point of T.

Theorem 3.15. (Banach-Picard)

If $T : X \longrightarrow X$ is a contraction mapping on a complete metric space (X, d), then there is exactly one solution $x \in X$.

Proof. The proof is constructive,meaning that we will explicitly construct a sequence converging to the fixed point.Let x_0 be any point in X. We define a sequence (x_n) in X by

$$x_{n+1} = Tx_n \qquad \text{for} \quad n \geq 0.$$

To simplify the notation,we often omit the parentheses around the argument of a map.We denote the nth iterate of T by T^n, so that $x_n = T^n x_0$. First,we show that (x_n) is a cauchy sequence.If $n \geq m \geq 1$,then from 3.3 and the triangle inequality,we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &\leq c^m d(T^{n-m} x_0, x_0) \\ &\leq c^m [d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) \\ &\quad + \cdots d(Tx_0, x_0)] \\ &\leq c^m \left[\sum_{k=0}^{n-m-1} \right] d(x_1, x_0) \\ &\leq c^m \left[\sum_{k=0}^{\infty} \right] d(x_1, x_0) \\ &\leq \left(\frac{c^m}{1-c} \right) d(x_1, x_0), \end{aligned}$$

wich implies that (x_n) is cauchy .Since X is complete, x_n converges to a limit $x \in X$. The fact that the limit x is a fixed point of T follows from the continuity of T:

$$Tx = T \lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} x_{n+1} = x.$$

Finally,if x and y are two fixed points, then

$$0 \leq d(x, y) = d(Tx, Ty) \leq cd(x, y).$$

Since $c < 1$, we have $d(x, y) = 0$, so $x = y$ and the fixed point is unique. □

Theorem 3.16. (Fatou's Lemma) *Let $f_n : \mathbb{R}^N$ be (nonnegative)Lebesgue measurable functions. Then*

$$\liminf_{n \longrightarrow \infty} \int_{\mathbb{R}} f_n \, d\mu \geq \int_{\mathbb{R}} \liminf_{n \longrightarrow \infty} f_n \, d\mu$$

3.3 Local existence

In this section, we give the local existence of the problem 3.1-3.2. To state our results, we introduce an auxiliary function

$$\psi(t, x) := A \frac{\langle x \rangle^{2-\alpha}}{(1+t)^{1+\beta}}, \tag{3.4}$$

with

$$A = \frac{(1+\beta)}{(2-\alpha)^2 (2+\delta)}, \quad \delta > 0. \tag{3.5}$$

This type of weight function was first introduced by Ikehata and Tanizawa [24].

Lemma 3.7. *Let $u(t,x)$ be solution to problem 3.1-3.2 on $[0,T_m)$. Then for all $t \in [0,T_m)$ it is true that*

$$\|e^{\psi} Du(t,.)\| \leqslant CI_0 + C\left(\sup_{[0,t]}(s+1)^{\delta}\|e^{\gamma\psi(s,.)}u(s,.)\|_{p+1}\right)^{(p+1)/2},$$

where

$$I_0 = \int_{\mathbb{R}^N} e^{\psi(0,x)}\left(u_1 + |\nabla u_0| + |u_0|\right) dx,$$

and

$$1 \geqslant \gamma \geq 2/(p+1), \quad \delta \geq 0, \quad D = (\partial_t, \nabla),$$

with $C = C_{\delta,\gamma} \geq 0$ is a constant, which depends on δ and γ.

Proof. We multiply 3.1 by $e^{2\psi}u_t$, then it holds that

$$\frac{\partial}{\partial t}\left(\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2) - \frac{e^{2\psi}}{p+1}|u|^p\right) - \nabla(e^{2\psi}u_t\nabla u) + e^{\psi}\left(\phi(x,t) - \frac{|\nabla\psi|^2}{-\psi_t} - \psi_t\right)u_t^2 + \frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2 = -\frac{2\psi_t}{p+1}e^{2\psi}|u|^p u. \tag{3.6}$$

Integrating over $[0,t]\times\mathbb{R}^N$ and we obtain

$$\int_0^t\int_{\mathbb{R}^N}\frac{\partial}{\partial s}\left(\frac{e^{2\psi}}{2}(|u_s|^2 + |\nabla u|^2) - \frac{e^{2\psi}}{p+1}|u|^p u\right) dxds - \int_0^t\int_{\mathbb{R}^N}\nabla(e^{2\psi}u_s\nabla u)dxds \leqslant -\frac{2}{p+1}\int_0^t\int_{\mathbb{R}^N}\psi_s e^{2\psi}|u|^p u\, dxds.$$

Since

$$\int_0^t\int_{\mathbb{R}^N}\nabla(e^{2\psi}u_s\nabla u)\, dxds = 0.$$

we find the following estimate with some constant $C \geq 0$,

$$\|e^{\psi}Du(t)\|^2 \leqslant CI_0^2 + C\|e^{(2/p+1)\psi}u(t)\|_{p+1}^{p+1} + C\int_0^t\int_{\mathbb{R}^N}|\psi_s|e^{(2-\gamma(p+1))\psi}e^{\gamma(p+1)\psi}|u|^{p+1}dxds.$$

Thus we see

$$\|e^{\psi}Du(t)\|^2 \leqslant CI_0^2 + C\|e^{(2/p+1)\psi}u(t)\|_{p+1}^{p+1} + C\int_0^t\left(\max_{x\in\mathbb{R}^N}\Upsilon(s,x)\right)\|e^{\gamma\psi(s,.)}u(s,.)\|_{p+1}^{p+1}\, ds, \tag{3.7}$$

where

$$\Upsilon(s,x) = |\psi_s(s,x)|e^{(2-\gamma(p+1))\psi(s,x)}, \quad \gamma \geq \frac{2}{p+1}.$$

Thus it follows

$$\max_{x\in\mathbb{R}^N}\Upsilon(s,x) \leqslant \frac{C_\gamma}{1+s}. \tag{3.8}$$

Now let us show the desired estimate. In fact, from 3.7 and 3.8 one has

$$\begin{aligned}\|e^{\psi(t,.)}Du(t)\|^2 &\leqslant CI_0^2 + C\|e^{\gamma\psi(t,.)}u(t)\|_{p+1}^{p+1} + C_\gamma\int_0^t \frac{1}{s+1}\|e^{\gamma\psi(s,.)}u(s)\|_{p+1}^{p+1}ds\\ &\leqslant CI_0^2 + C\left\{\sup_{[0,t]}(1+s)^{\delta}\|e^{\gamma\psi(s,.)}u(s)\|_{p+1}\right\}^{p+1}\\ &+ C_\gamma\int_0^t \frac{1}{(1+s)^{1+\delta(p+1)}}\left\{\sup_{[0,t]}(1+s)^{\delta}\|e^{\gamma\psi(s,.)}u(s)\|_{p+1}\right\}^{p+1} ds\\ &\leqslant CI_0^2 + C_{\gamma,\delta}\left\{\sup_{[0,t]}(1+s)^{\delta}\|e^{\gamma\psi(s,.)}u(s)\|_{p+1}\right\}^{p+1},\end{aligned}$$

where we have used the fact

$$\int_0^\infty \frac{1}{(1+s)^{1+\delta(p+1)}}ds = C_\delta < +\infty.$$

This completes the proof of 3.7. □

Lemma 3.8. *Let $\partial(q) = N(\frac{1}{2} - \frac{1}{q})$ and $0 \leqslant \partial(q) < 1$, and let $0 < \sigma \leqslant 1$. If $v \in H^1_\psi(\mathbb{R}^N)$, then it is true that*

$$\|e^{\sigma\psi(t,)}v\|_q \leqslant C_\sigma(1+t)^{(1-\partial(q))/2}\|\nabla v\|^\sigma,$$

for each $t \geqslant 0$, where $C_\sigma > 0$ is a constant.

We describe the local existence result:

Theorem 3.17. *Let $a \geqslant 0, \beta \in \mathbb{R}, 1 < p \leqslant \frac{N}{(N-2)}$ $(N \geqslant 3), 1 < p < \infty$ $(N = 1,2), \varepsilon > 0$, and $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ satisfying*

$$I_0^2 < \infty,$$

there exists a maximal existence time $T_\varepsilon > 0$ such that the problem has a unique solution $u \in X(T) := C^1([0,T], L^2) \cap C([0,T], H^1)$ satisfying

$$\sup_{[0,T]}[\|e^\psi \nabla u\| + \|e^\psi u_t\| + \|e^\psi u\|] < \infty.$$

Moreover, for any $T < T_\varepsilon$, in particular ,$T_\varepsilon < \infty$, then is true that

$$\limsup_{t\longrightarrow T_\varepsilon}[\| e^{\psi(t)}u(t,.) \| + \| e^{\psi(t)}\nabla u(t,.) \| + \| e^{\psi}u_t(t,.) \|] = +\infty.$$

Proof. For the proof we denote

$$B^\psi_{T,K} = \left\{v \in X(0,T)(\mathbb{R}^N); \|v\|^\psi_T \leqslant K\right\}, \quad K > 0, T > 0.$$

And

$$\|v\|^\psi_T = \sup_{[0,T]} = (\|e^\psi v_t\| + \|e^\psi \nabla v\| + \|e^\psi v\|).$$

For a fixed $v^\psi_{T,K}$,we define a mapping $\Phi : B^\psi_{T,K} \longrightarrow X_1(0,t)(\mathbb{R}^N)$ such that $u(t) = (\Phi v)(t)$ is a unique solution to problem :

$$u_{tt} - \Delta u + \phi(t,x)u_t = |v|^p, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N,$$

$$(u, u_t)(0,x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N.$$

Then as in the proof of lemma 3.7 it follows from 3.6 that

$$e^{2\psi} u_t |v|^p \geqslant \frac{d}{dt}\left\{ \frac{e^{2\psi}}{2}(|u_t|^2 + |\nabla u|^2) \right\} - div(e^{2\psi} u_t \nabla u),$$

so that from the integration by parts one has

$$E_{\psi,u}(t) \leqslant E_{\psi,u}(0) + \int_0^t \int_{\mathbb{R}^N} e^{2\psi(s,x)} u_t |v|^p \; dxds,$$

where

$$E_{\psi,u}(t) = \frac{1}{2}\int_{\mathbb{R}^N} e^{2\psi(t,x)} \left(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 \right) \; dx.$$

It follows from the Schwarz inequality 3.11 that

$$E_{\psi,u}(t) \leqslant E_{\psi,u}(0) + \sqrt{2} \int_0^t \left(\int_{\mathbb{R}^N} e^{2\psi(s,x)} |v(s,x)|^{2p} \; dx \right)^{1/2} E_{\psi,u}(s)^{1/2} ds.$$

The Gronwall type inequality 3.14 implies

$$E_{\psi,u}(t)^{1/2} \leqslant E_{\psi,u}(0)^{1/2} + \frac{1}{\sqrt{2}} \int_0^t \left(\int_{\mathbb{R}^N} e^{2\psi(s,x)} |v(s,x)|^{2p} dx \right)^{1/2} ds. \qquad 3.9$$

Since $v(t) \in H^1_{\psi(t)}(\mathbb{R}^N)$, we can apply Lemma 3.8 to 3.9 in order to derive

$$\int_{\mathbb{R}^N} e^{2\psi(s,x)} |v(s,x)|^{2p} dx \leqslant C_p(1+s)^{p(1-\partial(2p))} \|\nabla v(s)\|^{2(p-1)} \|e^{\psi(s)}\|^2$$
$$\leqslant C_p(1+s)^{p(1-\partial(2p))} K^{2p},$$

so that one obtains

$$E_{\psi,u}(t)^{1/2} \leqslant E_{\psi,u}(0)^{1/2} + C_p T(1+T)^{(2p-Np+N)/4} K^p. \qquad 3.10$$

