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Option: Stationary Problems

Theme

Results of existence of non-trivial weak solutions for a boundary value problem

Presented by Kamache Fares

Mrs Mesloub Fatiha	President	MCA	Larbi Tebessi University, Tebessa
Mr Guefaifia Rafik	Supervisor	MCA	Larbi Tebessi University, Tebessa
Mr Bouali Tahar	Co-Advisor	MCA	Larbi Tebessi University, Tebessa
Mr Aissaoui Adel	Examiner	Professor	Echahid Hamma Lakhdar University, Eloued
Mr Oussaeif Taki Eddine	Examiner	MCA	Larbi Ben M'hidi University, Oum El-Bouaghi
Mrs Degaichia Hakima	Examiner	MCA	Larbi Tebessi University, Tebessa

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Abstract

The objective of this thesis is to study and demonstrate the existence of at least three weak solutions for a certain class of boundary value problems for nonlinear fractional differential systems. The first part is devoted to the notions of functional analysis and also to the definitions used in this work, also it presents the fundamental theorems implemented to demonstrate the existence of the solutions. Then, the necessary background to familiarize the reader with fractional calculus and the main issues related to the research is provided. We demonstrate the existence of three weak solutions by the variational method and theorem of Bonanno and Marano for new class of fractional p-Laplacian boundary value systems. In the second part we prove the existence of the multiple solutions for perturbed nonlinear fractional p-Laplacian boundary value systems with two control parameters by using of the critical point theorem of Ricceri.

Key words: Nonlinear fractional; Dirichlet boundary value systems; p-Laplacian type; Variational method; Critical point theory.

Résumé

L'objectif de cette thèse est d'étudier et de démontrer l'existence d'au moins trois solutions faibles pour une certaine classe de problèmes aux limites pour les systèmes différentiels fractionnaires non linéaires. La première partie est consacrée aux notions d'analyse fonctionnelle ainsi qu'aux définitions utilisées dans ce travail, elle présente également les théorèmes fondamentaux mis en œuvre pour démontrer l'existence des solutions. Ensuite, le contexte nécessaire pour familiariser le lecteur avec le calcul fractionnaire et les principaux problèmes liés à la recherche est fourni. Nous démontrons l'existence de trois solutions faibles par la méthode variationnelle et le théorème de Bonanno et Marano pour une nouvelle classe de systèmes de valeurs aux limites fractionnaires p-Laplaciens. Dans la deuxième partie, nous prouvons l'existence des solutions multiples pour les systèmes de valeurs aux limites fractionnaires p-Laplaciens non linéaires perturbés avec deux paramètres de contrôle en utilisant le théorème du point critique de Ricceri.

Mots clés: Fractionnel non linéaire; Problèmes de valeur aux limites de dirichlet; Type p-laplacien; Méthode variationnelle; Théorie des points critiques

ملخص

الهدف من هذه الرسالة هو دراسة وإثبات وجود ثلاثة حلول ضعيفة على الأقل لفئة معينة من مشاكل القيمة الحدية للأنظمة التفاضلية الكسرية غير الخطية. الجزء الأول مخصص لمفاهيم التحليل الدالي وكذلك للتعريفات المستخدمة في هذا العمل، كما أنه يقدم النظريات الأساسية المطبقة لإثبات وجود الحلول. بعد ذلك يتم توفير الخلفية اللازمة لتعريف القارئ بحساب التفاضل والتكامل الكسري و القضايا الرئيسية المتعلقة بالبحث. كما أثبتنا وجود ثلاثة حلول ضعيفة من خلال طريقة التغيرية و نظرية بونانو و مارانو لفئة جديدة من أنظمة قيم حدود p-لابلاس. في الجزء الثاني تم اثبات وجود حلول متعددة لأنظمة قيم حدود p-لابلاس غير الخطية المضطربة مع معاملين تحكم باستخدام نظرية النقطة الحرجة لريسري .

الكلمات المفتاحية : كسور غير خطية؛ مشاكل القيم الحدية ديريكليت؛ مؤثرp-لابلاس ؛ طريقة التغيرية؛ نظرية النقطة الحرجة.

Acknowledgement and dedication

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At last, I dedicate this work to my parents, my wife and children and to all those interested in making research and spreading knowledge all around the world.

Contents

Abbreviations

FDE	Fractional differential equations
ODE	Ordinary differential equation
G-differentiable	Gâteaux differentiable
a.e	almost everywhere

Symbols

$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^*$	The set of natural numbers with zero included.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
Ω	Bounded domain in $\mathbb{R}$.
$L^p(\Omega)$	The space of measurable functions of power $p \in [0, +\infty[$integrable on Ω.
$L^\infty(\Omega)$	The space of measurable functions essentially bounded on Ω.
$AC(\Omega)$	The space of absolutely continuous functions on Ω.
$AC^n(\Omega)$	The space of functions f which have continuous derivatives on Ω up to order $(n-1)$.
$\Gamma(z)$	The Euler's Gamma function.
X	Banach space.
X^*	The dual space of X.
$\\|.\\|_X$	The norm in the space X.
A	A linear operator in X.
A^{-1}	Inverse operator A.
D^α	The Riemann-Liouville fractional derivatives of order α.
${}_aD_t^{-\alpha}$	The Left Riemann-Liouville fractional integrals.
${}_tD_b^{-\alpha}$	The right Riemann-Liouville fractional integrals.
${}_aD_t^{\alpha}$	The Left Riemann-Liouville fractional derivatives of order α.
${}_tD_b^{\alpha}$	The right Riemann-Liouville fractional derivatives of order α.
${}_a^CD_t^{\alpha}$	The Left Caputo fractional derivatives of order α.
${}_t^CD_b^{\alpha}$	The right Caputo fractional derivatives of order α.
E_α^p, E_β^p	Banach space.
Υ_X	Denote the class of all functionals $\phi : X \to \mathbb{R}$ that possess the following property: if $\{w_n\}$ is a sequence in X converging weakly to $w \in X$ and $\lim\limits_{n\to\infty} \inf\ \phi(w_n) \leq \phi(w)$

Introduction

Fractional differential equations can generally be seen as the study of differential equations with the fractional calculus application. With its use , the natural phenomena and mathematical models in several areas of science and engineering can be precisely described. The Fractional differential equations (FDE) have also many uses in different domains like engineering, physics, chemistry, biology, mechanics, biophysics, and other fields (see [18], [13], [24], [25], [26] and [32]). As a result, many improvements have been made in the theory of fractional calculus and fractional ordinary and partial differential equations ([6], [3], [34], [4], [5], [19], [7] and [44]). Several studies have explored the existence and different solutions for nonlinear fractional initial and boundary value problems through the use of several tools and techniques of nonlinear analysis (see for example [33], [39], [48], [9] and [28].

A FDE often has very many solutions, the conditions being less strict than in the case of an ordinary differential equation (ODE) with a single variable; the problems often make up boundary conditions which restrict the set of solutions. While the sets of solutions of an ordinary differential equation are parametrized by one of several parameters corresponding to the additional conditions, in the case of Partial differential equations, the boundary conditions are presented more in the form of a function, intuitively this means that the set of solutions is much larger, which is true in almost all problems. For linear FDE, various methods and techniques can be used as the fixed point theorems, critical point theory, the monotone iterative methods, the coincidence degree theory to get the solution.

Variational methods have emerged as one of the most effective analytic tools in the study of nonlinear equations. But there are other nonvariational techniques of use for nonlinear elliptic and parabolic FDE such as monotonicity and fixed point methods that played an important role in the study of nonlinear boundary value problems for a long time. The idea behind them is attempting to solve a given problem by looking for critical point theory which was very useful in determining the existence of solutions to complete differential equations with certain boundary conditions, see for example, in the extensive literature on the subject, classical books [32], [37], [47] and references appearing there. But so far, some problems have been created for fractional marginal value problems by exploiting this approach, where it is often very difficult to create a suitable space and a suitable function for fractional problems.

The aim of this thesis is to acquaint the reader with the greatly new result for the existence of three solution of nonlinear fractional elliptique problems involving the p-Laplacian operator type equations and systems. Chapter 1 of this thesis reviews some useful preliminary notions as Banach spaces and Monotone operator with giving the importent theorems to prove the multiplicity of solutions . In chapter 2 we present the basic technique from Calculus fractionnaire and methods used in our work for proving the existence results of different problems.

Chapters 3 and 4 are collection of published papers, each paper presents a chapter dealing with one main problem, and for each one of them we start by giving an introduction discussing its technical details and assumptions and a small historical review.

In chapter 3 (published in "Mathematics") [21], We obtain at least three weak solutions for a new class of p-Laplacian type nonlinear fractional systems according to two parameters by using variational methods combined with a critical point theory due to Bonano and Marano. Some necessary definitions and preliminary facts are presented for fractional calculus which are used to provde the availability of the weak solutions for the following system:

$$\begin{cases} {}_tD_T^{\alpha_i}\left(\frac{1}{w_i(t)^{p-2}}\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}u_i\left(t\right)\right)\right)+\mu\left|u_i\left(t\right)\right|^{p-2}u_i\left(t\right) \\ \\ =\lambda F_{u_i}\left(t,u_1\left(t\right),u_2\left(t\right),...,u_n\left(t\right)\right)\ \text{a.e.}\ t\in\left[0,T\right], \\ \\ u_i\left(0\right)=u_i\left(T\right)=0. \end{cases} \tag{1}$$

Chapter 4 (published in " j.Pseudo-Differ.Oper.Appl") [22], uses two control parameters to investigate a class of perturbed nonlinear fractional p-Laplacian differential systems, where we ensure the existence of three weak solutions by using the variational method and Ricceri's critical points theorems respecting some necessary conditions on the primitive function of nonlinear terms F_u and F_v for the following perturbed fractional differential system:

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ \ {}_0D_t^{\alpha}u(t)\right)\right)+\mu\left|u(t)\right|^{p-2}u(t) \\ =\lambda F_u\left(t,u(t),v(t)\right)+\delta G_u\left(t,u(t),v(t)\right)\ \text{ a.e. } t\in[0,T], \\ {}_tD_T^{\beta}\left(\frac{1}{w_2(t)^{p-2}}\Phi_p\left(w_2(t)\ \ {}_0D_t^{\beta}v(t)\right)\right)+\mu\left|v(t)\right|^{p-2}v(t) \\ =\lambda F_v\left(t,u(t),v(t)\right)+\delta G_v\left(t,u(t),v(t)\right)\ \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0,\quad v(0)=v(T)=0. \end{cases} \tag{2}$$

Chapter 1

Preliminary

1- L^p Space.

2- Banach Space.

3- Continuous function spaces.

4- Some inequalities.

5- Monotone operators.

6- Some elements of critical point theory.

7- Three critical points theorem.

1.1 L^p Spaces

Let Ω an open from $\mathbb{R}^n$, with **Lebesgue** measure dx. We denote by $L^1(\Omega)$ the space of functions that can be integrated into Ω with values in $\mathbb{R}$, we provide it with the standard

$$\|u\|_{L^1} = \int_\Omega |u(x)|\, dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int_\Omega |f(x)|^p\, dx < +\infty \right\}.$$

Let standard is

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p\, dx \right)^{\frac{1}{p}}.$$

We also define the space $L^\infty(\Omega)$

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable},\ \exists c > 0,\ \text{such as } |f(x)| \leq c \ \text{ a.e on } \Omega \}.$$

It will be provided with the sup-essentie standard

$$\|u\|_{L^\infty} = \underset{x\in\Omega}{ess\sup} |u(x)| = \inf \{ c;\ \ |u(x)| \leq c \quad \text{a.e on } \Omega \}.$$

1.2 Banach spaces

Definition 1.1 *[43]*

Let X be a vector space over $\mathbb{R}$. A real-valued function $\|.\|$ defined on X and satisfying the following conditions is called a norm:

i) $\|u\| \geq 0, \qquad \|u\| = 0$ *if and only if* $u = 0.$

ii) $\|\lambda u\| = |\lambda| \, \|u\|$, *for all* $u \in X$ *and* $\lambda \in \mathbb{R}.$

iii) $\|u + v\| \leq \|u\| + \|v\|$, $\forall u, v \in X.$

$(X, \|.\|)$, *vector space* X *equipped with* $\|.\|$ *is called a normed space.*

Definition 1.2 *[43]*

A normed space X *is called a Banach space, if its every Cauchy sequence is convergent, that is* $\|u_n - u_m\| \to 0$ *as* $n, m \to \infty$ $\forall u_n, u_m \in X$ *implies that* $\exists u \in X$ *such that* $\|u_n - u\| \to 0$ *as* $n \to \infty$.

1.3 Continuous function spaces

Definition 1.3 *[24]*

Let $\Omega = [0, T] \, (0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$ *and* $n \in \mathbb{N}$. *We denote by* $C^n(\Omega)$ *the space of functions* f *which are* m *times continuously differentiable on* Ω *with the norm:*

$$\|f\|_{C^n(\Omega)} = \sum_{k=0}^{n} \left\| f^{(k)} \right\|_{C^n(\Omega)} = \sum_{k=0}^{n} \max_{t \in \Omega} \left| f^{(k)}(t) \right|, \; n \in \mathbb{N}.$$

In particular, for $n = 0$, $C^0(\Omega) = C(\Omega)$ *in the space of continuous functions* f *on* Ω *with the norm:*

$$\|f\|_{C^n(\Omega)} = \max_{t \in \Omega} |f(t)|.$$

Definition 1.4 *[24]*

Let $\Omega = [0, T] \, (0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$. *We denote by* $AC(\Omega)$ *the space of primitive functions of integrable functions, that is to say :*

$$AC(\Omega) = \left\{ f \; / \exists \varphi \in L^1(\Omega) : f(t) = c + \int_0^t \varphi(s) \, ds \right\},$$

and we call $AC(\Omega)$ *the space of absolutely continuous functions on* Ω.

Definition 1.5 *[24]*

For $n \in \mathbb{N}^*$ we denote by $C_\mu^n(\Omega)$ the space of functions f which have continuous derivatives on Ω up to order $(n-1)$ and such that $f^{(n-1)} \in AC(\Omega)$ that is to say :

$$AC^n(\Omega) = \left\{ f : \Omega \to \mathbb{C},\ f^{(k)} \in C\left(\Omega\right),\ k \in \{0, 1, ..., n-1\},\ f^{(n-1)} \in AC(\Omega) \right\}.$$

In particular $AC^1(\Omega) = AC(\Omega)$.

A characterization of the functions of this space is given by the following lemma:

Lemma 1.1 *[24]*

A function $f \in AC^n(\Omega)$, $n \in \mathbb{N}^*$, if and only if it is represented as:

$$f\left(t\right) = \frac{1}{(n-1)!} \int_0^t \left(t-s\right)^{n-1} f^{(n)}\left(s\right) ds + \sum_{k=0}^{n-1} \frac{f^{(k)}\left(t\right)}{k!} t^k.$$

Lemma 1.2 *[50]* (*Lebesgue's dominated convergence theorem*)

Let Ω be a measurable set and let $\{f_n\}$ be a sequence of measurable functions such that $\lim\limits_{n\to\infty} f_n\left(x\right) = f\left(x\right)$ a.e. in Ω, and for every $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ a.e. in Ω, where g is integrable on Ω. Then

$$\lim_{n\to\infty} \int_\Omega f_n\left(x\right) dx = \int_\Omega f\left(x\right) dx.$$

Lemma 1.3 *[12]* (*Fatou's lemma*)

If $\{f_n\}$ is a sequence of nonnegative measurable functions on Ω, then

$$\int_\Omega \liminf_{n\to\infty}\ f_n\left(x\right) dx \leq \liminf_{n\to\infty} \int_\Omega f_n\left(x\right) dx.$$

1.4 Some inequalities

Hölder's inequality [17]

$\forall\ (u,v) \in L^p(\Omega) \times L^q(\Omega)$ we have

$$\left| \int_\Omega uv dx \right| \leq \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{q}},$$

where p and q are strictly positive linked by the relation $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$.

Inequalitie for Vectors [31]

Some special inequalities are helpful in the study of the p-Laplace operator. Expressions like

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle,$$

are needed, a and b denoting vectors in $\mathbb{R}^n$. As expected, the cases $p > 2$ and $p < 2$ are different. Let us begin with the identity

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle = \frac{|b|^{p-2} + |a|^{p-2}}{2} |b-a|^2 + \frac{\left(|b|^{p-2} - |a|^{p-2}\right)\left(|b|^2 + |a|^2\right)}{2},$$

which is easy to verify by a calculation. We can read off the following inequalities

1) If $p \geq 2$

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle \geq 2^{-1} \left(|b|^{p-2} + |a|^{p-2}\right) |b-a|^2$$

$$\geq 2^{p-2} |b-a|^p.$$

2) If $p \leq 2$

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle \leq \frac{1}{2} \left(|b|^{p-2} + |a|^{p-2}\right) |b-a|^2.$$

However, the second inequality in 1) cannot be reversed for $p \leq 2$, as the first one, not even with a poorer constant than 2^{p-2}.

1.5 Monotone operators

Definition 1.6 *[51]*

Let X be real Banach space, and let $A : X \to X^*$ be an operator.

i) *A is called monotone iff*

$$\langle Au - Av, u - v\rangle \geq 0 \text{ for all } u, v \in X.$$

ii) *A is called strictly monotone iff*

$$\langle Au - Av, u - v\rangle > 0 \text{ for all } u, v \in X \text{ with } u \neq v.$$

iii) *A is called strongly monotone iff there is a $c > 0$ such that*

$$\langle Au - Av, u - v\rangle \geq c \left\|u - v\right\|^2 \text{ for all } u, v \in X.$$

iv) *A is called uniformly monotone iff*

$$\langle Au - Av, u - v\rangle \geq a\left(\|u - v\|\right)\|u - v\| \text{ for all } u, v \in X,$$

where the continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to +oo$ as $t \to +oo$.

Definition 1.7 *[51]*

Let X be real Banach space, and let $A : X \to X^*$ be an operator. A is called hemicontinuous if for all u, $v \in X$, l'appliction $t \to \langle A(u + tv), v\rangle$ is continuous from $\mathbb{R}$ in $\mathbb{R}$.

Definition 1.8 *[51]*

Let X be real Banach space, and let $A : X \to X^*$ be an operator. A is called coercive iff

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

1.6 Some elements of critical point theory

Definition 1.9 *[23]*

Let ω be a part of a Banach space X and $J : \omega \to \mathbb{R}$. If $u \in \omega$ and $v \in X$ are such that for $t > 0$ quite small we have $u + tv \in \omega$ we say that J admits (at the point u) a derivative in the direction v if

$$\lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t},$$

exist. We will denote this limit by $J'_v(u)$

Definition 1.10 *[23]*

Let ω be a part of a Banach space X and $J : \omega \to \mathbb{R}$. If $u \in \omega$, we say that J is Gâteaux differentiable (or G-differentiable) at u, if there exists $l \in X'$ such that in each direction $v \in X$ where $J(u + tv)$ exists for $t > 0$ small enough, the directional derivative $J'_v(u)$ exists and we have

$$\lim_{t \to 0^+} \frac{F(u + tv) - F(u)}{t} = \langle l, v \rangle .$$

We write $J'(u) = l$.

