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Entitled

Optimal control of various linear PDEs with incomplete data

Presented by:
Achab Fatma
Boukouba Chayma

Before the jury:

Mr: **Faycel Merghadi**	MCA	University of Tébessa	President
Mr: **Nouri Boumaza**	MCA	University of Tébessa	Examiner
Mr: **Abdelhak Hafdallah**	MCB	University of Tébessa	Supervisor

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I dedicate this modest work:

*-To my father **Ali**, who I see millions of reasons why I need to be successful when I look at him.*

*-To the great women I see in my life my mother **Hayette**. I'm always proud of her being my mom.*

*-To my brother **Rami** and my sisters **Douaa** and **Israa** symbol of love and giving.*

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To all of them, I wish all the best

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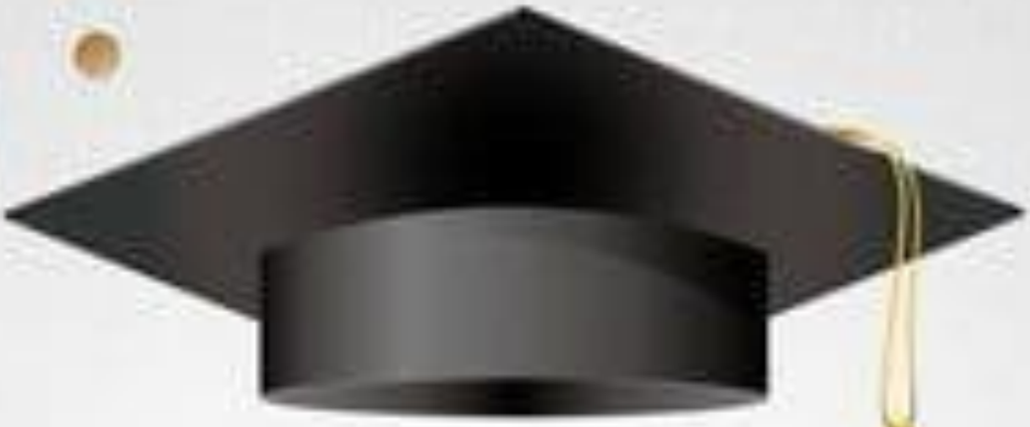

Dedication:

I dedicate this work to those who proudly carry their name my beloved parents who are a huge support in my life, who keep giving me with love and their prayers what is necessary so that I can achieve what I am today, all I am now thanks to them.

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Abstract

The aim of this memory is to study optimal control problem of some linear distributed systems with incomplete data. The no-regret control method seems to be the best-adapted method to solve it. Also, the averaged control is used to control systems depending on unknown parameter. The no-regret control limit of the sequence of low-regret controls will be characterized by an optimality system.

Keywords: Optimal control, incomplete data, no-regret control, low-regret control, averaged control, average no-regret control, electromagnetic wave equation.

Résumé

Le but de cette mémoire est d'étudier le problème de contrôle optimal de quelques systèmes linéaires distribués avec des données incomplètes. La méthode de contrôle sans regret est la meilleure méthode de résoudre ce type de problème. Aussi, le contrôle moyenne est utilisé pour contrôler des systèmes dépends d'un paramètre inconnus. Le contrôle sans regret est la limite d'une suite de contrôle à moindre regret sera caractérisé par un système d'optimalité.

Mots clés: Contrôle optimal, donnée incomplète, contrôle sans regret, contrôle à moindre regret, contrôle moyenne, contrôle sans regret moyenne, équation des ondes électromagnétiques.

الملخص

الهدف من هذه المذكرة هو دراسة التحكم الأمثل لبعض الأنظمة التوزيعية الخطية ذات معطيات غير المكتملة. طريقة التحكم دون ندم هي أفضل طريقة لحل هذا النوع من المسائل. أيضا يستخدم التحكم المتوسط للتحكم في الأنظمة التي تتعلق بوسيط مجهول. سيميز التحكم دون ندم نهاية متتالية التحكم منخفض الندم بنظام استمثالي.

الكلمات المفتاحية: التحكم الأمثل، معطيات غير مكتملة، التحكم دون ندم، التحكم المنخفض الندم، التحكم المتوسط، التحكم المتوسط دون ندم، معادلة الأمواج الكهرومغناطيسية

Contents

Notations & abbreviations

$\mathbb{R}$	Set of real numbers.
$\|.\|_H$	A norm in Banach space H.
$(.,.)_H$	A scalar product in Hilbert space H.
$\vert.\vert_H$	A semi-norm in H.
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$	The gradient operator.
div	Divergence.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(\mathcal{Y}, \mathcal{Z})$	The space of linear bounded operators from $\mathcal{Y}$ to $\mathcal{Z}$.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support in Q.
$\mathcal{D}'(Q)$	The dual space of $\mathcal{D}(Q)$.
$L^2(0, T, H)$	The bounded linear operator space.
a.e.	Almost every where.
PDE	Partial differential equation.
$\rightharpoonup$	Symbol of weak convergence.
iff	If and only if.

Introduction

Fields of science like Physics, Statistic, Chemistry, Biology or Population dynamics are modeled by using Partial Differential Equations (PDE), our goal is to pass from an initial state to a final desirable state while minimizing the objective function to control those phenomena. Therefore, we will discuss the optimal control problem which usually given by a dynamic system and some cost function to minimize.

The theory of optimal control appeared after the second world war (1950s) as a special topic within the discipline of a differential equation. At that time, two important works discovered, the first one is Dynamic Programming (Richard Bellman) which reduce the search for an optimal control function to finding the solution of PDE and the second is Pontryagin Maximum Principle (Hamilton - Jacobie - Bellman Equation) which gives a set of necessary conditions for the optimal control function.

Nowadays, the optimal control theory has become a part of our daily life, aiming to improve our quality of life and facilitate certain tasks. For example, in biomedical phenomena the human cells are affected by the X-rays energy, so we control the X-rays by displacing the wave to get the suitable energy for the cells can carry. In ecology, we also control the pollution, which includes reducing the effects of pollution with the help of control, in given situations rather than leaving them abandoned.

Sometimes when we modeling those phenomena, we don't have all the information related to it, such as the pollution problems we can't know the initial moment in which pollution occurred and this is what makes us in front of the problem of control with incomplete data.

This memory aims to study the optimal control problem for systems described by PDEs with incomplete data or missing data using the notions of no-regret control, the averaged control and the averaged no-regret control.

The notion of no-regret control is using to control systems such that the initial conditions, boundary conditions or second member of the equation are missing, introduced the first time by the famous mathematician Jacque Louis Lions [11] who inspired the idea from Savage in statistics [18], we associate with the no-regret control a sequence of low-regret controls defined by a quadratic perturbation. Then, we introduce the classic tools to prove the existence and uniqueness of the solution of the optimal control problems. Also, we prove that the sequence of low-regret control converges to the no-regret control which gives the optimality system of the no-regret control. Many scientists develop this notion like O. Nakoulima, A. Omran and J. Velin [15], Jacob and Omran [4], Hafdallah & Ayadi [2].

Moreover, the notion of averaged control is used to control systems depend on an unknown parameter that can be in the operator of the system, it was introduced the first time by Zuazua [20], and Lohéa & Zuazua [12].

In order to control systems which contains two kinds of missing data, the first is a missing boundary condition and the second is an unknown parameter we use the notion of averaged no-regret control which used by Hafdallah & Ayadi [5] to control an electromagnetic waves equation with an unknown velocity of propagation, also used by Mophou [14] to control parabolic equation. This concept has been generalized recently by A. Hafdallah [8].

Below we present the organization of our memory

In the first chapter, we present the optimal control in a distributed system with complete data. Thus, we prove the existence of the optimum and we give it characterization. Also, we give some examples in different types (elliptic, parabolic and hyperbolic equations).

In the second chapter, we consecrated to study the notion of no-regret control and low-regret control and we give its characterizations and example. Then, we present the idea of the averaged control.

In the third chapter, we give the optimal control problem of an electromagnetic waves through a medium since that we don't know their permeability and permittivity which causes the unknown velocity of propagation and with an unknown Dirichlet boundary condition. Under this condition, we use the notion of averaged no-regret control and we give their optimality system which is a limit to the optimality system of averaged low-regret control.

In the last chapter, we take the optimal control problem of a parabolic equation with missing boundary condition as an example.

Chapter 1

Optimal control of linear distributed systems : Preliminaries

During this chapter, we will study the classical theory of optimal control in distributed system[1] with complete data introduced by **J.Lions** in 1971. We begin our chapter by the presentation of our optimal control problem and we give it characterization then we finish by some examples.

1.1 Control Problem

A control problem consists to manipulate a system, with an input-output space. The input is the control can be a function in a boundary condition, an initial condition, a coefficient in a partial differential equation modeling the system, or any parameter in the equation, and the output is the state or the solution of the system or any information related to her.
An optimal control problem is an optimization problem with PDE constraint minimize a criterion depending on the observation of the state and on the control variable.
The theory of optimal control requires the following condition:
- Control u which belongs to the set of control.
- State $y(u)$ which to be controlled.
-Observation $z(u)$.
- Cost function $J(u)$ should be minimized.
Among the goals of this theory is to get the necessary condition to be u a minimum. Also, to get algorithms that approximate the minimum u with a necessary condition which is called 'variational inequality'.

[1] it means that the system defined by PDEs in infinite dimension space

1.2 Position of problem

Let $\mathcal{Y}$, $\mathcal{U}$ and $\mathcal{Z}$ be infinite dimensional Hilbert spaces of states, controls and observation resp. $\mathcal{U}_{ad}\subset \mathcal{U}$ is a subset of admissible controls supposed non empty, closed and convex.
Consider the following well-posed state equation related to the control $v \in \mathcal{U}_{ad}$

$$\mathcal{A}y = f + \mathcal{B}v. \tag{1.1}$$

Where $\mathcal{A} \in \mathcal{L}(\mathcal{Y})$ is a linear partial differential operator stationary or evolutionary (elliptic, parabolic and hyperbolic) modelling a distributed system makes an isomorphism on $\mathcal{Y}$, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the control operator.
Our optimal control problem consists in looking for a control function $u \in \mathcal{U}_{ad}$ which minimizes the following cost function

$$J(v) = \|\mathcal{C}y(v) - y_d\|^2_{\mathcal{Z}} + N\|v\|^2_{\mathcal{U}} \quad \forall v \in \mathcal{U}_{ad}, \tag{1.2}$$

where J is convex function from $\mathcal{U}_{ad} \subset \mathcal{U}$ to $\mathbb{R} \cup \{+\infty\}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ the observation operator and $N > 0$, y_d is the fixed observation in $\mathcal{Z}$.
i.e, our optimal control problem is

$$\begin{cases} \text{find } u \in \mathcal{U}_{ad} \text{ such that} \\ \quad J(u) = \inf\limits_{v\in\mathcal{U}_{ad}} J(v), \end{cases} \tag{1.3}$$

Theorem 1.1 ***(Existence and uniqueness of optimal control)***

Let $\mathcal{U}_{ad} \subset \mathcal{U}$ closed and nonempty, J is lower semicontinuous, bounded from below and coercive on $\mathcal{U}_{ad}$. Then there exists a minimizer for J on $\mathcal{U}_{ad}$. Moreover, if J is strictly convex the minimizer is unique.
Proof. 1.**Existence**
Since J is bounded from below

$$m = \inf_{v\in\mathcal{U}_{ad}} J(v),$$

Let (v_n) be a minimizing sequence[2] in $\mathcal{U}_{ad}$ since J is coercive, (v_n) is converge to $u \in \mathcal{U}_{ad}$, and J is lower semicontinuous, then

$$J(u) \leq \lim_{n\to+\infty} \inf_{v\in\mathcal{U}_{ad}} J(v_n) = \inf_{v\in\mathcal{U}_{ad}} J(v) = m,$$

[2] a minimizing sequence of the criterion J on the set $\mathcal{U}_{ad}$ is a sequence (v_n) such that $\lim\limits_{n\to+\infty} J(v_n) = \inf\limits_{v\in\mathcal{U}_{ad}} J(v)$

then u is a minimizer of J on $\mathcal{U}_{ad}$.

2.Uniqueness

Suppose that the problem (1.3) admits two distinct solutions u_1, u_2. We set $u = \frac{u_1+u_2}{2}$, due to strict convexity of J we get

$$J(u) < \frac{1}{2}J(u_1) + \frac{1}{2}J(u_2) = m,$$

we obtain a contradiction with the assumption that u_1, u_2 are two solutions of (1.3). Thus (1.3) admits a unique solution. ■

1.3 Characterization of the optimal control (optimality system)

A first order optimality condition gives:

$$J'(u)(v-u) \geq 0 \qquad \forall v \in \mathcal{U}_{ad},$$

we know that the cost function J is Gateaux-differentiable with

$$J'(u)(v-u) = \lim_{t\to 0}\frac{J(u+t(v-u)) - J(u)}{t} \text{ for every } v \in \mathcal{U}_{ad},$$

by a simple calculus we get

$$\begin{aligned} J(u+t(v-u)) &= J(u) + t^2\|\mathcal{C}y(v-u)\|^2_{\mathcal{Z}} + 2t(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}} + t^2N\|v-u\|^2_{\mathcal{U}} \\ &\quad +2tN(u, v-u)_{\mathcal{U}}, \end{aligned}$$

which gives

$$\frac{J(u+t(v-u)) - J(u)}{t} = t\|\mathcal{C}y(v-u)\|^2_{\mathcal{Z}} + 2(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}} + tN\|v-u\|^2_{\mathcal{U}} + 2N(u, v-u)_{\mathcal{U}},$$

make t tends to a 0 to get

$$J'(u)(v-u) = 2(\mathcal{C}^*(\mathcal{C}y(u) - y_d), y(v-u))_{\mathcal{Y}} + 2N(u, v-u)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

Let's introduce the adjoint state $p = p(u)$

$$\mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d),$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, then

$$\begin{aligned}(\mathcal{C}^*(\mathcal{C}y\,(u) - y_d), y(v-u))_{\mathcal{Y}} &= (\mathcal{A}^*p(u), y(v-u))_{\mathcal{Y}} = (p(u), \mathcal{A}y(v-u))_{\mathcal{Y}} \\ &= (p(u), \mathcal{B}(v-u))_{\mathcal{Y}} = (\mathcal{B}^*p(u), (v-u))_{\mathcal{U}},\end{aligned}$$

Hence,

$$J'\,(u)\,(v-u) = (\mathcal{B}^*p(u) + Nu, v-u)_{\mathcal{U}} \quad \forall v \in \mathcal{U}_{ad}.$$

Therefore the optimal control u is characterized by the following optimality system :

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^*p + Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}. \end{cases}$$

The first two equation must be associated with some appropriate boundary and initial condition and the pair $(u, p(u))$ called optimal pair.

Remark 1.1 *if $\mathcal{U}_{ad} = \mathcal{U}$ we have also*

$$J'\,(u)\,(v-u) \leq 0 \;\; \forall v \in \mathcal{U},$$

and with the previous condition we get

$$J'\,(u)\,(v-u) = 0 \;\; \forall v \in \mathcal{U},$$

therefore the optimality system become as following

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^*p(u) + Nu, v-u)_{\mathcal{U}} = 0 \qquad \forall v \in \mathcal{U}. \end{cases}$$

1.4 Examples

1-**Optimal control of an elliptic distributed system :**

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, consider the following Laplace equation with Newman boundary condition

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $f \in L^2(\Omega)$ is a source function and $v \in L^2(\Gamma)$ is a control function. Associate to (1.4) the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(\Omega)} + N\|v\|^2_{L^2(\Gamma)},$$

where $y(v)$ is the solution of (1.4), $y_d \in L^2(\Omega)$ is a fixed observation and $N > 0$.
Our optimal control consists a determine $u \in \mathcal{U}_{ad}$ that minimizing $J(v)$. That's why we have to search u solution of

$$\inf\left\{J(v,y) : (v,y) \in \mathcal{U}_{ad} \times H^1(\Omega) \text{ verifies } (1.4)\right\}.$$

In this case :
the state space $\mathcal{Y} = H^1(\Omega)$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Gamma)$.
the observation operator is canonical injection

$$\begin{array}{rccc} \mathcal{C}: & H^1(\Omega) & \longrightarrow & L^2(\Gamma) \\ & y & \longrightarrow & y\ , \end{array}$$

An optimality condition gives us

$$J'(u)(v-u) \geq 0 \forall v \in \mathcal{U}_{ad} \iff 2(y(u) - y_d, y(v-u))_{L^2(\Omega)} + 2N(u, v-u)_{L^2(\Gamma)} \geq 0 \forall v \in \mathcal{U}_{ad}.$$

Now, we introduce an adjoint state $p = p(u)$

$$\begin{cases} -\Delta p + p = y(u) - y_d & \text{in } \Omega, \\ \frac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

and using the second Green formula (see appendix Theorem 4.3) we get

$$\begin{aligned} (y(u) - y_d, y(v-u))_{L^2(\Omega)} &= (-\Delta p + p, y(v-u))_{L^2(\Omega)} \\ &= (p, -\Delta y(v-u) + y(v-u))_{L^2(\Omega)} \\ &\quad + \int_\Gamma (p\frac{\partial y}{\partial \nu}(v-u) - y(v-u)\frac{\partial p}{\partial \nu})d\Gamma \\ &= \int_\Gamma p(v-u)d\Gamma. \end{aligned}$$

Finally, the solution of (1.4) is characterized by the following optimality system:

$$\begin{cases} -\Delta y(u) + y(u) = f & \text{in}\Omega, \\ \frac{\partial y}{\partial \nu}(u) = u & \text{on } \Gamma, \\ -\Delta p + p = y(u) - y_d & \text{in } \Omega, \\ \frac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma, \\ \int_\Gamma (p + Nu)(v-u)d\Gamma \geq 0 & \forall v \in \mathcal{U}_{ad}. \end{cases}$$

2-Optimal control of a parabolic distributed system:

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, $T>0$, Consider time space cylinder $Q=\Omega\times[0,T]\,,\,\Sigma=\Gamma\times[0,T]$ and the Heat equation with Dirichlet boundary condition

$$\begin{cases} \frac{\partial y}{\partial t}-\Delta y=f+\chi_\omega v & \text{in } Q, \\ y=0 & \text{on } \Sigma, \\ y(x,0)=y_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $f\in L^2\left(Q\right), v\in L^2(0,T,L^2(w))$, χ_ω is the characteristic function of ω a bounded open of Ω and $y_0\in L^2(\Omega)$.

