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Thème

A fractional order pine wilt disease model with Caputo–Fabrizio derivative

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Résumé

Dans ce travail , on presénte le modèle mathématique de Caputo-Fabrizio de la dérivation fractionnelle de la rosacée (FPWD). Et on va étudier les caractéristiques de base du modèle, ainsi l'existence et l'unicité de la solution du modèle, sont vérifiées par la théorie du point fixe du Banach.on voit que les résulttats obtenus avec ce modèle fractionnaire de Caputo-Fabrizio sont mieux et donnent plus d'information que le modéle ordinnaire .

Les mots clés : Dérivé fractionnaire de Caputo – Fabrizio (CF); Maladie du flétrissement de la pinte; Mathématique modèle; Théorème du point fixe de Banach

Abstract

A Caputo–Fabrizio type fractional order mathematical model for the dynamics of pine wilt disease (FPWD) is presented. The basic properties of the model are investigated. The existence and uniqueness of the solution for the proposed FPWD model are given via the fixed point theorem. The non-integer order derivative provides more flexible and deeper information about the complexity of the dynamics of the proposed FPWD model than the integer order models established before .

Keywords: Caputo–Fabrizio (CF) fractional derivative; Pint wilt disease; Mathematical model; Fixed point theorem .

ملخص

نموذج Caputo-Fabrizio الرياضي للإشتقاق الكسري لديناميكية مرض الذبول الصنوبري (FPWD). يتم التحقق من الخصائص الأساسية للنموذج, وجود و وحدانية الحل للنموذج , وتتم هذه العملية عن طريق او بإستخدام نظرية النقطة الثابتة لبناخ . المشتق برتبة غير صحيحة يوفر معلومات أكثر مرونة وواقعية ومتعمقة حول التعقيدات و الروابط الموجودة في ديناميكية هذا المرض (FPWD) , وهي نتائج أفضل من دراستها بمشتق ذو رتبة صحيحة .

الكلمات المفتاحية: المشتق الكسوري نموذج Caputo-Fabrizio , مرض الذبول الصنوبري , نظرية النقطة الثابة لبناخ .

Contents

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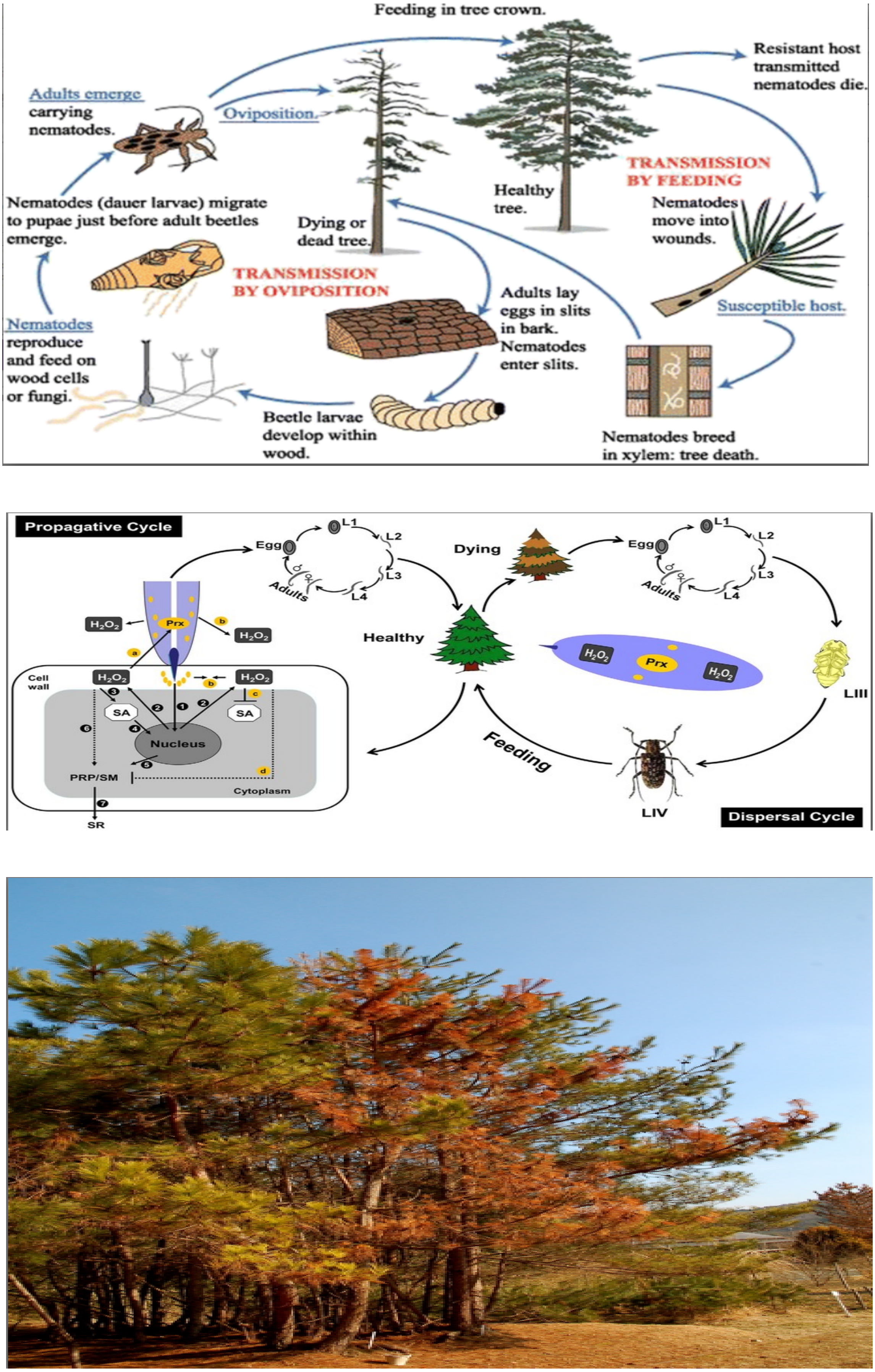

Figure 1: the Pine wilt disease PWD

Introduction

Mathematical models in epidemiology are used widely in order to understand better the dynamics of an infectious disease [1, 2]. The application of the mathematical models is not limited to only human diseases, but they are also widely applied in other phenomena of biological sciences, such as ecology, forest, etc. In the human life, forest has an important role, therefore, it is necessary to ensure the safety strategies to protect it from being infected with diseases. The forest provides greenery to the environment and pleasant atmosphere for humans. The pine wilt disease (PWD) infects pine trees and is one of the main threats to the ecosystem and forest. The PWD is considered to be the most destructive disease which damages pine trees in a short period of time, that is, a year or sometimes in a few months. The initial symptoms of the PWD include discoloration of needles, which turn from yellow to green then to reddish brown. The main agent of the disease is small worms, known as pinewood nematode (bursaphelenchus xylophilus), causing tree decline [3]. As the trees begin to die, they are attacked by insects, known as sawyers, which are species that transfer the nematode to healthy trees, which is one of the causes of pine wilt disease [4–6]. Native to North America, the PW nematode was introduced while the first epidemic of the PWD was accursed in Japan in 1905 [7], and it has spread in southern China, Korea, Taiwan, and other regions of Europe since the early 1980s [4–6, 8].

The PWD has three main organisms: the gymnosperm host, the pine wood nematode, and the insect vector. At the stage of primary transmission, dauer juveniles (JIV stage) of bursaphelenchus xylophilus are carried vertically in the tracheae of their beetle host to young twigs of susceptible trees, where they enter through resin canals in wounds made during maturation feeding by the insect [8].

Recently, some mathematical models have been presented to explore the dynamics of PWD consisting of a system of nonlinear differential equations. Lee and Kwang [9] explored the stability analysis of PWD and proposed some suitable controlling strategies for this disease. Khan et al. [10] introduced a model on PWD and its optimal control. A mathematical model with variable population and suggested optimal control was developed in [11]. Most recently, in [12] the dynamics of PWD with saturated incidence rate was explored. All of the above PWD models are restricted to classical integer order differential equations. In the present paper we consider a PWD model with saturated incidence rate in fractional environment using the CF derivative. First, we give an overview of recently published papers on fractional mathematical models using the CF derivative.

Fractional order models are more reliable and helpful in the real phenomena than the classical models due to hereditary properties and the description of memory [13, 14]. Also, in the real world explanation, the integer order derivative does not explore the dynamics between two different points. To deal with such failures of classical local differentiation, different concepts on differentiation with non-local or fractional orders have been developed in the existing literature. For instance, Riemann and Liouville introduced the concept of fractional orders differentiation in [14]. Recently, Caputo and Fabrizio [15] introduced a new derivative with fractional order based on the exponential kernel. The new CF fractional order derivative has been used successfully in modeling of various real phenomena. For example, a fractional Adams–Bashforth technique via the CF derivative was presented in [16]. A study of magnetohydrodynamic electroosmotic flow of Maxwell fluids with CF derivatives was carried out by Abdulh et al. [17]. In [18], the CF fractional derivative was used for numerical approach of the Fokker–Planck equation using Ritz approximation. A mathematical comparative analysis of RL and RC electrical circuits using AB and CF fractional derivatives was recently done in [19]. Mustafa et al.[20] explored the dynamics of the cancer treatment model with the CF fractional derivative. Recently, a new fractional model of hepatitis B virus in the CF derivative sense was presented in [21].