On the other hand, since

$$u(t,x) = u_0(x) + \int_0^t u_s(s,x)\, ds,$$

it follows that

$$e^{\psi(t,x)} u(t,x) = e^{\psi(t,x)} u_0(x) + \int_0^t e^{\psi(t,x)} u_s(s,x) \; ds,$$

so that from 3.10 we can estimate as follows:

$$\begin{aligned}\|e^{\psi(t,.)}u(t)\| &\leqslant \|e^{\psi(t,.)}u_0\| + \int_0^t \|e^{\psi(t,.)}u_s(s)\|\,ds \leqslant \|e^{\psi(0,.)}u_0\| + \int_0^t \|e^{\psi(s)}u_s(s)\|ds \\ &\leqslant \|e^{\psi(0,.)}u_0\| + \int_0^t \left(E_{\psi,u}(0)^{1/2} + C_pT(1+T)^{(2p-Np+N)/4}K^p\right)\,ds \\ &\leqslant \|e^{\psi(0,.)}u_0\| + E_{\psi,u}(0)^{1/2}T + C_pT^2(1+T)^{(2p-Np+N)/4}K^p.\end{aligned} \tag{3.11}$$

3.10 and 3.11 implies:

$$\begin{aligned}&\|e^{\psi(t)}u_t(t)\| + \|e^{\psi(t)}\nabla u(t)\| + \|e^{\psi(t)}u(t)\| \\ &\leqslant \|e^{\psi(0,.)u_0}\| + \|e^{\psi(0)}Du(0)\| + T\|e^{\psi(0)}Du(0)\| \\ &+ C_pT(1+T)^{1+((2p-Np+N)/4)}K^p.\end{aligned}$$

By taking $K > 0$ large enough such that

$$\|e^{\psi(0)}u(0)\| + \|e^{\psi(0)}Du(0)\| < \frac{k}{2}, \tag{3.12}$$

one arrives at the desired estimate:

$$\|u\|_T^{\psi} < K,$$

which implies that the mapping $\Phi : B^{\psi}_{T,K} \longrightarrow B^{\psi}_{T,K}$ is well-defined for large $K > 0$ and small $T > 0$. Next we shall prove that $\Phi : B^{\psi}_{T,K} \longrightarrow B^{\psi}_{T,K}$ becomes a contraction mapping ifone takes $T > 0$ further small enough. For this we take $u = \Phi(v)$, and $\overline{u} = \Phi(\overline{v})(v, \overline{v} \in B^{\psi}_{T,K})$. Then $w = u - \overline{u}$ satisfies

$$w_{tt} - \Delta w + w_t = |v|^p - |\overline{v}|^p \tag{3.13}$$

$$w(0,x) = w_t(0,x) = 0, \qquad x \in \mathbb{R}^N. \tag{3.14}$$

Then as in the proof of 3.6 one has

$$\|e^{\psi(t)}Dw(t)\|^2 \leqslant \int_0^t \int_{\mathbb{R}^N} e^{2\psi(s,x)}(|v(s)|^p - |\overline{v}(s)|^p)w_t(s,x)\,dxds.$$

Because ofthe mean value theorem one has

$$||v|^p - |\overline{v}|^p| \leqslant p|v - \overline{v}|(|v| + |\overline{v}|)^{p-1},$$

so that the Schwarz inequality 3.11 gives rise to the estimate:

$$\begin{aligned}\|e^{\psi(t)}Dw(t)\|^2 &\leqslant p\int_0^t\int_{\mathbb{R}^N} e^{2\psi(s,x)}|v(s)-\overline{v}(s)|(|v(s)|-|\overline{v}(s)|)^{p-1}|w_t(s,x)|\,dxds\\ &\leqslant p\int_0^t\left(\int_{\mathbb{R}^N} e^{2\psi(s,x)}w_t(s)^2dx\right)^{1/2}\\ &\quad\times\left(\int_{\mathbb{R}^N} e^{2\psi(s,x)}|v(s)-\overline{v}(s)|^2(|v(s)|+|\overline{v}(s)|)^{2(p-1)}\,dx\right)^{1/2}\,ds\\ &\leqslant p\int_0^t\|e^{\psi(s)}Dw(s)\|\left(\int_{\mathbb{R}^N} e^{2\psi(s)}|v(s)-\overline{v}(s)|^2(|v(s)|\right.\\ &\quad\left.-|\overline{v}(s)|)^{2(p-1)}\,dx\right)^{1/2}\,ds.\end{aligned} \tag{3.15}$$

Here it follows from the Hölder inequality 1.3 that

$$\begin{aligned}&\int_{\mathbb{R}^N} e^{2\psi(s,x)}|v(s)-\overline{v}(s)|^2(|v(s)|+|\overline{v}|)^{2(p-1)}dx\\ &\leqslant \|e^{\psi(s)/2}(v(s)-\overline{v}(s))\|_{2p}^2\|e^{\psi(s)/2(p-1)}(|v(s)|+|\overline{v}(s)|)\|_{2p}^{2(p-1)}.\end{aligned} \tag{3.16}$$

3.15 and 3.16 imply

$$\begin{aligned}&\|e^{\psi(t)}Dw(t)\|^2 \leq\\ &p\int_0^t\|e^{\psi(s)}Dw(s)\|\|e^{\psi(s)/2}(v(s)-\overline{v}(s))\|_{2p}\|e^{\psi(s)/2(p-1)}(|v(s)|\\ &\quad+|\overline{v}(s)|)\|_{2p}^{p-1}\,ds.\end{aligned}$$

By the Gronwall 3.14 inequality one obtains

$$\begin{aligned}\|e^{\psi(t)}Dw(t)\| \leqslant C_p\int_0^t &\|e^{\psi(s)/2}(v(s)-\overline{v}(s))\|_{2p}\\ &\times(\|e^{\psi(s)/2(p-1)}v(s)\|_{2p}+\|e^{\psi(s)/2(p-1)}\overline{v}(s)\|_{2p})^{p-1}\,ds.\end{aligned} \tag{3.17}$$

By Lemma 3.8 with $\sigma = 1/(2(p1)), q = 2p$ we have

$$\begin{aligned}\|e^{\psi(s)/2(p-1)}v(s)\|_{2p} &\leqslant C_p(1+s)^{((2-N)p+N)/4p}\|e^{\psi(s)}\nabla v(s)\|\\ &\leqslant C_pK(1+T)^{((2-N)p+N)/4p},\end{aligned}$$

it follows that

$$\|e^{\psi(s)/2}(v(s)-\overline{v}(s))\|_{2p} \leqslant C_p(1+T)^{((2-N)p+N)/4p}\|e^{\psi(s)}\nabla(v(s)-\overline{v}(s))\|.$$

Thus from 3.17 we find that

$$\begin{aligned}\|e^{\psi(t)}Dw(t)\| &\leqslant C_pK^{p-1}(1+T)^{\gamma}\int_0^t\|e^{\psi(s)}\nabla(v(s)-\overline{v}(s))\|\,ds\\ &\leqslant C_pK^{p-1}(1+T)^{\gamma}T\|v-\overline{v}\|_T^{\psi},\end{aligned} \tag{3.18}$$

where

$$\gamma = \frac{N-(N-2)p}{4} \geqslant 0.$$

Furthermore, since

$$w(t,x) = \int_0^t w_s(s,x)ds,$$

one has

$$\begin{aligned} \|e^{\psi(t,.)}w(t)\| &\leqslant \int_0^t \|e^{\psi(t,.)}w_s(s)\|\, ds \leqslant \int_0^t \|e^{\psi(s,.)}w_s(s)\, ds\| \leqslant \int_0^t \|e^{\psi(s)}Dw(s)\|\, ds \\ &\leqslant C_p K^{p-1}(1+T)^{\gamma}T^2\|v-\overline{v}\|_T^{\psi}. \end{aligned} \quad (3.19)$$

From 3.18 and 3.19 we can deduce

$$\|u-\overline{u}\|_T^{\psi} \leqslant C_p K^{p-1}(1+T)^{\gamma+1}T\|v-\overline{v}\|_T^{\psi}.$$

By taking $T > 0$ further small such that

$$C_p K^{p-1}(1+T)^{\gamma+1}T < \frac{1}{2},$$

one arrives at the crucial estimate:

$$\|u-\overline{u}\|_T^{\psi} \leqslant \frac{1}{2}\|v-\overline{v}\|_T^{\psi}, \quad (3.20)$$

which shows that $\Psi : B_{T,K}^{\psi} \longrightarrow B_{T,K}^{\psi}$ becomes a contraction mapping for large $K > 0$ satisfying 3.12 and small $T > 0$. Finally, let us define a sequence of solutions as follows:

$$\begin{aligned} &u^{(0)}(t,x) = u_0(x), \qquad u_0 \in B_{T,K}^{\psi}, \\ &u^{(n)}(t,x) = (\Psi u^{(n-1)})(t,x), \qquad n = 1,2,3,\cdots, \end{aligned}$$

and $u^{(n)}$ satisfies

$$\begin{aligned} &u_{tt}^{(n)}(t,x) - \Delta u^{(n)}(t,x) + u_t^{(n)}(t,x) = |u^{(n-1)}(t,x)|^p, \quad (t,x) \in (0,t)\times\mathbb{R}^N, \\ &u^{(n)}(0,x) = u_0(x), \quad u_{(t)}^{(n)}(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \end{aligned}$$

By 3.20, there exists a function $u \in X_1(0,T)(\mathbb{R}^N)$ such that

$$\begin{aligned} &u^{(n)} \longrightarrow u \in C([0,t];H^1(\mathbb{R}^N)), \\ &u_t^{(n)} \longrightarrow u_t \in C([0,t];L^2(\mathbb{R}^N)), \end{aligned}$$

as $n \longrightarrow \infty$, and so, u becomes the weak solution to 3.1-3.2 on $[0,T]$. Furthermore, we also have

$$\|e^{\psi(t)}\nabla u^{(n)}(t)\| + \|e^{\psi(t)}u_t^{(n)}(t)\| + \|e^{\psi(t)}u^{(n)}(t)\|, \quad (3.21)$$

for all $t \in [0,T]$. Let $\Psi \in C_0^\infty(\mathbb{R}^N)$ be fixed. Then, for $j = 1,2,3,\cdots,N$ one has

$$\begin{aligned} |\left(e^{\psi(t)}\frac{\partial u}{\partial x_j},\Psi\right)| &= |\left(\frac{\partial u}{\partial x_j}, e^{\psi(t)}\Psi\right)| \\ &\leqslant |\left(\frac{\partial u}{\partial x_j} - \frac{\partial u^{(n)}}{\partial x_j}, e^{\psi(t)}\Psi\right)| + |\left(\frac{\partial u^{(n)}}{\partial x_j}, e^{\psi(t)}\Psi\right)| \\ &\leqslant |\left(\frac{\partial u}{\partial x_j} - \frac{\partial u^{(n)}}{\partial x_j}, e^{\psi(t)}\Psi\right)| + \|e^{\psi(t)}\frac{\partial u^{(n)}}{\partial x_j}\|\|\Psi\|, \end{aligned} \tag{3.22}$$

