Control of Some Distributed Systems with Missing Data

By

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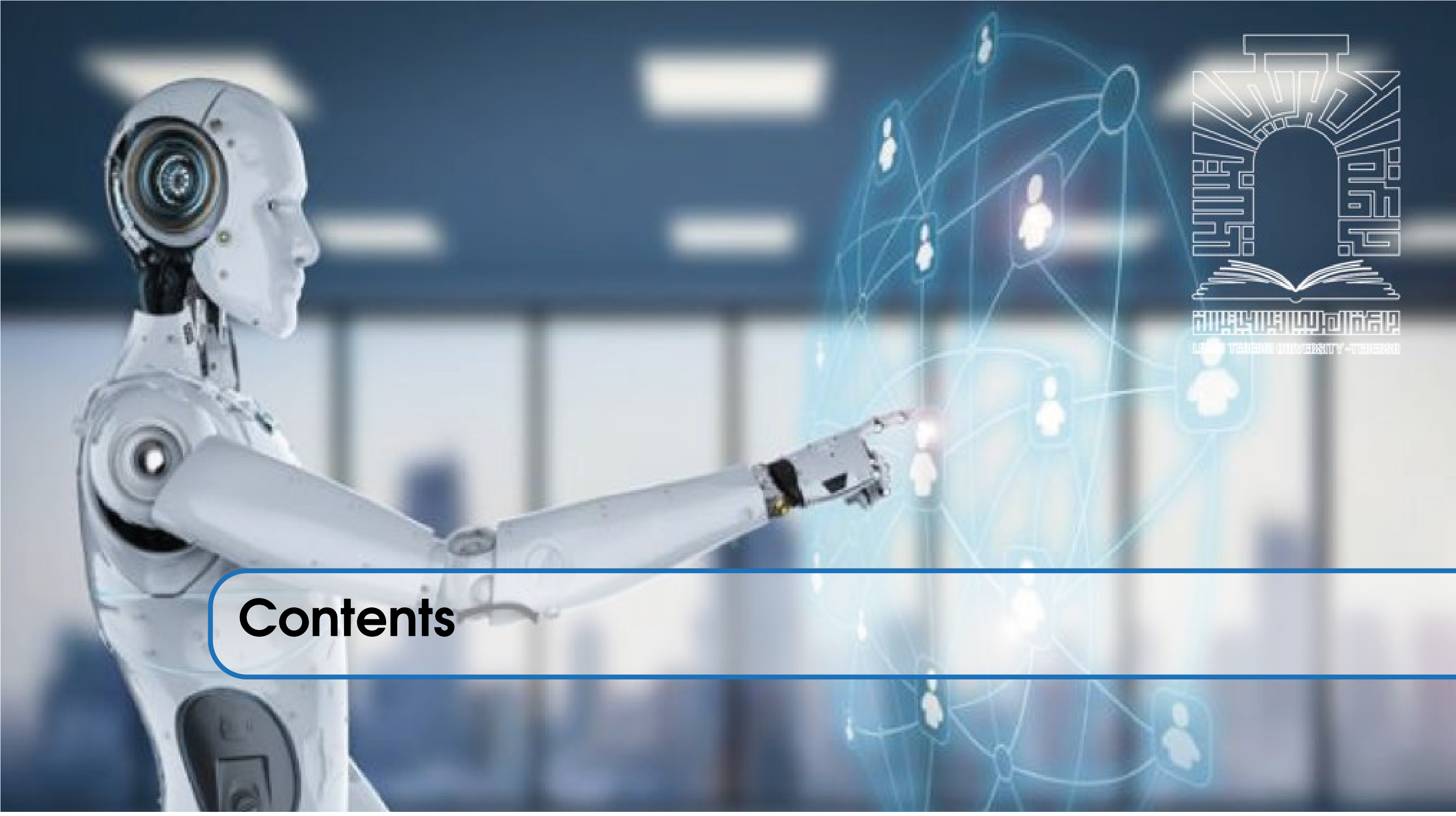

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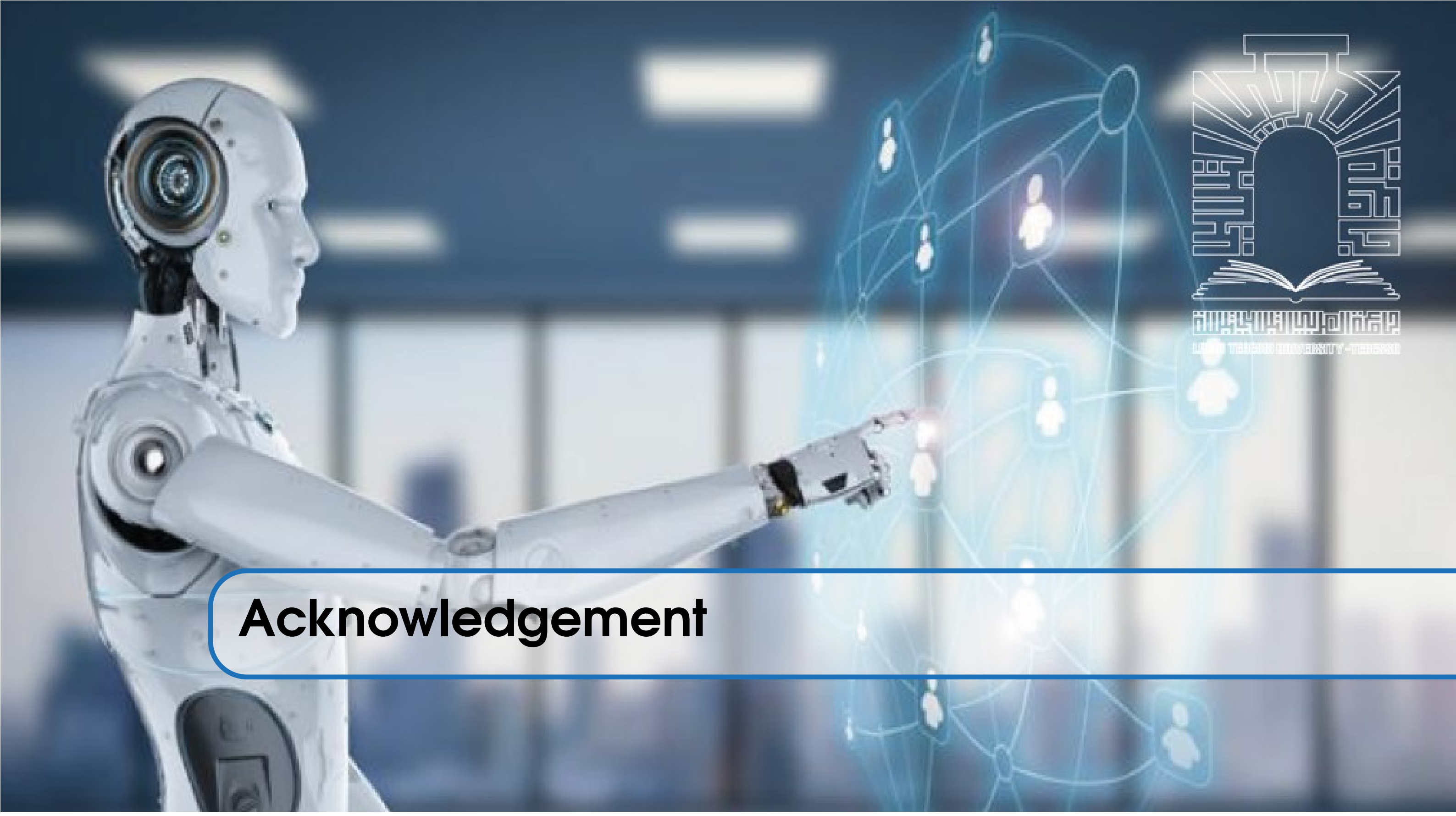
Dedication

To the two who gave me life...
To the two who gave me another life...
To the one who made me free...
To my sister who makes me great coffee...
To my brothers who I make them crazy...

"YOU CANNOT ALWAYS CONTROL WHAT GOES ON OUTSIDE.
BUT YOU CAN ALWAYS CONTROL WHAT GOES ON INSIDE."
WAYNE DYER



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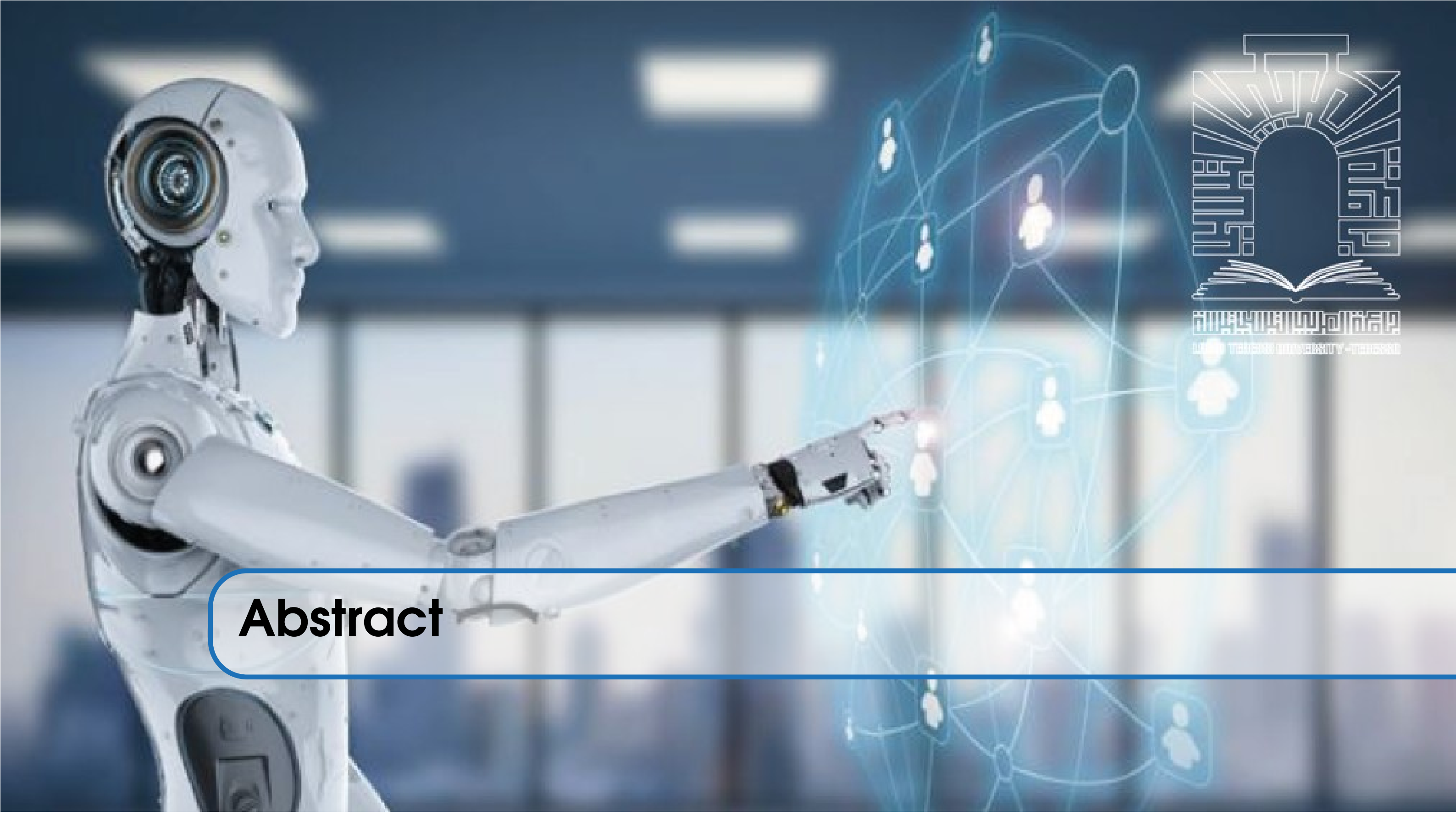


Abstract

Unusually, this thesis is divided into two different parts, the objective of the first part is to control the non-linear ODEs, the newly here is the combining the classic method of optimal control with the new concept of average control which is introduced by Zuazua, the modern notion is the average optimal control, thus, we up-date our cost function to an average cost function. So, thanks to one of the important optimality principles which is the Ponteryaguine Maximum Principle, we prove the uniqueness and the existence of the average optimal control, therefore, we arrive at the average optimal control characterization. To precise our results, we must use the shooting method for finding a simulation of that average optimal control.

The second part aims to control linear PDEs, where we combine the same notion of average control with the optimal control, we find a new average cost function. Because our distributed system has missing data, the way to characterize the optimality system changes, and it is divided into steps, first we describe the average no-regret control problem, then, using a quadratic perturbation to obtain average low-regret control, which helps us to find an average low-regret control characterization, finally, we can come back to the average no-regret control characterization.

The processed example in the first part is controlling the outbreak of an epidemic, To be precise, we study the control of an outbreak of COVID-19 in the city of Wuhan, China in December 2019. In the second part, we control an abstract hyperbolic-parabolic coupled system depending on an unknown parameter.

Keywords : Linear systems, Non-linear systems, Missing data, Distributed systems, Optimal control, Pontryaguine maximum principle, Shooting methods, Average control, Optimal average control, No-regret control, No-regret average control, Low-regret control, Low-regret average control, Mathematics modelling of COVID-19, Optimal control of COVID-19, The epidemic outbreak, Numerical analysis for COVID-19, Abstract systems, Abstract hyperbolic-parabolic systems, Coupled parameter, Optimality condition.

Frensh Abstract

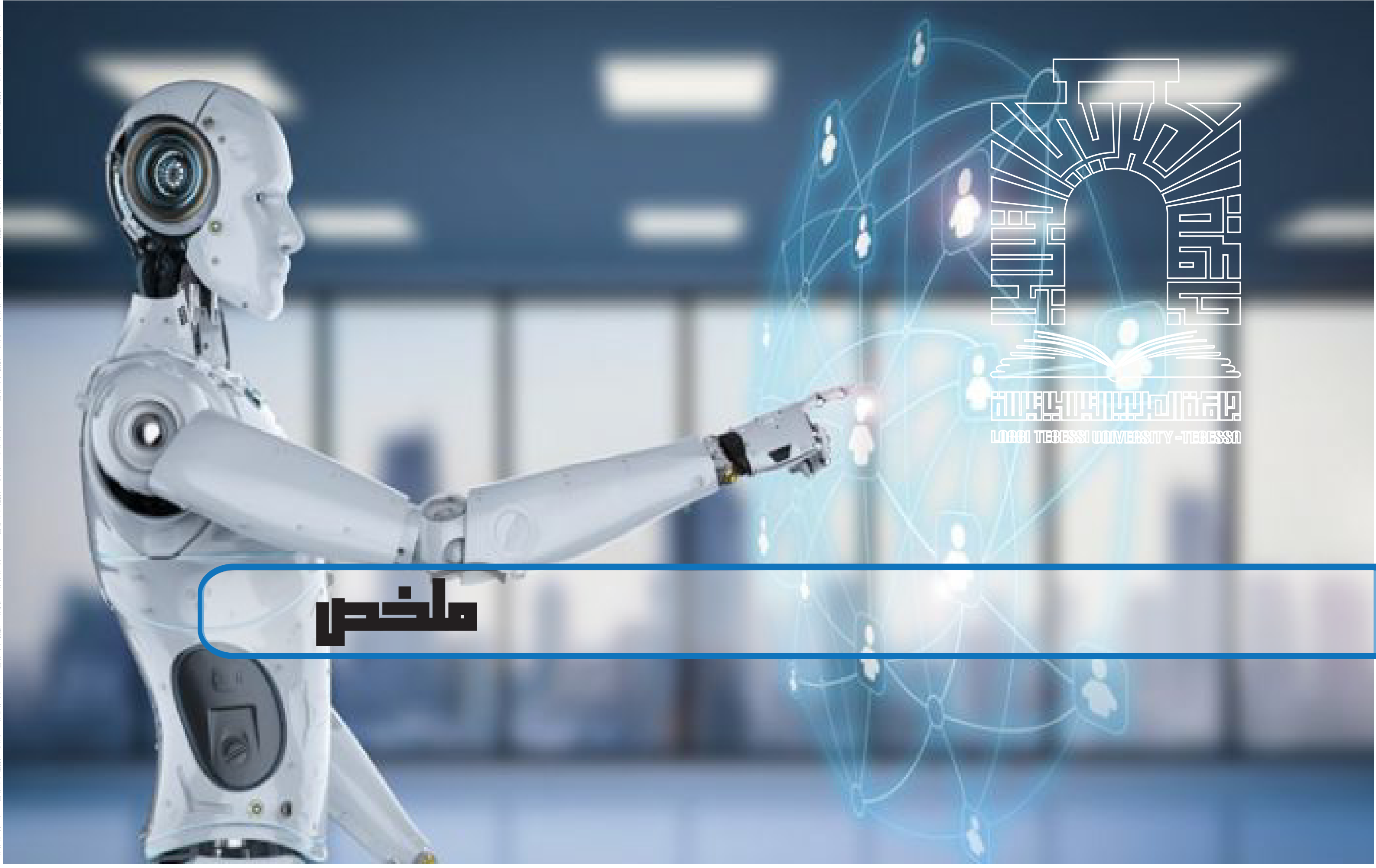
Exceptionnellement, cette thèse est divisée en deux parties différentes, l'objectif de la première partie est de contrôler les ODE non linéaires, la nouvelle ici est la combinaison de la méthode classique de contrôle optimal avec le nouveau concept de contrôle moyen qui est introduit par Zuazua, la notion moderne est le contrôle optimal moyen, ainsi, nous mettons à jour notre fonction de coût en fonction de coût moyenne. Ainsi, grâce à l'un des principes d'optimalité importants qui est le principe du maximum de Ponteryaguine, nous prouvons l'unicité et l'existence du contrôle optimal moyen, donc, nous arrivons à la caractérisation du contrôle optimal moyen. Pour préciser nos résultats, nous devons utiliser la méthode de tir pour trouver une simulation de ce contrôle optimal moyen.

La deuxième partie vise à contrôler les EDP linéaires, où l'on combine la même notion de contrôle moyen avec le contrôle optimal, on trouve une nouvelle fonction de coût moyenne. Parce que notre système distribué a des données manquantes, la façon de caractériser le système d'optimalité change, et il est divisé en étapes, d'abord nous décrivons le problème de contrôle moyen sans regret, puis, en utilisant une perturbation quadratique pour obtenir un contrôle moyen moindre regret, qui nous aide à trouver une caractérisation moyenne du contrôle moindre regret, enfin, nous pouvons revenir à la caractérisation moyenne du contrôle sans regret.

L'exemple traité dans la première partie est le contrôle de l'apparition d'une épidémie, plus précisément, nous étudions le contrôle d'une épidémie de COVID-19 dans la ville de Wuhan, en Chine en Décembre 2019. Dans la deuxième partie, nous contrôlons un système abstrait hyperbolique-parabolique couplé dépendant d'un paramètre inconnu.

Mots clés : Systèmes linéaires, Systèmes non linéaires, Données manquantes, Systèmes distribués, Contrôle optimal, Principe du maximum de Pontryaguine, Méthodes de tir, Contrôle moyen, Contrôle moyen optimal, Contrôle sans regret, Contrôle moyen sans regret, Contrôle moindre regret, Contrôle moyen moindre regret, Modélisation mathématique du COVID-19, Contrôle op-

timal du COVID-19, Analyse numérique du COVID-19, Systèmes abstraits, Systèmes abstraits hyperboliques paraboliques, Paramètre couplé, Condition d'optimalité.



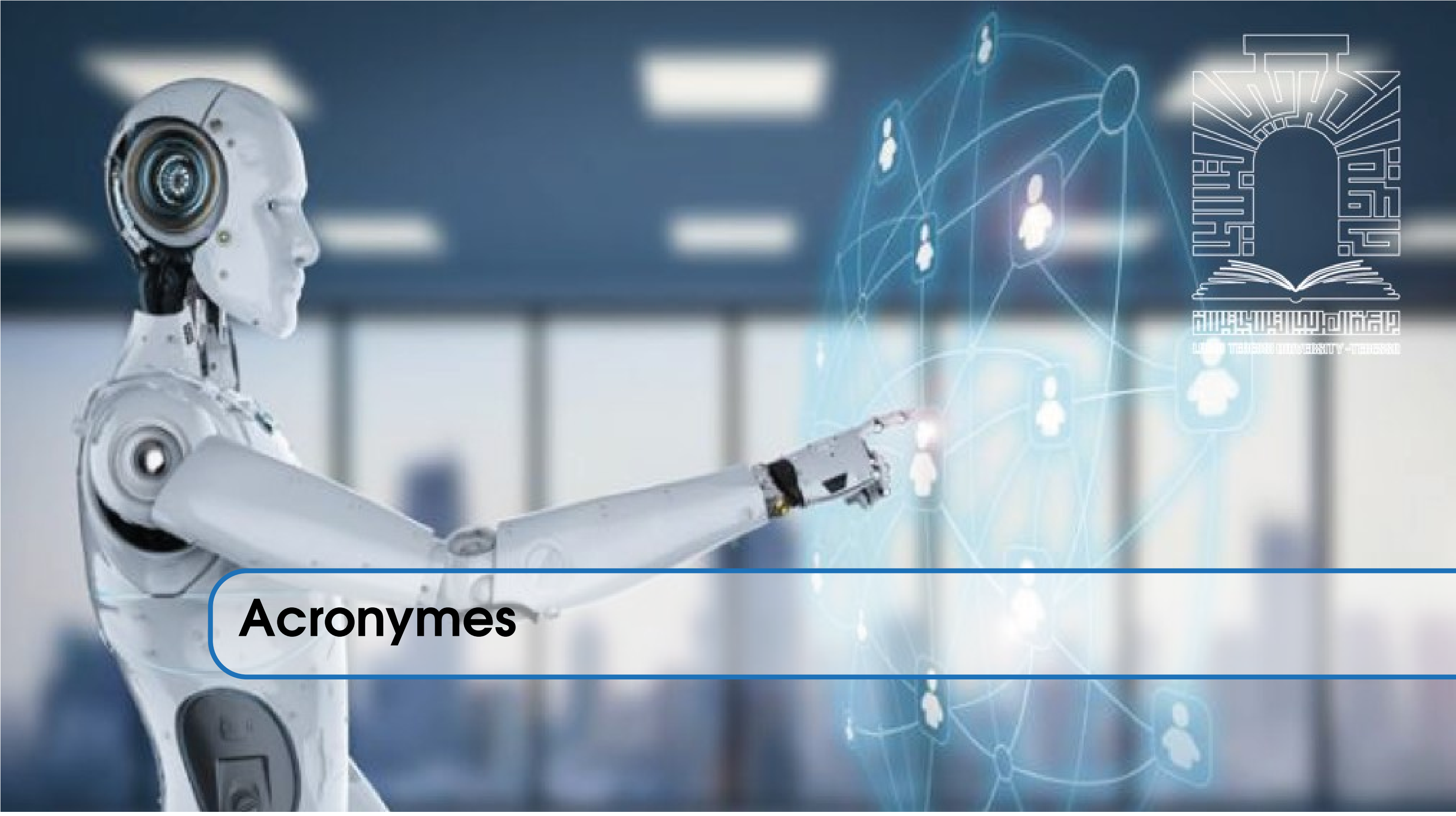
ملخص

بشكل غير عادي، يتم تقسيم هذه الأطروحة إلى جزئين مختلفين، والهدف من الجزء الأول هو التحكم في المعادلات التفاضلية العادية الغير خطية، والحديث هنا هو الجمع بين الطريقة الكلاسيكية للتحكم الأمثل مع المفهوم الجديد للتحكم المتوسط الذي قدمه Zuazua، المفهوم الحديث هو متوسط التحكم الأمثل، وبالتالي، نقوم بتحديث دالة التكلفة الخاصة بنا إلى دالة متوسط التكلفة. لذلك، بفضل أحد مبادئ الأمثلية الهامة وهو مبدأ Ponteryaguine الأقصى، نثبت وجود ووحدانية متوسط التحكم الأمثل، وبالتالي، نصل إلى توصيف متوسط التحكم الأمثل. لجعل نتائجنا دقيقة،استخدمنا طريقة التصوير لإيجاد محاكاة لمتوسط التحكم الأمثل.

يهدف الجزء الثاني إلى التحكم في المعادلات التفاضلية الجزئية الخطية، حيث نجمع بين نفس فكرة التحكم المتوسط والتحكم الأمثل، نجد دالة متوسط التكلفة الجديدة. نظرًا لأن الأنظمة التوزيعية ذات بيانات مفقودة،فإن طريقة توصيف نظام الأمثل تتغير، وهي مقسمة إلى خطوات، أولاً نصف مشكلة التحكم دون ندم المتوسطة، ثم باستخدام الاضطراب التربيعي للحصول على متوسط تحكم منخفض الندم، والذي يساعدنا في العثور على توصيف تحكم متوسط منخفض الندم، أخيرًا، يمكننا العودة إلى توصيف متوسط التحكم في عدم الندم.

المثال الذي تمت معالجته في الجزء الأول هو التحكم في تفشي الوباء، على وجه التحديد ندرس السيطرة على تفشي COVID19 في مدينة ووهان، الصين في ديسمبر 2019 في الجزء الثاني، نتحكم في نظام مجرد مقترن يعتمد على بيانات غير معروفة.

الكلمات الرئيسية: الأنظمة الخطية، الأنظمة غير الخطية، البيانات المفقودة، الأنظمة التوزيعية، التحكم الأمثل، مبدأ Ponteryaguine الأقصى، طريقة التصوير، متوسط التحكم،متوسط التحكم الأمثل ،التحكم بدون ندم، متوسط التحكم دون ندم، التحكم منخفض الندم،متوسط التحكم منخفض الندم، نمذجة الرياضيات لـ COVID19، التحكم الأمثل لـ COVID19، تفشي الوباء، التحليل العددي لـ COVID19، الأنظمة المجردة، أنظمة القطع القطعي المجردة، المعلمة المزدوجة، الحالة المثلى.



Acronymes

This table lists of the main acronyms used in this thesis.

Acronyms	Meaning
ODEs	Ordinary Differentials Equations.
PDEs	Partial Differentials Equations.
PMP	Ponteriaguine Maximum Principles.
COVID-19	Corona-virus Disease-2019.
TPBVP	Two Points Boundary value Problem.
P	Problem.
IVP	Initial Value Problem.
BHRP	Bats-Hosts-Reservoirs-People.
U_{ad}	The admissible control space.
a.e.	almost everywhere.
sup	The supremum.
inf	The infimum.
max	The maximun.
lim	The limit.
div	The divergence operator.

Notations

We introduce the necessary notations and definitions which are used in this thesis.

Notations	Meaning
N^*	The adjoint operator of the operator N.
$\partial\Omega = \Gamma$	Boundary of Ω.
χ_ω	The characteristic function of the set ω.
$\frac{\partial y}{\partial \upsilon} = \nabla y.\upsilon$	The co-normal derivative.
$\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})^T$	The gradient operator.
$\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$	The Laplace operator.
$\|.\|_X$	The norm of Banach space X.
$(.)_X$	A scalar product in Hilbert space X.
$\langle . \rangle_{X',X}$	Duality product between X and X'.
$\lvert . \rvert_X$	The semi-norm in X.
C^2	The functions class with continuous first and second derivative.
$D(\Omega)$	The space of functions in C^∞ with a compact support in Ω.
$D'(\Omega)$	The dual space of $D(\Omega)$.
$\mathscr{L}(X,Y)$	The space of linear bounded operators from X to Y.
$L^p(\Omega)$	Measurable functions f on Ω and $\int_\Omega \lvert f(y)\rvert^p dy < \infty$, $1 \leq p < \infty$.
$L^\infty(\Omega)$	Measurable functions f on Ω and $\exists c > 0 : \lvert f(y)\rvert < c$ a.e. on Ω.
$L^2([0,T],H)$	Space of L^2-integrable functions from $[0,T]$ to H.
$H^1(\Omega)$	$\left\{u \in L^2(\Omega),\ u_{x_i} \in L^2(\Omega),\ \forall i = \overline{1.n}\right\}$.
$H_0^1(\Omega)$	$H^1(\Omega)$, $u = 0$ on $\partial\Omega$.
$H^m(\Omega)$	$\left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\ \forall n \in \mathbb{N}^n,\ \lvert\alpha\rvert \leq m\right\}$.
$H^m(\Omega)$	$\left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\ \forall n \in \mathbb{N}^n,\ \lvert\alpha\rvert \leq m\right\}$.
$H^S(\Omega)$	$H^S(\Omega) = \left\{u \in L^2(\Omega),\ \int_\Omega \left(1 + \lvert\xi\rvert^2\right)^s \lvert\hat{u}\xi\rvert^2 d\xi < \infty\right\}$.
H'	The dual of H.
$\rightharpoonup$	The weak convergence symbol.

Figures List

This table lists of the main figures used in this thesis.

Tables List

This table lists of the main tables used in this thesis.

General Introduction

The theory of control makes it possible to bring a system from a given initial state to a certain final state by respecting certain criteria, this is the stage of carrying out the command. For example. Everyone balances a pendulum on their finger. On the other hand, it is much more difficult to balance on your finger an inverted double pendulum, Control theory allows everything to be done. But to effectively achieve this balance, it is better to have a good mathematical model and know how to solve the equations. A wing on which we act with the accelerator and brake pedals, and that we guide with the steering wheel is an example of a control system, a dynamic system on which we can act by means of a command represented by the brake.

A control system is a dynamic system on which one can act by means of command. To precisely, define the concept of the control system, it is necessary to use mathematical language. Each system has a specific structure, properties, and purposes. Note that this concept can describe both discrete and continuous transformations. This, therefore, makes it possible to model the operation of robots, adaptive systems with variable structures, ... By considering all these objects as control systems, we are interested in their behaviour and their functional characteristics, without necessarily attaching importance to their internal or intrinsic properties. Therefore, two control systems having, in some sense, the same behaviour and similar characteristics, are considered identical. Nowadays, automated systems are completely part of our daily life, the goal is to improve our quality of life and to facilitate certain tasks.

Historically, control theory is linked on the one hand with the calculus of variations as in [**51**] and on the other hand with the resolution of ordinary differential equations. For the first time, Johann Bernoulli submitted the brachistochrone[1] problem corresponding to the problem of the fastest trajectory between two points, in 1696, Leibniz, and his brother Jacques Bernoulli found the solution. The classic method for solving the problem is the calculus of variations. This is considered a pioneering result in the field of optimal control. This theory, which is an extension of the calculus of variations, deals with how to find a control law for a system, modelled by a set of differential equations describing the state and control trajectories, such that a certain optimality

[1] A curve between two points along which a body can move under gravity in a shorter time than for any other curve.

criterion is met. Optimal control problem solving started with the famous Pontryaguine Maximum Principle (PMP) [**35**], which provides a necessary condition for optimality.

Partial differential equations control theory has evolved a lot in recent years and is a growing area of research. Each year new profound results are demonstrated and new directions of research are taken. The control of conservation laws, linear or non-linear PDEs, the interactions between finite and infinite dimension both at the theoretical level and on the occasion of the passage of the continuum discrete during the numerical discretization of control problems, are recent sectors in full development, not to mention the growing interest in PDEs systems. The field of systems of coupled parabolic equations reveals many unexpected phenomena and opens up very varied fields of research in which many teams have embarked. These directions indicate challenges in research both at the theoretical and methodological level and at the level of applications and their digital processing. The question of effective control, from the theoretical problem to its implementation and its robustness is an equally important aspect from the application's viewpoint..

The area of control of infinite-dimensional systems PDEs has been under development since at least the 1960s. Some of the initial efforts that laid the foundations of the field in the late 1960 and early 1970 were in optimal control of linear PDEs systems. Part of this thesis deals with the optimal control of distributed systems with missing data, which drives us to think about the no-regret and the low-regret controls, where Jacques Louis Lions [**25**], [**28**], and [**29**] was the first luckiest mathematician who described and introduced those notions, especially for the problems with missing or incomplete data, collected with the new concept of the average control which was developed by the Great scientist Enrique Zuazua [**55**].

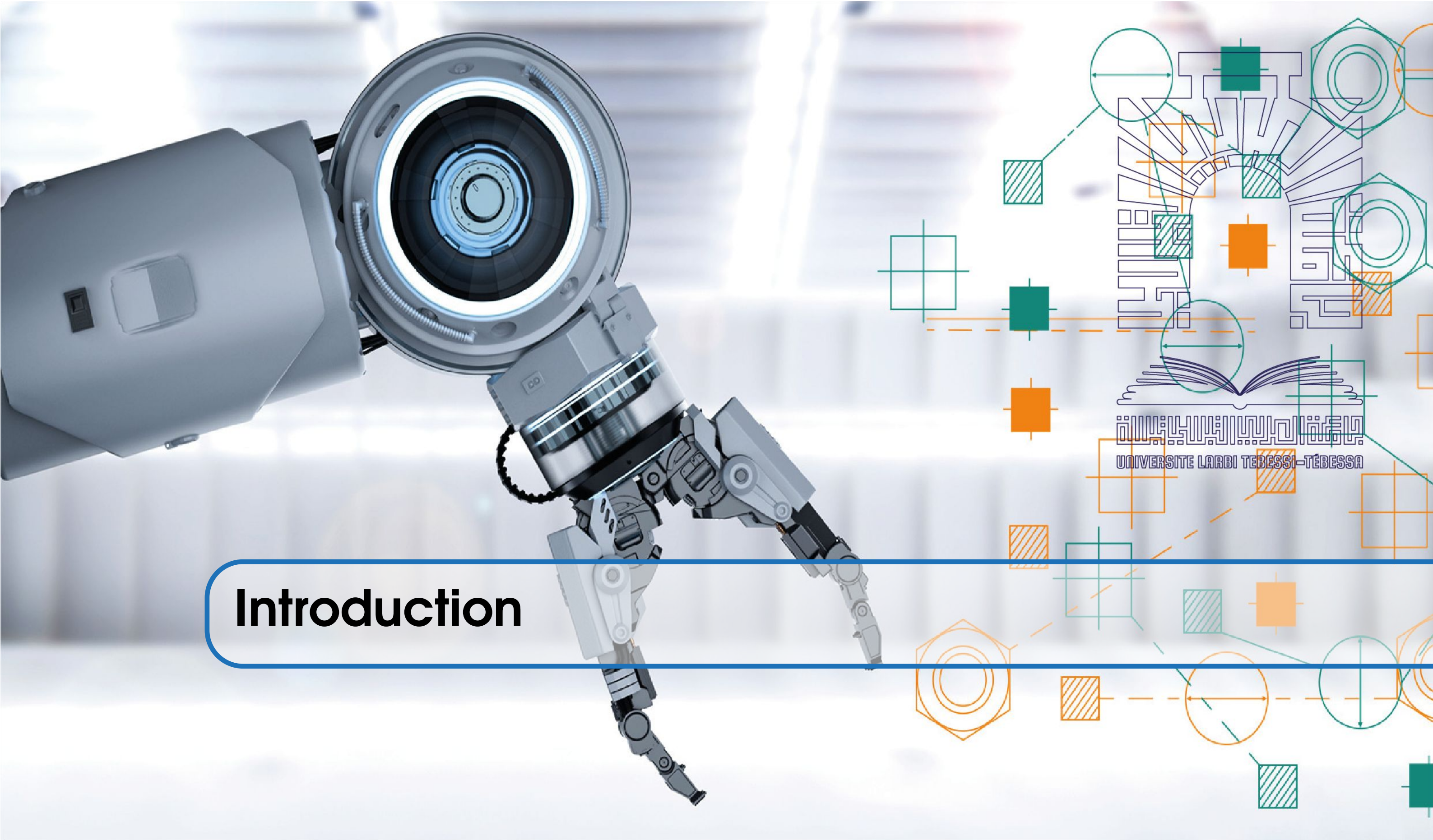
The lofty goal of this thesis is to control the average system state with respect to the unknown parameter, whatever the conditions of the missing data.

Actually, I started my scientific research in control theory on linear PDEs, but, because of the outbreak of COVID-19 in the whole world, during that time a lot of people died. As a researcher, my inquisitive sense woke me up, and push me into the fashion world of scientific research, and the result was a scientific article in a Scopus journal on controlling the outbreak of COVID-19 in China, which was control of nonlinear ODEs systems. What makes reading the thesis entertaining and interesting is that it is rich in two different axes which are divided into two parts organized as follows:

⋇ The first part will contain two chapters, the first chapter will present and discusses some concepts of optimal control in non-linear finite which is based on the PMP and consists in reducing the problem of control to a boundary value problem, then, we resolve it numerically by a shooting method, with some examples for different systems. The second chapter deals with the mathematical modelization of the propagation of the COVID-19 in the city of Wuhan, China. Afterword, this chapter introduces and proves the uniqueness and existence of the average optimal control using the maximum principle and obtaining the optimality systems of the epidemic COVID-19 and solves that system numerically the Shooting function $G(y_0)$ in MATLAB.

⋇ The second part presents and defines the different notions of optimal control of linear PDEs, it contains also two chapters, the first one discusses and introduces some concepts of optimal control and average control in infinite dimensions, with some examples in different systems (parabolic, hyperbolic, elliptic, and abstract coupled systems), it introduces the no-regret, low-regret control and the new notion of the average control, it shows also the existence and uniqueness of the optimal control. The other chapter will discuss the optimal control of an abstract system. also we will introduce new results about the average optimal control of an abstract hyperbolic-parabolic system depending on an unknown parameter with missing initial conditions.

