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About stability analysis in epidemic reaction-diffusion model

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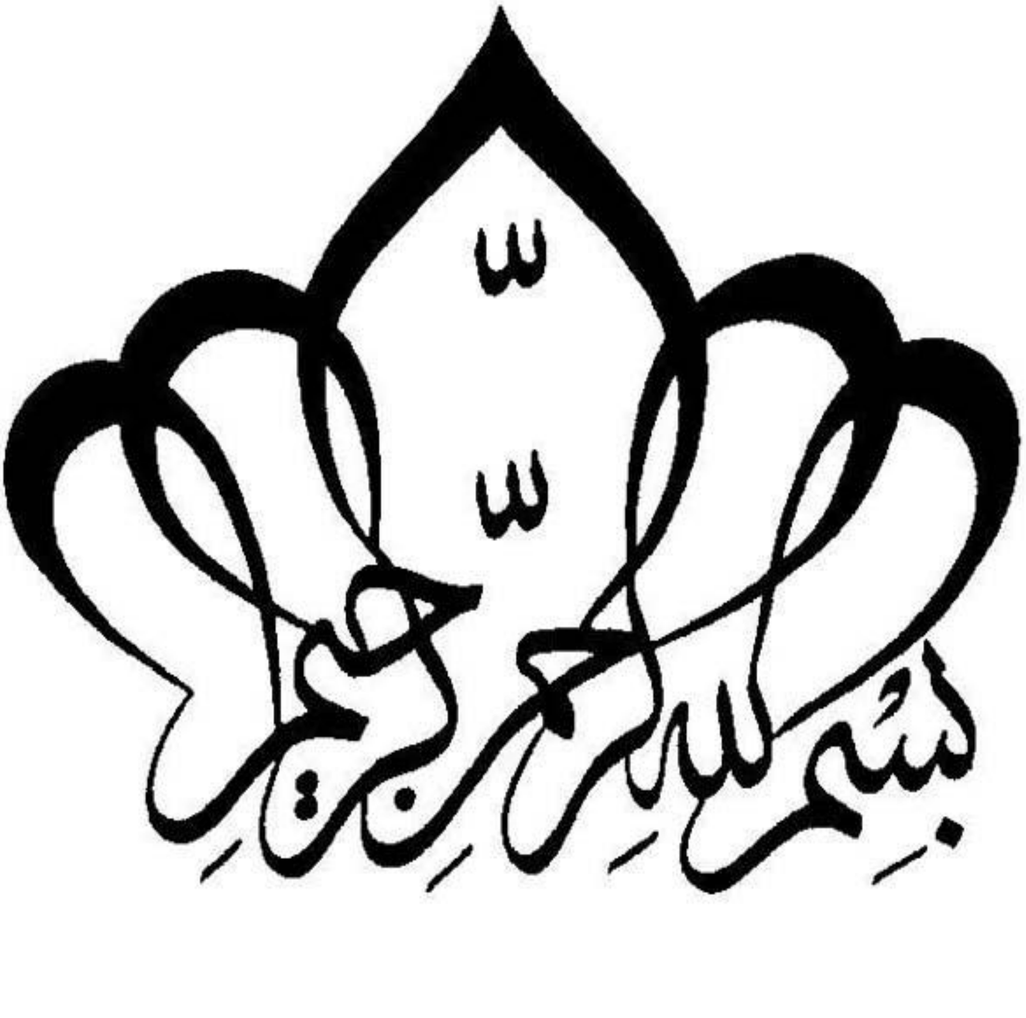
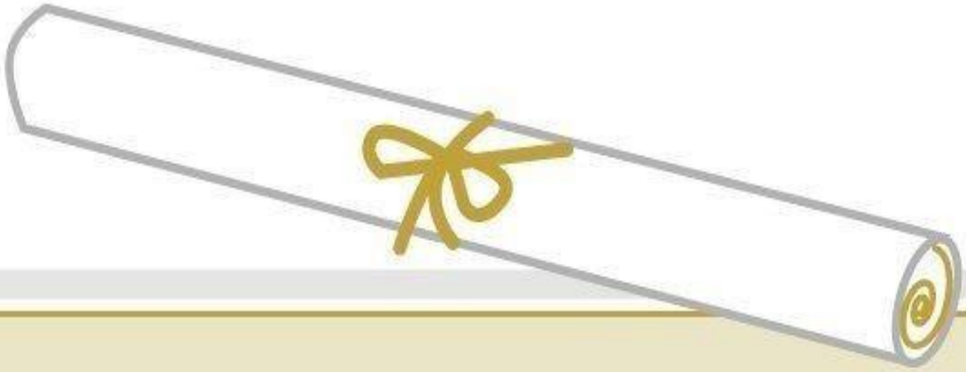

بسم الله الرحمن الرحيم

شكر و عرفان

نحمد الله عز وجل الذي وفقنا في إتمام هذا البحث العلمي و الذي ألهمنا الصحة والعافية والعزيمة، فالحمد لله حمدا كثيرا و الشكر له سبحانه أولا وأخيرا على امتنانه فهو القائل: { لئن شكرتم لأزيدنكم }

وقال الرسول صلى الله عليه وسلم: {من لا يشكر الناس لا يشكر الله }.

نتقدم بجزيل الشكر و التقدير إلى الأستاذ المشرف"سالم عبد المالك" على ما قدمه لنا من توجيهات و معلومات قيمة ساهمت في إثراء موضوع دراستنا من جوانب مختلفة. كما نتقدم بجزيل الشكر إلى أعضاء اللجنة المناقشة الموقرة ونرفع كلمة الشكر إلى كل معلم أفادنا بعلمه من أولى المراحل الدراسية حتى هذه اللحظة. كما نشكر كل من مد لنا يد العون من بعيد أو قريب ونشكر كل أساتذة وعمال القسم.



إهداء

مذكرتنا هذه ثمرة الجهد و النجاح بفضله تعالى، مهداة

إلى أعز الناس وأقربهم ألا وهما الوالدين الكريمين حفظهما

الله وأدامهما نورا لدربنا، اللذان كانا عونا وسندا لنا طيلة مشوارنا

الدراسي، و كان لدعائهما المبارك أعظم أثر

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نسأل الله العلي القدير أن ينفعنا بهذا العمل المتواضع ويمدنا

بتوفيقه.

Abstract

The purpose of this thesis is to study the local and global asymptotic stability of the nonnegative constant steady states of an epidemic reaction-diffusion system (susceptible-infectious) with a nonlinear incidence in the case of ordinary and partial differential equations depending on the basic reproduction number, with determining the linearity of the studied system in both cases . Where the local asymptotic stability is determined by the nature of the eigenvalues, but for the global asymptotic stability we use the Lyapunov method, in addition to illustrate the analytical results through numerical examples.

Key words: The reaction-diffusion system, nonlinear incidence, global and local asymptotic stability, equilibrium, basic reproduction number, Lyapunov function, linearization.

ملخص

الهدف من هذه المذكرة هو دراسة الاستقرار المقارب المحلي والكلي لحالات الثبات غير السالبة لنظام انتشار رد الفعل الوبائي (معرض للإصابة-مصاب) مع الحدوث غير الخطي في حالة المعادلات التفاضلية العادية و الجزئية اعتمادا على رقم التكاثر الأساسي مع تحديد خطية النظام المدروس في كلتا الحالتين حيث يتم تعيين الاستقرار المقارب المحلي من خلال طبيعة القيم الذاتية، لكن بالنسبة للاستقرار المقارب الكلي نستعمل طريقة ليابونوف، بالإضافة إلى توضيح النتائج التحليلية عن طريق الأمثلة العددية.

<u>الكلمات المفتاحية:</u> نموذج التفاعل والانتشار، حدوث غير خطي، الاستقرار المقارب الكلي والمحلي، التوازن، رقم الاستنساخ، دالة ليابونوف، التوسيع الخطي.

Résumé

Le but de cette thèse est d'étudier la stabilité asymptotique locale et globale de la non négative états stationnaires constants d'un système de réaction-diffusion épidémique (susceptible-infectieux) avec incidence non linéaire dans le cas d'équations aux dérivées ordinaires ou partielles dépendant du nombre de reproduction de base, avec détermination de la linéarité du système étudié dans les deux cas où la stabilité asymptotique locale est déterminé par la nature des valeurs propres, mais pour la stabilité asymptotique globale nous utilisons la méthode de Lyapunov, en plus d'illustrer les résultats analytiques par des exemples numériques.

Mots clés: Système de réaction-diffusion, incidence non linéaire, stabilité asymptotique globale et locale, équilibre, nombre de reproduction de base, fonction de Lyapunov, linéarisation.

Contents

Notations

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{\geq 0}$	Set of positive real numbers.
$\mathbb{R}^n$	Vectorial space of dimension of dimension $n \geq 1$.
$\mathbb{R}^2_{\geq 0}$	Set of positive real numbers of dimension $n = 2$.
Ω	Open bounded subset of $\mathbb{R}^n$.
$\overline{\Omega}$	The closing of Ω.
$\partial\Omega$	Smooth boundary.
$C\left(\overline{\Omega}\right)$	Continuous set of functions on $\overline{\Omega}$.
$C^1(\Omega, \mathbb{R})$	Space of the functions continuously differentiable to ordre 1 on Ω in $\mathbb{R}$.
$H^1(\Omega), H^2(\Omega)$	Sobolev spaces.
$L^2(\Omega)$	The space of square integrable functions on Ω.
$\frac{\partial u(x,t)}{\partial t}$	Partial derivative of u with respect to time t.
Δ	The laplacian operator.
∇	Gradient operator.
$\mathcal{L}$	The linearizing operator.
$\|\cdot\|_2$	The norm in L^2.
$\|\cdot\|_p$	The norm in L^p.
$\langle\ ,\ \rangle$	The euclidean scalar product.
R_0	The basic reproduction number.
$J(u, v)$	The jacobian matrix.
tr	Trace of matrix.
$\det$	Determinant of matrix.
ODEs	Ordinary differential equations.
PDEs	Partial differential equations.

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General Introduction

Throughout history, many epidemics have had major implications for human society, from killing large proportions of the world's population to making humans think of a solution to reduce them, mathematical modelling has been the way to do so, by modeling problems and analyzing them mathematically using non-linear differential equation systems.

The models have multiplied and become very widely used, but one of the most important of them is epidemiological modelling or so-called infectious disease modelling and is an important tool to improve our understanding of how a disease spreads and make an approximate guess by predicting its future course for the purpose of mitigating its effects, where scientists use a combination of mathematics and data to apply this modeling.

Daniel Bernoulli was the first to create a model to defend the practice of radical vaccination in 1760. In the 20th century, William Hammer and Ronald Ross introduced the Mass Action Act to explain epidemiological behavior. The 1920s saw the emergence of fragmented models, the most important of which was the Kermack-Mckendrick model (1927), which succeeded in predicting the behaviour of outbreaks in a very similar way to the behavior observed in many recorded epidemics, where it was considered a fixed set of only three sections: susceptible S(t), infected I(t) and recovery R(t), this model is known as SIR, as well as other models including: SIS, SERS, SI and so on.

This thesis aims to highlight the study of the local and global asymptotic stability of an epidemic reaction-diffusion SI (susceptible-infectious) model with a nonlinear incidence.

The subject of this study was divided into two sections, an analytical (theoretical) and an application section (numerical).

The theoretical has two chapters which are as follows:

Chapter one: Giving the most general form of the system of reaction-diffusion with initial concepts, theories and definitions related to local and total stability.

Chapter two: Customizing the study of the reaction-diffusion SI epidemic model with a non-

linear incidence for the global and local approach stability of both ODEs and PDEs.
The practical aspect includes:
chapter three: it confirms the analytical results by examples of data that are accompanied by numerical analysis and calculations.

Chapter 1

Stability theory

1.1 Introduction

Reaction-diffusion systems of partial differential equations play an important role in modeling real-life applications, which attracted the interest of scientists, including the scientist Alan Turing in 1952.

Reaction-diffusion systems basically represent the change in space and time of certain physical quantities as a result of two phenomena. The first phenomenon is reaction, which denotes the transformation of one quantity to another, while the second is diffusion and corresponds to the spatial spreading of the quantities.

The most general form of a reaction-diffusion system may be given by

$$\frac{\partial}{\partial t}U\left(t,x\right) = D\Delta U\left(t,x\right) + F(U\left(t,x\right)) \;\; x \in \Omega, \; t \geq 0, \tag{1.1}$$

where

$$U(t,x) = (u_1(t,x), u_2(t,x), ..., u_n(t,x))^T \tag{1.2}$$

is the unknown vector function, D is an $n \times n$ matrix of diffusion coefficients, and

$$F(u) = (f_1(u), f_2(u), ..., f_n(u))^T \tag{1.3}$$

is a functional representing the interaction.

In this chapter, we will present some preliminary concepts, with a mention of theories and definitions related to the local and global asymptotic stability of this system.