The classical integer order mathematical model is useful for a local dynamic system with no external forces. These models cannot therefore replicate the complexity of the dynamics of the communicable disease like PWD as the model can sometimes have a crossover behavior and this cannot be handled by the classical differential operators. Further, in the literature fractional order models provide a better fit to the real data for different diseases and other experimental work in fluid mechanics. For example, Diethelm [22] provided a good agreement to the real data of the 2009 dengue outbreak in Cape Verde using a noninteger order biological model instead of the ordinary one. A fractional order model for Ebola epidemic was applied to provide a suitable approximation to the real data on Ebola virus [23]. Makris et al. [24] used a fractional order Maxwell model to attain a better fit to the experimental work.

Therefore, motivated by the above work, in this paper, we aim to extend the recently published PWD model [12] to a fractional case by using the newly established derivative known as CF derivative of order $\tau \in (0, 1]$. The details of the remaining chapter of this paper are as follows:

we remember some definitions, general notions and fundamental theorems. We start by recall briefly some general notions, spaces, and the basic reproduction number , free and endemic equilibrum and The basic definition and results of fractional order derivative are stated in chapiter. 1. In chapiter. 2, we explore the model formulation, model equilibria, and the basic reproduction number. chapiter 3 deals with the existence of solution in the spread PWD disease model via the fixed point theorem. Also, the uniqueness of a model solution isobtained.

Finally, the concluding remarks are given in the conclusion Section.

Chapter
Preliminary knowledge

Preliminary knowledge

1.1 Preliminary

In this chapter, we present some preliminaries that will be used in the next chapters. This chapter is devoted to remember some definitions, general notions and fundamental theorems. We start by recall briefly some general notions, spaces, and the basic reproduction number , free and endemic equilibrum , then we give definitions of the new fractional derevative definition of caputo-fabrizio.

1.2 Functional analysis

1.2.1 L^p spaces

Definition 1.2.1 *Let* $I = [a, b]$ *provided with the Borel tribe and a measure on* (I, B_I). *For* $1 \leq p < \infty$, *We denote by* $L^p(I, x)$ *the set of measurable functions* $f : I \to \mathbb{R}$ *as*

$$\|f\|_p = \left(\int_I |f|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

It is clear that $L^1(I, x)$ *is a vector space. To obtain a similar result in the case* $p > 1$, *We need the following theorem.*

Definition 1.2.2 *We set*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ such that } |f(x)| \leq C \text{ on } \Omega\}.$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C; |f(x)| \leq C \text{ on } \Omega\}$$

the following remark implies that $\|.\|_\infty$ *is a norm*

Remark 1.2.1 *if* $f \in L^\infty$ *then we have and*

$$|f(x)| \leq \|f\|_\infty \ a.e \text{ on } \Omega \ .$$

indeed there exists a sequence C_n *such that* $C_n \to \|f\|_\infty$ *and for each n,* $|f(x)| \leq C_n$ *a e on* Ω. *there fore* $|f(x)| \leq C$ *for all* $x \in \Omega_n$. *With* $|E_N| = 0$ *We set* $E = \cup_{n=1}^\infty E_n$. *So that* $|E| = 0$ *and*

$$|f(x)| \leq C_n \ \forall n \quad \forall x \in \Omega$$

it follows that $|f(x)| \leq \|f\|_\infty \ \forall x \in \Omega$

Theorem 1.2.1 *Let $p, q \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. So for any measurable functions $f, g : I \to \mathbb{R}$ we have*

$$\left| \int_I f\, g\, dx \right| \leq \|f\|_p \, \|g\|_q \qquad \text{(Hölder).}$$
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad \text{(Minkowski).}$$

proof 1.2.1 *We first demonstrate the inequality of Hölder. Without loss of generality, we can suppose that $\|f\|_p = \|g\|_q = 1$. For every $x, y \geq 0$, we have*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Then

$$\left| \int_I f\, g\, dx \right| \leq \int_I |fg| \; dx \leq \int_I \left(\frac{|f|^p}{p} + \frac{|g|^q}{q} \right) \; dx = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = 1.$$

Let us now show Minkowski's inequality. We obtain

$$\begin{aligned} \|f + g\|_p^p &= \int_I |f + g|^p \; dx \leq \int_I |f + g|^{p-1} \left(|f| + |g| \right) \, dx \\ &\leq \quad \left(\int_I |f + g|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_I |f|^p \; dx \right)^{\frac{1}{p}} + \left(\int_I |g|^p \; dx \right)^{\frac{1}{p}}. \end{aligned}$$

This inequality immediately implies the desired result.

1.2.2 Sobolev space

Definition 1.2.3 *(Weak derivative) A functin $f \in L^1_{loc}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f \partial_i \phi \; dx = - \int_\Omega g_i \phi \; dx \text{ for all } \phi \in C_c^\infty(\Omega)$$

The function g_i is called the weak it's partial derivative of and is denoted by $\partial_i f$. Thus for weak derivative, the integration by parts formula

$$\int_\Omega f \partial_i \phi \; dx = \int_\Omega \partial_i f \phi \; dx$$

holds by definition for all $\phi \in C_c^\infty(\Omega)$. Since C_c^∞ is dense in $L^1_{loc}(\Omega)$. the weak derivative of a function. If it exists is unique up to pointwise almost everywhere equivalence moreover. The weak derivative of a continuously differentiable function agree with the pointwise derivative. The existence of a weak derivative is however. Not equivalent to the existence of a point wise derivative almost every where.

Definition 1.2.4 *suppose that Ω is an open set in $\mathbb{R}^n$, $k \in N$, and $1 \leq p \leq \infty$, the Sobolev space $w^{k.p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}^n$ such that*

$$\partial^a f \in L^p(\Omega) \text{ for } 0 \leq |a| \leq k.$$

We write $w^{k.2}(\Omega) = H^k(\Omega)$. the sobolev space $w^{k.p}(\Omega)$ is a banach space when equipped with the norm

$$\|f\|_{w^{k.p}(\Omega)} = \left(\sum_{|a| \leq k} \int_\Omega |\partial^a f|^p \ dx \right)^{\frac{1}{p}}$$

for $1 \leq p \leq \infty$ and

$$\|f\|_{w^{k.p}(\Omega)} = \max_{|a| \leq k} \sup_\Omega |\partial^a f|.$$

proposition 1.2.1 *if $f \in L^1_{loc}(\Omega)$ has weak partial derivative $\partial_i f \in L^1_{loc}$ and $\psi \in C^\infty$. Then ψf is weakly differentiable with respect to x_i and*

$$\partial_i(\psi f) = (\partial_i \psi) f + \psi(\partial_i f).$$

proof 1.2.2 *let $\phi \in C_c^\infty(\Omega)$ be any test function. Then $\psi\phi \in C_c^\infty$ and the weak differentiability of f implies that*

$$\int_\Omega f \partial_i(\psi\phi) \ dx = - \int_\Omega (\partial_i f) \psi\phi \ dx.$$

expanding $\partial_i(\psi\phi) = \psi(\partial_i\phi) + (\partial_i\psi)\phi$ in this equation and rearranging the result. We get

$$\int_\Omega \psi f(\partial_i\phi) \ dx = - \int_\Omega \left[(\partial_i\psi) f + \psi (\partial_i f) \right] \phi \ dx$$

thus. ψf is weakly differentiable and its weak derivative.

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$$\int_\Omega \psi f(\partial_i\phi) \ dx = - \int_\Omega \left[(\partial_i\psi) f + \psi (\partial_i f) \right] \phi \ dx$$

thus. ψf is weakly differentiable and its weak derivative.

Theorem 1.2.2 (basic properties of Sobolev spaces) • *Each Sobolev space $W^{k;p}(\Omega)$ is a Banach space*

Remark 1.2.2 *we know that $w^{k.2}(\Omega) = H^k(\Omega)$.*

Lemma 1.2.1 (convergence of weak derivatives) *Consider a sequence of functions $f_n \in L^1_{loc}(\Omega)$. For a fixed multi-index α, assume that each f_n admits the weak derivative $g_n = Df_n$. If $f_n \to f$ and $g_n \to g$ in $L^1_{loc}(\Omega)$, then $g = Df$.*

proof 1.2.4 *we need to show that the space $W^{k;p}(\Omega)$ is complete, hence it is a Banach space. Let (u_n) $1 \leq n$ be a Cauchy sequence in $W^{k;p}(\Omega)$. For any multi-index α with $|\alpha| \leq k$, the sequence of weak derivatives $D^\alpha u_n$ is Cauchy in $L^p(\Omega)$. Since the space $L^p(\Omega)$ is complete, there exist functions u and u, such that*

$$\|u_n - u\|_{L_p} \to 0 \quad , \|D^\alpha u_n - u_\alpha\|_{L_p} \to 0 \quad , for\, all \ |\alpha| \leq k$$

By Lemma 2.1 , the limit function u is precisely the weak derivative $D^\alpha u$. Since this holds for every multi-index α with $|\alpha| \leq k$, the convergence $u_n \to u$ holds in $W^{k;p}(\Omega)$. This completes the proof .