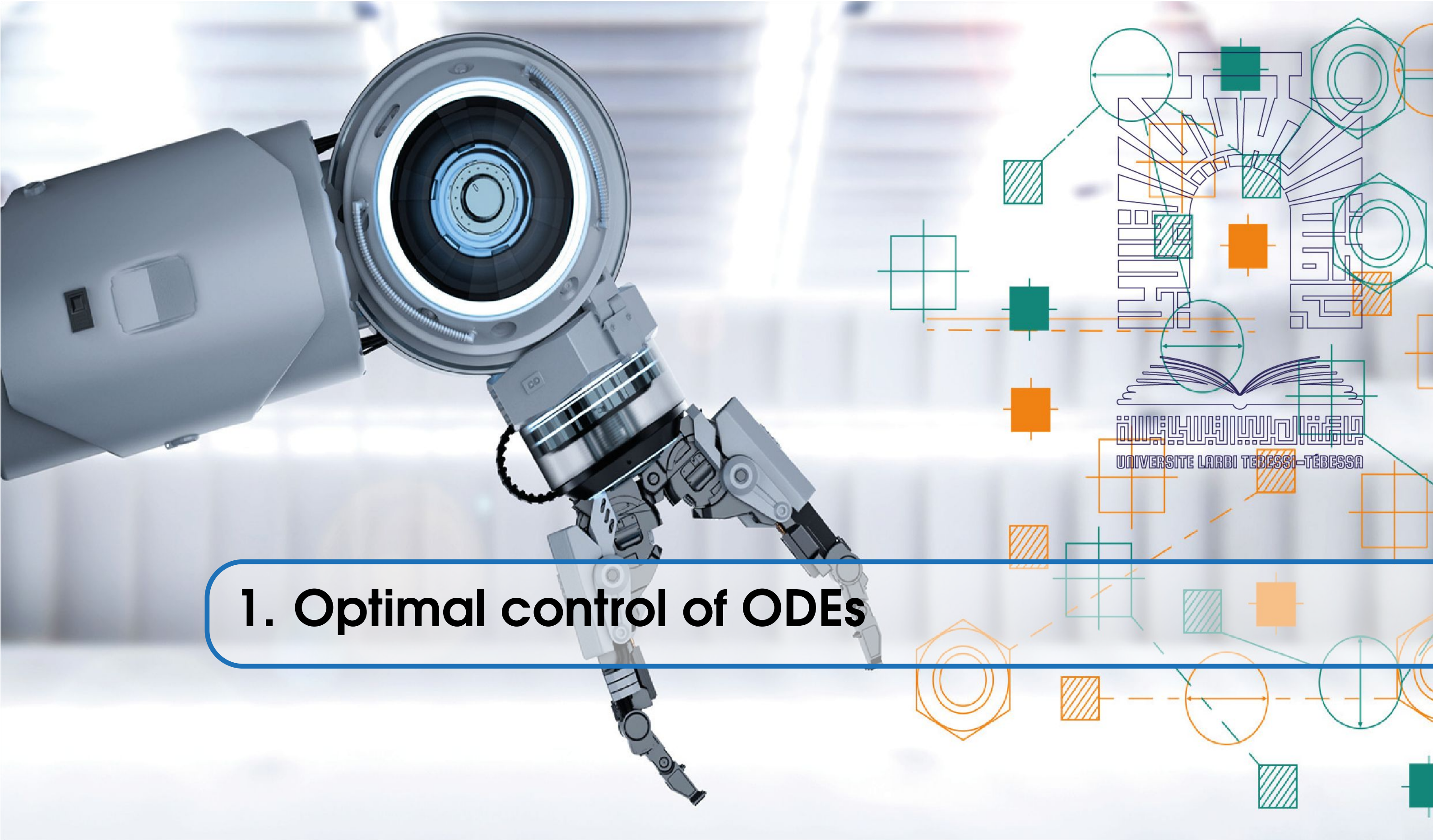
I Control of non-linear ODEs



Introduction

The optimal control is among the best methods of solving and improving the performance of control systems, at the same time, it always seeks the best and most favourable states possible. The problem of the optimal control consists in finding the control minimizing certain chosen criteria. Today the field of the application of optimal control extends from tracking to shape optimization. It is a very important part of engineering techniques. The object of this part is to formulate in a more general way the problem of the optimal control of systems and to present the theory and the practice of this form of control.

On the other hand, consist in applying the Ponteriaguine maximum principle (PMP)[see [35]], which gives necessary conditions of first-order optimality and results in a two boundary value problem. We then seek the trajectories verifying these conditions, which in practice amounts to seeking the zero of a certain shooting function associated with the original problem. These methods are both precise and fast, but on the other hand, they are very sensitive to initialization [36]. Also, this part shows a great and new application of the PMP is on the COVID-19 which is in charge of the current outbreak of pneumonia that began at the beginning of December 2019 near Wuhan City, Hubei Province, China.



1. Optimal control of ODEs

This chapter is dedicated to introducing the study of a resolution method of optimal control problems, which is precise and fast at the same time. It is based on the PMP and consists in reducing the problem of control to a boundary value problem, then, we resolve it numerically by a shooting method. This last consists of the search for zero of the associated shooting function. Also, it is rich with four sections which contain the optimization criterion, PMP, shooting method, and some applications of optimal control to different non-linear ODEs.

1.1 Average control

In some distributed systems, often parameters are not fully known, in this situation to control such kind of systems we look for controls independent of the unknown parameters. To control this systems Zuazua in [55] introduced the notion of "averaged control". The main the idea is to control the average of the state with respect to the unknown parameter instead of controlling the state itself.

1.2 Principals of optimal control

1.2.1 Diagram of the optimal control

The goal is to be able to control the system, therefore to bring the output to a certain value thanks to the action on the input while respecting certain constraints (Criteria to be minimized or maximized). The question is: what is the best control, given these constraints, leading to the desired output? The answer is Optimal control. The optimal control is a state feedback control, it is given by the next diagram

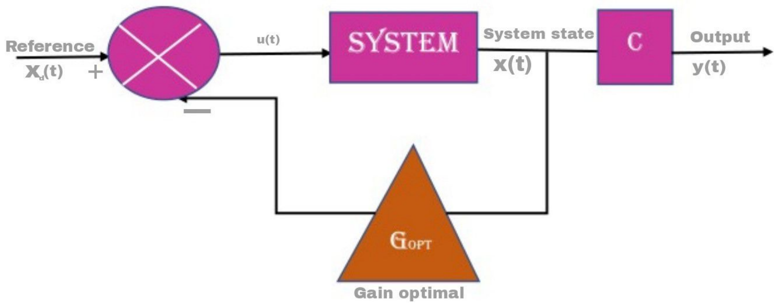


Figure 1.1 : Control diagram by state feedback.

The control law is given by the next relation

$$u(t) = x_d - G_{opt}.x(t) = \omega(t) - G_{opt}.x(t),$$

where :
• $\omega(t) = x_d$ represents the reference or desired state (can be vector or element).
• $x(t)$ represents the vector of system state variables.
• G_{opt} represents the optimal feedback or control gain (can be matrix vector).
The optimal control by state feedback is the control of the systems modelled in the state space. Its optimal design is obtained based on a cost function and an optimality criterion chosen in such a way that they make it possible to find the values of the optimal return gain G_{opt} among the different possible gains, that is to say, find the optimal control law among the different admissible controls $u(t)$ possible, the one that allows both :
• Check the given initial and final conditions.
• Satisfy various imposed constraints.
• Optimize a chosen criterion.

1.2.2 Optimization criterion

In many optimal control problems, where the objective is to determine and generate optimal solutions for a criterion of our choice, this criterion is represented as a sum of two terms :

$$J(x,u,t) = \int_{t_0}^{t_f} f^0(x(t),u(t),t)dt + g(x(t_f),t_f), \tag{1.1}$$

such that : f^0 and g are scalar functions which are given at the times t_0.
In equation (1.1), the first element of the left side the integral of the function $\int_{t_0}^{t_f} f^0(x(t),u(t),t)dt$ is evaluated along the trajectory y obtained in the output space for $t \in [t_0,t_f]$. Otherwise, it is the final element favoured in the criterion $J(x,u,t)$, it takes into account the initial, intermediate and final states.
The rest term $g(x(t_f),t_f)$ is given in the criterion. $J(x,u,t)$ is a function of the final state of the system, which represents the cost of the final deviation on the output and on the time compared to their desired values, therefore, this term does not intervene in the calculations. Consequently, the optimization of the criterion given by the function (1.1) amounts to that of the criterion

$$J = \int_{t_0}^{t_f} f^0(x(t),u(t),t)dt.$$

There are some different criteria as :
• Lagrange criterion

$$J = \int_{t_0}^{t_f} f^0(x(t),u(t),t)dt.$$

• Mayer criterion

$$J = g(x(t_f), t_f).$$

The functions f^0 and g are given scalar functions as indicated previously.
The choice of the criterion is very important and a command which minimizes a given criterion is not necessarily interesting if the criterion is badly chosen or does not take into account the physical constraints imposed or the desired performance of the system.
The general formulation of the optimization criterion can be presented in various aspects.
Depending on the properties of the considered system and the objective of the optimization problem, the classical forms correspond to the minimization problem :
• The energy provided by the implementation of the control.
• The horizon time $[t_0, t_f]$.
• The energy consumption.

Main optimization criteria

There are three families of fundamental problems in optimal control :

⋆ The control in minimum time

Minimum time control is a typical application of the maximum principle. This type is encountered, for example, in safety or manufacturing problems. The criterion used is then written as follows

$$J = \int_{t_0}^{t_f} dt.$$

⋆ The minimum consumption control

The criterion corresponds to the integral of a flow

$$J = \int_{t_0}^{t_f} |u|\, dt.$$

⋆ Minimum energy control

Which is the integral of a power

$$J = \int_{t_0}^{t_f} u^2 dt.$$

1.3 Optimality principles

To be able to determine an optimal control, we must choose, then, apply a principle of optimality. There are different principles, among which we declare the following criteria :
• The Pontryaguine Maximum Principle.
• The Bellman Optimality Principle.
• Euler Lagrange equations.
In the following, we will mention only The PMP, which is formulated by Pontriaguine and his collaborators in 1956, they gave the necessary optimality conditions, the same time, they make it possible to calculate the optimal trajectories.

1.3.1 The Pontryaguine maximum principle

From a global point of view, an optimal control problem is formulated on a set M. The general problem of optimal control is the next

Consider a general control system

$$\begin{cases} \dot{x}(t) & = & f(t,x(t)u(t)), \\ x(t_0) & = & x_0, \end{cases} \tag{1.2}$$

where :
• f is a class C^1 subset of $I \times V \times U \subset \mathbb{R}^n$, $I \subset \mathbb{R}$.
• $V \subset \mathbb{R}^n$ is an open subset.
• $U \subset \mathbb{R}^m$ an open subset of admissible controls, which is bounded and continuous piecewise on $(x_0, t_0) \in V \times I$.
Furthermore, we assume that the controls $u(.)$ belong to a subset of $L^1_{loc}(I, \mathbb{R}^m)$.
These assumptions ensure, for any control u, the existence and uniqueness on a maximal solution $x_u(t)$ on a set $J \subset I$, of the Cauchy problem (1.2). For ease of writing, we assume in the next that $t_0 = 0$.
For any control $u \in L^\infty_{loc}(I, \mathbb{R}^m)$, the associated trajectory $x_u(.)$ is defined on a maximal set $[0, t_\varepsilon(u)[$, where $t_\varepsilon(u) \in \mathbb{R}^+ \cup \{+\infty\}$. For example if $t_\varepsilon(u) < +\infty$ then the trajectory explodes in t_ε. For all $T \in I, T \geq 0$, we represent by U_{ad} the set of admissible controls on $[0,T]$, i.e. the set of controls such that the associated trajectory is well defined on $[0,T]$. In other words $T < t_\varepsilon(u)$.
Let f^0 be a function of class C^1 on $I \times V \times U$, and g a continuous function on V. For any control $u \in U_{ad}$ we define the cost function of the associated trajectory $x_u(.)$ over the interval $[0,T]$

$$J(T,u) = \int_0^T f^0(t, x_u(t), u(t))dt + g(T, x_u(T)). \tag{1.3}$$

Let M_0 and M_1 be two subsets of V. The optimal control problem is to determine the trajectories $x_u(.)$ solutions of

$$\dot{x}_u(t) = f(t, x_u(t), u(t)).$$

such that : $x_u(0) \in M_0$, $x_u(T) \in M_1$, and minimizing the cost function $J(T,u)$. We say that the optimal control problem has an unfixed final time if the final time T is free, otherwise, we speak of a fixed final time problem.
The Pontryaguine maximum principle is difficult to demonstrate in all its generality.

General statement

Theorem 1.3.1 We consider the next control system

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{in } \mathbb{R}^n,$$

where :
• $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is of class C^1.
• The controls are measurable and bounded sets defined on a $[0, t_\varepsilon(u)[\cup \mathbb{R}^+$ with values in $\Omega \subset \mathbb{R}^m$.
Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. We denote by U_{ad} the set of admissible controls u whose associated trajectories connect an initial point of M_0 to an end point of M_1 in time $t(u) < t_\varepsilon(u)$.
Moreover, we define the cost of a control u on $[0,t]$

$$J(T,u) = \int_0^t f^0(s, x_s(t), u_s(t)ds + g(t, x(t)),$$

where :
• $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are C^1.

- $x(.)$ is the solution trajectory of our differential equation associated with the control u.

We consider the next optimal control system
Determine a trajectory linking M_0 to M_1 and minimizing the cost function. The final time may or may not be fixed.
If the control $u \in U_{ad}$ associated with the trajectory $x(.)$ is optimal on $[0,T]$, then there exists an absolutely continuous application $p(.) : [0,T] \longrightarrow \mathbb{R}^n$ called the adjoint vector, and a real $p^0 \leq 0$, such that the pair $(p(.), p^0)$ is non-trivial, for almost any $t \in [0,T]$ we have

$$\begin{cases} \dot{x}(t) & = & \dfrac{dH}{dp}(t,x(t),p(t),p^0,u(t)), \\ \dot{p}(t) & = & -\dfrac{dH}{dx}(t,x(t),p(t),p^0,u(t)), \end{cases}$$

where, $H(t,x,p,p^0,u) = \langle p, f(t,x,u)\rangle + p^0 f^0(t,x,u)$ is the Hamiltonian of the system, and we have the next maximization condition almost everywhere on $[0,T]$

$$H(t,x(t),p(t),p^0,u(t)) = \max_{v\in\Omega} H(t,x(t),p(t),p^0,v).$$

If the final time to reach the target M_1 is not fixed, we have at the final time T the next condition

$$\max_{v\in\Omega} H(T,x(T),p(T),p^0,v) = -p^0 \frac{\partial g}{\partial t}(T,x(T)).$$

R If control u is continuous at time T , previous condition can be described as follows

$$H(T,x(T),p(T),p^0,u(T)) = -p^0 \frac{\partial g}{\partial t}(T,x(T)).$$

If moreover M_0 and M_1 (or just one of the two sets) are manifolds[1] of $\mathbb{R}^n$ having tangent spaces at $x(0) \in M_0$ and $x(T) \in M_1$, then the adjoint vector can be constructed in such a way as to satisfy the transversality conditions at both ends (or just one of the two)

$$p(0) \perp T_{x(0)} M_0, \tag{1.4}$$

and

$$p(T) - p^0 \frac{\partial g}{\partial x}(T,x(T)) \perp T_{x(T)} M_1. \tag{1.5}$$

R Under the conditions of the theorem, moreover, we have

$$\frac{d}{dt} H(t,x(t),p(t),p^0,u(t)) = \frac{\partial H}{\partial t}(t,x(t),p(t),p^0,u(t)), \ \ \forall t \in [0,T].$$

In particular, if the augmented system is autonomous, i.e., if f^0 and f do not depend on t, then H does not depend on t, and we have

$$\max_{v\in\Omega} H(t,x(t),p(t),p^0,v) = constant.$$

Then, note that this equality is valid on $[0,T]$ (indeed this function of t is Lipschitz).

[1] It is a topological space that locally resembles Euclidean space near each point.

Transversality conditions

Transverse conditions on the adjoint vector

In this paragraph, the final time to reach the target can be fixed or not. Let us rewrite conditions (1.4) and (1.5) in the following two important cases.

⋆ **Lagrange problem.** In this case ($g = 0$), we can write the cost function as follows

$$J(t,u) = \int_0^t f^0(s, x_s(t), u_s(t))ds.$$

The transversality conditions (1.4) and (1.5) on the adjoint vector are then written

$$p(0) \perp T_{x(0)}M_0 \quad , \quad p(T) \perp T_{x(T)}M_1.$$

⋆ **Mayer problem.** In this case ($f^0 = 0$) the cost function is written as bellow

$$J(t,u) = g(t, x(t)).$$

The transversality conditions (1.4) and (1.5) are not simplified a priori, but in the particular case where $M_1 = \mathbb{R}^n$. In other words the final point $x(T)$ is free, the condition (1.5) becomes

$$p(T) = p^0 \frac{\partial g}{\partial x}(T, x(T)).$$

If moreover g does not depend on time, we usually write $p(T) = p^0 \nabla(x(T))$. In other words, the deputy vector at the final time is equal to within the constant p^0, to the gradient of g taken at the final point.

Transversality condition on the Hamiltonian

The next condition

$$\max_{v \in \Omega} H(T, x(T), p(T), p^0, v) = -p^0 \frac{\partial g}{\partial t}(T, x(T))$$

is valid only if the final time to reach the target is not fixed. In this paragraph, therefore, we place ourselves in this case.

The only notable simplification of this condition is the case where a function g does not depend on time t, and the previous transversality condition on the Hamiltonian then becomes as bellow

$$\max_{v \in \Omega} H(T, x(T), p(T), p^0, v) = 0,$$

or, if u is continuous at time T,

$$H(T, x(T), p(T), p^0, u(T)) = 0,$$

In other words, the Hamiltonian cancels out at the final time.

1.4 Some applications of Optimal Control

1.4.1 Optimal control of a non-linear spring

Let us consider the next control system

$$\begin{cases} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) - 2x^3(t) + u(t), \end{cases}$$

where, we allow as controls all the piecewise continuous functions $u(t)$, such that, $|u(t)| \leq 1$.
The objective is to bring the spring from any initial position $(x_0, y_0 = \dot{x}_0)$ to its equilibrium position $(0,0)$ in minimal time t_*.
According to the application of the PMP, the Hamiltonian of the previous system is written as follows

$$H(x,p,u) = p_x y + p_y(-x - 2x^3 + u) + p^0.$$

If (x,p,u) is an extremal then we must have :

$$\begin{aligned}
\dot{p}_x &= -\frac{\partial H}{\partial x} = p_y(1+6x^2), \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -p_x.
\end{aligned}$$

note that, since the adjoint vector (p_x, p_y) must be non-trivial, p_y cannot vanish on a set (we would also have $(p_x = -\dot{p}_y = 0)$. Moreover, the maximization condition gives us

$$u(t) = sign(p_y(t)) \text{ where } p_y \text{ is a solution of } \left\{ \begin{array}{lcl} p_y(t) + p_y(t)(1+6x^2(t)) &=& 0, \\ p_y(t_*) &=& \cos\alpha, \\ \dot{p}_y(t_*) &=& \sin\alpha, \end{array} \right.$$

the parameter $\alpha \in [0, 2\pi[$ being undetermined.
by reversing the time $(t \to -t)$, it is clear that, our problem is equivalent to the minimal time problem for the following system

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& -y(t), \\ \dot{p}_y(t) &=& p_x(t), \\ \dot{y}(t) &=& x(t) + 2x^3(t) - sign(p_y(t)), \\ \dot{p}_x(t) &=& -p_y(t)(1+6x^2(t)), \end{array} \right.$$

$$\text{with :} \quad \left| \begin{array}{lllll} x(0) = 0 & , & x(t_*) = x_0 & , & p_y(0) = \cos\alpha, \\ y(0) = 0 & , & y(t_*) = y_0 & , & p_x(0) = \sin\alpha, \end{array} \right.$$

where, $\alpha \in [0, 2\pi[$ is to be determined.

1.4.2 Optimal transfer of computer files

A file of x_0 Mb must be transferred by the network. At each time t we can choose the transmission rate $u(t) \in [0,1]\, Mb/s$, but it costs $u(t)f(t)$, where, $f(.)$ is a known function. Furthermore, at the final time we have an additional cost γt_f^2, where $\gamma > 0$. The system is therefore :

$$\dot{x} = -u, \quad x(0) = x_0, \quad x(t_f) = 0.$$

We have to minimize the next cost function

$$J(t_f, u) = \int_{t_0}^{t_f} u(t) f(t) dt + \gamma t_f^2.$$

In this case we have

$$f^0 = uf \quad \text{and} \quad g = \gamma t^2.$$

The Hamiltonian is $H = -pu + p^0 fu$. Since $\dot{p} = 0$, we have $p(t) = Cste = p$. Furthermore

$$u(t) = \left\{ \begin{array}{lll} 0 & if & -p + p^0 f(t) < 0, \\ 1 & if & -p + p^0 f(t) > 0, \end{array} \right.$$

$u(t)$ is undetermined if $-p+p^0 f(t)=0$ on a subinterval. At the final time, we have

$$H(t_f) = -p^0\frac{\partial g}{\partial t} = -2p^0\gamma t_f,$$

hence,

$$-u(t_f)(p+f(t_f)) = -2p^0\gamma t_f.$$

If $p^0=0$, then necessarily $p\neq 0$, and $u(t)$ is constant, therefore necessarily $u(t)=1$, but then the above relation implies $p=0$, which is absurd. So, $p^0=-1$. It is clear that at the final time t_f we have $u(t_f)=1$ (otherwise u would not be optimal, because of the term γt_f^2), and therefore, $p=-2\gamma t_f - f(t_f)$. Ultimately

$$u(t) = \begin{cases} 0 & if \quad f(t) > 2\gamma t_f + f(t_f), \\ 1 & if \quad f(t) < 2\gamma t_f + f(t_f). \end{cases}$$

1.4.3 Optimal control of consumption and production

Assume that, we have a factory for which we can control production. As we construct the mathematical model by setting

$$x(t) = \text{ quantity produced at time } t.$$

Suppose, we consume a fraction of our production at each instant t, as well, we reinvest the remaining fraction. Note the fraction of remaining production at time t by $u(t)$.
$u(t)$ will be our control which we will also subject to the next constraint

$$0 \leq u(t) \leq 1 \text{ at each instant } t.$$

The production of our factory is governed by the dynamic next system

$$\begin{cases} \dot{x}(t) = ku(t)x(t), \\ x(t_0) = x_0 = 0, \end{cases}$$

where k is a constant that represents the growth rate of our reinvestment. Let us take as the next cost function

$$J(u(t)) = \int_0^T (1-u(t))x(t)dt$$

which means that, we seek to maximize the total consumption of the quantity produced. Our consumption at a given instant t is $(1-u(t))x(t)$.
We will now, seek to characterize an optimal control of our problem. For this we apply the PMP. We have $n=m=1$:

$$f(x(t),u(t)) = x(t)u(t), \;\; g=0, \;\; r(x,u) = (1-u)x.$$

Now, let us calculate the Hamiltonian of the previous system which reads as follows

$$\begin{aligned} H(x(t),p(t),u(t)) &= f(x(t),u(t))p(t) + r(x(t),u(t)) \\ &= x(t)u(t)p(t) + (1-u(t))x(t) \\ &= x(t) - u(t)x(t)(p(t)-1). \end{aligned}$$

The dynamic equation of the system is

$$\dot{x}(t) = H_p = \frac{\partial H}{\partial p} = u^*(t)x(t), \tag{1.6}$$

and the next adjoint equation (vector)

$$\dot{p}(t) = -H_x = -\frac{\partial H}{\partial x} = -1 + u^*(t)(p(t)-1). \tag{1.7}$$

Transversality conditions The final condition can be expressed as follows

$$p(T) = g_x(x(t)) = 0. \tag{1.8}$$

Finally, the maximum of the Hamiltonian

$$H(x(t), p(t), u^*(t)) = \max_{0 \le u \le 1} \{x(t) + u(t)x(t)(p(t) - 1)\}. \tag{1.9}$$

Use of the maximum principle We deduce the useful information from the equations of the system as well as the next equations : (1.6), (1.7), (1.8) and (1.9).
From (1.9), at any instant t, the value $u^*(t)$ must be chosen so as to maximize $u(p(t) - 1)$ for $0 \le u \le 1$. As $x(t) > 0$, then the extremal solution is equal

$$u^*(t) = \begin{cases} 0 & if \quad p(t) \le 1, \\ 1 & if \quad p(t) > 1. \end{cases}$$

Thus, it remains to know $p(.)$ to determine the optimal control $u^*(.)$. We are going to solve the next system

$$\begin{cases} \dot{p}(t) &= -1 - u^*(t)(p(t) - 1), \;\; t \in [0, T] \\ p(T) &= 0. \end{cases}$$

Since $p(T) = 0$, we conclude by continuity that, $p(t) \le 1$ for all t close enough to T, $t < T$. Consequently $u(t) = 0$ for these values of $t \in V(T)$.
So, $\dot{p}(t) = -1$, and thus $p(t) = T - t$ for the times t which are in this interval. We thus have $p(t) = T - t$ for the instants satisfying $p(t) \le 1$, and this occurs for $T - 1 \le t \le T$.
But, for times $t \le T - 1$, with t close to $T - 1$, we have $u(t) = 1$, and according to (1.7) we will have

$$\dot{p}(t) = -1 - (p(t) - 1) = -p(t).$$

As long as $p(T - 1) = 1$, we have $p(t) = e^{T-1-t} > 1$ for all times $(0 \le t \le T - 1)$. In particular, there are no commutation points above this set.
Therefore, we deduce that the optimal control is

$$u^*(t) = \begin{cases} 1 & if \quad 0 \le t \le t^*, \\ 0 & if \quad t^* \le t \le T. \end{cases}$$

For an optimal commutation time $t^* = T - 1$.

1.5 Numerical method in optimal control (shooting method)

There are two types :

* The direct methods consist of the discretization the control, the state, and reducing the problem to a non-linear optimization (programming) problem.

* The indirect methods consist in solving numerically, by shooting method, we apply the maximum principle to obtain a boundary value problem.

In this section, we focus just on the indirect method which is the shooting method.

1.5.1 Simple shooting method

Consider the optimal control problem (1.2)-(1.3), and first, suppose that, the final time t_f is fixed. The maximum principle award a necessary optimality condition, it asserts that, any optimal trajectory is the projection of an extremal. If we are able from the maximum condition, to express the extremal control as a function of $(x(t), p(t))$, then the extremal system is a differential system of the form $\dot{z}(t) = F(t, z(t))$, where : $z(t) = (x(t), p(t))$, the initial, the final and the transversality conditions, take the form $\mathscr{R}(z(0), z(t_f)) = 0$.

$$\begin{cases} \dot{z}(t) & = & F(t, z(t)), \\ \mathscr{R}(z(0), z(t_f)) & = & 0. \end{cases} \tag{1.10}$$

Let $z(t, z_0)$ be the solution of the next Cauchy problem

$$\begin{cases} \dot{z}(t) & = & F(t, z(t)), \\ z(0) & = & 0. \end{cases}$$

Let $G(z_0) = \mathscr{R}(z(0), z(t_f, z_0))$. The problem (1.10) is equivalent to

$$G(z_0) = 0.$$

It is the issue of determining a zero of the function G.
However, it may be preferable, when the final time is free, to use the transversality condition on the Hamiltonian.

1.5.2 Multiple shooting method

Compared to the simple shooting method, the multiple shooting method divides the set $[0, t_f]$ into N sets $[t_i, t_{i+1}]$, and is given as unknowns the values $z(t_i)$ at the start of each subset. Continuity conditions must be taken into account at each time t_i.
Consider a general optimal control problem. Applying the maximum principle reduces the problem to a limit value problem of the type

$$\dot{z}(t) = F(t, z(t)) = \begin{cases} F_0(t, z(t)) & \text{if} \quad t_0 \leq t < t_1, \\ F_1(t, z(t)) & \text{if} \quad t_1 \leq t < t_2, \\ \vdots & \\ F_s(t, z(t)) & \text{if} \quad t_s \leq t < t_f, \end{cases}$$

where $z = (x, p) \in \mathbb{R}^{2n}$ (p is the adjoint vector), and $t_1, t_2, \ldots, t_s \in [t_0, t_f]$ can be switching times. In the case of constraints on the state, we have boundary conditions on the state, adjoint vector and on the Hamiltonian if the final time is free.

R A priori the final time t_f is unknown. Moreover, in the multiple shooting method, the number s of switching must be fixed, it is determined when possible by a geometric analysis of the problem.

The multiple shooting method consists in subdividing the set $[t_0, t_f]$ into N subsets, the value of $z(t)$ at the beginning of each subset being unknown. More specifically, let $t_0 < \sigma_1 < \sigma_k < t_f$ be a fixed subdivision of the interval $[t_0, t_f]$. At any point σ_j the function z is continuous. We can consider σ_j as a fixed switching point, in which we have

$$\begin{cases} z(\sigma_j^+) = z(\sigma_j^-), \\ \sigma_j = \sigma_j^* = fixed. \end{cases}$$

Now, we define the nodes

$$\{\tau_1,...,\tau_m\} = \{t_0,t_f\} \cup \{\sigma_1,...,\sigma_k\} \cup \{t_1,...,t_s\}.$$

Finally, we are led to the next limit value problem

•

$$\dot{z}(t) = F(t,z(t)) = \begin{cases} F_0(t,z(t)) & \text{if} \quad \tau_1 \le t < \tau_2, \\ F_1(t,z(t)) & \text{if} \quad \tau_2 \le t < \tau_3, \\ \vdots & \\ F_{m-1}(t,z(t)) & \text{if} \quad \tau_{m-1} \le t < \tau_m. \end{cases}$$

• $\forall j \in \{2,...,m-1\}$, $r_j(\tau_j, z(\tau_j^-), z(\tau_j^+)) = 0$.
• $r_m(\tau_m, z(\tau_m), z(\tau_m)) = 0$.
Where $\tau_1 = t_0$ is fixed, $\tau_m = t_f$, and the r_j represent the previous interior or boundary conditions.

1.5.3 Resolution of TPBVP

General case Consider the next general optimal control problem at fixed times

$$\begin{cases} \min g(t_0,x(t_0),t_f,x(t_f)) + \int_{t_0}^{t_f} f^0(t,x(t),u(t)), & \text{"Objective."} \\ \dot{x}(t) = f(t,x(t),u(t)), & \text{"Dynamic."} \\ u(t) \in U \subset \mathbb{R}^m, & \text{"Admissible controls."} \\ \psi_0(t_0,x(t_0)) = 0 \in \mathbb{R}^{n_0}, & \text{"Initial conditions."} \\ \psi_f(t_f,x(t_f)) = 0 \in \mathbb{R}^{n_1}. & \text{"Final conditions."} \end{cases}$$

TPBV problem

The necessary optimality condition (PMP) leads us to a differential system with $2n$ equations, n_0+n_1 parameters (μ_0 and μ_1) and with $2n+n_0+n_1$ final and initial conditions as follows :

$$(P) \begin{cases} \dot{x}(t) & = & f(t,x(t),u(t)), \\ \dot{p}(t) & = & -f_x^t(t,x(t),u(t))p(t) - l_x(t,x(t),u(t)), \\ u(t) & = & h(p(t)), \\ (h_0,h_1)(x(t_0)) & = & (0,0), \\ p(t_0) & = & -\dfrac{\partial \phi}{\partial x_0}(x(t_0),x(t_f),\mu_0,\mu_1), \\ p(t_f) & = & \dfrac{\partial \phi}{\partial x_f}(x(t_f),x(t_f),\mu_0,\mu_1), \end{cases}$$

where, $u(t) = h(p(t))$ is given by the minimization of the Hamiltonian, and the function ϕ is given y :

$$\phi : (t_0,x_0,t_f,x_f,\mu_0,\mu_1) \to g(t_0,x_0,t_f,x_f) + (\psi_0(t_0,x_0)|\mu_0) + (\psi_0(t_f,x_f)|\mu_1),$$

by setting $y(t)$ the pair state, adjoint state $(y(t) = (x,p(t)))$ and ϕ the dynamics of the pair state, adjoint state given by the Hamiltonian system and by eliminating the parameters, μ_0 and μ_1. We give the next TPBV problem

$$(TPBVP) \begin{cases} y(t) = \phi(y) = \varphi(t,y(t)), & \text{"Almost everywhere on } [t_0,t_f].\text{"} \\ c_0(t_0,y(t_0)) = 0, & \text{"Boundary conditions in } t_0.\text{"} \\ c_f(t_f,y(t_f)) = 0. & \text{"Boundary conditions in } t_f.\text{"} \end{cases}$$

R These boundary conditions c_0 and c_f cores correspond to the transversality conditions mentioned above, which contain the initial and final conditions of (P), in addition to the conditions on the adjoint state p.

Initial value problem & shooting method

We will define the shooting method to solve this problem of two points boundary values Let $y(.,z)$ be the solution of the system with the following IVP initial value

$$(IVP)\left\{\begin{array}{rcll} y(t) & = & \varphi(t,y(t)), & \text{"Almost everywhere on } [t_0,t_f]." \\ y(t_0) & = & z. & \text{"Initial value ."} \end{array}\right.$$

we now introduce an application G called shooting function , which has the initial value z associates the value of the boundary conditions in t_f for the corresponding solution of (IVP), defined by

$$\begin{array}{rcl} G:\mathbb{R}^{2n} & \rightarrow & \mathbb{R}^{2n} \\ z & \mapsto & G(z)=\begin{pmatrix} R_0(z) \\ R_f(y(t_f,z)) \end{pmatrix} \end{array}$$

Finding a zero of the shooting function G is then equivalent to the problem solving $(TPBVP)$, and thus gives a solution of (P).

> **R** The numerical problem-solving algorithm TPBVP will be completely defined :
> • The resolution algorithm of $G(z)=0$.
> • The algorithm for integrating an initial-valued differential system to calculate the shooting function G.