Letting $n \longrightarrow \infty$ above, it follows from 3.22 that

$$|\left(e^{\psi(t)}\frac{\partial u(t)}{\partial x_j},\Psi\right)| \leqslant \left(\limsup_{n\longrightarrow\infty} \|e^{\psi(t)}\frac{\partial u^n(t)}{\partial x_j}\|\right)\|\Psi\| \leqslant K\|\Psi\|, \tag{3.23}$$

and similarly

$$\begin{aligned} |(e^{\psi(t)}u_t,\Psi)| &\leqslant \left(\limsup_{n\longrightarrow\infty}\|e^{\psi(t)}u_t(t)^{(n)}(t)\|\right)\|\Psi\| \leqslant K\|\Psi\|, \\ |(e^{\psi(t)}u(t),\Psi)| &\leqslant \left(\limsup_{n\longrightarrow\infty}\|e^{\psi(t)}u^{(n)}(t)\|\right)\|\Psi\| \leqslant K\|\Psi\|. \end{aligned} \tag{3.24}$$

By density, because of 3.24 - 3.23 one can observe that

$$e^{\psi(t)}\frac{\partial u(t)}{\partial x_j} \in L^2(\mathbb{R}^N), \qquad e^{\psi(t)}u_t(t) \in L^2(\mathbb{R}^N), \qquad e^{\psi(t)}u(t) \in \mathbb{L}^2(\mathbb{R}^N),$$

for each $t \in [0,T]$, and

$$\|e^{\psi(t)}\frac{\partial u(t)}{\partial x_j}\| \leqslant K, \qquad \|e^{\psi}u_t(t)\| \leqslant K, \qquad \|e^{\psi(t)}u(t)\| \leqslant K,$$

so that one has arrived at the estimates:

$$\|e^{\psi(t)}u(t)\| + \|e^{\psi(t)}\nabla u(t)\| + \|e^{\psi(t)}u_t(t)\| \leqslant (N+2)K,$$

for all $t \in [0,T]$. Note that because of 3.12 the (local) solution to 3.1 - 3.2 can be continued in time as long as the quantity $\|e^{\psi(t)}u(t)\| + \|e^{\psi(t)}\nabla u(t)\| + \|e^{\psi(t)}u_t(t)\|$ is finite. The uniqueness of a weak solution in $X_1(0,T)(\mathbb{R}^N)$ is standard. This completes the proof of theorem . □

3.4 Global existence

In this section, we give the global existence of 3.1 - 3.2.

Theorem 3.18. *If* $p > 1 + \frac{2}{N-a}$, *then there exists a small positive number* $\delta_0 > 0$ *such that for any* $0 < \delta \leqslant \delta_0$ *the following holds: If*

$$I_0^2 := \int_{\mathbb{R}^N} e^{2\psi(0,x)}\left(u_1^2 + |\nabla u_0|^2 + |u_0|^2\right)dx,$$

is sufficiently small, then there exists a unique $u \in C([0,\infty); H^1(\mathbb{R}^N)) \cap C^1([0,\infty); L^2(\mathbb{R}^N))$ *solution to* 3.1 *satisfying*

$$\int_{\mathbb{R}^N} e^{2\psi(t,x)}|u|^2\,dx \leqslant c_\delta(1+t)^{-(1+\beta)\frac{N-2a}{2-a}+\varepsilon}, \tag{3.25}$$

$$\int_{\mathbb{R}^N} e^{2\psi(0,x)}(|u_t|^2+|\nabla u|^2)\,dx \leqslant c_\delta(1+t)^{-(1+\beta)(\frac{N-a}{2-a}+1)+\varepsilon},$$

where

$$\varepsilon = \varepsilon(\delta) := \frac{3(1+\beta)(N-a)}{2(2-a)(2+\delta)}\delta, \tag{3.26}$$

and C_δ *is a constant depending on* δ.

Remark 4. *We do not assume that the data are compactly supported. Hence our result is an extension of the results of Ikehata et al. [25] to noncompactly supported data cases.*

Proof. We prove an a priori estimate for the following functional:

$$M(t) = \sup_{0\leqslant\tau<t}\Bigg[(1+\tau)^{B+1-\varepsilon}\int_{\mathbb{R}^N} e^{2\psi}(u_t^2+|\nabla u|^2)dx \\ +(1+\tau)^{B-\varepsilon}\int_{\mathbb{R}^N} e^{2\psi}\phi(x,t)u^2dx\Bigg], \tag{3.27}$$

where

$$B := \frac{(1+\beta)(N-a)}{2-a}+\beta,$$

and ε is given by 3.26.

From 3.4, 3.5, it is easy to see that

$$-\psi_t = \frac{1+\beta}{1+t}\psi, \tag{3.28}$$

$$\nabla\psi = A\frac{(2-a)\langle x\rangle^{-a}x}{(1+t)^{1+\beta}}, \tag{3.29}$$

$$\Delta\psi := \left(\frac{(1+\beta)(N-a)}{2(2-a)}-\delta_1\right)\frac{\phi(x,t)}{1+t}. \tag{3.30}$$

We also have

$$\begin{aligned}(-\psi_t)\phi(x,t) &= A(1+\beta)\frac{\langle x\rangle^{2-2a}}{(1+t)^{2+2\beta}} \\ &\geqslant \frac{(1+\beta)}{(2-a)^2A}A^2(2-a)^2\frac{\langle x\rangle^{-2a}|x|^2}{(1+t)^{2+2\beta}} \\ &= (2+\delta)|\nabla|^2.\end{aligned} \tag{3.31}$$

By multiplying 3.1 by $e^{2\psi}u_t$, it follows that

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right]-\nabla(e^{2\psi}u_t\nabla u)+e^{\psi}\left(\phi(x,t)-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t\right)u_t^2 \\ +\underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2}_{T_1} = \frac{\partial}{\partial_t}[e^{2\psi}F(u)]+2e^{2\psi}(-\psi_t)F(u), \tag{3.32}$$

where F is the primitive of f satisfying $F(0) = 0$, namely $F(u) = f(u)$. Using the Schwarz inequality 3.11 and 3.31, we can calculate

$$\begin{aligned} T_1 &= \frac{e^{2\psi}}{-\psi_t}(\psi_t^2|\nabla|^2 - 2\psi_t u_t \nabla u \nabla \psi + u_t^2 |\nabla|^2) \\ &\geqslant \frac{e^{2\psi}}{-\psi_t}\left(\frac{1}{5}\psi_t^2|\nabla u|^2 - \frac{1}{4}u_t^2|\nabla \psi|^2\right) \\ &\geqslant e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u|^2 - \frac{\phi(x,t)}{4(2+\delta)}u_t^2\right). \end{aligned} \tag{3.33}$$

From this and 3.31, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2)\right] - \nabla(e^{2\psi}u_t\nabla u) + e^{2\psi}\left\{\left(\frac{1}{4}\phi(x,t) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\} \\ &\leqslant \frac{\partial}{\partial_t}\left[e^{2\psi}F(u)\right] + 2e^{2\psi}(-\psi_t)F(u). \end{aligned} \tag{3.34}$$

By multiplying 3.34 by $(t_0 + t)^{B+1-\varepsilon}$, here $t_0 \geqslant 1$ is determined later, it follows that

$$\begin{aligned} &\frac{\partial}{\partial t}\left[(t_0 + t)^{B+1-\varepsilon}\frac{e^{2\psi}}{2}\right] - (B + 1 - \varepsilon)(t_0 - t)^{B-\varepsilon}\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2) \\ &- \nabla\left((t_0 - t)^{B+1-\varepsilon}e^{2\psi}u_t\nabla u\right) + e^{2\psi}(t_0 + t)^{B+1+\varepsilon}\left\{\left(\frac{1}{4}\phi(x,t) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\} \\ &\leqslant \frac{\partial}{\partial t}[(t_0 + t)^{B+1-\varepsilon}\, e^{2\psi}F(u) - (B + 1 - \varepsilon)(t_0 + t)^{B-\varepsilon}e^{2\psi}F(u) + 2(t_0 + t)^{B+1-\varepsilon}e^{2\psi}(-\psi_t)F(u). \end{aligned} \tag{3.35}$$

We put

$$E(t) = \int_{\mathbb{R}^N} e^{2\psi}(u_t^2 + |\nabla u|^2)dx; \quad E_\psi(t) = \int_{\mathbb{R}^N} e^{2\psi}(-\psi_t)(u_t^2 + |\nabla u|^2)dx,$$

$$J(t; g) = \int_{\mathbb{R}^N} e^{2\psi}gdx; \quad J_\psi(t; g) = \int_{\mathbb{R}^N} e^{2\psi}(-\psi_t)gdx.$$

Integrating 3.35 over the whole space, we have

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left[(t_0 + t)^{B+1-\varepsilon}E(t)\right] - \frac{1}{2}(B + 1 - \varepsilon)(t_0 + t)^{B+\varepsilon}E(t) \\ &+ \frac{1}{4}(t_0 - t)^{B+1-\varepsilon}J(t, \phi(t,x)u_t^2) + \frac{1}{5}(t_0 - t)^{B+1-\varepsilon}E_\psi(t) \\ &\leqslant \frac{\partial}{\partial_t}\left[(t_0 + t)^{B+1-\varepsilon}\int e^{2\psi}F(u)dx\right] + C(t_0 + t)^{B+1-\varepsilon}J_\psi(t; |u|^{p+1}) \\ &+ C(t_0 + t)^{B-\varepsilon}J(t; |u|^{p+1}). \end{aligned} \tag{3.36}$$

Therefore, we integrate on the interval $[0, t]$ and obtain the estimate for $(t_0 + t)^{B+1-\varepsilon}E(t)$, which is the first term of $M(t)$:

$$\begin{aligned}(t_0 - t)^{B+1-\varepsilon}E(t) - C\int_0^t (t_0+\tau)^{B-\varepsilon}E(\tau)d\tau + \int_0^t (t_0+\tau)^{B+1-\varepsilon}J(\tau;\phi(x,t)u_t^2)\\ + (t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau\\ \leqslant CI_0^2 + C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}) + C\int (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\ + C\int_0^t (t_0+t)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau.\end{aligned} \tag{3.37}$$

In order to complete the a priori estimate, however, we have to manage the second term of the inequality above whose sign is negative, and we also have to estimate the second term of $M(t)$. The following argument, which is little more complicated, can settle both these problems.
At first, we multiply 3.1 by $e^{2\psi}u$ and have

$$\begin{aligned}\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{\phi(x,t)}{2}u^2\right)\right] - \nabla(e^{2\psi}u\nabla u)\\ + e^{2\psi}\left\{|\nabla u|^2 + \left(-\psi_t + \frac{\beta}{2(1+t)}\right)\phi(x,t)u^2 + \underbrace{2u\nabla\psi\nabla u}_{T_2} - 2\psi_t uu_t - u_t^2\right\}\\ = e^2 uf(u).\end{aligned} \tag{3.38}$$

We calculate

$$\begin{aligned} e^{2\psi}T_2 &= 4e^{2\psi}u\nabla u\nabla\psi - 2e^{2\psi}u\nabla\psi\nabla u\\ &= 4e^{2\psi}u\nabla\psi\nabla u - \nabla(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2,\end{aligned}$$

and by 3.30 we can rewrite 3.38 to

$$\begin{aligned}\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{\phi(x,t)}{2}u^2\right)\right] - \nabla(e^{2\psi}(u\nabla u + u^2\nabla\psi))\\ + e^{2\psi}\left\{\underbrace{|\nabla u|^2 + 4u\nabla u\nabla\psi + ((-\psi_t)\phi(x,t) + 2|\nabla\psi|^2)u^2}_{T_3}\right.\\ \left. + (B - 2\delta_1)\frac{\phi(x,t)}{2(1+t)}u^2 - 2\psi_t uu_t - u_t^2 \leqslant e^{2\psi}uf(u).\right.\end{aligned} \tag{3.39}$$