Banach Fixed Point Theorem

Definition 1.2.5 *A Banach space is a vector space X over the field $\mathbb{R}$ of real numbers, or over the field $\mathbb{C}$ of complex numbers, which is equipped with a norm $\| . \|_x$, and which is complete with respect to the distance function induced by the norm, that is to say, for every Cauchy sequence $\{x_n\}$ in X, there exists an element x in X such that :*

$$\lim_{n\to\infty} x_n = x$$

Definition 1.2.6 (Lipschitz condition) *a function $f(t, y)$ satisfies aLipschitz condition in the variable y on a set $D \subset X$ if a constant $l > 0$ exists with:*

$$\|f(t, y_1) - f(t, y_2)\| \leq l \|y_1 - y_2\|$$

whenever $(t, y1), (t, y2)$ are in D. l is Lipschitz constant.

Definition 1.2.7 (Contraction) *Let X be a normed vector space, and $f : X \to X$: We will say that f is a contraction if there exists some $0 < k < 1$ such that $\|f(x); f(y)\| < k\|x; y\|$ for all $x; y \in X$. The inf of such k is called the contraction coefcient.*

Theorem 1.2.3 (Banach's Fixed Point Theorem) *Let X be a complete normed space, and f be a contraction on X . Then there exists a unique x^* such that $f(x^*) = x^*$. The Banach Fixed Point theorem is also called the contraction mapping theorem, and it is in general use to prove that an unique solution to a given equation exists. There are several examples of where Banach Fixed Point theorem can be used .*

1.2.3 basic reproduction number

In epidemiology, the basic reproduction number (sometimes called basic reproductive ratio, or incorrectly basic reproductive rate, and denoted R_0, r nought) of an infection can be thought of as the number of cases one case generates on average over the course of its infectious period, in an otherwise uninfected population.

This metric is useful because it helps determine whether or not an infectious disease can spread through a population. The roots of the basic reproduction concept can be traced through the work of Alfred Lotka, Ronald Ross, and others, but its first modern application in epidemiology was by George MacDonald in 1952, who constructed population models of the spread of malaria. and we have When

$$\mathrm{R}_0 < 1$$

the infection will die out in the long run. But if :

$$\mathrm{R}_0 > 1$$

the infection will be able to spread in a population.

Generally, the larger the value of R_0, the harder it is to control the epidemic. For simple models and a 100 effective vaccine, the proportion of the population that needs to be vaccinated to prevent sustained spread of the infection is given by $1 - \frac{1}{R_0}$. The basic reproduction number is affected by several factors including the duration of infectivity of affected patients, the infectiousness of the organism, and the number of susceptible people in the population that the affected patients are in contact with.

In populations that are not homogeneous, the definition of R_0 is more subtle. The definition must account for the fact that a typical infected individual may not be an average individual. As an extreme example, consider a population in which a small portion of the individuals mix fully with one another while the remaining individuals are all isolated. A disease may be able to spread in the fully mixed portion even though a randomly selected individual would lead to fewer than one secondary case. This is because the typical infected individual is in the fully mixed portion and thus is able to successfully cause infections. In general, if the individuals who become infected early in an epidemic may be more (or less) likely to transmit than a randomly chosen individual late in the epidemic, then our computation of R_0 must account for this tendency. An appropriate definition for R_0 in this case is "the expected number of secondary cases produced by a typical infected individual early in an epidemic"

1.2.4 endemic and free equilibrum

In epidemiology, an infection is said to be endemic in a population when that infection is constantly maintained at a baseline level in a geographic area without external inputs. For example, chickenpox is endemic (steady state) in the UK, but malaria is not. Every year, there are a few cases of malaria reported in the UK, but these do not lead to sustained transmission in the population due to the lack of a suitable vector (mosquitoes of the genus Anopheles). While it might be common to say that AIDS is "endemic" in Africa, meaning found in an area, this is a use of the word in its etymological, rather than epidemiological, form. AIDS cases in Africa are increasing, so the disease is not in an endemic steady state. It is correct to call the spread of AIDS in Africa an epidemic. For an infection that relies on person-to-person transmission to be endemic, each person who becomes infected with the disease must pass it on to one other person on average. Assuming a completely susceptible population, that means that the basic reproduction number (R_0) of the infection must equal 1. In a

population with some immune individuals, the basic reproduction number multiplied by the proportion of susceptible individuals in the population (S) must be 1. This takes account of the probability of each individual to whom the disease may be transmitted being susceptible to it, effectively discounting the immune sector of the population.
So, for a disease to be in an endemic steady state it is:

$$\mathrm{R}_0 \times S = 1\ .$$

In this way, the infection neither dies out nor does the number of infected people increase exponentially but the infection is said to be in an endemic steady state. An infection that starts as an epidemic will eventually either die out (with the possibility of it resurging in a theoretically predictable cyclical manner) or reach the endemic steady state, depending on a number of factors, including the virulence of the disease and its mode of transmission.
If a disease is in endemic steady state in a population, the relation above allows us to estimate the R0 (an important parameter) of a particular infection. This in turn can be fed into the mathematical model of an epidemic.

1.2.5 The Caputo-Fabrizio fractional derivative

Here, we give some basic definitions of the fractional calculus that will be used in the onward analysis of the model.see [16]

Definition 1.2.8 *Let $g \in H_1(a,b)$, with b greater than a, $\tau \in [0,1]$, then the CF fractional derivative [15] is given as :*

$$D_t^{\tau}\left(g\left(t\right)\right) = \frac{M(\tau)}{1-\tau}\int\limits_a^t g'(x)exp\left[-\tau\frac{t-x}{1-\tau}\right]dx \tag{1.1}$$

In Eq. (1) . $M\left(\tau\right)$ represents a normality with $M(0) = M(1) = 1$ [15]. However, if $g \notin H_1(a,b)$, then the following expression of the derivative is obtained:

$$D_t^{\tau}\left(g\left(t\right)\right) = \frac{\tau M(\tau)}{1-\tau}\int\limits_a^t \left(g\left(t\right) - g\left(x\right)\right)\exp\left[-\tau\frac{t-x}{1-\tau}\right]dx \tag{1.2}$$

Remark 1.2.3 $\sigma = \frac{1-\tau}{\tau} \in [0,\infty), \tau = \frac{1}{1+\sigma} \in [0,1]$ *then Eq. (2) gives the following form:*

$$D_t^{\tau}\left(g\left(t\right)\right) = \frac{N(\sigma)}{\sigma}\int\limits_a^t g'\left(x\right)\exp\left[-\frac{t-x}{\sigma}\right]dx\ ,\ \ N\left(0\right) = N\left(\infty\right) = 1 \tag{1.3}$$

Moreover,

$$\lim_{\sigma\to 0} \tfrac{1}{\sigma} \exp\left[-\tfrac{t-x}{\sigma}\right] = \delta\left(x-t\right) \tag{1.4}$$

Nieto and Losada [25] give the following definition of the integral.

Let $0 < \tau < 1$, then the fractional integral of the function g having order τ is given below.

$$\mathrm{I}_t^{\tau}\left(g\left(t\right)\right) = \tfrac{2(1-\tau)}{(2-\tau)M(\tau)} g\left(t\right) + \tfrac{2\tau}{(2-\tau)M(\tau)} \int\limits_0^t g(t)ds, \;\; t \geq 0 \tag{1.5}$$

Remark 1.2.4 *From Definition 2, we have*

$$\frac{2\left(1-\tau\right)}{\left(2-\tau\right)M\left(\tau\right)} + \frac{2\tau}{\left(2-\tau\right)M\left(\tau\right)} = 1 \tag{1.6}$$

which implies $M\left(\tau\right) = \frac{2}{2-\tau}$, $0 < \tau < 1$*. In view of (6), a new Caputo derivative of order* $0 < \tau < 1$ *is suggested by Nieto and Losada [25], given as follows:*

$$D_t^{\tau}\left(g\left(t\right)\right) = \tfrac{1}{1-\tau} \int\limits_0^t g'\left(x\right) \exp\left[\tau \tfrac{t-x}{1-\tau}\right] dx \tag{1.7}$$

The CF derivative [15], given in the above definitions, has been recently used in the mathematical modeling of HBV [21], Maxwell fluid with slip effects [26], and diabetes model [27].