Our optimal control problem consists in looking for a control function $u\in L^2(0,T,L^2(\omega))$ which minimizes the following cost function

$$J(v)=\|y(v)-y_d\|^2_{L^2(Q)}+N\,\|v\|^2_{L^2(0,T,L^2(\omega))}\,,$$

with $y(v)$is the solution of (1.5), $y_d\in L^2\left(Q\right), N>0$. So, we want to characterize the solution of the following optimal control

$$\inf\left\{J(v,y):(v,y)\in\mathcal{U}_{ad}\times L^2(0,T,H^1_0(\Omega))\text{ verifies (1.5)}\right\}.$$

In this case

the state space $\mathcal{Y}=L^2(0,T,H^1_0(\Omega))$, the observation space $\mathcal{Z}=L^2(\Omega)$ and the control space $\mathcal{U}=L^2(0,T,L^2(\omega))$.

the observation operator is canonical injection

$$\begin{aligned} \mathcal{C}:\ L^2(0,T,H^1_0(\Omega)) &\longrightarrow L^2(\Omega) \\ y &\longrightarrow y\ , \end{aligned}$$

An optimality condition gives us

$$J'(u)(v-u)=2(y(u)-y_d,y(v-u))_{L^2(Q)}+2N(u,v-u)_{L^2(0,T,L^2(\omega))}\geq 0\quad \forall v\in\mathcal{U}_{ad}.$$

by introducing the adjoint state $p=p(u)$

$$\begin{cases} -\frac{\partial p}{\partial t}-\Delta p=y(u)-y_d & \text{in } Q, \\ p=0 & \text{on}\Sigma, \\ p(x,T)=0 & \text{in } \Omega. \end{cases}$$

and using second Green formula we get

$$\begin{aligned}
(y(u)-y_d, y(v-u))_{L^2(Q)} &= (-\frac{\partial p}{\partial t} - \Delta p, y(v-u))_{L^2(Q)} \\
&= \int_0^T \int_\Omega -\frac{\partial p}{\partial t} y(v-u)dxdt - \int_0^T \int_\Omega \Delta p y(v-u)dxdt \\
&= \int_0^T \int_\Omega p\left(\frac{\partial y}{\partial t}(v-u) - \Delta y(v-u)\right) dxdt - \int_\Omega [py(v-u)]_0^T \\
&\quad + \int_0^T \int_\Gamma p(\frac{\partial y}{\partial \nu}(v-u) - y(v-u)\frac{\partial p}{\partial \nu})d\Gamma dt \\
&= \int_0^T \int_\Omega p\chi_\omega (v-u)dxdt - \int_\Omega p(T)y(v-u)\,(T)\,dx \\
&\quad + \int_0^T \int_\Gamma p\frac{\partial y}{\partial \nu}(v-u)d\Gamma dt \\
&= \int_0^T \int_\omega p(v-u)dxdt,
\end{aligned}$$

Hence,

$$J'(u)(v-u) = 2\int_0^T \int_\omega (p+Nu)(v-u)dxdt \geq 0 \ \ \forall v \in \mathcal{U}_{ad}.$$

Finally, the solution of (1.5) is characterized by the following optimality system

$$\begin{cases}
\frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(x,0) = y_0(x) & \text{in } \Omega, \\
-\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d & \text{in } Q, \\
p = 0 & \text{on} \Sigma, \\
p(x,T) = 0 & \text{in } \Omega, \\
\int_0^T \int_\omega (p+Nu)(v-u)dxdt \geq 0 & \forall v \in \mathcal{U}_{ad}.
\end{cases}$$

3-Optimal control of a hyperbolic distributed system :

The notions Ω, Γ, Σ, Q and the assumptions on Ω and Γ are the same of example (2). Consider wave equation with Dirichlet boundary condition

$$\begin{cases}
\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 & \text{in } Q, \\
y = v & \text{on } \Sigma, \\
y(x,0) = y_0(x), \ \ \frac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega,
\end{cases} \tag{1.6}$$

where $v \in L^2(0,T,L^2(\Gamma))$, $y_0 \in L^2(\Omega)$

Our optimal control problem consists in looking for a control function $u \in L^2(\Sigma)$ which minimizes the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(Q)} + \|y(v)(T) - y_d(T)\|^2_{L^2(\Omega)} + N\|v\|^2_{L^2(\Sigma)},$$

with $y(v)$ is the solution of (1.6), $y_d \in L^2(Q)$, $N > 0$. So we want to characterize the solution of the following optimal control problem

$$\inf\left\{J(v,y) : (v,y) \in L^2(\Sigma) \times L^2(0,T,H^1(\Omega)) \text{ verifies (1.6)}\right\}.$$

In this case
the state space $\mathcal{Y} = L^2(0,T,H^1(\Omega))$, the observation space $\mathcal{Z} = L^2(Q) \times L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Sigma)$.
The observation operator is

$$\begin{aligned} \mathcal{C}: \ L^2(0,T,H^1_0(\Omega)) &\longrightarrow L^2(Q) \times L^2(\Omega) \\ y &\longrightarrow \begin{pmatrix} y(v) \\ y(v)(T) \end{pmatrix}, \end{aligned}$$

an optimality condition gives us

$$\begin{gathered} J'(u)(v-u) = 2(y(u) - y_d, y(v-u))_{L^2(Q)} + 2(y(u)(T) - y_d(T), y(v-u)(T))_{L^2(\Omega)} \\ +2N(u, v-u)_{L^2(\Sigma)} = 0 \ \ \forall v \in \mathcal{U}. \end{gathered}$$

by introducing adjoint state $p = p(u)$

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} - \Delta p = y(u) - y_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,T) = 0, \ \frac{\partial p}{\partial t}(x,T) = -(y(u)(T) - y_d(T)) & \text{in } \Omega. \end{cases}$$

and using second Green formula and integrate by part we get

$$\begin{aligned} (y(u) - y_d, y(v-u))_{L^2(Q)} &= \left(\frac{\partial^2 p}{\partial t^2} - \Delta p, y(v-u)\right)_{L^2(Q)} \\ &= \int_0^T \int_\Omega \left(\frac{\partial^2 p}{\partial t^2} - \Delta p\right) y(v-u)\,dxdt \\ &= \int_0^T \int_\Omega p\left(\frac{\partial^2 y}{\partial t^2}(v-u) - \Delta y(v-u)\right) dxdt \\ &\quad + \int_\Omega \left[\frac{\partial p}{\partial t} y(v-u) - p\frac{\partial y(v-u)}{\partial t}\right]_0^T dx \\ &\quad + \int_0^T \int_\Gamma \left(p\frac{\partial y}{\partial \nu}(v-u) - y(v-u)\frac{\partial p}{\partial \nu}\right) d\Gamma dt \end{aligned}$$

$$= \int_{\Omega} \left[-p(T) \frac{\partial y}{\partial t} (v-u)(T) \right] dx + \int_0^T \int_{\Gamma} -\frac{\partial p}{\partial \nu}(v-u) d\Gamma dt$$

Hence, the optimality condition is given by

$$J'(u)(v-u) = 2\int_0^T \int_{\Gamma} (-\frac{\partial p}{\partial \nu} + Nu)(v-u) d\Gamma dt \geq 0 \ \ \forall v \in \mathcal{U}_{ad}.$$

Finally, the solution of (1.6) is characterized by the following optimality system

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = 0 & \text{in } Q, \\ y = u & \text{on } \Sigma, \\ y(x,0) = y_0(x) \ \ , \frac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial^2 p}{\partial t^2} - \Delta p = y(u) - y_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,T) = 0 \, , \frac{\partial p}{\partial t}(x,T) = -(y(u)(T) - y_d(T)) & \text{in } \Omega, \\ \int_0^T \int_{\Gamma} (-\frac{\partial p}{\partial \nu} + Nu)(v-u) d\Gamma dt = 0 & \forall v \in \mathcal{U}. \end{cases}$$

Chapter 2

Optimal control of linear distributed system with incomplete data

In this chapter, we study the optimal control for a linear distributed system with incomplete data [1] this leads to define the notion of no-regret control introduced the first time by **J.Lions (1992)**, which associate to there a sequence of low-regret control and prove that it converges to the no-regret control, then we characterize them via optimality systems and we give example. Also, we represent the notion of averaged control making by **Zuazua (2014)** to control systems depending on an unknown parameter.

2.1 Statement of the problem

We keep the same spaces and operators definitions that we defined in the last chapter, the difference here is the presence of missing data. For this reason, we define a new Hilbert space of uncertainties denoting by G, and we will denote by $\beta \in \mathcal{L}(G, \mathcal{Y})$ the operator of the missing data. For $f \in \mathcal{Y}$ the abstract equation related to the control $v \in \mathcal{U}_{ad}$ and the uncertainty $g \in G$ is given by

$$\mathcal{A}y(v, g) = f + \mathcal{B}v + \beta g. \tag{2.1}$$

The equation (2.1) is well posed in $\mathcal{Y}$ and her solution $y(v, g)$, which associate to her the following cost function:

$$J(v, g) = \| \mathcal{C}y(v, g) - y_d \|_{\mathcal{Z}}^2 + N \| v \|_{\mathcal{U}}^2 \qquad \forall v \in \mathcal{U}_{ad}, \forall g \in G, \tag{2.2}$$

[1] means that the initial conditions, boundary conditions, source function or some of the parameters in the main operator in the system are unknown.

as usual, we are concerned with the optimal control of (2.1) and (2.2) is to search u solution of:

$$\inf_{v\in\mathcal{U}_{ad}} J(v,g) \quad \forall g \in G,$$

but in this case, we can't apply the classical approach method because our goal to get an optimal control independently to g. So en thought to take

$$\inf_{v\in\mathcal{U}_{ad}} \left(\sup_{g\in G} J(v,g)\right), \tag{2.3}$$

but G is an infinite dimensional space we can get $\sup_{g\in G} J(v,g) = +\infty$ and by the way the problem has no sense, to avoid this difficulty **J.Lions** introduce the concept of "*No-regret control*".

Remark 2.1 *If $G = \{0\}$ then $J(v,g) = J(v,0)$. Therefore, the problem (2.3) becomes a classical problem of optimal control*

$$\begin{cases} \text{find } u \in \mathcal{U}_{ad} \text{ such that} \\ J(u) = \inf_{v\in\mathcal{U}_{ad}} J(v). \end{cases}$$

To avoid difficulties arise when we get $\sup_{g\in G} J(v,g) = +\infty$, **J.Lions** thought to take only controls such that $\forall v \in \mathcal{U}_{ad}$

$$J(v,g) \leq J(0,g) \qquad \forall g \in G, \tag{2.4}$$

i.e,

$$J(v,g) - J(0,g) \leq 0 \qquad \forall g \in G.$$

Thus, we can say that $\sup_{g\in G} (J(v,g) - J(0,g))$ exists.

2.2 The no-regret control notion

Definition 2.1 *[15]We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for (2.1) and (2.2) if u solves*

$$\inf_{v\in\mathcal{U}_{ad}} \left(\sup_{g\in G} (J(v,g) - J(0,g))\right).$$

Lemma 2.1 *For every $u \in \mathcal{U}_{ad}$ and $g \in G$ we have:*

$$J(v,g) - J(0,g) = J(v.0) - J(0,0) + 2\left(S(v), g\right)_G, \tag{2.5}$$

where $S(v) = \beta^* \xi(v)$ and $\xi(v)$ defined for $v \in \mathcal{U}_{ad}$ by

$$\mathcal{A}^* \xi(v) = \mathcal{C}^* \mathcal{C}(y(v, 0) - y(0, 0)).$$

Proof. $\mathcal{A}$ is a linear operator in $\mathcal{Y}$, so:

$$\begin{cases} y(v.g) = y(v, 0) + y(0, g) - y(0, 0) \\ y(0, g) = y(0, 0) + y(0, g) - y(0, 0) \end{cases}$$

with $y(v, 0)$ and $y(0, g)$ are a solution of (2.1) when $g = 0$ and $v = 0$ resp.
By the definition of $J(v, g)$ one obtain

$$\begin{aligned} J(v, g) &= \|\mathcal{C}(y(v, 0) + y(0, g) - y(0, 0)) - y_d\|^2_{\mathcal{Z}} + N \|v\|^2_{\mathcal{U}} \\ &= J(v, 0) + \|\mathcal{C}(y(0, g) - y(0, 0)\|^2_{\mathcal{Z}} + 2(\mathcal{C}y(v, 0) - y_d, \mathcal{C}(y(0, g) - y(0, 0)))_{\mathcal{Z}}, \end{aligned}$$

and

$$\begin{aligned} J(0, g) &= \|\mathcal{C}(y(0, 0) + y(0, g) - y(0, 0)) - y_d\|^2_{\mathcal{Z}} \\ &= J(0, 0) + \|\mathcal{C}(y(0, g) - y(0, 0))\|^2_{\mathcal{Z}} + 2(\mathcal{C}(y(0, 0) - y_d, \mathcal{C}(y(0, g) - y(0, 0)))_{\mathcal{Z}}, \end{aligned}$$

then

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(\mathcal{C}^* \mathcal{C}(y(v, 0) - y(0, 0)), y(0, g) - y(0, 0))_{\mathcal{Y}}.$$

Introduce an adjoint state $\xi(v)$ given by $\mathcal{A}^* \xi(v) = \mathcal{C}^* \mathcal{C}(y(v, 0) - y(0, 0))$ to write

$$\begin{aligned} J(v, g) - J(0, g) &= J(v, 0) - J(0, 0) + 2(\mathcal{A}^* \xi(v), y(0, g) - y(0, 0))_{\mathcal{Y}} \\ &= J(v, 0) - J(0, 0) + 2(\xi(v), \mathcal{A}(y(0, g) - y(0, 0)))_{\mathcal{Y}} \\ &= J(v, 0) - J(0, 0) + 2(\xi(v), \beta g)_{\mathcal{Y}} = J(v, 0) - J(0, 0) + 2(\beta^* \xi(v), g)_G \\ &= J(v, 0) - J(0, 0) + 2(S(v), g)_G \text{ where } S(v) = \beta^* \xi(v). \end{aligned}$$

■

2.3 The low-regret control

From (2.5) we have that:

$$\sup_{g \in G} (J(v, g) - J(0, g)) = J(v, 0) - J(0, 0) + \sup_{g \in G} (2S(v), g)_G,$$

is realized only for the no-regret control v if $v \in K$, where K is a closed subspace of $\mathcal{U}_{ad}$ given by

$$K = \{v \in \mathcal{U}_{ad},\ (S(v), g) = 0\ \forall g \in G\}.$$

The main difficulty here is to characterize the set K. To avoid that, we relax the problem by adding a quadratic term to (2.4)

$$J(v, g) \leq J(0, g) + \gamma \|g\|_G^2,\ \gamma > 0,$$

then

$$J(v, g) - J(0, g) - \gamma \|g\|_G^2 = J(v, 0) - J(0, 0) + 2(S(v), g)_G - \gamma \|g\|_G^2,$$

which implies

$$\sup_{g \in G}(J(v, g) - J(0, g) - \gamma \|g\|_G^2) = J(v, 0) - J(0, 0) + \sup_{g \in G}(2(S(v), g)_G - \gamma \|g\|_G^2),$$

use Legendre transform (see appendix Definition 4.4)[19] to obtain

$$\sup_{g \in G}(J(v, g) - J(0, g) - \gamma \|g\|_G^2) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(v)\|_G^2.$$

Hence, the problem (2.5) becomes for all $\gamma > 0$

$$\begin{cases} \text{find } u_\gamma \in \mathcal{U}_{ad} \text{ such that} \\ J^\gamma(u_\gamma) = \inf_{v \in \mathcal{U}_{ad}} J^\gamma(v). \end{cases} \tag{2.6}$$

where the new cost function is given by

$$J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(v)\|_G^2. \tag{2.7}$$

Now, we can define the low-regret by

Definition 2.2 *[15]We say that $u_\gamma \in \mathcal{U}_{ad}$ is a low-regret control for (2.1) and (2.2) if u solves*

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in G}(J(v, g) - J(0, g) - \gamma \|g\|_G^2,\ \gamma > 0).$$

Theorem 2.1 ***(Low-regret control: existence and uniqueness)***

The problem (2.1) and (2.6) with (2.7) has a unique solution u_γ.