1.5.4 Application of the shooting method

In our case, we solve the optimal problem by considering the initial state x_0 the origin t by taking a final state $x_1=(0,-1)$ of the space X.
There exists a function $z(t)=(x(t),p(t))$ defined in X with value in U, the final, the initial, and the transversality conditions are set in the next form

$$h(z(t),p(t_f)).$$

Defines all the optimal trajectories, and according to the principle of maximum we have

$$u(t)=sign(p_y(t)).$$

The knowing of the shooting function makes it possible to consider that this optimal problem is entirely solved mathematically.
We have the next two points boundary value problem

$$(TPBVP)\left\{\begin{array}{rcl} (\dot{x},\dot{y},\dot{p}_x,\dot{p}_y)(t) & = & (y(t),u(t),0,-p_x)(t), \\ (x(t_0),y(t_0)) & = & (0,0), \\ (x(t_f),y(t_f)) & = & (0,-1), \end{array}\right.$$

Assume that

$$z(t)=(x(t),y(t),p_x(t),p_y(t))=(z_1(t),z_2(t),z_3(t),z_4(t)).$$

Solving the problem $(TPBVP)$ is then equivalent to finding a zero of the equation $G(z)=0$ where the function G is the shooting function associated with our problem and is defined by

$$\begin{array}{rcl} G:\mathbb{R}^4 & \rightarrow & \mathbb{R}^4 \\ z & \mapsto & G(z)=z(t,0,h), \end{array}$$

such that, $z(t,0,h)$ is the solution of the next system

$$\begin{cases} (\dot{z}_1,\dot{z}_2,\dot{z}_3,\dot{z}_4)(t) &= (z_2(t),u(t),0,-z_3(t)), \\ (z_1,z_2)(0) &= (0,0), \\ (z_3,z_4)(0) &= (h_1,h_2)\in\mathbb{R}\times\mathbb{R}. \end{cases}$$

Let $z(t,0,0,h_1,h_2)$ be a solution of our system at time t with the initial and final conditions $(0,0,h_1,h_2)$.
In this example we have that, the final time t_f is free, so, we must have

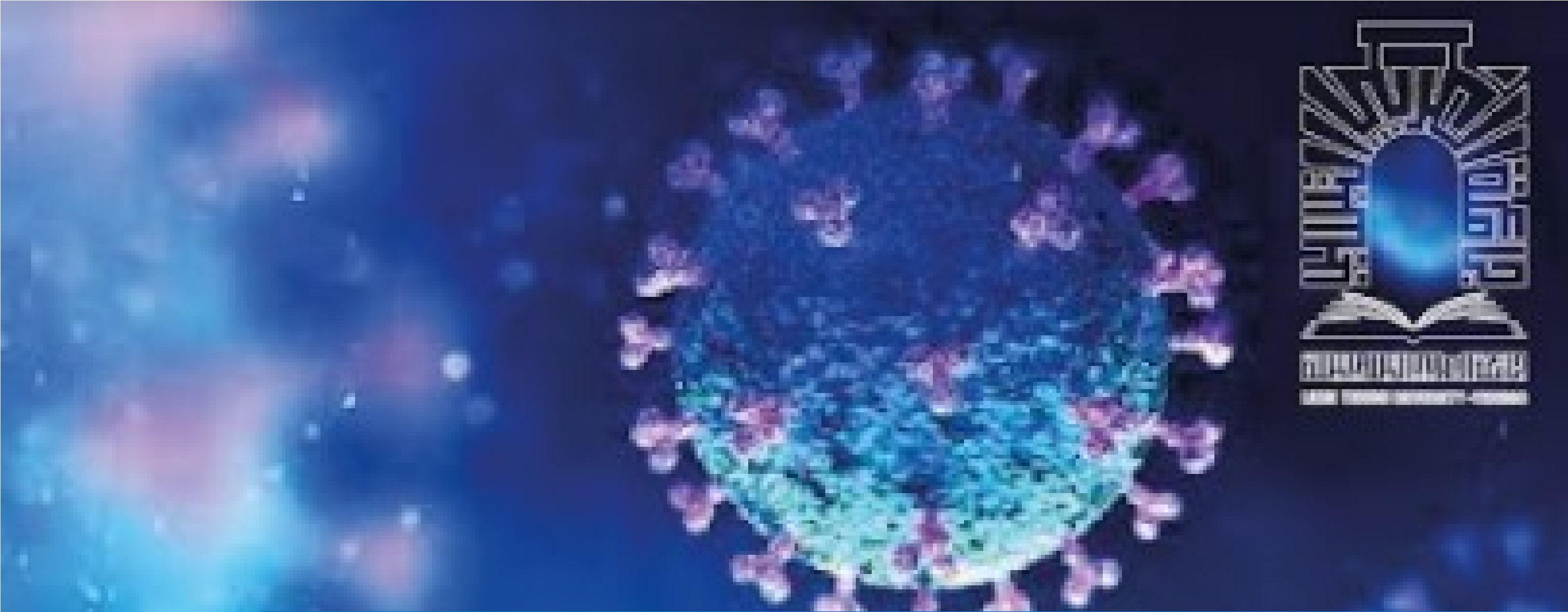
$$z(t_f,0,0,h_1,h_2)=\begin{pmatrix} z_1(t_f,0,0,h_1,h_2) \\ z_2(t_f,0,0,h_1,h_2) \\ z_3(t_f,0,0,h_1,h_2) \\ z_4(t_f,0,0,h_1,h_2) \end{pmatrix}=\begin{pmatrix} 0 \\ -1 \\ . \\ . \end{pmatrix}.$$

We define the next shooting function

$$G(z)=\begin{pmatrix} z_1(t_f) \\ z_2(t_f)+1 \\ H(t_f)=0 \end{pmatrix}=\begin{pmatrix} z_1(t_f)-0 \\ z_2(t_f)-(-1) \\ z_4(t_f)z_3(t_f)+\left|z_4(t_f)\right|-1 \end{pmatrix}.$$

Since $(x_f,y_f)=(0,-1)$ and $\max H(t)=z_4(t)z_3(t)+|z_4(t)|-1$, in this case we have $p^0=-1$, and the fact that, the final time t_f is free, therefore,

$$H(t_f)=z_4(t_f)z_3(t_f)+\left|z_4(t_f)\right|-1=0.$$



2. Control of COVID-19

A novel pathogenic virus corona-virus (CoV) called 'COVID-19', '2019-nCoV' or '2019 novel corona-virus' by WHO[1] is in charge of the current outbreak of pneumonia that started at the month of December 2019 in Wuhan, China.

The first part of this chapter presents the outbreak of COVID-19 with the help of a mathematical model using just the non-linear ODEs. The spread of the disease has been on the increase across the globe for some time with no end in sight. The second section treats the optimal average control into COVID-19 systems, which is based on the precedent given modelling. As a result, we arrive at the required characterization of the optimal control. It can be simulated by Matlab software by using the shooting method.

2.1 COVID-19 Mathematics modelling

This section aimed to present a mathematical model of propagation of the virus, in other words we present a Bats-Hosts-Reservoir-People (BHRP) transmission network model. Since the Bats-Hosts-Reservoir network was hard to explore obviously and public concerns were focusing on the transmission from Seafood Wholesale Market (reservoir) to people, the model was simplified as Reservoir-People (RP) transmission network model.

2.2 The BHRP transmission network model

The BHRP transmission network model was posted to bioRxiv on 19 January 2020 (see [**8**]). We assumed that the virus transmitted among the bats, and then transmitted to unknown hosts (probably some wild animals). The hosts were hunted and sent to the seafood market which was defined as the reservoir of the virus. People exposed to the market got the risks of the infection Figure 2.2. The BHRP transmission network model was based on the next assumptions :

[1] World Health Organization.

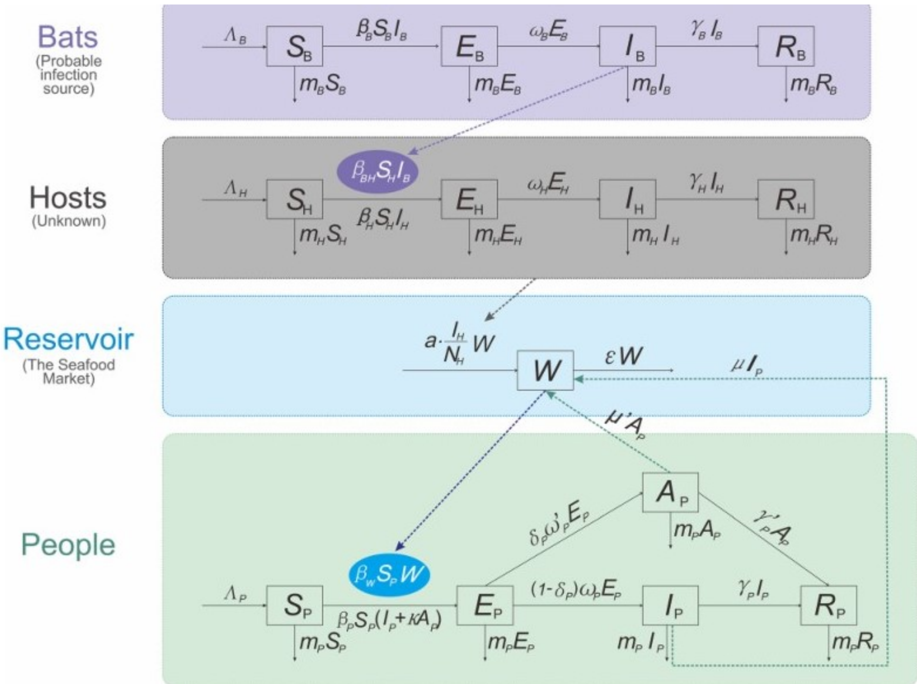


Figure 2.1: Flowchart of the BHRP transmission network model.

• The bats were divided into four compartments :

▷ I_B are infected bats.
▷ S_B are susceptible bats, where it will be infected through sufficient contact with I_B.
▷ E_B are exposed bats.
▷ R_B are removed bats.

Where :

⋆ n_B and m_B are the birth and the death rate of bats.
⋆ $\Lambda_B = n_B \times N_B$ is the number of the newborn bats, such that, N_B is the total number of bats.
⋆ $1/\omega_B$ is the incubation period of bat infection.
⋆ $1/\gamma_B$ is the infectious period of bat infection.
⋆ β_B is the transmission rate.

• The hosts were also divided into four compartments :

▷ I_H are infected hosts .
▷ S_H are susceptible hosts, it will be infected through sufficient contact with I_B and I_H.
▷ E_H are exposed hosts .
▷ R_H are removed hosts .

Where :

⋆ n_H and m_H are the birth and the death rate of hosts.
⋆ $\Lambda_H = n_H \times N_H$,such that, N_H is the total number of hosts.
⋆ $1/\omega_H$ is the incubation period of host infection.
⋆ $1/\gamma_H$ is the infectious period of host infection.
⋆ β_{BH} and β_H are the transmission rates.

• The SARS-CoV-2 in reservoir (the seafood market) was denoted as W.

Let us suppose that :

▷ a is the retail purchases rate of the hosts in the market.

▷ I_H/N_H is the prevalence of SARS-CoV-2 in the purchases.
▷ aWI_H/N_H is the rate of the COVID-19 in W imported form the hosts, such that, N_H is the total number of hosts. ▷ μ_P and μ'_P are the rate of the symptomatic and the asymptomatic infected people could export the virus into W.
The virus in W will subsequently leave the W compartment at a rate of εW, where $1/\varepsilon$ is the lifetime of the virus.

- The people were divided into five compartments :

▷ I_P are the symptomatic infected people.
▷ S_P are the susceptible people, it will be infected through sufficient contact with W and I_P
▷ E_P are the exposed people .
▷ A_P are the asymptomatic infected people.
▷ R_P are the removed people, which including recovered and death people.
Where :
⋆ n_P and m_P are the birth and the death rate of people.
⋆ $\Lambda_P = n_P \times N_P$, such that, N_P is the total number of people.
⋆ $1/\omega_P$ and $1/\omega'_P$ are the incubation and the latent period of human infection.
⋆ $1/\gamma_P$ and $1/\gamma'_P$ are the infectious period of I_P and A_P.
⋆ δ_P is the proportion of asymptomatic infection.
⋆ β_W and β_P are the transmission rates.
⋆ The transmissibility of A_P was κ times that of I_P, where, $0 \leq \kappa \leq 1$.

$$(BHRP)\begin{cases}
\frac{dS_B}{dt} &= \Lambda_B - m_B S_B - \beta_B S_B I_B, \\
\frac{dE_B}{dt} &= \beta_B S_B I_B - \omega_B E_B - m_B E_B, \\
\frac{dI_B}{dt} &= \omega_B E_B - m_B E_B - (\gamma_B + m_B) I_B, \\
\frac{dR_B}{dt} &= \gamma_B I_B - m_B R_B, \\
\frac{dS_H}{dt} &= \Lambda_H - m_H S_H - \beta_{BH} S_H I_B, \\
\frac{dE_H}{dt} &= \beta_{BH} S_H I_B + \beta_H S_H I_H - \omega_H E_H - m_H E_H, \\
\frac{dI_H}{dt} &= \omega_H E_H - m_H E_H - (\gamma_H + m_H) I_H, \\
\frac{dR_H}{dt} &= \gamma_H I_H - m_H R_H, \\
\frac{dS_P}{dt} &= \Lambda_P - m_P S_P - \beta_P S_P (I_P + \kappa A_P) - \beta_W S_P W, \\
\frac{dE_P}{dt} &= \beta_P S_P (I_P + \kappa A_P) + \beta_W S_P W - (1-\delta_P)\omega_P E_P - \delta_P \omega'_P E_P - m_P E_P, \\
\frac{dI_P}{dt} &= (1-\delta_P)\omega_P E_P - (\gamma_P + m_P) I_P, \\
\frac{dA_P}{dt} &= \delta_P \omega'_P E_P - (\gamma'_P + m_P) A_P, \\
\frac{dR_P}{dt} &= \gamma_P I_P + \gamma'_P A_P - m_P R_P, \\
\frac{dW}{dt} &= aW \frac{I_H}{N_H} + \mu_P I_P + \mu'_P A_P - \varepsilon W.
\end{cases}$$

The next table presents the parameters of BHRP model

Parameter	Description
n_P	The birth rate parameter of people.
n_B	The birth rate parameter of bats.
n_H	The birth rate parameter of hosts.
m_P	The people death rate.
m_B	The bats death rate.
m_H	The hosts death rate.
$1/w_P$	The incubation period of people.
$1/w_H$	The bats incubation period.

$1/w_B$	The bats incubation period.
$1/w'_P$	The people latent period.
$1/\gamma_P$	The infectious period of people symptomatic infection.
$1/\gamma_B$	The bats infectious period.
$1/\gamma_H$	The hosts infectious period.
$1/\gamma'_P$	The infectious period of people asymptomatic infection.
β_P	The transmission rate from I_P to S_p.
β_B	The transmission rate from I_B to S_B.
β_H	The transmission rate from I_H to S_H.
β_W	The transmission rate from W to S_p.
β_{BH}	The transmission rate from I_B to S_H.
a	The retail purchases rate of the hosts in the market.
κ	The multiple of the transmissibility of A_P to that of I_P.
μ_P	The shedding coefficients from I_P to W.
μ'_P	The shedding coefficients from A_P to W.
$1/\varepsilon$	The virus lifetime in W.
δ_P	The proportion of people asymptomatic infection rate.

Table 2.1 : The parameters definition of the BHRP model.

2.2.1 The transmission network model of simplified reservoir-people

We suppose that, the COVID-19 could be imported to the seafood market in a short time. Then, we have the next assumptions :

- The Bats-Host transmission network was neglected.
- Let us put the initial value of W as an impulse function, based on the previous studies on simulating importation ([**10**], [**52**]), it gives as follows

$$Importation = impulse(n, t_0, t_i).$$

Then, the next PR model was clarified from the BHRP model

$$\begin{cases} \dfrac{dS_P}{dt} &= \Lambda_P - m_P S_P - \beta_P S_P (I_P + \kappa A_P) - \beta_w S_P W, \\ \dfrac{dE_P}{dt} &= \beta_p S_P (I_P + \kappa A_P) + \beta_w S_P W - (1-\delta_P) w_P E_P - \delta_P w'_P E_P - m_P E_P, \\ \dfrac{dI_P}{dt} &= (1-\delta_P) w_P E_P - (\gamma_P + m_P) I_P, \\ \dfrac{dA_P}{dt} &= \delta_P w'_P E_P - (\gamma'_P + m_P), \\ \dfrac{dR_P}{dt} &= \gamma_P I_P + \gamma'_P A_P - m_P R_P, \\ \dfrac{dW}{dt} &= \mu_P I_P + \mu'_P A_P - \varepsilon W. \end{cases}$$

The natural population death and birth rates were at a relatively low level, during the outbreak period. According to the previous study (see [**9**]). m_P and n_P refer to the rate of people travelling out from Wuhan City and travelling into Wuhan City, respectively. In our model, people and viruses have different dimensions, therefore, we used the next sets to perform the normalization :

$$\begin{array}{llll} s_P = \dfrac{S_P}{N_P} & , \quad i_P = \dfrac{I_P}{N_P} & , \quad a_P = \dfrac{A_P}{N_P} & , \quad r_P = \dfrac{R_P}{N_P}, \\ w_P = \dfrac{\varepsilon W}{\mu_P N_P} & , \quad \mu'_P = c\mu_P & , \quad b_P = \beta'_P N_P & , \quad b_W = \dfrac{\mu_P \beta_W N_P}{\varepsilon}. \end{array}$$

In the normalization, we have, c is a parameter refers to the relative shedding coefficient of A_P compared to I_P. The normalized model of RP is converted as follows

$$\begin{cases} \frac{ds_P}{dt} &= n_P - m_P s_P - b_p s_P(i_P + k a_P) - b_w s_P, \\ \frac{de_P}{dt} &= b_p s_P(i_P + k a_P) + b_w s_P w - (1-\delta_P)\omega_P e_P - \delta_P \omega_P' e_P - m_P e_P, \\ \frac{di_P}{dt} &= (1-\delta_P)\omega_P e_P - (\gamma_P + m_P) i_P, \\ \frac{da_P}{dt} &= \delta_P \omega_P' e_P - (\gamma_P' + m_P) a_P, \\ \frac{dr_P}{dt} &= \gamma_P i_P + \gamma_P' a_P - m_P r_P, \\ \frac{dw}{dt} &= \varepsilon(i_P + c a_P - w), \end{cases}$$

Where : n_p, m_p, w_p, w_p', γ_p, γ_p', δ_p, ε, k and c are known parameters while, b_p and b_w are unknown parameters in the set $[0,1]$.

2.3 Optimal control of COVID-19

This section presents the controlled the propagation of the COVID-19 in the society by using the average optimal control on the propagation equations of the virus, so our control was represented by the optimal control in free time and we reached very good results attached with fees to stabilize the propagation of the disease in time.

2.3.1 Problem statement

Mathematical models are a powerful tools for investigating the infectious diseases dynamics and their control.Optimal control theory is used to propose the most effective strategy to minimize the number of individuals infected during the infection while effectively balancing the vaccination used to models with various cost scenarios.

During the outbreak period epidemic corona-virus , the natural birth rate and death rate in the population was at a relatively low level. However, people were commonly travelling into and out from Wuhan city mainly due to the Chinese New Year holiday.

In this model, viruses and people have different dimensions (see [**8**]), thus, we use the next sets to perform the normalizations :

$$\left| \begin{array}{lcl} s &=& \frac{S}{N}, \\ e &=& \frac{E}{N}, \\ i &=& \frac{I}{N}, \end{array} \right. \qquad \left| \begin{array}{lcl} a &=& \frac{A}{N}, \\ r &=& \frac{R}{N}, \\ w &=& \frac{\varepsilon W}{\mu N'}, \end{array} \right. \qquad \left| \begin{array}{lcl} \mu' &=& c\mu, \\ b_p &=& \beta_p N, \\ b_w &=& \frac{\mu \beta_w N}{\varepsilon}. \end{array} \right.$$

such that :
- S is the susceptible people.
- E is the exposed people.
- I is the symptomatic infected people.
- A is the asymptomatic infected people.
- R is the removed people (death and recovered).
- εW is the rate of the virus in compartment W, such that, $\frac{1}{\varepsilon}$ is the virus lifetime.
- μ is the shedding coefficients from I to W.
- μ' is the shedding coefficients from A to W.
- β_p is the average transmission rate from S to I.
- β_w is the average transmission rate from S to W.
- N is the people total number.

Let us consider the next mathematical model of the COVID-19 outbreak, which is a clarified model from BHRP to the next RP model

$$\begin{cases} \frac{ds}{dt} &= n - ms - b_p s(i+ka) - b_w sw, \\ \frac{de}{dt} &= b_p s(i+ka) + b_w sw - (1-\delta)we - \delta w' e - me, \\ \frac{di}{dt} &= (1-\delta)we - (\gamma+m)i, \\ \frac{da}{dt} &= \delta w' e - (\gamma' + m)a, \\ \frac{dr}{dt} &= \gamma i + \gamma' a - mr, \\ \frac{dw}{dt} &= \varepsilon(i + ca - w). \end{cases} \tag{2.1}$$

Where :

- n is the people birth rate.
- m is the people death rate.
- c is the relative shedding coefficient of A_p compared to I_P.
- k is the multiple of the transmissibility of A_p to that of I_P.
- δ is the people proportion of asymptomatic infection rate.
- $\frac{1}{w'}$ is people the latent period.
- $\frac{1}{\gamma}$ is the infectious period of people symptomatic infection of .
- $\frac{1}{\gamma'}$ is the infectious period of people asymptomatic infection.
- $\frac{1}{w}$ is the people incubation period.

2.3.2 Average optimal control existence

This section deals with the intervention strategy through vaccination. Here we consider a campaign of vaccination over a time fixed period $[0,T]$. The vaccine leads susceptible individuals to the recovered class. So, we introduce the control $u(t)$ of the system as follows

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \end{cases} \tag{2.2}$$

where, $u(t)$ is a measurable function and $x(t)$ is the system state.
The average optimal control problem aims to look for the associated state variable $x(.)$ and the control $u(t)$ which maximizes or minimizes the cost function J, such that,

$$J(T,u) = \int_0^t \int_0^t g(T, x(t, \alpha_1, \alpha_2))\, d\alpha_1 d\alpha_2 + \int_0^T f^0(t, x(t), u(t))\, dt,\ \alpha_1, \alpha_2 \in [0,1], \tag{2.3}$$

where, $f^0 \in C^1(I \times U \times V)$ and f is continuous in V. We suppose that, I_1, I_2 two subsets of $I \in [0,T]$. The problem of average optimal control aims to determine the trajectories $x_u(.)$ solutions of the next system

$$\begin{cases} \dot{x}(t) = f(t, x_u(t), u(t)), \\ x_u(0) \in I_1, \\ x_u(T) \in I_2, \\ \min J(t, x(t), u(t)). \end{cases}$$

R We say that the optimal average control problem is at non-fixed final time if T is free.

Definition 2.3.1 [49] Let $T>0$, the input-output application in time T of the controlled system initialized at x_0 is the application

$$\begin{array}{rcl} E_T : U & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & x_u(t), \end{array}$$

where, U is the admissible controls set of u, where, its associated trajectory is defined on $[0,T]$.

Definition 2.3.2 [49] Let $u(.)$ be a defined control on $I=[0,T]$, where, the associated trajectory $x_u(.)$ issue $x(0)=x_0$ is defined on I. We say that the control $u(.)$ is singular on I if the differential in the sense of Frechet [a] E_T is not subjective otherwise we say it is regular.

[a] Let V and W be normed vector spaces, and $U \subseteq V$ be an open subset of V. A function $f: U \to W$ is called Frechet differentiable at $x \in U$ if there exists a bounded linear operator $A: V \to W$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h)-f(x)-Ah\|_W}{\|h\|_V} = 0.$$

Proposition 2.3.1 Let $u(.)$ be a singular control on I for the system (2.2) and $x(.)$ be the associated singular trajectory, then there is an absolutely continuous application $\lambda : [0,T] \to \mathbb{R}^n / \{0\}$ called the adjoint vector, such that the following equations are checked that

$$\begin{cases} -\dfrac{\partial H(t,x(t),u(t),\lambda(t))}{\partial x_i} & = & \lambda_i'(t) & , & i=\overline{1,n}, \\ \dfrac{\partial H(t,x(t),u(t),\lambda(t))}{\partial u} & = & 0 & , & \forall t \in [0,T]. \end{cases}$$

where, H is the Hamiltonian of the system (2.2) given by

$$H(t,x(t),u(t),\lambda(t)) = \lambda_0 f^0(t,x(t),u(t)) + \lambda(t) f(t,x_u(t),u(t)).$$

Theorem 2.3.2 — Principle Maximum of Pontryguine (PMP). Suppose that $f,g \in C^1$ are convex on u, we assume that : $u^*(t)$ is the optimal control of (2.3), $\lambda(t)$ is continuous differential function with $\lambda(t) \geq 0$ for all t, and, $x^*(t)$ be the associated state, suppose that,

$$\forall t \in [0,T], \ H(t,x^*(t),u^*(t),\lambda(t)) = 0,$$

So,

$$H(t,x^*(t),u^*(t),\lambda(t)) \leq H(t,x(t),u(t),\lambda(t)).$$

Proof. The goal here is to minimize the total number of infected people and the cost associated with vaccination during the vaccination campaign.
The system (2.1) becomes as follows

$$\begin{cases} \dfrac{ds}{dt} & = & n - ms(t) - b_p s(t)(i(t)+ka(t)) - b_w s(t)w(t) - u(t)s(t), \\ \dfrac{de}{dt} & = & b_p s(t)(i(t)+ka(t)) + b_w s(t)w(t) - (1-\delta)we(t) - \delta w' e(t) - me(t), \\ \dfrac{di}{dt} & = & (1-\delta)we(t) - (\gamma+m)i(t), \\ \dfrac{da}{dt} & = & \delta w' e(t) - (\gamma'+m)a(t), \\ \dfrac{dr}{dt} & = & \gamma i(t) + \gamma' a(t) - mr(t) + u(t)s(t), \\ \dfrac{dw}{dt} & = & \varepsilon(i(t)+ca(t)-w(t)). \end{cases} \tag{2.4}$$

Where : $n_p, m_p, w_p, w'_p, \gamma_p, \gamma'_p, \delta_p, \varepsilon, k$ and c are known parameters while, b_p and b_w are unknown parameters in the set $[0,1]$.
The optimal control problem is to minimize, in a fixed time T, the cost

$$J(T,u) = \alpha \int_0^1 \int_0^1 e_p\left(T, b_p, b_w\right) db_p db_w + \int_0^T u^2\left(t\right) dt,\ \alpha > 0.$$

By considering the system (2.4) with the non-negative initial data

$$\begin{array}{lllll} s_p(0) = s_0 & , & e_p(0) = e_0 & , & i_p(0) = i_0, \\ a_p(0) = a_0 & , & r_p(0) = r_0 & , & w(0) = w_0. \end{array}$$

We define the optimal control problem as bellow

$$\begin{cases} J(u) = \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w + \int_0^T u^2(t)dt \longrightarrow min_u\ , \alpha > 0, \\ \frac{ds(t)}{dt} = n - ms(t) - b_p(t)(i(t) + ka(t)) - b_w s(t)w(t) - u(t)s(t) \\ \frac{de(t)}{dt} = b_p s(t)(i(t) + ka(t)) + b_w s(t)w(t) - (1-\delta)we(t) - \delta w'e(t) - me(t), \\ \frac{di(t)}{dt} = (1-\delta)we(t) - (\gamma + m)i(t), \\ \frac{da(t)}{dt} = \delta w'e(t) - (\gamma' + m)a(t), \\ \frac{dr(t)}{dt} = \gamma i(t) + \gamma' a(t) - mr(t) + u(t)s(t), \\ \frac{dw(t)}{dt} = \varepsilon(i(t) + ca(t) - w(t)), \\ s(0) = s_0, e(0) = e_0, i(0) = i_0, a(0) = a_0, r(0) = r_0, w(0) = w_0. \\ t \in [0,T],\ \ 0 \le u(t) \le \lambda, b_p, b_w \in [0,1]. \end{cases} \tag{2.5}$$

We have considered a quadratic cost function on control, which is the simplest and widest non-linear representation of the cost of vaccination.
The parameter λ is a weight parameter describing the comparative importance of the two terms in the functional. For example, a high value of λ means that it is more important to reduce the burden of the disease than to reduce the costs of vaccination. ■

2.3.3 Principal maximum application

Theoretical resolution The optimal control problem (2.5)) is our initial problem. hence, before characterizing the optimal control, let us define the Hamiltonian as below

$$\begin{aligned} H\left(s,e,i,a,r,w,P_s,P_e,P_i,P_a,P_r,P_w,u,t\right) &= p^0 u^2 \\ &+ P_s\left(n - ms - b_p s\left(i + ka\right) - b_w sw - us\right) \\ &+ P_e\left(b_p s\left(i + ka\right) + b_w sw - \left(1-\delta\right) we - \delta w'e - me\right) \\ &+ P_i\left(\left(1-\delta\right) we - \left(\gamma + m\right) i\right) \\ &+ P_a\left(\delta w'e - \left(\gamma' + m\right) a\right) \\ &+ P_r\left(\gamma i + \gamma' a - mr + us\right) \\ &+ P_w\left(\varepsilon\left(i + ca - w\right)\right), \end{aligned}$$

so,

$$\begin{aligned} H\left(s,e,i,a,r,w,P_s,P_e,P_i,P_a,P_r,P_w,u,t\right) &= p^0 u^2 \\ &+ nP_s - msP_s - b_p siP_s - kb_p saP_s - b_w swP_s - usP_s \\ &+ b_p siP_e + kb_p saP_e + b_w swP_e - \left(1-\delta\right) weP_e - \delta w'eP_e - meP_e \\ &+ \left(1-\delta\right) weP_i - \left(\gamma + m\right) iP_i \\ &+ \delta w'P_a - \left(\gamma' + m\right) aP_a \\ &+ \gamma iP_r + \gamma' aP_r - mrP_r + usP_r \\ &+ \varepsilon iP_w + \varepsilon caP_w - \varepsilon wP_w \end{aligned}$$

where : $P_{s_p},\ P_{e_p},\ P_{i_p},\ P_{a_p},\ P_{r_p}$ and P_w are the adjoints vectors.
Then by applying the PMP we obtain the next adjoints equations :

$$\begin{cases} \dot{P}_s &= -\dfrac{\partial H}{\partial s} = mP_s + b_p iP_s + kb_p aP_s + b_w wP_s + uP_s - b_p iP_e - kb_p aP_e - b_w wP_e - uP_r, \\ \dot{P}_e &= -\dfrac{\partial H}{\partial e} = (1-\delta)\, wP_e + \delta w' P_e + mP_e - (1-\delta)\, wP_i - \delta w' P_a, \\ \dot{P}_i &= -\dfrac{\partial H}{\partial i} = b_p siP_s - b_p sP_e + (\gamma_p + m_p)\, P_i - \gamma P_r - \varepsilon P_w, \\ \dot{P}_a &= -\dfrac{\partial H}{\partial a_p} = kb_p sP_s - kb_p sP_e - \varepsilon c P_w, \\ \dot{P}_r &= -\dfrac{\partial H}{\partial r} = mP_r, \\ \dot{P}_w &= -\dfrac{\partial H}{\partial w} = b_w sP_s - b_w sP_e + \varepsilon P_w, \end{cases} \tag{2.6}$$

with the next transversality conditions :

$$\begin{cases} \dot{P}_s &= p^0 \dfrac{\partial}{\partial s} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = 0, \\ \dot{P}_e &= p^0 \dfrac{\partial}{\partial e} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = p^0 \partial \alpha, \\ \dot{P}_i &= p^0 \dfrac{\partial}{\partial i} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = 0, \\ \dot{P}_a &= p^0 \dfrac{\partial}{\partial a} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = 0, \\ \dot{P}_r &= p^0 \dfrac{\partial}{\partial r} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = 0, \\ \dot{P}_w &= p^0 \dfrac{\partial}{\partial w} \alpha \int_0^1 \int_0^1 e(t, b_p, b_w) db_p db_w = 0. \end{cases} \tag{2.7}$$

Now, we maximize the Hamiltonian H

$$\max_{0 \le u \le \lambda} H,$$

which is equivalent to maximizing the functional ϕ such that,

$$\phi(u) = p^0 u^2 - u(P_s - P_r)s.$$

Necessary condition.
Looking for the roots of its derivative

$$\phi'(u) = 0 \iff 2p^0 u - (P_s - P_r)s = 0 \iff u^* = \frac{(P_s - P_r)s}{-2p^0}.$$

assume that $p^0 = -1/2$, we find

$$u^* = (P_s - P_r)s.$$

Sufficient condition
It is clear that,

$$\phi''(u) = 2p^0 = -1 < 0 \implies u^* = (P_s - P_r)s$$

is a maximum.
By taking into account the limits on u, the characterization of the optimal average control is

$$u^*(t) = \begin{cases} 0, & (P_s(t) - P_r(t))(t) < 0, \\ (P_s + P_r)s, & 0 \leq (P_s(t) - P_r(t))s(t) \leq \lambda, \\ \lambda, & (P_s(t) - P_r(t))s(t) > \lambda, \end{cases} \tag{2.8}$$

we can rewrite it in the next abbreviated formula

$$u^*(t) = \max\{\min\{(P_s(t) + P_r(t))s(t), \lambda\}, 0\}. \tag{2.9}$$

The average optimal control and state are found by solving the average optimality system which includes the state system (2.4), the initial conditions, the assistant system (2.6), the conditions of transversalities (2.7) and the characterization of optimal control (2.8).

Two Points Boundary value problem The *PMP* gives us a necessary condition of optimality and leads us to a two points boundary value problem

$$(TPBVP)\begin{cases} \dot{s}(t) &= n - ms(t) - b_p s(t)(i(t) + ka(t)) & \\ & - \; b_w s(t)w(t) - u(t)s(t), & s(0) = s_0, \\ \dot{e}(t) &= b_p s(t)(i(t) + ka(t)) + b_w s(t)w(t) & \\ & - \; (1-\delta)w_p e(t) - \delta w_p' e(t) - me(t), & e(0) = e_0, \\ \dot{i}(t) &= (1-\delta)w_p e(t) - (\gamma + m)i(t), & i(0) = i_0, \\ \dot{a}(t) &= \delta w_p' e(t) - (\gamma' + m)a(t), & a(0) = a_0, \\ \dot{r}(t) &= \gamma i(t) + \gamma' a(t) - mr(t) + u(t)s(t), & r(0) = r_0, \\ \dot{w}(t) &= \varepsilon(i(t) + ca(t) - w(t)), & w(0) = w_0, \\ \dot{P}_s &= mP_s + b_p iP_s + kb_p aP_s + b_w wP_s + uP_s & \\ & - \; b_p iP_e - kb_p aP_e - b_w wP_e - uP_r, & P_s(T) = 0, \\ \dot{P}_e &= (1-\delta)w_p P_e + \delta w_p' P_e + mP_e & \\ & - \; (1-\delta)w_p P_i - \delta w_p' P_a, & P_e(T) = -\frac{1}{2}\alpha, \\ \dot{P}_i &= b_p siP_s - b_p sP_e + (\gamma + m)P_i - \gamma P_r - \varepsilon P_w, & P_i(T) = 0, \\ \dot{P}_a &= kb_p sP_s - kb_p sP_e - \varepsilon cP_w, & P_a(T) = 0, \\ \dot{P}_r &= mP_r, & P_r(T) = 0, \\ \dot{P}_w &= b_w sP_s - b_w sP_e + \varepsilon P_w, & P_w(T) = 0, \end{cases}$$

with

$$u^*(t) = \begin{cases} 0, & (P_s(t) - P_r(t))s(t) < 0, \\ (P_s - P_r)s, & 0 \leq (P_s(t) - P_r(t))s(t) \leq \lambda, \\ \lambda, & (P_s(t) - P_r(t))s(t) > \lambda. \end{cases}$$

We assume that,

$$y=\begin{pmatrix} s\\ e\\ i\\ a\\ r\\ w\\ P_s\\ P_e\\ P_i\\ P_a\\ P_r\\ P_w \end{pmatrix}=\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8\\ y_9\\ y_{10}\\ y_{11}\\ y_{12} \end{pmatrix},$$

then, we have

$$\left\{\begin{array}{lcll}
\dot{y}_1 & = & n-my_1-b_py_1\left(y_3+ky_4\right)-b_wy_1y_6-u\left(t\right)y_1, & y_1\left(0\right)=y_1^0,\\
\dot{y}_2 & = & b_py_1\left(y_3+ky_4\right)+b_wy_1y_6-\left(1-\delta\right)w_py_2-\delta w_p^{'}y_2-my_2, & y_2\left(0\right)=y_2^0,\\
\dot{y}_3 & = & \left(1-\delta\right)w_py_2-\left(\gamma+m\right)y_3, & y_3\left(0\right)=y_3^0,\\
\dot{y}_4 & = & \delta w_p^{'}y_2-\left(\gamma^{'}+m\right)a_p, & y_4\left(0\right)=y_4^0,\\
\dot{y}_5 & = & \gamma y_3+\gamma^{'}y_4-my_5+u\left(t\right)y_1, & y_5\left(0\right)=y_5^0,\\
\dot{y}_6 & = & \varepsilon\left(y_3+cy_4-y_6\right), & y_6\left(0\right)=y_6^0,\\
\dot{y}_7 & = & my_7+b_py_3y_7+kb_py_4y_7+b_wy_6y_7+uy_7 & \\
 & - & b_py_3y_8-kb_py_4y_8-b_wy_6y_8-uy_{11}, & y_7\left(T\right)=0,\\
\dot{y}_8 & = & \left(1-\delta\right)w_py_8+\delta w_p^{'}y_8+my_8-\left(1-\delta_p\right)w_py_9-\delta w_p^{'}y_{10}, & y_8\left(T\right)=-\dfrac{1}{2}\alpha,\\
\dot{y}_9 & = & b_py_1y_3y_7-b_py_1y_8+\left(\gamma+m\right)y_9-\gamma y_{11}-\varepsilon y_{12}, & y_9\left(T\right)=0,\\
\dot{y}_{10} & = & kb_py_1y_7-kb_py_1y_8-\varepsilon cy_{12}, & y_{10}\left(T\right)=0,\\
\dot{y}_{11} & = & my_{11}, & y_{11}\left(T\right)=0,\\
\dot{y}_{12} & = & b_wy_1y_7-b_wy_1y_8+\varepsilon y_{12}, & y_{12}\left(T\right)=0,
\end{array}\right. \tag{2.10}$$

with

$$u^*\left(t\right)=\begin{cases} 0, & \left(y_7-y_{11}\right)y_1<0,\\ \left(y_7-y_{11}\right)y_1, & 0\leq\left(y_7-y_{11}\right)y_1\leq\lambda,\\ \lambda, & \left(y_7-y_{11}\right)y_1>\lambda,\end{cases}$$

the two points boundary value problem are equivalent to $\dot{y}=F\left(t,y\right)$ with the next conditions

$$Y\left(0\right)=\begin{pmatrix} s_0\\ e_0\\ i_0\\ a_0\\ r_0\\ w_0\end{pmatrix},\ \ y\left(T\right)=\begin{pmatrix} P_s\left(T\right)\\ P_e\left(T\right)\\ P_i\left(T\right)\\ P_a\left(T\right)\\ P_r\left(T\right)\\ P_w\left(T\right)\end{pmatrix},$$

The Cauchy problem associated with our problem is

$$\dot{y} = F(t,y), y(0) = y_0 = \begin{pmatrix} s_0 \\ e_0 \\ i_0 \\ a_0 \\ r_0 \\ w_0 \\ P_s(0) \\ P_e(0) \\ P_i(0) \\ P_a(0) \\ P_r(0) \\ P_w(0) \end{pmatrix}.$$

The solution of our system is written as follows

$$y(T,y_0) = y(T) = \begin{pmatrix} s(T) \\ e(T) \\ i(T) \\ a(T) \\ r(T) \\ w(T) \\ P_s(T) \\ P_e(T) \\ P_i(T) \\ P_a(T) \\ P_r(T) \\ P_w(T) \end{pmatrix} = \begin{pmatrix} s(T) \\ e(T) \\ i(T) \\ a(T) \\ r(T) \\ w(T) \\ 0 \\ -\frac{1}{2}\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving the problem (TPBVP) is equivalent to find a zero of the shooting function $G(y_0)$ defined by

$$G(y_0) = y(T,y_0) - y(T).$$

2.3.4 Numerical analysis

In this section, our optimality system will be numerically solved by the Shooting function. To resolve the system of differential equations we must find the zero of the shooting function $G(y_0)$, to make t possible, we use the "fsolve" command in Matlab. The algorithm of integration of a differential system with initial value is carried out using the command ode of Matlab.The following program allows us to display the solutions to the problem.
As in [**12**], we assume that, the parameter $\alpha = 100$.