It follows from 3.29 that

$$\begin{aligned} T_3 &= |\nabla u|^2 + 4u\nabla u\nabla\psi + \left\{\left(1 - \frac{\delta}{3}(-\psi_t)\phi(x,t) + 2|\nabla\psi|^2\right\}u^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t)u^2\right.\\ &\geqslant |\nabla u|^2 + 4u\nabla u\nabla\psi + \left(4 + \frac{\delta}{3} - \frac{\delta^2}{3}\right)|\nabla\psi|^2u^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t)u^2\\ &= \left(1 - \frac{4}{4+\delta_2}\right)|\nabla u|^2 + \delta_2|\nabla\psi|^2u^2 \mid \frac{2}{\sqrt{4+\delta_2}}\nabla u + \sqrt{4+\delta_2}u\nabla\psi\mid^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t)u^2\\ &\geqslant \delta_3(|\nabla u|^2 + |\nabla|^2u^2) + \frac{\delta}{3}(-\psi_t)\phi(x,t)u^2,\end{aligned}$$

where $\delta_2 := \frac{\delta}{6} - \frac{\delta^2}{6} \quad \delta_3 := \left(1 - \frac{4}{4+\delta_2}, \delta_2\right)$. Thus, we obtain

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{\phi(x,t)}{2}u^2\right)\right] - \nabla(e^{2\psi}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\delta_3|\nabla u|^2$$
$$+ e^{2\psi}\left(\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t) + (B - 2\delta_1)\frac{\phi(x,t)}{2(1+t)}\right)u^2 + e^{2\psi}(-2\psi_t uu_t - u_t^2) \quad (3.40)$$
$$\leqslant e^{2\psi}uf(u).$$

Following , related to the size of $1 + |x|^2$ and the size of $(1+t)^2$, we divide the space $\mathbb{R}^N$ into two different zones $\Omega(t; K, t_0)$ and $\Omega^c(t; k, t_0)$, where

$$\Omega = \Omega(t; k, t_0) := \{x \in \mathbb{R}^N; (t_0 + t)^2 \geqslant K + |x|^2\},$$

and $\Omega^c := \mathbb{R}^N \backslash \Omega(t; k, t_0)$ with $K \geqslant 1$ determined later. Since $\phi(x,t)(t + t0)^{(\alpha+\beta)}$ in the domain Ω, we multiply 3.34 by $(t_0 + t)^{\alpha+\beta}$ and obtain

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(t_0+t)^{\alpha+\beta}(u_t^2 + |\nabla u|^2)\right] - \nabla(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u) + e^{2\psi}\left[\left(\frac{1}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right)\right.$$
$$\left. + (t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2 + e^{2\psi}\left[\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right]|\nabla u|^2 \quad (3.41)$$
$$\leqslant \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)] - \frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u) + 2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u).$$

Let υ be a small positive number depending on δ, which will be chosen later.
By 3.41 $+\upsilon\times$ 3.40, we have

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2 + \upsilon uu_t + \frac{\upsilon\phi(x,t)}{2}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] - \nabla(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u)$$
$$+ \upsilon e^{2\psi}(u\nabla + u^2\nabla u + u^2\nabla\psi)) + e^{2\psi}\left[\left(\frac{1}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} - \upsilon\right) + (t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2$$
$$+ e^{2\psi}\left[\upsilon\delta_3 - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} + \frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2$$
$$+ e^{2\psi}\upsilon\left[\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t) + (B - 2\delta_1)\frac{\phi(x,t)}{2(1+t)}\right]u^2 + 2\upsilon e^{2\psi}(-\psi_t)uu_t$$
$$\leqslant \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)] - \frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u) + 2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u) + \upsilon e^{2\psi}uf(u). \quad (3.42)$$

By the Schwarz inequality, the last term of the left hand side in the above inequality can be estimated as

$$|2\upsilon(-\psi_t uu_t)| \leqslant \frac{\upsilon\delta}{3}(-\psi_t)\phi(x,t)u^2 + \frac{3\upsilon}{\delta}(-\psi_t)(t_0+t)^{\alpha+\beta}u_t^2.$$

Thus, we have

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu uu_t+\frac{\nu\varphi(x,t)}{2}u^2+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]-\nabla(e^{\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u$$
$$+\nu e^{2\psi}(u\nabla u+u^2\nabla\psi))+e^{\psi}\left[\left(\frac{1}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}-\nu\right)+\left(1-\frac{3\nu}{\delta}\right)(t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2$$
$$+e^{2\psi}\left[\nu\delta_3-\frac{\alpha+\beta}{2(t_0-t)^{1-\alpha-\beta}}+\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2+e^{2\psi}\left[\nu\left(\delta_3|\nabla\psi|^2+(B-\delta_1)\frac{\varphi(x,t)}{2(1+t)}\right)\right]u^2$$
$$\leqslant\frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)]-\frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u)+2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u)+\nu e^{2\psi}uf(u).$$
3.43

Now we choose the parameters ν and t_0 such that

$$\begin{cases}\frac{1}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}-\nu\geqslant C_0, & \text{if } 1-\frac{3\nu}{\delta}\geqslant C_0,\\ \nu\delta_3-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\geqslant C_0, & \text{if } \nu\delta_3\geqslant C_0,\ \frac{1}{5}\geqslant C_0,\end{cases}$$

hold for some constant $c_0>0$. This is possible because we first determine ν sufficiently small depending on δ and then we choose t_0 sufficiently large depending on ν. Therefore, integrating 3.43 on Ω, we obtain the following energy inequality:

$$\frac{d}{dt}\overline{E}_{\psi}(t;\Omega(t;K,t_0))-N_1(t)-M_1(t)+H_{\psi}(t;\Omega(t;K,t_0))\leqslant P_1, \tag{3.44}$$

where

$$\overline{E}_{\psi}(t;\Omega):=E_{\psi}(t;\Omega(t;K,t_0))$$
$$:=\int_{\Omega}e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu uu_t+\frac{\nu\varphi(x,t)}{2}u^2+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)dx,$$

$$N_1(t):=\int_{S^{n-1}}e^{2\psi}\left[\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu uu_t+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2+\frac{\nu\varphi(x,t)}{2}u^2\right]_{|x|=\sqrt{(t_0+t)^2-K}}$$
$$\times[(t_0+t)^2-K]^{(n-1)/2}d\partial\frac{d}{dt}\sqrt{(t_0+t)^2-K},$$

$$M_1(t):=\int_{\partial\Omega}(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u+\nu e^{2\psi}(u\nabla u+u^2\nabla\psi))\vec{n}\,dS,$$

$$H_{\psi}(t;\Omega)=H_{\psi}(t;\Omega(t;K,t_0))$$
$$:=C_0\int_{\Omega}e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))(u_t^2+|\nabla|^2)dx$$
$$+\nu(B-2\delta_1)\int_{\Omega}\frac{e^{2\psi}\varphi(t,x)}{2(1+t)}u^2dx,$$

and

$$P_1 := \frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx\right] - \int_{S^{n-1}}(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)$$
$$\times[(t_0+t)^2-K]^{(n-1)/2}d\partial\frac{d}{dt}\sqrt{(t_0+t)-K} + C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.$$

Here $\vec{n}$ denotes the unit outer normal vector of $\partial\Omega$. We note that by $\nu \leqslant 1/4$ and

$$|\nu u u_t| \leqslant \frac{\nu\phi(x,t)}{4}u^2 + \nu(t_0+t)^{\alpha+\beta}u_t^2,$$

it follows that

$$C\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_\Omega e^{2\psi}\phi(x,t)u^2dx$$
$$\leqslant \overline{E}_\psi(t;\Omega(t;K,t_0)) \leqslant C\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + C\int_\Omega e^{2\psi}\phi(x,t)u^2dx,$$

for some constants $c>0$ and $C>0$. Next, we derive an energy inequality in the domain Ω^c. We use the notation

$$\langle x\rangle_K := (K+|x|^2)^{1/2}.$$

Since $\phi(x,t) \geqslant \langle x\rangle_K^{-(\alpha+\beta)}\text{in}\Omega^c(t;K,t_0)$ we multiply 3.34 by $\langle x\rangle_K^{\alpha+\beta}$ and obtain

$$\frac{\partial}{\partial_t}\left[\frac{e^{2\psi}}{2}\langle x\rangle_K^{\alpha+\beta}(u_t^2+|\nabla|^2)\right] - \nabla(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u) + e^{2\psi}\left(\frac{1}{4}+(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right)u_t^2$$
$$+\frac{1}{5}e^{2\psi}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}|\nabla u|^2 + (\alpha+\beta)e^{2\psi}\langle x\rangle_K^{\alpha+\beta-2}xu_t\nabla u \tag{3.45}$$
$$\leqslant \frac{\partial}{\partial t}[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)] + 2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u).$$

By 3.45 + $\hat{\nu}\times$ 3.40, here $\hat{\nu}$ is a small positive parameter determined later, it follows that

$$\frac{\partial}{\partial t}\left[\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}\phi(x,t)}{2}u^2 + \frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] - \nabla(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u$$
$$+\nu e^{2\psi}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\left[\frac{1}{4}-\hat{\nu}+(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2 + e^{2\psi}\left[\hat{\nu}\delta_3 + \frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2$$
$$+e^{2\psi}\left[\hat{\nu}\left(\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)\phi(x,t) + (B-2\delta_1)\frac{\phi(x,t)}{2(1+t)}\right)\right]u^2$$
$$+e^{2\psi}\underbrace{[(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}xu_t\nabla u - 2\hat{\nu}\psi_t uu_t]}_{T_4}$$
$$\leqslant \frac{\partial}{\partial_t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right] + 2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u) + \hat{\nu}e^{2\psi}uF(u). \tag{3.46}$$

The terms T_4 can be estimated as

$$|(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}xu_t\nabla u| \leqslant \frac{\hat{\nu}\delta_3}{2}|\nabla u|^2 + \frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}u_t^2,$$

$$|2\hat{\nu}(-\psi_t)uu_t| \leqslant \frac{\hat{\nu}\delta}{3}(-\psi_t)\phi(x,t)u^2 + \frac{3\hat{\nu}}{\delta}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}u_t^2.$$

From this we can rewrite 3.46 as

$$\begin{aligned}
&\frac{\partial}{\partial_t}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}\phi(x,t)}{2}|\nabla u|^2\right)\right] - \nabla(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u \\
&+ \hat{\nu}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\left[\left(\frac{1}{4} - \hat{\nu} - \frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}\right) + \left(1 - \frac{3\hat{\nu}}{\delta}\right)(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2 \\
&+ e^{2\psi}\left[\frac{\hat{\nu}\delta_3}{2} + \frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2 + e^{2\psi}\left[\hat{\nu}\left(\delta_3|\nabla\psi|^2 + (B-2\delta_1)\frac{\phi(x,t)}{2(1+t)}\right)\right]u^2 \\
&\leqslant \frac{\partial}{\partial_t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right] + 2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u) + \hat{\nu}e^{2\psi}f(u).
\end{aligned} \tag{3.47}$$