Proof. 1.**Existence**

We have that:

$$J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(v)\|_G^2 \qquad \forall v \in \mathcal{U}_{ad},$$

which implies that

$$J^{\gamma}(v) \geq -J(0,0) = constant,$$

i.e, $\inf\limits_{v\in\mathcal{U}_{ad}} J^{\gamma}(v)$ exists.
we denote by $d_{\gamma} = \inf\limits_{v\in\mathcal{U}_{ad}} J^{\gamma}(v)$. Let a minimizing sequence (v_n^{γ}) verifying

$$\lim_{n\to\infty} J^{\gamma}(v_n^{\gamma}) = \inf_{v\in\mathcal{U}_{ad}} J^{\gamma}(v) = d_{\gamma},$$

we have that :

$$-J(0,0) \leq J^{\gamma}(v_n^{\gamma}) = J(v_n^{\gamma},0) - J(0,0) + \frac{1}{\gamma}\left\|\beta^{*}\zeta(v_n^{\gamma})\right\|_G^2 \leq d_{\gamma} + 1,$$

which implies that

$$\|Cy(v_n^{\gamma},0) - y_d\|_{\mathcal{Z}}^2 + N\|v_n^{\gamma}\|_{\mathcal{U}}^2 + \frac{1}{\gamma}\|\beta^{*}\zeta(v_n^{\gamma})\|_G^2 \leq d_{\gamma} + J(0,0) + 1 = C_{\gamma}.$$

we deduce that

$$\|v_n^{\gamma}\|_{\mathcal{U}} \leq C_{\gamma}, \tag{2.8.a}$$

$$\|Cy(v_n^{\gamma},0) - y_d\|_{\mathcal{Z}} \leq C_{\gamma}, \text{ implies } \|Cy(v_n^{\gamma},0)\|_{\mathcal{Z}} \leq C_{\gamma}, \tag{2.8.b}$$

$$\|\beta^{*}\zeta(v_n^{\gamma})\|_G \leq C_{\gamma}\sqrt{\gamma}, \tag{2.8.c}$$

where C_{γ} is a constant independent of n.
From $(2.8.a)$ we deduce that (v_n^{γ}) is bounded in compact space $\mathcal{U}_{ad}$ then we can extracting a subsequence still denoting by (v_n^{γ}) converges weakly to u_{γ} in $\mathcal{U}_{ad}$, due to isomorphism of $\mathcal{A}$ we deduce that $y(v_n^{\gamma},0)$ converge weakly to $y(u_{\gamma},0)$ in $\mathcal{Y}$.
The cost function $J^{\gamma}(v)$ is a lower semi continuous

$$J^{\gamma}(u_{\gamma}) \leq \lim_{n\to\infty} \inf_{v\in\mathcal{U}_{ad}} J^{\gamma}(v_n^{\gamma}) = \inf_{v\in\mathcal{U}_{ad}} J^{\gamma}(v) = d_{\gamma},$$

$J^{\gamma}(u_{\gamma}) = \inf\limits_{v\in\mathcal{U}_{ad}} J^{\gamma}(v)$.

2.Uniqueness

Suppose that the problem (2.6) admits two distinct solutions u_{γ}^1, u_{γ}^2. We set $u_{\gamma} = \frac{u_{\gamma}^1+u_{\gamma}^2}{2}$, due to strict convexity of J we get

$$J^{\gamma}(u_{\gamma}) < \frac{1}{2}J^{\gamma}\left(u_{\gamma}^1\right) + \frac{1}{2}J^{\gamma}\left(u_{\gamma}^2\right) = d_{\gamma},$$

we obtain a contradiction with the assumption that u_{γ}^1, u_{γ}^2 are two solutions of (2.6). Thus (2.6) admits a unique solution. ■

Theorem 2.2 *The unique low-regret control $u_\gamma$is converge weakly when γ tends to 0 to the unique no-regret control u in $\mathcal{U}_{ad}$.*

Proof. Let u_γ be a low-regret control in $\mathcal{U}_{ad}$ then for all $v \in \mathcal{U}_{ad}$

$$J^\gamma(u_\gamma) \leq J^\gamma(v),$$

from the definition of $J^\gamma(v)$ we find

$$J(u_\gamma,0) - J(0,0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2 \leq J(v,0) - J(0,0) + \frac{1}{\gamma}\|\beta^*\zeta(v)\|_G^2 \quad \forall v \in \mathcal{U}_{ad},$$

which implies

$$J(u_\gamma,0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2 \leq J(v,0) + \frac{1}{\gamma}\|\beta^*\zeta(v)\|_G^2 \quad \forall v \in \mathcal{U}_{ad},$$

we choose $v = 0$ to find :

$$J(u_\gamma,0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2 = \|\mathcal{C}y(u_\gamma,0) - y_d\|_{\mathcal{Z}}^2 + N\|u_\gamma\|_{\mathcal{U}}^2 + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2 \leq J(0,0) = constant,$$

then

$$\|u_\gamma\|_{\mathcal{U}} \leq C, \quad (2.9.a)$$

$$\|\mathcal{C}y(u_\gamma,0)\|_{\mathcal{Z}} \leq C, \quad (2.9.b)$$

$$\|\beta^*\zeta(u_\gamma)\|_G \leq \sqrt{\gamma}C, \quad (2.9.c)$$

where C is a constant independent of γ.
We deduce from $(2.9.a)$ that (u_γ) is bounded in $\mathcal{U}_{ad}$ then we can extract a subsequence still be denoting (u_γ) converges weakly to $u \in \mathcal{U}_{ad}$.
It's clear that for every $v \in \mathcal{U}_{ad}$

$$J(v,g) - J(0,g) - \gamma\|g\|_G^2 \leq J(v,g) - J(0,g) \qquad \forall g \in G,$$

i.e,

$$J(v,g) - J(0,g) - \gamma\|g\|_G^2 \leq \sup_{g\in G}(J(v,g) - J(0,g))\, \forall g \in G,$$

from another side we have

$$J(u_\gamma,g) - J(0,g) - \gamma\|g\|_G^2 \leq J(v,g) - J(0,g) - \gamma\|g\|_G^2,$$

so

$$J(u_\gamma,g) - J(0,g) - \gamma\|g\|_G^2 \leq \sup_{g\in G}(J(v,g) - J(0,g)) \qquad \forall g \in G,$$

when γ tend to 0 we obtain:

$$J(u,g) - J(0,g) \leq \sup_{g\in G}\left(J(v,g) - J(0,g)\right) \qquad \forall g \in G,$$

which means that

$$\sup_{g\in G}\left(J(u,g) - J(0,g)\right) = \inf_{v\in\mathcal{U}_{ad}}\left\{\sup_{g\in G}\left(J(v,g) - J(0,g)\right)\right\}.$$

In conclusion, u is a no-regret control. ■

Characterization of the low-regret control

A first order optimality condition gives

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) \geq 0 \ \ \forall v \in \mathcal{U}_{ad},$$

where

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = \lim_{h\to 0}\frac{J\left(u_\gamma + h\left(v-u_\gamma\right)\right) - J\left(u_\gamma\right)}{h} \ \ \forall v \in \mathcal{U}_{ad},$$

we have

$$\begin{aligned}\frac{J\left(u_\gamma + t\left(v-u_\gamma\right)\right) - J^{\gamma}(u_\gamma)}{h} &= h\left\|\mathcal{C}y(v-u_\gamma,0)\right\|^2_{\mathcal{Z}} + hN\left\|v-u_\gamma\right\|^2_{\mathcal{U}} + \frac{h}{\gamma}\left\|S\left(v-u_\gamma\right)\right\|^2_G \\ &+2(\mathcal{C}y(u_\gamma,0) - y_d, \mathcal{C}y(v-u_\gamma,0))_{\mathcal{Z}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} \\ &+\frac{2}{\gamma}(S(u_\gamma), S\left(v-u_\gamma\right))_G,\end{aligned}$$

make h tends to 0 to get

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = 2(\mathcal{C}y(u_\gamma,0) - y_d, \mathcal{C}y(v-u_\gamma,0))_{\mathcal{Z}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\left(v-u_\gamma\right))_G.$$

From linearity of the operator $\mathcal{C}$ in $\mathcal{Z}$, we get:

$$\begin{aligned}J^{\gamma\prime}(u_\gamma)(v-u_\gamma) &= 2(\mathcal{C}y(u_\gamma,0) - y_d, \mathcal{C}y(v,0) - \mathcal{C}y\left(u_\gamma,0\right))_{\mathcal{Z}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\left(v-u_\gamma\right))_G \\ &= 2(\mathcal{C}^*\left(\mathcal{C}y(u_\gamma,0) - y_d\right), y(v,0) - y\left(u_\gamma,0\right))_{\mathcal{Y}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\left(v-u_\gamma\right))_G,\end{aligned}$$

thanks to linearity

$$y(v,0) - y(u_\gamma,0) = y\left(v-u_\gamma,0\right) - y\left(0,0\right),$$

then

$$
\begin{aligned}
J^{\gamma\prime}(u_\gamma)(v-u_\gamma) &= 2(\mathcal{C}^*\left(\mathcal{C}y(u_\gamma,0)-y_d\right), y(v-u_\gamma,0)-y\left(0,0\right))_{\mathcal{Y}} \\
&\quad +2N(u_\gamma, v-u_\gamma)_{\mathcal{U}}+\frac{2}{\gamma}(S(u_\gamma), S\left(v-u_\gamma\right))_G.
\end{aligned}
$$

We recall the adjoint state defined previously by $\mathcal{A}^*\xi(u_\gamma)=\mathcal{C}^*\mathcal{C}(y(u_\gamma,0)-y(0,0))$, then

$$
(S(u_\gamma), S\left(v-u_\gamma\right))_G=(\beta^*\xi(u_\gamma),\beta^*\xi(v-u_\gamma))_G=(\beta\beta^*\xi(u_\gamma),\xi(v-u_\gamma))_{\mathcal{Y}}.
$$

Also, we define the new state $\rho_\gamma=\rho(u_\gamma)$ by

$$
\mathcal{A}\rho_\gamma=\frac{1}{\gamma}\beta\beta^*\xi(u_\gamma),
$$

this leads to the following equality

$$
\begin{aligned}
(\mathcal{A}\rho_\gamma,\xi(v-u_\gamma))_{\mathcal{Y}} &= (\rho_\gamma,\mathcal{A}^*\xi(v-u_\gamma))_{\mathcal{Y}}=(\rho_\gamma,\mathcal{C}^*\mathcal{C}(y(v-u_\gamma,0)-y(0,0)))_{\mathcal{Y}} \\
&= (\mathcal{C}^*\mathcal{C}\rho_\gamma, y(v-u_\gamma,0)-y(0,0))_{\mathcal{Y}},
\end{aligned}
$$

introducing the new adjoint state $p_\gamma=p(u_\gamma)$ by

$$
\mathcal{A}^*p_\gamma=\mathcal{C}^*(\mathcal{C}y_\gamma-y_d)+\mathcal{C}^*\mathcal{C}\rho_\gamma,
$$

to find

$$
\begin{aligned}
(\mathcal{A}^*p_\gamma, y\left(v-u_\gamma,0\right)-y(0,0))_{\mathcal{Y}} &= (p_\gamma,\mathcal{A}(y\left(v-u_\gamma,0\right)-y(0,0)))_{\mathcal{Y}} \\
&= (p_\gamma,\mathcal{B}\left(v-u_\gamma\right))_{\mathcal{Y}} \\
&= (\mathcal{B}^*p_\gamma, v-u_\gamma)_{\mathcal{U}},
\end{aligned}
$$

Hence, the optimality condition is given by

$$
J^{\gamma\prime}(u_\gamma)(v-u_\gamma)=(\mathcal{B}^*p_\gamma+Nu_\gamma, v-u_\gamma)_{\mathcal{U}}\geq 0\ \ \forall v\in\mathcal{U}_{ad},
$$

Finally, the low-regret control is characterized by the following optimality system:

$$
\left\{
\begin{array}{c}
\mathcal{A}y_\gamma=f+Bu_\gamma, \\
\mathcal{A}^*\xi_\gamma=\mathcal{C}^*\mathcal{C}(y_\gamma-y(0,0)), \\
\mathcal{A}\rho_\gamma=\frac{1}{\gamma}\beta\beta^*\xi_\gamma, \\
\mathcal{A}^*p_\gamma=\mathcal{C}^*(\mathcal{C}y_\gamma-y_d)+\mathcal{C}^*\mathcal{C}\rho_\gamma, \\
(\mathcal{B}^*p_\gamma+Nu_\gamma, v-u_\gamma)_{\mathcal{U}}\geq 0\ \ \forall v\in\mathcal{U}_{ad}.
\end{array}
\right. \tag{2.10}
$$

where $y\left(u_\gamma,0\right)=y_\gamma$, $\xi(u_\gamma)=\xi_\gamma$.

Characterization of no-regret control

To get the optimality system of no-regret control we pass to limit when γ tends to 0 in the system (2.10)

$$\begin{cases} \mathcal{A}y = f + Bu, \\ \mathcal{A}^{*}\zeta = \mathcal{C}^{*}\mathcal{C}y\left(u,0\right) - y_{d}, \\ \mathcal{A}\rho = \beta\lambda, \lambda \in G, \\ \mathcal{A}^{*}p = \mathcal{C}^{*}\left(\mathcal{C}y\left(u,0\right) - y_{d}\right) + \mathcal{C}^{*}\mathcal{C}\rho, \\ \left(\mathcal{B}^{*}p + Nu, v - u\right)_{\mathcal{U}} \geq 0 \;\; \forall v \in \mathcal{U}_{ad}. \end{cases} \tag{2.11}$$

where $y\left(u,0\right) = y$, $\xi(u) = \xi$.

2.4 Example

Optimal control of an elliptic distributed system with missing Newmann boundary condition:

Let Ω be an open bounded set of $\mathbb{R}^{n}$ with smooth boundary Γ. Consider the following elliptic equation :

$$\begin{cases} -\Delta y + y = f + v & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = g & \text{on } \Gamma, \end{cases} \tag{2.12}$$

where $v \in L^{2}\left(\Omega\right)$, $g \in G = L^{2}\left(\Gamma\right)$, $f \in L^{2}\left(\Omega\right)$ and $y\left(v,g\right) \in H^{\frac{3}{2}}\left(\Omega\right) \subset L^{2}\left(\Omega\right)$ is the unique solution of this system depend on v and g. Associate to (2.12) the following cost function:

$$J\left(v,g\right) = \left|y\left(v,g\right) - y_{d}\right|^{2}_{L^{2}(\Gamma)} + N\left\|v\right\|^{2}_{L^{2}(\Omega)} \forall g \in G, \tag{2.13}$$

where $y_{d} \in L^{2}\left(\Gamma\right)$, $N > 0$ and $|.|_{L^{2}(\Gamma)}$ denote the semi norm in $L^{2}\left(\Gamma\right)$.
Here, we have that: $\mathcal{Y} = L^{2}\left(\Omega\right)$ is the state space, $\mathcal{U} = L^{2}\left(\Omega\right)$ is the control space, $\mathcal{Z} = L^{2}\left(\Gamma\right)$ is the observation space, $G = L^{2}\left(\Gamma\right)$ is the uncertainties space, the observation operator $\mathcal{C}$:

$$\begin{aligned} \mathcal{C}: \; L^{2}\left(\Omega\right) &\longrightarrow L^{2}\left(\Gamma\right) \\ y &\longrightarrow y\left|_{\Gamma}\right. \end{aligned}$$

β : is the uncertainties operator

$$\begin{aligned} \beta: \; L^{2}\left(\Gamma\right) &\longrightarrow L^{2}\left(\Omega\right) \\ g &\longrightarrow y\left(0,g\right) \end{aligned}$$

where $y\left(0,g\right)$ is solution of (2.12) when $v = 0$.