Chapter

A fractional order pine wilt disease model with Caputo–Fabrizio derivative

A fractional order pine wilt disease model with Caputo–Fabrizio derivative

2.1 Model formulation

Here, in this section, we extend the PWD model [12] to fractional order using a CF derivative of order $\tau \in [0, 1]$. The classical integer order PWD model is formulated by the following nonlinear system of differential equations:

$$\begin{aligned}
\frac{dS_H}{dt} &= \Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H, \\
\frac{dE_H}{dt} &= \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H - \delta_H E_H, \\
\frac{dI_H}{dt} &= \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H, \\
\frac{dI_H}{dt} &= \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V, \\
\frac{dE_V}{dt} &= \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\
\frac{dI_V}{dt} &= \delta_V E_V - \gamma_V I_V.
\end{aligned} \tag{2.1}$$

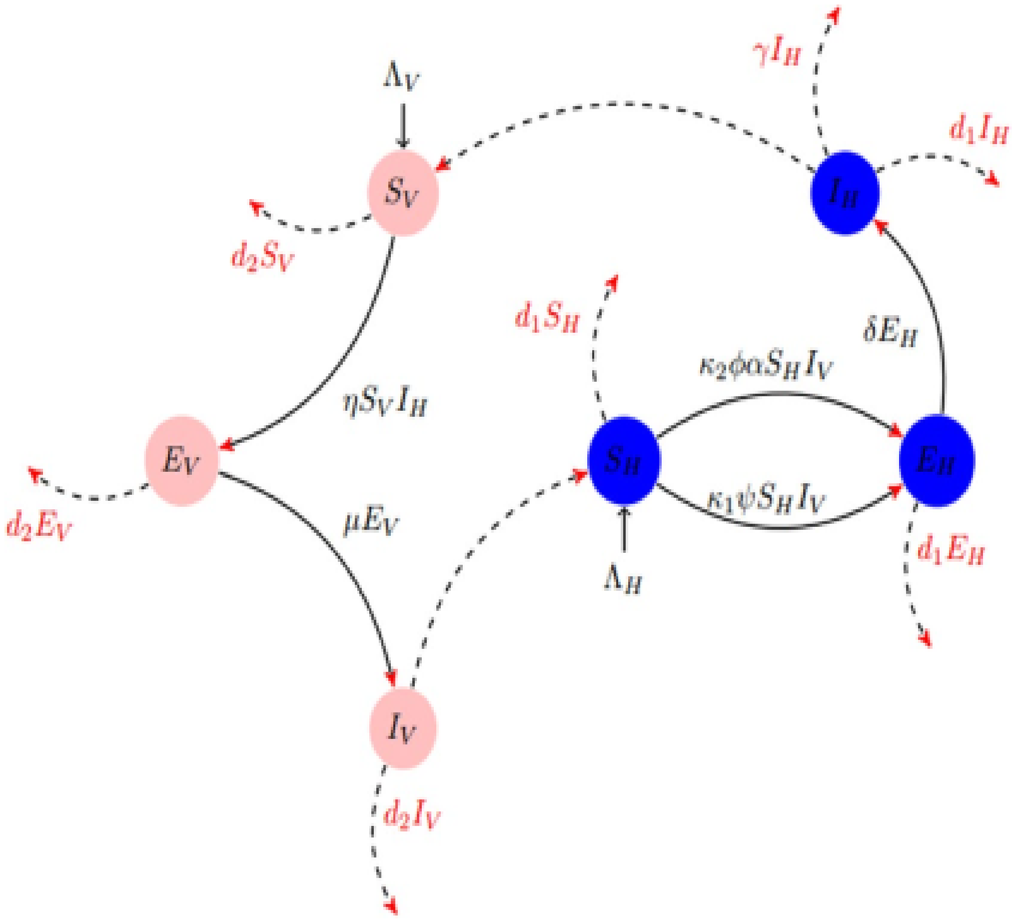

Figure 2.1: Flow chart for the transmission for the Pine wilt disease PWD

In the above model (1), the total host population (pine trees) is denoted by $N_H(t)$. It is subdivided into three classes: susceptible $S_H(t)$, exposed $E_H(t)$, and infected $I_H(t)$ pine trees.
The total vector population (beetles) is further divided into three subclasses: susceptible vector $S_V(t)$, exposed vectors $E_V(t)$, and infected vector $I_V(t)$. The recruitment rates of pine trees and vector population are denoted by π_H and π_V , respectively. The rate of contact between suspectable trees and infected vectors is K_1, while K_2 is the contact rate between suspectable trees and infected vectors when the nematode is transmitted by the infected vector at oviposition. The natural death rates of pine trees and vector population are denoted by parameters γ_H and γ_H , respectively. The natural death rate of pine trees which are uninfected through beetles is denoted by parameter ψ. The constants of saturation are θ_1 and θ_2. The exposed pine trees join the infected class at the rate δ_H while the transfer rate of an exposed vector to become an infected vector is denoted by δ_V . The parameter β_1 is the contact rate of a suspectable vector with infected pine trees. We reformulate the classical PWD model (1) by replacing the ordinary integer order derivative by the new CF fractional derivative and it can be written as follows:

$$
\begin{aligned}
{}_0^{CF}D_t^{\tau} S_H &= \Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H,\\
{}_0^{CF}D_t^{\tau} E_H &= \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H - \delta_H E_H,\\
{}_0^{CF}D_t^{\tau} I_H &= \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H,\\
{}_0^{CF}D_t^{\tau} S_H &= \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V,\\
{}_0^{CF}D_t^{\tau} E_V &= \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V,\\
{}_0^{CF}D_t^{\tau} I_V &= \delta_V E_V - \gamma_V I_V.
\end{aligned}
\tag{2.2}
$$

The initial conditions involved in (2) are :

$$S_H(0) = c_1,$$
$$E_H(0) = c_2,$$
$$I_H(0) = c_3,$$
$$S_V(0) = c_4,$$
$$E_V(0) = c_5,$$
$$\textit{and } \mathrm{I}_V(0) = c_6.$$

2.2 Equilibria and basic reproduction number

Model (2) has a disease free equilibrium $E_0 = (S_H^o, 0, 0, S_V^o, 0, 0)$ and is obtained by solving the system :

$${}_0^{CF}D_t^{\tau} S_H = {}_0^{CF}D_t^{\tau} F_H = {}_0^{CF}D_t^{\tau} I_H = {}_0^{CF}D_t^{\tau} S_V = {}_0^{CF}D_t^{\tau} E_V = {}_0^{CF}D_t^{\tau} I_V = 0 \ ,$$

and is given by:

$$E^0 = \left(\tfrac{\Pi_H}{\gamma_H}, 0, 0, \tfrac{\Pi_V}{\gamma_V}, 0, 0\right) \ .$$

The model (2) has a unique endemic equilibrium, denoted by E_1, given by

$$\begin{aligned}
S_H^* &= \frac{\Pi_H\left(1+\theta_1 I_V^*\right)}{\gamma_H + I_V^*\left(K_1 + K_2\psi + \gamma_H\theta_1\right)},\\
E_H^* &= \frac{\Pi_H K_1 I_V^*}{\left(\gamma_H+\delta_H\right)\left(K_1 I_V^* + K_2\psi I_V^* + \gamma_H + \theta_1\gamma_H I_V^*\right)},\\
I_H^* &= \frac{\Pi_H I_V^*\left(K_1\psi\gamma_H + K_1\delta_H + K_2\psi\delta_H\right)}{\gamma_H\left(\gamma_H+\delta_H\right)\left(K_1 I_V^* + K_2 I_V^* + \gamma_H + \theta_1\gamma_H I_V^*\right)},\\
S_V^* &= \frac{\Pi_V\left(1+\theta_2 I_V^*\right)}{\gamma_V + I_H^*\left(\beta_1 + \gamma_V\theta_1\right)},\\
E_V^* &= \frac{\Pi_H K_1 I_V^*}{\left(\gamma_V+\delta_V\right)\left(\gamma_V + I_V^*\left(\beta_1+\theta_2\gamma_V\right)\right)},\\
I_V^* &= \frac{\Pi_V\beta_1\delta_V I_H^*}{\gamma_V\left(\gamma_V+\delta_V\right)\left(\gamma_H + I_H^*\left(\beta_1+\theta_2\gamma_V\right)\right)}.
\end{aligned}$$

The basic reproduction number $\mathcal{R}_0$ is obtained by using the next generation technique [28] and is given as follows:

$$\mathcal{R}_0 = \sqrt{\tfrac{\delta_V\beta_1 S_H^0 S_V^0\left(K_1\delta_H + K_2\psi(\gamma_H+\delta_H)\right)}{\gamma_H\gamma_V(\gamma_H+\delta_H)(\gamma_V+\delta_V)}} \ .$$

Hence, we state the following theorem.