The main program of the function simple shoot.

```
clc ; format long ;
global X0;
X0=[800,50,20,30,0,0.0002];
P0=[1,1,1,1,1,1];
n=0.0018; m=0.0018; k=0.5; delta=0.5; c=0.5;
epsilon=0.1; gamma=0.1724; g = 0.1724;
w=0.1923; v=0.1923;
alpha=100;
tf=3;
options=optimset('Display','iter','LargeScale','on');
[P0tf,FVAL,EXITFLAG] = fsolve(@G,[P0,tf],options);
EXITFLAG;
options=odeset('AbsTol',1e-9,'RelTol',1e-9);
[t,y] =
 ode45(@sys,[X0,P0tf(1),P0tf(2),P0tf(3),P0tf(4),P0tf(5),P0tf(6)],options);
if (y(:,7)-y(:,11))'*y(:,1)<0
y(:,13)=0;
elseif (y(:,7)-y(:,11))'*y(:,1)>1
y(:,13)=1;
else
y(:,13)=(y(:,7)-y(:,11))'*y(:,1);
end
plot(t,y(:,13),'r');
title('u(t) Trajectory');
grid;
figure
hold on
plot(t,y(:,1)); plot(t,y(:,2));plot(t,y(:,3));
plot(t,y(:,4));plot(t,y(:,5));plot(t,y(:,6));
title('S(t),E(t),I(t),A(t),R(t),W(t) Trajectories');
legend('S(t)','E(t)','I(t)','A(t)','R(t)','W(t)');
grid;
hold off
```

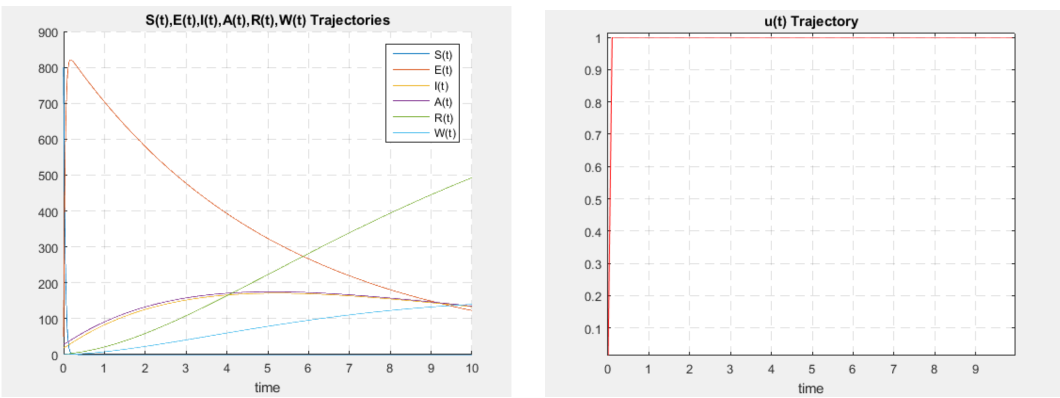
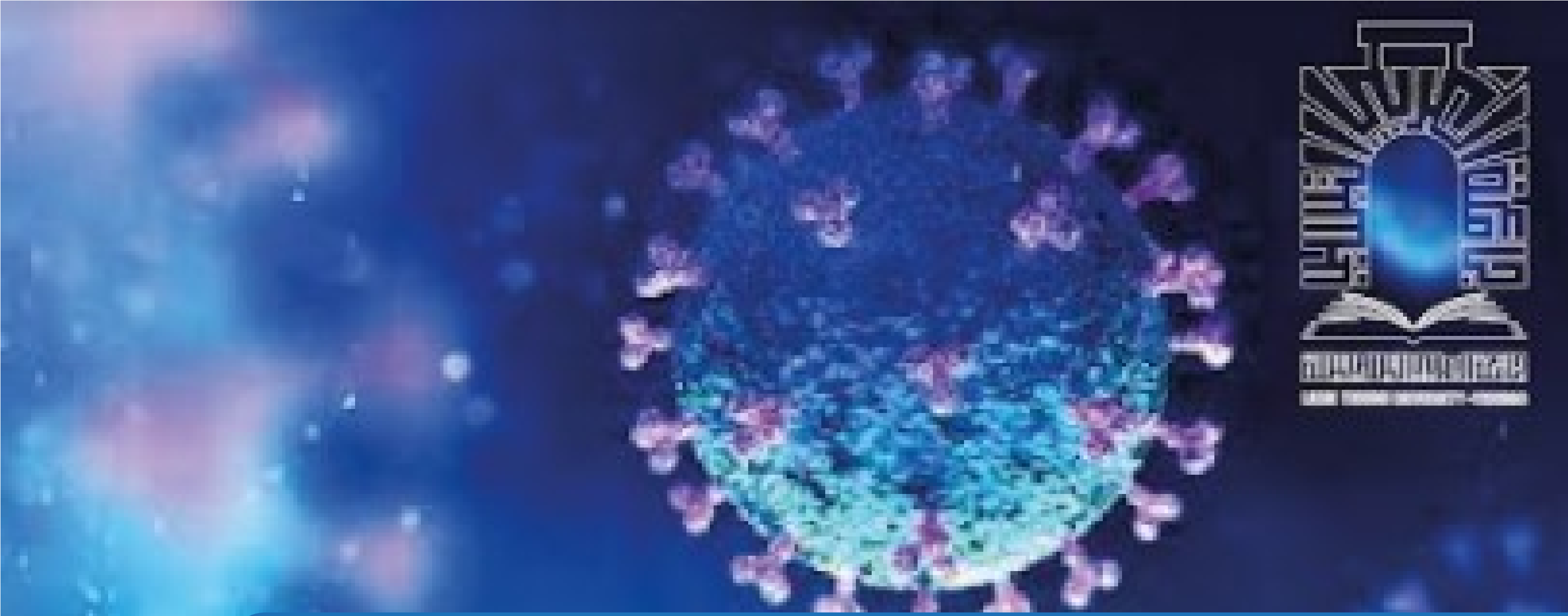


Figure 2.2 : Solutions and Control trajectories.

First of all, denote that the vaccination rate stabilizes at the highest possible value, either from the start or after a very short time, regardless of the value of α. therefore we find that vaccinating at the highest possible rate as soon as possible is essential to control an epidemic.
By increasing the parameter $(\alpha = 100)$ the vaccination rate takes the maximum value $(u = 1)$ from the first months, we see that, the number of exposed and susceptible people is decreasing over time till it cancelled, while the number of infected people (symptomatic or asymptomatic) takes stable state from the fourth month before it starts increasing in the seventh going to no-case state.



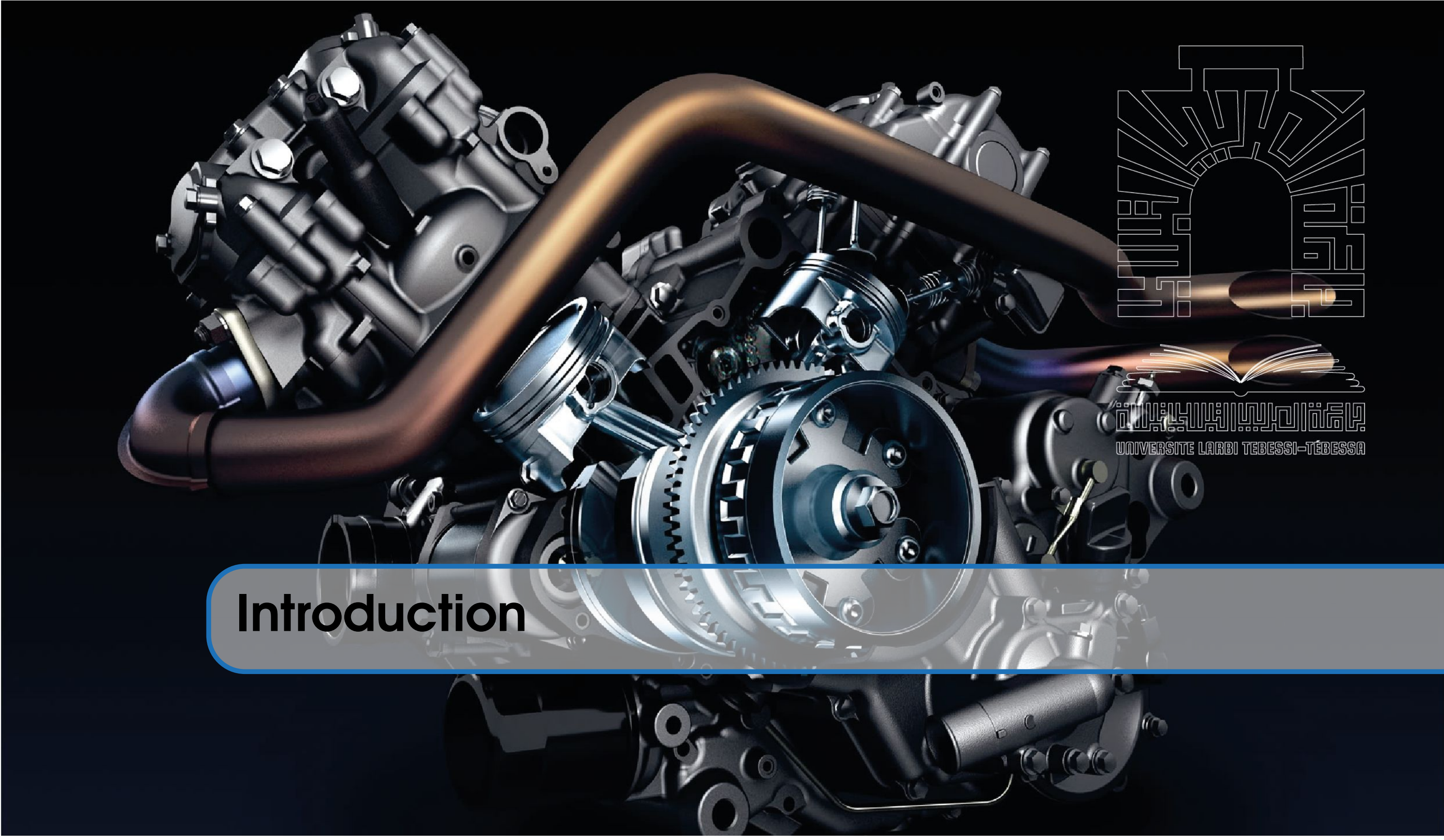
Conclusion

This part showed us that the Pontryaguine maximum principle gives necessary conditions of optimality which allow calculating the optimal trajectories, and also its application can be quite complex in practice. on the other hand, it introduced a numerical method in optimal control. It is an indirect method based on the Maximum Principle, which allows the command to be expressed as a function of the state and the adjoint vector. We, therefore, obtain a limit value problem, because we have initial and final conditions, which we can solve numerically by a shooting method.

This part ended with the presentation and discussion of new results which have tried to consider an optimal control problem of the new pandemic which is sweeping the world. We have modelled the problem, the goal of which is to determine the rate of vaccination needed to stop the spread of the epidemic while minimizing some cost. For the resolution of our problem, we first started by applying theoretically the principle of the maximum of Pontryaguine which gives the necessary conditions of optimality of the first order, then numerically with implementation under Matlab software using the simple shooting method. This allowed us to determine an optimal vaccination strategy.

II

Control of linear PDEs

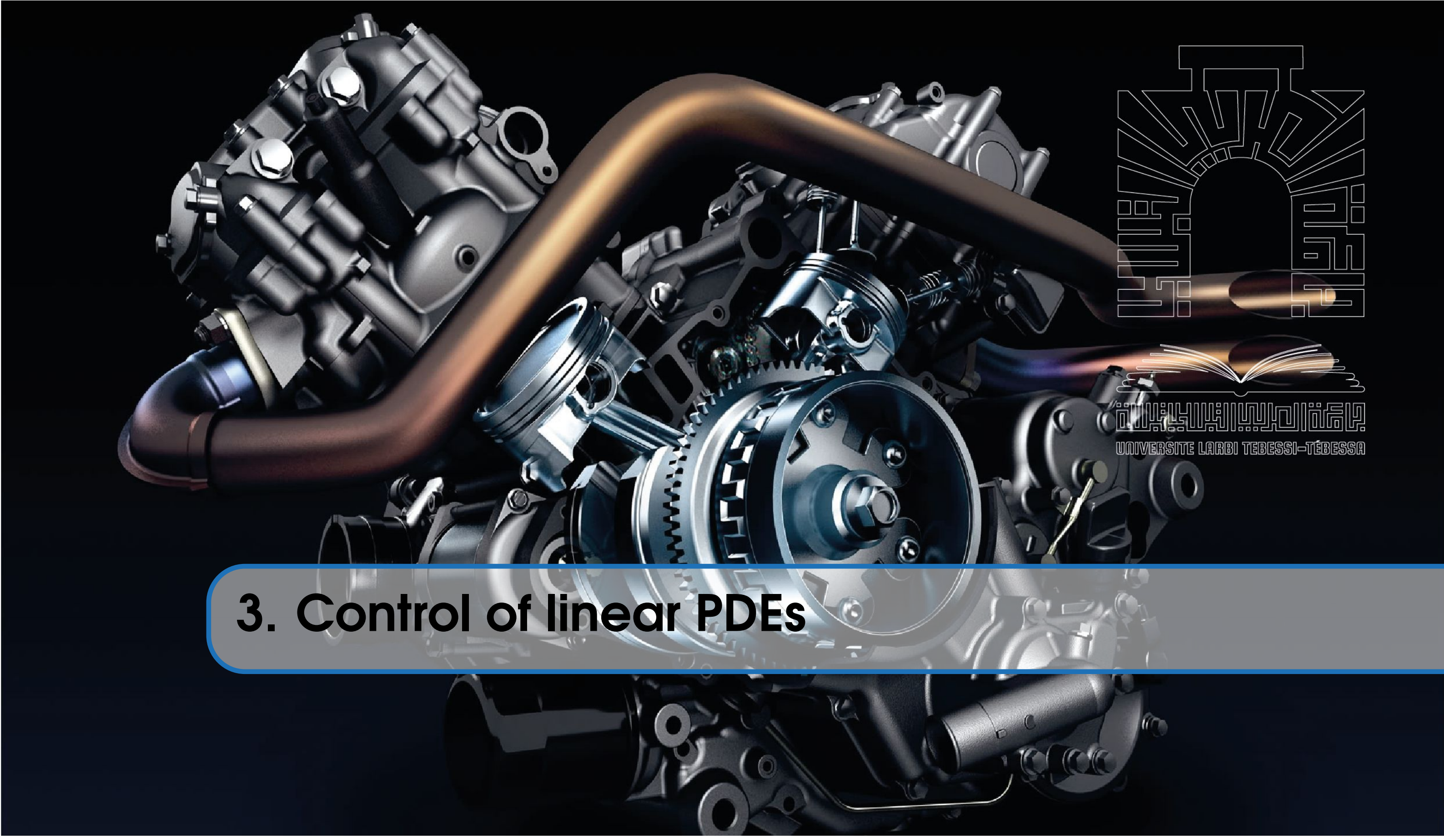


Introduction

To control a system is to try to make it behave according to our wishes, at the least possible cost, in a way that is compatible with safety, regulations, and ethics. We have a problem of optimal control of a rigorous and pragmatic state approach, here our objective (cost function) is to create recognized educational works at a given time t. Another example is the optimal control problem to make performance high effective digital tools the goal here is to precise in analysis and synthesis and speed implementation.

Those two optimal control problems have to follow a certain algorithm that contains some constraints and conditions named by the optimality conditions, which drive us to find a characterization of our optimal control.

In contrast to the preceding part where the main attention was paid to control problems governed by ODEs, this part aims to presents and define the different notions of optimal control of linear PDEs, the chapter one aw specialized to introduce the no-regret and low-regret control those twins notions applied just on a problems with missing data and the new notion of the average control such that it applied on an optimal control elliptic problem, also we can not forget some examples of optimal control ot different kind of PDEs (parabolic,elliptic, hyperbolic and hyperbolic-parabolic coupled system which is a new study example), all of those examples are just an introduction the the next chapter wherever we present a base article talk about the optimal control of an abstract systems with missing initial conditions, here is the start to the new results of an optimal control of an abstract hyperbolic-parabolic coupled systems with missing initial conditions.



3. Control of linear PDEs

This chapter is dedicated to introducing in the second section some properties of the cost functional, after that the third section contains some examples of the method of optimal control applied on an hyperbolic, parabolic and elliptic systems, this chapter rich also by a applied this method on an abstract hyperbolic-parabolic coupled system, which is a new work may be published later.

Without forgetting this chapter introduces and explains also the notions of no-regret and low-regret control which act only on the problem with missing or incomplete data. The last important concept presented here is that concept of Zuazua [55] called the *average control*, here it introduces just the average optimal control applied on an elliptic system. All of that helps with some important basics which are obtained in the first section.

3.1 Basics

The pieces of information under this section are taken from the reference [**5**].

While diving into this part of thesis and somewhere, you will find some transforms or passages made only with the help of the next basics :

Definition 3.1.1 A Banach space is a Complete normed space for distance $d(u,v) = \|u-v\|$.

Definition 3.1.2 — Quotient space E/M, $[x]$ or $\tilde{x}$. E normed space, M closed subspace of E, E/M vector $[x]$ if and only if

- $[x] = \{x+y, y \in M\}$.
- $[x+\lambda x'] = [x] + \lambda\,[x']$.
- $\|[x]\| = \inf \|x+y\|\,, y \in M$.

Proposition 3.1.1 If F is a Banach space, $\mathscr{L}(E,F)$ endowed with $|||.|||$ is a Banach space.

Definition 3.1.3 The set $\mathscr{L}(E,R)$ is the dual space of E, denoted by E'.

Theorem 3.1.2 Let $K \subset H$ be non-empty closed convex subset. Then for all $u \in H$, there exists $v \in K$ unique, such that,

$$\|u-v\| = \inf_{w\in K} \|u-w\| = \min_{w\in K} \|u-w\|.$$

Moreover, v characterized by

$$v \in K, \langle u-v, w-v\rangle \leq 0, \forall w \in K.$$

Definition 3.1.4 v is called the projection of u onto K and denoted $P_K u$.

Proposition 3.1.3 For all $(u_1,u_2) \in H^2$, $\|P_K u_1 - P_K u_2\| \leq \|u_1 - u_2\|$.

Corollary 3.1.4 Let $M \subset H$ vector subspace closed and $u \in H$. Then $v = P_M u$ characterized by

$$v \in M, \langle u-v, w\rangle = 0, \forall w \in M.$$

Theorem 3.1.5 — Riesz . Let $\phi \in H'$. There exists a unique $u \in H$, such that,

$$v \in H, \phi(v) = \langle u, v\rangle.$$

Moreover,

$$\|u\| = \|\phi\|_{H'}.$$

Theorem 3.1.6 From any bounded sequence of H, we can extract a weakly convergent subsequence.

Stampacchia theorem.

Definition 3.1.5 A bilinear form $A : H \times H \to \mathbb{R}$ is :
- Continue if there is $C > 0$, such that, for all (u,v), $|A(u,v)| \leq C\|u\|\,\|v\|$.
- Coercive if there is $\alpha > 0$, such that, for all u, $A(u,u) \geq \alpha \|u\|^2$.

Theorem 3.1.7 Let A be bilinear continuous and coercive. Let K be non-empty closed convex. For $\phi \in H'$, there exists a unique $u \in K$, such that,

$$\forall v \in K,\ A(u, v-u) \geq \phi(v-u).$$

If A is symmetric, then u characterized by

$$u \in K,\ \frac{1}{2}A(u,u) - \phi(u) = \min_{v\in K}\left(\frac{1}{2}A(v,v) - \phi(v)\right).$$

Definition 3.1.6 — Convex functions. Assume that $\Omega \subset \mathbb{R}^n$ an open convex subset, the function $f : \Omega \to \mathbb{R}^n$ is a convex, for all $x,\ y \in \Omega$ and $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Definition 3.1.7 — Strictly convex functions. Assume that $\Omega \subset \mathbb{R}^n$ an open convex subset, the function $f : \Omega \to \mathbb{R}^n$ is a strictly convex, for all $x, y \in \Omega, x \neq y$ and $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

Theorem 3.1.8 — Convex functions characterization. Assume that $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C^1(\mathbb{R}^n)$ and $\langle .,. \rangle$ the inner scalar product of $\mathbb{R}^n$. The next assumptions are equivalent :

- f is a convex function.
- $\forall x, y \in \mathbb{R}^n$ the next inequality is checked

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$

- $\forall x, y \in \mathbb{R}^n$ the next inequality is verified

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0.$$

- If $f \in C^2(\mathbb{R}^n)$, we have, $\forall x, v \in \mathbb{R}^n$ the next inequality is verified

$$\left\langle \nabla^2 f(x) v, \ v \right\rangle \geq 0.$$

Proposition 3.1.9 The space $W^{1,p}(\Omega)$ is :

- A Banach space for $1 \leq p \leq \infty$.
- A separable space for $1 \leq p < \infty$.
- A reflective space for $1 < p < \infty$.

Definition 3.1.8 Let $1 \leq p \leq \infty$. $W_0^{1,p}(\Omega)$ means the closing of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$, it is endowed with the norm induced by $W^{1,p}(\Omega)$ is a separable Banach space, it is moreover reflexive for $1 < p < \infty$. We denote

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space H_0^1 is a Hilbert space endowed with the scalar product of H^1

Lemma 3.1.10 When $\Omega = \mathbb{R}^N$ we know that, $C_0^1(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. Consequently,

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N).$$

Corollary 3.1.11 Another characterization of $W_0^{1,p}(\Omega)$, if is in class C^1 and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $1 \leq p < \infty$, then, the following properties are equivalent :

- $u = 0$ on $\partial\Omega$.
- $u \in W_0^{1,p}(\Omega)$.

Definition 3.1.9 — Dual space of $W_0^{1,p}(\Omega)$. We denote by, $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$, $(1 \leq p < \infty)$ avec $\frac{1}{p} + \frac{1}{p'} = 1$. We note that, $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$.
We can identify L^2 and its dual, therefore we have the inclusions

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),$$

with continuous injections.

Theorem 3.1.12 — Rellich-Kondrachov. If we assume that Ω is a bounded open-set of class C^1, we have :

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} C\left(\overline{\Omega}\right) & p > N, \\ L^q(\Omega), \ \forall q \in [1, p^*[\ if & p < N, \\ L^q(\Omega), \ \forall q \in [1, \infty[\ if & p = N. \end{cases}$$

with a compact injections.

The weak convergence in the spaces L^p

Definition 3.1.10 Let Ω an open subset of $\mathbb{R}^n$.
• If $1 \leq p < \infty$. we say that a sequence u_ν converges weakly to u in $L^p(\Omega)$ if u, $u_\nu \in L^p(\Omega)$ and if,

$$\lim_{\nu \to \infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x) = 0, \forall \varphi \in L^{p'}(\Omega).$$

We denote in this case $u_\nu \rightharpoonup u$ in L^p.
• If $p = \infty$, we say that the sequence u_ν converges $(*)$-weakly towards u in $L^\infty(\Omega)$ if u, $u_\nu \in L^\infty(\Omega)$ and if,

$$\lim_{\nu \to \infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x) = 0, \ \forall \varphi \in L^1(\Omega).$$

We denote that, $u_\nu \rightharpoonup^* u$ in L^∞.

Theorem 3.1.13 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The next properties are checked :
• If $u_\nu \to u$, then, $u_\nu \rightharpoonup u$ in L^p , for all $1 \leq p < \infty$.
• If $1 \leq p \leq \infty$ and if $u_\nu \rightharpoonup u$ in L^p , then, there exists a constant $C > 0$, such that,

$$\|u_\nu\|_{L^p} \leq C \text{ and } \|u\|_{L^p} \leq \liminf_{\nu \longrightarrow \infty} \|u_\nu\|_{L^p}.$$

• If $1 < p < \infty$ and if there exists a constant $C > 0$, such that, $\|u_\nu\|_{L^p} \leq C$, then, there is a subsequence $\{u_{\nu_i}\}$ and u in L^p, such that,

$$u_{\nu_i} \rightharpoonup u \text{ in } L^p.$$

• If $p = \infty$ and if there exists a constant $K > 0$, such that, $\|u_\nu\|_{L^\infty} \leq K$, then there is a subsequence $\{u_{\nu_i}\}$ and u in L^p, such that,

$$u_{\nu_i} \rightharpoonup^* u \text{ in } L^\infty.$$

Theorem 3.1.14 — Lebesgue Dominated Convergence. Let $(f_n(x))_{n=1}^\infty$ be a sequence of Lebesgue integrable functions that converge to a limit function f almost everywhere on I. Suppose that, there exists a Lebesgue integrable function g such that $|f_n| \leq g$ almost everywhere on I and for all $n \in N$. Then, f is Lebesgue integrable on I and

$$\lim_{n \to \infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

Theorem 3.1.15 — The Green formula. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and υ be the outward unit normal vector on $\Gamma = \partial\Omega$. Then, we have,
For $(u,v) \in H^1(\Omega) \times H^2(\Omega)$, we have the half Green formula

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \nabla v dx + \int_\Gamma u \frac{\partial v}{\partial \nu}\Gamma.$$

Theorem 3.1.16 — Fubini theorem. If $f(x,y)$ is continuous on the rectangular region

$$R = \{a \leq x \leq b,\ c \leq y \leq d\},$$

then, the equality

$$\iint_R f(x,y)d(x,y) = \int_a^b \int_c^d f(x,y)dxdy.$$

Gateaux derivative functions

Definition 3.1.11 Let $J : U \subset X \longrightarrow Y$ be an operator with Banach spaces X, Y and $U \neq 0$ open. J is called directionally differentiable at $x \in U$ if the limit

$$dJ(x,h) = \lim_{t\to 0^+} \frac{J(x+th) - J(x)}{t} \in Y$$

exists for all $h \in X$. J is called Gateaux differentiable at $x \in U$ if J is directionally differentiable at x and the directional derivative

$$J'(x) : h \in X \to dJ(x,h) \in Y$$

is bounded and linear, i.e., $J(x) \in \mathscr{L}(X,Y)$.

Theorem 3.1.17 Let X be a Banach space and $U \subset X$ be non-empty and convex. Furthermore, let $J : V \to R$ be defined on an open neighbourhood of U. Let u be a local solution of

$$\inf_{v\in U} J(v),$$

at which J is Gateaux-differentiable. Then the following optimality condition holds,

$$\langle J'(u), v-u\rangle_{X',X} \geq 0, \forall v \in U.$$

If J is convex on U, the last condition is necessary and sufficient for global optimality.

3.2 Minimization of convex functional

We remind here a number of well-known results in convex analysis. Let us consider a functional

$$\begin{array}{rcl} J : U_{ad} \subset U & \longrightarrow & \mathbb{R} \\ v & \longmapsto & J(v). \end{array}$$

We assume that :
- J is convex and continuous on U_{ad}.
- U_{ad} is a closed convex set in U.

• U is a reflexive Banach space on $\mathbb{R}$.
We have the next problem

$$\inf_{v \in U_{ad}} J(v).$$

Theorem 3.2.1 — Existence. [27] If we assume that,

$$J(v) \longrightarrow +\infty \text{ if } \|v\| \longrightarrow \infty,$$

then, there exists $u \in U_{ad}$, such that,

$$J(u) \leq J(v),\ \forall v \in U_{ad}, \tag{3.1}$$

Theorem 3.2.2 — Uniqueness. [27] If we assume that, the functional $v \longrightarrow J(v)$ is strictly convex, i.e., if

$$J((1-\lambda)u+\lambda v) < (1-\lambda)J(u)+\lambda J(v) \text{ if } 0 < \lambda < 1 \text{ and } u \neq v,$$

then, there is at most one u satisfying (3.1).

Let us give now more "analytic" conditions for (3.1) to hold true. If we assume that the function $v \longrightarrow J(v)$ is differentiable, then (3.1) is equivalent to

$$\forall v,\ u \in U_{ad},\ (J'(u), v-u) \geq 0$$

where,

$$(J'(u), \phi) = \frac{d}{d\gamma} J(u+\gamma\phi)|_{\gamma} = 0.$$

We assume that, J takes the next form

$$J(v) = J_1(v) + J_2(v).$$

where $J_1(v)$ is differentiable and $J_2(v)$ is continuous (not necessarily differentiable), and where both functions are convex, then, the previous equality is equivalent to

$$\forall v,\ u \in U_{ad},\ (J_1'(u), v-u) + J_2(v) - J_1(u) \geq 0.$$

■ **Example 3.1** Let us suppose that, U is a Hilbert space, we consider the next system

$$A(v) = Av - f,$$

where, $A \in L(U, U')$ and where f is given in U'. Then, if

$$(Av, v) \geq \alpha \|v\|^2_{U_{ad}},\ \alpha \geq 0,$$

there exists a unique $u \in U_{ad}$, such that,

$$\forall v \in U_{ad},\ (Au - f, v-u) \geq 0. \tag{3.2}$$

■

▶ For the proof see [**1**]

R If A is symmetric, then $Au - f$ is the derivative of

$$J(v) = \frac{1}{2}(Av, v) - (f, v),$$

if A is not symmetric, then (3.2) does not correspond to a minimization problem, but it is a useful tool.

3.3 Optimal control of linear distributed systems

Let Y, U and V be Hilbert spaces, let us consider the infinite dimensional optimal control problem written in the following abstract form

$$\inf J\left(v,y\right), \tag{3.3}$$

under the constraints

$$Ay = f + Bv, \tag{3.4}$$

where :
• $v \in U_{ad} \subset U$, U_{ad} is the set of admissible controls, which is a closed convex non-empty set of the space of controls U.
• The spaces Y and U are respectively the state spaces.
• A is a linear partial differential operator, isomorphism of $\mathscr{L}(Y)$.
• B is the control linear operator $\mathscr{L}\left(U,Y\right)$.
• J is a convex function from $Y \times U$ to $\mathbb{R} \cup \{\infty\}$.
Let Z be an observation Hilbert space and $C \in \mathscr{L}\left(Y,Z\right)$ an observation operator.
Consider the next quadratic cost function

$$J\left(v,y\right) = \|Cy(v) - y_d\|_Z^2 + \beta\left(v,v\right)_U \tag{3.5}$$

where :
• y_d is an observation given in Z it is the goal.
• $\beta \in \mathscr{L}\left(U\right)$ symmetric and positive.
Then, our optimal control problem consists in determining or characterizing the control u which minimizes J on U_{ad} i.e.,

$$\begin{cases} \text{Find } u \in U_{ad}, \\ J(u, y(u)) = \inf_{v \in U_{ad}} J(v, y), \\ Ay = f + Bv. \end{cases} \tag{3.6}$$

The pair $(u, y(u))$ verifies (3.6) is called the optimal pair.

Theorem 3.3.1 If the cost function J defined by (3.5) is Gateaux derivative, then the optimal control $u \in U_{ad}$ is unique and characterized by

$$\begin{cases} J'(u)(v-u) &= (Cy(u) - y_d, C(y(v) - y(u)))_Z + (\beta u, v-u)_U \geq 0, \forall v \in U_{ad}, \\ Ay(u) &= f + Bu. \end{cases} \tag{3.7}$$

Proof. J is strictly convex which implies the uniqueness of an optimum coercive since

$$J(v) \longrightarrow \infty \text{ when } \|v\|_U, \; v \in U_{ad},$$

which implies the existence of an optimum, lower semi-continuous[1] because it is continuous. The optimum is characterised by the next variational inequality

$$\forall v \in U_{ad}, \; J'(u)(v-u) \geq 0,$$

where,

$$\begin{aligned} J'(u)(v-u) &= \left.\frac{d}{dh}J(u+h(v-u))\right|_{h=0}, \\ &= (Cy(u)-y_d, C(y(v)-y(u)))_Z + (\beta u, v-u)_U, \; \forall v \in U_{ad}. \end{aligned}$$

■

R In the case where $U_{ad} = U$ the vector $v-u$ describes the entire space U, then (3.7) is written as follows

$$J'(u)(w) \geq 0, \; \forall w \in U_{ad},$$

this is the case without constraint on v.

Optimality system :

Let us introduce an adjoint state $p = p(u)$ of (3.4) given by

$$A^* p = C^*(Cy(u) - y_d),$$

where A^* and C^* are the adjoint operators of A and C respectively. and we have

$$(A^* p, y(v) - (u))_Y = (C^*(Cy(u) - y_d), y(v) - y(u))_Y,$$

On the other hand

$$(A^* p, y(v) - y(u))_Y = (p, Ay(v) - Ay(u))_Y = (p, B(v-u))_Y = (B^* p, v-u)_U,$$

Then, inequality (3.7) is equivalent to

$$\begin{cases} Ay(u) = f + Bu, \\ A^* p = C^*(Cy(u) - y_d), \\ (B^* p + \beta u, v-u)_U \geq 0, \; \forall v, \; u \in U_{ad}. \end{cases}$$

In the unconstrained case i.e. $U_{ad} = U$ the optimality system is written as follows

$$\begin{cases} Ay(u) &= f + Bu, \\ A^* p &= C^*(Cy(u) - y_d), \\ u &= \dfrac{-1}{\beta} B^* p, \; \forall u \in U_{ad}. \end{cases}$$

[1] Let $f : D \to \mathbb{R}$ and let $\bar{x} \in D$. We say that f is lower semi-continuous (l.s.c.) at $\bar{x}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, such that, $f(\bar{x}) - \varepsilon < f(x)$ for all $x \in B(\bar{x}, \delta) \cap D$.

3.3.1 Optimal control of hyperbolic distributed system

Let Ω be a bounded domain of $\mathbb{R}^n$, and let Γ its boundary be of class C^2, $T>0$, $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. Consider the next wave equation with Dirichlet condition

$$\left\{\begin{array}{rcll} y_{tt}-\Delta y &=& f+\chi_\omega v & \textit{in } Q,\\ (y,y_t)(x,0) &=& (y_0(x),y_1(x)) & \textit{in } \Omega,\\ y &=& 0 & \textit{on } \Sigma. \end{array}\right. \tag{3.8}$$

where :
- f is a source function in $L^2(Q)$.
- v is the control function supposed to be in $L^2(0,T;L^2(\Omega))$.
- χ_ω is the characteristic function of ω which is a given open subset of Ω.
- $(y_0,y_1)\in H_0^1(\Omega)\times L^2(\Omega)$.

Give an existence and uniqueness theorem for (3.8).

Theorem 3.3.2 For all $(f,y_0,y_1)\in(L^2(Q),H_0^1(\Omega),L^2(\Omega))$ the equation (3.8) has a weak solution y, moreover, the application

$$\begin{array}{rcl} L^2(Q),H_0^1(\Omega),L^2(\Omega) & \longrightarrow & C(0,T;H_0^1(\Omega))\cup C^1(0,T;L^2(\Omega))\\ (f,y_0,y_1) & \longmapsto & y(f,y_0,y_1), \end{array}$$

is linear and continuous.

Our optimal control problem consists in finding a control function $u\in L^2(0,T;L^2(\Omega))$ which minimizes the following quadratic cost function

$$\begin{array}{rcl} J(v,y) &=& \|y(v)-y_d\|^2_{L^2(Q)}+\|y(v)(T)-y_d(T)\|^2_{L2(\Omega)}\\ &+& \beta\,\|v\|^2_{L^2(0,T;L^2(\omega))}\,,\ v\in U_{ad}\subset L^2(0,T;L^2(\omega)), \end{array} \tag{3.9}$$

where :
- $\beta>0$.
- $y(v)$ is the solution of (3.6).
- $y_d\in L^2(Q)$ and $y_d(T)\in L^2(\Omega)$ is a desired states.

Here, we will say that, the control is internal or distributed. So, we will write an optimality system that characterizes the solution of

$$\inf\left\{J(v,y):(v,y)\in U_{ad}\times L^2(0,T;H_0^1(\Omega))\quad\text{verify}\quad(3.8)\right\}. \tag{3.10}$$

In this case:
- $Y=L^2(0,T;H_0^1(\Omega))$ is the state space.
- $Z=L^2(Q)$ is the observation space.
- $U=L^2(0,T;L^2(\Omega))$ is the control space.
- $C:L^2(0,T;H_0^1(\Omega))\to L^2(Q)$ is the canonical injection.