Definition 1.11 *[23]*

Let X be a Banach space, $\omega \in X$ an open space and $J \in C^1(\omega, \mathbb{R})$. We say that $u \in \omega$ is a critical point of J if $J'(u) = 0$ with $J'(u)$ is the G-differentiable of J at point. If u are not a critical point then we say that u is a regular point of J. If $c \in \mathbb{R}$, we say that c is a value critical of J, if there exists $u \in \omega$ such that $J(u) = c$ and $J'(u) = 0$. If c is not a critical value then we say that c is a regular value of J.

Definition 1.12 *[23]*

Let X be a Banach space, $F \in C^1(X, \mathbb{R})$ and a set of constraints:

$$S = \{v \in X : F(v) = 0\},$$

we suppose that for everything $u \in S$, we have $F'(v) \neq 0$. Si $J \in C^1(X, \mathbb{R})$ we say that $c \in \mathbb{R}$ is *value criticism of J on S if there exists $u \in S$, and $\lambda \in \mathbb{R}$ such that*

$$J(u) = c \text{ and } J'(u) = \lambda F'(u).$$

The point u is a critical point of J on S and the real one is called the Lagrange multiplier for the critical value c (or the critical point u).

When X is a functional space and the equation $J'(u) = \lambda F'(u)$ corresponds to an equation with partial derivatives, we say that $J'(u) = \lambda F'(u)$ is the Euler-Lagrange equation (or the Euler equation) satisfied by the critical point u on the constraint S.

Definition 1.13 *[23]*

Let X be a Banach space and ω is a part of X. A function $J : \omega \to \mathbb{R}$ is said to be weakly sequentially lower semi-continuous if for any sequence $(u_n)_n$ of ω weakly converging to $u \in \omega$ we have:

$$J(u) \leq \lim_{n\to\infty} \inf J(u_n).$$

Proposition 1.1 *[23]*

Let X be a reflexive Banach space, $K \subset X$ a closed convex and $J : K \to \mathbb{R}$ a weakly sequentially lower semi-continuous. Moreover, if K is unbounded, we assume that for any sequence $(u_n)_n$ of K such that kunk $\|u_n\| \to \infty$, we have $J(u_n) \to \infty$. Then J is bounded lower and it reaches its minimum i.e.

$$\exists u \in K, \ J(u) = \inf_{v\in K} J(v) = \min_{V\in K} J(v).$$

1.7 Three critical points theorem

We present a critical point theorem due to Bonanno and Marano and Ricceri's critical points theorems to prove the existence of at least three weak solutions.

Theorem 1.1 *[8]*

Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gateaux differentiable sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a *continuous inverse on X^*, bounded on bounded subsets of X, $\Psi : X \to \mathbb{R}$ a continuously Gateaux differentiable functional whose Gateaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that
(a_1) $\sup\limits_{\Phi(u) \leq r} \dfrac{\Psi(u)}{r} < \dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}$.

(a_2) *For each* $\lambda \in \Lambda_\lambda = \left] \dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup\limits_{\Phi(u) \leq r} \Psi(u)} \right[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for any $\lambda \in \Lambda_\lambda$, the functional $\Phi - \lambda\Psi$ has at least three critical point in X.

Theorem 1.2 *[38]*

Let X be a separable reflexive real Banach space, and let $\phi : X \to \mathbb{R}$ be a coercive sequentially weakly lower semicontinuous, C^1 functional belonging to Υ_X, bounded on each bounded subset of X, with derivative admitting a continuous inverse on X^*. Let $\psi : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that ϕ has a strict local minimum x_0 with $\phi(x_0) = \psi(x_0) = 0$.

Finally, setting

$$\delta_1 = \max\left\{0, \lim_{\|x\| \to +\infty} \sup \frac{\psi(x)}{\phi(x)}, \lim_{x \to x_0} \sup \frac{\psi(x)}{\phi(x)}\right\},$$

$$\delta_2 = \sup_{x \in \phi^{-1}(]0,+\infty[)} \frac{\psi(x)}{\phi(x)},$$

we suppose that $\delta_1 < \delta_2$.

Then, for each compact interval $[a, b] \subset (\frac{1}{\delta_2}, \frac{1}{\delta_1})$(with the conventions $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$), there exists $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta^* > 0$ such that, for each $\delta \in [0, \delta^*]$, the equation

$$\phi'(x) = \lambda\psi'(x) + \delta J'(x),$$

has at least three solutions in X with norms less than ϱ.

Theorem 1.3 *[39]*

Let X be a reflexive real Banach space, and let $I \subset \mathbb{R}$ be an interval.

Let $\phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, C^1 functional bounded on each bounded subset of X, with derivative admitting a continuous inverse on X^*. Let $-\psi : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that

$$\lim_{\|x\| \to +\infty} (\phi(x) - \lambda\psi(x)) = +\infty,$$

for all $\lambda \in I$ and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\phi(x) + \lambda\rho - \psi(x)) < \inf_{x \in X} \sup_{\lambda \in I} (\phi(x) + \lambda\rho - \psi(x)) .$$

Then there exist a nonempty open set $\Lambda \subset I$ and a positive number with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $-J : X \to \mathbb{R}$ with compact derivative, there exists $\delta^* > 0$ such that, for each $\delta \in [0, \delta^*]$, the equation

$$\phi^{'}(x) - \lambda\psi^{'}(x) - \delta J^{'}(x) = 0,$$

has at least three solutions in X with norms less than ϱ .

Chapter 2

Fractional calculus

1- Introduction.

2- Special function.

3- Fractional integral in the sense of Riemann-Liouville.

4- Fractional derivative in the sense of Riemann-Liouville.

5- Fractional derivative in the sense of Caputo.

6- Some fractional derivation properties in the sense of Riemann-Liouville.

7- Examples.

This section is devoted to the presentation of certain elements of fractional calculation. We start with general introduction on adequate fractional calculus as well as special function, then we recall the definitions and some properties of the integral and fractional derivatives within the meaning of Riemann-Liouville.

2.1 Introduction

The objective of fractional calculus is to generalize traditional derivatives to non-integer orders. As it is well known, many dynamic systems are best characterized by a dynamic fractional order model, generally based on the notion of differentiation or integration of the non-whole order. The origins of fractional calculus date back to the end of the 17th century, starting from some speculations by GW Leibniz concerning the study conducted in 09/30/1695, on the sibnification of $\frac{d^n f}{dt^n}$ si $n = \frac{1}{2}$. Since then, many mathematicians contributed to the development of this theory, we cite among others PS. LAPLACE, J.B.J. FOURIER, N.H.ABEL, J. LIOUVILLE.

2.2 Special function

In this paragraph we present definitions and some properties for the Gamma function.

2.2.1 Gamma function

The Gamma function was introduced by the Swiss mathematician Leonhard Euler (1707-1783) with the aim of generalizing the factorial of non-integer values. Later, due to its great importance, it has been studied by other eminent mathematicians such as Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), KarlWeierstrass (1815-1897), Charles Hermite (1822-1901) and many others. The Gamma function belongs to the category of special transcendent functions and we will see that some famous mathematical constants occur in his study. It also appears in various fields, such as asymptotic series. Euler's Gamma function is a basic function of fractional calculus. This function generalizes the factorial $n!$.

Definition 2.1 *[36]*

One of the basic functions of fractional calculus is Euler's Gamma function $\Gamma(z)$. *The Gamma Function* $\Gamma(z)$ *is defined by the following integral:*

$$\Gamma(z) = \int_{0}^{+\infty} t^{z-1} e^{-t} dt, \tag{2.1}$$

with $\Gamma(1) = 1$, $\Gamma(0_{+}) = +\infty$ *is a strictly decreasing function for* $0 < z \leq 1$.

2.2.2 Some properties of the Gamma function

An important property of the Gamma function $\Gamma(z)$ is the following recurrence relation:

$$\Gamma(z+1) = z\Gamma(z).$$

That we can demonstrate it by integration by parts

$$\Gamma(z+1) = \int_{0}^{+\infty} t^{(z+1)-1} e^{-t} dt = \int_{0}^{+\infty} t^{z} e^{-t} dt = \left[-te^{-t}\right]_{0}^{+\infty} + z \int_{0}^{+\infty} t^{z-1} e^{-t} dt = z\Gamma(z).$$

Euler's Gamma function generalizes the factorial because $\Gamma(n+1) = n!;\ \forall n \in \mathbb{N}$, indeed $\Gamma(1) = 1$, we get:

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1! \\
\Gamma(3) &= 2\Gamma(2) = 2.1! = 2! \\
\Gamma(4) &= 3\Gamma(3) = 3.2! = 3! \\
\Gamma(5) &= 4\Gamma(4) = 4.3! = 4! \\
\Gamma(6) &= 5\Gamma(5) = 5.4! = 5! \\
\vdots \quad & \quad \vdots \quad \quad \vdots \quad \quad \vdots \\
\Gamma(n+1) &= n\Gamma(n) = n.(n-1)! = n!.
\end{aligned}$$

Let's also calculate $\Gamma\left(\frac{1}{2}\right)$. We pose $u=\sqrt{t}$ and so $t=u^2$ and $dt=2udu$ and we get

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{+\infty}\frac{e^{-t}}{\sqrt{t}}dt=\int_0^{+\infty}\frac{e^{-u^2}}{u}2udu=2\int_0^{+\infty}e^{-u^2}du=\sqrt{\pi}.$$

Maximum digits, the numerical values of some of these constants are:

$$\Gamma\left(\frac{1}{2}\right)=1.77245385090551602729816748334...$$
$$\Gamma\left(\frac{1}{3}\right)=2.67893853470774763365569294097...$$
$$\Gamma\left(\frac{1}{4}\right)=3.62560990822190831193068515587...$$
$$\Gamma\left(\frac{1}{5}\right)=4.59084371199880305320475827593...$$

$$\Gamma(0.6)=1.48919224881281710239433338832...$$
$$\Gamma(0.65)=1.38479510202651000285376452479...$$
$$\Gamma(0.7)=1.29805533264755778568117117915...$$
$$\Gamma(0.8)=1.16422971372530337363632093827...$$

2.3 Fractional integral in the sense of Riemann-Liouville

Definition 2.2 *[50](Left and right Riemann-Liouville fractional integrals).*

Let $j=[a,b]$ $(-\infty<a<b<+\infty)$ be a finite interval of $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals ${}_aD_t^{-\alpha}u(t)$ and ${}_tD_b^{-\alpha}u(t)$ of ordre $\alpha\in\mathbb{R}^+$ are defined by

$$ {}_aD_t^{-\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,ds, \qquad t>a, \qquad \alpha>0, \tag{2.2}$$

and

$$ {}_tD_b^{-\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\,ds, \qquad t<b, \qquad \alpha>0, \tag{2.3}$$

respectively, provided the right-hand sides are pointwise defined on $[a,b]$. When $\alpha=n\in\mathbb{N}$, the *definitions (2.2) and (2.3) coincide with the n-th integrals of the*

$$ {}_{a}D_{t}^{-n}u\left(t\right)=\frac{1}{\left(n-1\right)!}\int\limits_{a}^{t}\left(t-s\right)^{n-1}u\left(s\right)ds, $$

and

$$ {}_{t}D_{b}^{-n}u\left(t\right)=\frac{1}{\left(n-1\right)!}\int\limits_{t}^{b}\left(s-t\right)^{n-1}u\left(s\right)ds. $$

2.4 Fractional derivative in the sense of Riemann-Liouville

Definition 2.3 *[50](Left and right Riemann-Liouville fractional derivatives).*

The left and right Riemann-Liouville fractional derivatives ${}_{a}D_{t}^{\alpha}u\left(t\right)$ *and* ${}_{t}D_{b}^{\alpha}u\left(t\right)$ *of ordre* $\alpha\in\mathbb{R}^{+}$ *are defined by*

$$ {}_{a}D_{t}^{\alpha}u\left(t\right):=\frac{d^{n}}{dt^{n}}\ {}_{a}D_{t}^{\alpha-n}u\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}\left(\int\limits_{a}^{t}\left(t-s\right)^{n-\alpha-1}u\left(s\right)ds\right),\qquad t>a, \tag{2.4} $$

and

$$ {}_{t}D_{b}^{\alpha}u\left(t\right):=\left(-1\right)^{n}\frac{d^{n}}{dt^{n}}\ {}_{t}D_{b}^{\alpha-n}u\left(t\right)=\frac{\left(-1\right)^{n}}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}\left(\int\limits_{t}^{b}\left(t-s\right)^{n-\alpha-1}u\left(s\right)ds\right),\qquad t<b, \tag{2.5} $$

respectively, where $n=[\alpha]+1$, $[\alpha]$ *means the integer part of* α. *In particular, when* $\alpha=n\in\mathbb{N}^{*}$

$$ {}_{a}D_{t}^{0}u\left(t\right)=\ {}_{t}D_{b}^{0}u\left(t\right)=u\left(t\right), $$

$$ {}_{a}D_{t}^{n}u\left(t\right)=u^{(n)}\left(t\right)\ \text{ and }\ {}_{t}D_{b}^{n}u\left(t\right)=\left(-1\right)^{n}u^{(n)}\left(t\right), $$

where $u^{(n)}\left(t\right)$ *is the usual derivative of* $u\left(t\right)$ *of order* n. *If* $0<\alpha<1$, *then*

$$ {}_{a}D_{t}^{\alpha}u\left(t\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\left(\int\limits_{a}^{t}\left(t-s\right)^{-\alpha}u\left(s\right)ds\right),\qquad t>a, \tag{2.6} $$

and

$$ {}_tD_b^{\alpha}u\left(t\right)=-\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\left(\int\limits_t^b\left(t-s\right)^{-\alpha}u\left(s\right)ds\right),\qquad t<b. \tag{2.7}$$

The left and right Caputo fractional derivatives are defined via above Riemann-Liouville fractional derivatives.

2.5 Fractional derivative in the sense of Caputo

Definition 2.4 *[50](Left and right Caputo fractional derivatives).*

The left and right Caputo fractional derivatives ${}_a^CD_t^{\alpha}u(t)$ *and* ${}_t^CD_b^{\alpha}u(t)$ *of order* $\alpha\in\mathbb{R}^+$ *are defined by:*

$$ {}_a^CD_t^{\alpha}u\left(t\right)={}_aD_t^{\alpha}\left[u\left(t\right)-\sum_{k=0}^{n-1}\tfrac{u^{(k)}(a)}{k!}\left(t-a\right)^k\right],$$

and

$$ {}_t^CD_b^{\alpha}u\left(t\right)={}_tD_b^{\alpha}\left[u\left(t\right)-\sum_{k=0}^{n-1}\tfrac{u^{(k)}(b)}{k!}\left(b-t\right)^k\right],$$

respectively, where

$$n=n=[\alpha]+1,\ \textit{for}\ \alpha\notin\mathbb{N}^*,\ n=\alpha\ \textit{for}\ \alpha\in\mathbb{N}^*.$$

In particular, when $0<\alpha<1$, *then*

$$ {}_a^CD_t^{\alpha}u\left(t\right)={}_aD_t^{\alpha}\left(u\left(t\right)-u\left(a\right)\right),$$

and

$$ {}_t^CD_b^{\alpha}u\left(t\right)={}_tD_b^{\alpha}\left(u\left(t\right)-u\left(b\right)\right).$$

The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations.

Proposition 2.1 *[50]*

i) *If* $\alpha\notin\mathbb{N}^*$ *and* $u(t)$ *is a function for which the Caputo fractional derivatives* ${}_a^CD_t^{\alpha}u(t)$ *and* ${}_t^CD_b^{\alpha}u(t)$ *of order* $\alpha\in\mathbb{R}^+$ *exist together with the Riemann-Liouville fractional derivatives*

${}_aD_t^{\alpha}u\left(t\right)$ *and* ${}_tD_b^{\alpha}u\left(t\right)$, *then*

$${}_a^CD_t^{\alpha}u\left(t\right) = {}_aD_t^{\alpha}u\left(t\right) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}\left(t-a\right)^{k-\alpha},$$

and

$${}_t^CD_b^{\alpha}u\left(t\right) = {}_tD_b^{\alpha}u\left(t\right) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)}\left(b-t\right)^{k-\alpha},$$

where $n = [\alpha]+1$. *In particular, when* $0<\alpha<1$, *we have*

$${}_a^CD_t^{\alpha}u\left(t\right) = {}_aD_t^{\alpha}u\left(t\right) - \frac{u(a)}{\Gamma(1-\alpha)}\left(t-a\right)^{-\alpha},$$

and

$${}_t^CD_b^{\alpha}u\left(t\right) = {}_tD_b^{\alpha}u\left(t\right) - \frac{u(b)}{\Gamma(1-\alpha)}\left(b-t\right)^{-\alpha}.$$

ii) *If* $\alpha = n \in \mathbb{N}^*$ *and the usual derivative* $u^{(n)}(t)$ *of order* n *exists, then* ${}_a^CD_t^{\alpha}u(t)$ *and* ${}_t^CD_b^{\alpha}u(t)$ *are represented by*

$${}_a^CD_t^{n}u\left(t\right) = u^{(n)}\left(t\right) \text{ and } {}_t^CD_b^{n}u\left(t\right) = \left(-1\right)^n u^{(n)}\left(t\right).$$

2.6 Some fractional derivation properties in the sense of Riemann-Liouville

The derivation operator and integration by parts in the Riemann-Liouville sense has the properties summarized in the following propositions:

Proposition 2.2 *[36]*

For $n-1<\alpha\leq n$, $m-1<\beta\leq m$ *we have :*

1) *The Left and right Riemann-Liouville fractional operator is linear*

$${}_aD_t^{\alpha}\left(\lambda u\left(t\right)+\mu v\left(t\right)\right) = \lambda\,{}_aD_t^{\alpha}u\left(t\right) + \mu\,{}_aD_t^{\alpha}v\left(t\right),$$
$${}_tD_b^{\alpha}\left(\lambda u\left(t\right)+\mu v\left(t\right)\right) = \lambda\,{}_tD_b^{\alpha}u\left(t\right) + \mu\,{}_tD_b^{\alpha}v\left(t\right).$$

2) *In general*

$$ {}_aD_t^{\alpha}\left({}_aD_t^{\beta}u(t)\right) \neq {}_aD_t^{\beta}\left({}_aD_t^{\alpha}u(t)\right), $$

$$ {}_tD_b^{\alpha}\left({}_tD_b^{\beta}u(t)\right) \neq {}_tD_b^{\beta}\left({}_tD_b^{\alpha}u(t)\right). $$

Proof [36]

For the left and right Riemann-Liouville fractional derivatives ${}_aD_t^{\alpha}u(t)$ and ${}_tD_b^{\alpha}u(t)$ of ordre $n-1<\alpha\leq n$ are defined by (2.4) and (2.5) we have:

$$
\begin{aligned}
{}_aD_t^{\alpha}(\lambda u(t)+\mu v(t)) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\alpha-1}(\lambda u(t)+\mu v(t))\,ds\right) \\
&= \frac{\lambda}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\alpha-1}u(t)\,ds\right)+\frac{\mu}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\alpha-1}v(t)\,ds\right) \\
&= \lambda\ {}_aD_t^{\alpha}u(t)+\mu\ {}_aD_t^{\alpha}v(t).
\end{aligned}
$$

and

$$
\begin{aligned}
{}_tD_b^{\alpha}(\lambda u(t)+\mu v(t)) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_t^b (t-s)^{n-\alpha-1}(\lambda u(t)+\mu v(t))\,ds\right) \\
&= \frac{\lambda(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_t^b (t-s)^{n-\alpha-1}u(t)\,ds\right)+\frac{\mu(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_t^b (t-s)^{n-\alpha-1}v(t)\,ds\right) \\
&= \lambda\ {}_tD_b^{\alpha}u(t)+\mu\ {}_tD_b^{\alpha}v(t).
\end{aligned}
$$

Proposition 2.3 *[36]*

For $\alpha>0$, $t>0$ we have

$$ {}_aD_t^{\alpha}\left({}_aD_t^{-\alpha}u(t)\right)=u(t), $$

$$ {}_tD_b^{\alpha}\left({}_tD_b^{-\alpha}u(t)\right)=u(t). $$

Proof Let $\alpha = n \geq 1$, we have

$$\begin{aligned} {}_aD_t^{\alpha}\left({}_aD_t^{-\alpha}u\left(t\right)\right) &= \frac{d^n}{dt^n}\left(\int_a^t \frac{\left(t-s\right)^{n-1}}{\left(n-1\right)!}u\left(s\right)ds\right) \\ &= \frac{d}{dt}\int_a^t u\left(s\right)ds = u\left(t\right). \end{aligned}$$

Suppose now that $n-1 \leq \alpha < 1$ and use the rule of composition of fractional integrals in the sense of Riemann-Liouville. So we have :

$${}_aD_t^{\alpha}\left({}_aD_t^{-\alpha}u\left(t\right)\right) = {}_aD_t^{-(n-\alpha)}\left({}_aD_t^{-\alpha}u\left(t\right)\right),$$

from where

$$\begin{aligned} {}_aD_t^{\alpha}\left({}_aD_t^{-\alpha}u\left(t\right)\right) &= \frac{d^n}{dt^n}\left\{{}_aD_t^{-(n-\alpha)}\left({}_aD_t^{-\alpha}u\left(t\right)\right)\right\} \\ &= \frac{d^n}{dt^n}\left\{{}_aD_t^{-n}u\left(t\right)\right\} = u\left(t\right). \end{aligned}$$

The second formula is shown in the same way.