Theorem 2.2.1 *The FPWD model (2) has a unique endemic equilibrium if* $\mathcal{R}_0 > 1$*. see[29]*

Chapter

Existence and uniqueness of FPWD model

3.1 Existence and uniqueness of FPWD model

This section describes the existence of model solutions by using fixed point theory. We use the fractional integral operator in [25] on (2) to obtain

$$\begin{aligned}
S_H(t) - S_H(0) &= {}_0^{CF}I_t^{\tau}\left\{\Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H\right\},\\
E_H(t) - E_H(0) &= {}_0^{CF}I_t^{\tau}\left\{\frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H E_H - \delta_H E_H\right\},\\
I_H(t) - I_H(0) &= {}_0^{CF}I_t^{\tau}\left\{\frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H\right\},\\
S_V(t) - S_V(0) &= {}_0^{CF}I_t^{\tau}\left\{\Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V\right\},\\
E_V(t) - E_V(0) &= {}_0^{CF}I_t^{\tau}\left\{\frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V\right\},\\
I_V(t) - I_V(0) &= {}_0^{CF}I_t^{\tau}\left\{\delta_V E_V - \gamma_V I_V\right\}.
\end{aligned} \tag{3.1}$$

Applying the theorem in [25]:

Theorem 3.1.1 *Let $0<\alpha<1$, $T>0$ and $\varphi\ :[0,T]\times\mathbb{R}\to\mathbb{R}$ a continuous function such that there exits $L>0$ satisfying,*

$$|\varphi(t,s_1) - \varphi(t,s_2)| \leq L\,|s_1 - s_2| \quad \textit{for all}\ \ s_1, s_2 \in \mathbb{R}$$

If $(a_\alpha + b_\alpha T)L < 1$, then the initial value problem given by

$$\begin{aligned}
&{}^{CF}D^{\alpha} f(t) = \varphi(t, f(t)) \quad ,\ t\in[0;T]\\
&f(0) = f_0 \ \in \mathbb{R}
\end{aligned}$$

has a unique solution in H^1 .

we obtain

$$
\begin{aligned}
S_H(T)-S_H(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\Pi_H-\frac{K_1S_HI_V}{1+\theta_1I_V}-\frac{K_2\psi S_HI_V}{1+\theta_1I_V}-\gamma_HS_H\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\Pi_H-\frac{K_1S_HI_V}{1+\theta_1I_V}-\frac{K_2\psi S_HI_V}{1+\theta_1I_V}-\gamma_HS_H\right\}dy, \\
E_H(t)-E_H(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\frac{K_1S_HI_V}{1+\theta_1I_V}-\gamma_HE_H-\delta_HE_H\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\frac{K_1S_HI_V}{1+\theta_1I_V}-\gamma_HE_H-\delta_HE_H\right\}dy, \\
I_H(t)-I_H(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\frac{K_2\psi S_HI_V}{1+\theta_1I_V}-\gamma_HI_H+\delta_HE_H\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\frac{K_2\psi S_HI_V}{1+\theta_1I_V}-\gamma_HI_H+\delta_HE_H\right\}dy, \\
S_V(T)-S_V(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\Pi_V-\frac{K_1S_VI_H}{1+\theta_1I_H}-\gamma_VS_V\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\Pi_V-\frac{\beta_1S_VI_H}{1+\theta_1I_H}-\gamma_VS_V\right\}dy, \\
E_V(t)-E_V(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\frac{\beta_1S_VI_H}{1+\theta_2I_V}-\gamma_VE_V-\delta_VE_V\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\frac{\beta_1S_VI_H}{1+\theta_2I_H}-\gamma_VE_V-\delta_VE_V\right\}dy, \\
I_V(t)-I_V(0) &= \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left\{\delta_VE_V-\gamma_VI_V\right\} \\
&\quad+\frac{2\tau}{(2-\tau)M(\tau)}\int\limits_0^t\left\{\delta_VE_V-\gamma_VI_V\right\}dy.
\end{aligned}
\tag{3.2}
$$

For simplicity, we replace as follows:

$$\begin{aligned}
F_1(t, S_H) &= \Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H, \\
F_2(t, E_H) &= \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H E_H - \delta_H E_H, \\
F_3(t, I_H) &= \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H, \\
F_4(t, S_V) &= \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V, \\
F_5(t, E_V) &= \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\
F_6(t, I_V) &= \delta_V E_V - \gamma_V I_V.
\end{aligned} \tag{3.3}$$

Theorem 3.1.2 *The kernels $F_1, F_2, F_3, F_4, F_5,$ and F_6 fulfill the Lipschitz condition and contraction if the following inequality holds:*

$$0 \leq (K_1 + K_1 \psi) e + \gamma_H < 1$$

proof 3.1.1 *Here, we start from F_1 . Suppose S and S_1 are two functions, then we assess the following:*

$$\|F_1(t, S_H) - F_1(t, S_{1H})\| =$$
$$\left\| -\frac{K_1 I_V}{1+\theta_1 I_V}\{S_H(t) - S_H(t_1)\} - \frac{K_2 \psi I_V}{1+\theta_1 I_V}\{S_H(t) - S_H(t_1)\} - \gamma_H \{S_H(t) - S_H(t_1)\} \right\| \tag{3.4}$$

Using the triangular inequality on Eq. (4), we obtain

$$\begin{aligned}
\|F_1(t, S_H) - F_1(t, S_{1H})\| &\leq \left\| \frac{K_1 I_V}{1+\theta_1 I_V}\{S_H(t) - S_H(t_1)\} \right\| + \left\| \frac{K_2 \psi I_V}{1+\theta_1 I_V}\{S_H(t) - S_H(t_1)\} \right\| \\
&\quad + \gamma_H \|\{S_H(t) - S_H(t_1)\}\| \\
&\leq \left\{ \frac{K_1 I_V}{1+\theta_1 I_V}\|I_V(t)\| + \frac{K_2 \psi I_V}{1+\theta_1 I_V}\|I_V(t)\| + \gamma_H \right\} \|\{S_H(t) - S_H(t_1)\}\|
\end{aligned}$$

because $\dfrac{K_2 \psi}{1+\theta_1 I_V} < K_2 \psi$ *and* $\dfrac{K_1}{1+\theta_1 I_V} < K_1$ *we get :*

$$\begin{aligned}
&\leq (K_1 + K_2 \psi)\|I_V\| + \gamma_H \|\{S_H(t) - S_H(t_1)\}\| \\
&\leq ((K_1 + K_2 \psi) e + \gamma_H) \|\{S_H(t) - S_H(t_1)\}\| \\
&\leq \mu_1 \|\{S_H(t) - S_H(t_1)\}\| .
\end{aligned} \tag{3.5}$$

Taking $\mu_1 = (K_1 + K_2 \psi) e + \gamma_H$ where $\|I_H(t)\| \leq e$ is a bounded function, we get

$$\|F_1(t, S_H) - F_1(t, S_{1H})\| \leq \mu_1 \|S_H(t) - S_H(t_1)\| \tag{3.6}$$

Hence, the Lipschitz condition is fulfilled for F_1*, and if in addition* $0 \leq (K1 + K2)e + \gamma_H < 1$*, then it is also a contraction.*

For the second kernel F_2 *we have :*

$$\|F_2(t, E_H) - F_2(t, E_{1H})\| = \left\| \frac{K_1 I_V}{1 + \theta_1 I_V} \{E_H(t) - E_H(t_1)\} - \gamma_H \{E_H(t) - E_H(t_1)\} - \right.$$

$$\left. \delta_H \{E_H(t) - E_H(t_1)\}, \right.$$

$$\leq \left\| \frac{K_1 I_V}{1 + \theta_1 I_V} \{E_H(t) - E_H(t_1)\} \right\| + \gamma_H \|\{E_H(t) - E_H(t_1)\}\| + \delta_H \|\{E_H(t) - E_H(t_1)\}\|,$$

$$\leq \frac{K_1 I_V}{1 + \theta_1 I_V} \|\{E_H(t) - E_H(t_1)\}\| + \gamma_H \|\{E_H(t) - E_H(t_1)\}\| + \delta_H \|\{E_H(t) - E_H(t_1)\}\|,$$

$$\leq \left\{ \frac{K_1 I_V}{1 + \theta_1 I_V} + \gamma_H + \delta_H \right\} \|\{E_H(t) - E_H(t_1)\}\|,$$

$$\leq \mu_2 \|\{E_H(t) - E_H(t_1)\}\|. \tag{3.7}$$

Taking $\mu_2 = (K_1 + \delta_H \psi) e + \gamma_H$ *where* $\|I_H(t)\| \leq e$ *is a bounded function, we get*

$$\|F_2(t, E_H) - F_2(t, E_{1H})\| \leq \mu_2 \|E_H(t) - E_H(t_1)\|. \tag{3.8}$$

For the remaining cases, in a similar way the Lipschitz conditions are given as follows:

$$\begin{aligned}
\|F_3(t, I_H) - F_2(t, I_{1H})\| &\leq \mu_3 \|I_H(t) - I_H(t_1)\|, \\
\|F_4(t, S_V) - F_2(t, S_{1V})\| &\leq \mu_4 \|S_V(t) - S_V(t_1)\|, \\
\|F_5(t, E_V) - F_2(t, E_{1V})\| &\leq \mu_5 \|E_V(t) - E_V(t_1)\|, \\
\|F_6(t, I_V) - F_2(t, I_{1V})\| &\leq \mu_6 \|I_V(t) - I_V(t_1)\|.
\end{aligned} \tag{3.9}$$

Using notations for kernels, Eq. (2) becomes

$$\begin{aligned}
S_H(t) &= S_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_1(t, S_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_1(y, S_H))\,dy,\\
E_H(t) &= E_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_2(t, E_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_2(y, E_H))\,dy,\\
I_H(t) &= I_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_3(t, I_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_3(y, I_H))\,dy,\\
S_V(t) &= S_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_4(t, S_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_4(y, S_V))\,dy,\\
E_V(t) &= E_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_5(t, E_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_5(y, E_V))\,dy,\\
I_V(t) &= I_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_6(t, I_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t (F_6(y, I_V))\,dy
\end{aligned} \tag{3.10}$$

