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Fractional-integer order chaotic systems synchronization

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بسم الله الرحمن الرحيم

شكـر و عرفــان

نحمد الله عز وجل الذي وفقنا في اتمام هذا البحث العلمي والذي ألهمنا الصحه والعافيه والعزيمه، فالحمد لله حمدا كثيرا،

نتقدم بجزيل الشكر والتقدير الى الاستاذ الدكتور المشرف حناشي فارح على كل ما قدمه لنا من توجيهات ومعلومات قيمه ساهمت في اطراء موضوع دراستنا في جوانبها المختلفه كما نتقدم بجزيل الشكر الى اعضاء لجنه مناقشه الموقره دون نسيان مديري ومعلمي ومتعلمي التعليم الثانوي ومديريه التربيه والتعليم لولايه تبسة ،

ولا ننسى تقديم الشكر الجزيل لكل الاساتذه المحترمين والاستاذات بجامعة الشيخ العربي التبسي تبسة .

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الى كل من وقف معنا و ساندنا من قريب او من بعيد نقول شكرا دمتم في حياتنا ....

الى كل من خاول تثبيطنا .... زادتنا محاولاتكم الفاشلة تحفيزا و اصرار على الوصول ....

الى اصحاب الدعوات البريئة لنا بالنجاح .... وفقكم الله و زادكم علما و أدبا ......

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حديفة و حمزة

Abstract

In this work, we consider the problem of the synchronization between the fractional-order chaotic system and the chaotic system of integer order. Based on suitable controllers and the stability theory of linear integer order systems, the synchronization between the fractional-order chaotic system and the chaotic system of integer order is achieved. Finally, The corresponding simulation results are provided to demonstrate the effectiveness of the proposed method in Matlab.

Keywords: Dynamical system, chaos, strange attractor, chaotic system Integer-order system, Fractional-order system, Controllers, Synchronization

Resumé

Dans ce travail, nous considérons le problème de la synchronisation entre le système chaotique d'ordre fractionnaire et le système chaotique d'ordre entier.
Sur la base de contrôleurs appropriés et de la théorie de la stabilité des systèmes linéaires d'ordre entier, la synchronisation entre le système chaotique d'ordre fractionnaire et le système chaotique d'ordre entier est réalisée.
Enfin, les résultats de simulation correspondants sont fournis pour démontrer l'efficacité de la méthode proposée dans Matlab.
Mots clés : Système dynamique, chaos, attracteur étrange, système chaotique, Système d'ordre entier, Système d'ordre fractionnaire, Contrôleurs, Synchronisation.

الـمـلـخص

في هذا العمل، تطرقنا إلى مشكلة مزامنة الأنظمة الحركية الفوضويةٌ ذات الرتب الكسرية و الصحيحة، من خلال إيجاد قانون تحكم معين وتطبيق نظرية استقرار خاصة بالأنظمة الحركية الخطية ذات الرتب الصحيحة.
في الأخير، تم التحقق من النتائج باستخدام المحاكاة العددية من خلال إستعمال برنامج ماتلاب.
الكلمات المفتاحية: نظام ديناميكي، فوضى، جاذب غريب، نظام حركي برتب صحيحة، نظام حركي برتب كسرية

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General Introduction

In 1963, Lorenz identified the first three-dimensional chaotic system [43], and since then, many more have been discovered, including the Rossler system, Chen system, Liu system, Zhu system, Sprott system, and Vaidyanathan system, and others. Chaos has been researched in science, mathematics, engineering, and a variety of other fields as a significant nonlinear phenomena. Because of its potential uses in a variety of industries.

Chaos synchronization is an interesting phenomenon of nonlinear dynamical systems and it may occur when two or more chaotic systems are coupled or one chaotic system drives the other. The synchronization of chaotic systems was rst given by Pecora and Carroll [28] in 1960, and after which it has been intensively studied due to its potential applications in various elds viz., ecological system, physical system, chemical system, secure communications etc $[24-35]$. In recent years various types of synchronization have been investigated such as complete synchronization, anti-synchronization, lag synchronization, adaptive synchronization, projective synchronization, function projective synchronization etc $[19-32]$ and also di erent schemes have been successfully applied to chaos synchronization viz., linear and nonlinear feedback control method, active control method, adaptive control method, sliding mode control method, backstepping method,... etc.

In recent years, fractional order systems have become a hot research field demonstrated by many researchers such as$[24-27]$. Synchronization of chaos in fractional order differential systems has attracted increasing attention due to its powerful potential applications in different fields such as in secure communication, telecommunications, cryptography$[14-33]$. Several types of synchronization methods have been proposed and developed for fractional order systems. These include, adaptive control [29], sliding mode control$[25-32]$, active control technique $[15-19]$, function projective synchronization [38] and modified projective synchronization[10], hyprid projective synchronization [41], and others. However, the results in synchronization between the fractional-order systems and the integer-order systems are limited because they have not been extensively studied. Up to now, only a few works have been given to investigate this kind of synchronization, such as in$[18-36]$.

In this work, we consider the problem of the synchronization between the fractional-order chaotic system and the chaotic system of integer order. Based on suitable controllers and the stability theory of linear integer order systems, the synchronization between the fractional-order chaotic system and the chaotic system of integer order is achieved. Finally, The corresponding simulation results are provided to demonstrate the effectiveness of the proposed method in Matlab.

The following three chapters make up this thesis:

In Chapter 1, we give some definitions and prelimanaries about: dynamical systems , Chaos, Choatic systems, synchronization and types of synchronization. Also,Basic definitions and properties of fractional derivative are given with numerical method for solving fractional differential equations.

In chapter 02, we have discussed some examples of integer orders and fractional orders choatic systems.

In chapter 3, we present the study of the synchronization between two 3D and 4D fractional-integer orders chaotic systems .

Chapter 1

Preliminaries

1.1 Introduction

In Chapter, we introduce some prelimanaries about dynamical systems and Chaos theory, Choatic systems, synchronization and types of synchronization. Also,Basic definitions and properties of fractional derivative are given with numerical method for solving fractional differential equations.

1.2 Dynamic systems

Dynamical systems represent phenomena that evolve in space and/or time. They are developed and specialized during the nineteenth century.These systems come from Biology, Physics,Chemistry,or the social sciencesthe. dynamic system is the subject that provides mathematical tools for its analysis. its classified into two categories: **Discrete time dynamic system**

and **Continuous time dynamic system.**

1.2.1 Continuous dynamic systems

A dynamic system in a continuous time is represented by a system of different equations of the form:

$$\dot{x_t} = k(x,t)\ ;\ x \in R^n,\ t \in R^+$$

with $K : R^n \times R^+ \to R^n$ denotes the dynamics of the system.

1.2.2 Discrete dynamic systems

If a system only takes its values at regularly distributed points, it is said to be discrete or merely discrete [46] Its mathematical representation is given as follows.:

$$\begin{cases} x(k+1) = f(x(k)), \\ x(k_0) = x(0), \end{cases}$$

with k is a discrete momen, k_0 is the first discrete time and $x(0)$ is the vector of initial states

1.3 Phase portrait

Our first approach to chaos has made us realize the difficulty of finding exact solutions or even approaching nonlinear equations and this brings us the search for a representation that would allow us more simply access quality solutions. This is what presents the space of the phases [22]: It consists of an abstract space containing concrete information in geometric form. The variables that are the basis of the construction of this space are real quantities and each point corresponds to a well-determined physical situation . The space must contain any information on the dynamics of the studied system.

1.4 The Poincaré section

It is a tool frequently used to study dynamic systems and in particular periodic trajectories. Making a section of Poincaré cuts the trajectory in the space of the phases, in order to study the intersections of this trajectory (in dimension three, for example), with a plane. We then spend a continuous time dynamic system a discrete time dynamic system. Mathematicians have of course demonstrated that the properties of the system are preserved after the realization of a section of Poincaré judiciously chosen. Using this method, the dimension d of the initial problem in the form of a differential system is reduced by one unit with the application in dimension $d-1$

1.5 Chaos

The logistic map exhibits in stunning fashion a phenomenon which, for most functions, is only partially understood: the chaotic behavior of orbits of a dynamical system. There are many possible definitions of chaos, ranging from measure-theoretic notions of randomness in ergodic theory to the topological approach we will adopt here. Before we define chaos, we have to have some preliminary definitions (see[30]).

Definition 1.1 *A dynamic system is called a chaotic system if there is at least one chaotic attractor.*

Definition 1.2 *A dynamic system is called a chaotic system if is has at least one positif Lyapunouv exponents.*

1.6 Properties of chaotic systems

There is a set of properties that summarize the characteristics observed in chaotic systems. They are considered as mathematical criteria which dene chaos. The most popular are:

Definition 1.3 *Let V be a set. $f : V \rightarrow V$ is said to be chaotic on V if f has the following three properties:*

1· *f has sensitive dependence on initial conditions*

2· *The periodic points of f are dense in V*

3· *f is topologically transitive*

Definition 1.4 *$f : J \rightarrow J$ is expansive if there exists $v > 0$ such that, for any $x, y \in J$, $x \neq y$, there exists $n \geq 0$ such that $|f^n(x) - f^n(y)| > v$.*

Expansiveness differs from sensitive dependence in that all nearby points eventually separate by at least v

Definition 1.5 *aU of V subset is dense in$V$ if $\overline{U} = V$*

1.6.1 No periodicity

A system of chaotic behaviour develops into an orbit that never repeats itself.That is, orbits are never periodic

1.6.2 determinism

Determinism means that the system is nonrandom and does not have stochastic or input parameters. This property is proper to all systems whose evolution is defined by a set of differential equations or equations to differences. In the phenomenal randoms, it is absolutely impossible to predict the trajectory of any particle. On the contrary, and though they appear, At first sight, chaotic dynamical systems are governed by some equations representing the phenomenal, but whose solutions are sensitive to the initial conditions . The irregular behaviour observed in chaotic systems is due to the intrinsic non-linearity of the system rather than noise .

1.6.3 broadband spectrum

The Fourier spectrum for a chaotic signal is a broadband spectrum, similar to white noise . Figure (1.2) presents the spectrum of logistical function evolving in a chaotic regime

1.6.4 The strange attractor

Trajectories of a chaotic dynamic system are attracted to a so-called strange attractor. The latter is characterized by:

a)-a zero volume

b)-exponentially rapid separation of the original near trajectories;

c)- fractale dimension (non integer) The creation of a strange attraction is related to the existence of two processes:

-Stretching, responsible for instability and sensitivity to initial conditions

-The folding, responsible for the strange and fractal side of the attractor**.**

1.7 Synchronization

1.7.1 Synchronisation Methodes

This section is devoted to the presentation of various methods of synchronization most efficient and the most encountered

Definition 1.6 *synchronization between tow system if the trajectory of the response system trakes the trajectory of the drive system in time*

Definition 1.7 *Let* $\dot{x} = F(x;t)$ *be the drive (chaotic, hyperchaotic) system, and* $\dot{y} = G(y;t) + U$ *be the response system, where* $x = (x_1(t), x_2(t), ..., x_n(t))^T, y = (y_1(t), y_2(t), ..., y_m(t))^T, U = (u_1, u_2, ..., u_n)^T$ *is a controller to be determined later.*

- Synchronization is achieved if $\lim_{t\to+\infty} \|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y - x$,
- Anti-synchronization is occur if $\lim_{t\to+\infty} \|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y + x$,

- Function projective synchronization is achieved if $\lim_{t\to+\infty}\|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y - h(x)x$, $h(x) = (h_1(x_1), h_2(x_2), ..., h_n(x_n))$,

- Inverse function projective synchronization is achieved if $\lim_{t\to+\infty}\|e\| = 0$, $e \in \mathbb{R}^n$ with $e = y - h(y)y$, $h(x) = (h_1(y_1), h_2(y_2), ..., h_n(y_n))$,h is a scaling function matrix.