Definition 2.3 *We say that u is a no-regret control for* (2.12) *and* (2.13) *iff u is solution of:*

$$\inf_{v\in L^2(\Omega)}\left(\sup_{g\in L^2(\Gamma)} J\left(v,g\right)-J\left(0,g\right)\right).$$

We need the characterization of a no-regret control. Therefore, for all $v\in L^2\left(\Omega\right)$ and $g\in L^2\left(\Gamma\right)$, we have :

$$J\left(v,g\right)=J(v,0)+\left|y\left(0,g\right)-y\left(0,0\right)\right|^2_{L^2(\Gamma)}+2(y\left(v,0\right)-y_d,y\left(0,g\right)-y\left(0,0\right))_{L^2(\Gamma)},$$

and

$$J\left(0,g\right)=J\left(0,0\right)+\left|y\left(0,g\right)-y\left(0,0\right)\right|^2_{L^2(\Gamma)}+2(y\left(0,0\right)-y_d,y\left(0,g\right)-y\left(0,0\right))_{L^2(\Gamma)},$$

so:

$$J\left(v,g\right)-J\left(0,g\right)=J(v,0)-J\left(0,0\right)+2(y\left(v,0\right)-y\left(0,0\right),y\left(0,g\right)-y\left(0,0\right))_{L^2(\Gamma)}.$$

Let's introduce the adjoint state $\xi=\xi(u)$

$$\begin{cases} -\Delta\zeta+\zeta=0 & \text{in } \Omega,\\ \frac{\partial\zeta}{\partial\nu}=y\left(u,0\right)-y\left(0,0\right) & \text{on } \Gamma,\end{cases}$$

using the second Green formula(see appendix Theorem 4.3), we obtained:

$$\begin{aligned}(-\Delta\zeta+\zeta,y\left(0,g\right)-y\left(0,0\right))_{L^2(\Omega)} &= \int\limits_{\Omega}\left(-\Delta\zeta+\zeta\right)\left(y\left(0,g\right)-y\left(0,0\right)\right)dx\\ &= \int\limits_{\Gamma}\zeta g d\Gamma.\end{aligned}$$

So:

$$J\left(v,g\right)-J\left(0,g\right)=J(v,0)-J\left(0,0\right)+2(\zeta,g)_{L^2(\Gamma)}.$$

Let's define the low-regret control

Definition 2.4 *We say that u_γ is a low-regret control for* (2.12) *and* (2.13) *iff u_γ is solution of:*

$$\inf_{v\in L^2(\Omega)}\left(\sup_{g\in L^2(\Gamma)} J\left(v,g\right)-J\left(0,g\right)-\gamma\left\|g\right\|^2_{L^2(\Gamma)}\right).$$

We have

$$\begin{aligned}\sup_{g\in L^2(\Gamma)}\left(J(v,g)-J(0,g)-\gamma\|g\|^2_{L^2(\Gamma)}\right) &= J(v,0)-J(0,0)+\sup_{v\in L^2(\Gamma)}\left(2(\zeta,g)_{L^2(\Gamma)}-\gamma\|g\|^2_{L^2(\Gamma)}\right)\\ &= J(v,0)-J(0,0)+\frac{1}{\gamma}\|\zeta(v)\|^2_{L^2(\Gamma)}.\end{aligned}$$

Hence, we define the new following cost function related to the problem of low regret given by

$$J^{\gamma}(v)=J(v,0)-J(0,0)+\frac{1}{\gamma}\|\zeta(v)\|^2_{L^2(\Gamma)}. \tag{2.14}$$

Then our problem optimal becomes

$$\begin{cases}\text{find } u_\gamma\in\mathcal{U} \text{ such that}\\ J^{\gamma}(u_\gamma)=\inf\limits_{v\in\mathcal{U}}J^{\gamma}(v).\end{cases} \tag{2.15}$$

Theorem 2.3 ***(The existence and uniqueness of a low-regret control)***

The problem (2.12) and (2.15) with(2.14) has a unique solution u_γ.

Proof. The cost function $J^{\gamma}(v)$ is coercive and strictly convex in $L^2(\Omega)$ which implies the existence and uniqueness of u_γ. ■

Characterization of the low-regret control:

A first optimality condition gives us :

$$J^{\gamma'}(u_\gamma)\left(v-u_\gamma\right)=0 \qquad \forall v\in L^2(\Omega),$$

i.e,

$$\begin{aligned}J^{\gamma'}(u_\gamma)\left(v-u_\gamma\right) &= 2(y(u_\gamma,0)-y_d, y(v-u_\gamma,0)-y(0,0))_{L^2(\Gamma)}+2N(u_\gamma,v-u_\gamma)_{L^2(\Omega)}\\ &\quad+\frac{2}{\gamma}(\zeta(u_\gamma),\zeta(v-u_\gamma))_{L^2(\Gamma)}=0 \quad \forall v\in L^2(\Omega),\end{aligned}$$

Introduce the state $\rho_\gamma=\rho(u_\gamma)$ by

$$\begin{cases}-\Delta\rho_\gamma+\rho_\gamma=0 & \text{in } \Omega,\\ \frac{\partial\rho_\gamma}{\partial\nu}=\frac{1}{\gamma}\zeta(u_\gamma) & \text{on } \Gamma,\end{cases}$$

multiply the first equation of the last one by $\zeta(v-u_\gamma)$ and apply second Green formula (see appendix Theorem 4.3) to get :

$$\begin{aligned}\left(-\Delta\rho_\gamma+\rho_\gamma,\zeta(v-u_\gamma)\right)_{L^2(\Omega)} &= \int_\Omega\left(-\Delta\rho_\gamma+\rho_\gamma\right)\zeta(v-u_\gamma)\,dx\\ &= \int_\Gamma\left(\rho_\gamma\left(y(v,0)-y(0,0)\right)-\zeta(v-u_\gamma)\frac{\partial\rho_\gamma}{\partial\nu}\right)d\Gamma,\end{aligned}$$

Hence,

$$
\begin{aligned}
J^{\gamma'}(u_\gamma)\left(v-u_{_\gamma}\right) &= (\rho_\gamma + y(u_\gamma,0) - y_d, y(v-u_\gamma,0) - y(0,0))_{L^2(\Gamma)} + N(u_{_\gamma}, v-u_\gamma)_{L^2(\Omega)} \\
&= 0 \ \forall v \in \mathcal{U}.
\end{aligned}
$$

We introduce another adjoint state $p_\gamma = p(u_\gamma)$ given by:

$$
\begin{cases}
-\Delta p_\gamma + p_\gamma = 0 & \text{in } \Omega, \\
\frac{\partial p_\gamma}{\partial \nu} = \rho_\gamma + y(u_\gamma,0) - y_d & \text{on } \Gamma.
\end{cases}
$$

Again, we have

$$
\begin{aligned}
(-\Delta p_\gamma + p_\gamma, y(v-u_\gamma,0) - y(0,0))_{L^2(\Omega)} &= \int\limits_\Omega (-\Delta p_\gamma + p_\gamma)(y(v-u_\gamma,0) - y(0,0))\,dx \\
&= \int\limits_\Omega p_\gamma((-\Delta + I)(y(v-u_\gamma,0) - y(0,0)))dx \\
&\quad + \int\limits_\Gamma (p_\gamma \frac{\partial (y(v-u_\gamma,0) - y(0,0))}{\partial \nu} \\
&\quad - (y(v-u_\gamma,0) - y(0,0)) \frac{\partial p_\gamma}{\partial \nu})d\Gamma,
\end{aligned}
$$

then:

$$
J^{\gamma'}(u_\gamma)\left(v-u_{_\gamma}\right) = (p_{_\gamma} + Nu_{_\gamma}, v-u_\gamma)_{L^2(\Omega)} = 0, \ \forall v \in L^2(\Omega).
$$

i,e.

$$
\begin{aligned}
p_{_\gamma} + Nu_{_\gamma} &= 0 \text{ in } L^2(\Omega), \\
p_{_\gamma} + Nu_{_\gamma} &= 0 \text{ a.e in } \Omega.
\end{aligned}
$$

Then the low-regret control is characterized by the following optimality system:

$$\begin{cases} -\Delta y\left(u_{\gamma},0\right)+y\left(u_{\gamma},0\right)=f+u_{\gamma} & \text{in } \Omega,\\ \frac{\partial y\left(u_{\gamma},0\right)}{\partial \nu}=0 & \text{on } \Gamma,\\ -\Delta \zeta\left(u_{\gamma}\right)+\zeta\left(u_{\gamma}\right)=0 & \text{in } \Omega,\\ \frac{\partial \zeta_{\gamma}}{\partial \nu}=y\left(u_{\gamma},0\right)-y\left(0,0\right) & \text{on } \Gamma,\\ -\Delta \rho_{\gamma}\left(u_{\gamma}\right)+\rho_{\gamma}\left(u_{\gamma}\right)=0 & \text{in } \Omega,\\ \frac{\partial \rho_{\gamma}}{\partial \nu}=\frac{1}{\gamma}\zeta\left(u_{\gamma}\right) & \text{on } \Gamma,\\ -\Delta p_{\gamma}\left(u_{\gamma}\right)+p_{\gamma}\left(u_{\gamma}\right)=0 & \text{in } \Omega,\\ \frac{\partial p_{\gamma}}{\partial \nu}=\rho_{\gamma}+y\left(u_{\gamma},0\right)-y_{d} & \text{on } \Gamma,\\ p_{\gamma}+Nu_{\gamma}=0 & \text{a.e in } \Omega. \end{cases}$$

To get a no-regret control characterization we pass to limit when $\gamma \to 0$ in the last system we obtain:

$$\begin{cases} -\Delta y\left(u,0\right)+y\left(u,0\right)=f+u & \text{in } \Omega,\\ \frac{\partial y\left(u,0\right)}{\partial \nu}=0 & \text{on } \Gamma,\\ -\Delta \zeta\left(u\right)+\zeta\left(u\right)=0 & \text{in } \Omega,\\ \frac{\partial \zeta}{\partial \nu}=y\left(u,0\right)-y\left(0,0\right) & \text{on } \Gamma,\\ -\Delta \rho\left(u\right)+\rho\left(u\right)=0 & \text{in } \Omega,\\ \frac{\partial \rho}{\partial \nu}=\lambda & \text{on } \Gamma,\\ -\Delta p\left(u\right)+p\left(u\right)=0 & \text{in } \Omega,\\ \frac{\partial p}{\partial \nu}=\rho+y\left(u,0\right)-y_{d} & \text{on } \Gamma,\\ p+Nu=0 & \text{a.e in } \Omega. \end{cases}$$

with the following limits

$$\begin{aligned} \lim_{\gamma\to 0} u_{\gamma} &= u,\ \lim_{\gamma\to 0} y\left(u_{\gamma},0\right)=y\left(u,0\right),\ \lim_{\gamma\to 0}\xi_{\gamma}=\xi,\\ \lim_{\gamma\to 0}\rho\left(u_{\gamma}\right) &= \rho\left(u\right),\ \lim_{\gamma\to 0}\frac{1}{\gamma}\zeta\left(u_{\gamma}\right)=\lambda\in G,\ \lim_{\gamma\to 0}p\left(u_{\gamma}\right)=p\left(u\right). \end{aligned}$$

2.5 Averaged control in distributed systems

Average control is a method making by Zuazua [20] to control a distributed system depending on an unknown parameter. The idea of this method is not controlling the state is to control the average of the state.

Chapter 3

Optimal control of electromagnetic waves with missing data

This chapter is devoted to the study of the optimal control problem for an electromagnetic waves that penetrate a medium with some unknown physical properties as there velocity of propagation and missing boundary condition. For this purpose, Hafdallah & Ayadi use the concept of averaged no-regret control and averaged low-regret control. We show the existence and uniqueness of averaged low-regret control and show that it converges to the averaged no-regret control. Then, we give the optimality system that characterizes the controls.

3.1 Description of problem

Let $n = 1$, 2 or 3, and Ω be an open bounded domain in $\mathbb{R}^n$, with smooth boundary Γ, for $T > 0$ we set the cylindric time space $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the following controlled hyperbolic equation which modelling the propagation of electromagnetic waves in a medium with missing parameter σ belongs to $[\sigma_1, \sigma_2]$ represent the velocity of propagation and unknown Dirichlet boundary condition:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \sigma^2 \Delta y = v & \text{in } Q, \\ y = g & \text{on } \Sigma, \\ y(x, 0) = 0, \frac{\partial y}{\partial t}(x, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where g is an unknown function belongs to $L^2(\Sigma)$, $\mathcal{U}_{ad}$ is a non-empty closed convex subset of $L^2(Q)$. The control $v \in \mathcal{U}_{ad}$ and the function g are independent of σ. According to the data, we know that system (3.1) admits a unique solution

$$y(v, g, \sigma) = y(v, g, \sigma; x, t) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)),$$

which depend continuously on σ. We associate to the problem the following cost function:

$$J(v, g, \sigma) = \|y(v, g; \sigma) - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \tag{3.2}$$

where y_d is the desired state in $L^2(Q)$, $N > 0$. We are concerned with the optimal control of the problem (3.1) and (3.2). i.e, we want to solve

$$\inf_{v \in \mathcal{U}_{ad}} J(v, g, \sigma), \text{ for every } g \in L^2(\Sigma) \text{ and } \sigma \in [\sigma_1, \sigma_2], \tag{3.3}$$

since the function g is unknown, the optimal control problem (3.3) has no sense. So, we look for a solution to the following minimizing problem

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in L^2(\Sigma)} J(v, g, \sigma) \ \forall \sigma \in [\sigma_1, \sigma_2].$$

In this case, it's possible to get $\sup_{g \in L^2(\Sigma)} J(v, g, \sigma) = +\infty$, so we use the idea of **J.Lions** to look only for controls $v \in \mathcal{U}_{ad}$ such that

$$J(v, g; \sigma) \leq J(0, g, \sigma), \ \forall g \in L^2(\Sigma), \ \forall \sigma \in [\sigma_1, \sigma_2], \tag{3.4}$$

and for the unknown parameters σ we use the concept of averaged control. So, we substitute the state by it's average concerning the unknown parameter σ in the cost function (3.2) to get

$$J(v, g) = \left\| \int_{\sigma_1}^{\sigma_2} y(v, g, \sigma)\, d\sigma - z_d \right\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \tag{3.5}$$

where z_d is an averaged desired state observation in $L^2(Q)$.

3.2 Averaged no-regret control and averaged low-regret control: definitions

Definition 3.1 *[5]We say that $u \in \mathcal{U}_{ad}$ is an averaged no-regret control for (3.1) with (3.5) if u is a solution of*

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in L^2(\Sigma)} (J(v, g) - J(0, g)).$$

Let's try to isolate g to get a control independent to the missing condition.

Lemma 3.1 *Let $v \in \mathcal{U}_{ad}$ and $g \in L^2(\Sigma)$, we have*

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2(\sigma_1-\sigma_2)\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gd\Gamma dt, \tag{3.6}$$

where $\xi=\xi(v)\in C([0,T],H_0^1(\Omega))\cap C^1([0,T],L^2(\Omega))$ be solution of

$$\begin{cases}\frac{\partial^2\xi}{\partial t^2}-\Delta\xi=\frac{1}{t}\int_{\sigma_1}^{\sigma_2}y(v,0;\sigma)\,d\sigma & \text{in } Q,\\ \xi=0 & \text{on } \Sigma,\\ \xi(x,T)=0,\ \frac{\partial\xi(v)}{\partial t}(x,T)=0 & \text{in } \Omega.\end{cases} \tag{3.7}$$

Proof. Let us consider $y(v,0;\sigma)$ and $y(0,g;\sigma)$ be a solution of (3.1) where $g=0$ and $v=0$ resp, and we have

$$y(v,g,\sigma)=y(v,0;\sigma)+y(0,g;\sigma),$$

from the definition of $J(v,g)$ and by a simple calculus, we get

$$\begin{aligned}J(v,g) &= J(v,0)+J(0,g)+J(0,0)+2\left(\int_{\sigma_1}^{\sigma_2}y(0,g;\sigma)\,d\sigma-y_d,y_d\right)_{L^2(Q)}\\ &+2\left(\int_{\sigma_1}^{\sigma_2}y(v,0;\sigma)\,d\sigma-y_d,\int_{\sigma_1}^{\sigma_2}y(0,g;\sigma)\,d\sigma\right)_{L^2(Q)},\end{aligned}$$

Hence,

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2\left(\int_{\sigma_1}^{\sigma_2}y(v,0;\sigma)\,d\sigma,\int_{\sigma_1}^{\sigma_2}y(0,g;\sigma)\,d\sigma\right)_{L^2(Q)}.$$

Now, we need the system that describes $\int_{\sigma_1}^{\sigma_2}y(0,g;\sigma)\,d\sigma$, we pose

$$t=\sigma t\Longrightarrow\sigma=1$$

to get

$$y(0,g,\sigma;x,t)=Y(0,g,1;x,\sigma t),$$

where $Y(x,t)$ is a solution of

$$\begin{cases}\frac{\partial^2 Y}{\partial t^2}-\Delta Y=0 & \text{in } Q,\\ Y(x,\sigma t)=g(x,t) & \text{on } \Sigma,\\ Y(x,0)=0,\ \frac{\partial Y}{\partial t}(x,0)=0 & \text{in } \Omega,\end{cases}$$

then

$$\int_{\sigma_1}^{\sigma_2} y(0,g,\sigma;x,t)\,d\sigma = \int_{\sigma_1 t}^{\sigma_2 t} Y(0,g,1;x,\sigma t)\,\frac{dt}{t} = \frac{Z(x,\sigma_2 t) - Z(x,\sigma_1 t)}{t},$$

where $Z(x,t) = \int_0^t Y(0,g,1;x,s)\,ds$ is solution of

$$\begin{cases} \frac{\partial^2 Z}{\partial t^2} - \Delta Z = 0 & \text{in } Q, \\ Z(x,t) = \int_0^t Y(x,s)\,ds & \text{on } \Sigma, \\ Z(x,0) = 0, \frac{\partial Z}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

that is

$$\begin{aligned} J(v,g) - J(0,g) &= J(v,0) - J(0,0) + 2\left(\int_{\sigma_1}^{\sigma_2} y(v,0;\sigma)\,d\sigma, \int_{\sigma_1}^{\sigma_2} y(0,g;\sigma)\,d\sigma\right)_{L^2(Q)} \\ &= J(v,0) - J(0,0) + 2\left(\frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(v,0;\sigma)\,d\sigma, Z(x,\sigma_2 t) - Z(x,\sigma_1 t)\right)_{L^2(Q)}. \end{aligned}$$