The following recursive formula is presented:

$$\begin{aligned}
S_{Hn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_1\left(t, S_{H(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_1\left(y, S_{H(n-1)}\right)\right)dy,\\
E_{Hn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_2\left(t, E_{H(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_2\left(y, E_{H(n-1)}\right)\right)dy,\\
I_{Hn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_3\left(t, I_{H(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_3\left(y, I_{H(n-1)}\right)\right)dy,\\
S_{Vn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_4\left(t, S_{V(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_4\left(y, S_{V(n-1)}\right)\right)dy,\\
E_{Vn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_5\left(t, E_{V(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_5\left(y, E_{V(n-1)}\right)\right)dy,\\
I_{Vn}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_6\left(t, I_{V(n-1)}\right) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_6\left(y, I_{V(n-1)}\right)\right)dy.
\end{aligned} \tag{3.11}$$

with the initial conditions given below

$$
\begin{aligned}
S_H^0(t) &= S_H(t), \\
E_H^0(t) &= E_H(t), \\
I_H^0(t) &= I_H(t), \\
S_V^0(t) &= S_V(t), \\
E_V^0(t) &= E_V(t), \\
I_V^0(t) &= I_V(t).
\end{aligned} \tag{3.12}
$$

The difference of the successive terms is calculated as follows:

$$
\begin{aligned}
w_{1n}(t) = S_{Hn}(t) - S_{H(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_1\left(t, S_{H(n-1)}\right) - F_1\left(t, S_{H(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_1\left(y, S_{H(n-1)}\right) - F_1\left(y, S_{H(n-2)}\right)\right) dy, \\
w_{2n}(t) = E_{Hn}(t) - E_{H(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_2\left(t, E_{H(n-1)}\right) - F_2\left(t, E_{H(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_2\left(y, E_{H(n-1)}\right) - F_2\left(y, E_{H(n-2)}\right)\right) dy, \\
w_{3n}(t) = I_{Hn}(t) - I_{H(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_3\left(t, I_{H(n-1)}\right) - F_3\left(t, I_{H(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_3\left(y, I_{H(n-1)}\right) - F_3\left(y, I_{H(n-2)}\right)\right) dy, \\
w_{4n}(t) = S_{Vn}(t) - S_{V(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_4\left(t, S_{V(n-1)}\right) - F_4\left(t, S_{V(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_4\left(y, S_{V(n-1)}\right) - F_4\left(y, S_{V(n-2)}\right)\right) dy, \\
w_{5n}(t) = E_{Vn}(t) - E_{V(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_4\left(t, E_{V(n-1)}\right) - F_4\left(t, E_{V(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_4\left(y, E_{V(n-1)}\right) - F_4\left(y, E_{V(n-2)}\right)\right) dy, \\
w_{6n}(t) = I_{Vn}(t) - I_{V(n-1)}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_6\left(t, I_{V(n-1)}\right) - F_6\left(t, I_{V(n-2)}\right)\right) \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_6\left(y, I_{V(n-1)}\right) - F_4\left(y, I_{V(n-2)}\right)\right) dy,
\end{aligned} \tag{3.13}
$$

Notice that

$$\begin{cases} S_{Hn}(t) = \sum_{i=1}^{n} w_{1i}(t), \\ E_{Hn}(t) = \sum_{i=1}^{n} w_{2i}(t), \\ I_{Hn}(t) = \sum_{i=1}^{n} w_{3i}(t), \\ S_{Vn}(t) = \sum_{i=1}^{n} w_{4i}(t), \\ E_{Vn}(t) = \sum_{i=1}^{n} w_{5i}(t), \\ I_{Vn}(t) = \sum_{i=1}^{n} w_{6i}(t). \end{cases} \tag{3.14}$$

On continuing the same process, we assess

$$\begin{aligned} \|w_{1n}(t)\| &= \left\|S_{Hn}(t) - S_{H(n-1)}(t)\right\| \\ &= \left\| \frac{2(1-\tau)}{(2-\tau)M(\tau)} \times \left(F_1\left(t, S_{H(n-2)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)} \right. \\ &\quad \left. \times \int_0^t \left(F_1\left(y, S_{H(n-1)}\right) - F_1\left(y, S_{H(n-2)}\right)\right) dy \right\|. \end{aligned} \tag{3.15}$$

Using the triangular inequality, Eq. (15) is simplified to

$$\begin{aligned} \left\|S_{Hn}(t) - S_{H(n-1)}(t)\right\| \leq & \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\|\left(F_1\left(t, S_{H(n-1)}\right) - F_1\left(t, S_{H(n-2)}\right)\right)\right\| \\ & + \frac{2\tau}{(2-\tau)M(\tau)} \left\| \int_0^t \left(F_1\left(y, S_{H(n-1)}\right) - F_1\left(y, S_{H(n-2)}\right)\right) dy \right\|. \end{aligned} \tag{3.16}$$

As the kernel fulfills the Lipschitz condition,

$$\begin{aligned} \left\|S_{Hn}(t) - S_{H(n-1)}(t)\right\| \leq & \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \left\|S_{H(n-1)} - S_{H(n-2)}\right\| \\ & + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 \int_0^t \left\|S_{H(n-1)} - S_{H(n-2)}\right\| dy. \end{aligned} \tag{3.17}$$

then we have

$$\begin{aligned} \|w_{1n}(t)\| \leq & \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \left\|w_{1(n-1)}\right\| \\ & + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 \int_0^t \left\|w_{1(n-1)}(y)\right\| dy. \end{aligned} \tag{3.18}$$

for the second case we have :

$$\begin{aligned}\|w_{2n}(t)\| &= \left\|E_{Hn}(t) - E_{H(n-1)}(t)\right\| \\ &= \left\|\frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_2\left(t, E_{H(n-2)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)}\right. \\ &\quad \left.\times\int_0^t \left(F_2\left(y, E_{H(n-1)}\right) - F_2\left(y, E_{H(n-2)}\right)\right) dy\right\|.\end{aligned} \tag{3.19}$$

Using the triangular inequality, Eq. (19) is simplified to

$$\begin{aligned}\left\|E_{Hn}(t) - E_{H(n-1)}(t)\right\| \leq{}& \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|\left(F_2\left(t, E_{H(n-1)}\right) - F_2\left(t, E_{H(n-2)}\right)\right)\right\| \\ &+ \frac{2\tau}{(2-\tau)M(\tau)}\left\|\int_0^t \left(F_2\left(y, E_{H(n-1)}\right) - E_1\left(y, E_{H(n-2)}\right)\right) dy\right\|\end{aligned} \tag{3.20}$$

As the kernel fulfills the Lipschitz condition,

$$\begin{aligned}\left\|E_{Hn}(t) - E_{H(n-1)}(t)\right\| \leq{}& \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_2\left\|E_{H(n-1)} - E_{H(n-2)}\right\| \\ &+ \frac{2\tau}{(2-\tau)M(\tau)}\mu_2\int_0^t \left\|E_{H(n-1)} - E_{H(n-2)}\right\| dy,\end{aligned} \tag{3.21}$$

then we have

$$\begin{aligned}\|w_{2n}(t)\| \leq{}& \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_2\left\|w_{2(n-1)}\right\| \\ &+ \frac{2\tau}{(2-\tau)M(\tau)}\mu_2\int_0^t \left\|w_{2(n-1)}(y)\right\| dy,\end{aligned} \tag{3.22}$$

for the third case we have :

$$\begin{aligned}\|w_{3n}(t)\| &= \left\|I_{Hn}(t) - I_{H(n-1)}(t)\right\| \\ &= \left\|\frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_3\left(t, I_{H(n-2)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)}\right. \\ &\quad \left.\times\int_0^t \left(I_2\left(y, I_{H(n-1)}\right) - F_3\left(y, I_{H(n-2)}\right)\right) dy\right\|.\end{aligned} \tag{3.23}$$

Using the triangular inequality, Eq. (23) is simplified to

$$\begin{aligned}\left\|I_{Hn}(t) - I_{H(n-1)}(t)\right\| \leq{}& \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|\left(F_3\left(t, I_{H(n-1)}\right) - F_3\left(t, I_{H(n-2)}\right)\right)\right\| \\ &+ \frac{2\tau}{(2-\tau)M(\tau)}\left\|\int_0^t \left(F_3\left(y, I_{H(n-1)}\right) - I_1\left(y, I_{H(n-2)}\right)\right) dy\right\|\end{aligned} \tag{3.24}$$

As the kernel fulfills the Lipschitz condition,

$$\left\|I_{Hn}(t)-I_{H(n-1)}(t)\right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_3\left\|I_{H(n-1)}-I_{H(n-2)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_3\int_0^t \left\|I_{H(n-1)}-I_{H(n-2)}\right\| dy\,, \tag{3.25}$$

then we have

$$\|w_{3n}(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_3\left\|w_{3(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_3\int_0^t \left\|w_{3(n-1)}(y)\right\| dy\,. \tag{3.26}$$

for the forth case we have :

$$\begin{aligned}\|w_{4n}(t)\| &= \left\|S_{Vn}(t)-S_{V(n-1)}(t)\right\| \\ &= \left\|\frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_4\left(t,S_{V(n-2)}\right)\right)+\frac{2\tau}{(2-\tau)M(\tau)}\right. \\ &\quad\left.\times\int_0^t \left(F_4\left(y,S_{V(n-1)}\right)-F_4\left(y,S_{V(n-2)}\right)\right)dy\right\|.\end{aligned} \tag{3.27}$$