1.7.2 Active control Method

The application of active control for the synchronization of chaotic systems, was proposed by Bai and Lonngren [15], it is an effective technique which has shown its power not only for the synchronization of identical systems, but also for the synchronization of non-identical systems. Moreover, this method offers a remarkable simplicity for the implementation of the algorithm [24, 25]. Consider two chaotic systems to be synchronized, master and slave, defined by:

$$\frac{dx(t)}{dt} = F(x(t)), \tag{1.1}$$

and

$$\frac{dy(t)}{dt} = G(y(t)) + U, \tag{1.2}$$

where $x, y \in C^1(\mathbb{R}, \mathbb{R}^n)$, are the states of the master and slave systems, respectively $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^n$, and $U = (u_i)_{1\leq i\leq n}$, is a controller to be determined. For the two systems to synchronize, the error between the trajectories of the two systems must converge towards zero when time tends towards infinity. This error is obtained as follows:

$$e(t) = y(t) - x(t), \tag{1.3}$$

so the error system is given by

$$\begin{aligned} \frac{de(t)}{dt} &= \frac{dy(t)}{dt} - \frac{dx(t)}{dt} \\ &= G(y(t)) - F(x(t)) + U. \end{aligned} \tag{1.4}$$

If we can write the quantity $G(y(t)) - F(x(t))$ as follows:

$$G(y(t)) - F(x(t)) = Ae(t) + N(x(t), y(t)), \tag{1.5}$$

the error system can be expressed as follows:

$$\frac{de(t)}{dt} = Ae(t) + N(x(t), y(t)) + U, \tag{1.6}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix and N a nonlinear function. The controller U is offered as follows:

$$U = V - N(x(t), y(t)), \tag{1.7}$$

where V is the active controller, defined by:

$$V = -Le(t), \tag{1.8}$$

where L is an unknown control matrix. We therefore obtain the final formula for the error:

$$\frac{de(t)}{dt} = (A - L)e(t). \tag{1.9}$$

So the problem of synchronization between the master system $(1, 1)$ and the slave system $(1, 2)$ is transformed into a problem of zero-stability of the system $(1, 9)$. Now, the following Theorem is an immediate result of the theory of the stability of discrete linear dynamical systems.

Theorem 1.1 *The master system $(1, 1)$ and the slave system $(1, 2)$ are globally synchronized under the control law $(1, 7)$, if and only if the control matrix L is chosen such that the real part of the eigenvalues of $A - L$ is negative*

1.8 Fractional calculus

Basic definitions and properties of fractional derivative/integrals are given below $[2, 14, 31]$.

Definition 1.8 *A real function $f(x)$, $t > 0$ is said to be in space C_α, $\alpha \in \mathbb{R}$ if there exists a real number $p\,(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in [0, \infty)$.*

Definition 1.9 *A real function $f(x)$, $t > 0$ is said to be in space C_α^m, $m \in \mathbb{N} \bigcup \{0\}$if $f^{(m)} \in C_\alpha$.*

Definition 1.10 *Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann–Liouville integral of order μ, $\mu > 0$ is given by*

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, d\tau, \quad t > 0 \tag{1.10}$$

Definition 1.11 *The Caputo fractional derivative of f (t) is given as:-The Caputo fractional derivative of $f(t)$ is given as:*

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(\tau - n)^{q-n+1}} d\tau,$$

for $n-1<q\leq n, n\in N, t>0.\Gamma(\cdot)$ *is the gamma function.*

Definition 1.12 *The Laplace transform formula for the Caputo fractional derivative is as follows:*

$$L\left(D_t^q f\left(t\right)\right)=s^q F\left(s\right)-\sum_{k=0}^{n-1} s^{q-k-1} f^{(n)}\left(0\right),\left(q>0, n-1<q\leq n\ \right)$$

Particularly, when $0<q\leq 1$*, we have:*

$$L\left(D_t^q f\left(t\right)\right)=s^q F\left(s\right)-s^{q-1} f\left(0\right)$$

Also, The Laplace transform of the fractional integral of order Qsatisfies:

$$L(D_t^{-q} f(t))=s^{-q} F\left(s\right),\left(q>0\right)$$

with $F\left(s\right)=L(f(t))$

1.9 Stability of fractional order systems

In the stability theory of linear systems with invariant time and derivatives of integer order, we well know that a system is stable if the roots of the polynomial cacharacteristic are strictly negative real parts, therefore they located on the left half of the complex plan. Moreover, in the case of linear fractional systems in time invariant, the definition of stability is different from integer-order systems. Indeed, fractional or non-integer order systems can have roots in half right of the complex plane and be stable. The most well-known stability criterion for non-integer systems is that of Matignon.In particular, it relates, in to commensurable non-integer systems of order α including between 0 and 2. This criterion is based on the study of poles. Let the following dierential system be :

$$^cD^\alpha x(t)=f(t;x(t)); \tag{1.11}$$

where $0<\alpha<1; x=(x_1,...,x_n)\ ^T\in R^n; f: R\times R^n\rightarrow R^n$ a continuous function and $^cD^\alpha$denotes the derivative of Caputo.

1.9.1 Equilibrium point

Let's take the system (1.2), with initial condition $x(t_0)=x_0$.To evaluate the equilibrium points of the system (1.2), it suffices to solve the equation :$^cD^\alpha x(t)=0$

If x_ε is a solution of the equation, then :

$$f(x_\varepsilon) = 0$$

1.9.2 Stability of autonomous linear systems.

In this section, the stability of an autonomous linear fractional differential system is studied. D.Matignon gave in his article in 1996 the following theorem, we start by giving a stability result in the very simple case of a commensurable autonomous linear fractional differential system.

Theorem 1.2 *The following fractional-order autonomous linear system :*

$$\begin{cases} D^\alpha x(t) = Ax(t) \\ x(t_0) = 0 \end{cases} \tag{1.12}$$

such as : $x \in R^n, 0 < \alpha < 1$*and* $A \in R^n \times R^n$

is locally asymptotically stable if and only if : $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$ for everything $i = 1, 2, ..., n.$ This system is stable if and only if : $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$ for everything $i = 1, 2, ..., n.$ and the critical eigenvalues that satisfy $|\arg(\lambda_i)| = \alpha\frac{\pi}{2}$ have geometric multiplicity1,where λ_i , $i = 1, 2, ..., n.$are the eigenvalues of the matrix A.

An extension of this theorem to the case $1 < \alpha < 2$ was given as follows :

Theorem 1.3 *The system* (1.12)*is locally asymptotically stable if and only if*

$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$for everything$i = 1, 2, ..., n.$and $1 < \alpha < 2.$

Corollary 1.1 *Suppose that* $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq ... \neq \alpha_n$ *and all* α *i's are numbers rational between* 0*and* $1.$*Let* m *be the least common multiple of the denominators* u_i *of* α_i $(i = 1, 2, ..., n)$*where* $\alpha_i = \frac{v_i}{u_i}; v_i$ *and* $u_i \in Z^+$*with* $i = 1, 2, ..., n$ *and by posing* $\rho = \frac{1}{m}$ *so the system* (1.12) *is asymptotically stable if* $|\arg(\lambda_i)| > \rho\frac{\pi}{2},$

for all the roots λ of the following characteristic equation :

$$det(diag([\lambda^{m\alpha_1}, ..., \lambda^{m\alpha_n}]) - A) = 0.$$

This corollary says that in the case of rational orders the characteristic equation can be transformed into an integer-order polynomial equation.

1.10 Numerical method for solving fractional differential equations

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDE). A modification of Adams–Bashforth–Moulton algorithm is proposed by Diethelm. To solve FDEs. Consider for $\alpha \in (m-1, m]$ the initial value problem (IVP)

$$D^{\alpha} y(t) = f(t, y(t)), \quad 0 \leq t \leq T, \tag{1.13}$$

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, ..., m-1. \tag{1.14}$$

The IVP (1.13) and (1.14) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau. \tag{1.15}$$

Consider the uniform grid $\{t_n = nh / n = 0, 1, ..., N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$, $j = 1, 2, ..., n$ and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) \tag{1.16}$$

$$+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)), \tag{1.17}$$

Where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & if \;\; j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & if \;\; 1 \leq j \leq n, \\ 1, & if \;\; j = n+1. \end{cases}$$

The preliminary approximation $y_h^p(t_{n+1})$ is called predictor and is given by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)) \tag{1.18}$$

where

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right). \tag{1.19}$$

Error in this method is

$$\max_{j=0,1,\dots N} |y(t_j) - y_h(t_j)| = O(h^p), \tag{1.20}$$

where $p = \min(2, 1+\alpha)$.

1.11 Chaos synchronization between fractional order Lorenz and Liu system

In this section, we study the synchronization between the fractional order Lorenz and Liu systems.

1.11.1 Systems description for chaos synchronization

The fractional order Lorenz system is described by

$$\begin{cases} D^{\alpha}x = \sigma(y-x), \\ D^{\alpha}y = rx - y - xz, \\ D^{\alpha}z = xy - \mu z, \end{cases} \tag{1.21}$$

where $\sigma = 10$ is the Prandtl number, $r = 28$ is the Rayleigh number over the critical Rayleigh number and $\mu = 8/3$ gives the size of the region approximated by the system. The minimum effective dimension for this system is 2.97. A fractional version of the chaotic system by Liu et al is studied in [20] described by:

$$\begin{cases} D^{\alpha}x = -ax - ey^2, \\ D^{\alpha}y = by - kxz \\ D^{\alpha}z = -cz + mxz, \end{cases} \tag{1.22}$$

where $a = 1$, $e = 1$, $b = 2.5$, $k = 4$, $c = 5$, $m = 4$.The lowest value of α for which the system exhibits chaos is given by 0.92 [20].