A first-order optimality condition gives us

$$\begin{array}{rcl} (y(u)-y_d,y(v)-y(u))_{L^2(Q)} &+& (y(u)(T)-y_d(T),y(v)-y(u))_{L^2(\Omega)}\\ &+& (\beta u,v-u)_{L^2(0,T;L^2(\omega))}\geq 0,\ \forall v\in U_{ad}. \end{array}$$

Now, let us look for a state $p=p(u)$, such that,

$$\begin{array}{rcl} (y(u)-y_d,y(v)-y(u))_{L^2(Q)} &+& (y(u)(T)-y_d(T),y(v)-y(u))_{L^2(\Omega)}\\ &=& (p,v-u)_{L^2(0,T;L^2(\omega))}\geq 0,\ \forall v\in U_{ad}. \end{array}$$

Let us introduce the adjoint state $p = p(u)$ through

$$\left\{\begin{array}{rcll} p_{tt} - \Delta p & = & y(u) - y_d & in\ Q, \\ (p, p_t)(x, T) & = & (0, y(u)(T) - y_d(T)) & in\ \Omega, \\ p & = & 0 & on\ \Sigma. \end{array}\right.$$

Theorem 3.3.3 The optimal pair $(u, y(u))$ solution of (3.8) is characterized by the next optimality system

$$\left\{\begin{array}{lcll} y_{tt} - \Delta y & = & f + \chi_\omega & in\ Q, \\ p_{tt} - \Delta p & = & y(u) - y_d & in\ Q, \\ (y(u), y_t(u))(x, 0) & = & (y_0(x), y_1(x)) & in\ \Omega, \\ (p, p_t)(x, T) & = & (0, y(u)(T) - y_d(T)) & in\ \Omega, \\ (y, p) & = & (0, 0) & on\ \Sigma. \end{array}\right.$$

with the next optimality condition

$$(p + \beta u, v - u)_{L^2(0,T;L^2(\omega))} \geq 0, \ \forall v \in U_{ad}.$$

R If $U_{ad} = L^2(0, T; L^2(\omega))$, the optimal pair $(u, y(u))$ is characterized by the next optimality system

$$\left\{\begin{array}{lcll} y_{tt} - \Delta y & = & f + \chi_\omega u & in\ Q, \\ p_{tt} - \Delta p & = & y(u) - y_d & in\ Q, \\ (y(u), y_t(u))(x, 0) & = & (y_0(x), y_1(x)) & in\ \Omega, \\ (p, p_t)(x, T) & = & (0, y(u)(T) - y_d(T)) & in\ \Omega, \\ (y, p) & = & (0, 0) & on\ \Sigma. \end{array}\right.$$

with the next optimality condition

$$u = -\frac{1}{\beta} p(u) \ \in L^2(0, T; L^2(\omega)).$$

3.3.2 Optimal control of parabolic system

Let Ω a bounded domain of $\mathbb{R}^n$, Γ its boundary be of class C^2, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.
Consider the next heat equation with Dirichlet condition

$$\left\{\begin{array}{rcll} y_t - \Delta y & = & f + \chi_\omega v & in\ Q, \\ y(x, 0) & = & y_0(x) & in\ \Omega, \\ y & = & 0 & on\ \Sigma. \end{array}\right. \tag{3.11}$$

where :
- f is a source function in $L^2(Q)$.
- v is the control function assumed in $L^2(0, T; L^2(\Omega))$.
- χ_ω is the characteristic function of ω an open subset of Ω.
- $y_0 \in L^2(\Omega)$.

Before introducing the optimal control problem, we recall some results of the existence and uniqueness for the solution for (3.11).

Theorem 3.3.4 Consider the next system

$$\begin{cases} w_t - \Delta w &= \xi & in\ Q, \\ w(x,0) &= w_0(x) & in\ \Omega, \\ w &= 0 & on\ \Sigma. \end{cases}$$

For all $\xi \in L^2(Q)$ and $w_0 \in L^2(\Omega)$, the previous system admits a weak unique solution in $L^2(0,T;L^2(\Omega))$, moreover, the operator

$$\begin{array}{rcl} L^2(Q) \times L^2(\Omega) & \to & W(0,T;H_0^1(\Omega),H^{-1}(\Omega)^a) \\ (\xi, w_0) & \mapsto & w(\xi, w_0), \end{array}$$

with,

$$W(0,T;H_0^1(\Omega),H^{-1}(\Omega)) = \left\{ w \in L^2(0,T;H_0^1(\Omega)),\ w_t \in L^2(0,T;H^{-1}(\Omega)) \right\}$$

is continuous.

[a] The dual space of $H_0^1(\Omega)$.

Our optimal control problem deals with finding a control function $u \in L^2(0,T;L^2(\omega))$ which minimizes the next quadratic cost function

$$J(v,y) = \|y(v) - y_d\|^2_{L^2(Q)} + \beta \|v\|^2_{L^2(0,T;L^2(\omega))},\ v \in U_{ad} \subset L^2(0,T;L^2(\omega)).$$

Where :
- $\beta > 0$.
- $y(v)$ is the solution of (3.11).
- $y_d \in L^2(Q)$ is a desired state.

Here, we will say that, the control is internal or distributed. Then, we want to write an optimality system that characterizes the solution of

$$\inf \left\{ J(v,y) : (v,y) \in U_{ad} \times L^2(0,T;H_0^1(\Omega)) \quad \text{verify} \quad (3.11) \right\} \tag{3.12}$$

In this case:
- $Y = L^2(0,T;H_0^1(\Omega))$ is the state space.
- $Z = L^2(Q)$ is the observation space.
- $U = L^2(0,T;L^2(\Omega))$ is the control space.
- $C : L^2(0,T;H_0^1(\Omega)) \to L^2(Q)$ is the canonical injection.

A first-order optimality condition gives us

$$(y(u) - y_d, y(v) - y(u))_{L^2(Q)} + (\beta u, v-u)_{L^2(0,T;L^2(\omega))} \geq 0,\ \forall v \in U_{ad}.$$

Now let us look for a state $p = p(u)$, such that,

$$(y(u) - y_d, y(v) - y(u))_{L^2(Q)} = (p, v-u)_{L^2(0,T;L^2(\omega))} \geq 0,\ \forall v \in U_{ad}.$$

Let us introduce the adjoint state $p = p(u)$ through

$$\begin{cases} -p_t - \Delta p &= y(u) - y_d & in\ Q, \\ p(x,T) &= 0 & in\ \Omega, \\ p &= 0 & on\ \Sigma. \end{cases}$$

then, by using the Green formula

$$\begin{array}{rcl}\int_0^T \int_\Omega (y(u)-y_d)(y(v)-y(u))dxdt &=& \int_0^T \int_\Omega (-p_t-\Delta p)\,(y(v)-y(u))dxdt \\ &=& \int_0^T \int_\Omega p\left(-\frac{\partial}{\partial t}-\Delta\right)(y(v)-y(u))dxdt \\ &=& \int_0^T \int_\omega p(v-u)dxdt.\end{array}$$

So, we can write the first-order optimality as follows

$$(p+\beta u, v-u)_{L^2(0,T;L^2(\omega))} \geq 0, \ \forall v \in U_{ad}.$$

Theorem 3.3.5 The optimal pair $(u, y(u))$ solution of (3.12) is characterized by the following optimality system

$$\left\{\begin{array}{rcll} y_t-\Delta y &=& f+\chi_\omega v & in\ Q, \\ -p_t-\Delta p &=& y(u)-y_d & in\ Q, \\ y(x,0) &=& y_0(x) & in\ \Omega, \\ p(x,T) &=& 0 & in\ \Omega, \\ (y,p) &=& (0,0) & on\ \Sigma, \end{array}\right.$$

with the next optimality condition

$$(p+\beta u, v-u)_{L^2(0,T;L^2(\omega))} \geq 0, \ \forall v \in U_{ad}.$$

R If $U_{ad} = L^2(0,T;L^2(\omega))$, the optimal pair $(u, y(u))$ is characterised by the next optimality system

$$\left\{\begin{array}{rcll} y_t-\Delta y &=& f+\chi_\omega v & in\ Q, \\ -p_t-\Delta p &=& y(u)-y_d & in\ Q, \\ y(x,0) &=& y_0(x) & in\ \Omega, \\ p(x,T) &=& 0 & in\ \Omega, \\ (y,p) &=& (0,0) & on\ \Sigma, \\ u &=& \frac{1}{\beta}p & in\ L^2(0,T;L^2(\omega)). \end{array}\right.$$

3.3.3 Optimal control of elliptic system

Let Ω a bounded domain of $\mathbb{R}^n$, assume that Γ its boundary which is class C^2. Consider the next Laplace equation with Newmann condition

$$\left\{\begin{array}{rcll} -\Delta y+y &=& f & in\ \Omega, \\ \frac{\partial y}{\partial \nu} &=& v & on\ \Gamma. \end{array}\right. \tag{3.13}$$

Where :

- f is a source function in $L^2(\Omega)$.
- v is the control function in $L^2(\Gamma)$.

Before presenting the optimal control problem, we recall some results for the existence and uniqueness of a solution for (3.13).

Theorem 3.3.6 For all $f \in L^2(\Omega)$ and $v \in L^2(\Gamma)$, the equation (3.13) admits a unique weak solution in $H^1(\Omega)$, moreover, the application

$$\begin{array}{rcl} L^2(\Omega)\times L^2(\Gamma) & \to & H^{\frac{3}{2}}(\Omega) \\ (f,v) & \mapsto & y(f,v) \end{array}$$

is linear continuous.

Our optimal control problem deals with finding a control function $u \in L^2(\Gamma)$ which minimizes the next quadratic cost function

$$J(v,y) = \|y(v) - y_d\|^2_{L^2(\Omega)} + \beta\,\|v\|^2_{L^2(\Gamma)},\ v \in U_{ad} \subset L^2(\Gamma). \tag{3.14}$$

Where :
- $\beta > 0$.
- $y(v)$ is the solution of (3.19).
- $y_d \in L^2(\Omega)$ is a desired state.

Here, we will say that the control acts at the border. So, we must write an optimality system that characterizes the solution of

$$\inf\left\{J(v,y) : (v,y) \in U_{ad} \times H^1(\Omega) \text{ verify (3.11)}\right\}.$$

In this case :
- $Y = H^1(\Omega)$ is the state space.
- $Z = L^2(\Omega)$ is the observation space.
- $U = L^2(\Gamma)$ is the control space.
- U_{ad} is the set of admissible controls, is a set closed convex non-empty of the control space $L^2(\Gamma)$.
- $C : H^1(\Omega) \to L^2(\Omega)$ is the canonical injection.

A first-order optimality condition gives us

$$(y(u) - y_d, y(v) - y(u))_{L^2(\Omega)} + (\beta u, v-u)_{L^2(\Gamma)} \ \geq 0,\ \ \forall v \in U_{ad}.$$

Now, let us look for a state $p = p(u)$, such that,

$$(y(u) - y_d, y(v) - y(u))_{L^2(\Omega)} = (p, v-u)_{L^2(\Gamma)},\ \forall v \in U_{ad}.$$

It is clear that, this equality will be linked to the Green formula. Let us define a state $p = p(u) \in H^1(\Omega)$ solution of

$$\left\{\begin{array}{rcll} -\Delta p + p & = & y(u) - y_d & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \nu} & = & 0 & \text{on } \Gamma. \end{array}\right.$$

In this case according to the Green formula

$$\begin{array}{rcl} \int_\Omega (y(u) - y_d)(y(v) - y(u))\,dx & = & \int_\Omega (-\Delta p + p)(y(v) - y(u))\,dx \\ & = & \int_\Omega p\left(-\Delta(y(v) - y(u)) + y(v) - y(u)\right)dx \\ & + & \int_\Gamma \left(p\frac{\partial(y(v)-y(u))}{\partial \nu} - (y(v) - y(u))\frac{\partial p}{\partial \nu}\right) d\Gamma \end{array}$$

and the first-order optimality condition is written as follows

$$(p + \beta u, v - u)_{L^2(\Gamma)} \ \geq 0,\ \ \forall v \in U_{ad},$$

Theorem 3.3.7 The optimal pair $(u,y(u))$ solution of (3.14) is characterized by the next optimality system

$$\begin{cases} -\Delta y(u)+y(u) &= f & \text{in } \Omega, \\ -\Delta p+p &= y(u)-y_d & \text{in } \Omega, \\ \dfrac{\partial y(u)}{\partial \nu} &= u & \text{on } \Gamma, \\ \dfrac{\partial p}{\partial \nu} &= 0 & \text{on } \Gamma, \end{cases}$$

with the next optimality condition

$$(p+\beta u, v-u)_{L^2(\Gamma)} \geq 0, \ \ \forall v \in U_{ad}, \ where, \ \ (p,y(u)) \in H^{\frac{3}{2}}(\Omega)\times H^1(\Omega).$$

R If $U_{ad}=L^2(\Gamma)$, the optimal pair $(u,y(u))$ is characterized by the next optimality system

$$\begin{cases} -\Delta y(u)+y(u) &= f & in\ \Omega, \\ -\Delta p+p &= y(u)-y_d & in\ \Omega, \\ \dfrac{\partial y(u)}{\partial \nu} &= u & on\ \Gamma, \\ \dfrac{\partial p}{\partial \nu} &= 0 & on\ \Gamma, \\ u &= -\dfrac{1}{\beta}p & in\ \Gamma, \end{cases}$$

where,

$$(p,y(u)) \in H^{\frac{3}{2}}(\Omega)\times H^1(\Omega).$$

3.3.4 Optimal control of an abstract coupled system

Consider the next abstract hyperbolic-parabolic coupled system

$$\begin{cases} y_{tt}+L_1 y+\alpha M\theta &= f+Bv, \\ \theta_t+L_2\theta+\alpha N y_t &= 0, \\ (y,y_t,\theta)(0) &= (y_0,y_1,\theta_0). \end{cases} \tag{3.15}$$

Here :
• $t\in(0,T)$, $T>0$.
• U, H_1, H_2 and H_3 be a real separable Hilbert spaces.
• y and θ are functions with values in H_1 and H_2 respectively.
• $f\in L^2(0,T;H_1)$, L_1, L_2 are unbounded self adjoint positive definite operators acting in H_1 and H_2, with the domains D_{L_1} and D_{L_2} [2].
Let us define on D_{L_1} and D_{L_2}, respectively, the next norms :

$$\|.\|_{D_{L_1}} = \|L_1.\|_{H_1} \text{ and } \|.\|_{D_{L_2}} = \|L_2.\|_{H_2},$$

then, D_{L_1} and D_{L_2} become Banach spaces.
Denote by $D_{L_1^{1/2}}$ and $D_{L_2^{1/2}}$ [3] the domain of the operators $L_1^{1/2}$ and $L_2^{1/2}$ equipped with the next scalar product

$$\langle .,..\rangle_1 = \left(L_1^{1/2}.,L_1^{1/2}..\right)_1 \text{ and } \langle .,..\rangle_2 = \left(L_2^{1/2}.,L_2^{1/2}..\right)_2,$$

[2] D_{L_1} and D_{L_2} are dense in H_1 and H_2.
[3] Separable Hilbert spaces.

with the corresponding norms equipped with the next scalar product

$$\|.\|_{D_{L_1^{1/2}}} = \left\|L_1^{1/2}.\right\|_{H_1} \text{ and } \|.\|_{D_{L_2^{1/2}}} = \left\|L_2^{1/2}.\right\|_{H_2}.$$

where :

$$D_{L_1^{1/2}} \subset D_N \text{ and } D_{L_2^{1/2}} \subset D_M.$$

We assume that :

- M and N are linear operators acting, respectively, from D_M into H_1 and from D_N into H_2, where, D_M and D_N are linear sets, such that,

$$D_{L_2} \subset D_M,\ M \in \mathscr{L}(D_{L_2}, H_1), \quad \text{and} \quad N \in \mathscr{L}(D_{L_1^{1/2}}, H_2)$$

- $B \in \mathscr{L}(U, H_1)$ is the control operator.
- $g = (y_0, y_1, \theta_0)$ are known initial conditions belonging at least to the spaces $H = D_{L_1^{1/2}} \times H_1 \times H_2$,
- α is a coupled parameter in $(0,1)$.
- v is a distributed control vector in $U_{ad} \subset U$ is a convex, non-empty and closed sub-set.

Give an existence and uniqueness theorem for (3.15)

Theorem 3.3.8 For all $f \in L^1(0,T;H_1),\ g \in H$ the equation (3.15) has a unique weak solution,

$$(y,\theta) \in W^{1,\infty}(0,T;H_1) \cap L^\infty\left(0,T;D_{L_1^{1/2}}\right) \times L^\infty(0,T;H_2) \cap L^2\left(0,T;D_{L_2^{1/2}}\right).$$

This theorem is basically proved by well-known methods [**22**]. The exact weak solution to problem (3.15) is constructed as a weak limit of the sequence of its Galerkin approximate solutions in the corresponding function space.

Our optimal control problem consists in finding a control function $u \in U_{ad}$ which minimizes the next quadratic cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(0,T;H_1)} + \|\theta(v) - \theta_d\|^2_{L^2(0,T;H_2)} + \beta\|v\|^2_U,\ v \in U_{ad} \tag{3.16}$$

Such that, $y_d,\ \theta_d$ are given observations and $\beta > 0$.

Here, we will write an optimality system that characterizes the solution of

$$\inf\left\{J(v) : (v,y,\theta) \in U_{ad} \times L^2(0,T;H_1) \times L^2(0,T;H_2) \text{ verify (3.15)}\right\}. \tag{3.17}$$

A first-order optimality condition gives us

$$\begin{aligned}(y(u) - y_d, y(v) - y(u))_{L^2(0,T;H_1)} &+ (\theta(u) - \theta_d, \theta(v) - \theta(u))_{L^2(0,T;H_2)} \\ &+ (\beta u, v - u)_U \geq 0,\ \forall v \in U_{ad}.\end{aligned}$$

Now, let us look for a state

$$(p,q) = (p(u), q(u)) \in W^{1,1}(0,T;H_1) \cap L^1\left(0,T;D_{L_1^{1/2}}\right) \times W^{1,1}(0,T;H_2) \cap L^2\left(0,T;D_{L_2^{1/2}}\right).$$

Such that,

$$\begin{cases} p_{tt} + L_1 p - \alpha N^* q_t &= y(u,0) - y(0,0), \\ -q_t + L_2 q + \alpha M^* p &= \theta(u,0) - \theta(0,0), \\ (p, p_t, q)(T) &= (0,0,0). \end{cases} \tag{3.18}$$

Furthermore (p,q) is a solution of (3.18), which gives

$$\begin{array}{ll} & \int_0^T (y(u)-y_d, y(v)-y(u))_{H_1} dt + \int_0^T (\theta(u)-\theta_d, \theta(v)-\theta(u))_{H_2} dt \\ = & \int_0^T (p_{tt}+L_1 p-\alpha N^* q_t, y(v)-y(u))_{H_1} dt + \int_0^T (-q_t+L_2 q+\alpha M^* p, \theta(v)-\theta(u))_{H_2} dt \\ = & \int_0^T (p, B(v-u))_{H_1} dt = (B^* p, v-u)_U . \end{array}$$

So, the first-order optimality condition is written

$$(B^* p+\beta u, v-u)_U \geq 0,\ \forall v \in U_{ad}.$$

Accordingly, which is lead us to the next theorem.

Theorem 3.3.9 The optimal triple $(y(u), \theta(u), u)$ solution of (3.17) is characterised by the next optimality system

$$\left\{ \begin{array}{|lll} y_{tt}+L_1 y+\alpha M\theta & = & f+Bv, \\ \theta_t+L_2\theta+\alpha N y_t & = & 0, \\ (y, y_t, \theta)(0) & = & (y_0, y_1, \theta_0). \\ p_{tt}+L_1 p-\alpha N^* q_t & = & y(u,0)-y(0,0), \\ -q_t+L_2 q+\alpha M^* p & = & \theta(u,0)-\theta(0,0), \\ (p, p_t, q)(T) & = & (0,0,0). \end{array} \right. \tag{3.19}$$

with the next optimality condition

$$(B^* p+\beta u, v-u)_U \geq 0,\ \forall v \in U_{ad}.$$

R If $U_{ad}=U$, the optimal triple $(u, y(u), \theta(u))$ is characterized by the next optimality system

$$\left\{ \begin{array}{|lll} y_{tt}+L_1 y+\alpha M\theta & = & f+Bv, \\ \theta_t+L_2\theta+\alpha N y_t & = & 0, \\ (y, y_t, \theta)(0) & = & (y_0, y_1, \theta_0). \\ p_{tt}+L_1 p-\alpha N^* q_t & = & y(u,0)-y(0,0), \\ -q_t+L_2 q+\alpha M^* p & = & \theta(u,0)-\theta(0,0), \\ (p, p_t, q)(T) & = & (0,0,0). \end{array} \right. \tag{3.20}$$

with the next optimality condition

$$u=-\frac{1}{\beta} B^* p(u) \in U_{ad}.$$

3.4 No-regret and low-regret control

This section presents the concept of no-regret control for distributed systems with missing data, which was introduced by J.Lions in [**31**], then, was developed by O. Nakolima, A. Omrane et J. Valin in [**40**], [**41**], and [**42**].
A no-regret control is associated with a sequence of controls with the low-regrets defined by a quadratic perturbation. We show here that, the perturbed system which corresponds to a sequence of standard control problems converges towards the no-regret control for which we obtain an optimality system.

3.4.1 Statement of the problem

Let us consider the next controlled abstract equation with missing data

$$Ay(v,g) = f + Bv + Ng, \tag{3.21}$$

Where :
- V is a real Hilbert space with dual V'.
- A is an operator in $\mathscr{L}(V,V')$.
- U[4] is the controls space.
- $B \in \mathscr{L}(U,V')$.
- G is a non-empty closed subspace of the Hilbert space of uncertainties F.
- N an operator in $\mathscr{L}(F,V')$.

For every uncertainty $g \in G$, the equation (3.21) is well-posed in V', it has a unique solution $y = y(v,g)$.
We consider the next cost function which associated to (3.21)

$$J(v,g) = \|Cy - y_d\|_Z^2 + \beta \|v\|_U^2, \ \forall v \in U_{ad}, \tag{3.22}$$

where, $y_d \in Z$ and $\beta > 0$. In this case, the most important for us is the optimal control problem

$$\inf_{v\in U_{ad}} J(v,g), \ \forall g \in G, \tag{3.23}$$

with respect to (3.21), when G is an infinite dimensional space, this problem has no sense, hence, the celebrated mathematician J. L. Lions thought to take

$$\inf_{v\in U_{ad}} \left(\sup_{g\in G} J(v,g) \right), \tag{3.24}$$

Because $\sup_{g\in G} J(v,g) = +\infty$, $J(v,g)$ has not an upper bound.

R When $G = \{0\}$, then, $J(v,g) = J(v,0)$. Therefore, the problem (3.23) becomes a standard optimal control problem, it means, find $\inf_{v\in U_{ad}} J(v,0)$.

3.4.2 No-regret control

To avoid difficulty arises in (3.24) Lions thought [5] to look only for controls v, such that,

$$J(v,g) \le J(0,g), \ \forall g \in G. \tag{3.25}$$

Note that, the optimal control verifies the last equality, otherwise the optimum is $u = 0$.

Definition 3.4.1 [**31**] We say that $u \in U_{ad}$ is a no-regret control for (3.21)-(3.22) if u solves

$$\inf_{v\in U_{ad}} \sup_{g\in G} (J(v,g) - J(0,g)). \tag{3.26}$$

R The problem (3.26) is defined only for controls, such that,

$$\sup_{g\in G} (J(v,g) - J(0,g)) < \infty. \tag{3.27}$$

[4] Hilbert space.
[5] This idea was originated in statistics in [Savage, L.J., 1972. The foundations of statistics, 2nd edition. Dover.].

Lemma 3.4.1 [42] we have for every $(v,g) \in U_{ad} \times G$

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle S(v), g\rangle_{G',G}. \tag{3.28}$$

Where, $S(v) = N^*\xi(v)$ and ξ solves

$$A^*\xi(v) = C^*C(y(v,0) - y(0,0)).$$

Proof. A is an isomorphism[6] so $y(v,g) = y(v,0) + y(0,g) - y(0,0)$, then,

$$\begin{aligned} J(v,g) - J(0,g) &= J(v,0) - J(0,0) + 2(C(y(v,0) - y(0,0)), C(y(0,g) - y(0,0)))_Z \\ &= J(v,0) - J(0,0) + 2(C^*C(y(v,0) - y(0,0)), y(0,g) - y(0,0))_{V'}. \end{aligned} \tag{3.29}$$

Introduce $\xi(v)$ given by

$$A^*\xi(v) = C^*C(y(v,0) - y(0,0)),$$

we can write (3.28) as follows

$$\begin{aligned} J(v,0) - J(0,0) &= J(v,0) - J(0,0) + 2(A^*\xi(v), y(0,g) - y(0,0))_{V'} \\ &= J(v,0) - J(0,0) + 2(\xi(v), A(y(0,g) - y(0,0)))_{V'} \\ &= J(v,0) - J(0,0) + 2(\xi(v), Ng)_{V'} \\ &= J(v,0) - J(0,0) + 2\langle N^*\xi(v), g\rangle_{G',G}, \end{aligned}$$

the last equation leads to (3.28). ■

R

- By (3.28) we can see that, the condition (3.27) holds if and only if $v \in K$, where,

$$K = \left\{ v \in U_{ad} : \langle S(v), g\rangle_{G',G} = 0 \ \ \forall g \in G \right\}$$

is a closed subspace of U. Then, u is a no-regret control if and only if $u \in K$.

- The notion of no-regret control could be generalized to no-regret control related to any a fixed control $u_0 \in U_{ad}$, i.e., we want controls v, such that,

$$J(v,g) \le J(u_0,g) \text{ for every } g \in G.$$

Definition 3.4.2 We say that $u \in U_{ad}$ is a no-regret control related to $u_0 \in U_{ad}$ for (3.21)-(3.22) if u solves

$$\inf_{v\in U_{ad}} \sup_{g\in G} (J(v,g) - J(u_0,g)). \tag{3.30}$$

When we want to characterize the set K, the main difficulty with no-regret control arises, for this reason, we will relax the no-regret control by a sequence of controls called low-regret controls.

3.4.3 Low-regret control

We aim to relax (3.25) by making some quadratic perturbation on $J(0,g)$ (see [31]). In other words, we search controls v, such that,

$$\forall g \in G, \ \beta > 0 \ \ J(v,g) \le J(0,g) + \gamma \|g\|_G^2.$$

[6] An isomorphism between two structured sets is a one-to-one application that preserves the structure, and whose converse also preserves the structure.

Definition 3.4.3 [31] We say that $u_\gamma \in U_{ad}$ is a low-regret control for (3.21)-(3.22) if u solves

$$\inf_{v\in U_{ad}} \sup_{g\in G} \left(J(v,g) - J(0,g) - \gamma \|g\|_G^2 \right), \gamma > 0. \tag{3.31}$$

Take (3.28) into account to get the equivalence between (3.31) and

$$\inf_{v\in U_{ad}} \left(J(v,0) - J(0,0) + \sup_{g\in G} \left(2\langle S(v), g\rangle_{G',G} - \gamma \|g\|_G^2 \right) \right),$$

thanks to the Legendre transform [7] for

$$\sup_{g\in G} \left(2\langle S(v), g\rangle_{G',G} - \gamma \|g\|_G^2 \right) = \frac{1}{\gamma} \|S(v)\|_G^2,$$

then,

$$\inf_{v\in U_{ad}} \mathscr{J}^\gamma(v), \tag{3.32}$$

where,

$$\mathscr{J}^\gamma(v) = J(v,g) - J(0,0) + \frac{1}{\gamma} \|S(v)\|_G^2. \tag{3.33}$$

The low-regret control existence and uniqueness

Proposition 3.4.2 The problem (3.21)-(3.32)-(3.33) has a unique solution u_γ.

Proof. It's clear that,

$$\mathscr{J}^\gamma(v) \geq -J(0,0), \forall v \in U_{ad},$$

then $d_\gamma = \inf_{v\in U_{ad}} \mathscr{J}^\gamma(v)$ exists.
Let (v_n^γ) be a minimizing sequence, such that, $d_\gamma = \lim_{n\longrightarrow\infty} \mathscr{J}^\gamma(v_n^\gamma)$, we have

$$-J(0,0) \leq \mathscr{J}^\gamma(v_n^\gamma) = J(v_n^\gamma, 0) - J(0,0) + \frac{1}{\gamma} \|S(v_n^\gamma)\|_G^2 \leq d_\gamma + 1, \tag{3.34}$$

which gives the bounds

$$\begin{aligned} \left\|v_n^\gamma\right\|_U &\leq C_\gamma, \\ \frac{1}{\sqrt{\gamma}} \left\|S(v_n^\gamma)\right\|_G &\leq C_\gamma, \\ \left\|Cy(v_n^\gamma)\right\|_Z &\leq C_\gamma, \end{aligned} \tag{3.35}$$

where, C_γ is a constant independent of n. From (3.35) we deduce that, there exists $u \in U_{ad}$, such that,

$$v_n^\gamma \longrightarrow u \text{ weakly in } U_{ad}.$$

∎

[7] It is a mathematical operation which, schematically, transforms a function defined by its value at a point into a function defined by its tangent.

Theorem 3.4.3 The sequence of low-regret controls converges weakly in U_{ad} when $\gamma \longrightarrow 0$ to the unique no-regret control u solution to (3.21)-(3.22).

Proof. Let u_γ be the unique low-regret control solution to (3.21)-(3.32)-(3.33). Then,

$$J(u_\gamma,0)-J(0,0)+\frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq J(v,0)-J(0,0)+\frac{1}{\gamma}\|S(v)\|_G^2, \forall v\in U_{ad},$$

take $v=0$ to get

$$J(u_\gamma,0)-J(0,0)+\frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq J(0,0) = \text{constant}.$$

Remember the definition of $J(v,g)$ in (3.22) to find

$$\left\|Cy(u_\gamma,0)-y_d\right\|_Z^2+\beta\left\|u_\gamma\right\|_U+\frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq C, \tag{3.36}$$

where C is a constant independent of γ. From (3.36) we deduce that (u_γ) is bounded in U_{ad}, then, there exists a subsequence still be denoted (u_γ) converges weakly to $u\in U_{ad}$. Let us prove that u is the unique no-regret control solution to (3.21)-(3.22) as follows

$$J(u,g)-J(u,0)-\gamma\|g\|_G^2 \leq J(v,g)-J(0,g),\ \forall (v,g)\in U_{ad}\times G,$$

then,

$$J(u,g)-J(u,0)-\gamma\|g\|_G^2 \leq \sup_{g\in G}\left(J(v,g)-J(0,g)\right),$$

pass to limit $n\longrightarrow 0$ to get

$$J(u,g)-J(u,0) \leq \sup_{g\in G}\left(J(v,g)-J(0,g)\right),$$

which means that, u is a no-regret control. ■

Optimality system of low-regret control

In the next proposition, we give an optimality system characterizing low-regret control u_γ.

Proposition 3.4.4 The low-regret control u_γ, solution to (3.21)-(3.32)-(3.33) is characterized by the next optimality system

$$\begin{cases} Ay_\gamma &= f+Bu_\gamma, \\ A^*\xi_\gamma &= C^*C(y_\gamma - y(0,0)), \\ A\rho_\gamma &= \dfrac{1}{\gamma}NN^*\xi_\gamma, \\ A^*p_\gamma &= C^*(Cy-y_d)+C^*C\rho_\gamma, \end{cases} \tag{3.37}$$

with the next optimality condition

$$(B^*p_\gamma+\beta u_\gamma, v-u_\gamma)) \geq 0,\ \forall v\in U_{ad}.$$

Proof. Let u_γ be solution to (3.21)-(3.32) and (3.33). A first order necessary condition gives for every $v\in U_{ad}$

$$\begin{aligned} \left(\mathscr{J}^{\gamma\prime}(u_\gamma), v-u_\gamma\right)_U &= (C^*(Cy(u_\gamma,0)-y_d), y(v-u_\gamma,0)-y(0,0))_Z \\ &+ \beta(u_\gamma, v-u_\gamma)_U+\frac{1}{\gamma}S(u_\gamma, S(v-u_\gamma))_G \geq 0. \end{aligned} \tag{3.38}$$

Denote $y_\gamma = y(u_\gamma,0)$, $\xi_\gamma(v) = NS(v)$, by definition we have

$$A^*\xi_\gamma = C^*C(y_\gamma - y(0,0)).$$

Also, let ρ_γ be the solution of

$$A\rho_\gamma = \frac{1}{\gamma}NN^*\xi_\gamma.$$

Now, introduce the adjoint state $p_\gamma = p(u_\gamma,0)$ defined by

$$A^*p_\gamma = C^*(Cy_\gamma - y_d) + C^*C\rho_\gamma,$$

then,

$$\begin{aligned}
\frac{1}{\gamma}(S(u_\gamma),S(v-u_\gamma)_G &= \frac{1}{\gamma}(N^*\xi_\gamma(u_\gamma),N^*\xi(v-u_\gamma))_G \\
&= \frac{1}{\gamma}(NN^*\xi_\gamma(u_\gamma),\xi(v-u_\gamma))_{V'} \\
&= (A\rho_\gamma,\xi(v-u_\gamma))_{V'} \\
&= (\rho_\gamma,A^*\xi(v-u_\gamma))_{V'} \\
&= (\rho_\gamma,C^*C(y(v-u_\gamma,0)-y(0,0)))_{V'} \\
&= (C^*C\rho_\gamma,y(v-u_\gamma,0)-y(0,0))_{V'} \\
&= (A^*p_\gamma - C^*(Cy_\gamma - y_d),y(v-u_\gamma,0)-y(0,0))_{V'} \\
&= (p_\gamma,A(y(v-u_\gamma,0)-y(0,0)))_{V'} \\
&- (C^*(Cy_\gamma - y_d),y(v-u_\gamma,0)-y(0,0))_{V'}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{1}{\gamma}(S(u_\gamma),S(v-u_\gamma)_G &= (p_\gamma,B(v-u_\gamma)-y(0,0)))_{V'} \\
&- (C^*(Cy_\gamma - y_d),y(v-u_\gamma,0)-y(0,0))_{V'},
\end{aligned}$$

Hence, we conclude that, the optimality condition (3.38) is equivalent to

$$\forall v \in U_{ad},\ \ (B^*p_\gamma + \beta u_\gamma, v-u_\gamma)_U \geq 0.$$

■

Optimality system of no-regret control

Let us introduce

P : orthogonal projection operator of F on G,

then, $v \in K$ if and only if

$$PN^*\xi(v) = 0, \tag{3.39}$$

Finding a no-regret control u is equivalent to

$\inf J(0,g)$, v subject to $(PN^*\xi(v)) = 0)$.