Then, we introduce the adjoint state $\xi = \xi(v)$ define by (3.7) and use the second Green formula, we obtain:

$$\begin{aligned} \left(\frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(v,0;\sigma)\,d\sigma, Z(x,\sigma_2 t) - Z(x,\sigma_1 t)\right)_{L^2(Q)} &= \left(\frac{\partial^2 \xi}{\partial t^2} - \Delta\xi, Z(x,\sigma_2 t) - Z(x,\sigma_1 t)\right)_{L^2(Q)} \\ &= -\int_0^T \int_\Gamma \frac{\partial \xi}{\partial \nu} \int_{\sigma_1 t}^{\sigma_2 t} g(x,t)\,ds\,d\Gamma\,dt \\ &= (\sigma_1 - \sigma_2)\int_0^T \int_\Gamma t\frac{\partial \xi}{\partial \nu} g\,d\Gamma\,dt, \end{aligned}$$

we conclude that

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(\sigma_1 - \sigma_2)\int_0^T \int_\Gamma t\frac{\partial \xi}{\partial \nu} g\,d\Gamma\,dt.$$

■

Now, we consider the averaged no-regret control

$$\inf_{v\in\mathcal{U}_{ad}}\sup_{g\in L^2(\Sigma)}\left(J\left(v,g\right)-J\left(0,g\right)\right),$$

from (3.6) the problem is equivalent to the following one

$$\inf_{v\in\mathcal{U}_{ad}}\left(J\left(v,0\right)-J\left(0,0\right)+\left(\sigma_1-\sigma_2\right)\sup_{g\in L^2(\Sigma)}\left(2\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gdxdt\right)\right).$$

So, the problem it's defined only for

$$\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gd\Gamma dt=0,$$

it means that the no regret controls depend only to the set structure of K define by

$$K=\left\{v\in\mathcal{U}_{ad}\text{ such that }\left(\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gd\Gamma dt\right)=0,\ \forall g\in L^2\left(\Sigma\right)\right\}.$$

This set is hard to characterize, so we relax the problem by adding a quadratic perturbation to (3.4) to get

$$\inf_{v\in\mathcal{U}_{ad}}\sup_{g\in L^2(\Sigma)}\left(J\left(v,g\right)-J\left(0,g\right)-\gamma\left\|g\right\|^2_{L^2(\Sigma)}\right),\ \gamma>0.$$

Then

$$\sup_{g\in L^2(\Sigma)}\left(J\left(v,g\right)-J\left(0,g\right)-\gamma\left\|g\right\|^2_{L^2(\Sigma)}\right)=J\left(v,0\right)-J\left(0,0\right)+\sup_{g\in L^2(\Sigma)}\left(2\left(\sigma_1-\sigma_2\right)\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gd\Gamma dt\right.$$

$$\left.-\gamma\left\|g\right\|^2_{L^2(\Sigma)}\right),\ \gamma>0,$$

using the Legendre transform (see appendix definition 4.4) we get

$$\sup_{g\in L^2(\Sigma)}\left(J\left(v,g\right)-J\left(0,g\right)-\gamma\left\|g\right\|^2_{L^2(\Sigma)}\right)=J\left(v,0\right)-J\left(0,0\right)+\frac{\sigma_2-\sigma_1}{\gamma}\left\|t\frac{\partial\xi}{\partial\nu}\right\|^2_{L^2(\Sigma)}.$$

Therefore, we are front in a classical problem of optimal control independently to the function g and the parameters σ define by

$$\left\{\begin{array}{c}\text{find } u_\gamma\in\mathcal{U}_{ad}\text{ such that}\\ J^\gamma\left(u_\gamma\right)=\inf_{v\in\mathcal{U}_{ad}}J^\gamma\left(v\right),\ \forall\gamma>0,\end{array}\right.\tag{3.8}$$

where

$$J^{\gamma}(v) = J(v,0) - J(0,0) + \frac{\sigma_2 - \sigma_1}{\gamma} \left\| t \frac{\partial \xi}{\partial \nu} \right\|^2_{L^2(\Sigma)}. \tag{3.9}$$

This leads us to define the notion of averaged low-regret control.

Definition 3.2 *[5]We say that $u_\gamma \in \mathcal{U}_{ad}$ is an averaged low-regret control for (3.1) and (3.5) if u is a solution of*

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in L^2(\Sigma)} \left(J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Sigma)} \right).$$

Theorem 3.1 ***(The averaged low-regret control: Existence and uniqueness)***

There exists a unique averaged low-regret control u_γ solution to $(3.1) - (3.8)$ and (3.9).

Proof. 1-**Existence:**

From the definition of $J^{\gamma}(v)$, It's clear that

$$J^{\gamma}(v) \geq -J(0,0),$$

it means that $\inf\limits_{v \in \mathcal{U}_{ad}} J^{\gamma}(v) = d_\gamma$ exists. Let (v_n) be a minimizing sequence satisfying:

$$\lim_{n \to \infty} J^{\gamma}(v_n) = \inf_{v \in \mathcal{U}_{ad}} J^{\gamma}(v_n) = d_\gamma.$$

Moreover, we have

$$-J(0,0) \leq J^{\gamma}(v_n) = J(v_n,0) - J(0,0) + \frac{\sigma_2 - \sigma_1}{\gamma} \left\| t \frac{\partial \xi_n}{\partial \nu} \right\|^2_{L^2(\Sigma)} \leq d_\gamma + 1,$$

which implies

$$\|v_n\|_{L^2(Q)} \leq C_\gamma, \tag{3.10.1}$$

$$\left\| t \frac{\partial \xi_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_\gamma (\sigma_2 - \sigma_1) \sqrt{\gamma}, \tag{3.10.2}$$

$$\left\| \int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma \right\|_{L^2(Q)} \leq C_\gamma. \tag{3.10.3}$$

In the other side, we have the following energy estimate [13]

$$\left\| \frac{\partial y_n}{\partial t} \right\|^2_{L^\infty(0,T;L^2(\Omega))} + \sigma^2 \|y_n\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq \|v_n\|_{L^2(Q)} \leq C_\gamma,$$

where C_γ is a constant independent of n.

Which gives

$$\left\|\frac{\partial y_n}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C_\gamma, \tag{3.11.1}$$

$$\|y_n\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C_\gamma, \tag{3.11.2}$$

where $y_n = y(v_n, 0; \sigma)$ solution of

$$\begin{cases} \frac{\partial^2 y_n}{\partial t^2} - \sigma^2 \Delta y_n = v_n & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ y_n(x,0) = 0, \frac{\partial y_n}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.12}$$

From $(3.10.1)$ and (3.12) we get

$$\left\|\frac{\partial^2 y_n}{\partial t^2} - \sigma^2 \Delta y_n\right\|_{L^2(Q)} \leq C_\gamma, \tag{3.13}$$

Then from $(3.10.1)$, $(3.10.3)$, $(3.11.2)$ we can extracting a subsequences still denoted (v_n), (y_n), $(\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma)$
such that

$$\begin{aligned} v_n &\rightharpoonup u_\gamma \text{ weakly in } L^2(Q), \\ \int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma &\rightharpoonup z_\gamma \text{ weakly in } L^2(Q), \\ y_n &\rightharpoonup y_\gamma \text{ weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right). \end{aligned}$$

Because of continuous embedding of $L^\infty(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ into $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$ respectively, and by the continuity of y with respect to the data, we conclude

$$\begin{aligned} y_n &\rightharpoonup y(u_\gamma, 0, \sigma) \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right), \\ z_\gamma &= \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma, \end{aligned}$$

due to $(3.11.1)$ we deduce $\frac{\partial y_n}{\partial t} \rightharpoonup f_1$ weakly in $L^2(Q)$, and $\frac{\partial y_n}{\partial t} \rightharpoonup \frac{\partial y_\gamma}{\partial t}$ in $D'(Q)$ by the uniqueness of limit, we deduce

$$\frac{\partial y_n}{\partial t} \rightharpoonup \frac{\partial y_\gamma}{\partial t} \text{ weakly in } L^2(Q),$$

due to (3.13) we deduce $\frac{\partial^2 y_n}{\partial t^2} - \sigma^2 \Delta y_n \rightharpoonup f_2$ weakly in $L^2(Q)$, and $\frac{\partial^2 y_n}{\partial t^2} - \sigma^2 \Delta y_n \rightharpoonup \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma$ in $D'(Q)$ by the uniqueness of limit, we deduce

$$\frac{\partial^2 y_n}{\partial t^2} - \sigma^2 \Delta y_n \rightharpoonup \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma \text{ weakly in } L^2(Q).$$

In view to the initial condition, we deduce

$$y_\gamma(x,0)=0,\ \frac{\partial y_\gamma}{\partial t}(x,0)=0 \text{ in } \Omega.$$

Now, we prove that $y_\gamma=0$ in Σ, let $\phi\in D^{'}(Q)$ such that $\phi(x,T)=\frac{\partial\phi}{\partial t}(x,T)=0$ in Ω, $\phi=0$ on Σ, multiply (3.12) by ϕ we get

$$\int_0^T\int_\Omega\left(\frac{\partial^2 y_n}{\partial t^2}-\sigma^2\Delta y_n\right)\phi dxdt=\int_0^T\int_\Omega v_n\phi dxdt,$$

integrate by part and use the second Green formula

$$\int_0^T\int_\Omega y_n\left(\frac{\partial^2\phi}{\partial t^2}-\sigma^2\Delta\phi\right)dxdt+\sigma^2\int_0^T\int_\Gamma\left(y_n\frac{\partial\phi}{\partial\nu}-\phi\frac{\partial y_n}{\partial\nu}\right)d\Gamma dt+\int_\Omega\left(\left[\frac{\partial y_n}{\partial t}\phi-y_n\frac{\partial\phi}{\partial t}\right]_0^T\right)dx=\int_0^T\int_\Omega v_n\phi dxdt,$$

this last one beomes

$$\int_0^T\int_\Omega\left(\frac{\partial^2\phi}{\partial t^2}-\sigma^2\Delta\phi\right)y_n dxdt=\int_0^T\int_\Omega v_n\phi dxdt,$$

passing to limit $\gamma\to 0$

$$\int_0^T\int_\Omega\left(\frac{\partial^2\phi}{\partial t^2}-\sigma^2\Delta\phi\right)y_\gamma dxdt=\int_0^T\int_\Omega u_\gamma\phi dxdt,$$

integrate by part again and use the second Green formula, we get

$$\int_0^T\int_\Omega\left(\frac{\partial^2 y_\gamma}{\partial t^2}-\sigma^2\Delta y_\gamma\right)\phi dxdt+\sigma^2\int_0^T\int_\Gamma\left(\phi\frac{\partial y_\gamma}{\partial\nu}-y_\gamma\frac{\partial\phi}{\partial\nu}\right)d\Gamma dt=\int_0^T\int_\Omega u_\gamma\phi dxdt,$$

implies

$$\int_0^T\int_\Gamma y_\gamma\frac{\partial\phi}{\partial\nu}d\Gamma dt=0,$$

which means that $y_\gamma=0$ a.e in Σ.

Then, we know that

$$\frac{\partial^2\xi_n}{\partial t^2}-\Delta\xi_n=\frac{1}{t}\int_{\sigma_1}^{\sigma_2}y(v_n,0;\sigma)\,d\sigma \text{ in } Q,$$

multiply this last one by $\frac{\partial\xi_n}{\partial t}$ and apply Green formula

$$\begin{aligned}\int_0^T\int_\Omega t\left(\frac{\partial^2\xi_n}{\partial t^2}\frac{\partial\xi_n}{\partial t}-\Delta\xi_n\frac{\partial\xi_n}{\partial t}\right)dxdt &= \int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y(v_n,0;\sigma)\,d\sigma\right)\frac{\partial\xi_n}{\partial t}dxdt,\\ \frac{1}{2}\int_0^T\int_\Omega t\frac{d}{dt}\left[\left|\frac{\partial\xi_n}{\partial t}\right|^2+|\nabla\xi_n|^2\right]dxdt &= \int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y(v_n,0;\sigma)\,d\sigma\right)\frac{\partial\xi_n}{\partial t}dxdt,\end{aligned}$$

integrate by parts with respect to time variable

$$\frac{1}{2}\int_0^T\int_\Omega\left[\left|\frac{\partial\xi_n}{\partial t}\right|^2+\left|\nabla\xi_n\right|^2\right]dxdt=-\int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y\left(v_n,0;\sigma\right)d\sigma\right)\frac{\partial\xi_n}{\partial t}dxdt,$$

we take the absolute value and use Cauchy–inequality (see appendix Proposition 4.2) in the second part of inequality to obtain

$$\int_0^T\int_\Omega\left[\left|\frac{\partial\xi_n}{\partial t}\right|^2+\left|\nabla\xi_n\right|^2\right]dxdt\le\int_0^T\int_\Omega\left[\left|\frac{\partial\xi_n}{\partial t}\right|^2+\left|\int_{\sigma_1}^{\sigma_2}y\left(v_n,0;\sigma\right)d\sigma\right|^2\right]dxdt,$$

which gives

$$\int_0^T\int_\Omega\left|\nabla\xi_n\right|^2dxdt\le\int_0^T\int_\Omega\left|\int_{\sigma_1}^{\sigma_2}y\left(v_n,0;\sigma\right)d\sigma\right|^2dxdt\le C_\gamma\Longrightarrow\left\|\xi_n\right\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)}\le C_\gamma.$$

i.e,

$$\xi_n\rightharpoonup\xi_\gamma \text{ weakly in } L^2\left(0,T;H_0^1\left(\Omega\right)\right),$$

from (3.10.3) we get

$$t\left(\frac{\partial^2\xi_n}{\partial t^2}-\Delta\xi_n\right)\rightharpoonup f_3 \text{ weakly in } L^2\left(Q\right),$$

Also, we have

$$t\left(\frac{\partial^2\xi_n}{\partial t^2}-\Delta\xi_n\right)\rightharpoonup t\left(\frac{\partial^2\xi_\gamma}{\partial t^2}-\Delta\xi_\gamma\right) \text{ in } D'\left(Q\right),$$

i.e,

$$t\left(\frac{\partial^2\xi_n}{\partial t^2}-\Delta\xi_n\right)\rightharpoonup t\left(\frac{\partial^2\xi_\gamma}{\partial t^2}-\Delta\xi_\gamma\right) \text{ weakly in } L^2\left(Q\right),$$

and for the reset of condition we use the same method used for the system of y_γ.
Since, the cost function $J^\gamma\left(v\right)$ is lower semi-continous

$$J^\gamma\left(u_\gamma\right)\le\lim_{n\to+\infty}\inf_{v\in\mathcal{U}_{ad}}J^\gamma\left(v_n\right)=\inf_{v\in\mathcal{U}_{ad}}J^\gamma\left(v\right)=d_\gamma,$$

then u_γ is a minimizer of $J^\gamma\left(v\right)$.

2-Uniqueness

It follows from the strict convexity of J^γ. ■

3.2.1 Characterization of the averaged low-regret control

A first order optimality condition for $J^{\gamma}(v)$ gives:

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma)\geq 0 \ \ \forall v\in\mathcal{U}_{ad},$$

by a simple calculus, we get

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma)=\left(\int_{\sigma_1}^{\sigma_2} y(u_\gamma,0;\sigma)\,d\sigma - z_d, \int_{\sigma_1}^{\sigma_2} y(v-u_\gamma,0;\sigma)\,d\sigma\right)_{L^2(Q)} + N(u_\gamma, v-u_\gamma)_{L^2(Q)}$$

$$+\frac{(\sigma_2-\sigma_1)}{\gamma}\left(t\frac{\partial\xi}{\partial\nu}(u_\gamma), t\frac{\partial\xi}{\partial\nu}(v-u_\gamma)\right)_{L^2(\Sigma)}\geq 0\ \forall v\in\mathcal{U}_{ad}.$$

Let's introduce a new state $\rho_\gamma=\rho(u_\gamma)$ given by

$$\begin{cases} \frac{\partial^2\rho_\gamma}{\partial t^2}-\Delta\rho_\gamma=0 & \text{in } Q,\\ \rho_\gamma=-\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu} & \text{on } \Sigma,\\ \rho_\gamma(x,0)=0,\ \frac{\partial\rho_\gamma}{\partial t}(x,0)=0 & \text{in } \Omega,\end{cases}$$

multiply the first equation of the last one by $\xi(v-u_\gamma)$ and apply the second Green formula, we obtain

$$\begin{aligned}\left(\frac{\partial^2\rho_\gamma}{\partial t^2}-\triangle\rho_\gamma, \xi(v-u_\gamma)\right)_{L^2(Q)} &= \int_0^T\int_\Omega\left(\frac{\partial^2\xi}{\partial t^2}(v-u_\gamma)-\triangle\xi(v-u_\gamma)\right)\rho_\gamma dxdt\\ &+\int_\Omega\left[\frac{\partial\rho_\gamma}{\partial t}\xi(v-u_\gamma)-\rho_\gamma\frac{\partial\xi}{\partial t}(v-u_\gamma)\right]_0^T dx+\int_0^t\int_\Gamma(\frac{\partial\xi}{\partial\nu}(v-u_\gamma)\rho_\gamma\\ &-\frac{\partial\rho_\gamma}{\partial\nu}\xi(v-u_\gamma))d\Gamma dt\\ &= \int_0^T\int_\Omega\frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(v-u_\gamma,0,\sigma)\,d\sigma\rho_\gamma dxdt\\ &\int_0^T\int_\Gamma\frac{\partial\xi_\gamma}{\partial\nu}(v-u_\gamma)\rho_\gamma dxdt\end{aligned}$$