Proposition 2.4 *[36]*

Let $0 < \alpha < 1$ and $a < t < b$. Then

$$\int_a^t \left[{}_aD_t^{\alpha}f\left(s\right)\right]g\left(s\right)ds = \int_a^t f\left(s\right)\left[{}_sD_t^{\alpha}g\left(s\right)\right]ds,$$

$$\int_t^b \left[{}_sD_b^{\alpha}f\left(s\right)\right]g\left(s\right)ds = \int_t^b f\left(s\right)\left[{}_tD_s^{\alpha}g\left(s\right)\right]ds.$$

Specifically,

$$\int_a^b \left[{}_aD_b^{\alpha}f\left(s\right)\right]g\left(s\right)ds = \int_a^b f\left(s\right)\left[{}_sD_b^{\alpha}g\left(s\right)\right]ds.$$

Proof

$$
\begin{aligned}
\int_a^t \left[{}_aD_t^{\alpha} f(s)\right] g(s)\, ds &= \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{d}{ds} \left(\int_a^s (s-\tau)^{-\alpha} f(\tau)\, d\tau \right) g(s)\, ds \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\int_a^s (s-\tau)^{-\alpha} f(\tau)\, d\tau \right) g'(s)\, ds \\
&\quad + \left[\frac{1}{\Gamma(1-\alpha)} g(s) \int_a^s (s-\tau)^{-\alpha} f(\tau)\, d\tau \right]_{s=a}^{s=t} \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\int_\tau^t (s-\tau)^{-\alpha} g'(s)\, ds \right) f(\tau)\, d\tau \\
&\quad + g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f(\tau)\, d\tau \\
&= \int_a^t f(\tau) \left[{}_\tau^C D_t^{\alpha} g(\tau)\right] d\tau + g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f(\tau)\, d\tau \\
&= \int_a^t f(\tau) \left[{}_\tau D_t^{\alpha} g(\tau) - g(t) \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\right] d\tau + g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f(\tau)\, d\tau \\
&= \int_a^t f(\tau) \left[{}_\tau D_t^{\alpha} g(\tau)\right] d\tau \\
&= \int_a^t f(s) \left[{}_s D_t^{\alpha} g(s)\right] ds.
\end{aligned}
$$

The second formula is shown in the same way.

2.7 Examples

Some examples about derivation operator the Riemann-Liouville sense

Example 2.1 *The derivative of $f(t) = t^b$ in the sense of Riemann-Liouville. Let $\alpha > 0$ such* *that* $n-1<\alpha<n$ *and* $b>-1$,

$$D^{\alpha}t^{b}=\frac{\Gamma\left(b+1\right)}{\Gamma\left(b+n-\alpha+1\right)}D^{\alpha}t^{b+n-\alpha}. \tag{2.8}$$

Taking into account

$$\begin{aligned}D^{\alpha}t^{b+n-\alpha} &= \left(b+n-\alpha\right)\left(b+n-\alpha-1\right)...\left(b-\alpha+1\right)t^{b-\alpha}\\ &= \frac{\Gamma\left(b+n-\alpha+1\right)}{\Gamma\left(b-\alpha+1\right)}t^{b-\alpha}.\end{aligned} \tag{2.9}$$

We substitute the result (2.8), in the formula (2.9), to obtain:

$$\begin{aligned}D^{\alpha}t^{b} &= \frac{\Gamma\left(b+1\right)}{\Gamma\left(b-\alpha+1\right)}\frac{\Gamma\left(b+n-\alpha+1\right)}{\Gamma\left(b-\alpha+1\right)}t^{b-\alpha}\\ &= \frac{\Gamma\left(b+1\right)}{\Gamma\left(b-\alpha+1\right)}t^{b-\alpha}.\end{aligned}$$

So the fractional derivative in the sense of Riemann-Liouville of the function $f(t)=t^{b}$ *is given by:*

$$D^{\alpha}t^{b}=\frac{\Gamma\left(b+1\right)}{\Gamma\left(b-\alpha+1\right)}t^{b-\alpha}. \tag{2.10}$$

In particular, if $b=0$ *and* $\alpha>0$, *the Riemann fractional derivative-Liouville of a constant function* $f(t)=C$ *is non-zero, its value is:*

$$D^{\alpha}C=\frac{C}{\Gamma\left(1-\alpha\right)}t^{-b}.$$

Example 2.2 *The derivative of* $f(t)=\left(t-a\right)^{b}$ *in the sense of Riemann-Liouville. Let* $\alpha>0$ *such that* $n-1<\alpha<n$ *and* $b>-1$,

$$D^{\alpha}\left(t-a\right)^{b}=\frac{\Gamma\left(b+1\right)}{\Gamma\left(b+n-\alpha+1\right)}D^{\alpha}\left(t-a\right)^{b+n-\alpha}, \tag{2.11}$$

taking into account

$$\begin{aligned} D^{\alpha}(t-a)^{b+n-\alpha} &= (b+n-\alpha)(b+n-\alpha-1)\ldots(b-\alpha+1)(t-a)^{b-\alpha} \\ &= \frac{\Gamma(b+n-\alpha+1)}{\Gamma(b-\alpha+1)}(t-a)^{b-\alpha}, \end{aligned} \tag{2.12}$$

we substitute the result (2.11), in the formula (2.12), to obtain:

$$\begin{aligned} D^{\alpha}(t-a)^{b} &= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)}\frac{\Gamma(b+n-\alpha+1)}{\Gamma(b-\alpha+1)}(t-a)^{b-\alpha} \\ &= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)}(t-a)^{b-\alpha}, \end{aligned}$$

so the fractional derivative in the sense of Riemann-Liouville of the function $f(t)=(t-a)^{b}$ *is given by:*

$$D^{\alpha}(t-a)^{b} = \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)}(t-a)^{b-\alpha}. \tag{2.13}$$

Example 2.3 *The derivative of* $w_1(t)$ *in the sense of Riemann-Liouville where:*

$$w_1(t) = \begin{cases} \frac{\Gamma(2-\alpha)c_1}{\epsilon T}t, & t\in[0,\epsilon T[\,, \\ \Gamma(2-\alpha)c_1, & t\in[\epsilon T,(1-\epsilon)T]\,, \\ \frac{\Gamma(2-\alpha)c_1}{\epsilon T}(T-t), & t\in\,](1-\epsilon)T,T]\,, \end{cases}$$

is $D^{\alpha}w_1(t)$ *Taking into account (2.10) and (2.13) we have :*

$$\begin{aligned} D^{\alpha}\frac{\Gamma(2-\alpha)c_1}{\epsilon T}t &= \frac{\Gamma(2-\alpha)c_1}{\epsilon T}\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha} = \frac{c_1}{\epsilon T}t^{1-\alpha} \\ D^{\alpha}\Gamma(2-\alpha)c_1 &= \frac{c_1}{\epsilon T}\left(t^{1-\alpha}-(t-\epsilon T)^{1-\alpha}\right) \\ D^{\alpha}\frac{\Gamma(2-\alpha)c_1}{\epsilon T}(T-t) &= \frac{c_1}{\epsilon T}\left(t^{1-\alpha}-(t-\epsilon T)^{1-\alpha}-\left(t-(t-\epsilon T)^{1-\alpha}\right)\right), \end{aligned}$$

so the fractional derivative in the sense of Riemann-Liouville of the function $w_1(t)$ *is given by:*

$$D^{\alpha}w_1(t) = \begin{cases} \frac{c_1}{\epsilon T}t^{1-\alpha}, \ t\in[0,\epsilon T[\,, \\ \frac{c_1}{\epsilon T}\left(t^{1-\alpha}-(t-\epsilon T)^{1-\alpha}\right), \ t\in[\epsilon T,(1-\epsilon)T]\,, \\ \frac{c_1}{\epsilon T}\left(t^{1-\alpha}-(t-\epsilon T)^{1-\alpha}-\left(t-(t-\epsilon T)^{1-\alpha}\right)\right), \ t\in\,](1-\epsilon)T,T]\,. \end{cases} \tag{2.14}$$

Chapter 3

Existence of weak solutions for a new class of fractional p-Laplacian boundary value systems

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1- Introdution to the problem.

2- Definition and ratings.

3- Result of existence of at least three solutions.

4- Examples.

3.1 Introduction to the problem

In this chapter, at least three weak solutions were obtained for a new class of non-linear p-Laplace systems according to two parameters by using variational methods combined with a critical point theory due to Bonano and Marano. Some necessary definitions and preliminary facts are introduced for fractional calculus which are used to ensure the existence of three weak solutions for the following system:

$$\begin{cases} {}_tD_T^{\alpha_i}\left(\frac{1}{w_i(t)^{p-2}}\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}u_i\left(t\right)\right)\right)+\mu\left|u_i\left(t\right)\right|^{p-2}u_i\left(t\right) \\ \\ =\lambda F_{u_i}\left(t,u_1\left(t\right),u_2\left(t\right),...,u_n\left(t\right)\right)\ \text{ a.e. } t\in\left[0,T\right], \\ \\ u_i\left(0\right)=u_i\left(T\right)=0, \end{cases} \tag{3.1}$$

where

$$\phi_p\left(s\right)=\left|s\right|^{p-2}s,\ p>1,\ w_i\left(t\right)\in L^{\infty}\left[0,T\right],$$

with $w_i^0=ess\ \inf_{[0,T]}w_i\left(t\right)>0$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order $0<\alpha_i\leq 1$ respectively, for $1\leq i\leq n$, λ is positive parameter, and $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is measurable function with respect to $t\in[0,T]$ for every $(x_1,x_2,...,x_n)\in\mathbb{R}^n$ and are C^1 with respect to $(x_1,x_2,...,x_n)\in\mathbb{R}^n$.

For $t\in[0,T]$, F_{u_i} denote the partial derivative of F with respect to u_i, respectively,

(H_0) $\alpha_i\in(0;1]$ for $1\leq i\leq n$.

$(H1)$ $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a function such that $F\left(.,u_1,u_2,...,u_n\right)$ is continuous in $[0,T]$ for every $(u_1,u_2,...,u_n)\in\mathbb{R}^n$, $F\left(t,.,.,...,.\right)$ is a C^1 function in $\mathbb{R}^2$.

For $[0,T]\subseteq\mathbb{R}$, let $C\left([0,T],\mathbb{R}\right)$ be the real space of all continuous functions with norm $\|x\|_\infty=\max\limits_{t\in[0,T]}|x(t)|$,and $L^p\left([0,T],\mathbb{R}\right)(1\leq p<\infty)$ be the space of functions for which the p^{th} power of the absolute value is Lebesgue integrable with norm $\|x\|_{L^p}=\left(\int\limits_0^T|x(t)|^p\,dt\right)^{\frac{1}{p}}$.

3.2 Definitions and ratings

Definition 3.1 *[24]*

Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha_i > 0$ for a function u are defined by

$$
{}_aD_t^{\alpha_i} u(t) := \frac{d^n}{dt^n}\, {}_aD_t^{\alpha_i - n} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha_i-1} u(s)\, ds,
$$

and

$$
{}_tD_b^{\alpha_i} u(t) := (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\alpha_i - n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha_i-1} u(s)\, ds,
$$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \leq \alpha_i < n$ and $n \in \mathbb{N}^*$.

Here, $\Gamma(\alpha_i)$ is the standard gamma function given by

$$
\Gamma(\alpha_i) := \int_0^{+\infty} z^{\alpha_i - 1} e^{-z} dz.
$$

Setting $AC^n([a,b],\mathbb{R})$ the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC^n([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings being $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we denote $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Definition 3.2 *[28]*

Let $0 < \alpha_i \leq 1$, for $1 \leq i \leq n$, $1 < p < \infty$. The fractional derivative space

$$
E_{\alpha_i}^p = \left\{ u(t) \in L^p([0,T],\mathbb{R}) \;\; {}_aD_t^{\alpha_i} u(t) \in L^p([0,T],\mathbb{R}),\, u(0) = u(T) = 0 \right\},
$$

is a Banche space.

Then, for any $u \in E^p_{\alpha_i}$, we can define the weighted norm for $E^p_{\alpha_i}$ as

$$\|u\|_{\alpha_i} = \left(\int_0^T |u(t)|^p \, dt + \int_0^T w_i(t) \left| {}_a D_t^{\alpha_i} u(t) \right|^p dt \right)^{\frac{1}{p}}. \tag{3.2}$$

Multiplying (3.1) by any $v_i(t) \in E^p_{\alpha_i}$, and integrating, yields

$$\begin{cases} \int_0^T \sum_{i=1}^n \left({}_t D_T^{\alpha_i} \left(\frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \right) \right) v_i(t) \, dt + \mu \int_0^T \sum_{i=1t}^n |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt \\ = \lambda \int_0^T \sum_{i=1t}^n F_{u_i}(t, u_1(t), u_2(t), ..., u_n(t)) v_i(t) \, dt. \end{cases} \tag{3.3}$$

Then, combining Definition 3.1, Definition 2.4, Proposition 2.3 and Proposition 2.3, he left side of (3.3) can be transferred into

$$\begin{aligned} & \int_0^T \sum_{i=1}^n \left({}_t D_T^{\alpha_i} \left(\frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \right) \right) v_i(t) \, dt + \mu \int_0^T \sum_{i=1t}^n |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt \\ = & -\int_0^T v_i(t) \, d\left[{}_t D_T^{\alpha_i - 1} \left(\frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \right) \right] + \mu \int_0^T |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt \\ = & \int_0^T \frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \ {}_0 D_t^{\alpha_i - 1} v_i'(t) \, dt + \mu \int_0^T |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt \\ = & \int_0^T \frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \ {}_0^C D_t^{\alpha_i} v_i(t) \, dt + \mu \int_0^T |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt \\ = & \int_0^T \frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t) \ {}_0 D_t^{\alpha_i} u_i(t) \right) \ {}_0 D_t^{\alpha_i} v_i(t) \, dt + \mu \int_0^T |u_i(t)|^{p-2} u_i(t) v_i(t) \, dt. \end{aligned}$$

In what follows, we will give the definition of weak solution for (3.1), which is based on the discussion mentioned above.

Definition 3.3 *[13]*

We mean by a weak solution of system (3.1), any $u=(u_1,u_2,...,u_n)\in X$ *such that for all* $v=(v_1,v_2,...,v_n)\in X$

$$\begin{aligned}&\int_0^T\sum_{i=1}^n\frac{1}{w_i(t)^{p-2}}\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right){}_0D_t^{\alpha_i}v_i(t)\,dt\\&+\mu\int_0^T\sum_{i=1}^n|u_i(t)|^{p-2}u_i(t)\,v_i(t)\,dt\\&-\lambda\int_0^T\sum_{i=1}^n F_{u_i}\left(t,u_1(t),u_2(t),...,u_n(t)\right)v_i(t)\,dt=0.\end{aligned}$$

Lemma 3.1 *[13]*

Let $0<\alpha_i\le 1$, *for* $1\le i\le n$, $1<p<\infty$. *For any* $u\in E^p_{\alpha_i}$ *we have*

$$\|u_i\|_{L^p}\le\frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\|{}_0D_t^{\alpha_i}u_i\|_{L^p},$$

moreover, if $\alpha_i>\dfrac{1}{p}$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$, *then*

$$\|u_i\|_\infty\le\frac{T^{\alpha_i}}{\Gamma(\alpha_i)\left((\alpha_i-1)q+1\right)^{\frac{1}{q}}}\|{}_0D_t^{\alpha_i}u_i\|_{L^p}. \tag{3.4}$$

From Lemma 3.1, *we easily observe that*

$$\|u_i\|_{L^p}\le\frac{T^{\alpha_i}\left(\int_0^T w_i(t)\,|{}_0D_t^{\alpha_i}u(t)|^p\,dt\right)^{1/p}}{\Gamma(\alpha_i+1)}, \tag{3.5}$$

for $0<\alpha_i\le 1$, *and*

$$\|u_i\|_\infty\le\frac{T^{\alpha_i-\frac{1}{p}}\left(\int_0^T w_i(t)\,|{}_0D_t^{\alpha_i}u(t)|^p\,dt\right)^{1/p}}{\Gamma(\alpha_i)\,(w_i^0)^{\frac{1}{p}}\left((\alpha_i-1)q+1\right)^{\frac{1}{q}}}, \tag{3.6}$$

for $\alpha_i>\dfrac{1}{p}$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$.

By using (3.5), the norm of (3.2) is equivalent to

$$\|u\|_{\alpha_i} = \left(\int_0^T w_i(t) \left| {}_aD_t^{\alpha_i} u\,(t) \right|^p dt \right)^{\frac{1}{p}}, \ \forall u \in E_{\alpha_i}^p. \tag{3.7}$$

Throughout this paper, we let X be the Cartesian product of the n spaces $E_{\alpha_i}^p$ for $1 \le i \le n$, i.e, $X = E_{\alpha_1}^p \times E_{\alpha_2}^p \times ... \times E_{\alpha_n}^p$ equipped with the norm

$$\|u\| = \sum_{i=1}^n \|u_i\|_{E_{\alpha_i}^p}, \ u = (u_1, u_2, ..., u_n),$$

where $\|u_i\|_{E_{\alpha_i}^p}$ is defined in (3.7).