Approach by a penalty argument and define

$$J_\varepsilon(v) = J(v,0) + \frac{1}{\varepsilon}\|PN^*\xi(v)\|_F^2,\ \varepsilon > 0, \tag{3.40}$$

and consider the next problem

$$\inf_{v \in U_{ad}} J_\varepsilon(v), \tag{3.41}$$

this problem has a unique solution u_ε, such that,

$$u_\varepsilon \longrightarrow u \text{ in } U_{ad}.$$

Set

$$y(u_\varepsilon) = y_\varepsilon,\ \xi(u_\varepsilon) = \xi_\varepsilon,\ \lambda_\varepsilon = \frac{1}{\varepsilon} PN^* \xi_\varepsilon.$$

The control u_ε is characterized by

$$(Cy_\varepsilon - y_d)_Z + \beta(u_\varepsilon, v - u_\varepsilon)_U = (\lambda_\varepsilon, PN^*\xi(v - u_\varepsilon))_F \geq 0,\ \forall v \in U_{ad}. \tag{3.42}$$

Likewise,

$$Ay_\varepsilon = Bu_\varepsilon,\ A^*\xi_\varepsilon = C^*Cy_\varepsilon,$$

Introduce

$$A^* p_\varepsilon = C^*(Cy_\varepsilon - y_d) + C^*C\rho_\varepsilon,\ A\rho_\varepsilon = N\lambda_\varepsilon.$$

Then,

$$\begin{array}{rl} & (A^* p_\varepsilon, y(v - u_\varepsilon, 0))_{V'} + (Ap_\varepsilon, \xi(v - u_\varepsilon))_{V'} \\ = & (Cy_\varepsilon - y_d, Cy(v - u_\varepsilon, 0))_Z + (C\rho_\varepsilon, Cy(v - u_\varepsilon, 0))_Z + (N\lambda_\varepsilon, \xi(v - u_\varepsilon))_F \\ = & (p_\varepsilon, B(v - u_\varepsilon))_{V'} + (\rho_\varepsilon, A^*\xi(v - u_\varepsilon))_{V'} \\ = & (p_\varepsilon, B(v - u_\varepsilon))_{V'} + (C\rho_\varepsilon, Cy(v - u_\varepsilon, 0))_Z. \end{array}$$

Optimality condition (3.42) is reduced to

$$\forall v \in U_{ad},\ (B^* p_\varepsilon + \beta u_\varepsilon, v - u_\varepsilon)_U \geq 0. \tag{3.43}$$

A difficulty lies in obtaining a priori estimate on λ_ε. Introduce $\hat{p_\varepsilon},\ \sigma_\varepsilon$, such that,

$$\begin{array}{rcll} A^*\hat{p_\varepsilon} & = & C^*(Cy_\varepsilon - y_d), & p_\varepsilon \in V', \\ A^*\sigma_\varepsilon & = & C^*C\rho_\varepsilon, & \sigma_\varepsilon \in V'. \end{array}$$

Then, $p_\varepsilon = \hat{p_\varepsilon} + \sigma_\varepsilon$. Make $p_\varepsilon \longrightarrow \infty$, since $u_\varepsilon \longrightarrow u$ in U_{ad}.
We also know that, $y_\varepsilon \longrightarrow y, \xi_\varepsilon \longrightarrow \xi$ and $\hat{p_\varepsilon} \longrightarrow \hat{p}$ all in V', with

$$Ay = Bu,\ A^*\xi = C^*Cy,\ A^*\hat{p} = C^*(Cy - y_d).$$

Now, the optimality condition (3.43) is equivalent to

$$\forall v \in U_{ad},\ (B^*\hat{p_\varepsilon} + B^*\sigma_\varepsilon + \beta u_\varepsilon, v - u_\varepsilon)_U \geq 0.$$

When $\varepsilon \longrightarrow 0$ we get

$$\forall v \in U_{ad},\ (B^*\hat{p} + B^*\sigma + \beta u, v - u)_U \geq 0. \tag{3.44}$$

Consider

$$A\rho = Ng,\ A^*\sigma = C^*C\rho,$$

then, introduce

$$|||g||| = \|B^*\sigma\|_U \,. \tag{3.45}$$

The previous equality defines a semi-norm on $\mathring{G}$[8], also we construct the quotient space still denoted by $\mathring{G}$ associated to $g_1 \sim g_2$ if and only if $|||g_1||| = |||g_2|||$ and we define $\hat{G}$[9] with respect to the norm $|||.|||$ topology.
Then,

$$\lambda_\varepsilon \text{ remains in a bounded set of } \hat{G}. \tag{3.46}$$

Now, we can announce the next theorem characterizing no-regret control for (3.21)-(3.22)

Theorem 3.4.5 Suppose that (3.46) holds, then the no-regret control u solution to (3.21)-(3.22) is characterized by the next optimality system

$$\begin{cases} Ay &= f + Bu, \\ A^*\xi &= C^*C(y - y(0,0)), \\ A\rho &= N\lambda, \ \lambda \in \hat{G}. \\ A^* &= C^*(Cy - y_d) + C^*C\rho, \end{cases} \tag{3.47}$$

with the next optimality condition

$$(B^*p + \beta u, v - u)_U \geq 0, \ \forall v \in U_{ad}.$$

3.5 Average control

The average control is only one of the possibilities to deal with a more general problem of robust control for parameter-dependent systems. Zuazua in [**55**] analysed the problem of controlling systems submitted to parametrized perturbations, either infinite or finite dimensional ones, (PDEs or ODEs), depending on unknown parameters in a deterministic manner. he looked for a control, independent of the values of these parameters, that needs to be designed to perform well, in an average sense to be made precise, wherein he focused on the steady-state version and consider the averaged optimal in the context of a quadratic minimization problem for an elliptic equation depending on a parameter. Zuazua addresses new problems of average control. As we will see in the next example, the key is the identification of the corresponding adjoint state as the average of the parameter-dependent adjoint states.

Average optimal control of elliptic equation

Assume that, $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded domain, with smooth boundary, and ω is an open non-empty subset of Ω. Consider the next elliptic equation

$$\begin{cases} -div(a(x,\alpha)\nabla y) &= u(x)1_\omega \quad \text{in} \quad \Omega, \\ y &= 0 \qquad\quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.48}$$

The diffusivity coefficients $a(x,\alpha)$ are taken to be scalar, are assumed to be measurable in x, bounded above and below by a positive constant, and to depend on the uncertainty parameter $\alpha \in (0,1)$ in a measurable manner. Under these conditions, given $u \in L^2(\Omega)$, for each value of α

[8] The interior of the set G.
[9] $\hat{G}$ the completion of G on the quotient space.

there is a unique solution $y = y(x,\alpha) \in H_0^1(\Omega)$.
We are interested in the control of the average state

$$z(x) = \int_0^1 y(x,\alpha)d\alpha \in H_0^1(\Omega).$$

Given a target $z_d \in L^2(\Omega)$, consider the quadratic optimal control problem consisting on minimizing the next functional

$$J(u) = \frac{1}{2}\left[\|z - z_d\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\omega)}\right]. \tag{3.49}$$

By the Direct Method of the Calculus of Variations (DMCV) [**5**], it is easy to see that the minimizer of J in $L^2(\omega)$ exists and it is unique. This is due to the fact that J is continuous, convex and coercive in a Hilbert space. Uniqueness is due to strict convexity.
Let us denote by $u^* \in L^2(\omega)$ the minimizer of J. We now focus on the identification of u^* through an optimality system given by the next theorem

Theorem 3.5.1 [Zuazua] The unique optimal control u^* for the average optimal control problem consisting in the minimization of the functional J in (3.49) is given by

$$u^* = -\xi^* \text{ in } \omega, \tag{3.50}$$

where,

$$\xi^*(x) = \int_0^1 \varphi^*(x,\alpha)d\alpha,$$

and the adjoint system

$$\begin{cases} -div(a(x,\alpha)\nabla\varphi^*) &= z^* - z_d \quad \text{in} \quad \Omega, \\ \varphi^* &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.51}$$

(y^*, φ^*) being the unique solution of the optimality system

$$\begin{cases} -div(a(x,\alpha)\nabla y^*) = -\int_0^1 \varphi^*(x,\alpha)d\alpha 1_\omega, \\ -div(a(x,\alpha)\nabla\varphi^*) = \int_0^1 y^*(x,\alpha)d\alpha - z_d, \\ y^*|_{\partial\Omega} = \varphi^*|_{\partial\Omega} = 0. \end{cases} \tag{3.52}$$

Proof. The Euler-Lagrange equations characterizing the property that u^* minimizes J, using the Gateaux derivative of J at u^* in the direction v, leads to

$$\langle DJ(u^*), \delta u\rangle = \int_\Omega (z^* - z_d)\delta z dx + \int_\omega u^*\delta u dx = 0, \tag{3.53}$$

where, δz is the derivative of z with respect to u, in the direction of δu. It is characterized by the average

$$\delta z(x) = \int_0^1 \delta y(x,\alpha)d\alpha \tag{3.54}$$

and $\delta y(x,\alpha)$ is the derivative of the state $y(x,\alpha)$ with respect to u, which is the solution of the system

$$\begin{cases} -div(a(x,\alpha)\nabla(\delta y)) &= \delta u(x)1_\omega \quad \text{in} \quad \Omega, \\ \delta y &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.55}$$

Let us now introduce the α-dependent adjoint state $\varphi^*(x,\alpha)$ solution of

$$\begin{cases} -div(a(x,\alpha)\nabla\varphi^*) &= z^* - z_d \quad \text{in} \quad \Omega, \\ \varphi^* &= 0 \qquad\quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.56}$$

and the corresponding average

$$\xi^*(x) = \int_0^1 \varphi^*(x,\alpha)d\alpha. \tag{3.57}$$

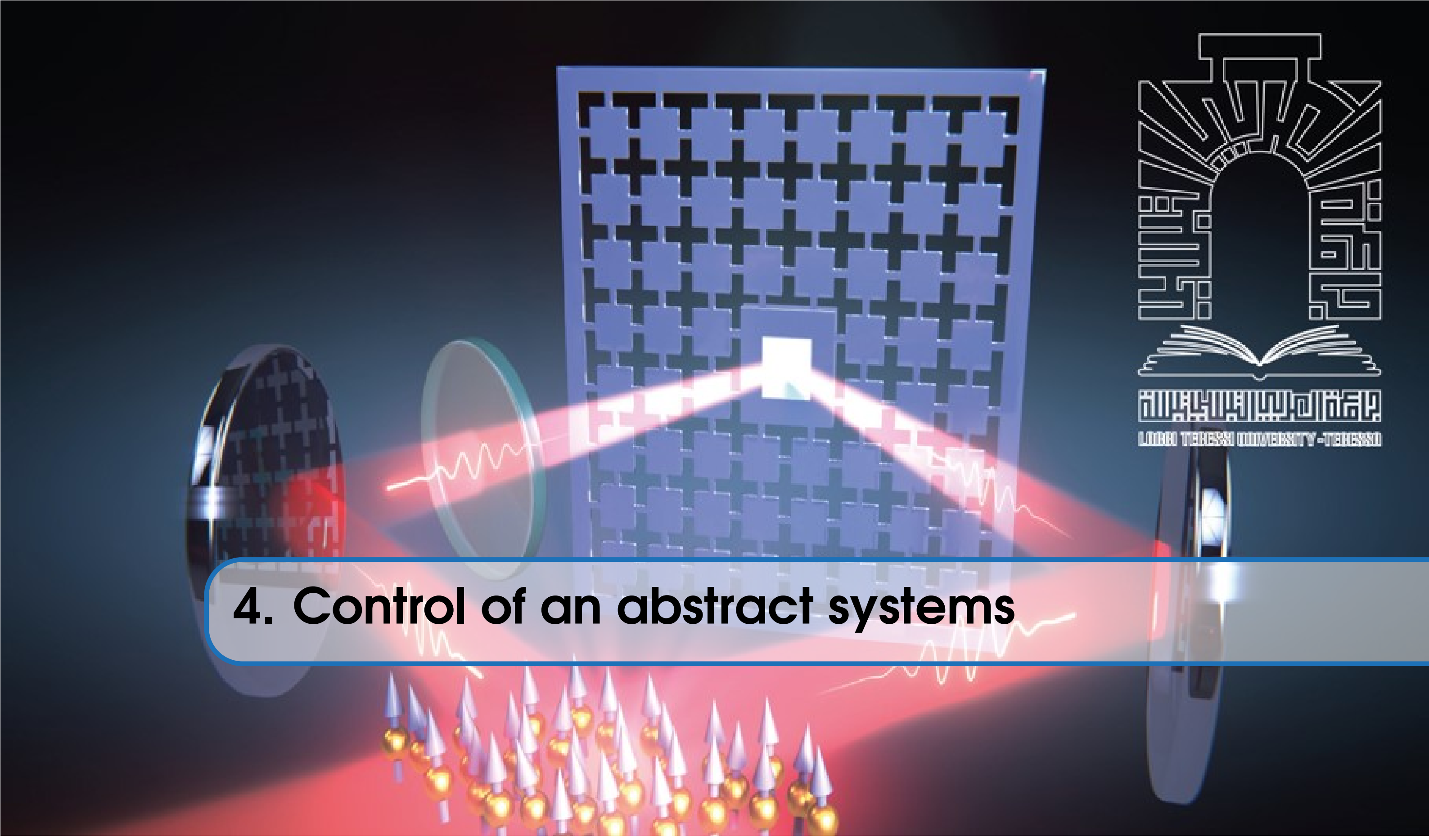
The key computation is the next

$$\begin{aligned} \int_\Omega (z^* - z_d)\delta z dx &= \int_\Omega (z^* - z_d)\int_0^1 \delta y d\alpha dx = \int_0^1 \int_\Omega (z^* - z_d)\delta y dx d\alpha \\ &= \int_0^1 \int_\Omega -div(a(x,\alpha)\nabla\varphi^*)\delta y dx d\alpha \\ &= \int_0^1 \int_\Omega -div(a(x,\alpha)\nabla(\delta y))\varphi^* dx d\alpha \\ &= \int_0^1 \int_\omega \delta u \varphi^* dx d\alpha + \int_\omega \delta u \xi^* dx. \end{aligned} \tag{3.58}$$

equation (3.53) then take to mean

$$\int_\omega \delta u \xi^* dx + \int_\omega \delta u \xi^* dx = 0, \tag{3.59}$$

which shows that the optimal control u^* is as in (3.50).
The optimal control is thus given by (3.50), where ξ^* is given by (3.57), and $(y^*,\ \varphi^*)$ are the optimal state and adjoint solutions of the optimality system (3.52). This ends the proof. ■



4. Control of an abstract systems

This chapter presents a base article talk about the optimal control of an abstract system with missing initial conditions, also the second section deals with new results of an optimal control of an abstract hyperbolic-parabolic coupled systems with missing initial conditions.

4.1 Optimal control of an abstract system

4.1.1 Problem statement

Consider the next abstract operator-differential equation

$$A(\alpha)y = B(\alpha)v + Ng, \tag{4.1}$$

where :

- $\alpha \in (0,1)$ is the unknown parameter.
- $A(\alpha) \in L(H)$ is a partial differential operator isomorphic on a real Hilbert space of functions H.
- $B(\alpha) \in L(U,H)$ is a control operator.
- U is a Hilbert space of controls.
- $N \in L(G,H)$ where G is a Hilbert space of missing data.
- $v \in U$ is the control function.
- g is a missing data in the Hilbert space G.

Moreover, we suppose that the scalar product defined on H realizes the next property

$$\int_0^1 (\xi(\alpha), \psi)_H \, d\alpha = \left(\int_0^1 \xi(\alpha) \, d\alpha, \psi \right)_H, \ \forall \xi, \ \psi \in H, \ \alpha \in (0,1).$$

Note that, many Hilbert spaces like $L^2(\Omega)$, $H^1(\Omega)$ and $L^2\left(0,T;H_0^1(\Omega)\right)$ verify the above property. Suppose that (4.1) holds in H and denote by $y(v,g,\alpha)$ her unique solution depending on the control v, the missing data g and depends continuously on α and the operators $A(\alpha)$ and $B(\alpha)$ depend on α continuously.

Associate to (4.1) the cost quadratic function of the form [**13**]

$$J(v,g) = \left\| \int_0^1 y(v,g,\alpha) \, d\alpha - y_d \right\|_H^2 + \beta \|v\|_U^2, \ \forall v \in U, \ \beta > 0, \tag{4.2}$$

where, y_d is an observation in H.
Note that, for all (v,g) in $U\times G$, $\int_0^1 y(v,g,\alpha)\,d\alpha$ has a sense, this comes from the continuity of $y(v,g,\alpha)$ with respect to the unknown datum α, also $\int_0^1 y(v,g,\alpha)\,d\alpha \in H$.

Now, let us redefine of average no-regret control.

Definition 4.1.1 We say that, $u\in U$ is an average no-regret control for (4.1)-(4.2) if u is a minimizer of the next problem

$$\inf_{v\in U}\left(\sup_{g\in G}\left(J(v,g)-J(0,g)\right)\right). \tag{4.3}$$

Corollary 4.1.1 For all $v\in U$ and $g\in G$, we have

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2\left(N^*\int_0^1\varphi(v,\alpha)\,d\alpha,g\right)_G, \tag{4.4}$$

where $\varphi(v,\alpha)$ is a solution for

$$A^*(\alpha)\,\varphi(v,\alpha)=\int_0^1 y(v,0,\alpha)\,d\alpha \text{ in } H',\ \forall\alpha\in(0,1). \tag{4.5}$$

Where, $A^*(\alpha)$ and N^* are the adjoint operators of $A(\alpha)$ and N respectively.

Proof. By linearity in (4.1) and a simple calculus, we get

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2\left(\int_0^1 y(v,0,\alpha)\,d\alpha,\int_0^1 y(0,g,\alpha)\,d\alpha\right)_H.$$

Then, from (4.5) we get

$$\begin{aligned} J(v,g)-J(0,g) &= J(v,0)-J(0,0)+2\textstyle\int_0^1\left(\int_0^1 y(v,0,\alpha)\,d\alpha, y(0,g,\alpha)\right)_H d\alpha \\ &= J(v,0)-J(0,0)+2\left(N^*\textstyle\int_0^1\varphi(v,\alpha)\,d\alpha,g\right)_G. \end{aligned}$$

■

Now, we define the sequence of average low-regret control by making some quadratic perturbation on the definition of average no-regret control, this sequence is expected to be convergent to the unique average no-regret control.

Definition 4.1.2 Let $\gamma>0$, we say that, $u_\gamma\in U$ is an average low-regret control for (4.1)-(4.2) if u_γ is a minimizer of

$$\inf_{v\in U}\left(\sup_{g\in G}\left(J(v,g)-J(0,g)-\gamma\|g\|_G^2\right)\right). \tag{4.6}$$

By using the relations (4.4)-(4.5), and the Legendre-Fenchel transform (see [**3**]), we can transform the problem (4.6) in an optimal control problem independent of an unknown parameter α and the missing data g, gives as follows

$$\inf_{v\in U} J_\gamma(v) : J_\gamma(v)=J(v,0)-J(0,0)+\frac{1}{\gamma}\left\|N^*\int_0^1\varphi(v,\alpha)\,d\alpha\right\|_G^2. \tag{4.7}$$

4.1.2 Average no-regret and average low-regret control

Existence, uniqueness, and characterization

Let us start this section by giving an existence and uniqueness results for the average low-regret control.

Proposition 4.1.2 The problem of optimal control (4.7) has a unique solution u_γ.

Proof. For all $v \in U$, $J_\gamma(v) \geq -J(0,0)$ then, J_γ is lower bounded. Assume that $(v_n) \subset U$ is a minimizing sequence, where,

$$J_\gamma(v_n) \underset{n\to\infty}{\longrightarrow} \inf_{v\in U} J_\gamma(v) = J_\gamma^*,$$

then, taking n large enough to obtain

$$J(v_n,0) - J(0,0) + \frac{1}{\gamma}\left\|N^* \int_0^1 \varphi(v,\alpha)\, d\alpha\right\|_G^2 \leq J_\gamma^* + 1,$$

the last inequality implies the next bounds

$$\|v_n\|_U \leq C_\gamma, \tag{4.8}$$

$$\left\|\int_0^1 y(v_n,0,\alpha)\, d\alpha\right\|_H \leq C_\gamma, \tag{4.9}$$

$$\left\|N^* \int_0^1 \varphi(v_n,\alpha)\, d\alpha\right\|_G \leq \sqrt{\gamma} C_\gamma, \tag{4.10}$$

where, for every n

$$A(\alpha)\, y(v_n,0,\alpha) = B(\alpha)\, v_n, \tag{4.11}$$

and C_γ is a positive constant independent of n.
From (4.8), we conclude that, there exists a subsequence still denoted (v_n), where,

$$v_n \rightharpoonup u_\gamma \text{ in } U,$$

the continuity of data gives that, $y(v_n,0,\alpha)$ is also bounded in H, then,

$$y(v_n,0,\alpha) \rightharpoonup y_\gamma \text{ in } H, \tag{4.12}$$

by passing to limit in (4.11) and uniqueness of limit we prove that, $y_\gamma = y\left(u_\gamma,0,\alpha\right)$. In view of (4.9)-(4.12) and by the theorem of Lebesgue dominated convergence, we obtain

$$\int_0^1 y(v_n,0,\alpha)\, d\alpha \rightharpoonup \int_0^1 y\left(u_\gamma,0,\alpha\right) d\alpha \text{ in } H.$$

Moreover, we have

$$A^*(\alpha)\, \varphi(v_n,\alpha) = \int_0^1 y(v_n,0,\alpha)\, d\alpha \rightharpoonup \int_0^1 y\left(u_\gamma,0,\alpha\right) d\alpha = A^*(\alpha)\, \varphi\left(u_\gamma,\alpha\right) \text{ in } H,$$

as $A^*(\alpha)$ is an isomorphism, we have also

$$\varphi(v_n,\alpha) \rightharpoonup \varphi\left(u_\gamma,\alpha\right) \text{ in } H.$$

In a manner similar to the convergence of $y(v_n,0,\alpha)$ we claim that

$$\int_0^1 \varphi(v_n,\alpha)\, d\alpha \rightharpoonup \int_0^1 \varphi\left(u_\gamma,\alpha\right) d\alpha \text{ in } H,$$

and according to the continuity of N^*, we deduce the next convergence

$$N^* \int_0^1 \varphi(v_n,\alpha)\,d\alpha \rightharpoonup N^* \int_0^1 \varphi(v,\alpha)\,d\alpha \text{ in } G.$$

Because the functional $J_\gamma(v)$ is weak lower semi-continuous and strictly convex, the uniqueness of u_γ comes easily. ■

Theorem 4.1.3 The unique average low-regret control u_γ minimizer of (4.7) is characterized by the next optimality system

$$\begin{cases} A(\alpha)\,y_\gamma &= B(\alpha)\,u_\gamma, \\ A^*(\alpha)\,\varphi_\gamma &= \int_0^1 y_\gamma d\alpha, \\ A(\alpha)\,\rho_\gamma &= \frac{1}{\gamma} N^* \int_0^1 \varphi_\gamma d\alpha, \\ A^*(\alpha)\,p_\gamma &= \int_0^1 (\rho_\gamma + y_\gamma)\,d\alpha - y_d, \end{cases} \tag{4.13}$$

with the next optimality condition

$$\int_0^1 B^*(\alpha)\,p_\gamma d\alpha + \alpha u_\gamma = 0 \quad \text{in } U,$$

where, $y_\gamma = y(u_\gamma,0,\alpha)$, $\varphi_\gamma = \varphi(u_\gamma,\alpha)$, $\rho_\gamma = \rho(\alpha)$, $p_\gamma = p(\alpha)$ and $\alpha \in (0,1)$.

Proof. We start by giving a sufficient first order optimality condition [**24**] for (4.7)

$$\begin{aligned} &\left(\int_0^1 y(u_\gamma,0,\alpha)\,d\alpha - y_d, \int_0^1 y(v,0,\alpha)\,d\alpha\right)_H + \frac{1}{\gamma}\left(N^* \int_0^1 \varphi(u_\gamma,\alpha)\,d\alpha, N^* \int_0^1 \varphi(v,\alpha)\,d\alpha\right)_G \\ &+ \beta\,(u_\gamma,v)_U \geq 0, \ \forall v \in U. \end{aligned} \tag{4.14}$$

Introduce a new state ρ_γ solution to

$$A(\alpha)\,\rho_\gamma = \frac{1}{\gamma} NN^* \int_0^1 \varphi(u_\gamma,\alpha)\,d\alpha,$$

then,

$$\begin{aligned} \frac{1}{\gamma}\left(N^* \int_0^1 \varphi(u_\gamma,\alpha)\,d\alpha, N^* \int_0^1 \varphi(v,\alpha)\,d\alpha\right)_G &= \int_0^1 \left(A(\alpha)\,\rho_\gamma, \varphi(v,\alpha)\right)_H d\alpha \\ &= \int_0^1 \left(\rho_\gamma, \int_0^1 y(v,0,\alpha)\,d\alpha\right)_H d\alpha \\ &= \left(\int_0^1 \rho_\gamma d\alpha, \int_0^1 y(v,0,\alpha)\,d\alpha\right)_H. \end{aligned}$$

Hence, the optimality condition (4.14) is equivalent to

$$\left(\int_0^1 (\rho_\gamma + y(u_\gamma,0,\alpha))\,d\alpha - y_d, \int_0^1 y(v,0,\alpha)\,d\alpha\right)_H + \alpha\,(u_\gamma,v)_U \geq 0, \ \forall v \in U.$$

Again, we construct an adjoint state $p_\gamma = p(u_\gamma,\alpha)$ solution to

$$A^*(\alpha)\,p(u_\gamma) = \int_0^1 (\rho_\gamma + y(u_\gamma,0,\alpha))\,d\alpha - y_d,$$

and (4.14) is equivalent to

$$\left(\int_0^1 B^*(\alpha)\, p\left(u_\gamma\right) d\alpha + \beta u_\gamma, v\right)_U \geq 0, \forall v \in U,$$

since U is a linear space we also have

$$\left(\int_0^1 B^*(\alpha)\, p\left(u_\gamma\right) d\alpha + \beta u_\gamma, v\right)_U \leq 0, \forall v \in U.$$

The optimality system (4.13) follows. ■

The next theorem proves that, the average low-regret control sequence converges to the average no-regret control.

Theorem 4.1.4 The sequence of average low-regret controls u_γ converges weakly when $\gamma \to 0$ to the unique averaged no-regret control u minimizer of (4.3).

Proof. As u_γ is an average low-regret control, so, we have $\forall v \in U$,

$$J\left(u_\gamma, 0\right) - J(0,0) + \frac{1}{\gamma}\left\|N^* \int_0^1 \varphi\left(u_\gamma, \alpha\right) d\alpha\right\|_G^2 \leq J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|N^* \int_0^1 \varphi(v,\alpha)\, d\alpha\right\|_G^2,$$

take $v = 0$ to find

$$\left\|\int_0^1 y\left(u_\gamma, 0, \alpha\right) d\alpha - y_d\right\|_H^2 + \beta\left\|u_\gamma\right\|_U^2 + \frac{1}{\gamma}\left\|N^* \int_0^1 \varphi\left(u_\gamma, \alpha\right) d\alpha\right\|_G^2 \leq J(0,0),$$

from which we deduce the next bounds

$$\left\|u_\gamma\right\|_U \leq C, \tag{4.15}$$

$$\left\|\int_0^1 y\left(u_\gamma, 0, \alpha\right) d\alpha\right\|_H \leq C, \tag{4.16}$$

$$\left\|N^* \int_0^1 \varphi\left(u_\gamma, \alpha\right) d\alpha\right\|_G^2 \leq C, \tag{4.17}$$

where, C is a positive constant independent of γ, then, by (4.15) we find that there exists a subsequence still denoted u_γ, where,

$$u_\gamma \rightharpoonup u \text{ in } U.$$

It is still to prove that, u is an average no-regret control other words prove that, u is a solution for (4.3). It is easy to show that,

$$J(v,g) - J(0,g) - \gamma\|g\|_G^2 \leq J(v,g) - J(0,g), \ \forall (g,v) \in G \times U,$$

then,

$$J\left(u_\gamma, g\right) - J(0,g) - \gamma\|g\|_G^2 \leq \sup_{g \in G}\left(J(v,g) - J(0,g)\right), \ \forall v \in U,$$

make $\gamma \to 0$ to get

$$J(u,g) - J(0,g) \leq \sup_{g \in G}\left(J(v,g) - J(0,g)\right), \ \forall v \in U,$$

in other words u is an average no-regret control. ■

Finally, we can provide a full characterization for the average no-regret control via an optimality system.

Theorem 4.1.5 The average no-regret control $u = \lim\limits_{\gamma \to 0} u_\gamma$ minimizer of (4.3) is characterized by the next optimality system

$$\begin{cases} A(\alpha) y & = & B(\alpha) u, \\ A^*(\alpha) \varphi & = & \int_0^1 y(u,0,\alpha) \, d\alpha, \\ A(\alpha) \rho & = & \lambda, \\ A^*(\alpha) p & = & \int_0^1 (\rho + y(u,0,\alpha)) \, d\alpha - y_d, \end{cases} \tag{4.18}$$

with the next optimality condition

$$\int_0^1 B^*(\alpha) p d\alpha + \beta u = 0 \text{ in } U,$$

where,

$$\begin{array}{rcl} u_\gamma & \xrightarrow{\gamma \to 0} & u \text{ in } U, \\ \frac{1}{\gamma} N N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha & \xrightarrow{\gamma \to 0} & \lambda \text{ in } H. \end{array}$$

Proof. First of all, we recall H' the dual of H. The Theorem 4.1.4, let us know that $u_\gamma \rightharpoonup u$ in U, then, as $B(\alpha)$ is bounded, we find,

$$B(\alpha) u_\gamma \rightharpoonup B(\alpha) u \text{ in } H.$$

Also, by continuity of $y_\gamma = y(u_\gamma, 0, \alpha)$ converges weakly to $y = y(u,0,\alpha)$ in H, and from the continuity of $A(\alpha)$ we deduce that

$$A(\alpha) y_\gamma \rightharpoonup A(\alpha) y \text{ in } H,$$

from the limit uniqueness, we deduce

$$A(\sigma) y = B(\alpha) u \text{ in } H.$$

By a similar way, also, the isomorphic property of $A^*(\alpha)$, drive us to prove that

$$A^*(\alpha) \varphi = \int_0^1 y(u,0,\alpha) \, d\alpha \text{ in } H,$$

By contradiction reasoning and from (4.17), for every $\gamma < 1$, we have

$$\frac{1}{\gamma} \left\| N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha \right\|_G^2 \leq C \Rightarrow \frac{1}{\gamma} \left\| N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha \right\|_G \leq C.$$

In other words we have $N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha$ is bounded in H, then, as N is bounded, $\frac{1}{\gamma} N N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha$ is also bounded and

$$\frac{1}{\gamma} N N^* \int_0^1 \varphi(u_\gamma, \alpha) \, d\alpha \rightharpoonup \lambda \text{ in } H.$$

Likewise, $A(\alpha) \rho_\gamma$ is bounded and by isomorphic property of $A(\alpha)$ show that ρ_γ is bounded also converges to ρ, then,

$$A(\alpha) \rho_\gamma \rightharpoonup A(\alpha) \rho \text{ in } H,$$

then,

$$A(\alpha)\rho = \lambda.$$

Moreover, the boundness of y_γ and ρ_γ shows the boundness of $A^*(\alpha)p_\gamma$, therefore, p_γ is bounded in H, and

$$A^*(\alpha)p = \int_0^1 (\rho + y(u,0,\alpha))\,d\alpha - z_d \text{ in } H.$$

After all, we pass to limit in the variational inequality of (4.13), we use weak convergences of u_γ, p_γ to u, p respectively. As final result, we obtain the next characterization

$$\int_0^1 B^*(\alpha)\,p d\alpha + \alpha u = 0 \text{ in } U.$$

■

4.2 Optimal control of an abstract coupled system

This section treats the average optimal control of an abstract hyperbolic-parabolic system depending on a coupled parameter with missing initial conditions. We introduce the concept of average no-regret control and its approach to get a general description from our average optimal control to the optimality system.

4.2.1 Statement of the problem

Consider the next hyperbolic-parabolic system with missing initial conditions, such that,

$$\begin{cases} y_{tt} + L_1 y + \alpha M\theta &= f + Bv, \\ \theta_t + L_2\theta + \alpha N y_t &= 0, \\ (y, y_t, \theta)(0) &= (y_0, y_1, \theta_0). \end{cases} \tag{4.19}$$

Here $t \in (0,T),\ T > 0$,

- U, H_1, H_2 and H_3 are a real separable Hilbert spaces.
- y and θ are functions with values in H_1 and H_2 respectively.
- L_1, L_2 are unbounded self adjoint positive definite operators acting in H_1 and H_2, with the domains D_{L_1} and D_{L_2} respectively.
- D_{L_1} and D_{L_2} are dense in H_1 and H_2 respectively.

Let us define on D_{L_1} and D_{L_2}, respectively, the next norms

$$\|.\|_{D_{L_1}} = \|L_1.\|_{H_1} \text{ and } \|.\|_{D_{L_2}} = \|L_2.\|_{H_2},$$

then, D_{L_1} and D_{L_2} become Banach spaces.

- Denote by $D_{L_1^{1/2}}$ and $D_{L_2^{1/2}}$[1] the domain of the operators $L_1^{1/2}$ and $L_2^{1/2}$ equipped with the next scalar product

$$\langle .,..\rangle_1 = \left(L_1^{1/2}.,L_1^{1/2}..\right)_1 \text{ and } \langle .,..\rangle_2 = \left(L_2^{1/2}.,L_2^{1/2}..\right)_2,$$

with the corresponding norms equipped with the next scalar product

$$\|.\|_{D_{L_1^{1/2}}} = \left\|L_1^{1/2}.\right\|_{H_1} \text{ and } \|.\|_{D_{L_2^{1/2}}} = \left\|L_2^{1/2}.\right\|_{H_2}.$$

[1] Separable Hilbert spaces.

where :

$$D_{L_1^{1/2}} \subset D_N \text{ and } D_{L_2^{1/2}} \subset D_M.$$

• M and N are linear operators acting, respectively, from D_M into H_1 and from D_N into H_2, where, D_M and D_N are linear sets, such that,

$$D_{L_2} \subset D_M, \ M \in \mathscr{L}(D_{L_2}, H_1), \text{ and } N \in \mathscr{L}(D_{L_1^{1/2}}, H_2)$$

• $B \in \mathscr{L}(U, H_1)$ is the control operator.
• $g = (y_0, y_1, \theta_0)$ are unknown initial conditions belonging at least to the spaces $H = D_{L_1^{1/2}} \times H_1 \times H_2$.
• v is a distributed control vector in $U_{ad} \subset U$ is a convex, non-empty and closed sub-set.
• $f \in L^2(0,T;H_1)$.
• α is a coupled parameter in $(0,1)$.

For all missing initial conditions g the problem (4.19) has a unique solution (previous chapter Section 3.3), for more details see [**22**], [**53**] and [**54**], given by the next couple,

$$(y,\theta) = (y(v,g,\alpha), \theta(v,g,\alpha)) \in W^{1,\infty}(0,T;H_1) \cap L^\infty\left(0,T;D_{L_1^{1/2}}\right) \times L^\infty(0,T;H_2) \cap L^2\left(0,T;D_{L_2^{1/2}}\right).$$

Moreover, we assume that, the scalar product defined on H_1 and H_2 realizes the next properties,

$$\forall \lambda, \phi \in H_i, i = \{1,2\}, \alpha \in (0,1), \int_0^1 (\lambda(\alpha), \phi)_{H_i} \, d\alpha = \left(\int_0^1 \lambda(\alpha) \, d\alpha, \phi\right)_{H_i}.$$

4.2.2 Average no-regret and average low-regret control

The control v acts only on the hyperbolic equation but, we do not introduce any control on the parabolic equation. To make it possible, we define the next quadratic cost function associated to the coupled system (4.19) (see [**14**]), for all $(v,g) \in H \times U_{ad}$,

$$J(v,g) = \left\| \int_0^1 y(v,g,\alpha) \, d\alpha - y_d \right\|^2_{L^2(0,T;H_1)} + \left\| \int_0^1 \theta(v,g,\alpha) \, d\alpha - \theta_d \right\|^2_{L^2(0,T;H_2)} + \beta \|v\|^2_U. \quad (4.20)$$

Such that :
• $\int_0^1 y(v,g,\alpha) \, d\alpha \in L^2(0,T;H_1)$ and $\int_0^1 \theta(v,g,\alpha) \, d\alpha \in L^2(0,T;H_2)$ present the average of observations.
• y_d, θ_d are given observations.
• β is a positive real number.

Average no-regret control

We focus on controlling the average of the state, for that reason, we are interested in the next optimal control problem with missing initial conditions [**30**]

$$\inf_{v \in U_{ad}} J(v,g), \ \forall g \in H. \quad (4.21)$$

We should resolve this min − max problem

$$\inf_{v \in U_{ad}} \sup_{g \in H} J(v,g).$$

Lions presented the notion of no-regret control (Chapter 1, Section 4), where we only look for control, such that,

$$J(v,g) \leq J(0,g), \forall g \in H. \quad (4.22)$$

We are going to study this problem by combining the notions of low-regret and no-regret control with the average control concept introduced by Zuazua (Chapter 1, Section 5).