1.2 Preliminary Concepts

1.2.1 The space L^p

Definition 1.1 *Let Ω be a domain in $\mathbb{R}^n$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable function U defined on Ω for wich*

$$\int_\Omega |U(t,x)|^p \, dx < \infty.$$

If $U \in L^p(\Omega)$, we define its norm

$$\|U(t,x)\|_p = \left(\int_\Omega |U(t,x)|^p \, dx \right)^{\frac{1}{p}}.$$

Corollary 1.1 *$L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle U, U \rangle_{L^2} = \|U\|_{L^2}^2 = \int_\Omega U^2 dx.$$

1.2.2 Sobolev space

Definition 1.2 *The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$, the weak derivatives $D^\alpha u$ exists $D^\alpha u \in L^p(\Omega)$.Thus*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega), \; D^\alpha u \in L^p(\Omega), \; |\alpha| \leq k\}.$$

Definition 1.3 *We call Sobolev space of order 1 on Ω the space*

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{ v \in L^2(\Omega), \; \partial_{x_i} v \in L^2(\Omega), \; 1 \leq i \leq d \right\}.$$

1.2.3 Equilibrium point

Definition 1.4 ([13]) *A point $U^* \in \mathbb{R}^n$ is called an equilibrium point of (1.1) if*

$$F(U^*) = 0.$$

1.2.4 Stability and asymptotic stability

Definition 1.5 ([16]) *U^* is said to be stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|U(0,x) - U^*\| < \delta$, then $\|U(t,x) - U^*\| < \varepsilon$ for all $t \geq 0$.*

- U^* is unstable if it is not stable.

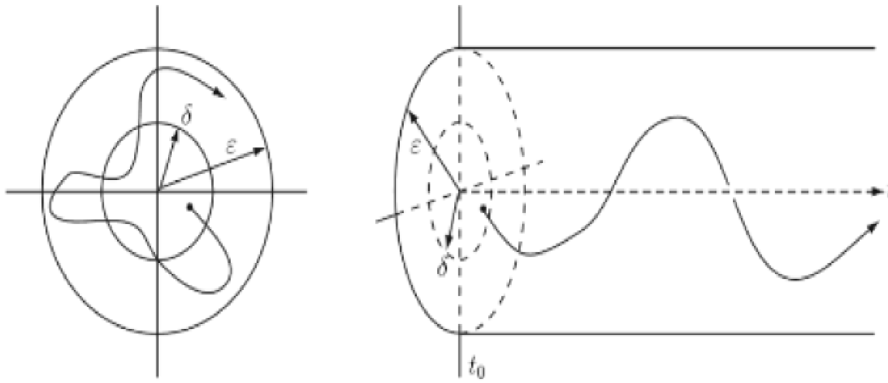

Figure 1.1: Stability of an equilibrium point.

Definition 1.6 ([16]) *U^* is said to be asymptotic stable if it is stable and there esists $\delta > 0$ such that whenever $\|U(0,x) - U^*\| < \delta$ then $\|U(t,x) - U^*\| \longrightarrow 0$ as $t \longrightarrow \infty$.*

In our thesis we will study two type of stability: local asymptotic stability and global asymptotic stability.

Local asymptotic stability [16]

The local asymptotic stability of a model at U^* is that the solution of the system must approach an equilibrium point under initial condition close to the equilibrium point; i.e. at U^* if there is a $\delta > 0$ such that $\|U(t,x) - U^*\| < \delta$ that implies $U(t,x) \longrightarrow U^*$ as $t \longrightarrow \infty$.

global asymptotic stability [16]

The global asymptotic stability of a model at U^* is that the solution of the system must approach to the equilibrium point under all initial condition; i.e. for every $U(t,x)$, we have $U(t,x) \longrightarrow U^*$ as $t \longrightarrow \infty$.

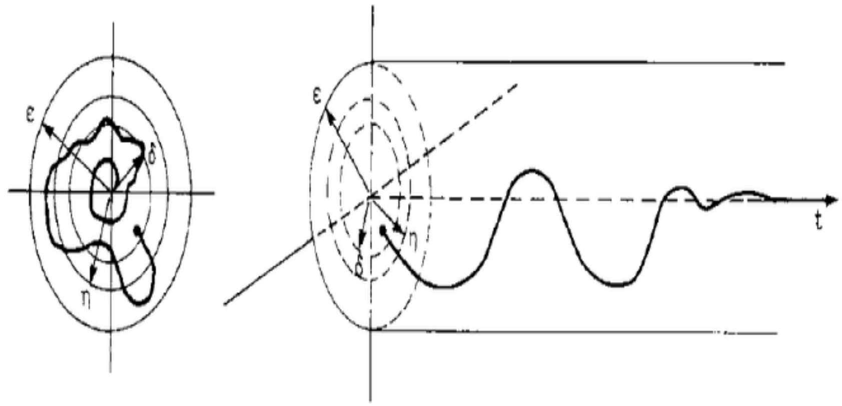

Figure 1.2: Asymptotic stability of an equilibrium point.

1.2.5 Non-negativity of solutions

Definition 1.7 ([8]) *Let $F : I \subseteq \overline{\mathbb{R}}_+^n \longrightarrow \mathbb{R}^n$.*
Then F is essentially nonnegative if $f_i(U) \geqslant 0$, for all $i = 1...n$ and $U \in \overline{\mathbb{R}}_+^n$ such that $u_i = 0$, where u_i denotes the i^{th} component of U.

Proposition 1.1 ([11]) *Suppose $I \subset \overline{\mathbb{R}}_+^n$. Then $\overline{\mathbb{R}}_+^n$ is an invariant set with respect to ODEs system if and only if F is essentially nonnegative.*

1.2.6 Intermediate value theorem

Theorem 1.1 ([14]) *Let $h(x)$ be a real-valued function which is continuous on the closed interval $[a, b]$. If k is any number between $h(a)$ and $h(b)$, then there exists at least one number $c \in [a, b]$ such that $h(c) = k$.*

The Intermediate value theorem can be used to determine whether there exists a solution to the equation $h(x) = k$ when $h(x)$ is a continuous function on a closed interval $[a, b]$.

Corollary 1.2 ([14]) *Let h be a real-valued function which is continuous on the closed interval $[a, b]$. If $h(a) \times h(b) < 0$, then there exists at least one number $c \in [a, b]$ such that $h(c) = 0$.*

Remark 1.1 ([12]) *if the function h is strictly monotonic and continuous on $[a, b]$ (i.e. strictly increasing or strictly decreasing) then the equation $h(x) = k$, has a unique solution.*

1.2.7 Eigenfunction

In mathematics, an *eigenfunction* of a linear operator P defined on some function space is any non-zero function Φ in that space that, when acted upon by P, is only multiplied by some scaling factor called an eigenvalue. As an equation, this condition can be written as

$$P\Phi = \lambda\Phi.$$

for some scalar eigenvalue λ. The solutions to this equation may also be subject to boundary conditions that limit the allowable eigenvalues and eigenfunctions.

1.2.8 Gronwall's Inequality

Theorem 1.2 ([10]) *Let $N(t)$ be a continuous nonnegative function such that*

$$N(t) \leq \alpha + \int_{t_0}^{t} (\beta N(s) + \gamma)\, ds, \text{ on } t \geq t_0,$$

where $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$. Then for $t \geq t_0$, $N(t)$ satisfies

$$N(t) \leq \alpha \exp(\beta (t - t_0)) + \frac{\gamma}{\beta}(\exp(\beta (t - t_0)) - 1).$$

1.2.9 Green Formula

[9] Let u, v are two function such that $u \in \boldsymbol{H}^2(\Omega)$ and $v \in \boldsymbol{H}^1(\Omega)$ then we have

$$\int_{\Omega} \Delta u v = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds - \int_{\Omega} \nabla u \nabla v dx.$$

1.3 Theories of local stability in case ODEs

In order to know the basic theories for the stability of the system ODEs, first we omitting the Laplacian operator Δ and setting the time derivative.
In general, system (1.1) can made up of two-component expressed in the following form

$$\frac{dU}{dt} = F(U), \tag{1.4}$$

where $F(U) = (f(u,v), g(u,v))^T$ (here, we changing notation).
We assume that the system (1.4) has as its equilibrium the point $(u^*, v^*) = (0,0)$, we get the linearity of this system at (u^*, v^*)

$$\frac{dU}{dt} = AU, \tag{1.5}$$

where

$$A = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix} |_{u=u^*,v=v^*} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } U = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.6}$$

and that If A is nonsingular. We are going to study the qualitative properties of the solutions to system (1.5), in particular their asymptotic behavior as $t \longrightarrow +\infty$. To know the asymptotic behavior we denote by λ_1 and λ_2 the eigenvalues of A, where this dependency is summarized in the following table [1]

Eigenvalues	Equilibrium
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1, \lambda_2 < 0$	Asymptotically stable node
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$	Unstable node
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 \cdot \lambda_2 < 0$	Unstable saddle
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha < 0$	Asymptotically stable node
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha > 0$	Unstable focus
$\lambda_{1,2} = \pm i\beta$,	Stable center

Table 1.1: The asymptotic behavior of solutions to the linear 2-component system (1.5) based on the nature of the eigenvalues of A

The first stability case, from the table 1.1, a linear system (1.5) is asymptotically stable if the real parts of the eigenvalues of A are negative. If at least one eigenvalue is positive or has a positive real part, then system (1.5) is unstable at $(u^*, v^*) = (0, 0)$.
The second stability case, the asymptotically stable node, can be guaranteed in the following theory

Theorem 1.3 ([4]) *The system* (1.5) *is locally asymptotically stable at the equilibrium* (u^*, v^*) *if and only if the trace of* A *is negative and its determinant is positive, i.e.*

$$\begin{cases} tr\,(J) = a_{11} + a_{22} < 0, \\ \det\,(J) = a_{11}a_{22} - a_{12}a_{21} > 0. \end{cases}$$

Definition 1.8 ([7]) *A subset* $D \subseteq \Omega$ *is an invariant set relative to* (1.4) *if* D *contains the orbits of all its points.*

Definition 1.9 ([7]) ***(A Positively Invariant Set)***

A positively invariant set is a set with the following properties: Given a dynamical system (1.4) *and trajectory* $U(t, U_0)$ *where* U_0 *is the initial point. Let* $D = \{U \in \mathbb{R}^n; N(U) = 0\}$ *where* N *is a real valued function. The set* D *is said to be positively invariant if* $U_0 \in D$ *implies that* $U(t, U_0) \in D$ *for all times* $t \geq 0$. *In other words, a solution that starts in* D *remains in* D *for all times* $t \geq 0$.

Definition 1.10 ([7]) ***(The Region of Attraction of the Equilibrium)***
Assume that $U = U^*$ *is an equilibrium point of* (1.4) *and let* N *be the solution of the system. The set*

$$D = \left\{ \zeta \in \Omega \diagup \lim_{t \longrightarrow \infty} \sup N(t, \zeta) = U^* \right\}$$

is called the region of attraction of the equilibrium U^*.

1.4 Theories of local stability in case PDEs

One of the common methods for studying the local asymptotic stability of the PDEs system is **the eigenfunction expansion method** [5]. It is important to recall some of the theory related to the eigenvalues of the Laplace operator.

1.4.1 Properties of the Eigenvalues of the Laplace Operator

Let us denote these eigenvalues by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \nearrow^{+\infty}$ and the corresponding normalized eigenfunctions in Ω by $\Phi_0, \cdots, \Phi_k, \cdots$. We assume Neumann boundary conditions. These eigenvalues and eigenfunctions satisfy the eigenvalue problem

$$-\Delta\Phi_k = \lambda_k \Phi_k \tag{1.7}$$

in Ω, with $\frac{\partial \Phi_k}{\partial v} = 0$ on $\partial\Omega$, and

$$\|\Phi_k\|_2 = \int_\Omega \Phi_k^2(x)\, dx = 1. \tag{1.8}$$

1.4.2 Local Stability

In general, system (1.1) can made up of two-component with a linearized reaction expressed in the following form

$$\frac{\partial}{\partial t} U - D\Delta U = J_0 U, \tag{1.9}$$

in the simplest case, D is assumed to be diagonal and cross–diffusion is neglected, i.e.

$$D = \begin{pmatrix} d_u & 0 \\ 0 & d_v \end{pmatrix}, \tag{1.10}$$

where d_u and d_v denote the diffusivity constants for substances u and v, respectively, and J_0 is the Jacobian matrix of the corresponding ODEs system evaluated at the equilibrium point.
Let denote the linearizing operator by $\mathcal{L} = J_0 U + D\Delta U$. Suppose $(\phi(x), \psi(x))$ is an eigenfunction of $\mathcal{L}$ corresponding to an eigenvalue ξ. We have

$$\mathcal{L}(\phi(x), \psi(x))^t = \xi(\phi(x), \psi(x))^t,$$

leading to

$$(\mathcal{L} - \xi I)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For simplification purposes, let us set

$$\begin{cases} \phi = \sum_{0\le i\le\infty, 1\le j\le m_i} a_{ij}\Phi_{ij}, \\ \psi = \sum_{0\le i\le\infty, 1\le j\le m_i} b_{ij}\Phi_{ij}. \end{cases}$$

We can now write $(\phi(x), \psi(x))^T = \sum_{0\le i\le\infty,1\le j\le m_i} (a_{ij}, b_{ij})^T \Phi_{ij}$.
This can be rearranged to the form

$$\sum_{0\le i\le\infty,1\le j\le m_i} (J_0 - \lambda_i D - \xi I)\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix}\Phi_{ij} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equilibrium point is locally asymptotically stable if all the eigenvalues of $\mathcal{L}$ have negative real parts.