Hence,

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma)=\left(\int_{\sigma_1}^{\sigma_2} y(u_\gamma,0,\sigma)\,d\sigma - z_d+\frac{(\sigma_2-\sigma_1)}{t}\rho_\gamma, \int_{\sigma_1}^{\sigma_2} y(v-u_\gamma,0,\sigma)\,d\sigma\right)_{L^2(Q)} + N(u_\gamma, v-u_\gamma)_{L^2(Q)}$$

Also, we introduce the adjoint state $p_\gamma=p(u_\gamma)$

$$\begin{cases} \frac{\partial^2 p_\gamma}{\partial t^2} - \sigma^2 \Delta p_\gamma = \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma - z_d + \frac{(\sigma_2-\sigma_1)}{t}\rho_\gamma & \text{in } Q, \\ p_\gamma = 0 & \text{on } \Sigma, \\ \frac{\partial p_\gamma}{\partial t}(x,T) = 0,\ p_\gamma(x,T) = 0 & \text{in } \Omega, \end{cases}$$

again we have

$$\begin{aligned} \left(\frac{\partial^2 p_\gamma}{\partial t^2} - \sigma^2 \Delta p_\gamma, \int_{\sigma_1}^{\sigma_2} y(v-u_\gamma, 0, \sigma)\, d\sigma\right)_{L^2(Q)} &= \int_0^T \int_\Omega \int_{\sigma_1}^{\sigma_2} \left(\frac{\partial^2 p_\gamma}{\partial t^2} - \sigma^2 \Delta p_\gamma\right) y(v-u_\gamma, 0, \sigma)\, d\sigma dx dt \\ &= \int_0^T \int_\Omega \int_{\sigma_1}^{\sigma_2} p_\gamma (v-u_\gamma)\, d\sigma dx dt, \end{aligned}$$

Then

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = \left(\int_{\sigma_1}^{\sigma_2} p_\gamma d\sigma + N u_\gamma, v - u_\gamma\right)_{L^2(Q)} \geq 0\ \forall v \in \mathcal{U}_{ad}.$$

Finally, the optimality system which characterized the low-regret control is :

$$\begin{cases} \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma = u_\gamma & \text{in } Q, \\ y_\gamma = 0 & \text{on } \Sigma, \\ y_\gamma(x,0) = 0,\ \frac{\partial y_\gamma}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial^2 \xi_\gamma}{\partial t^2} - \Delta \xi_\gamma = \frac{1}{t}\int\limits_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma & \text{in } Q, \\ & \text{on } \Sigma, \\ \xi_\gamma = 0 & \text{in } \Omega, \\ \xi_\gamma(x,T) = 0,\ \frac{\partial \xi_\gamma}{\partial t}(x,T) = 0 & \end{cases}$$

$$\begin{cases} \frac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta \rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma = -\frac{t^2}{\gamma}\frac{\partial \xi_\gamma}{\partial \nu} & \text{on } \Sigma, \\ \frac{\partial \rho_\gamma}{\partial t}(x,0) = 0,\ \rho_\gamma(x,0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial^2 p_\gamma}{\partial t^2} - \sigma^2 \Delta p_\gamma = \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma - z_d + \frac{(\sigma_2-\sigma_1)}{t}\rho_\gamma & \text{in } Q, \\ p_\gamma = 0 & \text{on } \Sigma, \\ \frac{\partial p_\gamma}{\partial t}(x,T) = 0,\ p_\gamma(x,T) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\left(\int_{\sigma_1}^{\sigma_2} p_\gamma d\sigma + N u_\gamma, v - u_\gamma\right)_{L^2(Q)} \geq 0\ \ \forall v \in \mathcal{U}_{ad}.$$

3.3 Characterization of the averaged no-regret control:

Below, we show the convergence of the averaged low-regret control to the averaged no-regret control. Then, we give the optimality system of the last one.

Proposition 3.1 *:The low-regret control sequences u_γ converges to the no-regret control u.*
Proof. *u_γ is the minimum of J^γ so*

$$J^\gamma(u_\gamma) = J(u_\gamma, 0) - J(0,0) + \frac{\sigma_2 - \sigma_1}{\gamma}\left\|t\frac{\partial \xi_\gamma}{\partial \nu}\right\|^2_{L^2(\Sigma)} \leq J^\gamma(0) = 0,$$

which implies

$$\left\|\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma - z_d\right\|^2_{L^2(Q)} + N\|u_\gamma\|^2_{L^2(Q)} + \frac{\sigma_2 - \sigma_1}{\gamma}\left\|t\frac{\partial \xi_\gamma}{\partial \nu}\right\|^2_{L^2(\Sigma)} \leq J(0,0) = \|z_d\|^2_{L^2(Q)},$$

this gives

$$\|u_\gamma\|_{L^2(Q)} \leq C, \tag{3.14.1}$$

$$\left\|\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma\right\|_{L^2(Q)} \leq C, \tag{3.14.2}$$

$$\left\|t\frac{\partial \xi_\gamma}{\partial \nu}\right\|_{L^2(\Sigma)} \leq C(\sigma_2 - \sigma_1)\sqrt{\gamma}, \tag{3.14.3}$$

where C is a constant independent of γ.
We have that

$$\begin{cases} \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma = u_\gamma & \text{in } Q, \\ y_\gamma = 0 & \text{on } \Sigma, \\ y_\gamma(x,0) = 0,\ \frac{\partial y_\gamma}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

multiply the first equation by $\frac{\partial y_\gamma}{\partial t}$ and integrate over $(0,t)$ to get

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{dt}\left[\left|\frac{\partial y_\gamma}{\partial t}\right|^2 + \sigma^2|\nabla y_\gamma|^2\right] dxds = \int_0^t\int_\Omega u_\gamma \frac{\partial y_\gamma}{\partial t} dxds,$$

use Cauchy inequality

$$\left\|\frac{\partial y_\gamma(t)}{\partial t}\right\|^2_{L^2(\Omega)} + \sigma^2\|\nabla y_\gamma(t)\|^2_{L^2(\Omega)} \leq \int_0^t\left(\|u_\gamma(s)\|^2_{L^2(\Omega)} + \left\|\frac{\partial y_\gamma}{\partial t}(s)\right\|^2_{L^2(\Omega)}\right) ds,$$

then, use Gronwall lemma (see appendix Lemma 4.2) we get

$$\left\|\frac{\partial y_\gamma(t)}{\partial t}\right\|^2_{L^2(\Omega)} + \sigma^2\|\nabla y_\gamma(t)\|^2_{L^2(\Omega)} \leq \|u_\gamma(s)\|^2_{L^2(\Omega)}\exp(t) \leq C,$$

implies

$$\left\| \frac{\partial y_\gamma(t)}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \;\; \|y_\gamma(t)\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C(\sigma), \tag{3.15}$$

Also, we have

$$\left\| \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma \right\|_{L^2(Q)} \leq C, \tag{3.16}$$

from $(3.14.1)$, $(3.14.2)$, (3.15) *and* (3.16)*by the same way in* ***Theorem 3.1,*** *we deduce that*

$$\begin{aligned} u_\gamma &\rightharpoonup u \text{ weakly in } L^2(Q), \\ y_\gamma &\rightharpoonup y \text{ weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ \frac{\partial^2 y_\gamma}{\partial t^2} - \sigma^2 \Delta y_\gamma &\rightharpoonup \frac{\partial^2 y}{\partial t^2} - \sigma^2 \Delta y \text{ weakly in } L^2(Q), \\ \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma &\rightharpoonup \int_{\sigma_1}^{\sigma_2} y(u, 0, \sigma)\, d\sigma \text{ weakly in } L^2(Q), \end{aligned} \tag{3.17}$$

and

$$y(x,0) = 0, \;\; \frac{\partial y}{\partial t}(x,0) = 0 \text{ in } \Omega,$$

and from (3.17)*, we get*

$$\begin{aligned} \left\| \frac{\partial \xi_\gamma}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} &\leq C, \;\; \|\xi_\gamma\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \\ \left\| \frac{\partial^2 \xi_\gamma}{\partial t^2} - \Delta \xi_\gamma \right\|_{L^2(Q)} &\leq C, \end{aligned}$$

then $\xi_\gamma \rightharpoonup \xi$ *weakly in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\frac{\partial^2 \xi_\gamma}{\partial t^2} - \Delta \xi_\gamma \rightharpoonup \frac{\partial^2 \xi}{\partial t^2} - \Delta \xi$ *weakly in* $L^2(Q)$.
Hence, $\xi = \xi(u) \in L^\infty(0,T;H_0^1(\Omega))$ *is the solution of*

$$\begin{cases} \frac{\partial^2 \xi}{\partial t^2} - \Delta \xi = \frac{1}{t} \int\limits_{\sigma_1}^{\sigma_2} y(u, 0; \sigma)\, d\sigma & \text{in } Q, \\ \xi = 0 & \text{on } \Sigma, \\ \xi(x,T) = 0, \;\; \frac{\partial \xi}{\partial t}(x,T) = 0 & \text{in } \Omega, \end{cases}$$

from $(3.14.3)$ *when* $\gamma \to 0$ *we get* $t\frac{\partial \xi_\gamma}{\partial \nu} \to \frac{\partial \xi}{\partial \nu} = 0$ *strongly in* $L^2(\Sigma)$. *Then* $\int\limits_0^T \int\limits_\Gamma t\frac{\partial \xi}{\partial \nu} g\, d\Gamma dt = 0$ *for every* $g \in L^2(\Sigma)$ *it means that* u *is a no-regret control.* ■

Theorem 3.2 *The averaged no-regret control corresponding to the state* $y(u,0)$ *characterized by the following optimality system*

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \sigma^2 \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0, \;\; \frac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial^2 \xi}{\partial t^2} - \Delta \xi = \frac{1}{t} \int_{\sigma_1}^{\sigma_2} y(u,0,\sigma) d\sigma & \textit{in } Q, \\ \xi = 0 & \textit{on } \Sigma, \\ \xi(x,T) = 0, \frac{\partial \xi}{\partial t}(x,T) = 0 & \textit{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial^2 \rho}{\partial t^2} - \Delta \rho = 0 & \textit{in } Q, \\ \rho = \lambda & \textit{on } \Sigma, \\ \rho(x,0) = 0, \ \frac{\partial \rho}{\partial t}(x,0) = 0 & \textit{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} - \Delta p = \int_{\sigma_1}^{\sigma_2} y(u,0,\sigma) d\sigma - z_d + \frac{(\sigma_2 - \sigma_1)}{t} \rho & \textit{in } Q, \\ p = 0 & \textit{on } \Sigma, \\ \frac{\partial p}{\partial t}(x,T) = 0, \ p(x,T) = 0 & \textit{in } \Omega, \end{cases}$$

and

$$\left(\int_{\sigma_1}^{\sigma_2} p(\sigma) \, d\sigma + Nu, v - u \right)_{L^2(Q)} \geq 0 \ \ \forall v \in \mathcal{U}_{ad}$$

where

$$\begin{aligned} u &= \lim_{\gamma \to 0} u_\gamma, \ y = \lim_{\gamma \to 0} y_\gamma, \ \xi = \lim_{\gamma \to 0} \xi_\gamma \\ \rho &= \lim_{\gamma \to 0} \rho_\gamma, \ p = \lim_{\gamma \to 0} p_\gamma, \ \lambda = \lim_{\gamma \to 0} \frac{t^2}{\gamma} \frac{\partial \xi}{\partial \nu} (u_\gamma) . \end{aligned}$$

Proof. We show in the last propositions convergence of y and ξ.ffor ρ_γ we have from $(3.14.3)$

$$\frac{1}{\gamma} \left\| t \frac{\partial \xi_\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C \Longrightarrow \frac{t^2}{\gamma} \frac{\partial \xi_\gamma}{\partial \nu} \in L^2(\Sigma) . \tag{3.18}$$

We use **Theorem 4.3** in [1] to get

$$\begin{aligned} \left\| \frac{\partial \rho_\gamma}{\partial t} \right\|_{L^\infty(0,T;H^{-1}(\Omega))} &\leq C, \left\| \rho_\gamma \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \\ \left\| \frac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta \rho_\gamma \right\|_{L^2(Q)} &\leq C, \end{aligned} \tag{3.19}$$

where C is a constant independent of γ.
We have from (3.19)

$$\rho_\gamma \rightharpoonup \rho \text{ weakly in } L^2(Q) ,$$

and $\frac{\partial^2 \rho_\gamma}{\partial t} - \Delta \rho_\gamma \rightharpoonup \frac{\partial^2 \rho}{\partial t} - \Delta \rho$ weakly in $L^2(Q)$ and from (3.18) we get $-\frac{t^2}{\gamma} \frac{\partial \xi_\gamma}{\partial \nu} \rightharpoonup \lambda$ weakly in $L^2(\Sigma)$ then

$$\begin{cases} \frac{\partial^2 \rho}{\partial t^2} - \Delta \rho = 0 & \text{in } Q, \\ \rho = \lambda & \text{on } \Sigma, \\ \rho(x,0) = 0, \frac{\partial \rho}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

It is easy to get

$$\begin{aligned}\left\|\frac{\partial p_\gamma}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} &\leq C,\ \|p_\gamma\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C,\\ \left\|\frac{\partial^2 p_\gamma}{\partial t^2} - \Delta p_\gamma\right\|_{L^2(Q)} &\leq C,\end{aligned}$$

again we get

$$\begin{aligned}p_\gamma &\rightharpoonup p \text{ weakly in } L^2(Q),\\ \frac{\partial^2 p_\gamma}{\partial t^2} - \sigma^2\Delta p_\gamma &\rightharpoonup \frac{\partial^2 p}{\partial t} - \sigma^2\Delta p \text{ weakly in } L^2(Q),\\ \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma - z_d + \frac{(\sigma_2-\sigma_1)}{t}\rho_\gamma &\rightharpoonup \int_{\sigma_1}^{\sigma_2} y(u, 0, \sigma)\, d\sigma - z_d + \frac{(\sigma_2-\sigma_1)}{t}\rho \text{ weakly in } L^2(Q).\end{aligned}$$

Hence,

$$\begin{cases}\frac{\partial^2 p}{\partial t^2} - \sigma^2\Delta p = \int_{\sigma_1}^{\sigma_2} y(u,0,\sigma)\, d\sigma - z_d + \frac{(\sigma_2-\sigma_1)}{t}\rho & \text{in } Q,\\ p = 0 & \text{on } \Sigma,\\ \frac{\partial p}{\partial t}(x,T) = 0,\ p(x,T) = 0 & \text{in } \Omega,\end{cases}$$

therefore, the optimality condition is given by

$$\left(\int_{\sigma_1}^{\sigma_2} p(\sigma)\, d\sigma + Nu, v - u\right)_{L^2(Q)} \geq 0\ \forall v \in \mathcal{U}_{ad}.$$

■

Chapter 4

Optimal control of heat equation with missing boundary condition

In this chapter, we consider the problem of a parabolic equation which describes the diffusion of the heat in a cylindrical domain. A mixed Dirichlet Newman boundary conditions are given on the face of the cylinder represented by a control v and unknown function g respectively, by the virtue of the last one leads us to optimal control for problem governed by a linear parabolic equation with missing data, which requires us to use the method of no-regret control that is well adapted in this case. Assuming that the no-regret control is associated with a sequence of the low-regret control. Besides, it converges weakly to the no-regret control we discussed together the optimality system describing the no-regret control.

4.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. We consider the time space cylinder $Q = \Omega \times (0, T)$, $\Sigma_1 = \Gamma_1 \times]0, T[$ and by $\Sigma_2 = \Gamma_2 \times]0, T[$ such that $]0, T[$ is the time interval we are looking at and $\Gamma_1 \cap \Gamma_2 = \varnothing$ and $\Gamma_1 \cup \Gamma_2 = \Gamma$.
We consider the following controlled heat equation

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}\left(a\left(x\right) \nabla y\right) = 0 & \text{in } Q, \\ \frac{\partial y}{\partial \nu_a} = v \text{ on } \Sigma_1, \ y = g & \text{on } \Sigma_2, \\ y\left(x, 0\right) = 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where

$$\frac{\partial y}{\partial \nu_a} = a\left(x\right) \nabla y.\nu,$$

ν is the outward unit normal vector and $a \in L^{\infty}(\Omega)$ is the coefficient of diffusions such that

$$0 < \alpha_1 \leq a(x) \leq \alpha_2, \text{ a.e in } \Omega.$$

The function g is unknown boundary belongs to $L^2(\Sigma_2)$, the control $v \in L^2(\Sigma_1)$. The problem (4.1) has a unique solution

$$y = y(g) = y(x,t) \in L^2\left(0,\ T;\ H^2(\Omega)\right) \subset L^2(Q).$$

We define the cost function related to the problem (4.1) as follows

$$J(v,g) = \|y - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(\Sigma_1)}, \tag{4.2}$$

where $N > 0$ is a constant and y_d is the fixed observation belongs to $L^2(Q)$. Our optimal control problem is written as follows

$$\inf_{v \in L^2(\Sigma_1)} J(v,g), \text{ for every } g \in L^2(\Sigma_2). \tag{4.3}$$

Solving the problem (4.3) leads to a control depending on g, this requires us to use the no-regret control method.