1.11.2 Fractional order Lorenz and Liu system synchronization via active control

Assuming that the Lorenz system drives the Liu system, we define the drive (master) and response (slave) systems as follows:

$$\begin{cases} D^{\alpha}x_1 = \sigma\left(y_1 - x_1\right), \\ D^{\alpha}y_1 = rx_1 - y_1 - x_1 z_1, \\ D^{\alpha}z_1 = x_1 y_1 - \mu z_1, \end{cases} \tag{1.23}$$

and

$$\begin{cases} D^{\alpha}x_2 = -ax_2 - ey_2^2 + u_1\left(t\right), \\ D^{\alpha}y_2 = by_2 - kx_2 z_2 + u_2\left(t\right), \\ D^{\alpha}z_2 = -cz_2 + mx_2 z_2 + u_3\left(t\right). \end{cases} \tag{1.24}$$

The unknown terms u_1, u_2, u_3 in (1.23) are active control functions to be determined. Define the error functions as:

$$e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1. \tag{1.25}$$

Eq. (1.23) together with (1.23) and (1.24) yields the error system:

$$\begin{cases} D^{\alpha}e_1 = -ae_1 - ax_1 - ee_2^2 - 2ee_2 y_1 - ey_1^2 - \alpha\left(y_1 - x_1\right) + u_1\left(t\right), \\ D^{\alpha}e_2 = be_2 + by_1 - ke_1\left(e_3 - z_1\right) - kx_1\left(z_1 + e_3\right) + y_1 + x_1\left(z_1 - r\right) + u_2\left(t\right), \\ D^{\alpha}e_3 = -ce_3 - cz_1 + me_1\left(e_2 + y_1\right) + mx_1\left(y_1 + e_2\right) + \mu z_1 - x_1 y_1 + u_3\left(t\right). \end{cases} \tag{1.26}$$

We define active control functions $u_i\left(t\right)$ as:

$$\begin{cases} u_1\left(t\right) = V_1\left(t\right) + ax_1 + ee_2^2 + 2ee_2 y_1 + ey_1^2 + \alpha\left(y_1 - x_1\right), \\ u_2\left(t\right) = V_2\left(t\right) - by_1 + ke_1\left(e_3 - z_1\right) + kx_1\left(z_1 + e_3\right) - y_1 - x_1\left(z_1 - r\right), \\ u_3\left(t\right) = V_3\left(t\right) + cz_1 - me_1\left(e_2 + y_1\right) - mx_1\left(y_1 + e_2\right) - \mu z_1 + x_1 y_1. \end{cases} \tag{1.27}$$

The terms $V_i\left(t\right)$ are linear functions of the error terms $e_i\left(t\right)$. With the choice of $u_i\left(t\right)$ given by (1.26) the error system (1.27) becomes:

$$\begin{cases} D^{\alpha}e_1 = -ae_1 + V_1\left(t\right) \\ D^{\alpha}e_2 = be_2 + V_2\left(t\right) \\ D^{\alpha}e_3 = -ce_3 + V_3\left(t\right) \end{cases} \tag{1.28}$$

The control terms $V_i\left(t\right) v$are chosen so that the system (1.28) becomes stable. There is not a unique choice for such functions.

We choose:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e3 \end{pmatrix}, \tag{1.29}$$

where A is a 3×3 real matrix, chosen so that for all eigenvalues λ_i of the system (1.28) the condition

$$|\arg(\lambda_i)| > \alpha\pi/2 \tag{1.30}$$

is satisfied.

If we choose

$$A = \begin{pmatrix} a-1 & 0 & 0 \\ 0 & -1-b & 0 \\ 0 & 0 & c-1 \end{pmatrix} \tag{1.31}$$

then the eigenvalues of the linear system (1.28) are $-1, -1$ and -1. Hence the condition (1.30) is satisfied for $\alpha < 2$. Since we consider only the values $\alpha \leq 1$, we get the required synchronization.

1.11.3 Simulation and results

Parameters of the Lorenz system are taken as $\sigma = 10$, $r = 28$, $\mu = 8/3$ and Liu system as $a = 1$, $e = 1$, $b = 2.5$, $k = 4$, $c = 5$, $m = 4$. The fractional order α is taken to be 0.99 for which both the systems are chaotic. The initial conditions for drive and response system are $x_1(0) = 10$, $y_1(0) = 5$, $z_1(0) = 10$, and $x_2(0) = 0.2$, $y_2(0) = 0$, $z_2(0) = 0.5$, respectively. Initial conditions for the error system are thus $e_1(0) = -9.8$, $e_2(0) = -5$, $e_3(0) = -9.5$. The errors $e_i(t)$ for the drive and response system are shown in Fig. (1.1).

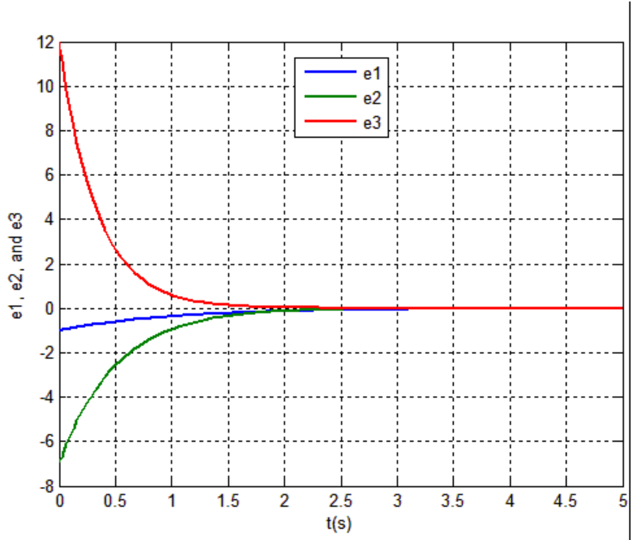


Figure1.1Time-History of the synchronization errors $e_i(t)$

1.12 Conclusion

This Chapter, contain some prelimanaries about dynamical systems and Chaos theory, Choatic systems, synchronization and types of synchronization. Also,Basic definitions and properties of fractional derivative are given with numerical method for solving fractional differential equations.

Chapter 2

Examples of chaotic system of integer orders and fractional orders

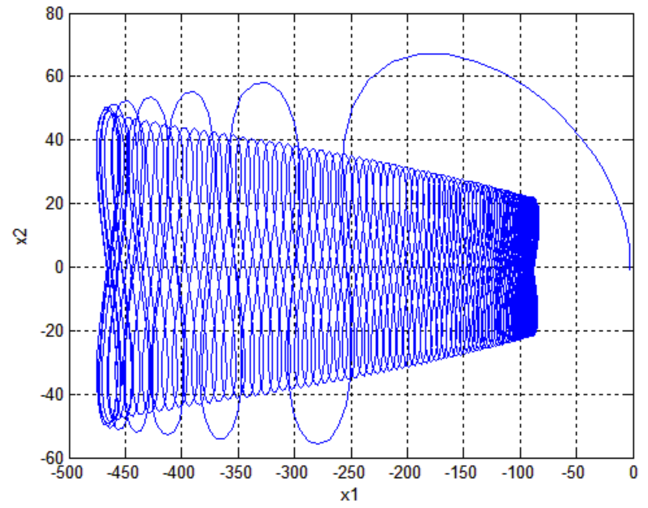
2.1 Introduction

In this Chapter, we give some integer orders and fractional orders chaotic systems in 3d.

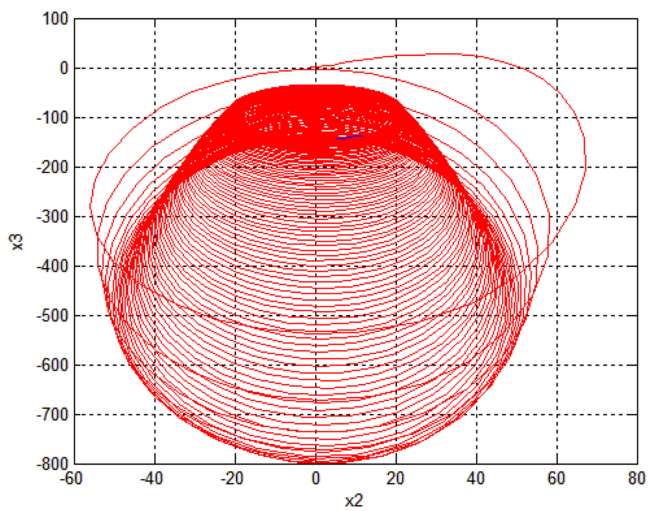
Example 2.1 *Condider the following integer-order Bhalekar system describs the drive system:*[4]

$$\begin{cases} \dot{x}_1 = -x_2^2 - \alpha x_1 \\ \dot{x}_2 = b(x_3 - x_1) \\ \dot{x}_3 = cx_2 - x_3 + x_1 x_2 \end{cases} \tag{2.1}$$

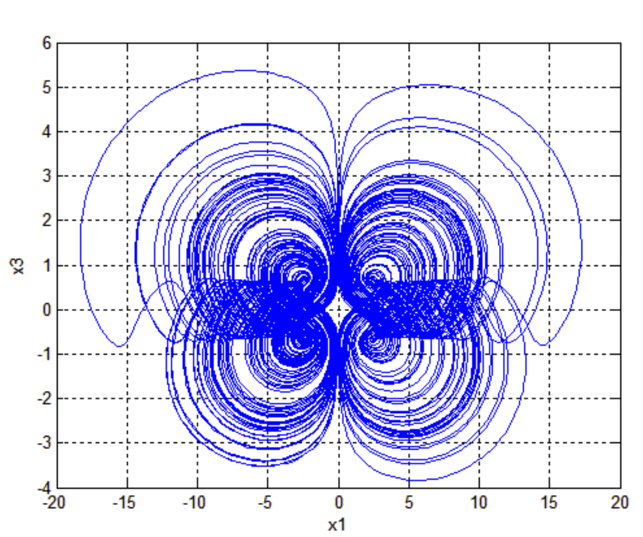
were $x = (x_1, x_2, x_3)$ is the system state vertor α, b, c are real parameters . It exhibits chaotic attractor for : $\alpha = 2.667, b = 10, c = 27.3$ The projection of the chaotic attractor which is shown in fig 1



$Figure 2.1$:2D view of the chaotic attractor of system (2,1) in (x_1, x_2) plane.



$Figure\ 2.2$: Projection on (x_2, x_3) plane of the chaotic attractor of system (2,1).



$Figure$ 2.3 :Projection on (x_1, x_3) plane of the chaotic attractor of system (2,1).

Example 2.2 *Condider the following integer-order Zhang system describs the drive system:*[44]

$$\begin{cases} \dot{x}_1 = -2x_1 + 10x_2x_3 \\ \dot{x}_2 = -6x_2^3 + 3x_1x_3 \\ \dot{x}_3 = 3x_3 - x_1x_2 \end{cases} \tag{2.2}$$

were $x = (x_1, x_2, x_3)$ is the system state vertor α, b, c are real parameters . It exhibits chaotic attractor for $\alpha = -2, b = -6, c = 27.3$.

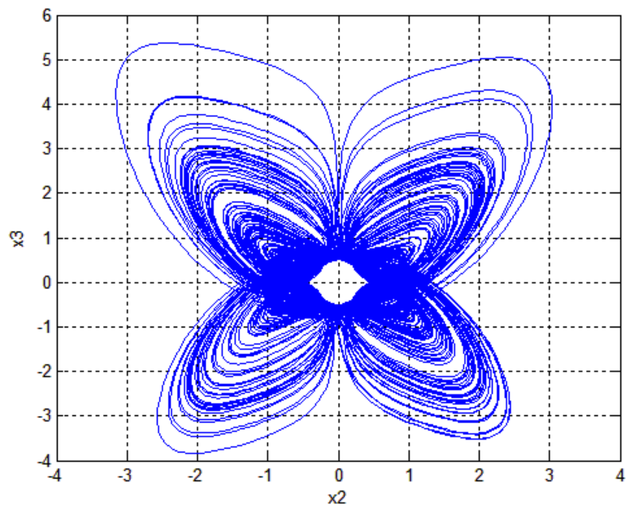


Figure 2.4 :Projection on (x_2, x_3) plane of the chaotic attractor of system (2,2).