Theorem 1.4 ([5]) *The equilibrium point of* (1.1) *is (locally) asymptotically stable if the equilibrium point of the linearized problem* (1.9) *is asymptotically stable.*

1.5 Theories of Global Asymptotic Stability

One of the important methods for studying global asymptotic stability is **the direct Lyapunov method**, which was developed by the Russian mathematician Aleksandr Lyapunov at the beginning of 1900's. We will describe this method and illustrate its applications.

1.5.1 The direct Lyapunov method

Definition 1.11 ([4]) *If $U^* \in \mathbb{R}^n$ is an equilibrium point of* (1.1) *and $\Omega \subseteq \mathbb{R}^n$ be an open set containing U^*. A real valued function $V \in \boldsymbol{C}^1(\Omega, \mathbb{R})$ is called a Lyapunov function for* (1.1) *if*

$$V(U) > V(U^*), \text{ for all } U \in \Omega,\ U \neq U^*, \tag{1.11}$$

and

$$\frac{dV\left(U\left(t\right)\right)}{dt} := \left\langle \nabla V\left(U\right), F\left(U\right)\right\rangle \leq 0, \text{ for all } U \in \Omega. \tag{1.12}$$

Theorem 1.5 ([4]) ***(Liapunov stabilty theorem)****.*

(i) If (1.1) has a Lyapunov function, then U^* is stable.

(ii) If one has that $\frac{dV(U(t))}{dt} < 0$, for all $U \neq U^*$, then U^* is asymptotically stable.

Theorem 1.6 ([7]) ***(LaSalle's theorem)***

Let $U = U^*$ be an equilibrium points and $\Omega \subset \mathbb{R}^n$ be a domain containing U^*. Let $V : \Omega \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $\frac{dV(U(t))}{dt} \leq 0$ in Ω. Let $M = \left\{u \in \Omega, \frac{dV(U(t))}{dt} = 0\right\} = \{u^*\}$. Then, U^* is asymptotially stable.

Remark 1.2 *If $\Omega = \mathbb{R}^n$, in the last theorem, U^* is globally asymptotially stable.*

Chapter 2

Global and local asymptotic stability of an epidemic reaction-diffusion model with a nonlinear incidence

To study disease dynamics, compartmental models have played an important role in eliminating the disease at the local and global levels, giving us simple equations to determine the number of people affected by an outbreak or to determine the size of the susceptible population. The original compartmental models have produced many different forms, for example: SI, is a classic model in mathematical epidemiology and the simplest form of all disease models, showing the spread of infectious disease in a population. Individuals are in simulation without immunity once infected and without treatment, so individuals remain infected throughout their lives, they remain in contact with susceptible populations.
In this chapter, we consider the following reaction-diffusion epidemic phenomena proposed in [6], with the nonlinear incidence $u\varphi(v)$, which is an extended version of the SI epidemic model.

2.1 System model

Systems of the form (1.1) appear naturally in many phenomena but we are interested in the following mathematical model

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u = \Lambda - \mu u - \lambda u\varphi(v) =: F(u,v) \;\text{ in } (0,\infty)\times\Omega, \\ \frac{\partial v}{\partial t} - d_2\Delta v = -\sigma v + \lambda u\varphi(v) =: G(u,v) \qquad \text{in } (0,\infty)\times\Omega. \end{cases} \tag{2.1}$$

We assume the initial conditions

$$u(0,x) = u_0(x),\; v(0,x) = v_0(x) \;\text{ in }\; \Omega, \tag{2.2}$$

where u_0, $v_0 \in C(\overline{\Omega})$, and impose homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0 \quad \text{on} \quad (0, \infty) \times \partial\Omega, \tag{2.3}$$

with υ being the unit outer normal to $\partial\Omega$. We will also assume that the initial conditions $u_0(x), v_0(x) \in \mathbb{R}_{\geq 0}$.

The constants d_1, $d_2 > 0$ are the diffusion coefficients and the constants parameters $\Lambda, \mu, \sigma, \lambda > 0$. The incidence function $\varphi(v)$ introduces a nonlinear relation between the two classes of individuals. We assume φ to be a continuously differentiable function on $\mathbb{R}^+$ satisfying

$$\varphi(0) = 0, \tag{2.4}$$

and

$$0 < v\varphi'(v) \leq \varphi(v) \text{ for all } v > 0. \tag{2.5}$$

This system may describe the transmission of a communicable disease between individuals such as HIV/AIDS.

2.2 Interpretation of model

Initially, the N population is confined to the study region Ω where there is no migration, which is expressed as $\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0$. To interpret the mathematical model, we rely on the scheme in the figure 2.1.

N is divided into two different categories known as compartments, with the first compartment comprising the susceptible population u and the second reflecting the infection population v, (i.e. $N = u + v$). Both u and v change with regard to time t and location x.

The susceptible people become an infected at a rate of λ, (the more v in moment t, the greater the λ), where the number of susceptibles is reduced by leaving a sensitive community and joining the category of infected individuals, so we have $-\lambda u\varphi(v)$ and $\lambda u\varphi(v)$. The number of births Λ in addition to the natural mortality rate should be included μ, both of which fall into the u category, so we have $\Lambda - \mu u$. The infection is supposed to leave an infectious layer at the rate σ of infection, quarantine and then die, so we have $-\sigma v$.

Susceptible Peoples actually become infected by mixing with infected people in some way, for example through exposure to coughing or sneezing and so on, the latter are called diffusion coefficients d_1 and d_2 to spread the disease spatially.

The main objective of modeling is to predict the number of infections in order to eliminate the disease by reducing transmission, i.e. we need to reduce the average number of secondary infec-

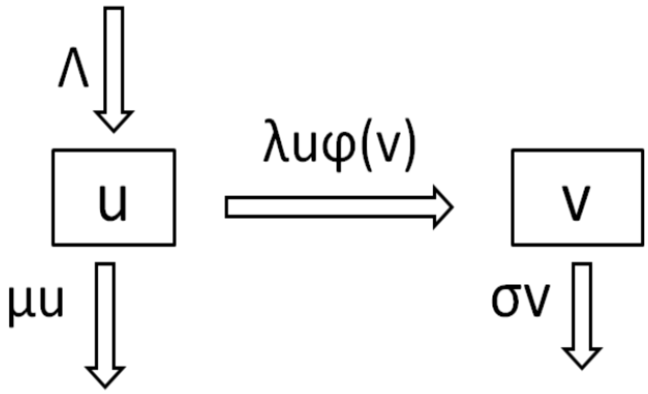

Figure 2.1: Epidemiological scheme of the proposed system model.

tions caused by a single infection to less than 1, which is known as the basic reproduction number R_0, which is an essential measure of disease biology and societal structure.

2.3 Preliminaries Properties of the Model

2.3.1 Positivity of the solution

Let us assume that the initial conditions $(u_0, v_0) \in \mathbb{R}^2_{\geq 0}$ .Note that for $(u, v) \in \mathbb{R}^2_{\geq 0}$. We have

$$\begin{cases} F(0, v) = \Lambda > 0, \\ G(u, 0) = 0 \text{ for all } v > 0, \end{cases} \tag{2.6}$$

which makes the function $(F, G)^T$ essentially nonnegative. Hence, the nonnegative quadrant $\mathbb{R}^2_{\geq 0}$, is an invariant set.

2.3.2 Absence of diffusion

By dropping the spatial variable, the proposed system reduces to the following system of **ODEs**:

$$\begin{cases} \frac{du}{dt} = F(u, v) \ \text{ in } (0, \infty), \\ \frac{dv}{dt} = G(u, v) \ \text{ in } (0, \infty), \end{cases} \tag{2.7}$$

with initial conditions

$$u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \tag{2.8}$$

Firstly, we study this system.

2.3.3 Invariant Regions

We define the invariant Regions , where we let $N = u + v$ and $\sigma_0 = \min(\sigma, \mu)$

$$D = \left\{ (u, v) : u, v \ \geq 0 \text{ and } u + v \leq \frac{\Lambda}{\sigma_0} \right\}.$$

The following proposition shows that D is an invariant region of system (2.7)-(2.8).

Proposition 2.1 ([6]) *The region D is non–empty, attracting and positively invariant.*

Proof. We start by summing the equations of system (2.7)-(2.8), which yield

$$\begin{aligned} \frac{d}{dt}(u+v) &= \frac{d}{dt}N(t) \\ &= F(u,v) + G(u,v) \\ &= \Lambda - \mu u - \sigma v \\ &\leq \Lambda - \sigma_0(u+v), \end{aligned}$$

so

$$\frac{d}{dt}N(t) \leq \Lambda - \sigma_0 N.$$

Integration of both sides

$$\begin{aligned} \int_0^t \frac{d}{ds}N(s)ds &\leq \int_0^t (\Lambda - \sigma_0 N(s))ds \\ N(t) - N(0) &\leq \Lambda t - \sigma_0 \int_0^t N(s)ds \\ N(t) &\leq N(0) + \Lambda t - \sigma_0 \int_0^t N(s)ds, \end{aligned}$$

application of the Gronwall's inequality

$$N(t) \leq N_0 e^{-\sigma_0 t} - \frac{\Lambda}{\sigma_0}(e^{-\sigma_0 t} - 1),$$

Substituting the value of N yields

$$(u+v)(t) \leq (u+v)(0)e^{-\sigma_0 t} + \frac{\Lambda}{\sigma_0}(1 - e^{-\sigma_0 t}), \text{ for } t \geq 0.$$

If the initial states satisfy $(u+v)(0) \leq \frac{\Lambda}{\sigma_0}$, then $(u+v)(t) \leq \frac{\Lambda}{\sigma_0}$ and

$$\lim_{t \longrightarrow \infty} \sup N(t) \leq \frac{\Lambda}{\sigma_0}.$$

As a result, region D is positively invariant and attracting within $\mathbb{R}^2_{\geq 0}$. Therefore, it is sufficient to consider the dynamics of the model within D as D is the biologically feasible region of the system where the existence and uniqueness results hold for the system . ■

2.3.4 Existence of equilibrium solutions

Theorem 2.1 ([6]) *Under conditions* (2.4)-(2.5)

▶ *System* (2.7)-(2.8) *admits always a disease-free equilibrium* $E_0 = (\frac{\Lambda}{\mu}, 0)$.

▶ *If* $R_0 > 1$, *System* (2.7)-(2.8)*has a endemic equilibrium* , $E^* = (u^*, v^*)$.

Proof. **First**, we calculate Equilibrium points :

The positive equilibria of model (2.7)-(2.8) satisfy :

$$\begin{cases} F(u,v) = \Lambda - \mu u - \lambda u\varphi(v) = 0, \\ G(u,v) = -\sigma v + \lambda u\varphi(v) = 0. \end{cases} \tag{2.9}$$

▷ Equilibrium E_0

If $u = 0$, it is easy to see that the system has no equilibrium.

If $v = 0$, $\Lambda - \mu u - \lambda u\varphi(0) = 0$ implies that $u = \frac{\Lambda}{\mu}$

So there's only equilibrium is : $E_0 = (\frac{\Lambda}{\mu}, 0)$.

▷ Equilibrium E^*

We have

$$F(u^*, v^*) + G(u^*, v^*) = \underbrace{\Lambda - \sigma v - \mu u^*}_{(*)} = 0.$$

By subtracting $F(u^*, v^*)$ out of $(*)$

$$F(u^*, v^*) - (*) = -\lambda u\varphi(v^*) + \sigma v = 0 \text{ implies that } v^* = \frac{\lambda u^*\varphi(v)}{\sigma},$$

we make up v^* in $(*)$

$$\Lambda - \sigma\left(\frac{\lambda u^*\varphi(v^*)}{\sigma}\right) - \mu u^* = 0 \text{ implies that } u^* = \frac{\Lambda}{\lambda\varphi(v^*) + \mu}.$$

Whose solution is

$$E^* = (u^*, v^*) = \left(\frac{\Lambda}{\lambda\varphi(v^*) + \mu}, \frac{\lambda u^*\varphi(v)}{\sigma}\right).$$

Second, we study the existence conditions of an endemic steady state in the case $v > 0$. We have from the second part of (2.9), and because $\lambda > 0$ and $\varphi(v) > 0$, we obtain

$$u = \frac{\sigma v}{\lambda\varphi(v)}.$$

Substituting this into the first equation yields

$$\Lambda - \mu\left(\frac{\sigma v}{\lambda\varphi(v)}\right) - \lambda\varphi(v)\left(\frac{\sigma v}{\lambda\varphi(v)}\right) = 0,$$

and from it

$$\Lambda - \sigma v - \frac{\mu\sigma v}{\lambda\varphi(v)} = 0$$

implies that

$$h(v) = 0 \text{ for any } v > 0,$$

where

$$h(v) = \frac{\Lambda\lambda\varphi(v)}{\sigma v\mu} - \frac{\lambda\varphi(v)}{u} - 1$$

is continuous for any $v > 0$, because

$$\lim_{v\to 0} h(v) = \lim_{v\to 0} \frac{\Lambda\lambda\varphi(v)}{\sigma v\mu} - \frac{\lambda\varphi(v)}{u} - 1,$$

by applying **L'Hopital's rule**

$$\begin{aligned} \lim_{v\to 0} h(v) &= \lim_{v\to 0} \frac{\Lambda\lambda}{\sigma\mu}\varphi'(v) - 1 - \frac{\lambda\varphi(v)}{u} \\ &= \frac{\Lambda\lambda}{\sigma\mu}\varphi'(0) - 1 = R_0 - 1. \end{aligned}$$