Definition 4.2.1 We say that $u \in U_{ad}$ is the average no-regret control for the system (4.19) if u is the solution of

$$\inf_{v\in U_{ad}} \sup_{g\in H} \left(J(v,g) - J(0,g)\right). \tag{4.23}$$

We rewrite the fundamental quantity in no-regret control definition as

Lemma 4.2.1 We have $\forall (v,g) \in U \times H$,

$$\left| \begin{array}{l} J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\left(S(v), g\right)_H, \\ S(v) = \left(\int_0^1 \alpha N^* \psi(0)\, d\alpha - \int_0^1 \varphi_t(0)\, d\alpha, \int_0^1 \varphi(0)\, d\alpha, \int_0^1 \psi(0)\, d\alpha\right), \end{array}\right. \tag{4.24}$$

and the next adjoint-coupled system

$$\left\{ \begin{array}{rcl} \varphi_{tt} + L_1\varphi - \alpha N^*\psi_t &=& \int_0^1 \left[y(v,0,\alpha) - y(0,0,\alpha)\right] d\alpha, \\ -\psi_t + L_2\psi + \alpha M^*\varphi &=& \int_0^1 \left[\theta(v,0,\alpha) - \theta(0,0,\alpha)\right] d\alpha, \\ (\varphi, \varphi_t, \psi)(T) &=& (0,0,0). \end{array}\right. \tag{4.25}$$

Its solution gives by the next couple

$$(\varphi,\psi) = (\varphi(v,\alpha), \psi(v,\alpha)) \in W^{1,1}(0,T;H_1) \cap L^1\left(0,T; D_{L_1^{1/2}}\right) \times W^{1,1}(0,T;H_2) \cap L^2\left(0,T; D_{L_2^{1/2}}\right).$$

Proof. We compensate the next values of $y(v,g,\alpha)$ and $\theta(v,g,\alpha)$

$$\left| \begin{array}{rcl} y(v,g,\alpha) &=& y(v,0,\alpha) + y(0,g,\alpha) - y(0,0,\alpha), \\ \theta(v,g,\alpha) &=& \theta(v,0,\alpha) + \theta(0,g,\alpha) - \theta(0,0,\alpha) \end{array}\right.$$

in the equality (4.20), so, we get

$$\begin{array}{rcl} J(v,g) &=& \left\| \int_0^1 y(v,0,\alpha) d\alpha - y_d \right\|^2_{L^2(0,T;H_1)} + \left\| \int_0^1 \theta(v,0,\alpha) d\alpha - \theta_d \right\|^2_{L^2(0,T;H_2)} + \beta \|v\|_U^2 \\ &+& 2\left(\int_0^1 y(v,0,\alpha)\, d\alpha - y_d, \int_0^1 \left[y(0,g,\alpha) - y(0,0,\alpha)\right] d\alpha\right)_{L^2(0,T;H_1)} \\ &+& 2\left(\int_0^1 \theta(v,0,\alpha)\, d\alpha - \theta_d, \int_0^1 \left[\theta(0,g,\alpha) - \theta(0,0,\alpha)\right] d\alpha\right)_{L^2(0,T;H_2)}. \end{array}$$

Then, we add the next terms y_d, θ_d, $-y_d$, and $-\theta_d$ to the previous quantity, that drive us to obtain

$$\begin{array}{rcl} J(v,g) &=& J(v,0) + J(0,g) \\ &+& \left(\int_0^1 y(0,0,\alpha)\, d\alpha - y_d, \int_0^1 y(0,0,\alpha)\, d\alpha - y_d\right)_{L^2(0,T;H_1)} \\ &+& \left(\int_0^1 \theta(0,0,\alpha)\, d\alpha - \theta_d, \int_0^1 \theta(0,0,\alpha)\, d\alpha - \theta_d\right)_{L^2(0,T;H_2)} \\ &-& 2\left(\int_0^1 y(0,g,\alpha)\, d\alpha - y_d, \int_0^1 y(0,0,\alpha)\, d\alpha - y_d\right)_{L^2(0,T;H_1)} \\ &-& 2\left(\int_0^1 \theta(0,g,\alpha)\, d\alpha - \theta_d, \int_0^1 \theta(0,0,\alpha)\, d\alpha - \theta_d\right)_{L^2(0,T;H_2)} \\ &+& 2\left(\int_0^1 y(v,0,\alpha)\, d\alpha - y_d, \int_0^1 \left[y(0,g,\alpha) - y(0,0,\alpha)\right] d\alpha\right)_{L^2(0,T;H_1)} \\ &+& 2\left(\int_0^1 \theta(v,0,\alpha)\, d\alpha - \theta_d, \int_0^1 \left[\theta(0,g,\alpha) - \theta(0,0,\alpha)\right] d\alpha\right)_{L^2(0,T;H_2)} \end{array}$$

we add and subtract the next quantities :

$$2\left(\int_0^1 y(0,0,\alpha)\, d\alpha - y_d, \int_0^1 \left[y(0,g,\alpha) - y(0,0,\alpha)\right] d\alpha\right)_{L^2(0,T;H_1)},$$

and

$$2\left(\int_0^1 \theta(0,0,\alpha)\,d\alpha-\theta_d,\int_0^1[\theta(0,g,\alpha)-\theta(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_2)}.$$

Now, we simplify the calculation, hence, we get

$$\begin{array}{rcl} J(v,g) & - & J(0,0)=J(v,0)+J(0,g) \\ & + & 2\left(\int_0^1[y(v,0,\alpha)-y(0,0,\alpha)]\,d\alpha,\int_0^1[y(0,g,\alpha)-y(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_1)} \\ & + & 2\left(\int_0^1[\theta(v,0,\alpha)-\theta(0,0,\alpha)]\,d\alpha,\int_0^1[\theta(0,g,\alpha)-\theta(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_2)}. \end{array}$$

Now, we assume that,

$$\pi(\alpha,g)=y(0,g,\alpha)-y(0,0,\alpha)\ \text{ and }\ \eta(\alpha,g)=\theta(0,g,\alpha)-\theta(0,0,\alpha).$$

So, it is obvious that (π,η) is a solution for the next coupled system

$$\left\{\begin{array}{rcl} \pi_{tt}+L_1\pi+\alpha M\eta & = & 0, \\ \eta_t+L_2\eta+\alpha N\pi_t & = & 0, \\ (\pi,\pi_t,\eta)(0) & = & (y_0,y_1,\theta_0). \end{array}\right.$$

After broaching the adjoint coupled system given by (4.25) and helps of the integration by part, we arrive to this formula

$$\begin{array}{rl} & \left(\int_0^1[y(v,0,\alpha)-y(0,0,\alpha)]\,d\alpha,\int_0^1\pi(\alpha,g)\,d\alpha\right)_{L^2(0,T;H_1)} \\ = & \int_0^1(\varphi_{tt}+L_1\varphi-\alpha N^*\psi_t,\pi)_{L^2(0,T;H_1)}\,d\alpha \\ = & \int_0^1(\varphi,\pi_{tt})_{L^2(0,T;H_1)}\,d\alpha-\int_0^1(y_0,\varphi_t(0))_{H_1}\,d\alpha \\ + & \int_0^1(y_1,\varphi(0))_{H_1}\,d\alpha+\int_0^1(\varphi,L_1\pi)_{L^2(0,T;H_1)}\,d\alpha \\ + & \int_0^1(\psi,\alpha N\pi_t)_{L^2(0,T;H_2)}\,d\alpha+\int_0^1(y_0,\alpha N^*\psi(0))_{H_2}\,d\alpha. \end{array}$$

Similar way shows that

$$\begin{array}{rl} & \left(\int_0^1[\theta(0,g,\alpha)-\theta(0,0,\alpha)]\,d\alpha,\int_0^1\eta(\alpha,g)\,d\alpha\right)_{L^2(0,T;H_2)} \\ = & \int_0^1(-\psi_t+L_2\psi+\alpha M^*\varphi,\eta)_{L^2(0,T;H_2)}\,d\alpha \\ = & \int_0^T\int_0^1(\psi,\eta_t)_{H_2}\,d\alpha dt+\int_0^1(\theta_0,\psi(0))_{H_2}\,d\alpha \\ + & \int_0^1(\psi,L_2\eta)_{L^2(0,T;H_2)}\,d\alpha+\int_0^1(\varphi,\alpha M\eta)_{L^2(0,T;H_1)}\,d\alpha. \end{array}$$

The identity (4.24) follows easily by collecting two last equalities. ■

The previous lemma, drive us to write (4.23) as follows

$$\inf_{v\in U_{ad}}\left(J(v,0)-J(0,0)+2\sup_{g\in H}(S(v),g)_H\right).$$

Due to the linearity structure of the space H, we are in front of one of these next two cases

$$\sup_{g\in H}(S(v),g)_H=+\infty,$$

or

$$\sup_{g\in H}(S(v),g)_H=0. \tag{4.26}$$

Consequently, if (4.26) holds, the average no-regret control exists. In other hands if

$$v\in\{v\in U_{ad}:(S(v),g)_H=0,\ \forall g\in H\},$$

this set is not easy to characterize, so, we ought to try to relax the no-regret control by a quadratic approximate, which drives us to define the low-regret control as a sequence, that is expected to converge to the no-regret control.

Average low-regret control

We make some quadratic perturbations on our missing initial condition g, afterword, we take only the control v to relax the problem (4.23), for any $\gamma > 0$ we have

$$J(v,g) - J(0,g) \leq \gamma \|g\|_H^2, \forall g \in H.$$

Definition 4.2.2 We say that $u \in U_{ad}$ is an average low-regret control for (4.19) if u is the solution of

$$\inf_{v \in U_{ad}} \sup_{g \in H} \left(J(v,g) - J(0,g) - \gamma \|g\|_H^2 \right). \tag{4.27}$$

R We can rewrite (4.27) by (4.24) as follows

$$\inf_{v \in U_{ad}} \left(J(v,0) - J(0,0) + \sup_{g \in H} \left(2(S(v),g)_H - \gamma \|g\|_H^2 \right) \right),$$

by the Legendre transformation, we get,

$$\sup_{g \in H} \left(2(S(v),g)_H - \gamma \|g\|_H^2 \right) = \frac{1}{\gamma} \|S(v)\|_H^2.$$

Then, our problem is equivalent to the following standard optimal control problem

$$J_\gamma(u^\gamma) = \inf_{v \in U_{ad}} \mathscr{J}^\gamma(v), \; \mathscr{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} \|S(v)\|_H^2. \tag{4.28}$$

Proposition 4.2.2 There exists a unique average low-regret control solution to (4.28) denoted by $u^\gamma \in U$.

Proof. First of all, we have,

$$\forall v \in U_{ad}, \; \mathscr{J}^\gamma(v) \geq -J(0,0),$$

this means that (4.28) has a solution.
We assume that, $\left(v_\gamma^n\right) \subset U_{ad}$ is a minimizing sequence, so, it means that,

$$\liminf_{n \to +\infty} \mathscr{J}^\gamma\left(v_\gamma^n\right) = \mathscr{J}^\gamma\left(u_\gamma\right) = d_\gamma, \tag{4.29}$$

then,

$$\mathscr{J}^\gamma\left(v_\gamma^n\right) = J\left(v_\gamma^n, 0\right) - J(0,0) + \frac{1}{\gamma} \left\|S(v_\gamma^n)\right\|_H^2 \leq d_\gamma + 1,$$

where the couple $\left(y_\gamma^n, \theta_\gamma^n\right) = \left(y\left(t, v_\gamma^n, 0, \alpha\right), \theta\left(t, v_\gamma^n, 0, \alpha\right)\right)$ solves the next system

$$\left| \begin{array}{lcl} \left(y_\gamma^n\right)_{tt} + L_1 y_\gamma^n + \alpha M \theta_\gamma^n & = & f + B v_\gamma^n, \\ \left(\theta_\gamma^n\right)_t + L_2 \theta_\gamma^n + \alpha N \left(y_\gamma^n\right)_t & = & 0, \\ \left(y_\gamma^n, \left(y_\gamma^n\right)_t, \theta_\gamma^n\right)(0) & = & (0,0,0). \end{array} \right. \tag{4.30}$$

There exists $C_\gamma > 0$, such that :

$$J\left(v_\gamma^n, 0\right) \leq C_\gamma \quad \text{and} \quad \left\|S(v_\gamma^n)\right\|_H \leq \sqrt{\gamma} C_\gamma.$$

Hence, using the definition of $J\left(v_\gamma^n,0\right)$ to get the next bounds :

$$\begin{array}{lll}
\left\|v_\gamma^n\right\|_U & \leq & C_\gamma, \quad (4.30.a)\\
\left\|S(v_\gamma^n)\right\|_H & \leq & \sqrt{\gamma}C_\gamma, \quad (4.30.b)\\
\left\|\int_0^1 y_\gamma^n(\alpha)\,d\alpha\right\|_{L^2(0,T;H_1)} & \leq & C_\gamma, \quad (4.30.c)\\
\left\|\int_0^1 \theta_\gamma^n(\alpha)\,d\alpha\right\|_{L^2(0,T;H_2)} & \leq & C_\gamma. \quad (4.30.d)
\end{array}$$

In view of (4.30) and $(4.30.a)$, there exists $C_\gamma > 0$ independent of n, such that :

$$\left\|y_\gamma^n\right\|_{L^2(0,T;H_1)} \leq C_\gamma, \tag{4.31}$$

$$\left\|\theta_\gamma^n\right\|_{L^2(0,T;H_2)} \leq C_\gamma. \tag{4.32}$$

Hence, there exists

$$(y^\gamma, \theta^\gamma, u^\gamma) \in L^2(0,T;H_1) \times L^2(0,T;H_2) \times U$$

sub-sequences extracted from (v^n), (y^n) and (θ^n) [2], such that :

$$v_\gamma^n \rightharpoonup u^\gamma \text{ in } U, \tag{4.33}$$

$$y_\gamma^n \rightharpoonup y^\gamma \text{ in } L^2(0,T;H_1), \tag{4.34}$$

$$\theta_\gamma^n \rightharpoonup \theta^\gamma \text{ in } L^2(0,T;H_2). \tag{4.35}$$

For all $n \geq 0$, we assume that :

$$z_\gamma^n(t) = \int_0^1 y_\gamma^n(t,\alpha)d\alpha,$$

and

$$k_\gamma^n(t) = \int_0^1 \theta_\gamma^n(t,\alpha)d\alpha.$$

In view of $(4.30.c)$ and $(4.30.d)$ there exists sub-sequences denoted by $z_\gamma \in L^2(0,T;H_1)$, $k_\gamma \in L^2(0,T;H_2)$, such that, $z_\gamma^n \rightharpoonup z_\gamma$ in $L^2(0,T;H_1)$ and $k_\gamma^n \rightharpoonup k_\gamma$ in $L^2(0,T;H_2)$ thanks to the Riesz representation and dominated convergence theorems, there exists a unique φ and ψ in $L^2(0,T;H_1)$ and $L^2(0,T;H_2)$ respectively.

For all $\alpha \in (0,1)$, we have

$$\begin{array}{lll}
\left(z_\gamma^n,\varphi\right)_{L^2(0,T;H_1)} & = & \left(\int_0^1 y_\gamma^n(\alpha)\,d\alpha,\varphi\right)_{L^2(0,T;H_1)} = \int_0^1 \left(y_\gamma^n(\alpha),\varphi\right)_{L^2(0,T;H_1)} d\alpha \\
& \rightarrow & \int_0^1 \left(y_\gamma(\alpha),\varphi\right)_{L^2(0,T;H_1)} d\alpha = \left(\int_0^1 y_\gamma(\alpha)\,d\alpha,\varphi\right)_{L^2(0,T;H_1)},
\end{array} \tag{4.36}$$

[2] Still recalled by (v_γ^n), (y_γ^n) and (θ_γ^n).

and

$$\begin{array}{rcl}\left(k_\gamma^n,\psi\right)_{L^2(0,T;H_2)} & = & \left(\int_0^1 \theta_\gamma^n(\alpha)\,d\alpha,\psi\right)_{L^2(0,T;H_2)} = \int_0^1 \left(\theta_\gamma^n(\alpha),\psi\right)_{L^2(0,T;H_2)} d\alpha \\ & \to & \int_0^1 \left(\theta_\gamma(\alpha),\psi\right)_{L^2(0,T;H_2)} d\alpha = \left(\int_0^1 \theta_\gamma(\alpha)\,d\alpha,\psi\right)_{L^2(0,T;H_2)}.\end{array} \tag{4.37}$$

In view of the fact that y_γ^n and θ_γ^n are bounded independently of α, for all φ and ψ respectively in $L^2(0,T;H_1)$ and $L^2(0,T;H_2)$ we have

$$\left(y_\gamma^n,\varphi\right)_{L^2(0,T;H_1)} \to \left(y^\gamma,\varphi\right)_{L^2(0,T;H_1)} \text{ and } \left(\theta_\gamma^n,\psi\right)_{L^2(0,T;H_2)} \to \left(\theta^\gamma,\psi\right)_{L^2(0,T;H_2)}. \tag{4.38}$$

Hence, through (4.36), (4.37) and the limit uniqueness ensure that

$$\int_0^1 y^\gamma(t,\alpha)\,d\alpha = z^\gamma(t) \text{ and } \int_0^1 \theta^\gamma(t,\alpha)\,d\alpha = k^\gamma(t). \tag{4.39}$$

The rest of the proof will be divided into three steps.

Step 1. We prove that $(u^\gamma,y^\gamma,\theta^\gamma)$ satisfies (4.30). Let D_1 and D_2 be the sets of functions with a compact support in C^∞ on H_1 and H_2 respectively, with duals D_1',D_2'.
We multiply the system (4.30) by $(\varphi,\psi)\in D_1\times D_2$. Using the integration by part, we obtain

$$\begin{array}{l}\int_0^T \left(y_\gamma^n,\varphi_{tt}+L_1\varphi-\alpha N^*\psi_t\right)_{H_1} dt \;+\; \int_0^T \left(\theta_\gamma^n,-\psi_t+L_2\psi+\alpha M^*\varphi\right)_{H_2} dt \\ \qquad = \int_0^T \left(f+Bv_\gamma^n,\varphi\right)_{H_1} dt.\end{array}$$

Letting $n\to\infty$ and using (4.33) , (4.34) and (4.35) we obtain

$$\begin{array}{l}\int_0^T \left(y^\gamma,\varphi_{tt}+L_1\varphi-\alpha N^*\psi_t\right)_{H_1} dt \;+\; \int_0^T \left(\theta^\gamma,-\psi_t+L_2\psi+\alpha M^*\varphi\right)_{H_2} dt \\ \qquad = \int_0^T \left(f+Bu^\gamma,\varphi\right)_{H_1} dt.\end{array}$$

Hence, after the integration by part we deduce that,

$$\begin{array}{rcll}\left(y_\gamma^n\right)_{tt}+L_1y_\gamma^n+\alpha M\theta_\gamma^n & \to & y_{tt}^\gamma+L_1y^\gamma+\alpha M\theta^\gamma & \text{in } D_1', \\ f+Bv_\gamma^n & \to & f+Bu^\gamma & \text{in } D_1', \\ \left(\theta_\gamma^n\right)_t+L_2\theta_\gamma^n+\alpha N\left(y_\gamma^n\right)_t & \to & \theta_t^\gamma+L_2\theta^\gamma+\alpha Ny_t^\gamma & \text{in } D_2',\end{array}$$

The limit uniqueness gives us

$$\begin{array}{rclcll}\left(y_\gamma^n\right)_{tt}+L_1y_\gamma^n+\alpha M\theta_\gamma^n & \rightharpoonup & f_1=y_{tt}^\gamma+L_1y^\gamma+\alpha M\theta^\gamma & \in & L^2(0,T;H_1), & (a) \\ \left(\theta_\gamma^n\right)_t+L_2\theta_\gamma^n+\alpha N\left(y_\gamma^n\right)_t & \rightharpoonup & f_2=\theta_t^\gamma+L_2\theta^\gamma+\alpha Ny_t^\gamma, & \in & L^2(0,T;H_2), & (b) \\ f+Bv_\gamma^n & \rightharpoonup & f_3=f+Bu^\gamma & \in & L^2(0,T;H_1). & (c)\end{array}$$

Therefore,

$$\begin{array}{rcll}\left(y_\gamma^n\right)_{tt}+L_1y_\gamma^n+\alpha M\theta_\gamma^n & \rightharpoonup & y_{tt}^\gamma+L_1y^\gamma+\alpha M\theta^\gamma & \text{in } L^2(0,T;H_1), \\ \left(\theta_\gamma^n\right)_t+L_2\theta_\gamma^n+\alpha N\left(y_\gamma^n\right)_t & \rightharpoonup & \theta_t^\gamma+L_2\theta^\gamma+\alpha Ny_t^\gamma & \text{in } L^2(0,T;H_2), \\ f+Bv_\gamma^n & \rightharpoonup & f+Bu^\gamma & \text{in } L^2(0,T;H_1).\end{array}$$

We deduce that

$$\left|\begin{array}{rcl} y_{tt}^\gamma+L_1y^\gamma+\alpha M\theta^\gamma & = & f+Bu^\gamma, \\ \theta_t^\gamma+L_2\theta^\gamma+\alpha Ny_t^\gamma & = & 0\end{array}\right. \tag{d}$$

From (4.33) and (4.34), using (d) we have that $y_t^\gamma\in L^2(0,T;H_1)$ implies that
$y^\gamma\in L^2(0,T;H_1)$, also $L_1y^\gamma\in L^2(0,T;H_1)$ and $M\theta^\gamma\in L^2(0,T;H_2)$, this gives us that

$$(y(0),y_t(0),\theta(0))\in H.$$

We multiply the system (4.30) by $(\varphi,\psi)\in D_1\times D_2$, where $\varphi(T)=\varphi_t(T)=\psi(T)=0$. Using the integration by part, we obtain

$$\begin{array}{rcl}\left(f+Bv_\gamma^n,\varphi\right)_{L^2(0,T;H_1)} & = & -\ \left(y(0),\alpha N^*\psi(0)-\varphi_t(0)\right)_{H_1}\\ & & -\ \left(y_t(0),\varphi(0)\right)_{H_1}-\left(\theta(0),\psi(0)\right)_{H_2}\\ & & +\ \left(y_\gamma^n,\varphi_{tt}+L_1\varphi-\alpha N^*\psi_t\right)_{L^2(0,T;H_1)}\\ & & +\ \left(\theta_\gamma^n,-\psi_t+L_2\psi+\alpha M^*\varphi\right)_{L^2(0,T;H_2)},\end{array}$$

by passing to the limit when $n\longrightarrow\infty$, while using (4.33), (4.34) and (4.35) gives

$$\begin{array}{rcl}\left(f+Bu_\gamma,\varphi\right)_{L^2(0,T;H_1)} & = & -\ \left(y(0),\alpha N^*\psi(0)-\varphi_t(0)\right)_{H_1}\\ & & -\ \left(y_t(0),\varphi(0)\right)_{H_1}-\left(\theta(0),\psi(0)\right)_{H_2}\\ & & +\ \left(y_\gamma,\varphi_{tt}+L_1\varphi-\alpha N^*\psi_t\right)_{L^2(0,T;H_1)}\\ & & +\ \left(\theta_\gamma,-\psi_t+L_2\psi+\alpha M^*\varphi\right)_{L^2(0,T;H_2)},\\ \forall(\varphi,\psi)\in D_1\times D_2 & , & \varphi(T)=\varphi_t(T)=\psi(T)=0.\end{array}$$

Now, let us integrate by part the previous identity, we obtain

$$\begin{array}{rcl}\left(f+Bu_\gamma,\varphi\right)_{L^2(0,T;H_1)} & = & \left(y^\gamma(0),\alpha N^*\psi(0)-\varphi_t(0)\right)_{H_1}\\ & + & \left(y_t^\gamma(0),\varphi(0)\right)_{H_1}+\left(\theta^\gamma(0),\psi(0)\right)_{H_2}\\ & + & \left(y_{tt}^\gamma+L_1y^\gamma+\alpha M\theta_t^\gamma,\varphi\right)_{L^2(0,T;H_1)}\\ & + & \left(\theta_t^\gamma+L_2\theta^\gamma+\alpha Ny_t^\gamma,\psi\right)_{L^2(0,T;H_2)},\\ \forall(\varphi,\psi)\in D_1\times D_2 & , & \varphi(T)=\varphi_t(T)=\psi(T)=0.\end{array}$$

Now, we have

$$\begin{array}{rcl}\left(y^\gamma(0)-y(0),\alpha N^*\psi(0)-\varphi_t(0)\right)_{H_1} & = & 0,\\ \left(y_t^\gamma(0)-y_t(0),\varphi(0)\right)_{H_1} & = & 0,\\ \left(\theta^\gamma(0)-\theta(0),\psi(0)\right)_{H_2} & = & 0,\end{array}$$

which implies that,

$$y^\gamma(0)=y(0),\ y_t^\gamma(0)=y_t(0),\ \theta^\gamma(0)=\theta(0).$$

So, we deduce that, the solution of (d) is characterized by the next couple

$$(y^\gamma,\theta^\gamma)=\left(y(\alpha,u^\gamma,0),\theta(\alpha,u^\gamma,0)\right),$$

with initial conditions

$$y^\gamma(0)=0,\ y_t^\gamma(0)=0 \text{ and } \theta^\gamma(0)=0.$$

Step 2. It is clear that,

$$\left(\varphi(v^n,\alpha),\psi(v^n,\alpha)\right)\in W^{1,1}(0,T;H_1)\cap L^1\left(0,T;D_{L_1^{1/2}}\right)\times W^{1,1}(0,T;H_2)\cap L^2\left(0,T;D_{L_2^{1/2}}\right)$$

is a solution of the next adjoint coupled system

$$\left\{\begin{array}{rcl}\varphi_{tt}(v^n,\alpha)+L_1\varphi(v^n,\alpha)-\alpha N^*\psi_t(v^n,\alpha) & = & \int_0^1\left[y(v^n,0,\alpha)-y(0,0,\alpha)\right]d\alpha,\\ -\psi_t(v^n,\alpha)+L_2\psi(v^n,\alpha)+\alpha M^*\varphi(v^n,\alpha) & = & \int_0^1\left[\theta(v^n,0,\alpha)-\theta(0,0,\alpha)\right]d\alpha,\\ (\varphi,\varphi_t,\psi)(T) & = & (0,0,0).\end{array}\right. \tag{4.40}$$

So, we will prove that, this couple $\left(\varphi_\gamma, \psi_\gamma\right) = \left(\varphi\left(u^\gamma, \alpha\right), \psi\left(u^\gamma, \alpha\right)\right)$ verifies the next system

$$\left| \begin{array}{rcl} \varphi_{tt}^\gamma\left(u^\gamma,\alpha\right) + L_1\varphi^\gamma\left(u^\gamma,\alpha\right) - \alpha N^*\psi_t^\gamma\left(u^\gamma,\alpha\right) & = & \int_0^1 \left[y\left(u^\gamma,0,\alpha\right) - y\left(0,0,\alpha\right)\right] d\alpha, \\ -\psi_t^\gamma\left(u^\gamma,\alpha\right) + L_2\psi^\gamma\left(u^\gamma,\alpha\right) + \alpha M^*\varphi^\gamma\left(u^\gamma,\alpha\right) & = & \int_0^1 \left[\theta\left(u^\gamma,0,\alpha\right) - \theta\left(0,0,\alpha\right)\right] d\alpha, \\ \left(\varphi^\gamma, \varphi_t^\gamma, \psi^\gamma\right)(T) & = & (0,0,0). \end{array} \right. \tag{4.41}$$

Multiply (4.40) by $\left(\varphi_t\left(v^n,\alpha\right), \psi\left(v^n,\alpha\right)\right)$ and we integrate by part, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\left\|\varphi_t\right\|_{H_1}^2 + \left\|\left(L_1\right)^{\frac{1}{2}}\varphi\right\|_{H_1}^2 - \left\|\psi\right\|_{H_2}^2\right] &+ \left\|\left(L_2\right)^{\frac{1}{2}}\psi\right\|_{H_2}^2 - \alpha\left(N^*\psi_t,\varphi_t\right)_{H_1} + \alpha\left(M^*\varphi,\psi\right)_{H_2} \\ &= \int_0^1 \left(\pi_n d\alpha, \varphi_t\right)_{H_1} + \int_0^1 \left(\eta_n d\alpha, \psi\right)_{H_2}. \end{aligned}$$

The Gronwall inequality and some estimations give us

$$\begin{array}{lclclclclcl} \left\|\left(L_1\right)^{\frac{1}{2}}\varphi\right\|_{H_1} & \leq & C_\gamma & , & \left\|\varphi_t\right\|_{H_1} & \leq & C_\gamma & , & \left\|\psi_t\right\|_{L^2(0,T;H_2)} & \leq & C_\gamma, \\ \left\|\left(L_2\right)^{\frac{1}{2}}\psi\right\|_{L^2(0,T;H_2)} & \leq & C_\gamma & , & \left\|\varphi_t\right\|_{L^2(0,T;H_1)} & \leq & C_\gamma & , & \left\|\psi\right\|_{H_2} & \leq & C_\gamma. \end{array}$$

Then, there exists $\left(\varphi\left(v^n,\alpha\right), \psi\left(v^n,\alpha\right)\right)$ subsequences converge weakly to $\left(\varphi_\gamma, \psi_\gamma\right)$ respectively in $L^2\left(0,T;H_1\right)$ and $L^2\left(0,T;H_2\right)$. In view of this we obtain the next convergences weakly in $\left(D'\left(H_1\right), D'\left(H_2\right)\right)$ respectively,

$$\begin{array}{rcl} \varphi_{tt}\left(v^n,\alpha\right) + L_1\varphi\left(v^n,\alpha\right) - \alpha N^*\psi_t\left(v^n,\alpha\right) & \rightharpoonup & \varphi_{tt}\left(u^\gamma,\alpha\right) + L_1\varphi\left(u^\gamma,\alpha\right) - \alpha N^*\psi_t\left(u^\gamma,\alpha\right), \\ -\psi_t\left(v^n,\alpha\right) + L_2\psi\left(v^n,\alpha\right) + \alpha M^*\varphi\left(v^n,\alpha\right) & \rightharpoonup & -\psi_t\left(u^\gamma,\alpha\right) + L_2\psi\left(u^\gamma,\alpha\right) + \alpha M^*\varphi\left(u^\gamma,\alpha\right). \end{array}$$

From (a) and (b), we deduce that,

$$\begin{array}{rcl} \varphi_{tt}\left(v^n,\alpha\right) + L_1\varphi\left(v^n,\alpha\right) - \alpha N^*\psi_t\left(v^n,\alpha\right) & \rightharpoonup & f_1 \text{ in } L^2\left(0,T;H_1\right), \\ -\psi_t\left(v^n,\alpha\right) + L_2\psi\left(v^n,\alpha\right) + \alpha M^*\varphi\left(v^n,\alpha\right) & \rightharpoonup & f_2 \text{ in } L^2\left(0,T;H_2\right). \end{array}$$

The limit uniqueness property gives us

$$\begin{array}{rcl} \varphi_{tt}\left(u^\gamma,\alpha\right) + L_1\varphi\left(u^\gamma,\alpha\right) - \alpha N^*\psi_t\left(u^\gamma,\alpha\right) & = & f_1 \text{ in } L^2\left(0,T;H_1\right), \\ -\psi_t\left(u^\gamma,\alpha\right) + L_2\psi\left(u^\gamma,\alpha\right) + \alpha M^*\varphi\left(u^\gamma,\alpha\right) & = & f_2 \text{ in } L^2\left(0,T;H_2\right). \end{array}$$

Passing the limit in (4.40), using $(4.30.c)$ and $(4.30.d)$, we deduce that,

$$\left\{ \begin{array}{rcl} \varphi_{tt}\left(u^\gamma,\alpha\right) + L_1\varphi\left(u^\gamma,\alpha\right) - \alpha N^*\psi_t\left(u^\gamma,\alpha\right) & = & \int_0^1 \left[y\left(u^\gamma,0,\alpha\right) - y\left(0,0,\alpha\right)\right] d\alpha, \\ -\psi_t\left(u^\gamma,\alpha\right) + L_2\psi\left(u^\gamma,\alpha\right) + \alpha M^*\varphi\left(u^\gamma,\alpha\right) & = & \int_0^1 \left[\theta\left(u^\gamma,0,\alpha\right) - \theta\left(0,0,\alpha\right)\right] d\alpha. \end{array} \right.$$

The initials conditions follow by reasoning similar to the first step.

Step 3. Since $v \to \mathscr{J}^\gamma(v)$ is a lower semi-continuous function, we have

$$\mathscr{J}^\gamma\left(u^\gamma\right) = \lim_{n\to+\infty} \inf \mathscr{J}^\gamma\left(v_n\right).$$

According to (4.29), we have the fact that

$$\mathscr{J}^\gamma\left(u^\gamma\right) = \inf_{v\in U} \mathscr{J}^\gamma(v).$$

To finalize the proof, the uniqueness of u^γ follows because $\mathscr{J}^\gamma$ is strictly convex. ■

Proposition 4.2.3 The average low-regret control u^γ is characterized by the next coupled systems

$$\left\{\begin{array}{lll} y^\gamma_{tt}+L_1y^\gamma+\alpha M\theta^\gamma &=& f+Bu^\gamma, \\ \theta^\gamma_t+L_2\theta^\gamma+\alpha N y^\gamma_t &=& 0, \\ (y^\gamma,y^\gamma_t,\theta^\gamma)(0) &=& (0,0,0). \end{array}\right. \quad (4.42.a)$$

$$\left\{\begin{array}{lll} \varphi^\gamma_{tt}+L_1\varphi^\gamma-\alpha N^*\psi^\gamma_t &=& \int_0^1[y^\gamma(u^\gamma,0,\alpha)-y^\gamma(0,0,\alpha)]\,d\alpha, \\ -\psi^\gamma_t+L_2\psi^\gamma+\alpha M^*\varphi^\gamma &=& \int_0^1[\theta^\gamma(u^\gamma,0,\alpha)-\theta^\gamma(0,0,\alpha)]\,d\alpha, \\ (\varphi^\gamma,\varphi^\gamma_t,\psi^\gamma)(T) &=& (0,0,0). \end{array}\right. \quad (4.42.b)$$

$$\left\{\begin{array}{lll} \rho^\gamma_{tt}+L_1^\gamma\rho^\gamma+\alpha M\sigma^\gamma &=& 0, \\ \sigma^\gamma_t+L_2^\gamma\sigma^\gamma+\alpha N\rho^\gamma_t &=& 0, \\ \rho^\gamma(0) &=& \frac{1}{\gamma}\int_0^1\alpha N^*\psi^\gamma(0)\,d\alpha-\frac{1}{\gamma}\int_0^1\varphi^\gamma_t(0)\,d\alpha, \\ \rho^\gamma_t(0) &=& \frac{1}{\gamma}\int_0^1\varphi^\gamma(0)\,d\alpha, \\ \sigma^\gamma(x,0) &=& \dfrac{1}{\gamma}\int_0^1\psi^\gamma(0)\,d\alpha. \end{array}\right. \quad (4.42.c)$$

$$\left\{\begin{array}{lll} p^\gamma_{tt}+L_1p^\gamma-\alpha N^*q^\gamma_t &=& \int_0^1[y^\gamma(u^\gamma,0,\alpha)-y^\gamma(0,0,\alpha)]\,d\alpha-y_d+\int_0^1\rho^\gamma d\alpha, \\ -q^\gamma_t+L_2q^\gamma+\alpha M^*p^\gamma &=& \int_0^1[\theta^\gamma(u^\gamma,0,\alpha)-\theta^\gamma(0,0,\alpha)]\,d\alpha-\theta_d+\int_0^1\sigma^\gamma d\alpha, \\ (p^\gamma,p^\gamma_t,q^\gamma)(T) &=& (0,0,0). \end{array}\right. \quad (4.42.d)$$

With

$$\left(\int_0^1 B^*p^\gamma d\alpha+\beta u^\gamma, v-u^\gamma\right)_U \geq 0,\ \forall v\in U_{ad}.$$

Proof. A first-order Euler necessary condition for (4.36) gives us
For all $v\in U_{ad}$,

$$\begin{array}{l} \mathscr{J}^{\gamma\prime}(u^\gamma)(v-u^\gamma) \\ =\left(\int_0^1 y^\gamma(u^\gamma,0,\alpha)\,d\alpha-y_d,\int_0^1[y(v-u^\gamma,0,\alpha)-y(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_1)} \\ +\left(\int_0^1\theta^\gamma(u^\gamma,0,\alpha)\,d\alpha-\theta_d,\int_0^1[\theta(v-u^\gamma,0,\alpha)-\theta(0,0;\alpha)]\,d\alpha\right)_{L^2(0,T;H_2)}+\beta\,(u^\gamma,v-u^\gamma)_U \\ +\frac{1}{\gamma}\left(\int_0^1\alpha N^*\psi(0)\,d\alpha-\int_0^1\varphi_t(0)\,d\alpha,\int_0^1\alpha N^*\psi(v-u^\gamma)(0)\,d\alpha-\int_0^1\varphi_t(v-u^\gamma)(0)\,d\alpha\right)_{H_1} \\ +\frac{1}{\gamma}\left(\int_0^1\varphi(0)\,d\alpha,\int_0^1\varphi(v-u^\gamma)(0)\,d\alpha\right)_{H_1}+\frac{1}{\gamma}\left(\int_0^1\psi(0)\,d\alpha,\int_0^1\psi(v-u^\gamma)(0)\,d\alpha\right)_{H_2}\geq 0. \end{array}$$

We see that, $(\rho^\gamma,\sigma^\gamma)=(\rho(u^\gamma,0),\sigma(u^\gamma,0))$ is a solution of $(4.42.c)$.
An easy computation shows that :

$$\begin{array}{ll} & \left(\int_0^1\rho^\gamma d\alpha,\int_0^1[y(v-u^\gamma,0,\alpha)-y(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_1)} \\ = & \int_0^1(\rho^\gamma_{tt},\varphi(v-u^\gamma))_{L^2(0,T;H_1)}\,d\alpha+\int_0^1(L_1\rho^\gamma,\varphi(v-u^\gamma))_{L^2(0,T;H_1)}\,d\alpha \\ + & \int_0^1(\rho^\gamma_t,\alpha N^*\psi(v-u^\gamma))_{L^2(0,T;H_1)}\,d\alpha+\int_0^1(\rho^\gamma(0),\alpha N^*\psi(v-u^\gamma)(0))_{H_1}\,d\alpha \\ + & \int_0^1(\rho^\gamma_t(0),\varphi(v-u^\gamma)(0))_{H_1}\,d\alpha-\int_0^1(\rho^\gamma(0),\varphi_t(v-u^\gamma)(0))_{H_1}\,d\alpha, \end{array}$$

and

$$\begin{array}{ll} & \left(\int_0^1\sigma^\gamma d\alpha,\int_0^1[\theta(v-u^\gamma,0,\alpha)-\theta(0,0,\alpha)]\,d\alpha\right)_{L^2(0,T;H_2)} \\ = & \int_0^1(\sigma^\gamma_t,\psi(v-u^\gamma))_{L^2(0,T;H_2)}\,d\alpha+\int_0^1(L_2\sigma^\gamma,\psi(v-u^\gamma))_{L^2(0,T;H_2)}\,d\alpha \\ + & \int_0^1(\alpha M\sigma^\gamma,\varphi(v-u^\gamma))_{H_2}\,d\alpha+\int_0^1(\sigma^\gamma(0),\psi(v-u^\gamma)(0))_{H_2}\,d\alpha. \end{array}$$