Lemma 3.2 *[28]*

For $0 < \alpha_i \le 1$ and $1 < p < \infty$, the fractional derivative space X is a reflexive separable Banach space.

Lemma 3.3 *[51]*

Let $A : X \to X^*$be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Assume $\{w_1, w_2 ...\}$is a basis in X. Then the following assertion holds: (d) Inverse operator. If A is strictly monotone, then the inverse operator $A^{-1} : X^* \to X$ exists. This operator is strictly monotone, demicontinuous and bounded. If A is uniformly monotone, then A^{-1} is continuous. If A is strongly monotone, then it is Lipschitz continuous.

3.3 Result of existence of at least three solution

In the present section ,the existence of multiple solutions for system (3.1) is examined by using Theorem 1.1. First and foremost, we define the functionals $\Phi, \Psi : X \to \mathbb{R}$ as

$$\Phi(u) = \frac{1}{p} \int_0^T \sum_{i=1}^n \left(w_i(t) \left| {}_0D_t^{\alpha_i} u_i(t) \right|^p + \mu \left| u_i(t) \right|^p \right) dt, \tag{3.8}$$

$$u = (u_1, u_2, ..., u_n) \in X,$$

and

$$\Psi(u)=\int_0^T F(t,u_1(t),u_2(t),...,u_n(t))\,dt. \tag{3.9}$$

Lemma 3.4 *[28]*

Let $0<\alpha_i\leq 1$, $u=(u_1,u_2,...,u_n)\in X$. Functionals Φ and Ψ are defined in (3.8) and (3.9). Then, $\Phi:X\to\mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functionl whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi:X\to\mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Proof For each $u=(u_1,u_2,...,u_n)\in X$, define $\Phi,\ \Psi:X\to\mathbb{R}$ as

$$\Phi(u)=\frac{1}{p}\int_0^T\sum_{i=1}^{n}\left(w_i(t)\left|{}_0D_t^{\alpha_i}u_i(t)\right|^p+\mu\left|u_i(t)\right|^p\right)dt,$$

and

$$\Psi(u)=\int_0^T F(t,u_1(t),u_2(t),...,u_n(t))\,dt.$$

Clearly, Φ and Ψ are continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u\in X$ are given by

$$\begin{aligned}\Phi'(u)(v)=&\int_0^T\sum_{i=1}^{n}\tfrac{1}{w_i(t)^{p-2}}\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right)\ {}_0D_t^{\alpha_i}v_i(t)\,dt\\ &+\mu\int_0^T\sum_{i=1}^{n}\left|u_i(t)\right|^{p-2}u_i(t)\,v_i(t)\,dt,\end{aligned} \tag{3.10}$$

for every $v=(v_1,v_2,...,v_n)\in X$.

In addition, according to (3.8), one has $\Phi(u)\geq\frac{1}{p}\ \|u\|_X^p$, which means that Φ is a coercive functional. Next, we claim that Φ' admits a continuous inverse on X^*.

Let $u=(u_1,u_2,...,u_n)\in X, v=(v_1,v_2,...,v_n)\in X$. Recalling (3.10), we get

$$\begin{aligned}\left\langle \Phi'(u)-\Phi'(v),u-v\right\rangle =&\int_0^T\sum_{i=1}^n \tfrac{1}{w_i(t)^{p-2}}\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right)\ {}_0D_t^{\alpha_i}(u-v)\,dt\\ &+\mu\int_0^T\sum_{i=1}^n\left|u_i(t)\right|^{p-2}u_i(t)(u-v)\,dt\\ &-\int_0^T\sum_{i=1}^n \tfrac{1}{w_i(t)^{p-2}}\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right)\ {}_0D_t^{\alpha_i}(u-v)\,dt\\ &+\mu\int_0^T\sum_{i=1}^n\left|v_i(t)\right|^{p-2}v_i(t)(u-v)\,dt.\end{aligned}\tag{3.11}$$

According to the well-known inequality

$$\begin{aligned}&\left(|s_1|^{p-2}s_1-|s_2|^{p-2}s_2\right)(s_1-s_2)\\ &\geq\begin{cases}|s_1-s_2|^p, & p\geq 2\\ \frac{|s_1-s_2|^2}{(|s_1|+|s_2|)^{2-p}}, & 1<p\leq 2.\end{cases}\end{aligned}\tag{3.12}$$

We have

$$\begin{aligned}&\left(\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right)-\phi_p\left(w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right)\right)\\ &\geq\begin{cases}\frac{1}{w_i(t)}\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)-w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right|^p, & p\geq 2\\ \frac{1}{w_i(t)}\frac{\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)-w_i(t)\ {}_0D_t^{\alpha_i}v(t)\right|^2}{\left(\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right|+\left|w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right|\right)^{2-p}}, & 1<p<2.\end{cases}\end{aligned}$$

Hence, when $1<p<2$, one has

$$\begin{aligned}&\int_0^T\sum_{i=1}^n\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)-w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right|^p dt\\ &\leq\left(\int_0^T\sum_{i=1}^n\frac{\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)-w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right|^2}{w_i(t)\left(\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right|+\left|w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right|\right)^{2-p}}dt\right)^{\frac{p}{2}}\\ &\left(\int_0^T\sum_{i=1}^n w_i(t)^{\frac{p}{2-p}}\left(\left|w_i(t)\ {}_0D_t^{\alpha_i}u_i(t)\right|+\left|w_i(t)\ {}_0D_t^{\alpha_i}v_i(t)\right|\right)^p dt\right)^{\frac{2-p}{2}},\end{aligned}\tag{3.13}$$

which means that

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \frac{\left|w_i(t)_0D_t^{\alpha_i}u_i(t)-w_i(t)_0D_t^{\alpha_i}v_i(t)\right|^2}{w_i(t)\left(\left|w_i(t)_0D_t^{\alpha_i}u_i(t)\right|+\left|w_i(t)_0D_t^{\alpha_i}v_i(t)\right|\right)^{2-p}}dt\\ &\geq \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\widetilde{w_1^0}} \left\|u_i-v_i\right\|_{\alpha_i}^2 \left(\left\|u_i\right\|_{\alpha_i}^p+\left\|v_i\right\|_{\alpha_i}^p\right)^{\frac{p-2}{p}}.\end{aligned} \tag{3.14}$$

Then, we deduce

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}v_i\left(t\right)\right)\ {}_0D_t^{\alpha_i}\left(u-v\right)\right)dt\\ &\geq \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\widetilde{w_1^0}} \left\|u_i-v_i\right\|_{\alpha_i}^2 \left(\left\|u_i\right\|_{\alpha_i}^p+\left\|v_i\right\|_{\alpha_i}^p\right)^{\frac{p-2}{p}} > 0.\end{aligned} \tag{3.15}$$

When $p \geq 2$, we get

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}v_i\left(t\right)\right)\ {}_0D_t^{\alpha_i}\left(u-v\right)\right)dt\\ &\geq \left(w_1^0\right)^{p-2}\left\|u_i-v_i\right\|_{\alpha_i}^p > 0.\end{aligned} \tag{3.16}$$

Then, combining with (3.15), yields

$$\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)\ {}_0D_t^{\alpha_i}v_i\left(t\right)\right)\left({}_0D_t^{\alpha_i}u_i-\ {}_0D_t^{\alpha_i}v_i\right)\right)dt > 0. \tag{3.17}$$

For every $1 < p < \infty$

Further, denote

$$A=\int_0^T \sum_{i=1}^n \left|u_i\left(t\right)\right|^{p-2}u_i\left(t\right)\left(u-v\right)dt+\int_0^T \sum_{i=1}^n \left|v_i\left(t\right)\right|^{p-2}v_i\left(t\right)\left(u-v\right)dt.$$

Then, reapplying inequality (3.12), we always have

$$A \geq \left\|u_i-v_i\right\|_{\alpha_i}^p > 0, \quad \text{for } p \geq 2$$

and

$$A \geq 2^{p-2}\left\|u_i-v_i\right\|_{L^p}^2\left(\left\|u_i\right\|_{L^p}^p+\left\|v_i\right\|_{L^p}^p\right)^{\frac{p-2}{p}} > 0, \text{ for } 1 < p < 2.$$

That is, $A > 0$ for every $1 < p < \infty$. Therefore, by using (3.11) and (3.17), the following inequality holds

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle > 0,$$

which means that Φ' is strictly monotone. Furthermore, in view of X being reflexive, for $u_n \to u$ in X strongly, as $n \to \infty$, one has $\Phi'(u_n) \rightharpoonup \Phi'(u)$ in X^* as $n \to \infty$.

Thus, we say that Φ' is demicontinuous. Then, according to lemma 3.2 and lemma 3.3, we obtain that the inverse operator $(\Phi')^{-1}$ of Φ' exists and is continuous.

Moreover, let

$$\|u\|_{\mu,\alpha_i}^p = \int_0^T \sum_{i=1}^n \left(w_i(t) \left| {}_0D_t^{\alpha_i} u_i(t) \right|^p + \mu \left| u_i(t) \right|^p \right) dt,$$

owing to the sequentially weakly lower semicontinuity of $\|u\|_{\mu,\alpha_i}^p$ we observe that Φ is sequentially weakly lower semicontinuous in X.

Considering the functional Ψ, we will point out that Ψ is a Gâteaux differentiable, sequentially weakly upper semicontinuous functional on X.

Indeed, for $u_n \subset X$, assume that $u_n \rightharpoonup u$ in X, i.e u_n uniformly converges to u on $[0, T]$ as $n \to \infty$. By using Fatou's lemma, one has

$$\begin{aligned} \lim_{n\to+\infty} \inf \ \Psi(u_n) &\leq \int_0^T \lim_{n\to+\infty} \inf F(t, u_n(t))\, dt \\ &= \int_0^T F(t, u_1(t), u_2(t), ..., u_n(t))\, dt = \Psi(u), \end{aligned}$$

whereas $u = (u_1, u_2, ..., u_n) \in X$, which implies that Ψ is sequentially weakly upper semicontinuous. Furthermore, since F is continuously differentiable with respect to u_i for almost every $t \in [0, T]$, then based on the Lebesgue control convergence theorem, we obtain that $\Psi'(u_n) \to \Psi'(u)$ strongly , that is Ψ' is strongly continuous on X. Hence, we confirm that Ψ' is a compact operator.

Moreover, it is easy to prove that the functional with the Gâteaux derivative $\Psi'(u) \in X^*$ at the point $u \in X$

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), u_2(t), ..., u_n(t)) v_i(t) \, dt, \tag{3.18}$$

for any $v = (v_1, v_2, ..., v_n) \in X$.The proof is completed.

In order to facilitate the proof of our main result, some notations are given.

Putting

$$\begin{aligned} k &: = \max_{1 \le i \le n} \left\{ \frac{T^{p\alpha_i - 1}}{(\Gamma(\alpha_i))^p w_i^0 ((\alpha_i - 1) q + 1)^{\frac{p}{q}}} \right\}, \\ \widetilde{k} &: = \max_{1 \le i \le n} \left\{ \frac{T^{p\alpha_i}}{(\Gamma(\alpha_i + 1))^p w_i^0} \right\}. \end{aligned}$$

Define

$$\pi(\sigma) = \left\{ u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n : \frac{1}{p} \sum_{i=1}^n |u_i|^p < \sigma \right\}.$$

Theorem 3.1 *Let $\frac{1}{p} < \alpha_i \le 1$, for $1 \le i \le n$. Assume that there exists a positive constant r and a function $u = (u_1, u_2, ..., u_n) \in X$ such that*

(i)

$$\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \mu \sum_{i=1}^n \|u_i\|_{L^p}^p > pr;$$

(ii)

$$\frac{\int_0^T \sup_{u \in \pi(kr)} F(t, u_1, u_2, ..., u_n) \, dt}{r} < \frac{p \int_0^T F(t, u_1, u_2, ..., u_n) \, dt}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \mu \sum_{i=1}^n \|u_i\|_{L^p}^p},$$

(iii)

$$\liminf_{|u| \to \infty} \frac{F(t, u_1, u_2, ..., u_n)}{\sum_{i=1}^n |u_i|^p} < \frac{\int_0^T \sup_{u \in \pi(kr)} F(t, u_1, u_2, ..., u_n) \, dt}{pr\widetilde{k}}.$$

Then, setting

$$\Lambda = \left] \frac{\sum\limits_{i=1}^{n} \|u_i\|_{\alpha_i}^{p} + \mu \sum\limits_{i=1}^{n} \|u_i\|_{L^p}^{p}}{p \int\limits_{0}^{T} F(t, u_1, u_2, ..., u_n)\, dt}, \frac{r}{\int\limits_{0}^{T} \sup\limits_{u \in \pi(kr)} F(t, u_1, u_2, ..., u_n)\, dt} \right[$$

for each $\lambda \in \Lambda$ system (3.1) admits at least three weak solutions in X.

Proof Considering Theorem 1.1 and lemma 3.4, in order to obtain that system (3.1) possesses at least three weak solutions in X, we only need to guarantee the assumptions (a_1) and (a_2) of Theorem 1.1 are satisfied. Choose $u_0 = (u_{01}, u_{02}, ..., u_{0n})$ and $u_1 = (u_{11}, u_{12}, ..., u_{1n})$ with $(u_{01}, u_{02}, ..., u_{0n}) = (0, 0, ..., 0)$. Due to (3.9) and (i), we get $\Psi(u_0) = 0$ and $\Phi(u_1) > r > 0$, which satisfy the requirement of Theorem 1.1. Then, combining (3.8) and (3.5), yields

$$\begin{aligned}
& \{u = (u_1, u_2, ..., u_n) \in X : \Phi(u) \leq r\} \\
= & \left\{ u = (u_1, u_2, ..., u_n) \in X : \frac{1}{p} \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^{p} + \frac{\mu}{p} \sum_{i=1}^{n} \|u_i\|_{L^p}^{p} \leq r \right\} \\
\subseteq & \left\{ u = (u_1, u_2, ..., u_n) \in X : \frac{1}{p} \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^{p} \leq r \right\} \\
\subseteq & \left\{ u = (u_1, u_2, ..., u_n) \in X : \sum_{i=1}^{n} \frac{(\Gamma(\alpha_i))^p\, w_1^0((\alpha_i - 1)\, q + 1)^{\frac{p}{q}}}{pT^{p\alpha_i - 1}} \|u_i\|_{\infty} \leq r \right\} \\
\subseteq & \left\{ u = (u_1, u_2, ..., u_n) \in X : \sum_{i=1}^{n} |u_i|^p \leq kpr \right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\sup_{\Phi(u) \leq r} \Psi(u) &= \sup_{\Phi(u) \leq r} \int_{0}^{T} F(t, u_1, u_2, ..., u_n)\, dt \\
&\leq \int_{0}^{T} \sup_{u \in \pi(kr)} F(t, u_1, u_2, ..., u_n)\, dt.
\end{aligned}$$

Then, the following inequality is obtained under condition (ii)

$$\sup_{\Phi(u)\leq r} \Psi(u) \leq \frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}{r} \leq \frac{p\int_0^T \sum_{i=1}^{n} F(t,u_1,u_2,...,u_n)\,dt}{\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p + \mu\sum_{i=1}^{n}\|u_i\|_{L^p}^p} = \frac{\Psi(u_1)}{\Phi(u_1)}.$$

Thus the hypothesis (a_1) of Theorem 1.1 holds.

On the other hand, taking (iii) into account, there exist constants $C,\varepsilon\in\mathbb{R}$ with

$$C < \frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}{r},$$

such that

$$F(t,u_1,u_2,...,u_n) \leq \frac{C}{p\widetilde{k}}\sum_{i=1}^{n}|u_i|^p+\varepsilon \tag{3.19}$$

for any $t\in[0,T]$ and $u=(u_1,u_2,...,u_n)\in X$, when $C>0$ by using (3.8), (3.19) and (3.4) yields

$$\begin{aligned}
\Phi(u)-\lambda\Psi(u) &= \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p + \frac{\mu}{p}\sum_{i=1}^{n}\|u_i\|_{L^p}^p - \lambda\int_0^T F(t,u_1,u_2,...,u_n)\,dt\\
&\geq \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p - \lambda\int_0^T F(t,u_1,u_2,...,u_n)\,dt\\
&\geq \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p\widetilde{k}}\int_0^T\sum_{i=1}^{n}\|u_i\|_{L^p}^p\,dt - \lambda T\varepsilon\\
&\geq \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p\widetilde{k}}\left(\sum_{i=1}^{n}\frac{T^{\alpha_i}}{(\Gamma(\alpha_i+1))^p\, w_i^0}\|u_i\|_{\alpha_i}^p\right) - \lambda T\varepsilon\\
&\geq \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p - \lambda T\varepsilon\\
&\geq \left(\frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p\right)\left(1-C\frac{r}{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}\right) - \lambda T\varepsilon.
\end{aligned}$$

That is

$$\lim_{\|u\|_X \to +\infty} \Phi(u) - \lambda \Psi(u) = +\infty.$$

Furthermore, analogous to the case of $C > 0$, we can deduce that $\Phi(u) - \lambda\Psi(u) \to +\infty$ as $\|u\|_X \to +\infty$ with $C \leq 0$. Hence, all the hypotheses of Theorem 1.1 hold, then, system (3.1) admits at least three weak solutions in X. The proof is completed.

For simplicity, before giving a corollary of Theorem 3.1, some notations are presented.

Let $0 < h < \frac{1}{2}$ we put

$$\begin{aligned} A_i(\alpha_i, h) &= \frac{1}{(hT)}\left[\int_0^{hT}\sum_{i=1}^{n} w_i(t)\, t^{(1-\alpha_i)p}\,dt + \int_{hT}^{(1-h)T}\sum_{i=1}^{n} w_i(t)\left[t^{1-\alpha_i} - (t-hT)^{1-\alpha_i}\right]^p dt\right. \\ &\quad \left. + \int_{(1-h)T}^{T}\sum_{i=1}^{n} w_i(t)\left[t^{1-\alpha_i} - (t-hT)^{1-\alpha_i} - (t-((1-h)T))^{1-\alpha_i}\right]^p dt.\right] \end{aligned} \tag{3.20}$$

Corollary 3.1 *Let $\frac{1}{p} < \alpha_i \leq 1$. Assume that there exist $\tau > 0$ and $\theta = (\theta_1, \theta_2, ..., \theta_n) \in \mathbb{R}^n$ with $\theta_1 > 0, \theta_2 > 0, ..., \theta_n > 0$ and $\tau \leq k\,\frac{\sum_{i=1}^{n} A_i(\alpha_i,h)\theta_i^p}{p}$, such that*

$(i)'$

$$\begin{aligned} F(t, u_1, u_2, ..., u_n) &\geq 0 \text{ for} \\ (t, u_1, u_2, ..., u_n) &\in ([0, hT] \cup [(1-h)T, T] \times [0, \theta]) \end{aligned}$$

$(ii)'$

$$\frac{\int_0^T \sup_{u\in\pi(kr)} F(t, u_1, u_2, ..., u_n)\,dt}{r} < \frac{p\int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\theta_1, \Gamma(2-\alpha_2)\theta_2, ..., \Gamma(2-\alpha_n)\theta_n)\,dt}{k\left(1+\mu\widetilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i, h)\,\theta_i^p},$$

(iii)'

$$\liminf_{|u|\to\infty} \frac{F(t,u_1,u_2,...,u_n)}{\sum_{i=1}^{n}|u_i|^p} \leq 0$$

for each

$$\lambda \in \Lambda' = \left] \frac{\left(1+\mu\widetilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i,h)\,\theta_i^p}{p\int_{hT}^{(1-h)T} F\left(t,\Gamma(2-\alpha_1)\theta_1,\Gamma(2-\alpha_2)\theta_2,...,\Gamma(2-\alpha_n)\theta_n\right)dt}, \frac{\tau}{k\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt} \right[, \tag{3.21}$$

thus, system (3.1) admits at least three weak solutions in X.