Now we calculate $\lim_{v\longrightarrow \frac{\Lambda}{\sigma_0}} h(v)$ by using $\sigma_0 = \min(\mu, \sigma)$, we have

$$\begin{aligned} \lim_{v\longrightarrow \frac{\Lambda}{\sigma_0}} h(v) &= h\left(\frac{\Lambda}{\sigma_0}\right) \\ &= \frac{\lambda\Lambda}{\sigma\mu}\frac{\sigma_0}{\Lambda}\varphi\left(\frac{\Lambda}{\sigma_0}\right) - \frac{\lambda}{\mu}\varphi\left(\frac{\Lambda}{\sigma_0}\right) - 1 \\ &= \frac{\lambda}{\sigma\mu}(\sigma_0 - \sigma)\varphi\left(\frac{\Lambda}{\sigma_0}\right) - 1 < 0. \end{aligned}$$

Hence for $R_0 > 1$

$$\lim_{v\to 0} h(v) h\left(\frac{\Lambda}{\sigma_0}\right) = (R_0 - 1) h\left(\frac{\Lambda}{\sigma_0}\right) < 0.$$

By applying **the intermediate value theorem,** there exist a real $v^* \in \left(0, \frac{\Lambda}{\sigma_0}\right)$. Using the condition (2.5), we find

$$\frac{dh}{dv}(v) = \frac{\Lambda\lambda\left[v\varphi'(v) - \varphi(v)\right] - \sigma\lambda v^2\varphi'(v)}{\sigma\mu v^2} < 0.$$

So, the function h decreases monotonically for all $v > 0$, then there exists a unique real $v^* \in \left(0, \frac{\Lambda}{\sigma_0}\right)$ such that $h(v^*) = 0$, which is implies the existence of a unique $u^* = \frac{\sigma v^*}{\lambda\varphi(v^*)}$.
The second equation of (2.9) has no solution in $\left(\frac{\Lambda}{\sigma_0}, +\infty\right)$ because

$$\max_{v\in(\frac{\Lambda}{\sigma_0}, 0)} h(v) \leq h\left(\frac{\Lambda}{\sigma_0}\right) < 0.$$

■

2.3.5 The basic reproduction number R_0

The basic reproduction number is a central concept indicating the threshold for the epidemiological model. To give his exact definition , Diekmann, Heesterbeek and Metz (1990) introduced the next generation operator, which is a positive linear operator by which R_0 can be defined as the spectral radius of this operator.

Van den Driessche and Watmough (2002) have similarly done so that the models are ODEs systems. The next generation operator is described in terms of matrices, in this case R_0 is the largest eigenvalue of a matrix that describes the next generation operator. To calculate the basic reproduction number by using **the next generation matrix method** [17], we now move to the steps of this method :

The whole population is divided into n compartments in which there are $m < n$ infected compartments. Let $i = 1, 2, \cdots, m$ be the numbers of infected individuals in the i^{th} infected compartment at time t.

▷ *First*, the ODEs system can be written as

$$\frac{dU}{dt} = \mathcal{F}(U) - \vartheta(U),$$

where

$$\mathcal{F}(U) = (\mathcal{F}_1(U), \mathcal{F}_2(U), \cdots, \mathcal{F}_m(U))^T$$

is the rate of appearance of new infections in compartment i, and

$$\vartheta(U) = (\vartheta_1(U), \vartheta_2(U), \cdots, \vartheta_m(U))^T,$$

the function ϑ has the following decomposition

$$\vartheta(U) = \vartheta^-(U) - \vartheta^+(U),$$

where ▶ ϑ^+ be the rate of transfer of individuals into compartment i by all other means.

▶ ϑ^- be the rate of transfer of individuals out of compartment i.

▷ *Second*, let E_0 be the disease-free equilibrium. We calcule the Jacobian matrices of $\mathcal{F}$ and ϑ in E_0, are

$$J(\mathcal{F}(E_0)) = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$J(\vartheta(E_0)) = \begin{pmatrix} z & 0 \\ v_1 & v_2 \end{pmatrix},$$

where s and z are the $m \times m$ matrices defined by

$$s = \left(\frac{\partial \mathcal{F}_i}{\partial x_j} (E_0) \right) \text{ and } z = \left(\frac{\partial \vartheta_i}{\partial x_j} (E_0) \right) \text{ with } 1 \leq i, j \leq m.$$

Further, s is non-negative, z is a non-singular M-matrix and all eigenvalues of v_2 have positive real part.

▷ **The next generation matrix method** is defined as

$$K = s z^{-1}.$$

▷ Now, the basic reproduction number is simply the spectral radius of K, i.e.

$$R_0 = \rho \left(s z^{-1} \right).$$

We apply this method to the system (2.7).
We rewrite the system (2.7)-(2.8) in vector form as

$$\begin{pmatrix} \frac{dv}{dt} \\ \frac{du}{dt} \end{pmatrix} = \begin{pmatrix} \lambda u \varphi(v) \\ 0 \end{pmatrix} - \begin{pmatrix} \sigma v \\ -\Lambda + \mu u + \lambda u \varphi(v) \end{pmatrix}.$$

We calculate the Jacobian matrices corresponding to vectors $\begin{pmatrix} \lambda u \varphi(v) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sigma v \\ -\Lambda + \mu u + \lambda u \varphi(v) \end{pmatrix}$ at the disease–free equilibrium E_0 are given, respectively, by

$$\begin{pmatrix} \frac{\lambda \Lambda}{\mu} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \sigma & 0 \\ \frac{\lambda \Lambda}{\mu} & \mu \end{pmatrix} = \begin{pmatrix} z & 0 \\ v_1 & v_2 \end{pmatrix}.$$

Where the next generation matrix is

$$K = \left(\frac{\lambda \Lambda}{\mu} \varphi'(0) \right) (\sigma)^{-1} = \frac{\lambda \Lambda}{\mu \sigma} \varphi'(0).$$

So

$$R_0 = \frac{\lambda \Lambda}{\mu \sigma} \varphi'(0).$$

2.3.6 The Local ODEs Stability

In this theorem, we will examine the local stability of the previously defined E_0 and E^* points.

Theorem 2.2 ([6]) *Always under conditions* (2.4)-(2.5)*, the two statements are achieved for* (2.7)-(2.8)*:*
(i) *If* $R_0 < 1$ *, the disease-free equilibrium solution* E_0 *is the only steady state of the system and is locally asymptotically stable.*
(ii) *If* $R_0 > 1$*,* E_0 *is unstable and the positive constant endemic steady state* E^* *is locally asymptotically stable .*

Proof. We calculate **Jacobian** matrix :

$$J(u,v) = \begin{pmatrix} -\lambda\varphi(v) - \mu & -\lambda u\varphi'(v) \\ \lambda\varphi(v) & \lambda u\varphi'(v) - \sigma \end{pmatrix}.$$

▶ **First** , We study the stability of E_0 for $R_0 < 1$ and $R_0 > 1$.
Evaluating $J(u,v)$ at E_0 with (2.4) in mind yields

$$J(E_0) = \begin{pmatrix} -\mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma \end{pmatrix}.$$

And from there we have $\lambda_1 = -\mu$ and $\lambda_2 = \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma$.
If $R_0 < 1$,it is easy to see that $\lambda_1 < 0$ and $\lambda_2 < 0$, leading to the asymptotic stability .
If $R_0 > 1$,it is easy to see that $\lambda_1 < 0$ but $\lambda_2 > 0$, leading to unstability.
▶**second,** We study the stability of $E^*$for $R_0 > 1$.
Evaluating $J(u,v)$ at E^* yields

$$J(E^*) = \begin{pmatrix} -\lambda\varphi(v^*) - \mu & -\lambda u^*\varphi'(v^*) \\ \lambda\varphi(v^*) & \lambda u^*\varphi'(v^*) - \sigma \end{pmatrix}.$$

The determinant and trace of the Jacobin can be given by

$$\begin{aligned} \det J(u^*,v^*) &= \lambda\sigma\varphi(v^*) + \mu\sigma - \mu\lambda u^*\varphi'(v^*), \\ \operatorname{tr}(J(u^*,v^*)) &= -(\lambda\varphi(v^*) + \mu) + \lambda u^*\varphi'(v^*) - \sigma. \end{aligned}$$

We have from u^* and v^*

$$\begin{cases} \Lambda = \lambda u^*\varphi(v^*) + \mu u^*, \\ \sigma = \frac{\lambda u^*\varphi(v^*)}{v^*}. \end{cases} \tag{2.10}$$

Using this, we get

$$\begin{aligned}\det J(u^*,v^*) &= \lambda\frac{\lambda u^*\varphi(v^*)}{v^*}\varphi(v^*)+\mu\frac{\lambda u^*\varphi(v^*)}{v^*}-\mu\lambda u^*\varphi'(v^*)\\ &= \frac{\lambda^2u^*(\varphi(v^*))^2}{v^*}+\mu\lambda u^*\left[\frac{\varphi(v^*)}{v^*}-\varphi'(v^*)\right],\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}(J(u^*,v^*)) &= -\frac{(\lambda\varphi(v^*)u^*+\mu u^*)}{u^*}-\frac{\lambda u^*\varphi(v^*)}{v^*}+\lambda u^*\varphi'(v^*)\\ &= -\frac{\Lambda}{u^*}-\lambda u^*\left[\frac{\varphi(v^*)}{v^*}-\varphi'(v^*)\right].\end{aligned}$$

From the condition (2.5)

$$\frac{\varphi(v^*)}{v^*}-\varphi'(v^*)>0 \text{ for all } v^*>0,$$

we obtain $\det\left(J(u^*,v^*)\right)>0$ and $\mathrm{tr}(J(u^*,v^*))<0$.

Hence, the equilibrium E^* is locally asymptotically stable. ■

Now, we study the system **PDEs** .

2.3.7 The local PDEs stability

Theorem 2.3 ([6]) *Assuming that the incidence function φ satisfies , the following statements hold for system*

(i) *If $R_0<1$, the disease-free equilibrium E_0 is locally asymptotically stable.*

(ii) *If $R_0>1$, the positive constant endemic steady equilibrium E^* is locally asymptotically stable.*

Proof. (i) **First**, we proof the stability of E_0 If $R_0<1$.

In the presence of diffusion, E_0 satisfies

$$\begin{cases} d_1\Delta u+\Lambda-\lambda u^*\varphi\left(v^*\right)-\mu u^*=0,\\ d_2\Delta v+\lambda u^*\varphi\left(v^*\right)-\sigma v^*=0,\end{cases}$$

with the Neumann boundary

$$\frac{\partial u}{\partial \upsilon}=\frac{\partial v}{\partial \upsilon}=0 \text{ on } \mathbb{R}^+\times\partial\Omega.$$

Through the properties of Laplace eigenvalues, which were mentioned in the first chapter.

Let us denote the eigenvalues of the $-\Delta$ by $0=\lambda_0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots\nearrow^{+\infty}$ over Ω with Neumann boundary conditions, where each λ_i has algebraic multiplicity $m_i\geq 1$, and let $\left(\Phi_{ij}\right)_{j=\overline{1,m_i}}$, be the corresponding normalized eigenfunctions. It is important to note that the set $\left(\Phi_{ij}\right)_{j=\overline{1,m_i}}$, forms a complete orthonormal basis in $L^2\left(\Omega\right)$.

Linearizing system (2.1) around E_0 by using (1.9)

$$\frac{\partial U}{\partial t}-\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\Delta U=\begin{pmatrix} -\mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & \lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma \end{pmatrix}U,$$

we get

$$\frac{\partial U}{\partial t}=\begin{pmatrix} d_1\Delta-\mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & d_2\Delta+\lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma \end{pmatrix}U \quad ,$$

then the linearizing operator may be given by

$$\mathcal{L}(E_0)=\begin{pmatrix} d_1\Delta-\mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & d_2\Delta+\lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma \end{pmatrix}.$$

Suppose $(\phi(x),\psi(x))$ is an eigenfunction of $\mathcal{L}$ corresponding to an eigenvalue ξ. By definition of eigenfunction in the chapter 1, we have

$$\mathcal{L}(\phi(x),\psi(x))^t=\xi(\phi(x),\psi(x))^t,$$

leading to

$$(\mathcal{L}-\xi I)\begin{pmatrix} \phi \\ \psi \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting for $\mathcal{L}$ yields

$$\begin{pmatrix} d_1\Delta-\mu-\xi & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & d_2\Delta+\lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma-\xi \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For simplification purposes, let us set

$$\begin{cases} \phi=\sum_{0\leq i\leq\infty,1\leq j\leq m_i} a_{ij}\Phi_{ij}, \\ \psi=\sum_{0\leq i\leq\infty,1\leq j\leq m_i} b_{ij}\Phi_{ij}. \end{cases}$$

We can now write

$$\sum_{0\leq i\leq\infty,1\leq j\leq m_i}(J_i(E_0)-\xi I)\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix}\Phi_{ij}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the matrix $J_i(E_0)$ is defined as

$$J_i(E_0)=\begin{pmatrix} -d_1\lambda_i-\mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & -d_2\lambda_i+\lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma \end{pmatrix}, \text{for all } i\geq 0.$$

The eigenvalues are

$$\begin{cases} r_{i1}=-d_1\lambda_i-\mu, \\ r_{i2}=-d_2\lambda_i+\lambda\frac{\Lambda}{\mu}\varphi'(0)-\sigma. \end{cases}$$

Since the Laplacian eigenvalues are positive, and from him

$\triangleright$ It's clear that r_{i1} have negative real parts.