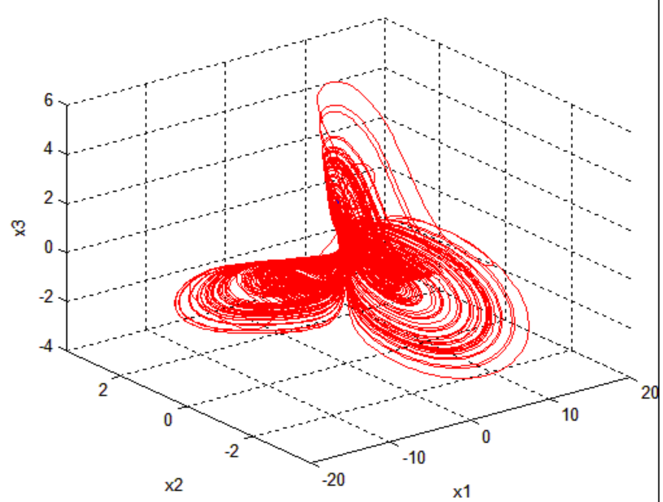
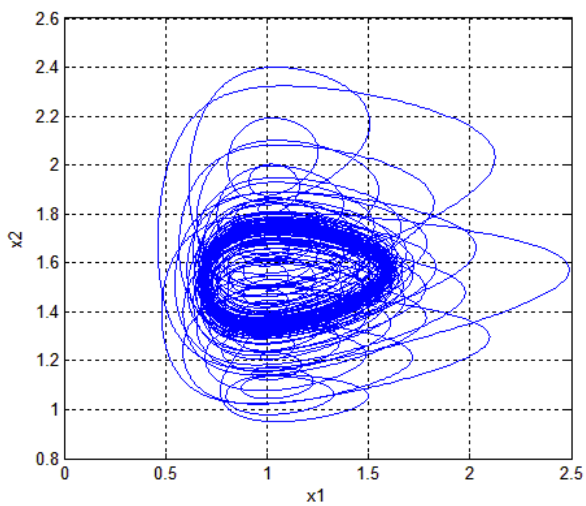


Figure 2.5 :3D view of the chaotic attractor of system (2,2) in (x_1, x_2, x_2) plane.

Example 2.3 *the fractional order Lotka-Voltra system is given as :*

$$\begin{cases} D_t^{q_1}x_1 = c_1x_1 - c_2x_1x_2 + c_5x_1^2 - c_6x_3x_1^2 \\ D_t^{q_2}x_2 = -c_3x_2 + c4x_1x_2 \\ D_t^{q_3}x_3 = -c_3x_3 + c_6x_3x_1^2 \end{cases} \tag{2.3}$$

this system shows chaotic behavior and the choatic attractor of the system is obtained for the values of the parameters $c_1 = c_2 = c_3 = c_4 = 1, c_5 = 2, c_6 = 2.7, c_t = 3$ *and initial condition* $(1, 1.4, 1)$ *and* $q = 0.95$.



$Figure$ 2.6 :Projection on (x_1, x_2) plane of the chaotic attractor of system (2,3).

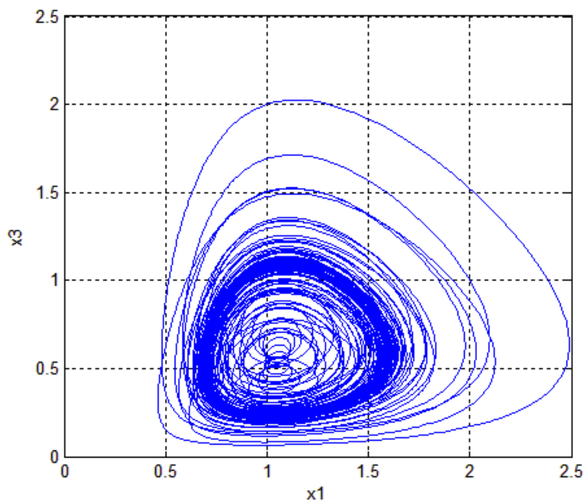


Figure 2.7 :Projection on (x_1, x_3) plane of the chaotic attractor of system (2,3).

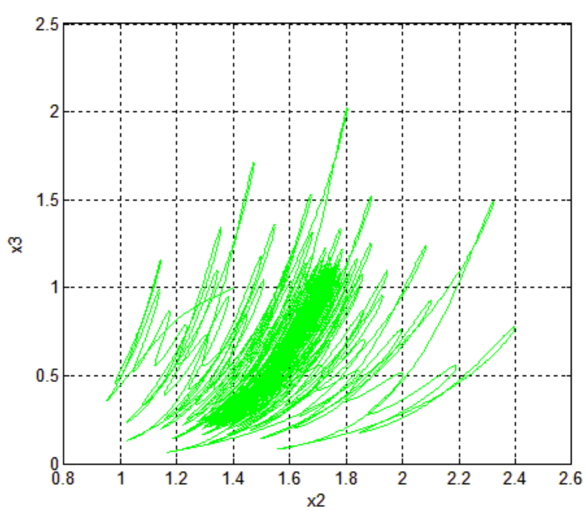
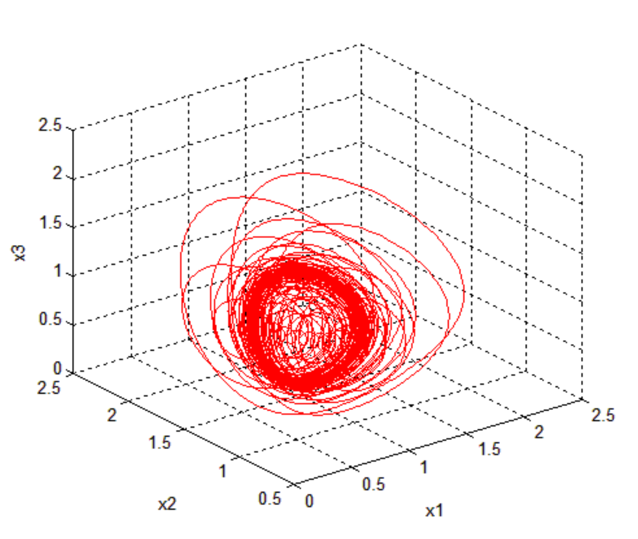


Figure 2.8 :Projection on (x_2, x_3) plane of the chaotic attractor of system (2,3).



$Figure$ 2.9 :3d view of the chaotic attractor of system (2,3).

Example 2.4 *the fractional order Newton leipnik system* [21] *was first studied in the year 2008 which is given by:*

$$\begin{cases} D_t^{q_1}x = -\alpha_1 x + y + 10yz \\ D_t^{q_2}y = -x - 0.4y + 5xz \\ D_t^{q_3}z = \alpha_2 w - 5xy.0 < q < 1. \end{cases} \quad (2.4)$$

where α_1 *and* α_2 *are the variable parameters and* $\alpha_2 \in (0, 0.8)$. *the chaotic attractor projections of this system are shown in figs. 10-14 for* $(\alpha_1, \alpha_2) = (0.40.175)$, $q_1 = q_2 = q_3 = 0.96$, *and initial conditions* $(x_0, y_0, z_0) = (0.19, 0, -0.18)$.

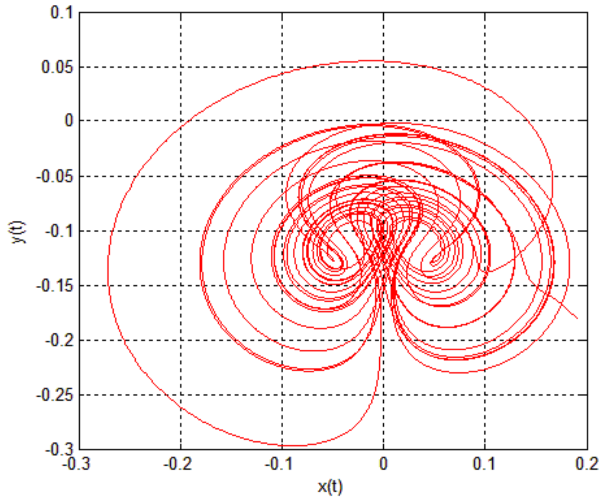
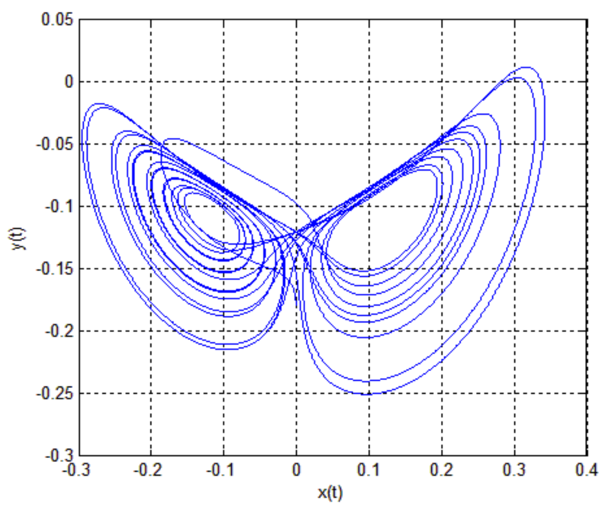


Figure 2.1: $Figure\ 2.11$:Projection on (x, z) plane of the chaotic attractor of system (2,4).



$Figure\ 2.10$:Projection on (x, y) plane of the chaotic attractor of system (2,4).

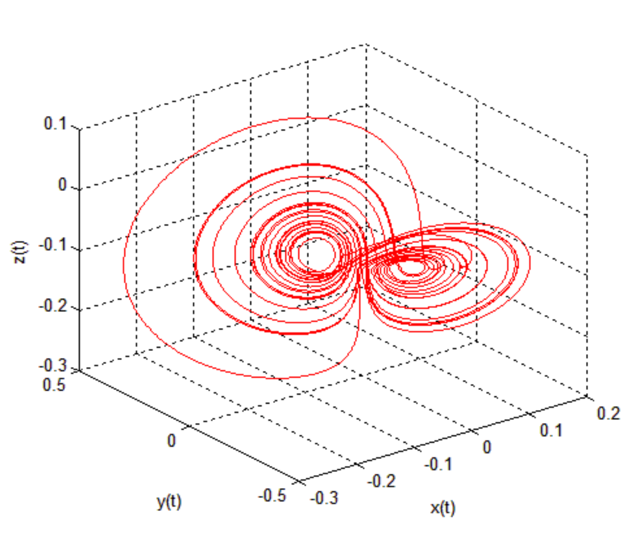


Figure 2.12 :3D view of the chaotic attractor of system (2,4).

2.2 Conclusion

This Chapter, contains some integer orders and fractional orders chaotic systems in 3d.