Proof Choose

$$U_i(t) = \begin{cases} \dfrac{\Gamma(2-\alpha_i)\theta_i}{hT}t, & t\in[0,hT[\,, \\ \Gamma(2-\alpha_i)\theta_i\ , & t\in[hT,(1-h)T]\,, \\ \dfrac{\Gamma(2-\alpha_i)\theta_i}{hT}(T-t\ ), & t\in](1-h)T,T]\,, \end{cases}$$

obviously $U_i(0)=U_i(T)=0$, $U_i(t)\in L^p[0,T]$. Owing to Definition 3.1, we derive,

$${}_0D_t^{\alpha_i}U_i(t) = \begin{cases} a_1(t)\,,\ t\in[0,hT[\,, \\ a_2(t)\,,\ t\in[hT,(1-h)T]\,, \\ a_3(t)\,,\ t\in](1-h)T,T]\,, \end{cases}$$

where

$$a_1(t) = \frac{\theta_i}{hT}t^{1-\alpha_i},\ a_2(t) = \frac{\theta_i}{hT}\left[t^{1-\alpha_i} - (t-(hT)^{1-\alpha_i}\right],$$

and

$$a_3(t) = \frac{\theta_i}{hT}\left[t^{1-\alpha_i} - (t-(hT)^{1-\alpha_i} - (t-(T-hT)^{1-\alpha_i}\right].$$

That is

$$\begin{aligned}
\|U\|_{\alpha_i}^{p} &= \int_0^T \sum_{i=1}^{n} w_i(t)\left|{}_0D_t^{\alpha_i}U_i(t)\right|^p dt \\
&= \int_0^{hT} \sum_{i=1}^{n} w_i(t)\left|{}_0D_t^{\alpha_i}U_i(t)\right|^p dt + \int_{hT}^{(1-h)T} \sum_{i=1}^{n} w_i(t)\left|{}_0D_t^{\alpha_i}U_i(t)\right|^p dt + \int_{(1-h)T}^{T} \sum_{i=1}^{n} w_i(t)\left|{}_0D_t^{\alpha_i}U_i(t)\right|^p dt \\
&= \sum_{i=1}^{n} A_i(\alpha_i,h)\,\theta_i^p,
\end{aligned}$$

where (3.20) is used. Hence $U=(U_1,U_2,...,U_n)\in X$.

Take $r=\dfrac{\tau}{k}$, then

$$\begin{aligned}
rk &= \tau \le k\,\frac{\sum_{i=1}^{n} A_i(\alpha_i,h)\,\theta_i^p}{p} \\
&= k\,\frac{\sum_{i=1}^{n}\|U_i\|_{\alpha_i}^{p}}{p} \le k\Phi(U),
\end{aligned}$$

for every $U=(U_1,U_2,...,U_n)\in X$.

Which means that

$$r \le \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^{p} + \frac{\mu}{p}\sum_{i=1}^{n}\|u_i\|_{L^p}^{p}.$$

Thus, the assumption (ii) of Theorem 3.1 holds.

On the other hand, based on (3.2) and (3.20), yields

$$\begin{aligned}
\Phi(U) &\le \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^{p} + \frac{\mu}{p}\sum_{i=1}^{n}\frac{T^{\alpha_i}}{(\Gamma(\alpha_i+1))^p\,w_i^0}\|u_i\|_{\alpha_i}^{p} \\
&\le \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^{p} + \frac{\mu}{p}\widetilde{k}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^{p} \\
&\le \frac{\left(1+\mu\widetilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i,h)\,\theta_i^p}{p}.
\end{aligned} \tag{3.22}$$

Then, from (3.22) and $(ii)'$, we can obtain the following inequality

$$\begin{aligned}
\frac{\int_0^T \sup_{u\in\pi(kr)} F\left(t,u_1,u_2,...,u_n\right)dt}{r} &= \frac{k\int_0^T \sup_{u\in\pi(kr)} F\left(t,u_1,u_2,...,u_n\right)dt}{\tau} \\
&< \frac{kp\int_{hT}^{(1-h)T} F\left(t,\Gamma\left(2-\alpha_1\right)\theta_1,\Gamma\left(2-\alpha_2\right)\theta_2,...,\Gamma\left(2-\alpha_n\right)\theta_n\right)dt}{k\left(1+\mu\widetilde{k}\right)\sum_{i=1}^{n} A_i\left(\alpha_i,h\right)\theta_i^p} \\
&\leq \frac{\int_{hT}^{(1-h)T} F\left(t,\Gamma\left(2-\alpha_1\right)\theta_1,\Gamma\left(2-\alpha_2\right)\theta_2,...,\Gamma\left(2-\alpha_n\right)\theta_n\right)dt}{\Phi\left(U\right)} \\
&\leq \frac{k\int_0^T F\left(t,u_1,u_2,...,u_n\right)dt}{\sum_{i=1}^{n}\left\|u_i\right\|_{\alpha_i}^p+\mu\sum_{i=1}^{n}\left\|u_i\right\|_{L^p}^p},
\end{aligned}$$

which means that the hypothesis (ii) of Theorem 3.1 is satisfied.

Furthermore, the condition (iii) of Theorem 3.1 holds under $(iii)'$ since $\Lambda' \subseteq \Lambda$ Theorem 3.1 is successfully employed to ensure the existence of at least three weak solutions for system (3.1), the proof is completed. ■

.

3.4 Examples

Now, we give the following two examples to illustrate the applications of our result.

Example 3.1 *Let* $p = 2$, $\alpha_1 = 0.8$, $\alpha_2 = 0.65$, $\mu = 1$, $w_1(t) = 1 + t^2$, $w_2(t) = 0.5 + t$, $T = 1$.

Then, system (3.1) gets the following form

$$\begin{cases} {}_tD_1^{0.8}\left((1+t^2)\ {}_0D_t^{0.8}u_1(t)\right) + u_1(t) = \lambda F_{u_1}(t, u_1(t), u_2(t)),\ t \in [0,1], \\ \\ {}_tD_1^{0.65}\left((0.5+t)\ {}_0D_t^{0.65}u_2(t)\right) + u_2(t) = \lambda F_{u_2}(t, u_1(t), u_2(t)),\ t \in [0,1], \\ \\ u_1(o) = u_1(1) = 0, u_2(o) = u_2(1) = 0. \end{cases}$$

Taking

$$U_1(t) = \Gamma(1.2)\, t(1-t), U_2(t) = \Gamma(1.35)\, t(1-t),$$

and

$$F(t, u_1(t), u_2(t)) = \left(1+t^2\right) G(u_1, u_2),$$

where

$$G(u_1,u_2) = \begin{cases} \left(u_1^2+u_2^2\right)^2,\ \ u_1^2+u_2^2 \le 1\ , \\ \\ 10\left(u_1^2+u_2^2\right)^{\frac{1}{2}} - 9\left(u_1^2+u_2^2\right)^{\frac{1}{3}},\ u_1^2+u_2^2 > 1. \end{cases}$$

Clearly, $F(t,0,0) = 0$, $w_1^0 = 1$ *and* $w_2^0 = 0.5$ *for any* $t \in [0,1]$.

By the direct calculation, we have

$$\max\left\{\frac{1}{(\Gamma(0.8))^2(2\times 0.8-1)}, \frac{1}{(\Gamma(0.65))^2 \times 0.5\,(2\times 0.65-1)}\right\} = k \approx 3.4764,$$

$$\max\left\{\frac{1}{(\Gamma(0.8+1))^2}, \frac{1}{(\Gamma(0.65+1))^2\times 0.5}\right\} = \tilde{k} \approx 2.4684,$$

and

$$\begin{aligned} {}_0D_t^{0.8}U_1(t) &= t^{0.2} - \frac{2\Gamma(1.2)}{\Gamma(2.2)} t^{1.2}, \\ {}_0D_t^{0.65}U_2(t) &= t^{0.35} - \frac{2\Gamma(1.35)}{\Gamma(2.35)} t^{1.35}. \end{aligned}$$

So that

$$\begin{aligned} \|U_1(t)\|_{0.8}^2 &\approx 0.19333, \|U_2(t)\|_{0.65}^2 \approx 0.078559 \\ \|U_1(t)\|_{L^2}^2 &\approx 0.028101, \|U_2(t)\|_{L^2}^2 \approx 0.026716. \end{aligned}$$

Take $r = 1 \times 10^{-4}$. *We easily obtain that*

$$\frac{1}{2}\left(\|U_1(t)\|_{0.8}^2 + \|U_2(t)\|_{0.65}^2\right) + \frac{1}{2}\left(\|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2\right) \approx 0.1632 > r,$$

which implies that the condition (i) *holds, and*

$$\begin{aligned} \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{r} &= \frac{16k^2 r}{3} \approx 0.006445 \\ &< \frac{2\int_0^1 F(t,U_1,U_2)\,dt}{\sum_{i=1}^{2}\|U_i\|_{\alpha_i}^2 + \sum_{i=1}^{2}\|U_i\|_{L^2}^2} \approx 0.0320085949, \end{aligned}$$

and

$$\begin{aligned} 0 = \lim_{|u_1|\to\infty,\,|u_2|\to\infty} \inf \frac{F(t,u_1,u_2)}{|u_1|^2+|u_2|^2} &< \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{2r\tilde{k}} \\ &\approx 0.001305, \end{aligned}$$

thus, conditions (ii) *and* (iii) *are satisfied. Then, in view of Theorem 3.1 for each* $\lambda \in]31.241, 155.159[$, *the system (3.1) has at least three weak solutions in* X.

Example 3.2 *Let $p=3,\ \alpha_1=0.8,\ \alpha_2=0.6,\ \mu=1,\ w_1(t)=1+t^2,\ w_2(t)=0.5+t$ and $T=1$. Then , system (3.1) gets the following form*

$$\begin{cases} {}_tD_1^{0.8}\left((1+t^2)\left({}_0D_t^{0.8}u_1(t)\right)\left|{}_0D_t^{0.8}u_1(t)\right|\right)+\left|u_1(t)\right|u_1(t)=\lambda F_{u_1}\left(t,u_1(t),u_2(t)\right),\ t\in[0,1], \\ \\ {}_tD_1^{0.6}\left((0.5+t)\left({}_0D_t^{0.6}u_2(t)\right)\left|{}_0D_t^{0.6}u_2(t)\right|\right)+\left|u_2(t)\right|u_2(t)=\lambda F_{u_2}\left(t,u_1(t),u_2(t)\right),\ t\in[0,1], \\ \\ u_1(0)=u_1(1)=0, u_2(0)=u_2(1)=0. \end{cases}$$

Taking

$$U_1(t)=\Gamma(1.2)\,t\,(1-t)\,, U_2(t)=\Gamma(1.4)\,t\,(1-t)\,,$$

and

$$F\left(t,u_1(t),u_2(t)\right)=(1+t)\,H\left(u_1,u_2\right),$$

where

$$H\left(u_1,u_2\right)=\begin{cases}\left(u_1^3+u_2^3\right)^2,\ u_1^3+u_2^3\leq 1, \\ \\ 10\left(u_1^3+u_2^3\right)^{\frac{1}{2}}-9\left(u_1^3+u_2^3\right)^{\frac{1}{3}},\ u_1^3+u_2^3>1.\end{cases}$$

Clearly, $F(t,0,0)=0,\ w_1^0=1$ and $w_2^0=0.5$ for any $t\in[0,1]$.

By the direct calculation, we have

$$\max\left\{\frac{1}{\left(\Gamma(0.8)\right)^3\left((0.8-1)\times 1.5+1\right)^2},\frac{1}{\left((\Gamma(0.6)\right)^3(0.6-1)\times 1.5+1)^2\times 0.5}\right\}=k\approx 3.7849,$$

$$\max\left\{\frac{1}{\left(\Gamma(0.8+1)\right)^3},\frac{1}{\left(\Gamma(0.6+1)\right)^3\times 0.5}\right\}=\tilde{k}\approx 2.803,$$

and

$$\begin{aligned} \left|{}_0D_t^{0.8}U_1(t)\right| &= \begin{cases} t^{0.2}-\frac{2\Gamma(1.2)}{\Gamma(2.2)}t^{1.2},\ t\in[0,0.6] \\ -\left(t^{0.2}-\frac{2\Gamma(1.2)}{\Gamma(2.2)}t^{1.2}\right),\ t\in(0.6,1], \end{cases} \\ \left|{}_0D_t^{0.6}U_2(t)\right| &= \begin{cases} t^{0.4}-\frac{2\Gamma(1.35)}{\Gamma(2.4)}t^{1.4},\ t\in[0,0.7] \\ -\left(t^{0.4}-\frac{2\Gamma(1.35)}{\Gamma(2.35)}t^{1.4}\right),\ t\in(0.7,1]. \end{cases} \end{aligned}$$

So that

$$\begin{aligned} \|U_1(t)\|_{0.8}^3 &\approx 0.09228, \|U_2(t)\|_{0.6}^3 \approx 0.0212 \\ \|U_1(t)\|_{L^3}^3 &\approx 0.0053, \|U_2(t)\|_{L^3}^3 \approx 0.005. \end{aligned}$$

Take $r = 1 \times 10^{-5}$. *We easily obtain that*

$$\begin{aligned} & \frac{1}{3}\left(\|U_1(t)\|_{0.8}^3 + \|U_2(t)\|_{0.65}^3\right) + \frac{1}{3}\left(\|U_1(t)\|_{L^3}^3 + \|U_2(t)\|_{L^3}^3\right) \\ \approx \ & 0.041 > r, \end{aligned}$$

which implies that the condition (i) *holds, and*

$$\begin{aligned} \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F\left(t, u_1, u_2\right) dt}{r} &= \frac{27k^2 r}{2} \approx 0.00193 \\ &< \frac{3\int_0^1 F\left(t, U_1, U_2\right) dt}{\sum_{i=1}^{2} \|U_i\|_{\alpha_i}^3 + \sum_{i=1}^{2} \|U_i\|_{L^3}^3} \approx 0.00656, \end{aligned}$$

and

$$0 = \lim_{u_1\to+\infty, u_2\to+\infty} \inf \frac{F\left(t, u_1, u_2\right)}{u_1^3 + u_2^3} < \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F\left(t, u_1, u_2\right) dt}{3r\tilde{k}} \approx 0.00023,$$

thus, conditions (ii) *and* (iii) *are satisfied. Then, in view of Theorem* 3.1 *for each* $\lambda \in\,]152.43, 518.13[$, *the system (3.1) admits at least three weak solutions in* X.

Chapter 4

Existence of three solutions for perturbed nonlinear fractional p-Laplacian boundary value systems with two control parameters

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1- Introdution to the problem.

2- Preliminary results.

3- Result of existence at least three solutions.

4- Examples..

4.1 Introduction to the problem

In this chapter, we use two control parameters to study a class of perturbed nonlinear fractional p-Laplacian differential systems, where we prove the existence of three weak solutions by using the variational method and Ricceri's critical points theorems respecting some necessary conditions on the primitive function of nonlinear terms F_u and F_v. To apply critical point theory to explore the existence of weak solutions for the following perturbed fractional differential system:

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\right)+\mu\left|u(t)\right|^{p-2}u(t) \\ =\lambda F_u\left(t,u(t),v(t)\right)+\delta G_u\left(t,u(t),v(t)\right) \text{ a.e. } t\in[0,T], \\ {}_tD_T^{\beta}\left(\frac{1}{w_2(t)^{p-2}}\Phi_p\left(w_2(t)\ {}_0D_t^{\beta}v(t)\right)\right)+\mu\left|v(t)\right|^{p-2}v(t) \\ =\lambda F_v\left(t,u(t),v(t)\right)+\delta G_v\left(t,u(t),v(t)\right) \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0, \quad v(0)=v(T)=0, \end{cases} \tag{4.1}$$

where λ,μ,δ are positive real parameters, $\alpha,\beta\in(0;1]$, ${}_0D_t^{\alpha}$, ${}_tD_T^{\alpha}$ and ${}_0D_t^{\beta}$, ${}_tD_T^{\beta}$ are the left and right Riemann-Liouville fractional derivatives of order α,β respectively.$\Phi_p(s)=|s|^{p-2}s, p>1, w_1(t), w_2(t)\in L^{\infty}[0,T]$ with $w_1^0=ess\ \inf_{[0,T]}w_1(t)>0$ and $w_2^0=ess\ \inf_{[0,T]}w_2(t)>0$.

$(F0)$ $F:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ is a function such that $F(\cdot,u,v)$ is continuous in$[0,T]$ for any (u,v) $\in\mathbb{R}^2$, $F(t,\cdot,\cdot)$ is a C^1 function in $\mathbb{R}^2$, and F_s is the partial derivative of F with respect to s ;

$(G0)$ $G:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ is measurable with respect to t for every$(u,v)\in\mathbb{R}^2$, continuously differentiable in $\mathbb{R}^2$ for a.e. $t\in[0,T]$, and G_u,G_v denote the partial derivatives of G that satisfy the following condition:

$$\underset{\sqrt{u^2+v^2}\leq\xi}{sup}\ \max\left\{\left|G_u(\cdot,u,v)\right|,\left|G_v(\cdot,u,v)\right|\right\}\in L^1([0,T]) \text{ for all } \xi>0, \tag{4.2}$$

we recall some basic notations and lemmas and construct a variational framework. Let X be a real Banach space, and let Υ_X denote the class of all functionals $\phi:X\to\mathbb{R}$ that possess the following property: if $\{w_n\}$ is a sequence in X converging weakly to $w\in X$ and $\lim\limits_{n\to\infty}$ inf $\phi(w_n)\leq\phi(w)$, then $\{w_n\}$ admits a subsequence converging strongly to w. For instance, if X is uniformly convex and $S:[0,+\infty)\to\mathbb{R}$ is a continuous strictly increasing function, then the functional $w \to S(\|w\|)$ belongs to the class Υ_X.

4.2 Preliminary results

Definition 4.1 *[24]*

Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u are defined by

$$
{}_aD_t^{\alpha} u(t) = \frac{d^n}{dt^n}\, {}_aD_t^{\alpha-n} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds,
$$

and

$$
{}_tD_b^{\alpha} u(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\alpha-n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha-1} u(s)\, ds,
$$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}^*$.

Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$
\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1} e^{-z} dz.
$$

Set $AC^n([a,b],\mathbb{R})$ the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC^n([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings having $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we signify $AC([a,b],\mathbb{R}) = AC^1([a,b],\mathbb{R})$.

Definition 4.2 *[28]*

Let $0 < \alpha \leq 1$, for $1 < p < \infty$. The fractional derivative space E_α^p is defined as

$$
E_\alpha^p = \left\{ u(t) \in L^p([0,T],\mathbb{R}) \middle|\ {}_0D_t^{\alpha} u(t) \in L^p([0,T],\mathbb{R}),\, u(0) = u(T) = 0 \right\},
$$

is a Banche space.

Then, for any $u \in E_\alpha^p$, we can define the weighted norm for E_α^p as

$$\|u\|_\alpha = \left(\int_0^T |u(t)|^p \, dt + \int_0^T w_1(t) \left|{}_0D_t^\alpha u\,(t)\right|^p dt\right)^{\frac{1}{p}} \tag{4.3}$$

Lemma 4.1 *[13]*

Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For any $u \in E_\alpha^p$ we have

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0D_t^\alpha u\right\|_{L^p}. \tag{4.4}$$

Also, if $\alpha > p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{\frac{1}{q}}} \left\|{}_0D_t^\alpha u\right\|_{L^p}. \tag{4.5}$$

From Lemma 4.1, we clearly observe that

$$\|u\|_{L^p} \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha+1)} \left(\int_0^T w_1(t) \left|{}_0D_t^\alpha u\,(t)\right|^p dt\right)^{1/p}, \tag{4.6}$$

for $0 < \alpha \leq 1$, and

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}} \left(\int_0^T w_1(t) \left|{}_0D_t^\alpha u\,(t)\right|^p dt\right)^{1/p}}{\Gamma(\alpha)\left(w_1^0\right)^{\frac{1}{p}} \left((\alpha-1)q+1\right)^{\frac{1}{q}}}, \tag{4.7}$$

for $\alpha > p$ and $\frac{1}{p} + \frac{1}{q} = 1$.

By using (4.6), the norm of (4.3) is equivalent to

$$\|u\|_\alpha = \left(\int_0^T w_1(t) \left|{}_0D_t^\alpha u\,(t)\right|^p dt\right)^{\frac{1}{p}}, \ \forall u \in E_\alpha^p. \tag{4.8}$$

For $0 < \beta \leq 1, 1 < p < \infty$ analogous to the space E_α^p we define the fractional derivative space E_β^p as

$$\left\{ v(t) \in L^p([0,T], \mathbb{R})\big|_0 D_t^\beta v\,(t) \in L^p([0,T], \mathbb{R}),\, v(0) = v(T) = 0 \right\},$$

is a Banche space.

Then, for any $v \in E_\beta^p$, the norm of E_β^p is defined by

$$\|v\|_\beta = \left(\int_0^T |v(t)|^p \, dt + \int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{\frac{1}{p}}, \ \forall v \in E_\beta^p \tag{4.9}$$

Similar with (4.6) and (4.7), we get

$$\|v\|_{L^p} \leq \frac{T^\beta \left(\int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{1/p}}{\Gamma(\beta+1) (w_2^0)^{\frac{1}{p}}} \tag{4.10}$$

for $0 < \beta \leq 1$, and

$$\|v\|_\infty \leq \frac{T^{\beta - \frac{1}{p}} \left(\int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{1/p}}{\Gamma(\beta) (w_2^0)^{\frac{1}{p}} ((\beta-1) q + 1)^{\frac{1}{q}}}. \tag{4.11}$$

for $\frac{1}{p} < \beta \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, based upon (4.10), the weighted norm

$$\|v\|_\beta = \left(\int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{\frac{1}{p}}, \tag{4.12}$$

is equivalent to (4.9), for every $v \in E_\beta^p$.

In the following discussion, for any $u \in E_\alpha^p$, $v \in E_\beta^p$ denote the space of $X = E_\alpha^p \times E_\beta^p$ with the norm

$$\|(u,v)\|_X = \left(\|u\|_\alpha^p + \|v\|_\beta^p \right)^{\frac{1}{p}}, \quad \forall (u,v) \in X,$$

where $\|u\|_\alpha$ and $\|v\|_\beta$ is defined in (4.8) and (4.12) respectively.

Lemma 4.2 *[28]*

For $0 < \alpha, \beta \leq 1$ and $1 < p < \infty$. The fractional derivative space X is a reflexive separable Banach space.

Consider the first equation with its boundary conditions of (4.1)

$$\begin{aligned} &{}_tD_T^\alpha \left(\tfrac{1}{w_1(t)^{p-2}} \Phi_p \left(w_1(t) \ {}_0D_t^\alpha u(t) \right) \right) + \mu |u(t)|^{p-2} u(t) \\ &= \lambda F_u(t, u(t), v(t)) + \delta G_u(t, u(t), v(t)) \ \text{ a.e. } t \in [0,T], \end{aligned} \tag{4.13}$$

Multiplying (4.13)by any $x(t) \in E_{\alpha}^{p}$, $v \in E_{\beta}^{p}$ and integrating, yields

$$\begin{aligned}&\int_0^T D_T^{\alpha}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\right)x(t)\,dt+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt\\&=\lambda\int_0^T F_u(t,u(t),v(t))+\delta G_u(t,u(t),v(t))\,x(t)\,dt.\end{aligned} \tag{4.14}$$

Then, combining Definition 4.1, Definition 2.4, Proposition 2.3 and Proposition 2.3, he left side of (4.14) can be transferred into

$$\begin{aligned}&\int_0^T {}_tD_T^{\alpha}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\right)x(t)\,dt+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt\\&=-\int_0^T x(t)\,d\left[{}_tD_T^{\alpha-1}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\right)\right]+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt\\&=\int_0^T \frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\ {}_0D_t^{\alpha-1}x'(t)\,dt+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt\\&=\int_0^T \frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\ {}_0^CD_t^{\alpha}x(t)\,dt+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt\\&=\int_0^T \frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\ {}_0D_t^{\alpha}x(t)\,dt+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt.\end{aligned}$$

Moreover, we can get similar results for the second equation of (4.1). In what follows, we will give the definition of weak solution for (4.1), which is based on the discussion mentioned above.

Definition 4.3 *[13]*

We say that $(u, v) \in X$ is a weak solution of (4.1). If the following identity holds for any $(x, y) \in X$ such that

$$\begin{aligned}&\int_0^T \frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha}u(t)\right)\ {}_0D_t^{\alpha}x(t)\,dt\\&+\int_0^T \frac{1}{w_2(t)^{p-2}}\Phi_p\left(w_2(t)\ {}_0D_t^{\alpha}v(t)\right)\ {}_0D_t^{\beta}y(t)\,dt\\&+\mu\int_0^T |u(t)|^{p-2}u(t)\,x(t)\,dt+\mu\int_0^T |v(t)|^{p-2}v(t)\,y(t)\,dt\\&-\lambda\int_0^T \left(F_u(t,u(t),v(t))\,x(t)+F_v(t,u(t),v(t))\,y(t)\right)dt\\&-\delta\int_0^T \left(G_u(t,u(t),v(t))\,x(t)+G_v(t,u(t),v(t))\,y(t)\right)dt=0.\end{aligned}$$

Lemma 4.3 *[51]*

Let $A : X \to X^*$be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X.Assume $\{w_1, w_2 ...\}$is a basis in X. Then the following assertion holds: (d) Inverse operator. if A is strictly monotone, then the inverse operator $A^{-1} : X^* \to X$ exist. This operator is strictly monotone,demicontinuous and bounded. If A is uniformly monotone, then A^{-1} is continuous. If A is strongly monotone, then is Lipschitz continuous.

Proposition 4.1 *[40]*

Let X be a nonempty set, and let ϕ, ψ be real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that

$$\phi(x_0) = \psi(x_0), \phi(x_1) > r \ , \qquad \sup_{x\in\phi^{-1}((-\infty;r])} \psi(x) < \rho < r\frac{\psi(x_1)}{\phi(x_1)}.$$

Then for each ρ satisfying

$$\sup_{x\in\phi^{-1}((-\infty;r])} \psi(x) < r\frac{\psi(x_1)}{\phi(x_1)}.$$

We have

$$\sup_{\lambda\geq 0} \inf_{x\in X} (\phi(x) + \lambda(\rho - \psi(x))) < \inf_{x\in X} \sup_{\lambda\geq 0} (\phi(x) + \lambda(\rho - \psi(x)))$$

Lemma 4.4 *[14]*

Let $0 < \alpha \leq 1$, for $1 < p < \infty$. Then, for any $f \in L^p([0, T], \mathbb{R})$,

$$\left\|{}_0D_\xi^{-\alpha} f\right\|_{L^p([0,T])} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p([0,T])}, \quad \textit{for } \xi \in [0,t], \ t \in [0,T],$$

wher ${}_0D_\xi^{-\alpha}$ is left Riemann–Liouville fractional integral of order α, and Γ is the gamma function.

Lemma 4.5 *[39]*

Assume that $\frac{1}{2} < \alpha \leq 1$*and the sequence* $\{u_n\}$ *converges weakly to* u *in* E_α^p : $u_k \rightharpoonup u$ *in* $C([0,T],\mathbb{R})$*, that is,* $\|u_k - u\|_\infty \to 0$ *as* $k \to \infty$.

4.3 Result of existence of at least three solution

In this part, we explore the existence of at least three weak solutions for problem (4.1) . For better understanding, we define the functionals $\phi, \psi, J : X \to \mathbb{R}$ as

$$\phi(u,v) := \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p \quad , \quad (u,v) \in X, \tag{4.15}$$

$$\psi(u,v) := \int_0^T F(t, u(t), v(t))\, dt, \tag{4.16}$$

$$J\ (u,v) := \int_0^T G(t, u(t), v(t))\, dt. \tag{4.17}$$

Clearly, ψ and J are well-defined continuously Gâteaux-differentiable functional at any $(u,v) \in X$, and their Gâteaux derivatives are

$$\psi'(u,v)(x,y) = \int_0^T \left(F_u(t, u(t), v(t))\, x(t) + F_v(t, u(t), v(t))\, y(t)\right) dt,$$

$$J\,'(u,v)(x,y) = \int_0^T \left(G_u(t, u(t), v(t))\, x(t) + G_v(t, u(t), v(t))\, y(t)\right) dt,$$

respectively, for every $(x,y) \in X$.

Lemma 4.6 .

The functional ϕ *is sequentially weakly lower semicontinuous and bounded on* X*, and* ϕ' *admits a continuous inverse on* X^*.

Proof Let $\{(u_n, v_n)\} \subset X, (u_n, v_n) \rightharpoonup (u, v)$ in X.From Lemma 4.5, (u_n, v_n) converges uniformly to (u, v) on $[0, T]$, and $\liminf_{n\to\infty} \|(u_n, v_n)\|_X \geq \|(u, v)\|_X$. Thus

$$\begin{aligned}
\liminf_{n\to\infty} \phi(u_n, v_n) &= \liminf_{n\to\infty} \left(\frac{1}{p} \|u_n\|_\alpha^p + \frac{1}{p} \|v_n\|_\beta^p \right) \\
&\geq \frac{1}{p} \|u\|_\alpha^p + \frac{1}{p} \|v\|_\beta^p = \phi(u, v).
\end{aligned}$$

So ϕ is a sequentially weakly lower semicontinuous functional.

Moreover, let Ω be a bounded subset of X, that is, there is a constant $c > 0$ such that $\|(u, v)\|_X \leq c$ for any $(u, v) \in \Omega$. By (4.6), (4.10) and Lemma 4.5, we have

$$\begin{aligned}
\phi(u, v) &= \frac{1}{p} \|u\|_\alpha^p + \frac{1}{p} \|v\|_\beta^p \\
&= \frac{1}{p} \left(\|u\|_\alpha^p + \|v\|_\beta^p \right) \\
&\leq \frac{c^p}{p}.
\end{aligned}$$

Hence ϕ is bounded on each bounded subset of X.

Next, we will show that $\phi' : X \to X^*$ admits a Lipschitz continuous inverse. Obviously, $\phi \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}
\langle \phi'(u, v), (x, y) \rangle &= \int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p\left(w_1(t)\ {}_0D_t^\alpha u(t)\right)\ {}_0D_t^\alpha x(t)\, dt \\
&\quad + \int_0^T \frac{1}{w_2(t)^{p-2}} \Phi_p\left(w_2(t)\ {}_0D_t^\alpha v(t)\right)\ {}_0D_t^\beta y(t)\, dt \\
&\quad + \mu \int_0^T |u(t)|^{p-2} u(t) x(t)\, dt + \mu \int_0^T |v(t)|^{p-2} v(t) y(t)\, dt \\
&= \langle \phi_1(u), x \rangle + \langle \phi_2(v), y \rangle,
\end{aligned}$$

where

$$\langle \phi_1(u), x\rangle = \int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p\left(w_1(t)\ {}_0D_t^{\alpha} u(t)\right)\ {}_0D_t^{\alpha} x(t)\, dt + \mu \int_0^T |u(t)|^{p-2} u(t)\, x(t)\, dt \qquad \forall x \in E_\alpha^p,$$

$$\langle \phi_2(v), y\rangle = \int_0^T \frac{1}{w_2(t)^{p-2}} \Phi_p\left(w_2(t)\ {}_0D_t^{\beta} v(t)\right)\ {}_0D_t^{\beta} y(t)\, dt + \mu \int_0^T |v(t)|^{p-2} v(t)\, y(t)\, dt,\ \forall y \in E_\beta^p.$$

For any u, $x \in E_\alpha^p$, it follows from (4.6), that

$$\begin{aligned}
\langle \phi_1(u) - \phi_1(x), u - x\rangle \quad = \quad & \int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p\left(w_1(t)\ {}_0D_t^{\alpha} u(t)\right)\ {}_0D_t^{\alpha}(u(t) - x(t))\, dt \\
& + \mu \int_0^T |u(t)|^{p-2} u(t)(u(t) - x(t))\, dt \\
& - \int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p\left(w_1(t)\ {}_0D_t^{\alpha} x(t)\right)\ {}_0D_t^{\alpha}(u(t) - x(t))\, dt \\
& + \mu \int_0^T |x(t)|^{p-2} x(t)(u(t) - x(t))\, dt.
\end{aligned}$$

According to the well-known inequality

$$\begin{aligned}
& \left(|s_1|^{p-2} s_1 - |s_2|^{p-2} s_2\right)(s_1 - s_2) \\
& \geq \begin{cases} |s_1 - s_2|^p, & p \geq 2 \\ \frac{|s_1 - s_2|^2}{(|s_1| + |s_2|)^{2-p}}, & 1 < p \leq 2 \end{cases}
\end{aligned} \tag{4.18}$$

We have

$$\begin{aligned}
& \left(\Phi_p\left(w_1(t)\ {}_0D_t^{\alpha} u(t)\right)\right)\ {}_0D_t^{\alpha}(u(t) - x(t)) \\
& \geq \begin{cases} \frac{1}{w_1(t)} \left|w_1(t)\ {}_0D_t^{\alpha} u(t)\right|^p, & p \geq 2 \\ \frac{1}{w_1(t)} \frac{\left|w_1(t)\,{}_0D_t^{\alpha} u(t)\right|^2}{\left(\left|w_1(t)\,{}_0D_t^{\alpha} u(t)\right|\right)^{2-p}}, & 1 < p < 2 \end{cases}
\end{aligned} \tag{4.19}$$

Hence, when $1 < p < 2$, one has

$$\begin{aligned}
&\int_0^T \left|w_1(t)\left({}_0D_t^\alpha u(t) - {}_0D_t^\alpha x(t)\right)\right|^p dt \\
&\le \left(\int_0^T \frac{\left|w_1(t)\ {}_0D_t^\alpha u(t) - {}_0D_t^\alpha x(t)\right|^2}{w_1(t)\left(\left|w_1(t){}_0D_t^\alpha u(t)\right| + \left|w_1(t){}_0D_t^\alpha x(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} \\
&\left(\int_0^T w_1(t)^{\frac{p}{2-p}} \left(\left|w_1(t)\ {}_0D_t^\alpha u(t)\right| + \left|w_1(t)\ {}_0D_t^\alpha x(t)\right|\right)^p dt\right)^{\frac{2-p}{2}},
\end{aligned} \tag{4.20}$$

which means that

$$\begin{aligned}
&\int_0^T \frac{\left|w_1(t)\ {}_0D_t^\alpha u_i(t) - w_1(t)\ {}_0D_t^\alpha x(t)\right|^2}{w_1(t)\left(\left|w_1(t)\ {}_0D_t^\alpha u(t)\right| + \left|w_1(t)\ {}_0D_t^\alpha x(t)\right|\right)^{2-p}} dt \\
&\ge \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\widetilde{w}_1^0} \|u - x\|_\alpha^2 \left(\|u\|_\alpha^p + \|x\|_\alpha^p\right)^{\frac{p-2}{p}}.
\end{aligned} \tag{4.21}$$

Then, we deduce

$$\begin{aligned}
&\int_0^T \left(\Phi_p\left(w_1(t)\ {}_0D_t^\alpha u(t)\right) - \Phi_p\left(w_1(t)\ {}_0D_t^\alpha x(t)\right)\ {}_0D_t^\alpha(u - x)\right) dt \\
&\ge \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\widetilde{w}_1^0} \|u - x\|_\alpha^2 \left(\|u\|_\alpha^p + \|x\|_\alpha^p\right)^{\frac{p-2}{p}} > 0.
\end{aligned} \tag{4.22}$$

When $p \ge 2$, we get

$$\begin{aligned}
&\int_0^T \left(\Phi_p\left(w_1(t)\ {}_0D_t^\alpha u(t)\right) - \Phi_p\left(w_1(t)\ {}_0D_t^\alpha x(t)\right)\ {}_0D_t^{\alpha_i}(u - x)\right) dt \\
&\ge \left(w_1^0\right)^{p-2} \|u - x\|_\alpha^p > 0.
\end{aligned} \tag{4.23}$$

Further, denote

$$A = \int_0^T |u(t)|^{p-2} u(t)(u - x)\, dt + \int_0^T |x(t)|^{p-2} x(t)(u - v)\, dt$$

Then, reapplying inequality (4.18), we always have

$$A \geq \|u-x\|_{\alpha}^{p} > 0, \quad \text{for } p \geq 2,$$

and

$$A \geq 2^{p-2}\left(\|u-x\|_{L^p}^{2}\left(\|u\|_{L^p}+\|x\|_{L^p}\right)^{\frac{p-2}{p}}\right) > 0, \ \text{for} 1<p<2$$

That is, $A>0$ for every $1<p<\infty$. Thus ϕ_1 is a uniformly monotone operator.

Similarly, it is easy to show that ϕ_2 is also a uniformly monotone operator. So ϕ' is uniformly monotone.

Furthermore, in view of X is reflexive,for $(u_n, v_n) \rightharpoonup (u, v)$ in X strongly, as $n \to \infty$, one has $\phi'(u_n, v_n) \rightharpoonup \phi'(u, v)$ in X^* as $n \to \infty$.