$\triangleright$ For $R_0 < 1$ implies that $\frac{\Lambda\lambda}{\mu}\varphi'(0) - \sigma < 0,$ and from it we find that r_{i2} have negative real parts.

Consequently, the disease-free equilibrium E_0 is locally asymptotically stable.

(ii) Now, The stability of the second steady state E^*.

We use the same method from the first steady state E_0.

Linearizing system (2.1) around E^*, we get

$$\mathcal{L}(E^*) = \begin{pmatrix} d_1\Delta - \lambda\varphi(v^*) - \mu & -\lambda u^*\varphi'(v^*) \\ \lambda\varphi(v^*) & d_2\Delta + \lambda u^*\varphi'(v^*) - \sigma \end{pmatrix}.$$

The matrix $J_i(E^*)$ is defined as

$$J_i(E^*) = \begin{pmatrix} -d_1\lambda_i - \lambda\varphi(v^*) - \mu & -\lambda u^*\varphi'(v^*) \\ \lambda\varphi(v^*) & -d_2\lambda_i + \lambda u^*\varphi'(v^*) - \sigma \end{pmatrix}, \text{for all } i \geq 0.$$

<u>Calculation the trace of</u> $J_i(E^*)$

$$\begin{aligned} tr(J_i(E^*)) &= -d_1\lambda_i - \lambda\varphi(v^*) - \mu - d_2\lambda_i + \lambda u^*\varphi'(v^*) - \sigma \\ &= -\lambda_i(d_1 + d_2) + tr(J(u^*, v^*)), \end{aligned}$$

we have $tr(J(u^*, v^*)) < 0$ so $tr(J_i(E^*)) < 0$ for all $i \geq 0$.

<u>Calculation the determinant of</u> $J_i(E^*)$

$$\begin{aligned} \det(J_i(E^*)) &= (-d_1\lambda_i - \lambda\varphi(v^*) - \mu)(-d_2\lambda_i + \lambda u^*\varphi'(v^*) - \sigma) + \lambda^2 u^*\varphi'(v^*)\varphi(v^*) \\ &= d_1d_2\lambda_i^2 + \lambda_i[-d_1\lambda u^*\varphi'(v^*) + \lambda d_2\varphi(v^*) + d_1\sigma + d_2\mu] + \mu\sigma - \mu\lambda u^*\varphi'(v^*) + \lambda\sigma\varphi(v^*) \\ &= d_1d_2\lambda_i^2 + \lambda_i H_0 + \det(J(u^*, v^*)), \end{aligned}$$

such as

$$H_0 = -d_1\lambda u^*\varphi'(v^*) + \lambda d_2\varphi(v^*) + d_1\sigma + d_2\mu,$$

using (2.10)

$$H_0 = -d_1\lambda u^*\varphi'(v^*) + \lambda d_2\varphi(v^*) + d_1\frac{\lambda u^*\varphi(v^*)}{v^*} + d_2\mu,$$

using (2.5) we obtain

$$\begin{aligned} H_0 &\geq -d_1\lambda u^*\frac{\varphi(v^*)}{v^*} + \lambda d_2\varphi(v^*) + d_1\frac{\lambda u^*\varphi(v^*)}{v^*} + d_2\mu \\ &= d_2(\lambda\varphi(v^*) + \mu) \\ &= d_2\frac{\Lambda}{\mu} > 0. \end{aligned}$$

From the above we have $\det\left(J\left(u^{*},v^{*}\right)\right)>0$, and from it we find that $\det\left(J_{i}\left(E^{*}\right)\right)>0$ for all $i\geq 0$.
Hence, E^{*} is locally asymptotically stable. ■

2.4 Global asymptotic stability

In this section, we study the global asymptotic stability of the steady state solutions E_0 and E^{*} for the system PDEs (2.1)-(2.3), which is based on the reproduction number R_0, this is when $R_0<1$ and $R_0>1$ by using an appropriate Lyapunov functional.

Lemma 2.1 ([2]) *Condition* (2.5) *implies that*

$$0<\frac{\varphi\left(v\right)}{v}<\varphi'\left(0\right)\ \ \textit{for all } v>0. \tag{2.11}$$

Proof. Of (2.5), which is equivalent to

$$\left(\frac{\varphi\left(v\right)}{v}\right)'=\frac{\varphi'\left(v\right)v-\varphi\left(v\right)}{v^{2}}\leq 0.$$

Therefore, the function $\frac{\varphi(v)}{v}$ is decreasing .
Now, for some $h\in\left(0,v\right)$, we have

$$\frac{\varphi\left(h\right)}{h}\geq\frac{\varphi\left(v\right)}{v},$$

implying that

$$\lim_{h\longrightarrow 0}\frac{\varphi\left(h\right)}{h}\geq\frac{\varphi\left(v\right)}{v},$$

which yields

$$\varphi'\left(0\right)\geq\frac{\varphi\left(v\right)}{v}.$$

■

Lemma 2.2 ([15]) *Given that* φ *satisfies criterion* (2.5) *and*

$$L\left(x\right)=x-1-\ln\left(x\right)\quad\textit{for all } x>0,$$

the inequality

$$L\left(\frac{\varphi\left(v\right)}{\varphi\left(v^{*}\right)}\right)\leq L\left(\frac{v}{v^{*}}\right),$$

where v^{*} *is the second component of the equilibrium point* E^{*}*, holds.*

Proof. The function $\frac{\varphi(v)}{v}$ is decreasing for all $v > 0$, from the condition (2.5).
We may separate the proof into two cases:
Case 1: Suppose $v \geq v^*$.
We have $\frac{\varphi(v)}{v}$is a decreasing function, This means that

$$\frac{\varphi(v)}{\varphi(v^*)} \leq \frac{v}{v^*},$$

and (2.5) implies that φ is non decreasing

$$\varphi(v) \geq \varphi(v^*),$$

consequently

$$1 \leq \frac{\varphi(v)}{\varphi(v^*)} \leq \frac{v}{v^*}.$$

We have $L'(x) = \frac{x-1}{x}$

x	0	1	$+\infty$
$L'(x)$	$-$	0	$+$
$L(x)$	$\searrow$	0	$\nearrow$

When $x > 1$ The function $L(x)$ is increasing, thus

$$L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) \leq L\left(\frac{v}{v^*}\right) \text{ for all } v \geq v^*.$$

Case 2: Suppose $0 < v < v^*$.
The function $\frac{\varphi(v)}{v}$ is decreasing for all $v > 0$, This means that

$$\frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*},$$

and φ is non-decreasing

$$\varphi(v) < \varphi(v^*).$$

We get

$$1 > \frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*} > 0.$$

When $0 < x < 1$ the function L is decreasing, thus

$$L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) < L\left(\frac{v}{v^*}\right) \text{ for all } v \geq v^*.$$

■

2.4.1 Global Asymptotic Stability with $R_0 < 1$

We consider the Lyapunov function

$$V_\theta(t) = \int_\Omega \left[uv + \frac{\theta}{2}\left(u - \frac{\Lambda}{\mu}\right)^2 + \frac{1}{2}v^2 + \frac{\Lambda}{\sigma}v \right] dx, \text{ where } \theta > 0.$$

Theorem 2.4 ([6]) *Assuming that* (2.5) *holds, if* $R_0 < 1$, *then* E_0 *is a globally asymptotically stable disease-free steady state for system* (2.1)-(2.3) *under the assumption*

$$\varphi'(0) \leq \frac{\mu + \sigma}{\lambda\left(\theta\frac{\Lambda}{\mu} + \frac{\mu}{\sigma}\right)}, \tag{2.12}$$

with

$$\theta \geq \frac{(d_1 + d_2)^2}{4d_1 d_2}. \tag{2.13}$$

Proof. We proof that V_θ is a Lyapunov function.

$i)$

$$\left.\begin{array}{l} V_\theta(t) > 0 \text{ for all } t > 0 \\ V_\theta(E_0) = 0 \end{array}\right\} \text{this implies that } V_\theta(t) > V_\theta(E_0).$$

$ii)$ We show if $\frac{d}{dt}V_\theta(t) \overset{?}{\leq} 0$.

Evaluating the derivative to $V_\theta(t)$ with respect to time

$$\begin{aligned} \frac{d}{dt}V_\theta(t) &= \frac{d}{dt}\left(\int_\Omega \left[uv + \frac{\theta}{2}\left(u - \frac{\Lambda}{\mu}\right)^2 + \frac{1}{2}v^2 + \frac{\Lambda}{\sigma}v \right] dx \right) \\ &= \int_\Omega \left(\frac{\partial u}{\partial t}v + u\frac{\partial v}{\partial t} \right) dx + \theta \int_\Omega \frac{\partial u}{\partial t}\left(u - \frac{\Lambda}{\mu}\right) dx + \int_\Omega v\frac{\partial v}{\partial t} dx + \frac{\Lambda}{\sigma}\int_\Omega \frac{\partial v}{\partial t} dx. \end{aligned}$$

Substituting the time derivatives with their values from (2.1)

$$\begin{aligned} \frac{d}{dt}V_\theta(t) &= \int_\Omega [d_1\Delta u + \Lambda - \mu u - \lambda u\varphi(v)]\, v + u\,[d_2\Delta v - \sigma v + \lambda u\varphi(v)]\, dx \\ &\quad + \theta \int_\Omega [d_1\Delta u + \Lambda - \mu u - \lambda u\varphi(v)]\left(u - \frac{\Lambda}{\mu}\right) dx + \int_\Omega v\,[d_2\Delta v - \sigma v + \lambda u\varphi(v)]\, dx \\ &\quad + \frac{\Lambda}{\sigma}\int_\Omega [d_2\Delta v - \sigma v + \lambda u\varphi(v)]\, dx, \end{aligned}$$

simplifying the resulting equation

$$\begin{aligned}\frac{d}{dt}V_{\theta}(t) &= d_1\int_{\Omega}\Delta uvdx+\Lambda\int_{\Omega}vdx-\mu\int_{\Omega}uvdx-\lambda\int_{\Omega}u\varphi(v)\,vdx+d_2\int_{\Omega}\Delta vudx-\sigma\int_{\Omega}vudx\\ &+\lambda\int_{\Omega}u^2\varphi(v)\,dx+\theta d_1\int_{\Omega}u\Delta udx+\Lambda\theta\int_{\Omega}udx-\theta\mu\int_{\Omega}u^2dx-\theta\lambda\int_{\Omega}u^2\varphi(v)\,dx\\ &-\theta\frac{\Lambda}{\mu}d_1\int_{\Omega}\Delta udx-\theta\int_{\Omega}\frac{\Lambda^2}{\mu}dx+\theta\Lambda\int_{\Omega}udx+\theta\frac{\Lambda}{\mu}\lambda\int_{\Omega}u\varphi(v)\,dx+d_2\int_{\Omega}v\Delta vdx\\ &-\sigma\int_{\Omega}v^2dx+\lambda\int_{\Omega}vu\varphi(v)\,dx+\frac{\Lambda}{\sigma}d_2\int_{\Omega}\Delta vdx-\Lambda\int_{\Omega}vdx+\frac{\Lambda}{\sigma}\lambda\int_{\Omega}u\varphi(v)\,dx.\end{aligned}$$

We apply Green's formula

$$\begin{aligned}\frac{d}{dt}V_{\theta}(t) &= -(d_1+d_2)\int_{\Omega}\nabla u\nabla vdx+\Lambda\int_{\Omega}vdx-\lambda\int_{\Omega}uv\varphi(v)\,dx+\lambda\int_{\Omega}u^2\varphi(v)\,dx\\ &-(\mu+\sigma)\int_{\Omega}uvdx-\theta d_1\int_{\Omega}|\nabla u|^2dx-\mu\theta\int_{\Omega}\left(u-\frac{\Lambda}{\mu}\right)^2dx-\lambda\theta\int_{\Omega}u^2\varphi(v)\,dx\\ &+\theta\frac{\Lambda}{\mu}\lambda\int_{\Omega}u\varphi(v)\,dx-d_2\int_{\Omega}|\nabla v|^2dx+\lambda\int_{\Omega}uv\varphi(v)\,dx-\sigma\int_{\Omega}v^2dx\\ &+\frac{\Lambda}{\sigma}\lambda\int_{\Omega}u\varphi(v)\,dx-\Lambda\int_{\Omega}vdx\\ &= -d_1\theta\int_{\Omega}|\nabla u|^2dx-(d_1+d_2)\int_{\Omega}\nabla u\nabla vdx-d_2\int_{\Omega}|\nabla v|^2dx+\lambda(1-\theta)\int_{\Omega}u^2\varphi(v)\,dx\\ &-(\mu+\sigma)\int_{\Omega}uvdx-\mu\theta\int_{\Omega}\left(u-\frac{\Lambda}{\mu}\right)^2dx-\sigma\int_{\Omega}v^2dx+\lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)\int_{\Omega}u\varphi(v)\,dx\\ &= I+J,\end{aligned}$$

such as the first part is

$$I=\int_{\Omega}-d_1\theta|\nabla u|^2-(d_1+d_2)\nabla u\nabla v-d_2|\nabla v|^2dx=-\int_{\Omega}Q(\nabla u,\nabla v)\,dx,$$

where

$$Q(\nabla u,\nabla v)=d_1\theta|\nabla u|^2+(d_1+d_2)\nabla u\nabla v+d_2|\nabla v|^2.$$