Chapter 3

Synchronization of two chaotic systems of integer order and fractional order

3.1 Introduction

In this chapter, we study the synchronization between the fractional-order chaotic system and the chaotic system of integer order in detail. We design suitable sub-controllers to achieve synchronization by using stability criteria of the integer-order linear system. Finally, the simulation results demonstrate the effectiveness of the proposed scheme.

3.1.1 Problem formulation

We consider the drive system given by:

$$\dot{x}_i(t) = f_i(X(t)),\ i = 1, ..., n. \tag{3.1}$$

Where: $X(t) = (x_1, x_2, ..., x_n)^T$ is the state vector of the system $(3,1)$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, ..., n$ are nonlinear functions, and as response system the system given by:

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + V_i,\ i = 1, ..., n. \tag{3.2}$$

Where: $Y(t) = (y_1, y_2, ..., y_n)^T$ is the state vector of the system $(3,2)$, $g_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, ..., n$ are nonlinear functions,$0 < q_i < 1$,$D_t^{q_i}$ is the Caputo fractionnal derivative of order q_i for $i = 1, ..., n$, V_i are controllers to be designed such as the system $(3,1)$and the system $(3,2)$ to be synchronized.

3.2 Synchronization of fractionel-integer order systems (3,1) and (3,2)

We decompose the controller V_i ,$i = 1, ..., n$, into two sub-controllers U_i and U_{ii} i.e., $V_i = U_i + U_{ii}$ for $i = 1, ..., n$, and propose the following form for the sub-controller U_i given by:

$$U_i = \left(D_t^{q_i-1} - 1\right) (\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \, , \; i = 1, ..., n. \tag{3.3}$$

By using $(3, 3)$, we can rewrite the slave system $(3, 2)$ as follows:

$$D_t^{q_i} y_i (t) = D_t^{q_i-1} \left(\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \right) + U_{ii}, \; i = 1, ..., n. \tag{3.4}$$

Applying a Laplace transform to the system $(3, 4)$, we find:

$$s^{q_i} Y_i (s) - s^{q_i - 1} y_i (0) = s^{q_i-1} L \left(\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \right) + L (U_{ii}) \, , \; i = 1, ..., n. \tag{3.5}$$

with $Y_i(s) = L(y_i(t))$. Multipling both sides of (9) by $s{-}^{(q_i-1)}$ and appling the inverse Laplace transform to the result:

$$L^{-1} (s Y_i (s) - y_i (0)) = \left(\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \right) + L^{-1} \left(s \, -^{(q_i-1)} L (U_{ii})\right) , \; i = 1, ..., n. \tag{3.6}$$

then

$$L^{-1} (L (D_t (y_i (t)))) = \left(\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \right) + L^{-1} \left(s \, -^{(q_i-1)} L (U_{ii})\right) , \; i = 1, ..., n. \tag{3.7}$$

By using the last propierty in lemma 2 we obtain:

$$D_t (y_i (t)) = \left(\sum_{j=1}^{n} b_{ij} y_j (t) + g_i (Y (t)) \right) + D_t^{1-q_i} (U_{ii}) \, , \; i = 1, ..., n. \tag{3.8}$$

then the problem synchronization (any type of synchronization) between the fractionnel-integer order systems $(3, 1)$ and $(3, 2)$ is reduce to another one between two integer-order systems $(3, 1)$

and $(3, 8)$. The state errors for (3,1) and (3,8) is:

$$e_i = y_i - x_i,\ i = 1, ..., n. \tag{3.9}$$

Consequently, the error dynamic system is given by:

$$\left\{ \dot{e}_i = \left(\sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) \right) + D_t^{1-q_i}(U_{ii}) - f_i(X(t)),\ i = 1, ..., n. \right. \tag{3.10}$$

In view of $(3, 10)$, we propose the sub-controller U_{ii} in the form:

$$U_{ii} = D_t^{q_i-1} \left(f_i(X(t)) - \sum_{j=1}^{n} b_{ij} x_j(t) - g_i(Y(t)) - \sum_{j=1}^{n} c_{ij} e_j(t) \right),\ i = 1, ..., n. \tag{3.11}$$

Theorem 3.1 *If we select the control matrix C such that $B - C$ is negative, then the two systems $(3, 1)$ and $(3, 2)$ are globally synchronized under the controllers $(3, 3)$ and $(3, 11)$.*

Proof. By inserting $(3, 11)$ into $(3, 10)$, we get:

$$\dot{e}_i = \left(\sum_{j=1}^{n} b_{ij} - c_{ij} \right) e_j,\ i = 1, ..., n. \tag{3.12}$$

The system $(3, 12)$ can expresed in matricial form as follows:

$$\dot{e} = (B - C)\, e \tag{3.13}$$

where: $B = (b_{ij}), C = (c_{ij})$ two $n \times n$ matrices and $e = (e_1, e_2, ..., e_n)^T$ is the errors vector of the system. If we chose matrix C such that $B - C$ is negative hence all the eigenvalues $\lambda_i, i = 1, 2, 3.$ of $(B - C)$ stay in the left-half plane i.e., $Re(\lambda_i) < 0$, which ensures according with the Lyapunov stability theory [12], that errors system $(3, 13)$ is asymptotically stable i.e., $\lim_{t \to +\infty} \|e\| = 0,\ e \in \mathbb{R}^3$ with $e = y - x$. Hence the synchronization between the system $(3, 1)$ and the system $(3, 2)$ is achieved. This completes the proof. ■

3.3 Exemple in 3D

Consider the following integer-order hyper chaotic Tuna dynamical system as a drive system :

$$\begin{cases} \dot{x_1} = x_2(x_3 - 1.3) \\ \dot{x_2} = -x_1(x_3 + 1.3) \\ \dot{x_3} = -x_2(1.3x_1 - x_2) - 4(x_3 - 1.3) \end{cases} \tag{3.14}$$

where : $x = (x_1, x_2, x_3)$ is the system state vector, a, b, c, d are real parameters. It exhibits chaotic attractor for ; $a = b = c = 1.3; d = 4; r = 5.2$ As response system ;we consider the controlled fractional order choatic system[5] given by :

$$\begin{cases} D_t^{q_1} y_1 = ay_1 - y_2y_3 + v_1 \\ D_t^{q_2} y_2 = -by_2 + y_1y_3 + v_2 \\ D_t^{q_3} y_3 = c - y_3 + y_1y_2 + dy_3y_2 + v_3 \end{cases} \tag{3.15}$$

where : $y = (y_1, y_2, y_3)$ is the system state vector, a, b, c, d are real parameters. when : $a = 0.7, b = 0.1, c = 0.001, d = 0.1$, this system exhibits chaotic attractor.

$$\begin{cases} U_1 = (D^{q_1-1} - 1)(0.7y_1 - y_2y_3) \\ U_2 = (D^{q_2-1} - 1)(-0.1y_2 + y_1y_3) \\ U_3 = (D^{q_3-1} - 1)(0.001 - y_3 + y_1y_2 + 0.1y_3y_2) \end{cases} \tag{3.16}$$

so the response system can be rwitten as followes :

$$\begin{cases} D_t^{q_1} y_1 = 0.7y_1 - y_2y_3 + U_1 + U_{11} \\ D_t^{q_2} y_2 = -0.1y_2 + y_1y_3 + U_2 + U_{22} \\ D_t^{q_3} y_3 = 0.001 - y_3 + y_1y_2 + 0.1y_3y_2 + U_3 + U_{33} \end{cases} \tag{3.17}$$

$$\begin{cases} D_t^{q_1} y_1 = 0,7y_1 - y_2y_3 - 0.7 + y_2y_3 + D_t^{q_1-1}(0.7y_1 - y_2y_3) + U_{11} \\ D_t^{q_2} y_2 = -0.1y_2 + y_1y_3 + 0.1y_2 - y_1y_3 + D^{q_2-1}(-0.1y_2 + y_1y_3) + U_{22} \\ D_t^{q_3} y_3 = 0.001 - y_3 + y_1y_2 + 0.1y_3y_2 - 0.001 + y_3 - y_1y_2 - 0.1y_3y_2 \\ \qquad + D_t^{q_3-1}(0.001 - y_3 + y_1y_2 + 0.1y_3y_2) + U_{33} \end{cases} \tag{3.18}$$

$$\begin{cases} D_t^{q_1} y_1 = D_t^{q_1-1}(0.7y_1 - y_2y_3) + U_{11} \\ D_t^{q_2} y_2 = D^{q_2-1}(-0.1y_2 + y_1y_3) + U_{22} \\ D_t^{q_3} y_3 = D_t^{q_3-1}(0.001 - y_3 + y_1y_2 + 0.1y_3y_2) + U_{33} \end{cases} \tag{3.19}$$

$$\begin{cases} \mathcal{L}\left[D_t^{q_1} y_1\right] = \mathcal{L}\left[D_t^{q_1-1}(0.7y_1 - y_2y_3)\right] + \mathcal{L}\left[U_{11}\right] \\ \mathcal{L}\left[D_t^{q_2} y_2\right] = \mathcal{L}\left[D^{q_2-1}(-0.1y_2 + y_1y_3)\right] + \mathcal{L}\left[U_{22}\right] \\ \mathcal{L}\left[D_t^{q_3} y_3\right] = \mathcal{L}\left[D_t^{q_3-1}(0.001 - y_3 + y_1y_2 + 0.1y_3y_2)\right] + \mathcal{L}\left[U_{33}\right] \end{cases} \tag{3.20}$$

$$\begin{cases} S^{q_1}y_1(s) - S^{q_1-1}y_1(0) = S^{q_1-1}\mathcal{L}(0.7y_1 - y_2y_3) + \mathcal{L}(U_{11}) \\ S^{q_2}y_2(s) - S^{q_2-1}y_2(0) = S^{q_2-1}\mathcal{L}(-0.1y_2 + y_1y_3) + \mathcal{L}(U_{22}) \\ S^{q_3}y_3(s) - S^{q_3-1}y_3(0) = S^{q_3-1}\mathcal{L}(0.001 - y_3 + y_1y_2 + 0.1y_3y_2) + \mathcal{L}(U_{33}) \end{cases} \tag{3.21}$$

Multiply both sides by : $S^{-q_i+1}, i = 1,3$:

$$\begin{cases} \mathcal{L}^{-1}\{sy_1(s) - y_1(0)\} = \mathcal{L}^{-1}\mathcal{L}(0.7y_1 - y_2y_3) + \mathcal{L}^{-1}(S^{-q_1+1}\mathcal{L}(U_{11})) \\ \mathcal{L}^{-1}\{Sy_2(s) - y_2(0)\} = \mathcal{L}^{-1}\mathcal{L}(-0.1y_2 + y_1y_3) + \mathcal{L}^{-1}(S^{-q_2+1}\mathcal{L}(U_{22})) \\ \mathcal{L}^{-1}\{Sy_2(s) - y_2(0)\} = \mathcal{L}^{-1}\mathcal{L}(0.001 - y_3 + y_1y_2 + 0.1y_3y_2) + \mathcal{L}^{-1}(S^{-q_3+1}\mathcal{L}(U_{33})) \end{cases} \tag{3.22}$$

$$\begin{cases} D_t(y_1(t)) = 0.7y_1 - y_2y_3 + D^{1-q_1}(U_{11}) \\ D_t(y_2(t)) = -0.1y_2 + y_1y_3 + D^{1-q_2}(U_{22}) \\ D_t(y_3(t)) = 0.001 - y_3 + y_1y_2 + 0.1y_3y_2 + D^{1-q_3}(U_{33}) \end{cases} \tag{3.23}$$