4.2 No-regret control :

Hence, we define the notion of no-regret control for the problem (4.1) as following:

Definition 4.1 *We say that $u \in L^2(\Sigma_1)$ is a no-regret control for (4.1) and (4.2) iff u is the solution of:*

$$\inf_{v \in L^2(\Sigma_1)} \sup_{g \in L^2(\Sigma_2)} \left(J(v,g) - J(0,g)\right).$$

Thanks to linearity in (4.1), it's easy to see that

$$y(v,g) = y(v,0) + y(0,g). \tag{4.4}$$

Remark 4.1 *It's clear that $y(0,0)$ is the trivial solution of $(4,1)$.*

Therefore, we shall isolate g to a form where the classical theory of optimal control can be applied, the method is shown in the following Lemma

Lemma 4.1 *For any $v \in L^2(\Sigma_1)$ and for any $g \in L^2(\Sigma_2)$, we have:*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) - 2(\frac{\partial \xi}{\partial \nu_a}, g)_{L^2(\Sigma_2)}, \tag{4.5}$$

where $\xi = \xi(x,t;v) \in L^2(Q)$ is solution of backword heat equation given by:

$$\begin{cases} -\frac{\partial \xi}{\partial t} - \operatorname{div}(a(x)\nabla\xi) = y(v,0) & \text{in } Q, \\ \frac{\partial \xi}{\partial \nu_a} = 0 \quad \text{on } \Sigma_1, \quad \xi = 0 & \text{on } \Sigma_2, \\ \xi(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.6}$$

Proof. We have

$$J(v,g) = \|y - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(\Sigma_1)},$$

by $(4,4)$ we get

$$J(v,g) = J(v,0) + J(0,g) - J(0,0) + 2(y(v,0), y(0,g))_{L^2(Q)}.$$

Consequently:

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(y(v,0), y(0,g))_{L^2(Q)}.$$

Now, we introduce an adjoint state $\xi = \xi(x,t;v)$ define by $(4,6)$, by the use of Green formula

$$\begin{aligned} (y(v,0), y(0,g))_{L^2(Q)} &= \left(-\frac{\partial \xi}{\partial t} - \operatorname{div}(a(x)\nabla\xi), y(0,g)\right)_{L^2(Q)} \\ &= -\int_0^T \int_{\Gamma_2} \frac{\partial \xi}{\partial \nu_a} g \, d\Gamma_2 dt, \end{aligned}$$

Then

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) - 2(\frac{\partial \xi}{\partial \nu_a}, g)_{L^2(\Sigma_2)}.$$

■

Now, we consider the no-regret control problem

$$\inf_{v \in L^2(\Sigma_1)} \sup_{g \in L^2(\Sigma_2)} (J(v,g) - J(0,g)),$$

from $(4,5)$ the problem is equivalent to the following one

$$\inf_{v \in L^2(\Sigma_1)} \left(J(v,0) - J(0,0) - \sup_{g \in L^2(\Sigma_2)} 2(\frac{\partial \xi}{\partial \nu_a}, g)_{L^2(\Sigma_2)} \right).$$

This leads us to define the low-regret control.

4.2.1 Low-regret control :

Definition 4.2 *Let $\gamma > 0$, we say that $u_\gamma \in L^2(\Sigma_1)$ is the low-regret control of* (4.1) *and* (4.2) *if it's a solution of*

$$\inf_{v \in L^2(\Sigma_1)} \sup_{g \in L^2(\Sigma_2)} \left(J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Sigma_2)} \right).$$

Hence, our optimal control problem will be as follows

$$\inf_{v \in L^2(\Sigma_1)} J^\gamma(v), \tag{4.7}$$

where

$$J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} \left\| \frac{\partial \xi}{\partial \nu_a} \right\|^2_{L^2(\Sigma_2)}.$$

Theorem 4.1 *There exist a unique low-regret control u_γ solution to* (4.1) *and* (4.7).

Proof. **1-Existence**

It's clear that $J^\gamma(v) \geq -J(0,0)$ implies $\inf\limits_{v \in L^2(\Sigma_1)} J^\gamma(v) = d_\gamma$ exists.

Let $(v_n) \subset L^2(\Sigma_1)$ be a minimizing sequence, since J^γ is coercive (v_n) is bounded i,e.

$$J^\gamma(v_n) \underset{n \to +\infty}{\rightarrow} \inf_{v \in L^2(\Sigma_1)} J^\gamma(v) = d_\gamma.$$

We have that

$$-J(0,0) \leq J^\gamma(v_n) = J(v_n,0) - J(0,0) + \frac{1}{\gamma} \left\| \frac{\partial \xi_n}{\partial \nu_a} \right\|^2_{L^2(\Sigma_2)} \leq d_\gamma + 1 = C_\gamma,$$

which gives the following bounds

$$\|v_n\|_{L^2(\Sigma_1)} \leq C_\gamma, \tag{4.8.1}$$

$$\|y(v_n,0)\|_{L^2(Q)} \leq C_\gamma, \tag{4.8.2}$$

$$\left\| \frac{\partial \xi_n}{\partial \nu_a} \right\|_{L^2(\Sigma_2)} \leq C_\gamma \sqrt{\gamma}, \tag{4.8.3}$$

where C_γ is a constant independent of n.

Consequently, from $(4.8.1)$ and $(4.8.2)$ we can extracting a subsequences still denoting by (v_n) and $y(v_n,0)$ such that

$$\begin{aligned} v_n &\rightharpoonup u_\gamma \text{ weakly in } L^2(\Sigma_1), \\ y(v_n,0) &\rightharpoonup y_\gamma \text{ weakly in } L^2(Q), \end{aligned}$$

and from the continuity of y with respect to data we deduce that

$$y(v_n,0) \rightharpoonup y(u_\gamma,0) \text{ weakly in } L^2(Q), \tag{4.9}$$

from the uniqueness of limit, we deduce that $y_\gamma = y(u_\gamma,0)$.
We know that $y_n = y(v_n,0)$ is solution of:

$$\begin{cases} \frac{\partial y_n}{\partial t} - \operatorname{div}(a(x)\nabla y_n) = 0 & \text{in } Q, \\ \frac{\partial y_n}{\partial \nu_a} = v_n \text{ on } \Sigma_1, \; y_n = 0 & \text{on } \Sigma_2, \\ y_n(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.10}$$

integrate over Q then we deduce that

$$\left\| \frac{\partial y_n}{\partial t} - \operatorname{div}(a(x)\nabla y_n) \right\|_{L^2(Q)} \leq C_\gamma,$$

it means that

$$0 = \frac{\partial y_n}{\partial t} - \operatorname{div}(a(x)\nabla y_n) \rightharpoonup f_1 = 0 \text{ weakly in } L^2(Q),$$

we have that

$$\frac{\partial y_n}{\partial t} - \operatorname{div}(a(x)\nabla y_n) \rightharpoonup \frac{\partial y_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla y_\gamma) \text{ weakly in } D'(Q),$$

from the uniqueness of limit we get:

$$\frac{\partial y_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla y_\gamma) = 0 \text{ in } Q.$$

from (4.9) we get

$$y_\gamma(x,0) = 0 \text{ in } \Omega.$$

The limit of boundary condition will be proved into two steps
First step. We know that

$$\begin{aligned} \|y_n\|_{L^2(Q)} &\leq C_\gamma, \\ y_n &\rightharpoonup y_\gamma \text{ weakly in } L^2(Q), \end{aligned}$$

from the continuity of trace operator, we have that

$$\left\| \frac{\partial y_n}{\partial \nu_a} \right\|_{L^2(\Sigma_1)} \leq c \|y_n\|_{L^2(Q)} \leq C_\gamma,$$

where c is arbitrary constant.
We deduce that

$$\frac{\partial y_n}{\partial \nu_a} \rightharpoonup g \text{ weakly in } L^2(\Sigma_1),$$

On the other hand

$$\frac{\partial y_n}{\partial \nu_a} = v_n \text{ on } \Sigma_1,$$

and

$$v_n \rightharpoonup u_\gamma \text{ weakly in } L^2(\Sigma_1),$$

according to the uniqueness of limit, we get

$$\frac{\partial y_\gamma}{\partial \nu_a} = u_\gamma \text{ on } \Sigma_1.$$

Second step. Now, we multiply the first equation in (4.10) by $\phi \in D(Q)$, where $\phi(T) = 0$ in Ω, $\frac{\partial \phi}{\partial \nu_a} = 0$ on Σ_1, $\phi = 0$ on Σ_2, and integrate over Q we get

$$\int_0^T\int_\Omega \left(\frac{\partial y_n}{\partial t} - \operatorname{div}(a(x)\nabla y_n)\right)\phi dxdt = \int_0^T\int_\Omega y_n\left(-\frac{\partial \phi}{\partial t} - \operatorname{div}(a(x)\nabla \phi)\right)dxdt - \int_{\Sigma_1} v_n\phi d\Sigma_1,$$

passing to limit we get:

$$\int_0^T\int_\Omega y_\gamma\left(-\frac{\partial \phi}{\partial t} - \operatorname{div}(a(x)\nabla \phi)\right)dxdt = \int_{\Sigma_1} u_\gamma\phi d\Sigma_1,$$

integrate another time

$$\int_{\Sigma_1} \frac{\partial y_\gamma}{\partial \nu_a}\phi d\Sigma_1 - \int_{\Sigma_2} y_\gamma\frac{\partial \phi}{\partial \nu_a}d\Sigma_2 = \int_{\Sigma_1} u_\gamma\phi d\Sigma_1,$$

Consequently

$$\int_{\Sigma_2} y_\gamma\frac{\partial \phi}{\partial \nu_a}d\Sigma_2 = 0,$$

finally

$$y_\gamma = 0 \text{ on } \Sigma_2.$$

In the other side, we have that $\xi_n = \xi(v_n)$ is a solution of the following adjoint problem

$$\begin{cases} -\frac{\partial \xi_n}{\partial t} - \operatorname{div}(a(x)\nabla \xi_n) = y_n & \text{in } Q, \\ \frac{\partial \xi_n}{\partial \nu_a} = 0 \text{ on } \Sigma_1,\ \xi_n = 0 & \text{on } \Sigma_2, \\ \xi_n(x,T) = 0 & \text{in } \Omega, \end{cases}$$

multiply the first equation by ξ_n and we integrate over Q

$$-\frac{1}{2}\int_0^T\int_\Omega \frac{d}{dt}|\xi_n|^2 dxdt + \alpha_1\int_0^T\int_\Omega |\nabla \xi_n|^2 dxdt \le \int_0^T\int_\Omega y_n\xi_n dxdt,$$

integrate by parts with respect to time variable

$$-\frac{1}{2}\int_{\Omega}(|\xi_n(T)|^2-|\xi_n(0)|^2)dx+\alpha_1\int_0^T\int_{\Omega}|\nabla\xi_n|^2\,dxdt\leq\int_0^T\int_{\Omega}y_n\xi_n dxdt,$$

we use Cauchy Schwartz inequality (see appendix Proposition 4.2) we get

$$\int_{\Omega}|\xi_n(0)|^2\,dx+2\alpha_1\int_0^T\int_{\Omega}|\nabla\xi_n|^2\,dxdt\leq\int_0^T\int_{\Omega}|y_n|^2\,dxdt+\int_0^T\int_{\Omega}|\xi_n|^2\,dxdt,$$

by using Poincare inequality (see appendix Proposition 4.3) we get

$$\int_{\Omega}|\xi_n(0)|^2\,dx+2\int_0^T\int_{\Omega}|\xi_n|^2\,dxdt\leq\int_0^T\int_{\Omega}|y_n|^2\,dxdt+\int_0^T\int_{\Omega}|\xi_n|^2\,dxdt,$$

implies that

$$\int_{\Omega}|\xi_n(0)|^2\,dx+\int_0^T\int_{\Omega}|\xi_n|^2\,dxdt\leq\int_0^T\int_{\Omega}|y_n|^2\,dxdt\leq C_\gamma,$$

we deduce

$$\|\xi_n\|_{L^2(Q)}\leq C_\gamma.$$

Hence, there exists a subsequence still be denoted (ξ_n) such that

$$\xi_n\rightharpoonup\xi_\gamma\text{ weakly in }L^2(Q)\,,\tag{4.11}$$

due to $(4.8.2)$, we have

$$\left\|-\frac{\partial\xi_n}{\partial t}-\operatorname{div}(a(x)\nabla\xi_n)\right\|_{L^2(Q)}\leq C_\gamma,$$

requires that there exists some subsequence converges weakly

$$-\frac{\partial\xi_n}{\partial t}-\operatorname{div}(a(x)\nabla\xi_n)\rightharpoonup g\text{ weakly in }L^2(Q)\,,$$

also, we have

$$-\frac{\partial\xi_n}{\partial t}-\operatorname{div}(a(x)\nabla\xi_n)\rightharpoonup-\frac{\partial\xi_\gamma}{\partial t}-\operatorname{div}\left(a(x)\nabla\xi_\gamma\right)\text{ weakly in }D'(Q)\,,$$

Then from the uniqueness of limit we get

$$g=-\frac{\partial\xi_\gamma}{\partial t}-\operatorname{div}\left(a(x)\nabla\xi_\gamma\right).$$

Now, we will show the convergences of the boundary and initial conditions, from (4.11) we deduce

$$\xi_{\gamma}(x,T)=0 \text{ in } \Omega.$$

For the limit of boundary condition will be proved in two steps

First step. We have

$$\begin{aligned} \|\xi_n\|_{L^2(Q)} &\leq C_{\gamma}, \\ \xi_n &\rightharpoonup \xi \text{ weakly in } L^2(Q), \end{aligned}$$

Also, we have that from the continuity of trace operator

$$\left\|\frac{\partial \xi_n}{\partial \nu_a}\right\|_{L^2(\Sigma_1)} \leq C_{\gamma},$$

we deduce that

$$\frac{\partial \xi_n}{\partial \nu_a} \rightharpoonup h \text{ weakly in } L^2(\Sigma_1),$$

from the continuity of the trace operator with respect to the data

$$\frac{\partial \xi_n}{\partial \nu_a} \rightharpoonup \frac{\partial \xi_{\gamma}}{\partial \nu_a} \text{ weakly in } L^2(\Sigma_1),$$

in other side, we have

$$\frac{\partial \xi_n}{\partial \nu_a}=0 \text{ on } \Sigma_1,$$

from the uniquness of limit

$$\frac{\partial \xi_{\gamma}}{\partial \nu_a}=0 \text{ on } \Sigma_1.$$

Second step. We follow the same method shown in the previous problem which describes the state y such that $\phi(0)=0$ in Ω, we obtain

$$\int_0^T\int_{\Omega} \xi_{\gamma}\left(\frac{\partial \phi}{\partial t}-\operatorname{div}(a(x)\nabla\phi)\right)dxdt=\int_0^T\int_{\Omega} y_{\gamma}\phi dxdt,$$

integrate another time, we get

$$\int_0^T\int_{\Omega} y_{\gamma}\phi dxdt+\int_{\Sigma_1}\frac{\partial \xi_{\gamma}}{\partial \nu_a}\phi d\Sigma_1-\int_{\Sigma_2}\xi_{\gamma}\frac{\partial \phi}{\partial \nu_a}d\Sigma_2=\int_0^T\int_{\Omega} y_{\gamma}\phi dxdt,$$

implies that

$$\int_{\Sigma_2}\xi_{\gamma}\frac{\partial \phi}{\partial \nu_a}d\Sigma_2=0,$$

Consequently

$$\xi_\gamma = 0 \text{ on } \Sigma_2.$$

Since The cost function $J^\gamma(v)$ is lower semi-continuous

$$J^\gamma(u_\gamma) \leq \lim_{n\to+\infty} \inf_{v\in L^2(\Sigma_1)} J^\gamma(v_n) = \inf_{v\in L^2(\Sigma_1)} J^\gamma(v) = d_\gamma.$$

Finally, u_γ is a minimizer of $J^\gamma(v)$.