Now we choose the parameters $\hat{\nu}$ and K in the same manner as before. Indeed taking $\hat{\nu}$ sufficiently small depending on δ and then choosing K sufficiently large depending on $\hat{\nu}$, we can obtain

$$\frac{1}{4} - \hat{\nu} - \frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}} \geqslant c_1, \quad 1 - \frac{3\hat{\nu}}{\delta} \geqslant c_1, \quad \nu\delta_3 \geqslant c_1, \quad \frac{1}{5} \geqslant c_1$$

for some constant $c_1 > 0$. Consequently, By integrating 3.47 on Ω^c, the energy inequality on Ω^c follows:

$$\frac{d}{dt}\overline{E}_\psi(t;\Omega^c(t;K,t_0)) + N_2(t) + M_2(t) + H_\psi(t;\Omega^c(t;K,t_0)) \leqslant P_2, \tag{3.48}$$

where

$$\begin{aligned}
\overline{E}_\psi(t;\Omega^c) &= \overline{E_\psi}(t;\Omega^c(t;K,t_0)) \\
&:= \int_{\Omega^c} e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}\phi(x,t)}{2}u^2 + \frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)dx,
\end{aligned}$$

$$\begin{aligned}
N_2(t) := &\int_{S^{n-1}}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}\phi(x,t)}{2}u^2 + \frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]_{|x|=\sqrt{(t_0+t)^2-K}} \\
&\times[(t_0+t)^2 - K]^{(n-1)/2}d\partial\frac{d}{dt}\sqrt{(t_0+t)^2-K},
\end{aligned}$$

$$M_2(t) := \int_{\partial\Omega^c}(e^{2\psi}\langle x\rangle_K^{\alpha+\beta)}u_t\nabla u + \hat{\nu}e^{2\psi}(u\nabla u + u^2\nabla\psi))\vec{n}\,dS,$$

$$\begin{aligned}
H_\psi(t;\Omega^c) &= H_\psi(t;\Omega^c(t;K,t_0)) \\
&:= c_1\int_\Omega(1 + \langle x\rangle_K^{\alpha+\beta}(-\psi_t))(u_t^2 + |\nabla u|^2)dx + \hat{\nu}(B-2\delta_1)\int_{\Omega_c}\frac{e^{2\psi}\phi(x,t)}{2(1+t)}u^2dx,
\end{aligned}$$

and

$$P_2 := \frac{d}{dt}\left[\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta)} F(u)dx\right] + \int_{S^{n-1}} \langle x\rangle_K^{\alpha+\beta)} e^{2\psi} F(u)|_{|x|=\sqrt{(t_0-t)^2-K}}$$
$$\times\ [(t_0+t)^2-K]^{(n-1)/2} d\vartheta \frac{d}{dt}\sqrt{(t_0+t)^2-K} + C\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.$$

In a similar way as for the case in Ω, we note that

$$c\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_{\Omega^c} e^{2\psi}\varphi(x,t)u^2dx \leqslant \overline{E}_\psi(t;\Omega^c(t;K,t_0))$$
$$\leqslant C\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + C\int_{\Omega^c} e^{2\psi}\varphi(x,t)u^2dx,$$

for some constants $c>0$ and $C>0$. We add the energy inequalities on Ω and Ω^c. We note that replacing ν and $\hat{\nu}$ by $\nu_0 := \min \nu, \hat{\nu}$, we can still have the inequalities 3.44 and 3.48, provided that we retake t_0 and Klarger. By (3.44 + 3.48)$\times(t_0+t)^{B-\varepsilon}$, we have

$$\frac{d}{dt}[(t_0+t)^{B-\varepsilon}(\overline{E_\psi}(t;\Omega)+\overline{E_\psi}(t;\Omega^c))]$$
$$-\underbrace{(B-\varepsilon)(t_0+t)^{B-1-\varepsilon}(\overline{E_\psi}(t;\Omega)+\overline{E_\psi}(t;\Omega^c))}_{T_5}+\underbrace{(t_0+t)^{B-\varepsilon}(H_\psi(t;\Omega)+H_\psi(t;\Omega^c)}_{T_6} \qquad 3.49$$
$$\leqslant (t_0+t)^{B-\varepsilon}(P_1+P_2),$$

here we note that

$$N_1(t)=N_2(t), \qquad M_1(t)=M_2(t)$$

on $\partial\Omega$ Since

$$|\hat{\nu}uu_t| \leqslant \frac{\nu_0\delta_4}{2}\varphi(x,t)u^2 + \frac{\nu_0}{2\delta_4}(t_0+t)^{\alpha+\beta}u_t^2$$

on Ω and

$$|\nu_0uu_t| \leqslant \frac{\nu_0\delta_4}{2}\varphi(x,t)u^2 + \frac{\nu_0}{2\delta_4}\langle x\rangle_K^{\alpha+\beta}u_t^2$$

on Ω^c, we have

$$-T_5+T_6 \geqslant (t_0+t)^{B-\varepsilon}I_1 + (t_0+t)^{B-\varepsilon}I_2, \qquad 3.50$$

where

$$I_1 := \int_\Omega e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4}\right)(t_0+t)^{\alpha+\beta}\right\}u_t^2$$
$$+e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}(t_0+t)^{\alpha+\beta}\right\}|\nabla u|^2dx$$
$$+\int_{\Omega^c} e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4}\right)\langle x\rangle_K^{\alpha+\beta}\right\}u_t^2$$
$$+e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\langle x\rangle_K^{\alpha+\beta}\right\}|\nabla u|^2dx$$
$$:= I_{1,1}+I_{1,2},$$

and

$$I_2 := \nu_0(B-2\delta_1-(1+\delta_4)(B-\varepsilon))\left(\int_\Omega+\int_{\Omega^c}\right)e^{2\psi}\frac{\phi(x,t)}{2(1+t)}u^2dx+\frac{c_2}{2}\int_{\mathbb{R}^N}e^{2\psi}(u_t^2+|\nabla u|^2)dx,$$

where $c_2 := \min(c_0, c_1)$. Recall the definition of ε and δ_1 (i.e. (3.26) and 3.30). A simple calculation shows $\varepsilon = 3\delta_1$. Choosing δ_4 sufficiently small depending on ε, we have

$$(t_0+t)^{B-\varepsilon}I_2 \geqslant c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbb{R}^N}e^{2\psi}\phi(x,t)u^2dx+\frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t)$$

for some constant $c_3 > 0$. Next, we prove that $I_1 \geqslant 0$. By noting that $\alpha+\beta<1$, it is easy to see that $I_{1,1} \geqslant 0$ if we retake t_0 larger depending on c_0, ν_0 and δ_4. To estimate $I_{1,2}$, we further divide the region Ω^c into

$$\Omega^c(t;K,t_0) = (\Omega^c(t;K,t_0)\cap\Sigma_L)\cup(\Omega^c(t;K,t_0)\cap\Sigma_L^c),$$

where

$$\Sigma_l := \{x\in\mathbb{R}^N; \langle x\rangle^{2-\alpha}\leqslant L(1+t)^{1+\beta}\},\quad \Sigma_L^c := \mathbb{R}^N\backslash\Sigma_L,$$

with $L >> 1$ determined later. First, since $K+|x|^2 \leqslant K(1+|x|^2) \leqslant KL^{2/(2\alpha)}(1+t)^{2(1+\beta)/(2-\alpha)}$ on $\Omega_c\cap\Sigma_L$, we have

$$\begin{aligned}&\frac{c_1}{2}(1+\langle x\rangle_K^{\beta+\alpha}(-\psi_t))-\frac{B-\varepsilon}{2(t_0-t)}\left(1+\frac{2\nu_0}{\delta_4}\right)\langle x\rangle_K^{\beta+\alpha}\\ &\geqslant \frac{c_1}{2}-\frac{B-\varepsilon}{2(t_0-t)}\left(1+\frac{2\nu_0}{\delta_4}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}}.\end{aligned}$$

We note that $-1+\frac{(1+\beta)(\alpha+\beta)}{2-\alpha} < 0$ by $\alpha+\beta<1$. Thus, we obtain

$$\frac{c_1}{2}-\frac{B-\varepsilon}{2(t_0-t)}\left(1+\frac{2\nu_0}{\delta_4}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}} \geqslant 0,$$

for large t_0 depending on L and K. Secondly, on $\Omega^c\cap\Sigma_L^c$, we have

$$\begin{aligned}&\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))-\frac{B-\varepsilon}{2(t_0-t)}\left(1+\frac{2\nu_0}{\delta_4}\right)\langle x\rangle_K^{\alpha+\beta}\\ &\geqslant\left\{\frac{c_1}{2}(1+\beta)\frac{\langle x\rangle_K^{2-\alpha}}{(1+t)^{2+\beta}}-\frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4}\right)\right\}\langle x\rangle_K^{\alpha+\beta}\\ &\geqslant\left\{\frac{c_1}{2}(1+\beta)\frac{L}{(1+t)}-\frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4}\right)\right\}\langle x\rangle_K^{\alpha+\beta}.\end{aligned}$$

Therefore one can obtain $I_{1,2} \geqslant 0$, provided that $L \geqslant \frac{B-\varepsilon}{c_1(1+\beta)}(1+\frac{2\nu_0}{\delta_4})$. Consequently, we have $I_1 \geqslant 0$. By 3.50 and what we mentioned above, it follows that

$$-T_5+T_6 \geqslant c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbb{R}^N}e^{2\psi}\phi(t,x)u^2dx+\frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t).$$

Therefore, we have

$$\frac{d}{d_t}[(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c))]+\frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t)+c_3(t_0+t)^{B-1-\varepsilon}J(t;\phi(t,x)u^2)$$
$$\leqslant (t_0+t)^{B-\varepsilon}(p_1+p_2). \tag{3.51}$$

Integrating 3.51 on the interval $[0,t]$, one can obtain the energy inequality on the whole space:

$$\begin{aligned}(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c)+\frac{c_2}{2}\int_0^t(t_0+\tau)^{B-\varepsilon}E(\tau)d\tau\\ +c_3\int_0^t(t_0+\tau)^{B-1-\varepsilon}J(\tau;\phi(\tau,x)u^2)d\tau\leqslant CI_0^2+\int_0^t(t_0+\tau)^{B-\varepsilon}(p_1+p_2)d\tau.\end{aligned} \tag{3.52}$$

By 3.52 $+\mu\times$ 3.37, here μ is a small positive parameter determined later, it follows that

$$\begin{aligned}&(t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega)+(t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega^c)+\int_0^t\frac{c_2}{2}(t_0+\tau)^{B-\varepsilon}E(\tau)d\tau-\mu C(t_0+\tau)^{B-\varepsilon}E(\tau)d\tau\\
&+c_3\int_0^t(t_0+\tau)^{B-1-\varepsilon}J(\tau;\phi(x,\tau)u_t^2)d\tau+\mu(t_0-t)^{B+1-\varepsilon}E(t)\\
&+\mu\int_0^t(t_0+\tau)^{B+1-\varepsilon}J(\tau;\phi(\tau,x)u_t^2)+(t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau\\
&\leqslant CI_0^2+P+C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})+C\int_0^t(t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\
&+C\int_0^t(t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau,\end{aligned} \tag{3.53}$$

where

$$P=\int_0^t(t_0+\tau)^{B-\varepsilon}(p_1+p_2)d\tau.$$

Now we choose μ sufficiently small; then we can rewrite 3.53 as

$$\begin{aligned}(t_0+t)^{B+1-\varepsilon}E(t)+(t_0+t)^{B-\varepsilon}J(t;\phi(x,t)u^2)\leqslant CI_0^2+P+C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})\\
+C\int_0^t(t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\
+C\int_0^t(t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau,\end{aligned} \tag{3.54}$$

We shall estimate the right hand side of 3.54. We need the lemma of Gagliardo-Nirenberg. holds.