So the state errors are:

$$\begin{cases} \dot{e_1} = 0.7y_1 - y_2y_3 + D^{1-q_1}(U_{11}) - x_2(x_3 - 1.3) \\ \dot{e_2} = -0.1y_2 + y_1y_3 + D^{1-q_2}(U_{22}) - x_1(x_3 + 1.3) \\ \dot{e_3} = 0.001 - y_3 + y_1y_2 + 0.1y_3y_2 + D^{1-q_3}(U_{33}) - (-x_2(1.3x_1 - x_2) - 4(x_3 - 1.3)) \end{cases} \tag{3.24}$$

we set:

$$\begin{cases} U_{11} = D^{q_1-1}(x_2x_3 - 1.3x_2 - 0.7x_1 + y_2y_3 - \sum_{i=1}^{3} C_{ij}e_j(t)), \\ U_{22} = D^{q_2-1}\left[(-x_1x_3 - 1.3x_1) + 0.1x_2 - y_1y_3) - \sum_{i=1}^{3} C_{ij}e_j(t)\right] \\ U_{33} = D^{q_3-1}\left[(-1.3x_1x_2 + x_2^2 - 4x_3 + 5.2) + x_3 - y_1y_2 - 0.1y_3y_2 - 0.001 - \sum_{i=1}^{3} C_{ij}e_j(t))\right] \end{cases} \tag{3.25}$$

The state errors is:

$$e_i = y_i - x_i, i = \overline{1.3} \tag{3.26}$$

we obtain:

$$\begin{cases} \dot{e_1} = 0.7y_1 - y_2y_3 + (x_2x_3 - 1.3x_2 - 0.7x_1 + y_2y_3 \\ \quad - \sum_{i=1}^{3} C_{1j}e_j(t)) - x_2(x_3 - 1.3) \\ \dot{e_2} = -0.1y_2 + y_1y_3 - x_1x_3 - 1.3x_1 + 0.1x_2 \\ \quad - \sum_{i=1}^{3} C_{2j}e_j(t) - y_1y_3 + x_1x_3 + 1.3x_1 \\ \dot{e_3} = 0.001 - y_3 + y_1y_2 + 0.1y_3y_2 + (-1.3x_1x_2 + x_2^2 - 4x_3 + 5.2) + x_3 - y_1y_2 - 0.1y_3y_2 - 0.001 \\ \quad - \sum_{i=1}^{3} C_{ij}e_j(t)) - (-x_2(1.3x_1 - x_2) - 4(x_3 - 1.3)) \end{cases} \tag{3.27}$$

gives:

$$\begin{cases} \dot{e_1} = 0.7e_1 - \sum_{i=1}^{3} C_{ij}e_j(t) \\ \dot{e_2} = -0.1e_2 - \sum_{i=1}^{3} C_{ij}e_j(t) \\ \dot{e_3} = -e_3 - \sum_{i=1}^{3} C_{ij}e_j(t) \end{cases} \tag{3.28}$$

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{pmatrix} = \begin{pmatrix} 0.7 - C_{11} & -C_{12} & -C_{13} \\ -C_{21} & -0.1 - C_{22} & -C_{23} \\ -C_{31} & -C_{32} & -1 - C_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{3.29}$$

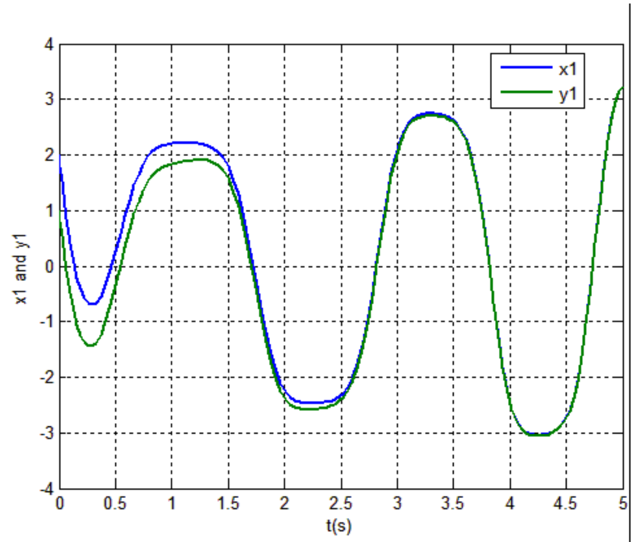
If the gain matrice is chosen as:

$$C = \begin{pmatrix} 1.7 & 0 & 0 \\ 0 & 1.9 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

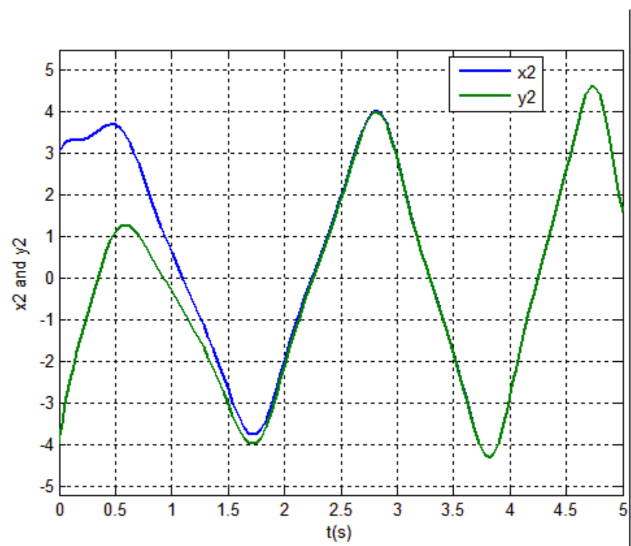
we get:

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{pmatrix} = \begin{pmatrix} 0.7 - C_{11} & -C_{12} & -C_{13} \\ -C_{21} & -0.1 - C_{22} & -C_{23} \\ -C_{31} & -C_{32} & -1 - C_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{3.30}$$

thus, $B - C$ is negative and all the eigenvalues $\lambda_i, i = 1, 2, 3.$ of $(B - C)$ stay in the left-half plane i.e., $Re(\lambda_i) < 0$, which ensures according with the Lyapunov stability theory [12], that errors system $(3, 30)$ is asymptotically stable i.e., $\lim_{t \to +\infty} \|e\| = 0$, $e \in \mathbb{R}^3$ with $e = y - x$. Hence the synchronization between the system $(3, 10)$ and the system $(3, 15)$ is achieved. In Figs. 3.1-3.4, the synchronization of the states of the master system $(3, 19)$ and slave system $(3, 20)$ is depicted, when the sub-control laws$(3, 25)$ are implemented.



$Figure 3.1$: Synchronization of the states x_1 and y_1



$Figure 3.2$: Synchronization of the states x_2 and y_2.

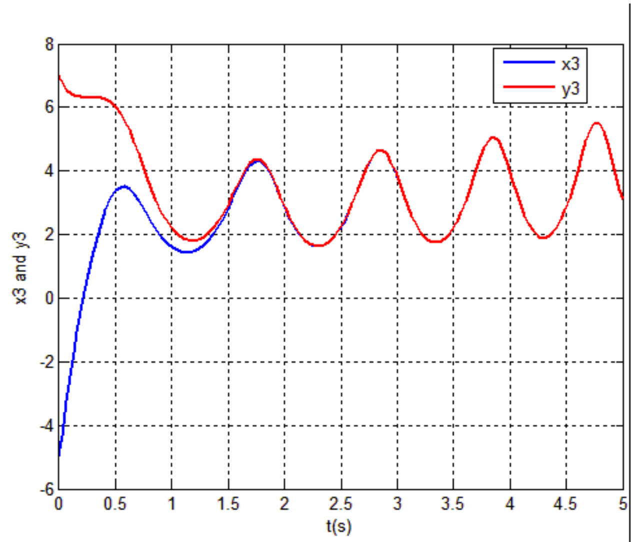


*Figure*3.3 : Synchronization of the states x_3 and y_3

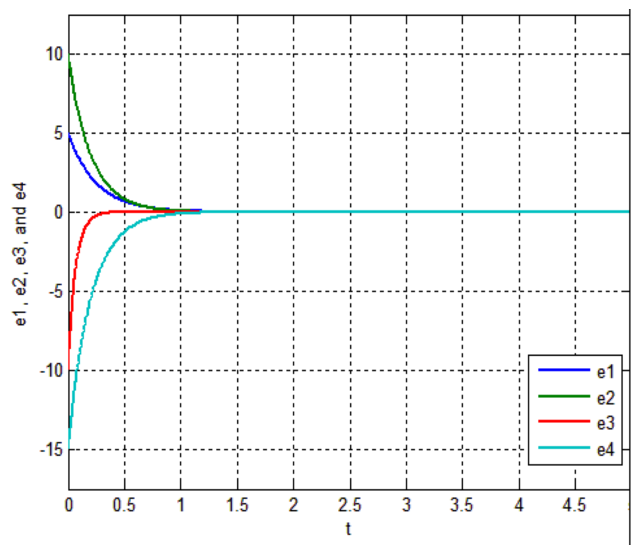


*Figure*3.4 : Time-History of the synchronization errors $e_1(t); e_2(t); e_3(t)$.

3.4 Exemple in 4D

Consider the following integer-order hyper choatic chen dynamical system dexrises the drive system :

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_4, \\ \dot{x_2} = dx_1 - x_1x_3 + cx_2, \\ \dot{x_3} = x_1x_2 - bx_3, \\ \dot{x_4} = x_2x_3 + 0.5x_4, \end{cases} \tag{3.31}$$

where : $x = (x_1, x_2, x_3, x_4)$ is the system state vector ; a,b,c,d,r are real parameters .it exhibits hyper chaotic attractor for : $a = 35; b = 3; c = 12, d = 7, r = 0.5.$

As reponse system : we consider the controlled fractional order hyper choatic lorenz dynamical system given by :

$$\begin{cases} D^{q_1}y_1 = \alpha(y_2 - y_1) + y_4 + u_1, \\ D^{q_2}y_2 = \gamma y_1 - y_2 - y_1y_3 + u_2, \\ D^{q_3}y_3 = y_1y_2 - \beta y_3 + u_3, \\ D^{q_4}y_4 = -y_2y_3 - \rho y_4 + u_4, \end{cases} \tag{3.32}$$

where: $y = (y_1, y_2, y_3, y_4)$ is the system state vector $\alpha, \beta, \gamma, \rho$ are real parameters, when: $\alpha = 10; \beta = \frac{8}{3}; \gamma = 28; \rho = 1$, $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$, and with fraction orders of the system: $q_1 = q_2 = q_3 = q_4 = 0.98$, it exhibit hyper -choatic attractor.