2-**Uniqueness**

It follows from the strict convexity of J^γ. ■

4.2.2 Optimality system of the low-regret control

A first order optimality condition for $J^\gamma(v)$ gives :

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = 0 \; \forall v \in L^2(\Sigma_1),$$

by a simple calculus, we get

$$\begin{aligned} J^{\gamma\prime}(u_\gamma)(v-u_\gamma) &= 2\left(y(u_\gamma,0)-y_d, y(v-u_\gamma,0)\right)_{L^2(Q)} + 2N\left(u_\gamma, v-u_\gamma\right)_{L^2(\Sigma_1)} \\ &\quad + \frac{2}{\gamma}\left(\frac{\partial\xi(u_\gamma)}{\partial\nu_a}, \frac{\partial\xi(v-u_\gamma)}{\partial\nu_a}\right)_{L^2(\Sigma_2)} = 0. \end{aligned}$$

Let's introduce the new state $\rho_\gamma = \rho(u_\gamma)$ given by

$$\begin{cases} \frac{\partial\rho_\gamma}{\partial t} - \operatorname{div}\left(a(x)\nabla\rho_\gamma\right) = 0 & \text{in } Q, \\ \frac{\partial\rho_\gamma}{\partial\nu_a} = 0 \text{ on } \Sigma_1, \;\; \rho_\gamma = -\frac{1}{\gamma}\frac{\partial\xi}{\partial\nu_a}(u_\gamma) & \text{on } \Sigma_2, \\ \rho_\gamma(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.12}$$

multiply the first equation of the last system by $\xi(v-u_\gamma)$ and apply the second Green formula, we obtain

$$\begin{aligned} \left(\frac{\partial\rho_\gamma}{\partial t} - \operatorname{div}\left(a(x)\nabla\rho_\gamma\right), \xi(v-u_\gamma)\right)_{L^2(Q)} &= \int_0^T\int_\Omega\left(\frac{\partial\rho_\gamma}{\partial t} - \operatorname{div}\left(a(x)\nabla\rho_\gamma\right)\right)\xi(v-u_\gamma)\,dxdt \\ &= \int_0^T\int_\Omega y(v-u_\gamma,0)\rho_\gamma\,dxdt, \end{aligned}$$

Hence,

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = 2\left(y(u_\gamma,0)-y_d+\rho_\gamma, y(v-u_\gamma,0)\right)_{L^2(Q)} + 2N\left(u_\gamma, v-u_\gamma\right)_{L^2(\Sigma_1)} = 0 \; \forall v \in L^2(\Sigma_1).$$

For the second time, we introduce an adjoint state $p_\gamma = p(u_\gamma)$

$$\begin{cases} -\frac{\partial p_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla p_\gamma) = y(u_\gamma, 0) - y_d + \rho_\gamma & \text{in } Q, \\ \frac{\partial p_\gamma}{\partial \nu_a} = 0 \text{ in } \Sigma_1, \; p_\gamma = 0 & \text{on } \Sigma_2, \\ p_\gamma(x, T) = 0 & \text{in } \Omega, \end{cases} \tag{4.13}$$

again we have

$$\begin{aligned} \left(-\frac{\partial p_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla p_\gamma), y(v - u_\gamma, 0)\right)_{L^2(Q)} &= \int_0^T \int_\Omega (-\frac{\partial p_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla p_\gamma))y(v - u_\gamma, 0)dxdt \\ &= \int_0^T \int_{\Gamma_1} p_\gamma(v - u_\gamma)dxdt, \end{aligned}$$

then

$$J^{\gamma\prime}(u_\gamma)(v - u_\gamma) = (p_\gamma + Nu_\gamma, v - u_\gamma)_{L^2(\Sigma_1)} = 0 \;\forall v \in L^2(\Sigma_1),$$

implies that

$$p_\gamma + Nu_\gamma = 0 \text{ a.e in } \Sigma_1.$$

Finally, the optimality system of the low-regret control is characterized by

$$\begin{cases} \frac{\partial y_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla y_\gamma) = 0 & \text{in } Q, \\ \frac{\partial y_\gamma}{\partial \nu_a} = u_\gamma \text{ on } \Sigma_1, \; y_\gamma = 0 & \text{on } \Sigma_2, \\ y_\gamma(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\frac{\partial \xi_\gamma}{\partial t} - \operatorname{div}\left(a(x)\nabla \xi_\gamma\right) = y(u_\gamma, 0) & \text{in } Q, \\ \frac{\partial \xi_\gamma}{\partial \nu_a} = 0 \text{ on } \Sigma_1, \; \xi_\gamma = 0 & \text{on } \Sigma_2, \\ \xi_\gamma(x, T) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \frac{\partial \rho_\gamma}{\partial t} - \operatorname{div}\left(a(x)\nabla \rho_\gamma\right) = 0 & \text{in } Q, \\ \frac{\partial \rho_\gamma}{\partial \nu_a} = 0 \text{ on } \Sigma_1, \; \rho_\gamma = -\frac{1}{\gamma}\frac{\partial \xi}{\partial \nu_a}(u_\gamma) & \text{on } \Sigma_2, \\ \rho_\gamma(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\frac{\partial p_\gamma}{\partial t} - \operatorname{div}(a(x)\nabla p_\gamma) = y(u_\gamma, 0) - y_d + \rho_\gamma & \text{in } Q, \\ \frac{\partial p_\gamma}{\partial \nu_a} = 0 \text{ on } \Sigma_1, \; p_\gamma = 0 & \text{on } \Sigma_2, \\ p_\gamma(x, T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$p_\gamma + Nu_\gamma = 0 \text{ a.e in } \Sigma_1.$$

4.3 Optimality system of the no-regret control

Before we start the characterization of the no-regret control we give the following proposition proving the convergence of the sequence of low-regret control to the no-regret control.

Proposition 4.1 *The low-regret control u_γ converge to the no-regret control u when γ tends to 0.*

Proof. u_γ is the minimum of $J^\gamma(v)$ then

$$J^\gamma(u_\gamma) \leq J^\gamma(0) = constant,$$

replacing $J^\gamma(u_\gamma)$ with their definition, we get

$$\|y(u_\gamma,0) - y_d\|^2_{L^2(Q)} + N\|u_\gamma\|^2_{L^2(\Sigma_1)} + \frac{1}{\gamma}\left\|\frac{\partial\xi(u_\gamma)}{\partial\nu_a}\right\|^2_{L^2(\Sigma_2)} \leq J(0,0) = C,$$

where C is a constant independent γ. This leads to the following bounds

$$\|u_\gamma\|_{L^2(\Sigma_1)} \leq C, \tag{4.14.a}$$

$$\|y(u_\gamma,0)\|_{L^2(Q)} \leq C, \tag{4.14.b}$$

$$\left\|\frac{\partial\xi(u_\gamma)}{\partial\nu_a}\right\|_{L^2(\Sigma_2)} \leq C\sqrt{\gamma}, \tag{4.14.c}$$

we deduce from $(4.14.a)$ and $(4.14.b)$ that we can extracting a subsequence still denoting by $(u_\gamma), y(u_\gamma,0)$ such that

$$\begin{aligned} u_\gamma &\rightharpoonup u \text{ weakly in } L^2(\Sigma_1), \\ y(u_\gamma,0) &\rightharpoonup y \text{ weakly in } L^2(Q), \end{aligned}$$

due to the continuity of y with respect to u, we deduce

$$y(u_\gamma,0) \rightharpoonup y(u,0) \text{ weakly in } L^2(Q),$$

from the uniqueness of limit $y = y(u,0)$.
As the same proof in the **Theorem 4.1**, we get the following system governed the state y and ξ.

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}(a(x)\nabla y) = 0 & \text{in } Q, \\ \frac{\partial y}{\partial\nu_a} = u \text{ on } \Sigma_1,\ y = 0 & \text{on } \Sigma_2, \\ y(x,0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\frac{\partial\xi}{\partial t} - \operatorname{div}(a(x)\nabla\xi) = y & \text{in } Q, \\ \frac{\partial\xi}{\partial\nu_a} = 0 \text{ on } \Sigma_1,\ \xi = 0 & \text{on } \Sigma_2, \\ \xi(x,T) = 0 & \text{in } \Omega. \end{cases}$$

due to $(4.14.c)$ and by passing to limit when $\gamma \to 0$ we get

$$\frac{\partial \xi\left(u_{\gamma}\right)}{\partial \nu_{a}} \rightarrow \frac{\partial \xi}{\partial \nu_{a}}=0 \text{ strongly in } L^{2}\left(\Sigma_{2}\right),$$

which implies that

$$\int_{\Sigma_{2}} \frac{\partial \xi}{\partial \nu_{a}} g d\Sigma_{2}=0.$$

Which means that u is the no-regret control. ■

Theorem 4.2 *The no-regret control u is characterised by the following optimality system:*

$$\left\{\begin{array}{ll}\frac{\partial y}{\partial t}-\operatorname{div}\left(a\left(x\right)\nabla y\right)=0 & \text{in } Q, \\ \frac{\partial y}{\partial \nu_{a}}=u \text{ on } \Sigma_{1},\ y=0 & \text{on } \Sigma_{2}, \\ y(x,0)=0 & \text{in } \Omega,\end{array}\right.$$

$$\left\{\begin{array}{ll}-\frac{\partial \xi}{\partial t}-\operatorname{div}\left(a\left(x\right)\nabla \xi\right)=y & \text{in } Q, \\ \frac{\partial \xi}{\partial \nu_{a}}=0 \text{ on } \Sigma_{1},\ \xi=0 & \text{on } \Sigma_{2}, \\ \xi\left(x,T\right)=0 & \text{in } \Omega,\end{array}\right.$$

$$\left\{\begin{array}{ll}\frac{\partial \rho}{\partial t}-\operatorname{div}\left(a(x)\nabla \rho\right)=0 & \text{in } Q, \\ \frac{\partial \rho}{\partial \nu_{a}}=0 \text{ on } \Sigma_{1},\ \rho=\lambda & \text{on } \Sigma_{2}, \\ \rho(x,0)=0 & \text{in } \Omega,\end{array}\right.$$

$$\left\{\begin{array}{ll}-\frac{\partial p}{\partial t}-\operatorname{div}\left(a(x)\nabla p\right)=y-y_{d}+\rho & \text{in } Q, \\ \frac{\partial p}{\partial \nu_{a}}=0 \text{ on } \Sigma_{1},\ p=0 & \text{on } \Sigma_{2}, \\ p(x,T)=0 & \text{in } \Omega,\end{array}\right.$$

with

$$p+Nu=0 \text{ a.e in } \Sigma_{1}.$$

where

$$\begin{aligned}\lim_{\gamma\to 0} u_{\gamma} &= u,\ \lim_{\gamma\to 0} y_{\gamma}=y,\ \lim_{\gamma\to 0} \xi_{\gamma}=\xi, \\ \lim_{\gamma\to 0} \rho_{\gamma} &= \rho,\ \lim_{\gamma\to 0} p_{\gamma}=p,\ \lim_{\gamma\to 0} \frac{1}{\gamma}\frac{\partial \xi}{\partial \nu}\left(u_{\gamma}\right)=\lambda.\end{aligned}$$

Proof. Actually, we have the systems that describe the state y and the adjoint state ξ in the previous propositions, now we need the systems that describe ρ and p shown in the following steps.

First step: Multiply the first equation in (4.12) by ρ_γ and integrate over Q we obtain

$$\int_0^T\int_\Omega \frac{\partial \rho_\gamma}{\partial t}\rho_\gamma dxdt - \int_0^T\int_\Omega \operatorname{div}\left(a(x)\nabla\rho_\gamma\right)\rho_\gamma dxdt = 0,$$

use Green formula

$$\frac{1}{2}\int_0^T\int_\Omega \frac{d}{dt}\left|\rho_\gamma\right|^2 dxdt + \int_0^T\int_\Omega a(x)\left|\nabla\rho_\gamma\right|^2 dxdt - \int_0^T\int_{\Gamma_1}\rho_\gamma\frac{\partial\rho_\gamma}{\partial\nu_a}d\Gamma_1 dt - \int_0^T\int_{\Gamma_2}\rho_\gamma\frac{\partial\rho_\gamma}{\partial\nu_a}d\Gamma_1 dt = 0,$$

implies that

$$\frac{1}{2}\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + \int_0^T\int_\Omega a(x)\left|\nabla\rho_\gamma\right|^2 dxdt + \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\frac{\partial\xi_\gamma}{\partial\nu_a}\frac{\partial\rho_\gamma}{\partial\nu_a}d\Gamma_2 dt = 0,$$

then

$$\frac{1}{2}\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + \alpha_1\int_0^T\int_\Omega\left|\nabla\rho_\gamma\right|^2 dxdt \leq -\frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\frac{\partial\xi_\gamma}{\partial\nu_a}\frac{\partial\rho_\gamma}{\partial\nu_a}d\Gamma_2 dt,$$

we take the absolute value and use Cauchy inequality we get

$$\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + 2\alpha_1\int_0^T\int_\Omega\left|\nabla\rho_\gamma\right|^2 dxdt \leq \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\left|\frac{\partial\xi_\gamma}{\partial\nu_a}\right|^2 d\Gamma_2 dt + \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\left|\frac{\partial\rho_\gamma}{\partial\nu_a}\right|^2 d\Gamma_2 dt,$$

from the continuity of trace operator, we obtain

$$\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + 2\alpha_1\int_0^T\int_\Omega\left|\nabla\rho_\gamma\right|^2 dxdt \leq \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\left|\frac{\partial\xi_\gamma}{\partial\nu_a}\right|^2 d\Gamma_2 dt + \int_0^T\int_\Omega\left|\rho_\gamma\right|^2 dxdt,$$

use Poincare inequality

$$\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + 2\int_0^T\int_\Omega\left|\rho_\gamma\right|^2 dxdt \leq \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\left|\frac{\partial\xi_\gamma}{\partial\nu_a}\right|^2 d\Gamma_2 dt + \int_0^T\int_\Omega\left|\rho_\gamma\right|^2 dxdt,$$

then

$$\int_\Omega\left|\rho_\gamma(T)\right|^2 dx + \int_0^T\int_\Omega\left|\rho_\gamma\right|^2 dxdt \leq \frac{1}{\gamma}\int_0^T\int_{\Gamma_2}\left|\frac{\partial\xi_\gamma}{\partial\nu_a}\right|^2 d\Gamma_2 dt \leq C,$$

requires that

$$\left\|\rho_\gamma\right\|_{L^2(Q)} \leq C.$$

There exists a subsequence such that

$$\rho_\gamma \rightharpoonup \rho \text{ weakly in } L^2(Q),$$

due to $(4.14.c)$ we have

$$\frac{1}{\gamma}\left\|\frac{\partial\xi(u_\gamma)}{\partial\nu_a}\right\|_{L^2(\Sigma_2)} \leq \frac{1}{\gamma}\left\|\frac{\partial\xi(u_\gamma)}{\partial\nu_a}\right\|^2_{L^2(\Sigma_2)} \leq C,$$

implies that exist $\lambda \in L^2(\Sigma_2)$ such that

$$\frac{1}{\gamma}\frac{\partial\xi(u_\gamma)}{\partial\nu_a} \rightharpoonup \lambda \text{ weakly in } L^2(\Sigma_2).$$

By passing to limit in (4.12) we get the system governed ρ

$$\begin{cases} \frac{\partial\rho}{\partial t} - \operatorname{div}(a(x)\nabla\rho) = 0 & \text{in } Q, \\ \frac{\partial\rho}{\partial\nu_a} = 0 \text{ on } \Sigma_1,\ \rho = \lambda & \text{on } \Sigma_2, \\ \rho(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Second step We know that $\rho_\gamma \in L^2(Q)$ and $y_\gamma - y_d \in L^2(Q)$ requires to $y_\gamma - y_d + \rho \in L^2(Q)$. As usual, we multiply the first equation in (4.13) by p_γ and use Green formula we get

$$\int_\Omega \left(|p_\gamma(T)|^2 - |p_\gamma(0)|^2\right)dx + 2\alpha_1\int_0^T\int_\Omega |\nabla p_\gamma|^2\,dxdt \leq \int_0^T\int_\Omega \left(\left|y_\gamma - y_d + \rho_\gamma\right|^2 + |p_\gamma|^2\right)dxdt,$$

use poincaré inequality

$$\|p_\gamma\|^2_{L^2(Q)} \leq \|p_\gamma(0)\|^2_{L^2(\Omega)} + \left\|y_\gamma - y_d + \rho_\gamma\right\|^2_{L^2(\Omega)},$$

Consequently,

$$\|p_\gamma\|_{L^2(Q)} \leq C$$

Then, there exists a subsequence such that

$$p_\gamma \rightharpoonup p \text{ weakly in } L^2(Q),$$

By the same way on the proof in **Theorem 4.1** and we pass to limit we get the system governed p

$$\begin{cases} -\frac{\partial p}{\partial t} - \operatorname{div}(a(x)\nabla p) = y - y_d + \rho & \text{in } Q, \\ \frac{\partial p}{\partial\nu_a} = 0 \text{ on } \Sigma_1,\ p = 0 & \text{on } \Sigma_2, \\ p(x,T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$p + Nu = 0 \text{ a.e in } L^2(\Sigma_1).$$

■

Appendices

Theorem 4.3 *(**Green Formulas**) Let $\Omega \subset \mathbb{R}^n$ a bounded and regular domain, and the normal vector unit to the outside $\Gamma = \partial\Omega$.So, for $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$ we have: the first Green's formula*

$$\int\limits_{\Omega} u\Delta v dx = -\int\limits_{\Omega} \nabla u \nabla v dx + \int\limits_{\Gamma} u\frac{\partial v}{\partial \nu} d\Gamma,$$

for $u, v \in H^2(\Omega)$ we have the second formula of Green's:

$$\int\limits_{\Omega} (u\Delta v - v\Delta u) dx = \int\limits_{\Gamma} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} d\Gamma.$$

Definition 4.3 *Let $U \in \mathbb{R}^n$ a convex set and $f : U \longrightarrow \mathbb{R}$ a strictly convex function in U if*

$$f(ty + (1-t)x) < tf(y) + (1-t)f(x) \;\; \forall x, y \in U, \forall t \in [0,1]$$

Definition 4.4 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper function, so Legendre transform f^* of f is a function of $E \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f^*(p) = \sup_{x \in E} ((p, x) - f(x))$$

Lemma 4.2 *(**Gronwall**)Let Ψ, G be continuous in $[0,T]$, with G nondecreasing and $\gamma > 0$. If*

$$\Psi(t) \leq G(t) + \gamma \int_0^t \Psi(s)\, ds \;\; \textit{for all } t \in [0,T],$$

then

$$\Psi(t) \leq G(t) \exp(\gamma t), \;\; \text{for all } t \in [0,T].$$

Proposition 4.2 ***(Cauchy inequality)***

Let a, b are any real numbers and p, q are real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the Cauchy inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right).$$

Proposition 4.3 ***(Poincare inequality)***

For $1 \leq p < \infty$, there exists a constant C such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

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