We have (2.13) implies that $4\theta d_1d_2\geq(d_1+d_2)^2$ this means that

$$\Delta=(d_1+d_2)^2-4d_1\theta d_2\leq 0,$$

which $Q(\nabla u,\nabla v)\,dx>0$, so

$$I\leq 0\,.$$

And the second part is

$$\begin{aligned} J &= \lambda(1-\theta)\int_\Omega u^2\varphi(v)\,dx - (\mu+\sigma)\int_\Omega uvdx - \sigma\int_\Omega v^2dx \; - \mu\theta\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dx \\ &\quad + \lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx\,. \end{aligned}$$

We have

$$\theta \geq \frac{(d_1+d_2)^2}{4d_1d_2} \geq 1 \text{ implies that } 1-\theta \leq 0,$$

so

$$J \leq \lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx - \mu\theta\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx \; - (\mu+\sigma)\int_\Omega uvdx.$$

Applying lemma 2.1

$$J \leq \lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)\int_\Omega uv\varphi'(0)\,dx - \mu\theta\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx \; - (\mu+\sigma)\int_\Omega uvdx$$

$$\leq \int_\Omega\left[\lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)\varphi'(0) - (\mu+\sigma)\right]uvdx - \mu\theta\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx,$$

under the assuming (2.12) implies that $\lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\mu}{\sigma}\right)\varphi'(0) \leq \mu+\sigma$, yields

$$J \leq -\mu\theta\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx \; \leq 0.$$

So we get $\frac{d}{dt}V_\theta(t) \leq 0$ for all $t \geq 0$.
As a result $V_\theta(t)$ is Lyapunov function, by Lyapunov's direct method, E_0 is globally asymptotically stable. ■

2.4.2 Global Asymptotic Stability with $R_0 > 1$

Theorem 2.5 ([6]) *Assuming that $u_0, v_0 > 0$ and (2.5) holds, if $R_0 > 1$, E^* is a globally asymptotically stable endemic steady-state for system (2.1)-(2.3).*

Proof. We consider the candidate Lyapunov function

$$V(t) = \int_\Omega\left[u^*L\left(\frac{u}{u^*}\right) + v^*L\left(\frac{v}{v^*}\right)\right]dx,$$

which is a positive definite and continuously differentiable function.

We have

$$L\left(\frac{u}{u^*}\right) = \frac{u}{u^*} - 1 - \ln\left(\frac{u}{u^*}\right),$$

so

$$\frac{d}{dt}L\left(\frac{u}{u^*}\right) = \frac{1}{u^*}\frac{du}{dt} - \frac{1}{u^*}\frac{du}{dt}\frac{u^*}{u} = \frac{1}{u^*}\left(1 - \frac{u^*}{u}\right)\frac{du}{dt}.$$

We proof that $\mathbf{\frac{d}{dt}V\left(t\right) \overset{?}{\leq} 0}$.

Evaluating the derivative to $V\left(t\right)$ with respect to time

$$\begin{aligned}
\frac{d}{dt}V\left(t\right) &= \int_\Omega u^*\frac{d}{dt}L\left(\frac{u}{u^*}\right)dx + \int_\Omega v^*\frac{d}{dt}L\left(\frac{v}{v^*}\right)dx \\
&= \int_\Omega u^*\frac{1}{u^*}\left(1 - \frac{u^*}{u}\right)\frac{du}{dt}dx + \int_\Omega v^*\frac{1}{v^*}\left(1 - \frac{v^*}{v}\right)\frac{dv}{dt}dx \\
&= \int_\Omega \left(1 - \frac{u^*}{u}\right)\frac{du}{dt}dx + \int_\Omega \left(1 - \frac{v^*}{v}\right)\frac{dv}{dt}dx.
\end{aligned}$$

Substituting the time derivatives with their values from (2.1)

$$\begin{aligned}
\frac{d}{dt}V\left(t\,\right) &= \int_\Omega \left(1 - \frac{u^*}{u}\right)\left[d_1\Delta u + \Lambda - \mu u - \lambda u\varphi\left(v\right)\right]dx + \int_\Omega \left(1 - \frac{v^*}{v}\right)\left[d_2\Delta v - \sigma v + \lambda u\varphi\left(v\right)\right]dx \\
&= d_1\int_\Omega \left(1 - \frac{u^*}{u}\right)\Delta u dx + \Lambda\int_\Omega \left(1 - \frac{u^*}{u}\right)dx - \lambda\int_\Omega \left(1 - \frac{u^*}{u}\right)u\varphi\left(v\right)dx \\
&\quad -\mu\int_\Omega \left(1 - \frac{u^*}{u}\right)u dx + d_2\int_\Omega \left(1 - \frac{v^*}{v}\right)\Delta v dx + \lambda\int_\Omega \left(1 - \frac{v^*}{v}\right)u\varphi\left(v\right)dx \\
&\quad -\sigma\int_\Omega \left(1 - \frac{v^*}{v}\right)v dx,
\end{aligned}$$

we apply Green's formula with Neumann boundaries

$$\begin{aligned}
\frac{d}{dt}V\left(t\right) &= -d_1\int_\Omega \nabla\left(1 - \frac{u^*}{u}\right)\nabla u dx + \Lambda\int_\Omega \left(1 - \frac{u^*}{u}\right)dx - \lambda\int_\Omega \left(1 - \frac{u^*}{u}\right)u\varphi\left(v\right)dx \\
&\quad -\mu\int_\Omega \left(1 - \frac{u^*}{u}\right)u dx - d_2\int_\Omega \nabla\left(1 - \frac{v^*}{v}\right)\nabla v dx + \lambda\int_\Omega \left(1 - \frac{v^*}{v}\right)u\varphi\left(v\right)dx \\
&\quad -\sigma\int_\Omega \left(1 - \frac{v^*}{v}\right)v dx \\
&= I + J,
\end{aligned}$$

where

$$\begin{aligned}
I &= -d_1\int_\Omega \nabla\left(1 - \frac{u^*}{u}\right)\nabla u dx - d_2\int_\Omega \nabla\left(1 - \frac{v^*}{v}\right)\nabla v dx \\
&= -\int_\Omega \left[d_1\frac{u^*}{u^2}\left|\nabla u\right|^2 + d_2\frac{v^*}{v}\left|\nabla v\right|^2\right]dx \;\leq 0\,.
\end{aligned}$$

And

$$J = \int_{\Omega} \left(1 - \frac{u^*}{u}\right) [\Lambda - \lambda u \varphi(v) - \mu u] \, dx + \int_{\Omega} \left(1 - \frac{v^*}{v}\right) [\lambda u \varphi(v) - \sigma v] \, dx,$$

using (2.10)

$$J = \int_{\Omega} \left(1 - \frac{u^*}{u}\right) [\lambda u^* \varphi(v^*) + \mu u^* - \lambda u \varphi(v) - \mu u] \, dx + \int_{\Omega} \left(1 - \frac{v^*}{v}\right) \left[\lambda u \varphi(v) - \frac{\lambda u^* \varphi(v^*)}{v^*} v\right] dx,$$

simplifying the resulting equation

$$\begin{aligned}
J &= \int_{\Omega} \left(1 - \frac{u^*}{u}\right) \lambda u^* \varphi(v^*) \, dx + \int_{\Omega} \left(1 - \frac{u^*}{u}\right) \mu u^* dx - \int_{\Omega} \left(1 - \frac{u^*}{u}\right) \lambda u \varphi(v) \, dx \\
&\quad - \int_{\Omega} \left(1 - \frac{u^*}{u}\right) \mu u dx + \int_{\Omega} \left(1 - \frac{v^*}{v}\right) \lambda u \varphi(v) \, dx - \int_{\Omega} \frac{\lambda u^* \varphi(v^*)}{v^*} v \left(1 - \frac{v^*}{v}\right) dx \\
&= \int_{\Omega} \left(1 - \frac{u^*}{u}\right) \left[1 - \frac{u \varphi(v)}{u^* \varphi(v^*)}\right] \lambda u^* \varphi(v^*) \, dx + \int_{\Omega} \left[\left(1 - \frac{u^*}{u}\right) \left(1 - \frac{u}{u^*}\right)\right] \mu u^* dx \\
&\quad + \int_{\Omega} \lambda u^* \varphi(v^*) \left[\frac{u \varphi(v)}{u^* \varphi(v^*)} - \frac{v}{v^*}\right] \left(1 - \frac{v^*}{v}\right) dx,
\end{aligned}$$

$$J = \int_{\Omega} [\mu u^* J_1 + \lambda u^* \varphi(v^*) J_2] \, dx,$$

where

$$\begin{aligned}
J_1 &= \left(1 - \frac{u^*}{u}\right) \left(1 - \frac{u}{u^*}\right) = 1 - \frac{u}{u^*} - \frac{u^*}{u} + 1 \\
&= 1 - \frac{u}{u^*} + \ln\left(\frac{u}{u^*}\right) - \ln\left(\frac{u}{u^*}\right) + 1 - \frac{u^*}{u} + \ln\left(\frac{u^*}{u}\right) - \ln\left(\frac{u^*}{u}\right) \\
&= -L\left(\frac{u}{u^*}\right) - L\left(\frac{u^*}{u}\right),
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \left(\frac{u \varphi(v)}{u^* \varphi(v^*)} - \frac{v}{v^*}\right) \left(1 - \frac{v^*}{v}\right) + \left(1 - \frac{u^*}{u}\right) \left(1 - \frac{u \varphi(v)}{u^* \varphi(v^*)}\right) \\
&= -\frac{u \varphi(v)}{u^* \varphi(v^*)} - \frac{v}{v^*} - \frac{u^*}{u} + \frac{\varphi(v)}{\varphi(v^*)} + 2 \\
&= 1 - \frac{u \varphi(v) v^*}{u^* \varphi(v^*) v} + \ln\left(\frac{u \varphi(v) v^*}{u^* \varphi(v^*) v}\right) - \ln\left(\frac{u \varphi(v) v^*}{u^* \varphi(v^*) v}\right) + 1 - \frac{v}{v^*} + \ln\left(\frac{v}{v^*}\right) - \ln\left(\frac{v}{v^*}\right) - 1 \\
&\quad + 1 - \frac{u^*}{u} + \ln\left(\frac{u^*}{u}\right) - \ln\left(\frac{u^*}{u}\right) - 1 + 1 + \frac{\varphi(v)}{\varphi(v^*)} - \ln\left(\frac{\varphi(v)}{\varphi(v^*)}\right) + \ln\left(\frac{\varphi(v)}{\varphi(v^*)}\right) \\
&= -L\left(\frac{u \varphi(v) v^*}{u^* \varphi(v^*) v}\right) - L\left(\frac{v}{v^*}\right) - L\left(\frac{u^*}{u}\right) + L\left(\frac{\varphi(v)}{\varphi(v^*)}\right).
\end{aligned}$$

Substituting in J we find

$$
\begin{aligned}
J &= -\mu u^* \int_\Omega L\left(\frac{u}{u^*}\right) + L\left(\frac{u^*}{u}\right) dx + \lambda u^* \varphi(v^*) \int_\Omega -L\left(\frac{u\varphi(v)\, v^*}{u^*\varphi(v^*)\, v}\right) - L\left(\frac{v}{v^*}\right) - L\left(\frac{u^*}{u}\right) \\
&\quad + L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) dx \\
&= -\mu u^* \int_\Omega L\left(\frac{u}{u^*}\right) + L\left(\frac{u^*}{u}\right) dx - \lambda u^* \varphi(v^*) \int_\Omega \left[L\left(\frac{u\varphi(v)\, v^*}{u^*\varphi(v^*)\, v}\right) + L\left(\frac{u^*}{u}\right)\right] dx \\
&\quad + \lambda u^* \varphi(v^*) \int_\Omega \left[L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) - L\left(\frac{v}{v^*}\right)\right] dx.
\end{aligned}
$$

We have the positivity of L and applying lemma 2.1, thus

$$J \leq 0.$$

Hence $\frac{d}{dt}V(t) \leq 0$.
As a result $V(t)$ is Lyapunov function, by Lyapunov's direct method, E^* is globally asymptotically stable. ■

Chapter 3

Numerical Examples

To clarify the results of the theories obtained in the second section, we will present in this section three numerical examples (from [6]) that illustrate and confirm the results of this study using the incidence function $u\varphi(v)$ with the employment of theorems 2.4 and 2.5 in order to evaluate the global asymptotic stability of the disease-free equilibrium E_0 at $R_0 < 1$ and equilibrium E^* at $R_0 > 1$.