The response system can be rewritten as follows :

$$\begin{cases} D^{q_1}y_1 = 10(y_2 - y_1) + y_4 + v_1 + v_{11}, \\ D^{q_2}y_2 = 28y_1 - y_2 - y_1y_3 + v_2 + v_{22}, \\ D^{q_3}y_3 = y_1y_2 - \frac{8}{3}y_3 + v_3 + v_{33}, \\ D^{q_4}y_4 = -y_2y_3 - y_4 + v_4 + v_{44}. \end{cases} \tag{3.33}$$

According to the above method, the sub-controller v is given as follows:

$$\begin{cases} v_1 = (D^{q_1-1} - 1)(10(y_2 - y_1) + y_4), \\ v_2 = (D^{q_2-1} - 1)(28y_1 - y_2 - y_1y_3), \\ v_3 = (D^{q_3-1} - 1)(y_1y_2 - \frac{8}{3}y_3), \\ v_4 = (D^{q_4-1} - 1)(-y_2y_3 - y_4), \end{cases} \tag{3.34}$$

so we get

$$\begin{cases} D^{q_1}y_1 = 10(y_2 - y_1) + y_4 - 10(y_2 - y_1) - y_4 + D^{q_1-1}(10(y_2 - y_1) + y_4) + v_{11}, \\ D^{q_2}y_2 = 28y_1 - y_2 - y_1y_3 - 28y_1 + y_2 + y_1y_3 + D^{q_2-1}(28y_1 - y_2 - y_1y_3) + v_{22}, \\ D^{q_3}y_3 = y_1y_2 - \frac{8}{3}y_3 - y_1y_2 + \frac{8}{3}y_3 + D^{q_3-1}(y_1y_2 - \frac{8}{3}y_3) + v_{33}, \\ D^{q_4}y_4 = -y_2y_3 - y_4 + y_2y_3 + y_4 + D^{q_4-1}(-y_2y_3 - y_4) + v_{44}, \end{cases} \tag{3.35}$$

$$\begin{cases} D^{q_1}y_1 = D^{q_1-1}(10(y_{2-}y_1) + y_4) + v_{11}, \\ D^{q_2}y_2 = D^{q_2-1}(28y_1 - y_2 - y_1y_3) + v_{22}, \\ D^{q_3}y_3 = D^{q_3-1}(y_1y_2 - \frac{8}{3}y_3) + v_{33}, \\ D^{q_4}y_4 = D^{q_4-1}(-y_2y_3 - y_4) + v_{44}. \end{cases} \tag{3.36}$$

By applying the laplace transform , we get :

$$\begin{cases} \mathcal{L}\left\{D^{q_1}y_1\right\} = \mathcal{L}\left\{D^{q_1-1}(10(y_2 - y_1) + y_4)\right\} + \mathcal{L}\left\{v_{11}\right\}, \\ \mathcal{L}\left\{D^{q_2}y_2\right\} = \mathcal{L}\left\{D^{q_2-1}(28y_1 - y_2 - y_1y_3)\right\} + \mathcal{L}\left\{v_{22}\right\}, \\ \mathcal{L}\left\{D^{q_3}y_3\right\} = \mathcal{L}\left\{D^{q_3-1}(y_1y_2 - \frac{8}{3}y_3)\right\} + \mathcal{L}\left\{v_{33}\right\}, \\ \mathcal{L}\left\{D^{q_4}y_4\right\} = \mathcal{L}\left\{D^{q_4-1}(-y_2y_3 - y_4)\right\} + \mathcal{L}\left\{v_{44}\right\}, \end{cases} \tag{3.37}$$

$$\begin{cases} S^{q_1}y_1(s) - S^{q_1-1}y_1(0) = S^{q_1-1}\mathcal{L}(10(y_2 - y_1) + y_4) + \mathcal{L}\left\{v_{11}\right\}, \\ S^{q_2}y_2(s) - S^{q_2-1}y_2(0) = S^{q_2-1}\mathcal{L}(28y_1 - y_2 - y_1y_3) + \mathcal{L}\left\{v_{22}\right\}, \\ S^{q_3}y_3(s) - S^{q_3-1}y_3(0) = S^{q_3-1}\mathcal{L}(y_1y_2 - \frac{8}{3}y_3) + \mathcal{L}\left\{v_{33}\right\}, \\ S^{q_4}y_4(s) - S^{q_4-1}y_4(0) = S^{q_4-1}\mathcal{L}(-y_2y_3 - y_4) + \mathcal{L}\left\{v_{44}\right\}. \end{cases} \tag{3.38}$$

Multiply both sides by : S^{-q_i+1}, $i = \overline{1,4}$:

$$\begin{cases} S^{-q_1+1} \cdot S^{q_1}y_1(s) - S^{-q_1+1} \cdot S^{q_1-1}y_1(0) = S^{-q_1+1} \cdot S^{q_1-1} \cdot \mathcal{L}(10(y_{2-}y_1) + y_4) + S^{-q_1+1} \cdot \mathcal{L}(v_{11}) \\ S^{-q_2+1} \cdot S^{q_2}y_2(s) - S^{-q_2+1} \cdot S^{q_2-1} \cdot y_2(0) = S^{-q_2+1} \cdot S^{q_2-1} \cdot \mathcal{L}(28y_1 - y_2 - y_1y_3) + S^{-q_2+1} \cdot \mathcal{L}(v_{22}) \\ S^{-q_3+1} \cdot S^{q_3} \cdot y_3(s) - S^{-q_3+1} \cdot S^{q_3-1} \cdot y_3(0) = S^{-q_3+1} \cdot S^{q_3-1} \cdot \mathcal{L}(y_1y_2 - \frac{8}{3}y_3) + S^{-q_3+1} \cdot \mathcal{L}(v_{33}) \\ S^{-q_4+1} \cdot S^{q_4} \cdot y_4(s) - S^{-q_4+1} \cdot S^{q_4-1}y_4(0) = S^{-q_4+1} \cdot S^{q_4-1} \cdot \mathcal{L}(-y_2y_3 - y_4) + \mathcal{L}(v_{44}) \end{cases} \tag{3.39}$$

By applying the inverse laplace transform , we get :

$$\begin{cases} \mathcal{L}^{-1}\left\{S \cdot y_1(s) - y_1(0)\right\} = \mathcal{L}^{-1}\mathcal{L}(10(y_2 - y_1) + y_4) + \mathcal{L}^{-1}(S^{-q_1+1} \cdot \mathcal{L}(v_{11})), \\ \mathcal{L}^{-1}\left\{S \cdot y_2(s) - y_2(0)\right\} = \mathcal{L}^{-1}\mathcal{L}(28y_1 - y_2 - y_1y_3) + \mathcal{L}\left\{S^{-q_2+1} \cdot \mathcal{L}(v_{22})\right\}, \\ \mathcal{L}^{-1}\left\{S \cdot y_3(s) - y_3(0)\right\} = \mathcal{L}^{-1}\mathcal{L}(y_1y_2 - \frac{8}{3}y_3) + \mathcal{L}^{-1}\left\{S^{-q_3+1}\mathcal{L}(v_{33})\right\}, \\ \mathcal{L}^{-1}\left\{S \cdot y_4(s) - y_4(0)\right\} = \mathcal{L}^{-1}\mathcal{L}(-y_2y_3 - y_4) + \mathcal{L}^{-1}\left\{S^{-q_4+1} \cdot \mathcal{L}(v_{44})\right\}, \end{cases} \tag{3.40}$$

so:

$$\begin{cases} D_t(y_1(t)) = 10(y_2 - y_1) + y_4 + D^{1-q_1}(v_{11}), \\ D_t(y_2(t)) = 28y_1 - y_2 - y_1y_3 + D^{1-q_2}(v_{22}), \\ D_t(y_3(t)) = y_1y_2 - \frac{8}{3}y_3 + D^{1-q_3}(v_{33}), \\ D_t(y_4(t)) = -y_2y_3 - y_4 + D^{1-q_4}(v_{44}). \end{cases} \tag{3.41}$$

The synchronization problem between $(3,31)$ and $(3,32)$ turns into another problem between $(3,31)$ and $(3,41)$:

The state errors for $(3,31)$ and $(3,41)$ is:

$$e_i = y_i - x_i, i = \overline{1.4} \tag{3.42}$$

gives

$$\begin{cases} \dot{e_1} = 10y_2 - 10y_1 + y_4 + D^{1-q_1}(v_{11}) - 35(x_2 - x_1) - x_4, \\ \dot{e_2} = 28y_1 - y_2 - y_1y_3 + D^{1-q_2}(v_{22}) - 7x_1 + x_1x_3 - 12x_2, \\ \dot{e_3} = y_1y_2 - \frac{8}{3}y_3 + D^{1-q_3}(v_{33}) - x_1x_2 + 3x_3, \\ \dot{e_4} = -y_2y_3 - y_4 + D^{1-q_4}(v_{44}) - x_2x_3 - 0.5x_4. \end{cases} \tag{3.43}$$

we choose $v_{ii}, i = \overline{1.4}$ as follows:

$$\begin{cases} v_{11} = D^{q_1-1}((-35x_1 + 35x_2 + x_4) - g_1 - (-10x_1 + 10x_2 + x_4) - \sum_{j=1}^{4} c_{1j}e_j(t)) \\ \quad = D^{q_1-1}(-25x_1 + 25x_2 - \sum_{j=1}^{4} c_{1j}e_j(t)), \\ v_{22} = D^{q_2-1}((7x_1 - x_1x_3 + 12x_2) - (28x_1 - x_2 - (x_1x_3) + (y_1y_3) - \sum_{j=1}^{4} c_{2j}e_j(t) \\ \quad = D^{q_2-1}(-21x_1 + 13x_2 + (y_1y_2 - x_1x_3) - \sum_{j=1}^{4} c_{2j}e_j(t)), \\ v_{33} = D^{q_3-1}((x_1x_2 - 3x_3) - ((x_1x_2) - \frac{8}{3}x_3) - \sum_{j=1}^{4} c_{3j}e_j(t) - y_1y_2) \\ \quad = D^{q_3-1}(-\frac{1}{3}x_3 + (x_1x_2) - \sum_{j=1}^{4} c_{3j}e_j(t) - y_1y_2), \\ v_{44} = D^{q_4-1}((x_2x_3 + 0.5x_4) - ((-x_2x_3 - x_4) - \sum_{j=1}^{4} c_{4j}e_j(t) + (y_2y_3)) \\ \quad = D^{q_4-1}(x_2x_3 + 1.5x_4 - \sum_{j=1}^{4} c_{4j}e_j(t) + y_2y_3), \end{cases} \tag{3.44}$$

so the response system$(3,41)$ is equivalent to the integer-order described by replace (v_{ii}) in