After the summation of two previous equations, the results come easy.
We denote that (p^γ, q^γ) is a solution of $(4.42.d)$, which gives

$$\begin{array}{rl} & \left(\int_0^1 \pi^\gamma d\alpha - y_d + \int_0^1 \rho^\gamma d\alpha, \int_0^1 \left[y\left(v-u^\gamma,0,\alpha\right) - y\left(0,0,\alpha\right)\right] d\alpha\right)_{L^2(0,T;H_1)} \\ + & \left(\int_0^1 \eta^\gamma d\alpha - \theta_d + \int_0^1 \sigma^\gamma d\alpha, \int_0^1 \left[\theta\left(v-u^\gamma,0,\alpha\right) - \theta\left(0,0,\alpha\right)\right] d\alpha\right)_{L^2(0,T;H_2)} \\ = & \left(p_{tt}^\gamma + L_1 p^\gamma - \alpha N^* q_t^\gamma, \int_0^1 \left[y\left(v-u^\gamma,0,\alpha\right) - y\left(0,0,\alpha\right)\right] d\alpha\right)_{L^2(0,T;H_1)} \\ + & \left(-q_t^\gamma + L_2 q^\gamma + \alpha M^* p^\gamma, \int_0^1 \left[\theta\left(v-u^\gamma,0,\alpha\right) - \theta\left(0,0,\alpha\right)\right] d\alpha\right)_{L^2(0,T;H_2)} \\ = & \left(p^\gamma, \int_0^1 B\left(v-u^\gamma\right) d\alpha\right)_{L^2(0,T;H_1)} = \left(\int_0^1 B^* p^\gamma d\alpha, v-u^\gamma\right)_U . \end{array}$$

Which ends the proof. ■

Average no-regret control Characterization

For the no-regret control, we should get some weak convergence of

$$\{y^\gamma, \varphi^\gamma, \rho^\gamma, p^\gamma; \psi^\gamma, \theta^\gamma, \sigma^\gamma, q^\gamma; u^\gamma\},$$

to some limits that characterize the no-regret control. Hence, we declare these results :

Proposition 4.2.4 There exists $C > 0$ independent of γ, such that :

$$\begin{array}{lcl} \|u^\gamma\|_U & \leq & C, \\ \left\|\int_0^1 y^\gamma\left(u^\gamma,0,\alpha\right) d\alpha\right\|_{L^2(0,T;H_1)} & \leq & C, \\ \left\|\int_0^1 \theta^\gamma\left(u^\gamma,0,\alpha\right) d\alpha\right\|_{L^2(0,T;H_2)} & \leq & C, \end{array} \tag{4.43.a}$$

$$\left\|\int_0^1 \alpha N^* \psi^\gamma(0)\, d\alpha - \int_0^1 \varphi_t^\gamma(0)\, d\alpha\right\|_{H_1} \leq \sqrt{\gamma} C, \tag{4.43.b}$$

$$\begin{array}{lcl} \left\|\int_0^1 \varphi^\gamma(0)\, d\alpha\right\|_{H_1} & \leq & \sqrt{\gamma} C, \\ \left\|\int_0^1 \psi^\gamma(0)\, d\alpha\right\|_{H_2} & \leq & \sqrt{\gamma} C, \end{array} \tag{4.43.c}$$

$$\|y^\gamma\|_{L^\infty(0,T;H_1)} \leq C,\ \|y_t^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \tag{4.44.a}$$

$$\|\theta^\gamma\|_{L^\infty(0,T;H_2)} \leq C, \|\theta^\gamma\|_{L^2(0,T;H_2)} \leq C, \tag{4.44.b}$$

$$\|\varphi^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \|\varphi_t^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \tag{4.45.a}$$

$$\|\psi^\gamma\|_{L^\infty(0,T;H_2)} \leq C, \|\psi^\gamma\|_{L^2(0,T;H_2)} \leq C, \tag{4.45.b}$$

$$\|\rho^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \|\rho_t^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \tag{4.46.a}$$

$$\|\sigma^\gamma\|_{L^\infty(0,T;H_2)} \leq C, \|\sigma^\gamma\|_{L^2(0,T;H_2)} \leq C, \tag{4.46.b}$$

$$\|p^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \|p_t^\gamma\|_{L^\infty(0,T;H_1)} \leq C, \tag{4.47.a}$$

$$\|q^\gamma\|_{L^\infty(0,T;H_2)} \leq C, \|q^\gamma\|_{L^2(0,T;H_2)} \leq C, \tag{4.47.b}$$

Proof. The function u_γ is a low-regret control, then,

$$\mathscr{J}^\gamma(u^\gamma) \leq \mathscr{J}^\gamma(0),$$

it means that,

$$J(u^{\gamma},0)-J(0,0)+\frac{1}{\gamma}\|S(u^{\gamma})\|_{H}^{2}\leq\frac{1}{\gamma}\|S(0)\|_{H}^{2}=0,$$

we deduce $(4.43.a)$, by contradiction, we get $(4.43.b)$. Multiplying the system $(4.42.a)$ by $\left(y_t^{\gamma},\theta^{\gamma}\right)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|y_t^{\gamma}\|_{H_1}^{2}+\left\|L_1^{\frac{1}{2}}y^{\gamma}\right\|_{H_1}^{2}+\|\theta^{\gamma}\|_{L^2(H_2)}^{2}\right]+\left\|L_2^{\frac{1}{2}}\theta^{\gamma}\right\|_{H_2}^{2}+\left(\alpha M\theta,y_t^{\gamma}\right)_{L^2(0,T;H_1)}$$
$$+\left(\alpha N y_t^{\gamma},\theta\right)_{L^2(0,T;H_2)}=\left(f+Bu^{\gamma},y_t^{\gamma}\right)_{L^2(0,T;H_1)}.$$

Now, let us integrate over $(0,t)$, use (4.41), and apply the Gronwall lemma, we obtain

$$\begin{array}{llllll}\sup_{t\in(0,T)}\|y_t^{\gamma}\|_{H_1} & \leq & C & , & \sup_{t\in(0,T)}\left\|L_1^{\frac{1}{2}}y^{\gamma}\right\|_{H_1} & \leq C,\\ \sup_{t\in(0,T)}\|\theta^{\gamma}\|_{H_2} & \leq & C & , & \sup_{t\in(0,T)}\left\|L_2^{\frac{1}{2}}\theta^{\gamma}\right\|_{H_2} & \leq C,\end{array}$$

The estimations of φ^{γ} and ψ^{γ} follow the same manner.
The same reasoning gives the estimations in $(4.45.a)$ and $(4.45.b)$.
For estimating of ρ^{γ} and σ^{γ}, let us consider the energy for all $t>0$,

$$E(t)=\frac{1}{2}\left[\|\rho_t^{\gamma}\|_{H_1}^{2}+\left\|L_1^{\frac{1}{2}}\rho^{\gamma}\right\|_{H_1}^{2}+\|\sigma^{\gamma}\|_{H_2}^{2}\right],$$

which satisfies

$$\frac{d}{dt}E(t)=-\left\|L_2^{\frac{1}{2}}\sigma^{\gamma}\right\|_{H_2}^{2}\leq 0,$$

then,

$$E(t)\leq E(0),\ \forall t\in(0,T).$$

Take into consideration $(4.43.b)$ and $(4.43.c)$ to conclude $(4.46.a)$ and $(4.46.b)$ in view of :

$$\begin{array}{ll} & \int_0^1\pi^{\gamma}d\alpha-y_d+\int_0^1\rho^{\gamma}d\alpha\in L^2(0,T;H_1),\\ \text{and} & \\ & \int_0^1\eta^{\gamma}d\alpha-\theta_d+\int_0^1\sigma^{\gamma}d\alpha\in L^2(0,T;H_2).\end{array}$$

Finally, we conclude $(4.47.a)$ and $(4.47.b)$ by the same manner of $(4.44.a),(4.44.b),(4.45.a)$ and $(4.45.b)$. ■

Proposition 4.2.5 The low-regret control u^{γ} converges in U to the no-regret control u defined by (4.23).

Proof. From (4.28) we have

$$J_{\gamma}(u^{\gamma})\leq J_{\gamma}(0)=0.$$

In view of the expression of J_{γ} giving by (4.29), implies that,

$$\left\|\int_0^1 y^{\gamma}d\alpha-y_d\right\|_{L^2(0,T;H_1)}^{2}+\left\|\int_0^1\theta^{\gamma}d\alpha-\theta_d\right\|_{L^2(0,T;H_2)}^{2}+\beta\|u^{\gamma}\|_{U}^{2}+\frac{1}{\gamma}\|S(u^{\gamma})\|_{H}^{2}\leq J(0,0).$$

Hence, we deduce that,

$$\|u^\gamma\|_U \leq \frac{\sqrt{J(0,0)}}{\sqrt{\beta}}, \tag{4.48}$$

$$\|S(u^\gamma)\|_H \leq \sqrt{\gamma}\sqrt{J(0,0)}, \tag{4.49}$$

$$\begin{aligned} \left\|\int_0^1 y^\gamma(\alpha)\,d\alpha\right\|_{L^2(0,T;H_1)} &\leq \sqrt{J(0,0)} + \|y_d\|_{L^2(0,T;H_1)}, \\ \left\|\int_0^1 \theta^\gamma(\alpha)\,d\alpha\right\|_{L^2(0,T;H_2)} &\leq \sqrt{J(0,0)} + \|\theta_d\|_{L^2(0,T;H_1)}. \end{aligned} \tag{4.50}$$

In light of $(4.42.a)$ and (4.48), there exists a constant $C > 0$ independent of γ, such that,

$$\|y^\gamma\|_{L^2(0,T;H_1)} \leq C, \ \ \|\theta^\gamma\|_{L^2(0,T;H_2)} \leq C. \tag{4.51}$$

Thus,

$$\exists u \in U, \ u^\gamma \rightharpoonup u \text{ in } U, \tag{4.52}$$

$$\begin{aligned} &\exists y \in L^2(0,T;H_1), \ y^\gamma \rightharpoonup y \text{ in } L^2(0,T;H_1), \\ \text{and} \quad & \\ &\exists \theta \in L^2(0,T;H_2), \ \theta^\gamma \rightharpoonup \theta \text{ in } L^2(0,T;H_2). \end{aligned} \tag{4.53}$$

Combining (4.52), (4.53) and $(4.42.a)$ to show that,

$$(y,\theta) = (y(\alpha,u,0), \theta(\alpha,u,0))$$

satisfies

$$\begin{cases} y_{tt} + L_1 y + \alpha M\theta &= f + Bu, \\ \theta_t + L_2\theta + \alpha N y_t &= 0, \\ (y, y_t, \theta)(0) &= (y_0, y_1, \theta_0). \end{cases} \tag{4.54}$$

Moreover, using $(4.50))$, (4.53) and the Lobesgue dominated convergence theorem to obtain :

$$\begin{aligned} &\int_0^1 y(\alpha,u^\gamma,0)\,d\alpha \rightharpoonup \int_0^1 y(\alpha,u,0)\,d\alpha \text{ in } L^2(0,T;H_1), \\ \text{and} \quad & \\ &\int_0^1 \theta(\alpha,u^\gamma,0)\,d\alpha \rightharpoonup \int_0^1 \theta(\alpha,u,0)\,d\alpha \text{ in } L^2(0,T;H_2). \end{aligned} \tag{4.55}$$

From (4.50) and $(4.42.b)$, we prove that there exists $C > 0$, such that,

$$\|S(u^\gamma)\|_H \leq C. \tag{4.56}$$

From $(4.43.b)$ and $(4.43.c)$, we know that :

$$\begin{array}{rcll} \int_0^1 \alpha N^* \psi^\gamma(0)\,d\alpha - \int_0^1 \varphi_t^\gamma(0)\,d\alpha & \to & 0 & \text{in} \quad H_1, \\ \int_0^1 \varphi^\gamma(0)\,d\alpha & \to & 0 & \text{in} \quad H_1, \\ \int_0^1 \psi^\gamma(0)\,d\alpha & \to & 0 & \text{in} \quad H_2. \end{array}$$

So,

$$S(u^\gamma) \longrightarrow S(u) = 0 \text{ in } H.$$

We deduce that,

$$\forall g \in H, \ (S(u), g)_H = 0,$$

It means that, u is the no-regret control of (4.19)-(4.20). ■

Theorem 4.2.6 The no-regret control u, solution of (4.23) is characterized by the next optimality coupled systems :

$$\begin{cases} y_{tt}+L_1 y+\alpha M\theta &= f+Bu, \\ \theta_t+L_2\theta+\alpha N y_t &= 0, \\ (y,y_t,\theta)(0) &= (0,0,0). \end{cases} \tag{4.57.a}$$

$$\begin{cases} \varphi_{tt}+L_1\varphi-\alpha N^*\psi_t &= \int_0^1 [y(u,0,\alpha)-y(0,0,\alpha)]\,d\alpha, \\ -\psi_t+L_2\psi+\alpha M^*\varphi &= \int_0^1 [\theta(u,0,\alpha)-\theta(0,0,\alpha)]\,d\alpha, \\ (\varphi,\varphi_t,\psi)(T) &= (0,0,0). \end{cases} \tag{4.57.b}$$

$$\begin{cases} \rho_{tt}+L_1\rho+\alpha M\sigma &= 0, \\ \sigma_t+L_2\sigma+\alpha N\rho_t &= 0, \\ (\rho,\rho_t,\sigma)(0) &= (\rho_0,\rho_1,\sigma_0). \end{cases} \tag{4.57.c}$$

$$\begin{cases} p_{tt}+L_1 p-\alpha N^* q_t &= \int_0^1 [y(u,0,\alpha)-y(0,0,\alpha)]\,d\alpha-y_d+\int_0^1 \rho\,d\alpha, \\ -q_t+L_2 q+\alpha M^* p &= \int_0^1 [\theta(u,0,\alpha)-\theta(0,0,\alpha)]\,d\alpha-\theta_d+\int_0^1 \sigma\,d\alpha, \\ (p,p_t,q)(T) &= (0,0,0). \end{cases} \tag{4.57.d}$$

With

$$\left(\int_0^1 B^* p\,d\alpha+\beta u, v-u\right)_U \geq 0,\ \forall v\in U_{ad},$$

and the next weak limits :

$$\begin{array}{llll} y=\lim_{\gamma\to 0} y^\gamma, & \theta=\lim_{\gamma\to 0}\theta^\gamma, & \varphi=\lim_{\gamma\to 0}\varphi^\gamma, & \psi=\lim_{\gamma\to 0}\psi^\gamma, \\ \rho=\lim_{\gamma\to 0}\rho^\gamma, & \sigma=\lim_{\gamma\to 0}\sigma^\gamma, & p=\lim_{\gamma\to 0} p^\gamma, & q=\lim_{\gamma\to 0} q^\gamma, \end{array}$$

$$\rho_0=\lim_{\gamma\to 0}\frac{1}{\gamma}\left(\int_0^1 \alpha N^*\psi^\gamma(0)\,d\alpha-\int_0^1\varphi_t^\gamma(0)\,d\alpha\right)\in H_1,$$
$$\rho_1=\lim_{\gamma\to 0}\frac{1}{\gamma}\int_0^1\varphi^\gamma(0)\,d\alpha\in H_1,$$
$$\sigma_0=\lim_{\gamma\to 0}\frac{1}{\gamma}\int_0^1\psi^\gamma(0)\,d\alpha\in H_2.$$

Proof. In the proof of Proposition 4.2.5, we have already proved that when $\gamma\to 0$, the sequence (u^γ) converges weakly to the no-regret control u in U, and we have obtained the coupled system $(4.57.a)$ given by the states y and θ.
The coupled systems are supported in the same way by the cases $\{\varphi,\rho,p;\psi,\sigma,q\}$ and by using of the estimates referred to in $(4.44.a)-(4.46.b)$.
Additionally, we deduce from $(4.43.b)$ and $(4.43.c)$ that :

$$\begin{array}{rcll} \frac{1}{\gamma}\left(\int_0^1\alpha N^*\psi^\gamma(0)\,d\alpha-\int_0^1\varphi_t^\gamma(0)\,d\alpha\right) & \rightharpoonup & \rho_0 & \text{in } H_1, \\ \frac{1}{\gamma}\int_0^1\varphi_t^\gamma(0)\,d\alpha & \rightharpoonup & \rho_1 & \text{in } H_1, \\ \frac{1}{\gamma}\int_0^1\psi^\gamma(0)\,d\alpha & \rightharpoonup & \sigma_0 & \text{in } H_2. \end{array}$$

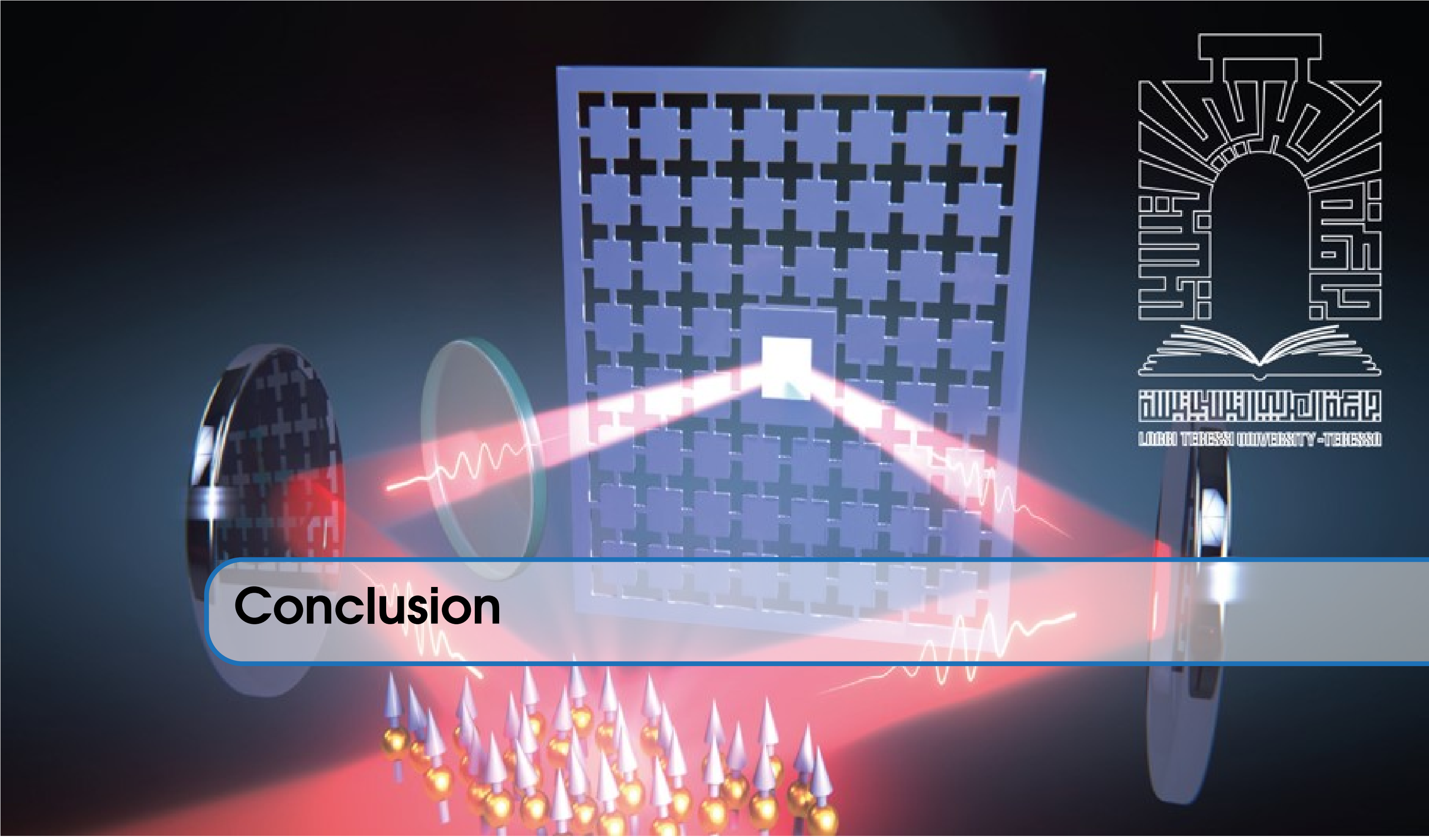
The weak convergence of p^γ to p in H_1 gives us the weak convergence of B^*p^γ to B^*p in U, thanks to (4.52) we get

$$\int_0^1 B^* p^\gamma d\alpha+\beta u_\gamma \rightharpoonup \int_0^1 B^* p\,d\alpha+\beta u \text{ in } U.$$

So,

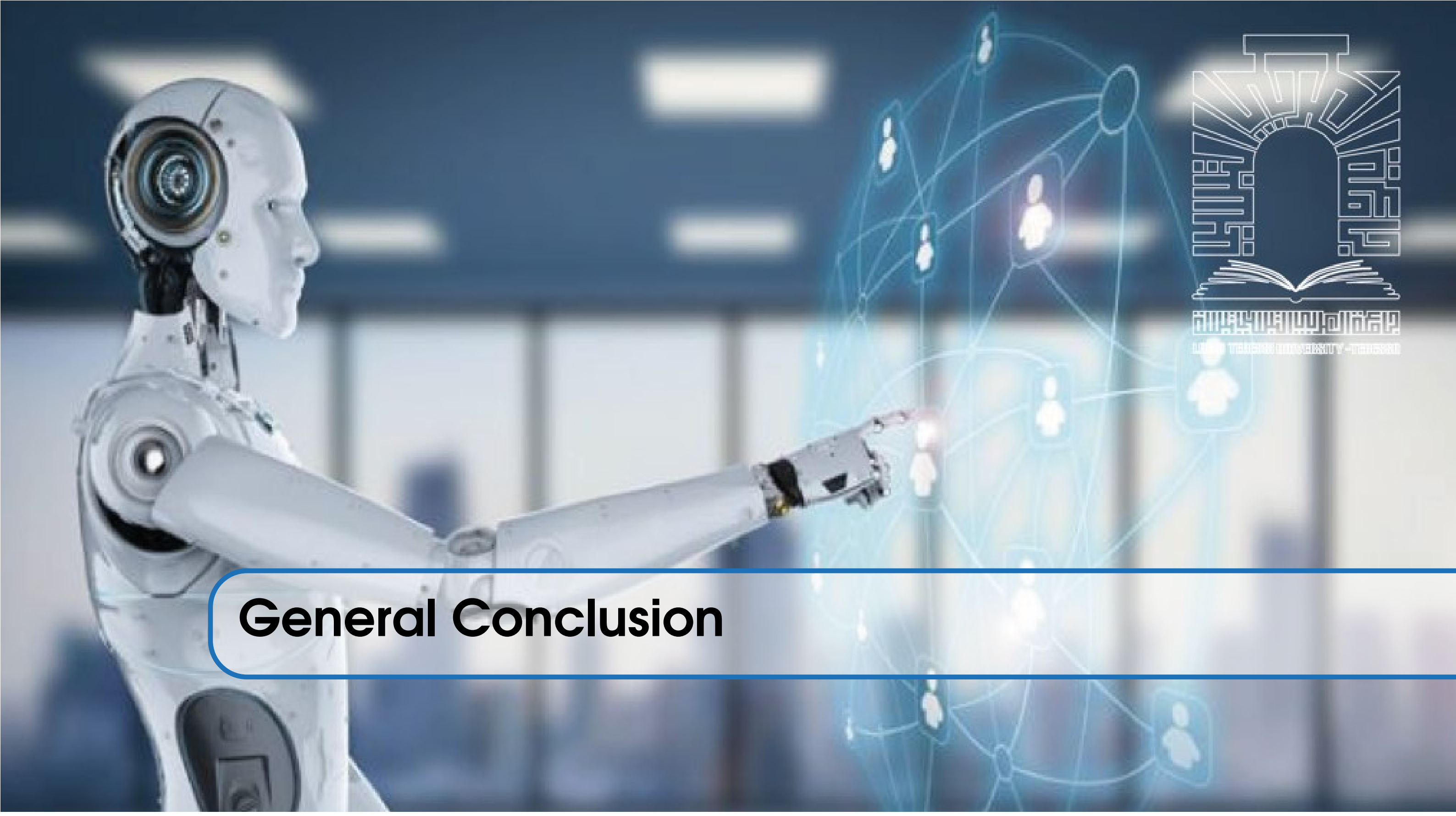
$$\left(\int_0^1 B^* p d\alpha + \beta u, v - u\right)_U \geq 0, \ \forall v \in U_{ad}.$$

■



Conclusion

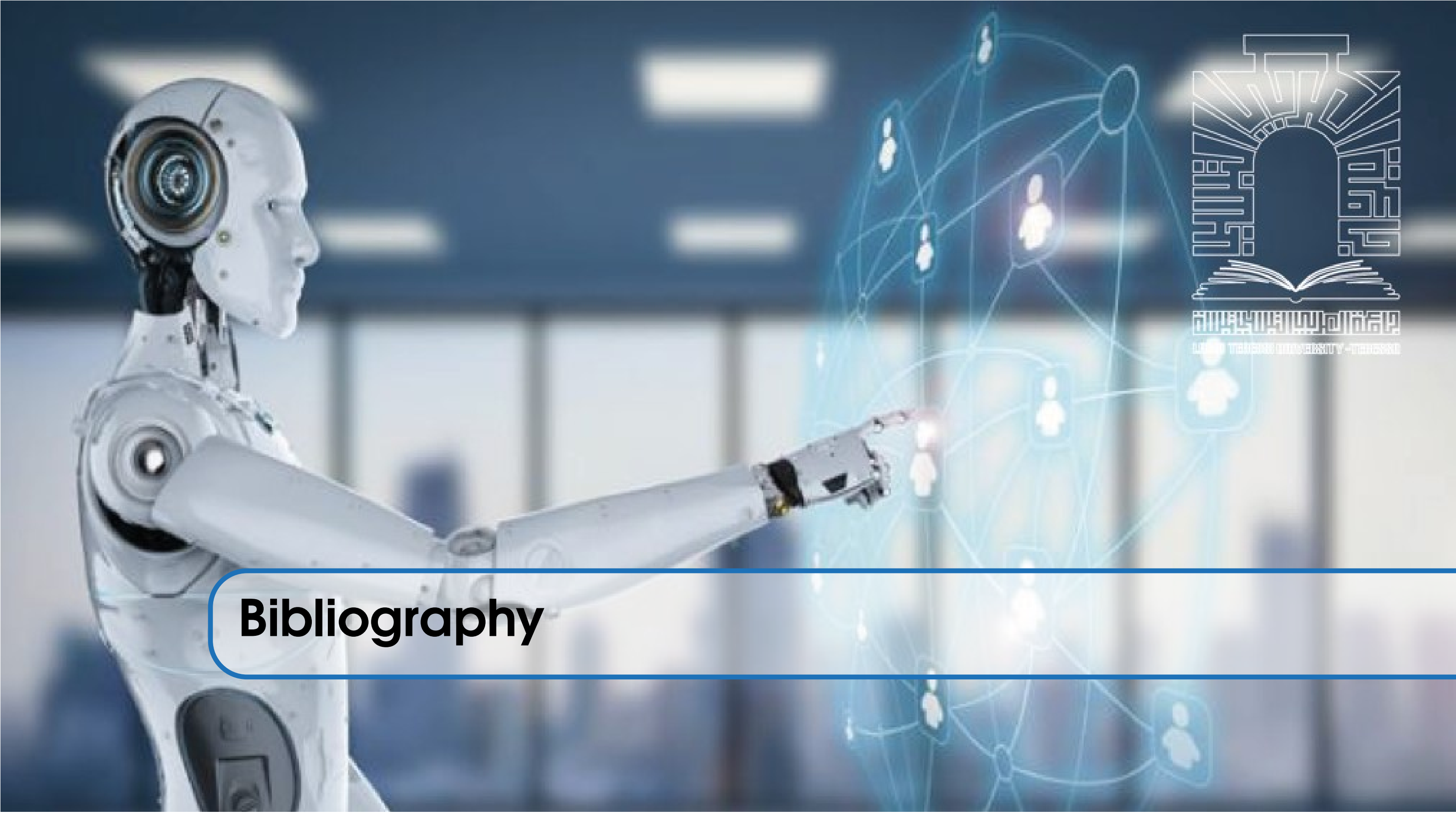
Concluding this part by important information which is the optimal control is a method to improve the solution of the different systems by giving an optimal solution implicitly in an optimality condition. Also, this part has shown that the average no-regret control could be worked to study optimal control problems for two problems first one is optimal control to general systems depending on an uncertainty parameter, the second one is optimal control of an abstract hyperbolic-parabolic system depending on a coupled parameter.



General Conclusion

This thesis aimed to identify in the first part of it dealt with the concept of PMP, which can be helped to study the optimal control of the non-linear Odes, that principle gives us a first-order optimally condition, resulting in two boundary value problems, afterword, we got trajectories that checked these conditions. At this level, the shooting method helped us to find the shooting functions zero ultimately, with Matlab software we described and discussed our results. The fun continued with the second part, where we applied the optimal control with a new dress which is the average notion on a linear distributed system with missing initial conditions, we can not arrive at the result which is an optimality system with an optimality condition without crossing with the no-regret control, because the characterization of it is so difficult we relaxed our no regret control problem by a quadratic perturbation and it became a low-regret control problem, as last moves, we had to play a little with the weak topology and embedding of spaces to find the characterization of the no-regret control problem (optimality systems with optimality condition).

Last but not least, in light of all the precedent results, my mind drove me to propose some lookouts for the advancement of future research works, so, here are some rich perspectives, in general, the precedent results lead us to think about double control or more at the same time act on the non-linear Odes and the linear PDEs, maybe it will be interesting results on the different abstract coupled systems, make it strong results by using the concept of the average control on both axes (ODEs/PDEs, linear/non-linear), further research must be made, as well, the numerical simulations s of optimality systems of PDEs. The development of the scientific need also prompts us to try to generalize the definition of the average control of Zuazua to the whole control theory, to ease the control problem solution. Without forgetting the optimal control of the reel life problems, robots, neural network, automate, fluid mechanics,...



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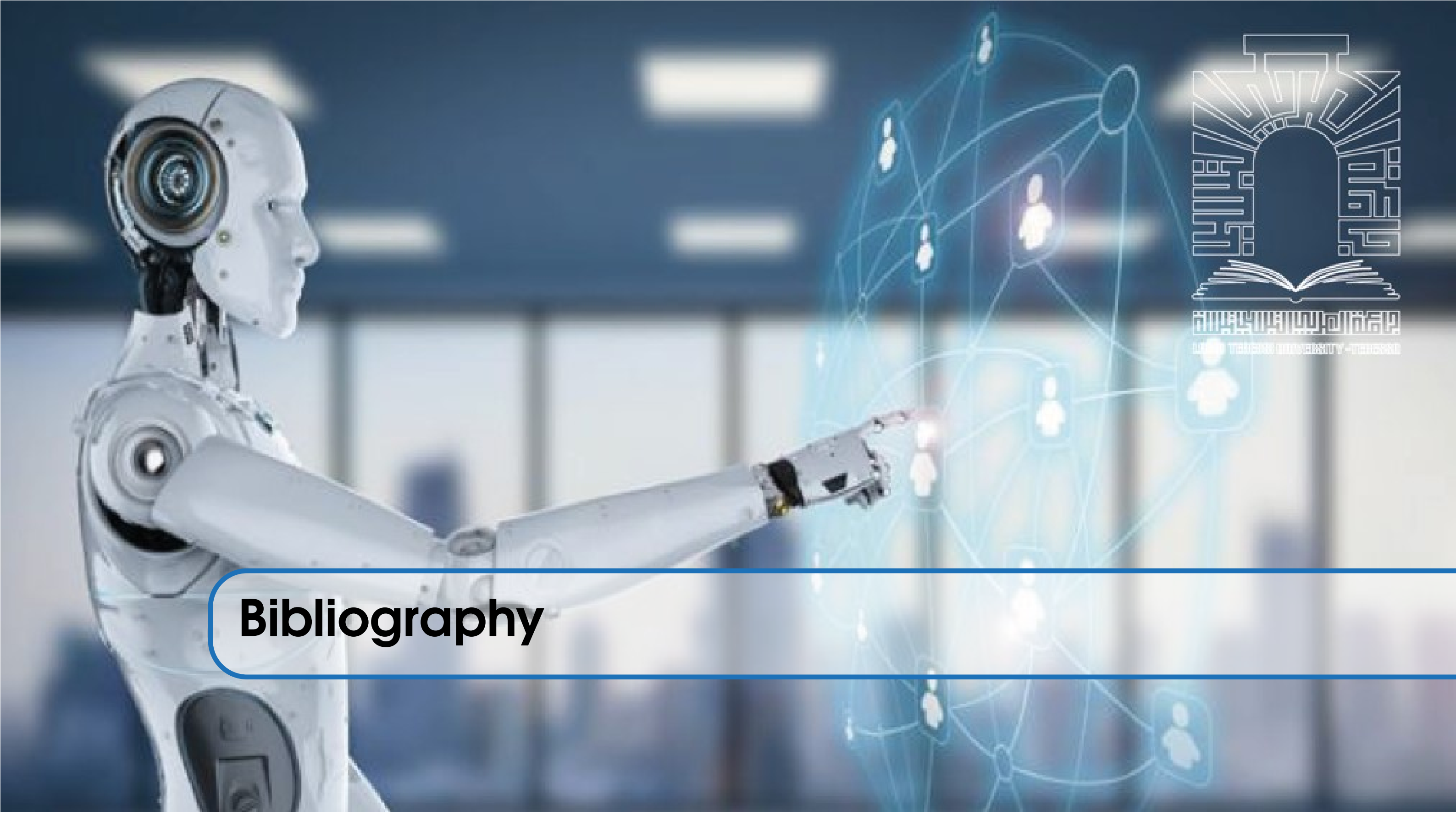

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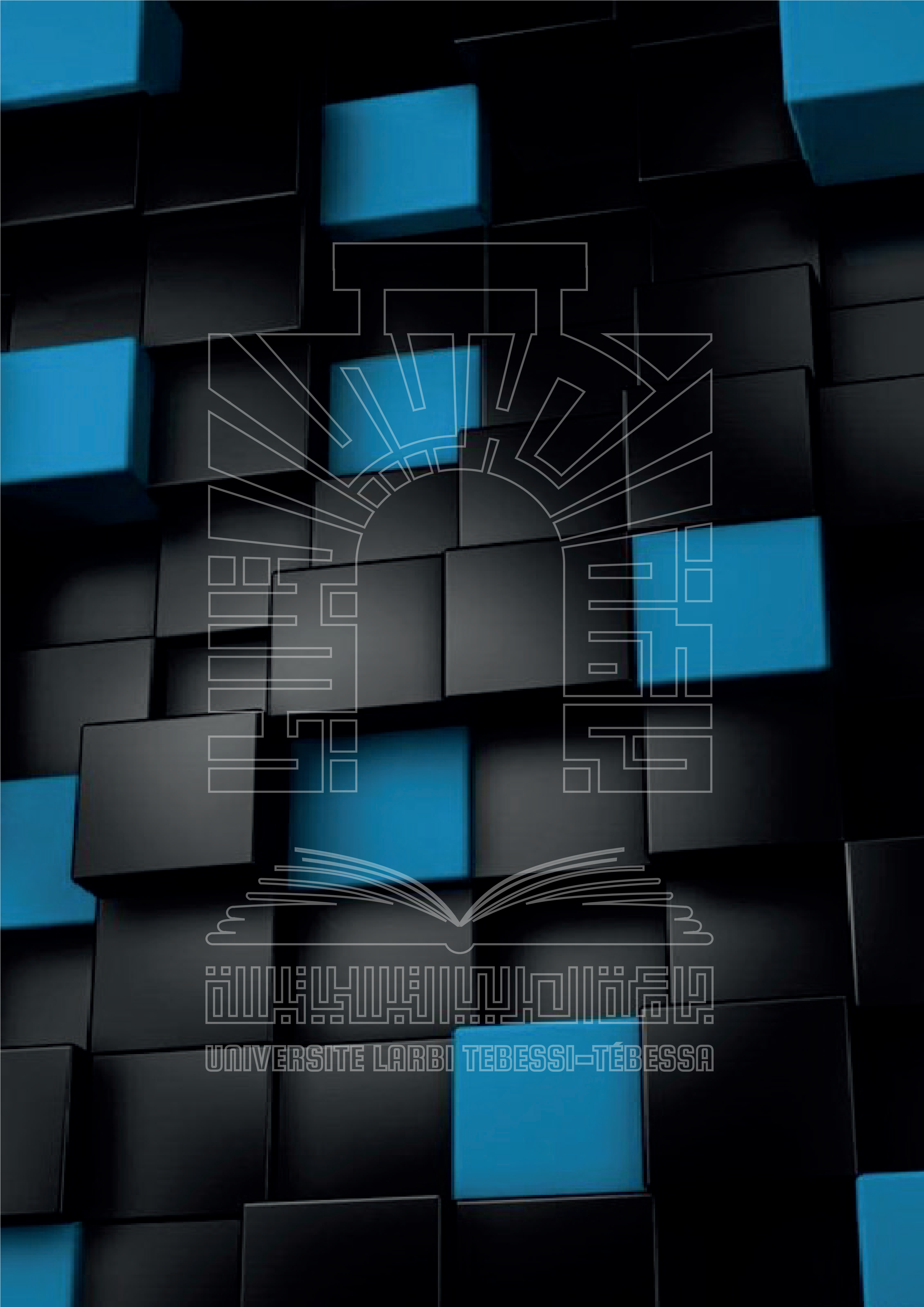
His domains of expertise in Applied Mathematics include Partial Differential Equations, Control Theory and Numerical Analysis.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