Thus, we say that ϕ' is demicontinuous. Then, according to lemma 4.3, we obtain that the inverse operator $(\phi')^{-1}$ of ϕ' exist and is continuous.

Moreover , let

$$\|u\|_{\mu,\alpha}^{p} = \int_0^T \left(w_1(t)\left|{}_0D_t^{\alpha}u(t)\right|^p + \mu\left|u(t)\right|^p\right)dt,$$

and

$$\|u\|_{\mu,\beta}^{p} = \int_0^T \left(w_2(t)\left|{}_0D_t^{\alpha}v(t)\right|^p + \mu\left|v(t)\right|^p\right)dt,$$

owing to the sequentially weakly lower semicontinuity of $\|u\|_{\mu,\alpha}^{p}$ and $\|u\|_{\mu,\beta}^{p}$we observe that ϕ is sequentially weakly lower semicontinuous in X. ∎

Lemma 4.7 .

The functionals ψ and J are continuously Gâteaux differentiable in X, and their derivatives ψ' , J' are compact.

Proof Considering the functional ψ, we will point out that ψ is a Gâteaux differentiable, sequentially weakly upper semicontinuous functional on X.Indeed, for $(u_n, v_n) \subset X$, assume that $(u_n, v_n) \rightharpoonup (u, v)$ in X, i.e (u_n, v_n) uniform converge to (u, v) on $[0, T]$ as $n \to \infty$.

Hence

$$\begin{aligned}\lim_{n\to+\infty}\inf\ \psi(u_n,v_n) &\leq \int_0^T \lim_{n\to+\infty}\inf F(t,u_n(t),v_n(t))\,dt \\ &= \int_0^T F(t,u(t),v(t))\,dt = \psi(u,v),\end{aligned}$$

which implies that ψ is sequentially weakly upper semicontinuous. Furthermore, since F is continuously differentiable with respect to u and v for almost every $t\in[0,T]$ .we have $F(t,u_n(t),v_n(t))\to F(t,u(t),v(t))$ as $n\to+\infty$. Then , based on the Lebesgue control convergence theorem, we obtain that $\psi'(u_n,v_n)\to\psi'(u,v)$ strongly, that is ψ' is strongly continuous on X. Hence, we confirm that ψ' is compact operator.

Moreover, it is easy to prove that the functional with the Gâteaux derivative $\psi'(u,v)\in X^*$at the point $(u,v)\in X$

$$\psi'(u,v)(x,y) := \int_0^T \left(F_u(t,u(t),v(t))\,x(t)+F_v(t,u(t),v(t))\,y(t)\right)dt \tag{4.24}$$

for any $(x,y)\in X$.

Analogously, we can deduce that $J'(u,v)$ is a compact operator for any $(u,v)\in X$.

The proof is completed.

In what follows, in order to facilitate the further discussion, we give some notation. Put

$$\begin{aligned} M &:= \max\left\{\frac{T^{p\alpha-1}}{(\Gamma(\alpha))^p\, w_1^0\,((\alpha-1)q+1)^{\frac{p}{q}}}, \frac{T^{p\beta-1}}{(\Gamma(\beta))^p\, w_2^0\,((\beta-1)q+1)^{\frac{p}{q}}}\right\}, \\ \lambda_1 &:= \inf\left\{\frac{\|u\|_\alpha^p+\|v\|_\beta^p}{p\int_0^T F(t,u,v)\,dt}, (u,v)\in X, \int_0^T F(t,u,v)\,dt>0\right\}, \\ \lambda_2 &:= \left(\max\left\{0, \limsup_{\|(u,v)\|_X\to+\infty}\frac{p\int_0^T F(t,u,v)\,dt}{\|u\|_\alpha^p+\|v\|_\beta^p}, \limsup_{(u,v)\to 0}\frac{p\int_0^T F(t,u,v)\,dt}{\|u\|_\alpha^p+\|v\|_\beta^p}\right\}\right)^{-1}. \end{aligned}$$

For a given constant $0 < \epsilon < \frac{1}{p}$, put

$$\begin{aligned} A(\alpha, \epsilon) &= \frac{1}{(\epsilon T)^p} \Bigg[\int_0^{\epsilon T} w_1(t)\, t^{(1-\alpha)p} dt + \int_{\epsilon T}^{(1-\epsilon)T} w_1(t) \left[t^{1-\alpha} - (t-\epsilon T)^{1-\alpha} \right]^p dt \\ &\quad + \int_{(1-\epsilon)T}^{T} w_1(t) \left[t^{1-\alpha} - (t-\epsilon T)^{1-\alpha} - (t - ((1-\epsilon)T))^{1-\alpha} \right]^p dt \Bigg] \end{aligned}$$

and

$$\begin{aligned} B(\beta, \epsilon) &= \frac{1}{(\epsilon T)^p} \Bigg[\int_0^{\epsilon T} w_2(t)\, t^{(1-\beta)p} dt + \int_{\epsilon T}^{(1-\epsilon)T} w_2(t) \left[t^{1-\beta} - (t-\epsilon T)^{1-\beta} \right]^p dt \\ &\quad + \int_{(1-\epsilon)T}^{T} w_2(t) \left[t^{1-\beta} - (t-\epsilon T)^{1-\beta} - (t - ((1-\epsilon)T))^{1-\beta} \right]^p dt \Bigg] \end{aligned}$$

$$\Delta_1 := \min\{A(\alpha,\epsilon), B(\beta,\epsilon)\}, \qquad \Delta_2 := \max\{A(\alpha,\epsilon), B(\beta,\epsilon)\}$$

for any $\sigma > 0$, we denote by $\Omega(\sigma)$ the set $\pi(\sigma) = \{(u,v) \in \mathbb{R}^2 : |u|^p + |v|^p < \sigma\}$

Theorem 4.1 *Assume that $(F0)$ hold. Moreover, assume that there exist a constant $\eta \geq 0$ and a function $\overline{w} = (u_1, v_1) \in X$ such that*

(i)

$$\max\left\{ \lim_{(u,v)\to(0,0)} \sup \frac{\max_{t\in[0,T]} F(t,u,v)}{|u|^p + |v|^p}, \lim_{|(u,v)|\to+\infty} \sup \frac{\max_{t\in[0,T]} F(t,u,v)}{|u|^p + |v|^p} \right\} \leq \eta,$$

(ii)

$$pTM\eta < \frac{\int_0^T F(t, u_1(t), v_1(t))\, dt}{\|u_1\|_\alpha^p + \|v_1\|_\beta^p},$$

Then, for any compact interval $[a_1, a_2] \subset (\lambda_1, \lambda_2)$, there exists a positive constant ϱ with the following property: for every $\lambda \in [a_1, a_2]$ and for two Carathéodory functions G_u, G_v satisfying $(G0)$, there is $\kappa > 0$ such that, for each $\delta \in [0, \kappa)$, problem (4.1) has at least three weak solutions with norms less than ϱ.

Proof Our aim is to apply Theorem 1.2 to our problem (4.1) by taking $X = E_\alpha^p \times E_\beta^p$ endowed with the norm $\|(u,v)\|_X$ defined before. Obviously, X is a separable reflexive Banach space. It follows from Lemmas 4.6 and 4.7 that the functional ϕ is sequentially weakly lower semicontinuous, with continuous Gâteaux derivative, and bounded on each bounded subset of X. ϕ' admits a continuous inverse m and ψ and J are continuously Gâteaux-differentiable functionals in X with compact derivatives.

It is easy to see that $\dfrac{1}{p}|u|^p + \dfrac{1}{p}|v|^p$ belongs to Υ_X. Now we prove that $\phi(u,v) \in \Upsilon_X$.

Let $\{(u_n,v_n)\} \subset X$, $(u_n,v_n) \rightharpoonup (u,v)$ in X, and

$$\liminf_{n\to\infty} \phi(u_n,v_n) \leq \phi(u,v).$$

By Lemma 4.5, (u_n,v_n) converges uniformly to (u,v) on $[0,T]$. Thus there exist constants $c_1, c_2 > 0$ such that $\|u_n\|_\infty \leq c_1$ and $\|v_n\|_\infty \leq c_2$ for any $n \in N$. Therefore $\phi \in \Upsilon_X$,

we have

$$\begin{aligned}\phi(u,v) &= \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p \\ &\geq \frac{1}{p}\left(\|u\|_\alpha^p + \|v\|_\beta^p\right),\end{aligned} \tag{4.25}$$

for all $(u,v) \in X$. So ϕ is coercive and has a strict local minimum $(u_0,v_0) = (0,0)$ with $\phi(u_0,v_0) = \psi(u_0,v_0) = 0$.

Fix $\varepsilon > 0$. According to (i), there exist σ_1, σ_2 with $0 < \sigma_1 < \sigma_2$ such that

$$F(t,u,v) \leq (\eta + \varepsilon)\left(|u|^p + |v|^p\right), \tag{4.26}$$

for all $t \in [0,T]$ and $|(u,v)| \in ([0,\sigma_1) \cup (\sigma_2,+\infty))$. In view of $(F0)$, $F(t,u,v)$ is bounded on $t \in [0,T]$ and $|(u,v)| \in [\sigma_1,\sigma_2]$, so we can choose $m_1, m_2 > 0$ and $\tau_1, \tau_2 > p$ such that

$$F(t,u,v) \leq (\eta + \varepsilon)\left(|u|^p + |v|^p + m_1|u|^{\tau_1} + m_2|u|^{\tau_2}\right),$$

for all $t \in [0,T]$ and $|(u,v)| \in [\sigma_1,\sigma_2]$, we have

$$\begin{aligned}\psi(u,v) &\leq (\eta+\varepsilon)\int_0^T \left(|u|^p+|v|^p\right)dt+\int_0^T \left(m_1|u|^{\tau_1}+m_2|u|^{\tau_2}\right)dt\\ &\leq (\eta+\varepsilon)TM\left(\|u\|_\alpha^p+\|v\|_\beta^p\right)+T\xi\left(\|u\|_\alpha^{\tau_1}+\|v\|_\beta^{\tau_2}\right),\end{aligned}$$

for all $(u,v)\in X$, where

$$\xi=\max\left\{m_1\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\sqrt[\frac{1}{p}]{w_1^0\left((\alpha-1)q+1\right)^{\frac{p}{q}}}}\right)^{\tau_1},m_2\left(\frac{T^{\beta-\frac{1}{p}}}{\Gamma(\beta)\sqrt[\frac{1}{p}]{w_2^0\left((\beta-1)q+1\right)^{\frac{p}{q}}}}\right)^{\tau_2}\right\}.$$

Hence

$$\lim_{(u,v)\to 0}\sup\frac{\psi(u,v)}{\phi(u,v)}\leq pTM(\eta+\varepsilon). \tag{4.27}$$

Furthermore, by (4.27) again, for any $(u,v)\in X\backslash\{(0,0)\}$, we have

$$\begin{aligned}\frac{\psi(u,v)}{\phi(u,v)} &= \frac{\int_{\|(u,v)\|\leq\sigma_2} F(t,u,v)\,dt}{\frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p}+\frac{\int_{\|(u,v)\|>\sigma_2} F(t,u,v)\,dt}{\frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p}\\ &\leq \frac{pT\sup_{t\in[0,T],|(u,v)|\in[0,\sigma_2]}F(t,u,v)}{\|u\|_\alpha^p+\|v\|_\beta^p}+\frac{pTM(\eta+\varepsilon)\left(\|u\|_\alpha^p+\|v\|_\beta^p\right)}{\|u\|_\alpha^p+\|v\|_\beta^p}\\ &\leq \frac{pT\sup_{t\in[0,T],|(u,v)|\in[0,\sigma_2]}F(t,u,v)}{\|u\|_\alpha^p+\|v\|_\beta^p}+pTM(\eta+\varepsilon),\end{aligned}$$

which implies that

$$\lim_{\|(u,v)\|_X\to+\infty}\frac{\psi(u,v)}{\phi(u,v)}\leq pTM(\eta+\varepsilon). \tag{4.28}$$

Since ε is arbitrary, combining with (4.27) and (4.28), we have

$$\delta_1=\max\left\{0,\lim_{(u,v)\to 0}\frac{\psi(u,v)}{\phi(u,v)},\lim_{\|(u,v)\|_X\to+\infty}\frac{\psi(u,v)}{\phi(u,v)}\right\}\leq pTM\eta$$

and

$$\begin{aligned}\delta_2 &= \sup_{(u,v)\in\phi^{-1}((0,+\infty))}\frac{\psi(u,v)}{\phi(u,v)} = \sup_{(u,v)\in X\ \backslash\{(0,0)\}}\frac{\psi(u,v)}{\phi(u,v)}\\ &\geq \frac{\int_0^T F\left(t,u_1,v_1\right)dt}{\frac{1}{p}\left\|u_1\right\|_\alpha^p+\frac{1}{p}\left\|v_1\right\|_\beta^p}\\ &> pTM\eta \geq \delta_1.\end{aligned}$$

Then, for each compact interval$[a_1,a_2] \subset (\lambda_1,\lambda_2)$, there is $\varrho > 0$ with the following property:for all $\lambda \in [a_1,a_2]$ and $G \in (G0)$, there is $\kappa > 0$ such that, for each $\delta \in [0,\kappa]$, problem (4.1) has at least three weak solutions with norms less than ϱ . ■

Theorem 4.2 *Assume that (F0), hold and there exist* $l,h \in L^1([0,T],\mathbb{R}^+)$,*three positive constants* σ,θ_1,θ_2, *and constant vector* $\bar{c} = (c_1,c_2) \in \mathbb{R}^2$, $c_1,c_2 > 0$, *with* $\sigma < pM\Delta_1(c_1^2+c_2^2)$ *and* $\theta_1,\theta_2 \in [0,p)$, *such that*

(A_1) $F\left(t,u,v\right) \geq 0$ *for all* $t \in [0,\epsilon T]\cup[(1-\epsilon)\,T,T]$, $|u| \leq \Gamma\left(2-\alpha\right)c_1$, *and* $|v| \leq \Gamma\left(2-\beta\right)c_2$;

(A_2) $|F\left(t,u,v\right)| \leq l\left(t\right)\left(|u|^{\theta_1}+|v|^{\theta_2}\right)+h(t)$ *for all* $(u,v) \in X$ *and a.e.* $t \in [0,T]$;

(A_3)

$$\max_{t\in[0,T],(u,v)\in\pi(\sigma)} F\left(t,u,v\right) < \frac{\sigma}{pMT}\frac{\int_{\epsilon T}^{(1-\epsilon)T} F\left(t,\Gamma\left(2-\alpha\right)c_1,\Gamma\left(2-\beta\right)c_2\right)}{\Delta_2(c_1^2+c_2^2)},$$

where $\pi\left(\sigma\right) = \{(u,v) \in \mathbb{R}^2 : |u|^p+|v|^p \leq \sigma\}$.

Then there exist an open interval $\Lambda \subset [0,+\infty)$ *and a positive constant* ϱ *with the following property: for every* $\lambda \in \Lambda$ *and for two carathéodory functions* G_u, G_v *satisfying* $(G0)$, *there is* $\kappa > 0$ *such that, for each* $\delta \in [0,\kappa)$, *problem (4.1) has at least three weak solutions with norms less than* ϱ.

Proof For any $\lambda \geq 0$ and $(u,v) \in X$, according to (4.25) and (A_2), we have

$$\begin{aligned}\phi(u,v)-\lambda\psi(u,v) &= \frac{1}{p}\left\|u\right\|_\alpha^p+\frac{1}{p}\left\|v\right\|_\beta^p-\lambda\int_0^T F\left(t,u\left(t\right),v(t)\right)dt\\ &\geq \frac{1}{p}\left\|u\right\|_\alpha^p+\frac{1}{p}\left\|v\right\|_\beta^p-\lambda\int_0^T l\left(t\right)\left(|u|^{\theta_1}+|v|^{\theta_2}\right)dt-\lambda\int_0^T h(t)dt\\ &\geq \frac{1}{p}\left(\left\|u\right\|_\alpha^p+\left\|v\right\|_\beta^p\right)-\lambda\theta\int_0^T l\left(t\right)\left(\left\|u\right\|_\alpha^{\theta_1}+\left\|v\right\|_\beta^{\theta_2}\right)dt-\lambda\int_0^T h(t)dt,\end{aligned}$$

where

$$\theta = \max\left\{ \left(\frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha) \sqrt[\frac{1}{p}]{w_1^0 \left((\alpha-1)q+1\right)^{\frac{p}{q}}}} \right)^{\theta_1}, \left(\frac{T^{\beta - \frac{1}{p}}}{\Gamma(\beta) \sqrt[\frac{1}{p}]{w_2^0 \left((\beta-1)q+1\right)^{\frac{p}{q}}}} \right)^{\theta_2} \right\}.$$

Since $\theta_1, \theta_2 \in [0, p)$, we have

$$\lim_{\|(u,v)\|_X \to +\infty} \phi(u,v) - \lambda\psi(u,v) = +\infty \quad \text{for all } \lambda \geq 0.$$

For every $r > 0$, by the definition of ϕ and (4.25) we have

$$\begin{aligned}
&\phi^{-1}((-\infty; r]) = \{(u,v) \in X : \phi(u,v) \leq r\} \\
&\subseteq \left\{ (u,v) \in X : \tfrac{1}{p}\|u\|_\alpha^p + \tfrac{1}{p}\|v\|_\beta^p \leq r \right\} \\
&\subseteq \left\{ (u,v) \in X : \|u\|_\alpha^p + \|v\|_\beta^p \leq pr \right\} \\
&\subseteq \left\{ (u,v) \in X : \tfrac{(\Gamma(\alpha))^p w_1^0((\alpha-1)q+1)^{\frac{p}{q}}}{T^{p\alpha-1}} \|u\|_\infty^p + \tfrac{(\Gamma(\beta))^p w_1^0((\beta-1)q+1)^{\frac{p}{q}}}{T^{p\beta-1}} \|v\|_\infty^p \leq pr \right\} \\
&\subseteq \{(u,v) \in X : |u|^p + |u|^p \leq Mpr, \text{ for all } t \in [0,T]\},
\end{aligned} \tag{4.29}$$

which implies that

$$\begin{aligned}
\sup_{(u,v)\in\phi^{-1}((-\infty;r])} \psi(u,v) &\leq \max_{(u,v)\in\pi(Mpr)} \psi(u,v) \\
&= \max_{(u,v)\in\pi(Mpr)} \int_0^T F(t,u,v)\,dt \\
&\leq T \max_{t\in[0,T],(u,v)\in\pi(Mpr)} \int_0^T F(t,u,v)\,dt.
\end{aligned}$$

Choose $\bar{w} = (u_1(t), v_1(t))$ with

$$u_1(t) = \begin{cases} \frac{\Gamma(2-\alpha)c_1}{\epsilon T} t, & t \in [0, \epsilon T[, \\ \Gamma(2-\alpha)c_1, & t \in [\epsilon T, (1-\epsilon)T], \\ \frac{\Gamma(2-\alpha)c_1}{\epsilon T}(T-t), & t \in](1-\epsilon)T, T], \end{cases}$$

and

$$v_1(t) = \begin{cases} \frac{\Gamma(2-\beta)c_2}{\epsilon T} t, & t \in [0, \epsilon T[\,, \\ \Gamma(2-\beta)\,c_2, & t \in [\epsilon T, (1-\epsilon)\,T]\,, \\ \frac{\Gamma(2-\beta)c_2}{\epsilon T}(T-t), & t \in\,](1-\epsilon)\,T, T]\,. \end{cases}$$