We first estimate $(t_0+t)^{B+1\varepsilon}J(t;|u|^{p+1})$. From the above lemma, we have

$$J(t;|u|^{p+1})\leqslant C\left(\int_{\mathbb{R}^N}e^{\frac{4}{p+1}\psi}u^2dx\right)^{(1-\sigma)(p+1)/2}\times\left(\int_{\mathbb{R}^N}e^{\frac{4}{p+1}\psi}|\nabla\psi|^2u^2dx+\int_{\mathbb{R}^N}e^{\frac{4}{p+1}\psi}|\nabla u|^2dx\right)^{\sigma(p+1)/2} \tag{3.55}$$

with $\sigma = \frac{n(p-1)}{2(p+1)}$. Since

$$\begin{aligned} e^{\frac{4}{p+1}\psi}u^2 &= (e^{2\psi}\phi(x,t)u^2)\phi(x,t)^{-1}e^{\left(\frac{4}{p+1}-2\right)\psi} \\ &\leqslant C(e^{2\psi}\phi(x,t)u^2)\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha}{2-\alpha}} e^{\left(\frac{4}{p+1}-2\right)\psi}\right]\times(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)} \\ &\leqslant C(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}e^{2\psi}\phi(x,t)u^2, \end{aligned}$$

and

$$\begin{aligned} e^{\frac{4}{p+1}\psi}|\nabla\psi|^2u^2 &\leqslant C\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^{2+2\beta}}e^{\frac{1}{2}(\frac{4}{p+1}-2)\psi}e^{\frac{1}{2}(\frac{4}{p+1}-2)\psi}e^{2\psi}u^2 \\ &\leqslant Ce^{\frac{1}{2}(\frac{4}{p+1}-2)\psi}e^{2\psi}\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{2-2\alpha}{2-\alpha}} e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}\right]\times(1+t)^{-2(1+\beta)+(1+\beta)(2-2\alpha)/(2-\alpha)}u^2 \\ &\leqslant C(1+t)^{-2(1+\beta)/(2-\alpha)}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}u^2 \\ &\leqslant C(1+t)^{-2(1+\beta)/(2-\alpha)}(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}e^{2\psi}\phi(x,t)u^2, \end{aligned}$$

we can estimate 3.55 as

$$\begin{aligned} J(t;|u|^{p+1}) \leqslant C(1+t)^{[\beta+(1+\beta)\alpha/(2-\alpha)](1-\sigma)(p+1)/2}J(t;\phi(x,t)u^2)^{(1-\sigma)(p+1)/2} \\ \times[(1+t)^{-1}J(t;\phi(x,t)u^2)+E(t)]^{\sigma(p+1)/2} \end{aligned}$$

and hence

$$(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}) \leqslant C\left((t_0+t)^{\gamma_1}M(t)^{(p+1)/2}+(t_0+t)^{\gamma_2}M(t)^{(p+1)/2}\right),$$

where

$$\begin{aligned} \gamma_1 &= B+1-\varepsilon+\left[\beta+(1+\beta)\frac{\alpha}{2-\alpha}\right]\frac{1-\sigma}{2}(p+1)-\frac{\sigma}{2}(p+1)-(B-\varepsilon)\frac{p+1}{2}, \\ \gamma_2 &= B+1-\varepsilon+\left[\beta+(1+\beta)\frac{\alpha}{2-\alpha}\right]\frac{1-\sigma}{2}(p+1)-(B-\varepsilon)\frac{1-\sigma}{2}(p+1)-(B+1-\varepsilon)\frac{\sigma}{2}(p+1). \end{aligned}$$

By a simple calculation it follows that if

$$p > 1+\frac{2}{N-\alpha},$$

then by taking ε sufficiently small (i.e. δ sufficiently small) both γ_1 and γ_2 are negative. We note that

$$\begin{aligned} J_\psi(t;|u|^{p+1}) &= \int_{\mathbb{R}^N} e^{2\psi}(-\psi_t)|u|^{p+1}dx \\ &\leqslant \frac{C}{1+t}\int_{\mathbb{R}^N} e^{(2+\rho)\psi}|u|^{p+1}dx, \end{aligned}$$

where ρ is a sufficiently small positive number. Therefore, we can estimate the terms

$$\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau$$

and

$$\int_0^t (t_0 - \tau)^{B-\varepsilon} J(\tau; |u|^{p+1}) d\tau$$

in the same manner as before. Noting that

$$p_1 + p_2 = \frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx + \int^{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\right]$$
$$+ C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx + C\int_{\Omega^c} e^{2\psi}(1+\langle X\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx,$$

we have

$$\begin{aligned}
p = &\int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+p_2)d\tau\\
\leqslant & CI_0^2 + C(t_0+\tau)^{B-\varepsilon}\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}F(u)dx + C(t_0+t)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\\
&+ C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_\Omega e^{2\psi}(t_0+\tau)^{\alpha+\beta}F(u)dxd\tau + C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_{\Omega_c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dxd\tau\\
&+ C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_\Omega e^{2\psi}(1+(t_0+\tau)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau\\
&+ C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau.
\end{aligned}$$

We calculate

$$\begin{aligned}
e^{2\psi}\langle x\rangle_K^{\alpha+\beta} &= e^{2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}}\langle x\rangle_K^{\alpha+\beta}\\
&\leqslant C2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha+\beta}{2-\alpha}}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}}\\
&\leqslant Ce^{(2+\rho)\psi}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}},
\end{aligned}$$

for small $\rho > 0$. Noting that $\frac{(\alpha+\beta)(1+\beta)}{2-\alpha} < 1$ and taking ρ sufficiently small, we can estimate the terms p in the same manner as estimating $(t_0+t)^{B+1\varepsilon}J(t;|u|^{p+1})$. Consequently, we have a priori estimate for $M(t)$:

$$M(t) \leqslant CI_0^2 + CM(t)^{(p+1)/2}. \quad 3.56$$

This shows that the local solution of 3.1 can be extended globally. We note that

$$e^{2\psi}\phi(x,t)(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}\beta},$$

with some constant $c > 0$. Then we have

$$\int_{\mathbb{R}^N} e^{2\psi}\phi(x,t)u^2dx(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}\int_{\mathbb{R}^N} u^2dx.$$

This implies the decay estimate of global solution 3.25 and completes the proof of Theorem 3.18. □

Remark 5. *Thus ,if $T_\varepsilon < +\infty$, 3.56 imply that*

$$\limsup_{t \longrightarrow T_\varepsilon} [\| e^{\psi(t)} u(t,.) \| + \| e^{\psi(t)} \nabla u(t,.) \| + \| e^{\psi} u_t(t,.) \|] < +\infty,$$

which contradicts the statement of latter part of 3.18.This shows $T_\varepsilon = +\infty$, and the desired decay estimates follows from 3.56.

IV

Chapter 4

Chapter 4: Life span of solutions to the semilinear wave equation with space–time dependent damping

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4.1 Intrduction

We consider the problem

$$u_{tt} - \Delta u + \varphi(t,x)u_t = |u|^p, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N,$$

with the initial condition

$$(u, u_t)(0,x) = \varepsilon(u_0, u_1)(x) \quad x \in \mathbb{R}^N.$$

Our aim is to obtain an estimate of the lifespan of solutions to 3.1. We recall some previous results for 3.1 . There are many results about global existence of solutions for 3.1 and many authors have tried to determine the critical exponent (see [26],[27] and the references therein). Here "critical" means that if $p_c < p$, all small data solutions of 3.1 are global; if $1 < p \leqslant p_c$, the local solution cannot be extended globally even for small data. In the constant coefficient case $\alpha = \beta = 0$, Todorova and Yordanov GA and Zhang [28] determined the critical exponent of 3.1 with compactly supported data as

$$p_c = 1 + \frac{2}{N}$$

This is also the critical exponent of the corresponding heat equation $-\Delta v + v_t = |v|^p$ and called the Fujita exponent (see [1]). We note that the proof by Todorova and Yordanov [29] also gives the same upper bound in the case $\beta = 0$, $1 < p < 1 + 1/n$. In this chapter we will improve the above result for all $1 < p < 1 + 2/n$ and give the sharp upper estimate. First, we define the solution of 3.1. We say that $u \in X(T)$ is a solution of 3.1 with initial data 3.2 on the interval $[0, T)$ if the identity

$$\begin{aligned} &\int_{[0,T)\times\mathbb{R}^N} u(t,x)(\partial_t^2\psi(t,x) - \Delta\psi(t,x) - \partial_t(\varphi(t,x)\psi(t,x)))\,dxdt \\ &\varepsilon\int_{\mathbb{R}^N} \{(\varphi(0,x)u_0(x) + u_1(x))\psi(0,x) - u_0(x)\partial_t\psi(0,x))\}\,dx \qquad (4.1) \\ &\quad + \int_{[0,T)\times\mathbb{R}^N} |u(t,x)|^p\psi(t,x)\,dxdt, \end{aligned}$$

holds for any $\psi \in (C_0^\infty \times \mathbb{R}^N)$.
We also define the lifespan for the local solution of 3.1-3.2 by

$$T_\varepsilon := \sup\{T \in (0,\infty];\ \text{there exists a unique solution } u \in X(T) \text{ of } 3.1 - 3.2\}.$$

4.1.1 Lower bound

Firstly we give a lower bound of life span to the solutions by the following result:

Proposition 4.2. *Let $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ be compactly supported and δ any positive number. We assume that $\alpha \in [0,1), \beta \in (-1,1), \alpha\beta \geqslant 0$ and $\alpha + \beta < 1$. Then there exists a constant $C = C(\delta, n, p, \alpha, \beta, u_0, u_1) > 0$ such that for any $\varepsilon > 0$, we have*

$$C\varepsilon^{-1/\kappa+\delta} \leqslant T_\varepsilon,$$

where

$$\kappa = \frac{2(1+\beta)}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{2}\right).$$

Proof. Multiplying Eq 3.1 by u_t, after integration by parts,the standard energy identity associated with the problem 3.1-3.2 gives

$$E_u(t) \leqslant E_u(0) + \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Let $t_0 > 0$, there exists $T \in (t_0; T_\varepsilon)$, which depends on $\varepsilon > 0$, such that for all $t \in [0;T]$

$$E_u(t) = \frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2),$$

and

$$E_u(t) \leqslant 2E_u(0),$$

where

$$E_u(0) = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2)\varepsilon^2.$$

By using *Gagliardo-Nirenberg* 3.6 we get

$$E_u(t) \leqslant E_u(0) + C\|\nabla u\|^{\partial/(p+1)}\|u\|^{(1-\partial)(p+1)},$$

where

$$\partial = \frac{N(p-1)}{2(p+1)},$$

and $C > 0$, using the *Poincare inequalities* 3.12 we get:

$$E_u(t) \leqslant E_u(0) + C\left((1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}+\varepsilon}\right)^{(1-\partial)(p+1)},$$

where

$$\|u\|^2 \leqslant C(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}+\varepsilon},$$

and
$$\varepsilon = \frac{3(1+\beta)(N-a)}{2(2-a)(2+\delta)}.$$
Thus we have
$$E_u(t) \leqslant E_u(0) + C(1+t)^{-(1+\beta)\frac{a}{2-a}+\varepsilon} 2E_u(t)^{\frac{p+1}{2}}.$$
Denote by T the first time $T > 0$ such that $E_u(T) = 2E_u(0)$. Since $E_u(0) < 2E_u(0)$, then $E_u(t) < 2E_u(0)$ for all $t \in [0;T)$. with $t = T$ we have
$$2E_u(0) \leqslant E_u(0) + C(1+T)^{-(1+\beta)\frac{a}{2-a}+\varepsilon} 2E_u(0)^{\frac{p+1}{2}}.$$
By solving this inequality with respect to T we find that the time T has the lower bound by $\varepsilon > 0$:
$$T \geqslant C\varepsilon^{-1/\kappa+\delta},$$
where
$$\kappa = \frac{2(1+\beta)}{2-a}\left(\frac{1}{p-1} - \frac{N-a}{2}\right).$$
This implies that by taking $\varepsilon > 0$ sufficiently small we can make a desired relation $t_0 < T < T_\varepsilon$, where Tm is the life span of the solution. Note that the T depends only on ε and $T \longrightarrow \infty$ when $\varepsilon \longrightarrow \infty$. □