3.1 First Example

In this example, we consider the function

$$\varphi(v) = \alpha v, \quad \text{for all } \alpha > 0,$$

by substituting in (2.1), we get the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = -\lambda \alpha u v + \Lambda - \mu u & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \lambda \alpha u v - \sigma v & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \; v(0,x) = v_0(x) & \text{on } \Omega, \\ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0, & \text{on } (0,\infty) \times \partial\Omega. \end{cases} \tag{3.1}$$

The system (3.1) is a special case of the system (2.1), and he is identical to the bird system. Conditions (2.4) and (2.5) are satisfied as

$$\begin{cases} \varphi(0) = 0, \\ \varphi'(v) = \alpha > 0, \\ \varphi'(0) = \alpha, \\ \alpha v = v\varphi'(v) \leq \varphi(v) = \alpha v. \end{cases}$$

Finding steady states for the system (3.1)

$$\begin{cases} -\lambda\alpha u^* v^* + \Lambda - \mu u^* = 0, \\ \lambda\alpha u^* v^* - \sigma v^* = 0, \end{cases} \tag{3.2}$$

we sum the first and second equations of (3.2)

$$\Lambda - \mu u^* - \sigma v^* \ = 0$$

this implies that

$$v^* = \frac{\Lambda - \mu u^*}{\sigma}.$$

We substitute in the first equation the value of v^*

$$\lambda\alpha u^* \left(\frac{\Lambda - \mu u^*}{\sigma}\right) - \Lambda + \mu u^* = 0,$$

we get a quadratic polynomial

$$\frac{-\lambda\alpha}{\sigma}(u^*)^2 + \left(\frac{\lambda\alpha\Lambda}{\sigma\mu} + 1\right) u^* - \frac{\Lambda}{\mu} = 0,$$

we calculate the discriminant of this equation

$$\Delta = \left(\frac{\lambda\alpha\Lambda}{\sigma\mu} - 1\right)^2 > 0.$$

The equation has two solutions

$$u_1^* = \frac{\Lambda}{\mu} \text{ this implies that } v_1^* = 0,$$

and

$$u_2^* = \frac{\sigma}{\lambda\alpha} \text{ this implies that } v_2^* = \frac{\Lambda}{\sigma} - \frac{\mu}{\lambda\alpha},$$

we have

$$R_0 = \frac{\lambda\Lambda}{\mu\sigma}\varphi'(0) = \frac{\lambda\Lambda}{\mu\sigma}\alpha,$$

so

$$v_2^* = \frac{\mu}{\lambda\alpha}(R_0 - 1).$$

The system (3.1) possesses two constant steady states

$$\begin{cases} E_0 = \left(\frac{\Lambda}{\mu}, 0\right), \\ E^* = \left(\frac{\sigma}{\lambda\alpha}, \frac{\mu}{\lambda\alpha}(R_0 - 1)\right). \end{cases}$$

• When $R_0 > 1$, the second steady state E^* exists and it is globally asymptotically stable.

• When $R_0 < 1$,the first steady state E_0 is globally asymptotically stable with $\frac{(d_1+d_2)^2}{4d_1d_2} \leq \theta \leq \frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda\alpha} - \frac{\Lambda}{\sigma}\right)$ when $d_1 \neq d_2$.

To obtain numerical solutions we use the information mentioned in the table 3.1

Set	**Figure**	$\mathbf{U_0}$	$\mathbf{V_0}$	$\mathbf{d_1}$	$\mathbf{d_2}$	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{\sigma}$	$\boldsymbol{\Lambda}$	$\boldsymbol{\mu}$	$\mathbf{R_0}$
Set 1	1	$4 + \frac{\cos(x)}{8}$	$0.6 + \frac{\sin(x)}{8}$	3	2	$\frac{9}{10}$	$\frac{2}{15}$	$\frac{1}{2}$	6	$\frac{3}{4}$	1.9200
Set 2	2	$0.4 + \frac{\cos(x)}{8}$	$2 + \frac{\sin(x)}{8}$	3	2	2	$\frac{1}{3}$	2	9	4	0.7500

Table 3.1: Simulation parameters for example 1

We assume a single spatial dimension with $\Omega = (0, 10)$.

• Figure 3.1 depicts the solution subject to parameter set 1, where $R_0 = 1.9200 > 1$, which by Theorem 2.5 means that $E^* = (4.1667, 5.75)$ is globally asymptotically stable.

• Figure 3.2 depicts the solution subject to parameter set 2, where $R_0 = 0.7500 < 1$. By Theorem 2.4 and given $\theta \in \left[\frac{25}{24}, 2\right]$, $E_0 = (2.25, 0)$ is globally asymptotically stable.

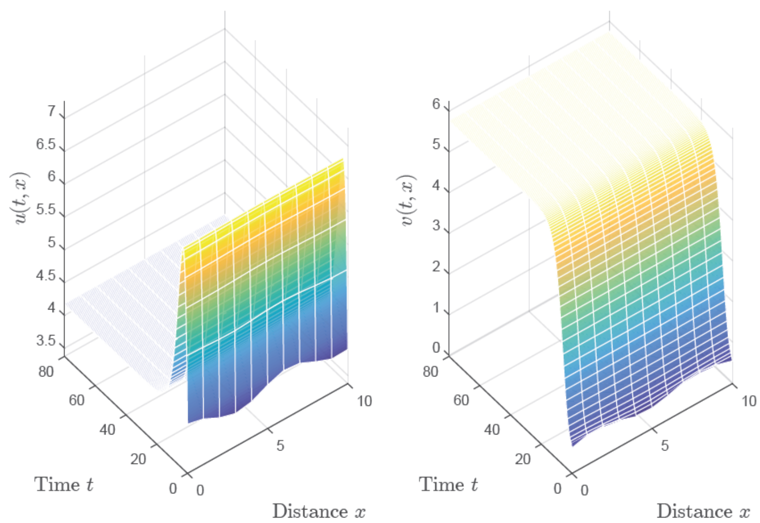

Figure 3.1: Numerical solutions of system (3.1) subject to the first set of parameters.

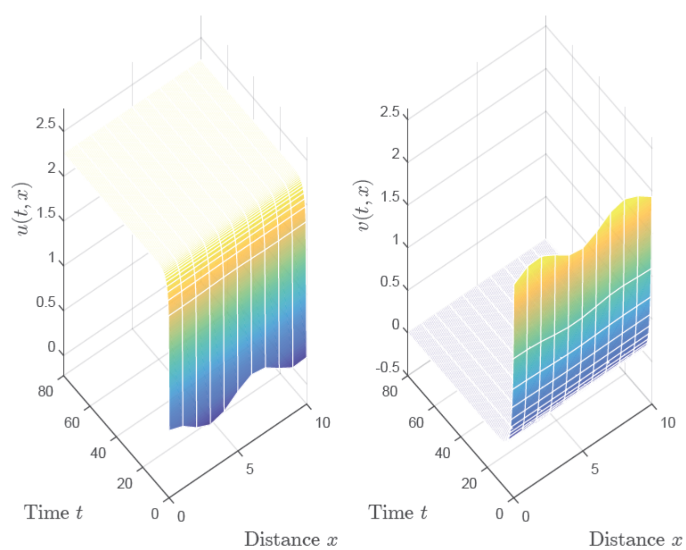

Figure 3.2: Numerical solutions of system (3.1) subject to the second set of parameters.

3.2 Second Example

In this example we consider the function

$$\varphi(v) = \frac{\alpha v}{1+kv}, \ \alpha > 0 \text{ and } k \geq 0. \tag{3.3}$$

The system is given (2.1) by substituting the considered function

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = -\lambda \frac{\alpha u v}{1+kv} + \Lambda - \mu u & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial v}{\partial t} - d_1 \Delta v = \lambda \frac{\alpha u v}{1+kv} - \sigma v & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) & \text{on } \Omega, \\ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{3.4}$$

Checking that the function (3.3) meets the conditions (2.4) and (2.5)

$$\varphi(0) = 0 \text{ and } \varphi(v) > 0 \text{ for all } v > 0.$$

Derive the function (3.3)

$$\varphi'(v) = \frac{\alpha}{(1+kv)^2}, \ \alpha > 0 \text{ and } \varphi'(0) = \alpha.$$

Finding steady states for the system (3.4)

▷ If $u = 0$, the system (3.4) has no equilibrium.

▷ If $v = 0$, then equilibrium is $E_0 = (\frac{\Lambda}{\mu}, 0)$.

Next, we find E^*

$$\lambda \frac{\alpha u^* v^*}{1+kv^*} - \sigma v^* = 0 \text{ implies that } u^* = \frac{\sigma(1+kv^*)}{\lambda\alpha}.$$

We have

$$\Lambda - \mu u^* - \sigma v^* = 0,$$

we replace u^* this equation

$$\sigma v^* = \Lambda - \mu\left(\frac{\sigma(1+kv^*)}{\lambda\alpha}\right),$$

we find

$$v^* = \frac{\mu\left(\frac{\alpha\lambda\Lambda - 1}{\mu\sigma} - 1\right)}{\alpha\lambda + \mu k} \text{ implies that } v^* = \frac{\mu(R_0 - 1)}{\alpha\lambda + \mu k}$$

so

$$E^* = \left(\frac{\sigma(1+kv^*)}{\lambda\alpha}, \mu\frac{(R_0 - 1)}{\alpha\lambda + \mu k}\right)$$

• E^* exists and it is globally asymptotically stable provided that the reproduction number $R_0 > 1$.
• E_0 is globally asymptotically stable when $R_0 < 1$ with $\frac{(d_1+d_2)^2}{4d_1d_2} \leq \theta \leq \frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda\alpha} - \frac{\Lambda}{\sigma}\right)$ when $d_1 \neq d_2$.
To obtain numerical solutions in we use the information mentioned in the table 3.2

Set	**Figure**	$\mathbf{U_0}$	$\mathbf{V_0}$	$\mathbf{d_1}$	$\mathbf{d_2}$	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{\sigma}$	$\boldsymbol{\Lambda}$	$\boldsymbol{\mu}$	**k**	$\mathbf{R_0}$
Set 1	3	$5 + \frac{\cos(x)}{9}$	$0.4 + \frac{\sin(x)}{10}$	$\frac{1}{2}$	4	$\frac{9}{10}$	$\frac{6}{13}$	$\frac{5}{4}$	$\frac{25}{6}$	$\frac{7}{8}$	4	1.5824
Set 2	4	$0.6 + \frac{\cos(x)}{9}$	$1.5 + \frac{\sin(x)}{10}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{9}{10}$	$\frac{6}{13}$	$\frac{5}{4}$	$\frac{25}{6}$	$\frac{7}{8}$	$\frac{9}{4}$	1.5824
Set 3	5	$0.4 + \frac{\cos(x)}{10}$	$0.3 + \frac{\sin(x)}{9}$	3	$\frac{4}{9}$	6	$\frac{1}{13}$	$\frac{9}{8}$	$\frac{11}{4}$	3	6	0.3761
Set 4	6	$0.8 + \frac{\cos(x)}{8}$	$3.2 + \frac{\sin\left(\frac{\pi}{2}x\right)}{8}$	3	$\frac{4}{9}$	6	$\frac{1}{13}$	$\frac{9}{8}$	$\frac{11}{4}$	3	$\frac{3}{7}$	0.3761

Table 3.2: Simulation parameters for example 2

• Figure 3.3 shows the **PDEs** solution obtained using parameter set 1 with $E^* = (4.5760, 0.1302)$. In this case, $R_0 = 1.5824 > 1$ and by Theorem 2.5, E^* is globally asymptotically stable.
• Figure 3.4 shows the **PDEs** solution obtained using parameter set 2 with $E^* = (4.4565, 0.2138)$. Since $R_0 = 1.5824 > 1$, E^* is globally asymptotically stable.
• Figure 3.5 shows the **PDEs** solution obtained using parameter set 3 with $E_0 = (0.9167, 0)$. In this case, $R_0 = 0.3761 < 1$ and using Theorem 2.4 with $\theta \in \left[\frac{961}{432}, 7.0833\right]$, E_0 is globally asymptotically stable.
• Figure 3.6 shows the **PDEs** solution obtained using parameter set 4 with $E_0 = (0.9167, 0)$. In this scenario, we again have $R_0 = 0.3761 < 1$ and with $\theta \in \left[\frac{961}{432}, 7.0833\right]$, E_0 is globally asymptotically stable.

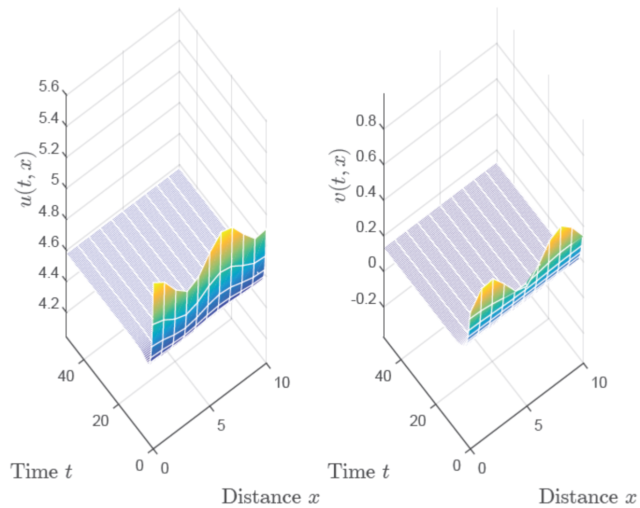

Figure 3.3: Numerical solutions of system (3.4) subject to the first set of parameters.