Clearly, $\bar{w}(0) = \bar{w}(T) = 0$ and $\bar{w} \in L^2[0,T]$. A direct calculation shows that

$$_0D_t^\alpha u_1(t) = \begin{cases} \frac{c_1}{\epsilon T} t^{1-\alpha},\ t \in [0, \epsilon T[\\ \frac{c_1}{\epsilon T}\left(t^{1-\alpha} - (t-\epsilon T)^{1-\alpha}\right),\ t \in [\epsilon T, (1-\epsilon)\,T] \\ \frac{c_1}{\epsilon T}\left(t^{1-\alpha} - (t-\epsilon T)^{1-\alpha} - \left(t - (t-\epsilon T)^{1-\alpha}\right)\right),\ t \in\,](1-\epsilon)\,T, T] \end{cases}$$

and

$$_0D_t^\beta v_1(t) = \begin{cases} \frac{c_2}{\epsilon T} t^{1-\beta},\ t \in [0, \epsilon T[\,, \\ \frac{c_2}{\epsilon T}\left(t^{1-\beta} - (t-\epsilon T)^{1-\beta}\right),\ t \in [\epsilon T, (1-\epsilon)\,T]\,, \\ \frac{c_2}{\epsilon T}\left(t^{1-\beta} - (t-\epsilon T)^{1-\beta} - \left(t - (t-\epsilon T)^{1-\beta}\right)\right),\ t \in\,](1-\epsilon)\,T, T]\,. \end{cases}$$

Furthermore,

$$\begin{aligned} \|u_1\|_\alpha^p &= \int_0^T w_1(t)\,|_0D_t^\alpha u_1(t)|^p\,dt \\ &= \int_0^{\epsilon T} w_1(t)\,|_0D_t^\alpha u_1(t)|^p\,dt + \int_{\epsilon T}^{(1-\epsilon)T} w_1(t)\,|_0D_t^\alpha u_1(t)|^p\,dt + \int_{(1-\epsilon)T}^{T} w_1(t)\,|_0D_t^\alpha u_1(t)|^p\,dt \\ &= pc_1^2 A(\alpha, \epsilon)\,, \end{aligned}$$

and

$$\begin{aligned} \|v_1\|_\beta^p &= \int_0^T w_2(t)\,|_0D_t^\alpha v_1(t)|^p\,dt \\ &= \int_0^{\epsilon T} w_2(t)\,|_0D_t^\alpha v_1(t)|^p\,dt + \int_{\epsilon T}^{(1-\epsilon)T} w_2(t)\,|_0D_t^\alpha v_1(t)|^p\,dt + \int_{(1-\epsilon)T}^{T} w_2(t)\,|_0D_t^\alpha v_1(t)|^p\,dt. \\ &= pc_2^2 B(\beta, \epsilon)\,. \end{aligned}$$

Thus, $w = (u_1(t), v_1(t)) \in X$, and

$$p\Delta_1 \left(c_1^2 + c_2^2\right) \leq \|u_1\|_\alpha^p + \|v_1\|_\beta^p \leq p\Delta_2 \left(c_1^2 + c_2^2\right).$$

Obviously, $\phi(0,0) = \psi(0,0) = 0$. Choose $r = \frac{\sigma}{pM}$. From $\sigma < pM\Delta_1(c_1^2 + c_2^2)$ and (4.29) we have

$$pMr = \sigma < pM\Delta_1(c_1^2 + c_2^2) \leq pM\ \phi(u_1, v_1),$$

which means that $\phi(u_1, v_1) > r$. According to (A_1) and $F(t, 0, 0) = 0$, we have

$$\int_0^T F(t, u_1, v_1)\, dt = \int\limits_0^{\epsilon T} + \int\limits_{\epsilon T}^{(1-\epsilon)T} + \int\limits_{(1-\epsilon)T}^{T} F(t, u_1, v_1)\, dt \geq \int\limits_{\epsilon T}^{(1-\epsilon)T} F(t, u_1, v_1)\, dt.$$

So

$$\begin{aligned}
r\frac{\psi(u_1, v_1)}{\phi(u_1, v_1)} &= r\frac{\int_0^T F(t, u_1, v_1)\, dt}{\frac{1}{p}\|u_1\|_\alpha^p + \frac{1}{p}\|v_1\|_\beta^p}, \\
&\geq r\frac{\int_{\epsilon T}^{(1-\epsilon)T} F(t, \Gamma(2-\alpha)c_1, \Gamma(2-\beta)c_2)}{\Delta_2(c_1^2 + c_2^2)}, \\
&= \frac{\sigma}{pMT}\frac{\int_{\epsilon T}^{(1-\epsilon)T} F(t, \Gamma(2-\alpha)c_1, \Gamma(2-\beta)c_2)}{\Delta_2(c_1^2 + c_2^2)}, \\
&> T \max_{t\in[0,T],(u,v)\in\pi(\sigma)} \int_0^T F(t, u, v)\, dt, \\
&= T \max_{t\in[0,T],(u,v)\in\pi(pMr)} \int_0^T F(t, u, v)\, dt \geq \sup_{(u,v)\in\phi^{-1}((-\infty;r])} \psi(u, v).
\end{aligned}$$

Thus we can fix ρ such that

$$\sup_{(u,v)\in\phi^{-1}((-\infty;r])} \psi(u, v) < \rho < r\frac{\psi(u, v)}{\phi(u, v)}.$$

By proposition 4.1 we have

$$\sup_{\lambda\in I} \inf_{x\in X} \left(\phi(x) + \lambda\rho - \psi(x)\right) < \inf_{x\in X} \sup_{\lambda\in I} \left(\phi(x) + \lambda\rho - \psi(x)\right).$$

So, according to Theorem 1.3, for each interval $\Lambda \subset [0, +\infty)$ and $\varrho > 0$ we have: for any $\lambda \in \Lambda$ and $G \in (G_0)$, there is $\kappa > 0$ such that, for each $\delta \in [0, \kappa)$, $\phi'(u, v) - \lambda\psi'(u, v) - \mu J\,'(u, v) = 0$, has at least three solutions in X with norms less than ϱ . Therefore problem (1.3) has at least three solutions in X with norms less than ϱ.

For the particular case of $F(t, u, v) = \varphi(t)f(u, v)$, where $\varphi(t) \in L^1([0, T]; \mathbb{R})\setminus \{0\}, f(u, v) \in C^1(\mathbb{R}^2, \mathbb{R})$, we can deduce the following two corollaries of Theorems 4.1 and 4.2, respectively. ■

Corollary 4.1 *Assume that there exist $\eta > 0$ and $\overline{w} = (u_1, v_1) \in \mathbb{R}^2\setminus \{(0, 0)\}$*

$(A_1)'$

$$\max_{t\in[0,T]} \varphi(t)\ .\ \max\left\{ \lim_{(u,v)\to 0} \sup \frac{f(u, v)}{|u|^p + |v|^p}, \lim_{|(u,v)|\to+\infty} \sup \frac{f(u, v)}{|u|^p + |v|^p} \right\} \le \eta,$$

$(A_2)'$

$$pTM\eta < \frac{f(u_1, v_1) \int_0^T \varphi(t)dt}{\|u_1\|_\alpha^p + \|v_1\|_\beta^p}.$$

Then, for any compact interval $[a_1, a_2] \subset (\lambda_1, \lambda_2)$, there exists a positive constant ϱ with the following property: for every $\lambda \in [a_1, a_2]$ and for two Carathéodory functions G_u, G_v satisfying $(G0)$, there is $\kappa > 0$ such that, for each $\delta \in [0, \kappa)$, problem (4.1) has at least three weak solutions with norms less than ϱ.

Corollary 4.2 *Assume that there exist five positive constants $l_0, h_0, \sigma, \theta_1, \theta_2$, and constant vector $\overline{c} = (c_1, c_2) \in \mathbb{R}^2$, $c_1, c_2 > 0$, with $\sigma < pM\Delta_1(c_1^2 + c_2^2)$ and $\theta_1, \theta_2 \in [0, p)$, such that*

$(A_1)''$ $\varphi(t)F(u, v) \ge 0$ *for all* $t \in [0, \epsilon T]\cup[(1-\epsilon)T, T]$, $|u| \le \Gamma(2-\alpha)c_1$, *and* $|v| \le \Gamma(2-\beta)c_2$,

$(A_2)''$ $|F(u, v)| \le l_0 \left(|u|^{\theta_1} + |v|^{\theta_2}\right) + h_0$ *for all* $(u, v) \in X$,

(A_3)

$$\max_{(u,v)\in\pi(\sigma)} F(u, v) < \frac{\sigma}{pM\|\varphi\|_{L^1}} \frac{F(\Gamma(2-\alpha)c_1, \Gamma(2-\beta)c_2) \int_{\epsilon T}^{(1-\epsilon)T} \varphi(t)dt}{\Delta_2(c_1^2 + c_2^2)},$$

where $\pi(\sigma) = \{(u, v) \in \mathbb{R}^2 : |u|^p + |v|^p < \sigma\}$.

Then there exist an open interval $\Lambda \subset [0, +\infty)$ and a positive constant ϱ with the following property: for every $\lambda \in \Lambda$ and for two carathéodory functions G_u, G_v satisfying $(G0)$, there is $\kappa > 0$ such that, for each $\delta \in [0, \kappa)$, problem (4.1) has at least three weak solutions with norms less than ϱ.

4.4 Examples

Now, we give the following two numerical examples to illustrate the applications of our result.

Example 4.1 *Let* $p = 2$, $\alpha = 0.8$, $\beta = 0.65$, $\mu = 1$, $w_1(t) = 1 + t^2$, $w_2(t) = 0.5 + t$, $T = 1$.

Then, system (4.1) becomes the following form

$$\begin{cases} {}_tD_1^{0.8}\left((1+t^2)\ {}_0D_t^{0.8}u\left(t\right)\right)+u\left(t\right) \\ = \lambda F_u\left(t,u\left(t\right),v\left(t\right)\right)+\delta G_u\left(t,u\left(t\right),v\left(t\right)\right),\ t\in[0,1] \\ {}_tD_1^{0.65}\left((0.5+t)\ {}_0D_t^{0.65}v\left(t\right)\right)+v\left(t\right) \\ = \lambda F_v\left(t,u\left(t\right),v\left(t\right)\right)+\delta G_v\left(t,u\left(t\right),v\left(t\right)\right),\ t\in[0,1] \\ u\left(0\right)=u(1)=0, v\left(0\right)=v(1)=0. \end{cases} \tag{4.30}$$

For all $(t, u, v) \in [0, 1] \times \mathbb{R}^2$,*Taking*

$$F\left(t,u,v\right)=10\left(1+3t^2\right)\left(u^2+v^2\right),$$

and

$$G\left(t,u,v\right)=\left(1+t^2\right)\left(|u|^{\frac{5}{4}}+|v|^{\frac{4}{3}}\right).$$

Clearly,$F(t, 0, 0) = G(t, 0, 0) = 0$, $w_1^0 = 1$ *and* $w_2^0 = 0.5$ *for all* $t \in [0, 1]$.

Conditions $(F0)$ *hold. By the direct calculation, we have*

$$\max\left\{\frac{1}{\left(\Gamma\left(0.8\right)\right)^2\left(2\times0.8-1\right)},\frac{1}{\left(\Gamma\left(0.65\right)\right)^2\times0.5\left(2\times0.65-1\right)}\right\}=M\approx3.4764.$$

Taking $\eta = \dfrac{1}{20}$, *we easily verify that* (i) *is satisfied. Moreover, we have* $\lambda_1 \geq \dfrac{1}{140}$ *and* $\lambda_2 \geq 2.8765$. *In fact,*

$$
\begin{aligned}
\lambda_1 &= \inf_{(u,v)\in X} \frac{\|u\|_{0.8}^2 + \|v\|_{0.65}^2}{20\int_0^1 (1+3t^2)\,dt\,(u^2+v^2)} \\
&= \frac{1}{20\int_0^1 (1+3t^2)\,dt} \inf_{(u,v)\in X} \frac{\|u_1\|_{0.8}^2 + \|v_1\|_{0.65}^2}{(u^2+v^2)} \\
&\geq \frac{1}{40} \inf_{(u,v)\in X} \frac{\|u_1\|_{0.8}^2 + \|v_1\|_{0.65}^2}{M\left(\|u_1\|_{0.8}^2 + \|v_1\|_{0.65}^2\right)} \\
&\geq \frac{1}{140},
\end{aligned}
$$

and $\lambda_2 \geq \frac{1}{2TM\eta} \approx 2.8765$*. On the other hand, choosing* $u_1(t) = \Gamma(1.2)\,t\,(1-t)$, $v_1(t) = \Gamma(1.35)\,t\,(1-t)$ *we have*

$$
\begin{aligned}
{}_0D_t^{0.8}u_1(t) &= t^{0.2} - \frac{2\Gamma(1.2)}{\Gamma(2.2)}t^{1.2}, \\
{}_0D_t^{0.65}v_1(t) &= t^{0.35} - \frac{2\Gamma(1.35)}{\Gamma(2.35)}t^{1.35}.
\end{aligned}
$$

So that

$$
\|u_1(t)\|_{0.8}^2 \approx 0.19333, \|v_1(t)\|_{0.65}^2 \approx 0.078559,
$$

and

$$
2TM\eta \approx 0.3476 < \frac{\int_0^1 F(t,u_1,v_1)\,dt}{\left(\|u_1\|_{0.8}^2 + \|v_1\|_{0.65}^2\right)} < 3.6889.
$$

Which implies that condition (ii) holds. Hence, by Theorem 4.1, for any compact interval $[a_1,a_2] \subset (\frac{1}{140}, 2.8765)$*, there exists a positive constant* ϱ *with the following property: for every* $\lambda \in [a_1,a_2]$ *, there is* $\kappa > 0$ *such that, for each* $\delta \in [0,\kappa)$*, problem (4.30) has at least three weak solutions with norms less than* ϱ*.*

Example 4.2 *Let $p = 2$, $\alpha = 0.8$, $\beta = 0.6$, $\mu = 1$, $w_1(t) = 1 + t^2$, $w_2(t) = 0.5 + t$, $T = 1$.*

Then system (4.1) becomes the following form

$$\begin{cases} {}_tD_1^{0.8}\left(\left(1+t^2\right)\ {}_0D_t^{0.8}u\left(t\right)\right) + u\left(t\right) \\ = \lambda F_u\left(t, u\left(t\right), v\left(t\right)\right) + \delta G_u\left(t, u\left(t\right), v\left(t\right)\right),\ t \in [0,1] \\ {}_tD_1^{0.6}\left(\left(0.5+t\right)\ {}_0D_t^{0.6}v\left(t\right)\right) + v\left(t\right) \\ = \lambda F_v\left(t, u\left(t\right), v\left(t\right)\right) + \delta G_v\left(t, u\left(t\right), v\left(t\right)\right),\ t \in [0,1] \\ u\left(0\right) = u(1) = 0, v\left(0\right) = v(1) = 0. \end{cases} \tag{4.31}$$

Moreover, for all $(t, u, v) \in [0,1] \times \mathbb{R}^2$, put $F\left(t, u, v\right) = \varphi\left(t\right)\left(|u|^{\frac{5}{4}} + |v|^{\frac{4}{3}}\right)$, where

$$\varphi\left(t\right) = \begin{cases} \frac{1}{4} - t\ ,\ \ t \in \left[0, \frac{3}{8}\right] \\ -\frac{1}{2} + t\ ,\ t \in \left[\frac{3}{8}, 1\right] \end{cases}$$

and

$$G\left(t, u, v\right) = t^2\left(|u|^{\frac{3}{2}} + |v|^{\frac{6}{5}}\right).$$

Clearly, $F\left(t, 0, 0\right) = 0$, $w_1^0 = w_2^0 = 1$, for any $t \in [0,1]$.

By the direct calculation, we have

$$\max\left\{\frac{1}{\left(\Gamma\left(0.8\right)\right)^2\left(2\times 0.8 - 1\right)}, \frac{1}{\left(\Gamma\left(0.6\right)\right)^2\left(2\times 0.6 - 1\right)}\right\} = M \approx 2.2548.$$

Letting $\epsilon = \dfrac{1}{4}$, we obtain $A\left(\alpha, \epsilon\right) = 1.3096$ and $B\left(\beta, \epsilon\right) = 0.4736$ Hence $\Delta_1 = 0.4736$ and $\Delta_2 = 1.3096$. Take $\sigma = \dfrac{1}{2}$, $c_1 = c_2 = \dfrac{1}{6}$, $l_0 = 1$, $h_0 > 0$, $\theta_1 = \dfrac{5}{4}$, and $\theta_2 = \dfrac{4}{3}$ Then all the conditions in Corollary 4.2 are satisfied. In fact, conditions $\left(A_1\right)''$ and $(A_2)''$ hold, and by direct *computation we have*

$$\frac{1}{2} = \sigma < 2M\Delta_1\left(c_1^2 + c_2^2\right) \approx 0.5932,$$

and

$$\begin{aligned}\max_{(u,v)\in\pi\left(\frac{1}{2}\right)} F(u,v) &\approx 0.1739 < \frac{9\sigma}{M\|\varphi\|_{L^1}} \cdot \frac{F\left(\Gamma(0.2), \Gamma(0.36)\right)\displaystyle\int_{\frac{1}{4}}^{\frac{3}{4}}\varphi(t)dt}{\Delta_2} \\ &\approx 1.703.\end{aligned}$$

Which implies that condition $(A_1)''$ holds. Hence, By Corollary 4.2 there exist an open interval $\Lambda \subset [0, +\infty)$ and a positive constant ϱ with the following property: for every $\lambda \in \Lambda$, there exists $\kappa > 0$ such that, for each $\delta \in [0, \kappa)$, problem (4.31) has at least three weak solutions with norms less than ϱ.

Conclusion and Percpective

Throughout this study, fractional differential equations have been carefully investigated in four chapters The importance of this paper rises in its application in many scientific and engineering fields such as models for various precesses in plasma physics, biology, medical science, chemistry as well as population dynamics, and control theory.In the first chapter, detailed theory has been presented to provide the necessary background information about the theoremes needed to understand the investigated problems in the other chapters. Then, chapter two has dealt with fractioal calculus and its relevance in this work. Furthermore, We could ensure the existence of at least three solutions for a class of fractional p-Laplacian differential systems in chapter three, note that some appropriate function spaces and variational methods were successfully created for the system (3.1). Chapter four, on the other hand, explains how building a variational framework and using some critical points in theorems of Ricceri is used to get other new existence results for at least three weak solutions in terms of different values of the two parameters λ, δ, taking into consideration that we have supposed the primitive function G of G_u and G_v to satisfy a general growth condition allowing us to apply a variational method. In addition, we have obtained the multiplicity results for two cases: where the primitive function F of F_u, F_v is asymptotically quadratic and where it is subquadratic as $|(u, v)| \to \infty$.

At last, further researches are recommended to enlarge this study and prove the existence of infinite number of solutions for the investigated problems.

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