4.1.2 Upper bound

Now,using the test function methode,we give an estimat of the life span of solution:

Theorem 4.19. *Let $a \in [0,1), \beta \in (-1,1), a\beta = 0$ and let $1 < p < 1+2/(N-a)$. We assume that the initial data $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ satisfy*
$$\langle x\rangle^{-a} Bu_0 + u_1 \in L^1(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} (\langle x\rangle^{-a} Bu_0(x) + u_1(x))\, dx > 0, \tag{4.2}$$
where
$$B = \left(\int_0^\infty e^{-\int_0^t (1+s)^{-\beta} ds} dt\right)^{-1}.$$
Then there exists $C > 0$ depending only on n, p, a, β and (u_0, u_1) such that T_ε is estimated as
$$T_\varepsilon \leqslant \begin{cases} \varepsilon^{-1/\kappa} & si\ 1 + a/(N-a) < p < 1 + 2/(N-a) \\ \varepsilon^{-(p-1)}(log(\varepsilon^{-1}))^{p-1} & si\ a > 0, p = 1 + a/(N-a) \\ \varepsilon^{-(p-1)} & si\ a > 0, 1 < p < 1 + a/(n-a). \end{cases}$$
for any $\varepsilon \in (0,1]$,
where
$$\kappa = \frac{2(1+\beta)}{2-a}\left(\frac{1}{p-1} - \frac{N-a}{2}\right).$$

Remark 6. *It is expected that the rate κ in Theorems 4.19 is sharp except for the case $a > 0$, $1 < p \leqslant 1 + a/(n-a)$ from Proposition 4.2.*

Remark 7. *The explicit form of* $\varphi = \langle x\rangle^{-\alpha}(1+t)^{-\beta}$ *is not necessary. Indeed, we can treat more general coefficients, for example,* $\varphi(t,x) = a(x)$ *satisfying* $a \in C(\mathbb{R}^N)$ *and* $0 \leqslant a(x) \leqslant \langle x\rangle^{-\alpha}$*, or* $\varphi(t,x) = b(t)$ *satisfying* $b \in C^1([0,\infty))$ *and* $b(t) \sim (1+t)^{-\beta}$*.*

Remark 8. *The same conclusion of Theorem 4.19 is valid for the corresponding heat equation* $-\Delta v + \varphi(t,x)v_t = |v|^p$ *in the same manner as our proof. Our proof is based on a test function method. Zhang [28] also used a similar way to determine the critical exponent for the case* $\alpha = \beta = 0$*. By using his method, many blow-up results were obtained for variable coefficient cases (see [30],[31]). However, the method of [28] was based on a contradiction argument and so upper estimates of the lifespan cannot be obtained. To avoid the contradiction argument, we use an idea by Kuiper. He obtained an upper bound of the lifespan for some parabolic equations . We note that to treat the time-dependent damping case, we also use a transformation of equation by Lin, Nishihara and Zhai [31] (see also [30]). At the end of this section, we explain some notation and terminology used throughout this paper. We put*

$$\|f\|_{L^p}(\mathbb{R}^N) := \left(\int_{\mathbb{R}^N} |f|^p\,dx\right)^{1/p}$$

We denote the usual Sobolev space by $H^1(\mathbb{R}^N)$ *For an interval* I *and a Banach space* X*, we define* $C^r(I,X)$ *as the Banach space whose element is an r-times continuously differentiable mapping from* I *to* X *with respect to the topology in* X*. The letter* C *indicates the generic constant, which may change from line to line. We also use the symbols* $\lesssim$ *and* $\backsim$*. The relation* $f \lesssim g$ *means* $f \leqslant Cg$ *with some constant* $C > 0$ *and* $f \backsim g$ *means* $f \lesssim g$ *and* $g \lesssim f$*.*

Proof. We first note that if $T_\varepsilon \leqslant C$, where C is a positive constant depending only on $n, p, \alpha, \beta, u_0, u_1$, then it is obvious that $T_\varepsilon \leqslant C\varepsilon^{-1/\kappa}$ for any $\kappa > 0$ and $\varepsilon \in (0,1]$. Therefore, once a constant $C = C(n,p,\alpha,\beta,u_0,u_1)$ is given, we may assume that $T_\varepsilon > C$. In the case $\beta \neq 0$, 3.1 is not divergence form and so we cannot apply the test function method. Therefore, we need to transform the equation 3.1 into divergence form. The following idea was introduced by Lin, Nishihara and Zhai [28]. Let $g(t)$ be the solution of the ordinary differential equation

$$\begin{cases} -g'(t) + (1+t)^{-\beta}g(t) = 1, \\ g(0) = B^{-1}. \end{cases} \tag{4.3}$$

The solution $g(t)$ is explicitly given by

$$g(t) = \exp^{\int_0^t (1+s)^{-\beta}ds}\left(B^{-1} - \int_0^t \exp^{-\int_0^\tau (1+s)^{-\beta}ds}\,d\tau\right).$$

By the de l'Hôpital theorem, we have

$$\lim_{t\to\infty}(1+t)^{-\beta}g(t) = 1,$$

and so $g(t) \sim (1+t)^\beta$. We note that $B = 1$ and $g(t) \equiv 1$ if $\beta = 0$. By the definition of $g(t)$, we also have $|g'(t)| \leqslant |(1+t)^{-\beta}g(t) - 1| \leqslant 1$.Multiplying the equation 3.1 by $g(t)$, we obtain the divergence form

$$(gu)_{tt} - \Delta(gu) - ((g' - 1)\langle x\rangle^{-\alpha}u)_t = g|u|^p, \tag{4.4}$$

here we note that $\alpha\beta = 0$. Therefore, we can apply the test function method to 4.2 We introduce the following test functions:

$$\phi(x) := \begin{cases} \exp(-1/(1-|x|^2)) & \text{if } (|x| < 1), \\ 0 & \text{if } (|x| > 1). \end{cases}$$

$$\eta(t) := \begin{cases} \frac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4))} + \exp(-1/(1-t^2)) & \text{si } 1/2 < t < 1, \\ 1 & \text{si } 0 \leqslant t \leqslant 1/2, \\ 0 & \text{si } t \geqslant 1. \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbb{R}^N), \eta \in C_0^\infty([0,\infty))$ and there exists a constant $C > 0$ such that for all $|x| < 1$ we have

$$\frac{|\nabla\phi|^2}{\phi(x)} \leqslant C.$$

Using this estimate, we can prove that there exists a constant $C > 0$ such that the estimate

$$|\Delta\phi(x)| \leqslant C\phi(x)^{1/p}, \tag{4.5}$$

is true for all $|x| < 1$. Indeed, putting $\varphi := \phi^{1/q}$ with $q = p/(p-1)$, we have $|\Delta\phi| = |\Delta\varphi^q| \leqslant |\Delta\varphi|\varphi^{q-1} + |\nabla\varphi|^2\varphi^{q-2} \leqslant \varphi^{q-1} = \phi^{1/p}$. In the same way, we can also prove that

$$|\eta^{'}| \leqslant C\eta^{1/p}, |\eta^{''}| \leqslant C\eta^{1/p}, \tag{4.6}$$

for $t \in [0,1)$

Let u be a solution on $[0, T_\varepsilon)$ and $\tau \in (\tau_0, T_\varepsilon)$, $R \geqslant R_0$ parameters, where $\tau \in [1, T_\varepsilon)$ are defined later. We define

$$\psi_{\tau,R}(t,x) := \eta_\tau(t)\phi_R(x) := \eta(t/\tau)\phi(x/R), \tag{4.7}$$

and

$$I_{\tau,R} := \int_{[0,\tau)\times B_R} g(t)|u(t,x)|^p\psi_{\tau,R}(t,x)dxdt,$$

$$J_R := \varepsilon\int_{B_R} (\langle x\rangle^{-a}Bu_0(x) + u_1(x))\phi_R(x)dx,$$

where $B_R = |x| < R$. Since $\psi_{\tau,R} \in C_0^\infty([0,T_\varepsilon)\times\mathbb{R}^N)$, and u is a solution on $[0, T_\varepsilon)$, we have

$$\begin{aligned} I_{\tau,R} + J_R &= \int_{[0,\tau)\times B_R} g(t)u\partial_t^2\psi_{\tau,R}dxdt - \int_{[0,\tau)\times B_R} g(t)u\Delta\psi_{\tau,R}dxdt \\ &+ \int_{[0,\tau)\times B_R} (g^{'}(t) - 1)\langle x\rangle^{-a}u\partial_t\psi_{\tau,R}dxdt \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Here we have used the property $\partial_t\psi(0,x) = 0$and substituted the test function $g(t)\psi(x,t)$ into the definition of solution 4.1.We note that for the corresponding heat equation, we have the same decomposition without the term K_1 and so we can obtain the same conclusion 6. We first estimate K_1. Par By the Hôolder inequality and 4.6,we have

$$\begin{aligned} K_1 &\leqslant \tau^{-2}\int_{[0,\tau)\times B_R} g(t)|u|\eta''(t/\tau)\phi_R(x)dxdt \\ &\leqslant C\tau^{-2}\int_{[\tau/2,\tau)\times B_R} g(t)|u|\eta(t)^{1/p}\phi_R(x)dxdt \\ &\leqslant \tau^{-2}I_{\tau,R}^{1/p}\left(\int_{\tau/2}^{\tau} g(t)dt\int_{B_R}\phi_R(x)dx\right)^{1/q} \\ &\leqslant C\tau^{-2+1/q}(1+\tau)^{\beta/q}R^{n/q}I_{\tau,R}^{1/p}. \end{aligned} \tag{4.8}$$

Using 4.5 and a similar calculation, we obtain

$$\begin{aligned} K_2 &\leqslant R^{-2}\int_{[0,\tau)\times B_R} g(t)|u||\Delta\phi(x/R)|\eta(t/\tau)dxdt \\ &\leqslant CR^{-2}\int_{[0,\tau)\times B_R} g(t)|u||\phi(x/R)|^{1/p}\eta(t/\tau)dxdt \\ &\leqslant CR^{-2}I_{\tau,R}^{1/p}\left(\int_0^{\tau} g(t)\eta(t/\tau)dt.\int_{B_R} 1dx\right)^{1/q} \\ &\leqslant C(1+\tau)^{(1+\beta)/q}R^{-2+n/q}I_{\tau,R}^{1/p}. \end{aligned} \tag{4.9}$$

For K_3 , using 4.6 and $|g'(t)-1|\lesssim C$, we have

$$\begin{aligned} K_3 &\leqslant \tau^{-1}\int_{[0,\tau)\times B_R} \langle x\rangle^{-\alpha}|u||\eta'(t/\tau)|\phi_R(x)dxdt \\ &\leqslant \tau^{-1}I_{\tau,R}^{1/p}\left(\int_{\tau/2}^{\tau} g(t)^{-q/p}dt.\int_{B_R}\langle x\rangle^{-\alpha q}\phi_R(x)dx\right)^{1/q} \\ &\leqslant C\tau^{-1+1/q}(1+\tau)^{-\beta/p}F_{p,\alpha}(R)I_{\tau,R}^{1/p}, \end{aligned} \tag{4.10}$$

where

$$F_{p,\alpha}(R) = \begin{cases} R^{-\alpha+n/q} & (\alpha q < n), \\ (\log(1+R))^{1/q} & (\alpha q = n), \\ 1 & (\alpha q > n). \end{cases}$$