$(3,43)$:

$$\begin{cases} D_t(y_1(t)) = 10(y_2 - y_1) + y_4 + (-25x_1 + 25x_2 - \sum_{j=1}^{4} c_{1j}e_j(t)), \\ D_t(y_2(t)) = 28y_1 - y_2 - y_1y_3 + (-21x_1 + 13x_2 + (y_1y_3 - x_1x_3) - \sum_{j=1}^{4} c_{2j}ej(t)), \\ Dt(y_3(t)) = y_1y_2 - y_4 + (-\frac{x_3}{3} - (y_1y_2 - x_1x_2) - \sum_{j=1}^{4} c_{3j}e_j(t)), \\ D_t(y_4(t)) = -y_2y_3 - y_4 + (x_2x_3 + 1.5x_4 + (y_2y_3) - \sum_{j=1}^{4} c_{4j}e_j(t)). \end{cases} \tag{3.45}$$

The error system is given by :

$$\begin{cases} \dot{e_1} = -10(y_1 - x_1) + 10(y_2 - x_2) + (y_4 - x_4) - \sum_{j=1}^{4} c_{1j}e_j(t) \\ \dot{e_2} = 28(y_1 - x_1) - (y_2 - x_2) - \sum_{j=1}^{4} c_{2j}e_j(t) \\ \dot{e_3} = -\frac{8}{3}(y_3 - x_3) - \sum_{j=1}^{4} c_{3j}e_j(t) \\ \dot{e_4} = -(y_4 - x_4) - \sum_{j=1}^{4} c_{4j}e_j(t) \end{cases} \tag{3.46}$$

i.e.,

$$\begin{cases} \dot{e_1} = -10e_1 + 10e_2 + e_4 - \sum_{j=1}^{4} c_{1j}e_j(t) \\ \dot{e_2} = 28e_1 - e_2 - \sum_{j=1}^{4} c_{2j}e_j(t) \\ \dot{e_3} = -\frac{8}{3}e_3 - \sum_{j=1}^{4} c_{3j}e_j(t) \\ \dot{e_4} = -e_4 - \sum_{j=1}^{4} c_{4j}e_j(t) \end{cases} \tag{3.47}$$

i.e.,

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \\ \dot{e_4} \end{pmatrix} = \begin{pmatrix} -10 - c_{11} & 10 - c_{12} & -c_{13} & 1 - c_{14} \\ 28 - c_{21} & -1 - c_{22} & -c_{23} & -c_{24} \\ -c_{31} & -c_{32} & -\frac{8}{3} - c_{33} & -c_{34} \\ -c_{41} & -c_{42} & -c_{43} & -1 - c_{44} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{3.48}$$

If the gain matrise is chosen as :

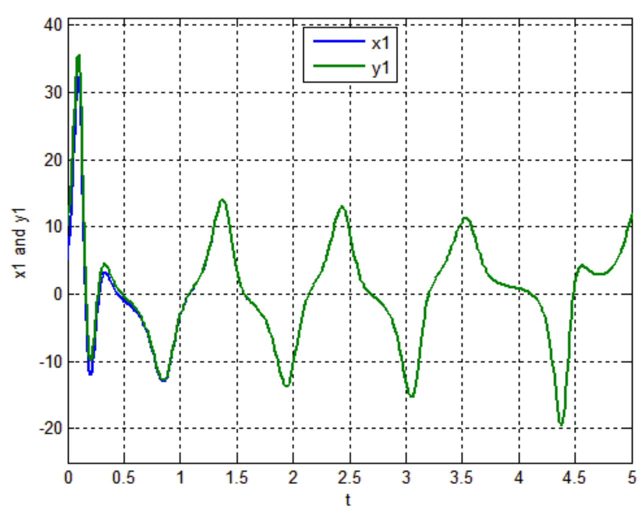
$$c = \begin{pmatrix} -6 & 10 & 0 & 1 \\ 28 & 4 & 0 & 0 \\ 0 & 0 & \frac{37}{3} & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{3.49}$$

Then (3.45); become in the form:

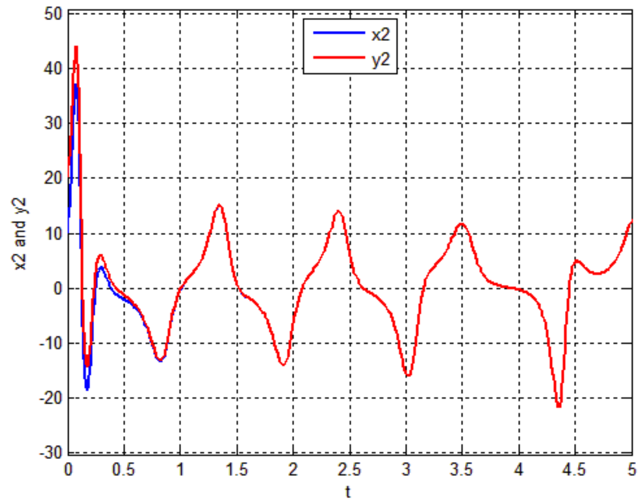
$$\begin{cases} D_t(y_1(t)) = 10(y_2 - y_1) + y_4 + (-25x_1 + 25x_2 + 6e_1 - 10e_2 - e_4). \\ D_t(y_2(t)) = 28y_1 - y_2 - y_1y_3 + (-21x_1 + 13x_2 - 28e_1 + y_1y_3 - x_1x_3 - 4e_2). \\ D_t(y_3(t)) = y_1y_2 - y_4 + (-\frac{x_3}{3} - \frac{37}{3}e_3 - y_1y_2 + x_1x_2). \\ D_t(y_4(t)) = -y_2y_3 - y_4 + (x_2x_3 + y_2y_3 + 1.5x_4 - 4e_4). \end{cases} \tag{3.50}$$

then the given eignvalues of the matrice $(B - C)$ are given by : $\lambda_1 = -4; \lambda_2 = -5; \lambda_3 = -15; \lambda_4 = -5$.

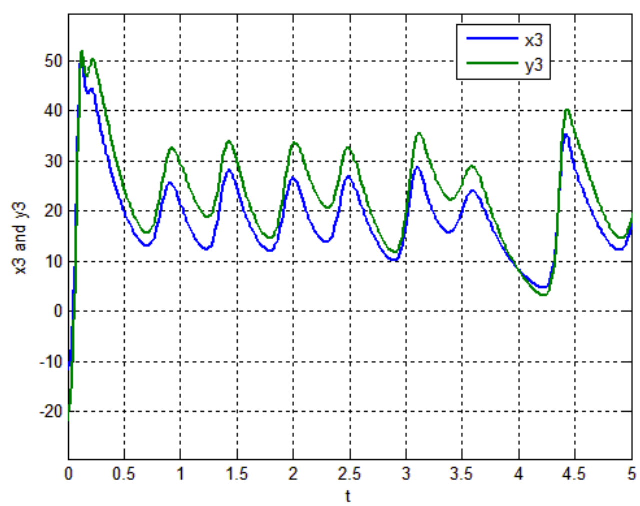
thus, $B - C$ is negative and all the eigenvalues $\lambda_i, i = 1, 2, 3, 4$. which ensures according with the Lyapunov stability theory, that zero solotion of errors system $(3, 48)$ is asymptotically stable i.e., $\lim_{t \to +\infty} \|e\| = 0$, $e \in \mathbb{R}^4$ with $e = y - x$. Hence the synchronization between the system $(3, 31)$ and the system $(3, 32)$ is achieved.



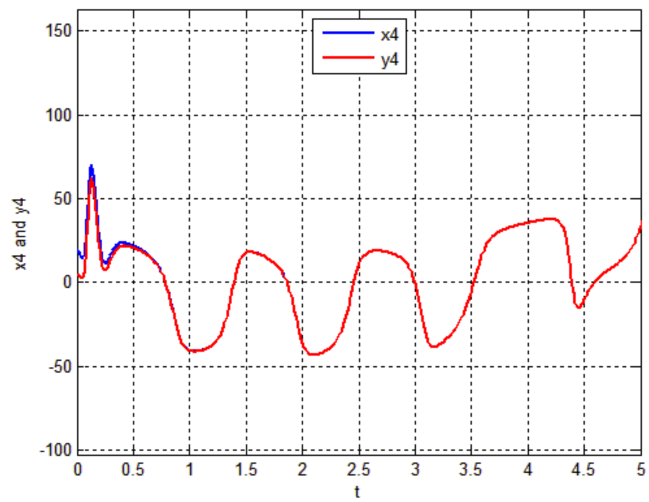
$Figure 3.5$:Synchronization of the states x_1 and y_1



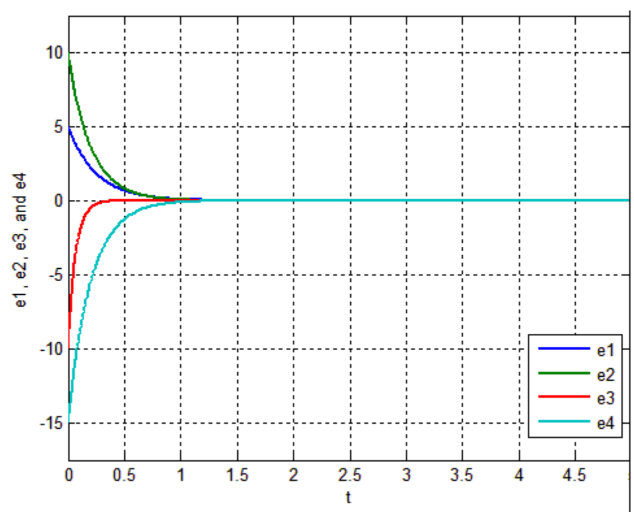
$Figure 3.6$:Synchronization of the states x_2 and y_2



$Figure 3.7$: Synchronization of the states x_3 and y_3



$Figure 3.8$: Synchronization of the states x_4 and y_4



$Figure 3.9$:Time-History of the synchronization errors $e_1(t); e_2(t); e_3(t); e_4(t)$.

3.5 Conclusion

In this chapter, we obtained suitable sub-controllers to achieve synchronization between the fractional-order chaotic system and the chaotic system of integer order by using stability criteria of the integer-order linear system with simulation results to demonstrate the effectiveness of the proposed scheme.

General conclusion

Chaos has been researched in science, mathematics, engineering, and a variety of other fields as a significant nonlinear phenomena. Because of its potential uses in a variety of industries, synchronization of chaotic systems has been a hot topic of research. Many synchronization solutions for chaotic systems have been developed in recent years.OYG method , backstepping design method, sliding mode control, passive control, nonlinear active control, projective synchronization, projective function synchronization, global synchronization,.. etc, were introduced and applied to chaotic and hyper-chaotic systems.

This work investigates the synchronization between fractional-order (chaotic, hyperchaotic) systems and integer-order (chaotic, hyperchaotic) systems. Based on the idea of the decomposition of the controller in the response system in two sub-controllers and the stability theory of the linear integer-order system, we design the effective controller to realize the synchronization between fractional-order and integer-order systems with two given examples in 3d and 4d.

This work is devided into 3 chapers, the first Chapter, contains some definitions and prelimanaries about: dynamical systems, Chaos, Choatic systems, synchronization and types of synchronization and some definitions and properties of fractional derivative with numerical method for solving fractional differential equations.

The second chapter, introduced some examples of integer orders and fractional orders choatic systems. The third chapter, present the study of the synchronization between two 3D and 4D fractional-integer orders chaotic systems. Finally, numerical simulations are given to demonstrate the effectiveness of the proposed method with numerical simulation.

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