Using the triangular inequality, Eq. (27) is simplified to

$$\left\|S_{Vn}(t)-S_{V(n-1)}(t)\right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|\left(F_4\left(t,S_{V(n-1)}\right)-F_4\left(t,S_{V(n-2)}\right)\right)\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\left\|\int_0^t \left(F_4\left(y,S_{V(n-1)}\right)-F_4\left(y,S_{V(n-2)}\right)\right)dy\right\|. \tag{3.28}$$

As the kernel fulfills the Lipschitz condition,

$$\left\|S_{Vn}(t)-S_{V(n-1)}(t)\right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_4\left\|S_{V(n-1)}-S_{V(n-2)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_4\int_0^t \left\|S_{V(n-1)}-S_{V(n-2)}\right\| dy\,. \tag{3.29}$$

then we have

$$\|w_{4n}(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_4\left\|w_{4(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_4\int_0^t \left\|w_{4(n-1)}(y)\right\| dy\,. \tag{3.30}$$

for the fifth case we have :

$$
\begin{aligned}
\|w_{5n}(t)\| &= \left\|E_{Vn}(t) - E_{V(n-1)}(t)\right\| \\
&= \left\|\frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_5\left(t, E_{V(n-2)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)}\right. \\
&\quad \left. \times \int_0^t \left(F_5\left(y, E_{V(n-1)}\right) - F_5\left(y, E_{V(n-2)}\right)\right) dy\right\| .
\end{aligned} \tag{3.31}
$$

Using the triangular inequality, Eq. (31) is simplified to

$$
\begin{aligned}
\left\|E_{Vn}(t) - E_{V(n-1)}(t)\right\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|\left(F_5\left(t, E_{V(n-1)}\right) - F_5\left(t, E_{V(n-2)}\right)\right)\right\| \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\left\|\int_0^t \left(F_5\left(y, E_{V(n-1)}\right) - E_1\left(y, E_{V(n-2)}\right)\right) dy\right\|
\end{aligned} \tag{3.32}
$$

As the kernel fulfills the Lipschitz condition,

$$
\begin{aligned}
\left\|E_{Vn}(t) - E_{V(n-1)}(t)\right\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_5\left\|E_{V(n-1)} - E_{V(n-2)}\right\| \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\mu_5\int_0^t \left\|E_{V(n-1)} - E_{V(n-2)}\right\| dy ,
\end{aligned} \tag{3.33}
$$

then we have

$$
\begin{aligned}
\|w_{5n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_5\left\|w_{5(n-1)}\right\| \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\mu_5\int_0^t \left\|w_{5(n-1)}(y)\right\| dy ,
\end{aligned} \tag{3.34}
$$

for the sixth case we have :

$$
\begin{aligned}
\|w_{6n}(t)\| &= \left\|I_{Vn}(t) - I_{V(n-1)}(t)\right\| \\
&= \left\|\frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_6\left(t, I_{V(n-2)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)}\right. \\
&\quad \left. \times \int_0^t \left(F_6\left(y, I_{V(n-1)}\right) - F_6\left(y, I_{V(n-2)}\right)\right) dy\right\| .
\end{aligned} \tag{3.35}
$$

Using the triangular inequality, Eq. (35) is simplified to

$$
\begin{aligned}
\left\|I_{Vn}(t) - I_{V(n-1)}(t)\right\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|\left(F_6\left(t, I_{V(n-1)}\right) - F_6\left(t, I_{V(n-2)}\right)\right)\right\| \\
&\quad + \frac{2\tau}{(2-\tau)M(\tau)}\left\|\int_0^t \left(F_6\left(y, I_{V(n-1)}\right) - I_1\left(y, I_{V(n-2)}\right)\right) dy\right\|
\end{aligned} \tag{3.36}
$$

As the kernel fulfills the Lipschitz condition,

$$\left\|I_{Vn}(t)-I_{v(n-1)}(t)\right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_6\left\|I_{V(n-1)}-I_{V(n-2)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_6\int_0^t\left\|I_{V(n-1)}-I_{V(n-2)}\right\|dy\,, \tag{3.37}$$

then we have

$$\|w_{6n}(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_6\left\|w_{6(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_6\int_0^t\left\|w_{6(n-1)}(y)\right\|dy\,. \tag{3.38}$$

we have :

$$\begin{aligned}
\|w_{1n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1\left\|w_{1(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_1\int_0^t\left\|w_{1(n-1)}(y)\right\|dy\,.\\
\|w_{2n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_2\left\|w_{2(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_2\int_0^t\left\|w_{2(n-1)}(y)\right\|dy\,.\\
\|w_{3n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_3\left\|w_{3(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_3\int_0^t\left\|w_{3(n-1)}(y)\right\|dy\,.\\
\|w_{4n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_4\left\|w_{4(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_4\int_0^t\left\|w_{4(n-1)}(y)\right\|dy\,.\\
\|w_{5n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_5\left\|w_{5(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_5\int_0^t\left\|w_{5(n-1)}(y)\right\|dy\,.\\
\|w_{6n}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_6\left\|w_{6(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_6\int_0^t\left\|w_{6(n-1)}(y)\right\|dy\,.
\end{aligned} \tag{3.39}$$

Now we state the theorem below.

Theorem 3.1.3 *The FPWD model (2.2) has exact coupled solutions if the conditions below hold. That is, we can find t_0 such that*

$$\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1 + + \frac{2\tau}{(2-\tau)M(\tau)}\mu_1 t_0 < 1$$

proof 3.1.2 *Since all the functions $S_H(t), E_H(t), I_H(t)$ and $S_V(t), E_V(t), I_V(t)$ are bounded, we have shown that the kernels fulfill the Lipschitz condition, thus on using of Eqs.*

(39) and by using the recursive method, we obtain the succeeding relation as follows:

$$\begin{aligned}
\|w_{1n}(t)\| &\leq \|S_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_1 t\right]^n, \\
\|w_{2n}(t)\| &\leq \|E_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_2\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_2 t\right]^n, \\
\|w_{3n}(t)\| &\leq \|I_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_3\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_3 t\right]^n, \\
\|w_{4n}(t)\| &\leq \|S_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_4\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_4 t\right]^n, \\
\|w_{5n}(t)\| &\leq \|E_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_5\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_5 t\right]^n, \\
\|w_{6n}(t)\| &\leq \|I_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_6\right) + \frac{2\tau}{(2-\tau)M(\tau)}\mu_6 t\right]^n.
\end{aligned} \tag{3.40}$$

Hence, the existence and continuity of the said solutions is proved. Furthermore, to ensure that the above function is a solution of Eq. (2.2), we proceed as follows:

$$\begin{aligned}
S_H(t) - S_H(0) &= S_{Hn}(t) - B_n(t), \\
E_H(t) - E_H(0) &= E_{Hn}(t) - C_n(t), \\
I_H(t) - I_H(0) &= I_{Hn}(t) - D_n(t), \\
S_V(t) - S_V(0) &= S_{Vn}(t) - F_n(t), \\
E_V(t) - E_V(0) &= E_{Vn}(t) - G_n(t), \\
I_V(t) - I_V(0) &= I_{Vn}(t) - H_n(t).
\end{aligned} \tag{3.41}$$

Therefore, we have

$$\begin{aligned}
&\|B_n(t)\| \\
&= \left\| \frac{2(1-\tau)}{(1-\tau)M(\tau)} \left(F_1(t, S_{Hn}) - F_1\left(t, S_{H(n-1)}\right)\right) + \frac{2\tau}{(2-\tau)M(\tau)} \right. \\
&\quad \left. \times \int_0^t \left(F_1(t, S_{Hn}) - F_1\left(t, S_{H(n-1)}\right)\right) dy \right\| \\
&\leq \frac{2(1-\tau)}{(1-\tau)M(\tau)}\mu_1 \left\|S_{Hn} - S_{H(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_1 \left\|S_{Hn} - S_{H(n-1)}\right\| t.
\end{aligned} \tag{3.42}$$

Using the process in a recursive manner gives

$$\|B_n(t)\| \leq \left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)}t\right)^{n+1} \mu_1^{n+1} a. \tag{3.43}$$

Then at t_0 we have

$$\|B_n(t)\| \leq \left(\tfrac{2(1-\tau)}{(2-\tau)M(\tau)} + \tfrac{2\tau}{(2-\tau)M(\tau)}t_0\right)^{n+1} \mu_1^{n+1} a. \tag{3.44}$$

By applying the limit on Eq. (44) as n tends to infinity, we get

$$\|B_n(t)\| \to 0$$

for the second case we have :

$$\begin{aligned}\|C_n(t)\| &= \left\| \frac{2(1-\tau)}{(1-\tau)M(\tau)}\left(F_2(t,E_{Hn}) - F_2(t,E_{H(n-1)})\right)\right. \\ &\quad \left. + \frac{2\tau}{(2-\tau)M(\tau)} \times \int_0^t (F_2(t,E_{Hn}) - F_2(t,E_{H(n-1)}))dy \right\|. \\ &\leq \frac{2(1-\tau)}{(1-\tau)M(\tau)}\mu_2 \left\|E_{Hn} - E_{H(n-1)}\right\| + \frac{2\tau}{(2-\tau)M(\tau)}\mu_2 \left\|E_{Hn} - E_{H(n-1)}\right\|\end{aligned} \tag{3.45}$$