Thus, putting

$$D(\tau,R) := \tau^{-(1+\beta)/p}\left(\tau^{-1+\beta}R^{q/n} + \tau^{1+\beta R^{-2+q/n}} + F_{p,\alpha}(R)\right),$$

and combining this with the estimates 4.8-4.10, we have

$$J_R \leqslant CD(\tau,R)I_{\tau,R}^{1/p} - I_{\tau,R}. \tag{4.11}$$

Now we use a fact that the inequality

$$ac^b - c \leqslant (1-b)b^{b/(1-b)}a^{1/(1-b)},$$

holds for all $a > 0$, $0 < b < 1$, $c \geqslant 0$.
We can immediately prove it by considering the maximal value of the function $f(c) =$

$ac^b - c$. From this and 4.11, we obtain

$$J_R \leqslant CD(\tau, R)^q. \tag{4.12}$$

On the other hand, by the assumption on the data and the Lebesgue dominated convergence theorem, there exist $C > 0$ and R_0 such that $J_R \geqslant C\varepsilon$ holds for all $R > R_0$. Combining this with 4.12, we have

$$\varepsilon \leqslant CD(\tau, R)^q, \tag{4.13}$$

for all $\tau \in (\tau_0, T_\varepsilon)$ and $R > R_0$
Now we define

$$\tau_0 := \max\left\{1, R_0^{(2-a)/(1+\beta)}\right\};$$

and we substitute

$$R = \begin{cases} \tau^{(1+\beta)/(2-a)} & (aq < n), \\ \tau & (aq = n), \\ 1 & (aq > n). \end{cases} \tag{4.14}$$

into 4.13. Here we note that $R > R_0$ is given by R 4.14. As was mentioned at the beginning of this section, we may assume that $\tau_0 < T_\varepsilon$. Finally, we have

$$\varepsilon \leqslant \begin{cases} \tau^{-\kappa} & (aq < n), \\ \tau^{-1/(p-1)} \log(1 + \tau) & (aq = n), \\ \tau^{-1/(p-1)} & (aq > n). \end{cases}$$

with

$$\kappa = \frac{2(1+\beta)}{2-a}\left(\frac{1}{p-1} - \frac{n-a}{2}\right).$$

We can rewrite this relation as

$$\tau \leqslant C \begin{cases} \varepsilon^{-1/\kappa} & \text{if } 1 + a/(n-a) < p < 1 + 2/(n-a), \\ \varepsilon^{-(p-1)}(\log(\varepsilon^{-1}))^{p-1} & \text{if } a > 0, p = 1 + a/(n-a), \\ \varepsilon^{-(p-1)} & \text{if } a > 0, 1 < p < 1 + a/(n-a). \end{cases}$$

Here we note that $\kappa > 0$ if and only if $1 < p < 1+2/(n-a)$ and that $aq = n$ is equivalent to $p = 1 + a/(n-a)$. Since τ is arbitrary in (τ_0, T_ε), we can obtain the conclusion of the theorem. □

Remark 9. *The results of Theorem 4.19 and Proposition 4.2 can be expressed by the following table :*

	$a = 0$	$\beta = 0$
p_c	$1 + \frac{2}{N}$	$1 + \frac{2}{N-a}$
$T_\varepsilon \leq$	$\varepsilon^{-1/\kappa}$	$\varepsilon^{-1/\kappa}$, $(1 + a/N - a < p < 1 + 2/(N-a))$ $\varepsilon^{-(p-1)}(log(\varepsilon^{-1}))^{p-1}$, $(p = 1 + (a/N - a))$ $\varepsilon^{-(p-1)}$, $(1 < p < a/N - a)$
$T_\varepsilon \geqslant$	$\varepsilon^{-1/\kappa+\delta}$	$\varepsilon^{-1/\kappa+\delta}$
κ	$(1+\beta)\left(\frac{1}{p-1} - \frac{2}{N}\right)$	$\frac{2}{2-a}\left(\frac{1}{p-1} - \frac{N-a}{2}\right)$

4.2 Blow-up of solutions

Theorem 4.20. *Let* $1 < p \leq 1 + \frac{2}{N-a}$. *Moreover, we assume that*

$$\int_{B_R} (\langle x\rangle^{-a} B u_0(x) + u_1(x)) dx > 0.$$

Then there is a blow-up solution.

Proof. Let R be a large parameter in $(0, \infty)$. We define the test function

$$\psi_R(t,x) := \eta_R(t)\phi_R(x) := \eta(t/R)\phi(x/R). \tag{4.15}$$

Suppose that u is a global solution with initial data (u_0, u_1) satisfying

$$\int_{B_R} (\langle x\rangle^{-a} B u_0(x) + u_1(x)) dx > 0.$$

Multipling equation 4.4 by 4.15 and integration by parts one can calculate

$$I_R := \int_{[0,R)\times B_R} g(t)|u(t,x)|^p \psi_R(t,x) dxdt,$$

$$V_R := \int_{B_R} (\langle x\rangle^{-a} B u_0(x) + u_1(x))\phi_R(x) dx.$$

Since $\psi_R \in C_0^\infty([0,T_\varepsilon) \times \mathbb{R}^N)$, and u is a solution on $[0, T_\varepsilon)$, we have

$$\begin{aligned} I_R + V_R &= \int_{[0,R)\times B_R} g(t) u \partial_t^2 \psi_R dxdt - \int_{[0,R)\times B_R} g(t) u \Delta\psi_R dxdt \\ &+ \int_{[0,R)\times B_R} (g'(t) - 1)\langle x\rangle^{-a} u \partial_t \psi_R dxdt \\ &= J_1 + J_2 + J_3. \end{aligned}$$

By the assumption on the data (u_0, u_1) it follows that

$$I_R < J_1 + J_2 + J_3.$$

Here we have used the property $\partial_t \psi(0,x) = 0$and substituted the test function $g(t)\psi(x,t)$ into the definition of solution 4.1 .We note that for the corresponding heat equation, we have the same decomposition without the term J_1 and so we can obtain the same conclusion 6. We first estimate J_1. By the Hôolder inequality and 4.6,we have

$$\begin{aligned} J_1 &\leqslant R^{-2}\int_{[0,R)\times B_R} g(t)|u|\eta''(t/R)\phi_R(x)dxdt \\ &\leqslant CR^{-2}\int_{[R/2,R)\times B_R} g(t)|u|\eta(t)^{1/p}\phi_R(x)dxdt \\ &\leqslant R^{-2}I_R^{1/p}\left(\int_{R/2}^{R} g(t)dt\int_{B_R}\phi_R(x)dx\right)^{1/q} \\ &\leqslant CR^{-2+1/q}(1+R)^{\beta/q}R^{n/q}I_R^{1/p}. \end{aligned} \tag{4.16}$$

Using 4.5 and a similar calculation, we obtain

$$\begin{aligned} J_2 &\leqslant R^{-2}\int_{[0,R)\times B_R} g(t)|u||\Delta\phi(x/R)|\eta(t/R)dxdt \\ &\leqslant CR^{-2}\int_{[0,R)\times B_R} g(t)|u||\phi(x/R)|^{1/p}\eta(t/R)dxdt \\ &\leqslant CR^{-2}I_R^{1/p}\left(\int_{0}^{R} g(t)\eta(t/R)dt.\int_{B_R} 1dx\right)^{1/q} \\ &\leqslant C(1+R)^{(1+\beta)/q}R^{-2+n/q}I_R^{1/p}. \end{aligned} \tag{4.17}$$

For J_3 , using 4.6 and $|g'(t)-1|\lesssim C$, we have

$$\begin{aligned} J_3 &\leqslant R^{-1}\int_{[0,R)\times B_R} \langle x\rangle^{-a}|u||\eta^{'}(t/\tau)|\phi_R(x)dxdt \\ &\leqslant R^{-1}I_R^{1/p}\left(\int_{R/2}^{R} g(t)^{-q/p}dt.\int_{B_R}\langle x\rangle^{-aq}\phi_R(x)dx\right)^{1/q} \\ &\leqslant CR^{-1+1/q}(1+R)^{-\beta/p}F_{p,a}(R)I_R^{1/p}, \end{aligned} \tag{4.18}$$

where

$$F_{p,a}(R)=\begin{cases} R^{-a+n/q} & (aq<n), \\ (\log(1+R))^{1/q} & (aq=n), \\ 1 & (aq>n). \end{cases}$$

Thus, putting

$$D(R):=R^{-(1+\beta)/p}\left(R^{-1+\beta}R^{q/n}+R^{1+\beta}R^{-2+q/n}+F_{p,a}(R)\right),$$

and combining this with the estimates 4.17-4.18, we have

$$I_R^{1/q}\leq CD(R). \tag{4.19}$$

We obtain by 4.19 the following estimation

$$I_{(R)}^{1-1/P}\leq C\left[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\right]. \tag{4.20}$$

Next we choose κ such that $\kappa = \max\{-\gamma_1, -\gamma_2, -\gamma_3\}$ so that

$$\kappa = \frac{2(1+\beta)}{2-a}\left(\frac{1}{p-1} - \frac{n-a}{2}\right).$$

Hence, we obtain

$$I_R \leq I_R^{1/p} C R^{-\kappa}. \tag{4.21}$$

If $1 < p < p_c$, by letting $R \longrightarrow 0$ we have $I_R \longrightarrow 0$ and hence $u = 0$, which contradicts the assumption on the data. If $p = p_c$, we have only $I_R \leq C$ with some constant C independent of R. This implies that $g(t)|u|^p$ is integrable on $(0, \infty) \times \mathbb{R}^N$ and hence

$$\lim_{R \longrightarrow \infty} (I_R^{1/p}) = 0.$$

By 4.21, we obtain $\lim_{R\longrightarrow\infty} I_R = 0$. Therefore, u must be 0.
This also leads a contradiction. □

Conclution

In view of our work, there have been some thought-provoking solutions which have Leads to results of the existence of the maximum time which is in dependence on the initial conditions sufficiently small. It is well known that there exists $T_\varepsilon > 0$ such that the problem prossesses a unique classical solution $u(t, x, \varepsilon)$ in $[0, T_\varepsilon)$, i.e., $u(t, x, \varepsilon) \in X$ is bounded in $[0, T_\varepsilon]$ for any $T' < T_\varepsilon$ and $\|u(t, x, \varepsilon)\|_Y \longrightarrow \infty$ when $t \longrightarrow T_\varepsilon$ if T_ε is finite. we call T_ε the life span of solution $u(t, x, \varepsilon)$ and say that $u(t, x, \varepsilon)$ blows up in finite time if $T_\varepsilon < \infty$.

In summary of our dissertation, or we have studied the problems arising from the sufficiently small initial conditions based on the results of estimating the maximum time of existence in a low-horizon domain. Based on the previous work, however, the latter also have the power to demonstrate a development which satisfies the initial conditions sufficiently large of which the problem is: We investigate the initial-boundary problem

$$u_t + (-\Delta)u = f(u), \quad (t, x) \in \Omega \times (0, \infty),$$
$$u = 0, \quad x \in \partial\Omega \times (0, \infty),$$
$$u(t, x) = \rho u_0(x), \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\rho > 0, u_0(x)$ is a nonnegative continuous function on $\overline{\Omega}$, $f(u)$ is a nonnegative superlinear continuous function on $[0, \infty]$.

We show that the life span (or blow-up time) of the solution of this problem, denoted by $T(\rho)$, satisfies $T(\rho) = \int_{\rho\|u_0\|_{\inf}} \frac{\partial u}{f(u)} + h.o.t$ as $\rho \longrightarrow \infty$. Moreover, when the maximum of u_0 is attained at a finite number of points in Ω, we can determine the higher-order term of $T(\rho)$ which depends on the minimal value of $|\Delta u_0|$ at the maximal points of u_0.

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