Using the process in a recursive manner gives

$$\|C_n(t)\| \leq \left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)}t\right)^{n+1} \mu_2^{n+1} a\,. \tag{3.46}$$

Then at t_0 we have

$$\|C_n(t)\| \leq \left(\tfrac{2(1-\tau)}{(2-\tau)M(\tau)} + \tfrac{2\tau}{(2-\tau)M(\tau)}t_0\right)^{n+1} \mu_2^{n+1} a\,. \tag{3.47}$$

By applying the limit on Eq. (47) as n tends to infinity, we get

$$\|C_n(t)\| \to 0$$

Similarly, we get

$$\|D_n(t)\| \to 0\ , \quad \|F_n(t)\| \to 0$$

$$\|G_n(t)\| \to 0 \quad , \quad \|H_n(t)\| \to 0\ .$$

For the uniqueness the system (2.2) solution, we take on contrary that there exists another solution of (2.2) given by $S_{1H}(t), E_{1H}(t), I_{1H}(t), S_{1V}(t), E_{1V}(t)$, and $I_{1V}\ (t)$. Then

$$\begin{aligned}S_H(t) - S_{1H}(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_1(t,S_H) - F_1(t,S_H)\right) + \frac{2\tau}{(2-\tau)M(\tau)} \\ &\quad \times \int_0^t \left(F_1(y,S_H) - F_1(y,S_H)\right)dy\end{aligned} \tag{3.48}$$

Taking norm on Eq. (48), we get

$$\begin{aligned}\|S_H(t) - S_{1H}(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left\|F_1(t,S_H) - F_1(t,S_H)\right\| + \frac{2\tau}{(2-\tau)M(\tau)} \\ &\quad \times \int_0^t \left\|F_1(y,S_H) - F_1(y,S_H)\right\|dy\,.\end{aligned} \tag{3.49}$$

By applying the Lipschitz condition of kernel, we have

$$\|S_H(t) - S_{1H}(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1 \|S_H(t) - S_{1H}(t)\| + \frac{2\tau}{(2-\tau)M(\tau)} \times \int_0^t \mu_1 t \|S_H(t) - S_{1H}(t)\| \, dy \,. \tag{3.50}$$

It gives

$$\|S_H(t) - S_{1H}(t)\| \left(1 - \tfrac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1 + \tfrac{2\tau}{(2-\tau)M(\tau)}\mu_1 t\right) \leq 0. \tag{3.51}$$

Theorem 3.1.4 *The model (2.2) solution will be unique if*

$$\left(1 - \tfrac{2(1-\tau)}{(2-\tau)M(\tau)}\mu_1 + \tfrac{2\tau}{(2-\tau)M(\tau)}\mu_1 t\right) > 0 \tag{3.52}$$

proof 3.1.3 *If condition (52) holds, then (51) implies that*

$$\|S_H(t) - S_{1H}(t)\| = 0$$

that's implies the equality betwen the two deferent soulution .

Conclusion

In the present work, we extended the PWD model [12] to fractional order using the Caputo–Fabrizio fractional derivative. The model equilibria and basic reproduction number are explored. The existence and uniqueness of the solution for the FPWD model with CF derivative are proved in detail. From theoritical points of view one can see that when fractional order of derivative decreases, the CF derivative provides more biologically feasible behavior about the dynamic of pine wilt disease. Therefore, we concluded that the newly fractional derivative is very useful for modeling such phenomena

Bibliography

[1] Wang, Y., Cao, J., Li, X., Alsaedi, A. : Edge-based epidemic dynamics with multiple routes of transmission on random networks. Nonlinear Dyn. 91(1), 403–420 (2018).

[2] Wang, Y., Jin, Z., Yang, Z., Zhang, Z.-K., Zhou, T., Sun, G.-Q.: Global analysis of an SIS model with an infective vector on complex networks. Nonlinear Anal., Real World Appl. 13(2), 543–557 (2012)

[3] Muhammad Altaf Khan, Saif Ullah, K.O. Okosun and Kamil Shah : A fractional order pine wilt disease model with Caputo–Fabrizio derivative .

[4] Mamiya, Y., Kiyohara, T.: Description of Bursaphelenchus lignicolus n. sp. (Nematoda: Aphelenchoididae) from pine wood and histopathology of nematode-infested trees. Nematologica 18(1), 120–124 (1972)

[5] Kiyohara, T., Tokushige, Y.: Inoculation experiments of a nematode, Bursaphelenchus sp., onto pine trees. J. Jpn. For. Soc. 53(6), 210–218 (1971)

[6] Mamiya, Y., Enda, N.: Transmission of bursaphelenchus lignicolus (nematoda: Aphelenchoididae) by monochamus alternatus (coleoptera: Cerambycidae). Nematologica 18(2), 159–162 (1972)

[7] Morimoto, K., Iwasaki, A.: Role of monochamus alternatus (Coleoptera: Cerambycidae) as a vector of Bursaphelenchus lignicolus (Nematoda: Aphelenchoididae). J. Jpn. For. Soc. 54(6), 177–183 (1972)

[8] Zhao, B.G., Futai, K., Sutherland, J.R., Takeuchi, Y.: Pine Wilt Disease. Springer, Berlin (2008)

[9] Rutherford, T., Mamiya, Y., Webster, J.: Nematode-induced pine wilt disease: factors influencing its occurrence and distribution. For. Sci. 36(1), 145–155 (1990)

[10] Lee, K.S.: Stability analysis and optimal control strategy for prevention of pine wilt disease. Abstr. Appl. Anal. 2014, Article ID 182680 (2014)

[11] Khan, M., Ali, K., Bonyah, E., Okosun, K., Islam, S., Khan, A.: Mathematical modeling and stability analysis of pine wilt disease with optimal control. Sci. Rep. 7(1), 1–19 (2017)

[12] Khan, M.A., Khan, R., Khan, Y., Islam, S.: A mathematical analysis of pine wilt disease with variable population size and optimal control strategies. Chaos Solitons Fractals 108, 205–217 (2018)

[13] Khan, M.A., Shah, K., Khan, Y., Islam, S.: Mathematical modeling approach to the transmission dynamics of pine wilt disease with saturated incidence rate. Int. J. Biomath. 11(03), 1–20 (2018)

[14] Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications (1999)

[15] Samko, S.G., Kilbas, A.A., Marichev, O.I., et al.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Yverdon (1993)

[16] Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 1–13 (2015)

[17] Owolabi, K.M., Atangana, A.: Analysis and application of new fractional Adams–Bashforth scheme with Caputo–Fabrizio derivative. Chaos Solitons Fractals 105, 111–119 (2017)

[18] Abdulhameed, M., Vieru, D., Roslan, R.: Magnetohydrodynamic electroosmotic flow of Maxwell fluids with Caputo–Fabrizio derivatives through circular tubes. Comput. Math. Appl. 74(10), 2503–2519 (2017)

[19] Firoozjaee, M., Jafari, H., Lia, A., Baleanu, D.: Numerical approach of Fokker–Planck equation with Caputo–Fabrizio fractional derivative using Ritz approximation. J. Comput. Appl. Math. 339(10), 367–373 (2018)

[20] Abro, K.A., Memon, A.A., Uqaili, M.A.: A comparative mathematical analysis of RL and RC electrical circuits via Atangana–Baleanu and Caputo–Fabrizio fractional derivatives. Eur. Phys. J. Plus 133(3), 113 (2018)

[21] Dokuyucu, M.A., Celik, E., Bulut, H., Baskonus, H.M.: Cancer treatment model with the Caputo–Fabrizio fractional derivative. Eur. Phys. J. Plus 133(3), 92 (2018)

[22] Ullah, S., Khan, M.A., Farooq, M.: A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative. Eur. Phys. J. Plus 133(6), 237 (2018)

[23] Diethelm, K.: A fractional calculus based model for the simulation of an outbreak of dengue fever. Nonlinear Dyn. 71(4), 613–619 (2013)

[24] Area, I., Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W., Torres, Á.: On a fractional order Ebola epidemic model. Adv. Differ. Equ. 2015(1), 278 (2015)

[25] Makris, N., Dargush, G., Constantinou, M.: Dynamic analysis of generalized viscoelastic fluids. J. Eng. Mech. 119(8), 1663–1679 (1993)

[26] Losada, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 87–92 (2015)

[27] Asif, N., Hammouch, Z., Riaz, M., Bulut, H.: Analytical solution of a Maxwell fluid with slip effects in view of the Caputo–Fabrizio derivative. Eur. Phys. J. Plus 133(7), 272 (2018)

[28] Singh, J., Kumar, D., Baleanu, D.: On the analysis of fractional diabetes model with exponential law. Adv. Differ. Equ. 2018(1), 231 (2018)

[29] Van den Driessche, P., Watmough, J.: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. 180(1–2), 29–48 (2002)