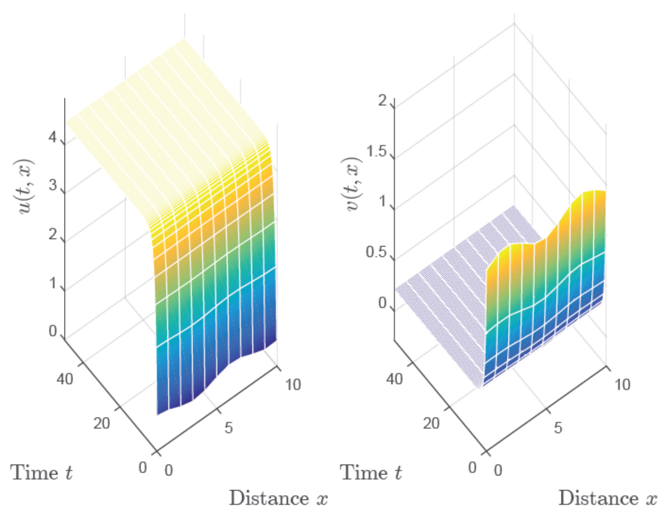

Figure 3.4: Numerical solutions of system (3.4) subject to the second set of parameters.

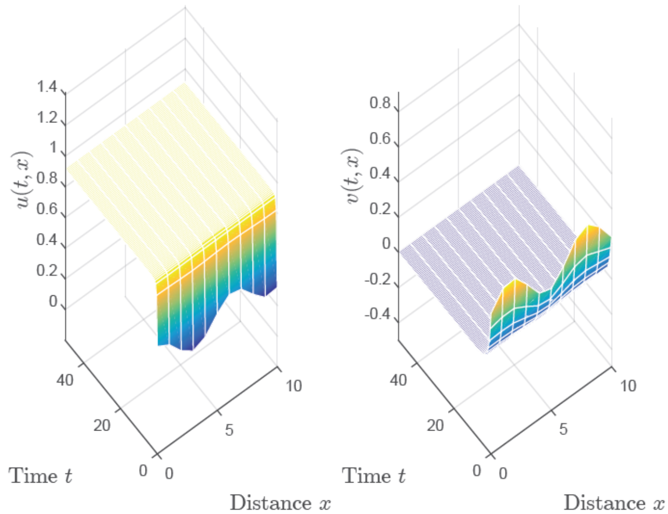

Figure 3.5: Numerical solutions of system (3.4) subject to the third set of parameters.

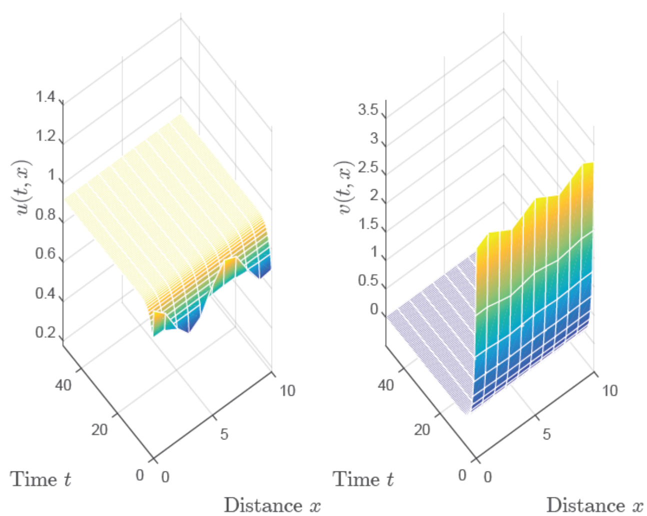

Figure 3.6: Numerical solutions of system (3.4) subject to the fourth set of parameters.

3.3 Third Example

In this example, we consider the function

$$\varphi(v) = \frac{kv}{1+\left(\frac{v}{\alpha}\right)}, \text{ for all } \alpha > 0 \text{ and } k > 0,$$

by substituting in (2.1), we get the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = -\lambda k \frac{v}{1+\left(\frac{v}{\alpha}\right)} u + \Lambda - \mu u & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \lambda k \frac{v}{1+\left(\frac{v}{\alpha}\right)} u - \sigma v & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x) & \text{on } \Omega, \\ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases} \tag{3.5}$$

Conditions (2.4) and (2.5) are satisfied as

$$\begin{cases} \varphi(0) = 0, \\ \varphi'(v) = \frac{k}{\left(1+\left(\frac{v}{\alpha}\right)\right)^2} > 0 \text{ for all } v \geq 0, \\ \varphi'(0) = k, \\ v\varphi'(v) = v\frac{k}{\left(1+\left(\frac{v}{\alpha}\right)\right)^2} \leq \varphi(v) = \frac{kv}{1+\left(\frac{v}{\alpha}\right)}. \end{cases}$$

Finding steady states for the system (3.5)

$$\begin{cases} -\lambda k \frac{v^*}{1+\left(\frac{v^*}{\alpha}\right)} u^* + \Lambda - \mu u^* & = 0 \\ \lambda k \frac{v^*}{1+\left(\frac{v^*}{\alpha}\right)} u^* - \sigma v^* & = 0 \end{cases} \tag{3.6}$$

▷ If $u^* = 0$, the system (3.5) has no equilibrium.

▷ If $v^* = 0$, then equilibrium is $E_0 = (\frac{\Lambda}{\mu}, 0)$.

Next, we find E^*.

From the second equation for (3.6) we extract

$$u^* = \frac{\sigma(\alpha + v^*)}{\lambda\alpha k}.$$

We sum the first and second equations of (3.6)

$$\Lambda - \mu u^* - \sigma v^* \ = 0,$$

we substitute the value of u^*

$$\Lambda - \mu\frac{\sigma(\alpha + v^*)}{\lambda\alpha k} - \sigma v^* \ = 0,$$

we find

$$v^* = \frac{\lambda k \Lambda \alpha - \mu\sigma\alpha}{\lambda\alpha k\sigma + \mu\sigma} = \frac{\mu\sigma\alpha\left(\frac{\lambda k\Lambda}{\sigma\mu} - 1\right)}{\sigma\left(\lambda\alpha k + \mu\right)}.$$

We have $R_0 = \frac{\lambda\Lambda}{\mu\sigma}k$ so

$$v^* = \mu\alpha\frac{(R_0 - 1)}{(\lambda k\alpha + \mu)}.$$

Ther are two steady states of system (3.5) are given by

$$\begin{cases} E_0 = \left(\frac{\Lambda}{\mu}, 0\right), \\ E^* = \left(\frac{\sigma(\alpha + v^*)}{\lambda\alpha k}, \mu\alpha\frac{(R_0-1)}{(\lambda k\alpha+\mu)}\right). \end{cases}$$

- E^* exists and it is globally asymptotically stable.
- E_0 is globally asymptotically stable if

$$\frac{(d_1 + d_2)^2}{4d_1d_2} \le \theta \le \frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda k} - \frac{\Lambda}{\sigma}\right) \text{ when } d_1 \neq d_2,$$

To obtain numerical solutions we use the information mentioned in the table 3.3

Set	Figure	U_0	V_0	d_1	d_2	λ	α	σ	Λ	μ	k	R_0
Set 1	7	$2.5+\frac{\cos(x)}{8}$	$5+\frac{\sin(x)}{8}$	$\frac{1}{2}$	3	$\frac{8}{15}$	$\frac{4}{5}$	$\frac{1}{2}$	9	$\frac{2}{7}$	$\frac{1}{3}$	11.2000
Set 2	8	$3.1+\frac{\cos(x)}{5}$	$1.3+\frac{\sin(x)}{5}$	2	1	$\frac{8}{15}$	$\frac{4}{5}$	$\frac{1}{2}$	9	$\frac{2}{7}$	$\frac{1}{3}$	11.2000
Set 3	9	$1.5+\frac{\cos(x)}{10}$	$2.6+\frac{\sin(x)}{10}$	$\frac{2}{3}$	$\frac{5}{4}$	10	$\frac{8}{5}$	$\frac{4}{3}$	7	$\frac{19}{12}$	$\frac{1}{27}$	1.2281
Set 4	10	$0.3+\frac{\cos(x)}{9}$	$1.5+\frac{\sin(x)}{10}$	$\frac{4}{7}$	3	$\frac{2}{9}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{7}{4}$	$\frac{14}{9}$	$\frac{3}{4}$	0.2813
Set 5	11	$0.3+\frac{\cos(x)}{9}$	$0.2+\frac{\sin(x)}{12}$	8	$\frac{15}{7}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	0.2000

Table 3.3: Simulation parameters for example 3

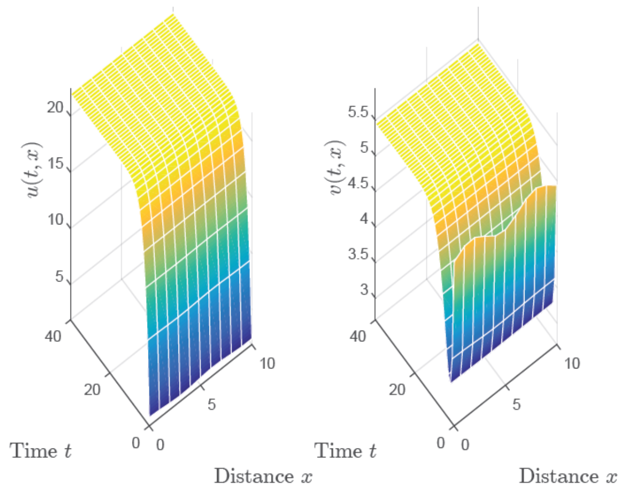

Figure 3.7: Numerical solutions of system (3.5) subject to the first set of parameters.

• Figure 3.7 shows the solution obtained using parameter set 1 with $E^* = (21.9659, 5.4481)$. In this case, $R_0 = 11.2 > 1$ and by Theorem 2.5, E^* is globally asymptotically stable.

• Figure 3.8 shows the solution obtained using parameter set 2 with $E^* = (21.9659, 5.4481)$. In this case, $R_0 = 11.2 > 1$ and E^* is globally asymptotically stable.

• Figure 3.9 shows the solution obtained using parameter set 3 with $E^* = (4.1974, 0.2655)$. In this case, $R_0 = 1.2281 > 1$ and E^* is globally asymptotically stable.

• Figure 3.10 shows the solution obtained using parameter set 4 with $E_0 = (1.1250, 0)$. In this case, $R_0 = 0.2813 < 1$ and by Theorem 2.4 with $\theta \in \left[\frac{625}{336}, 9.5185\right]$, E_0 is globally asymptotically stable.

• Figure 3.11 shows the solution obtained using parameter set 5 with $E_0 = (0.6667, 0)$. In this case, $R_0 = 0.2 < 1$ and by Theorem 2.4 with $\theta \in \left[\frac{5041}{3360}, 8\right]$, E_0 is globally asymptotically stable.

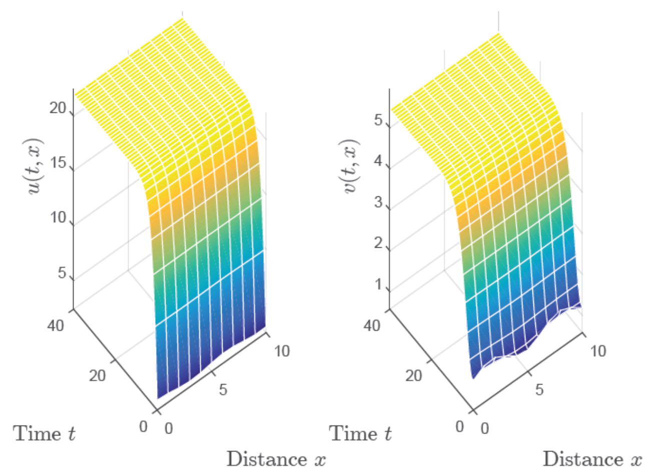

Figure 3.8: Numerical solutions of system (3.5) subject to the second set of parameters.

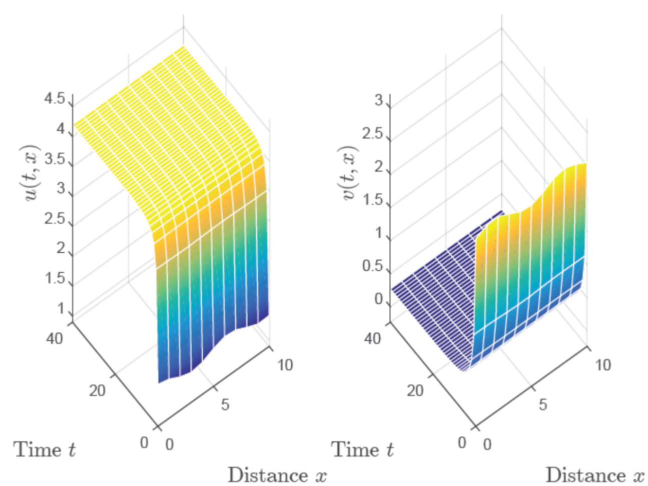

Figure 3.9: Numerical solutions of system (3.5) subject to the third set of parameters.

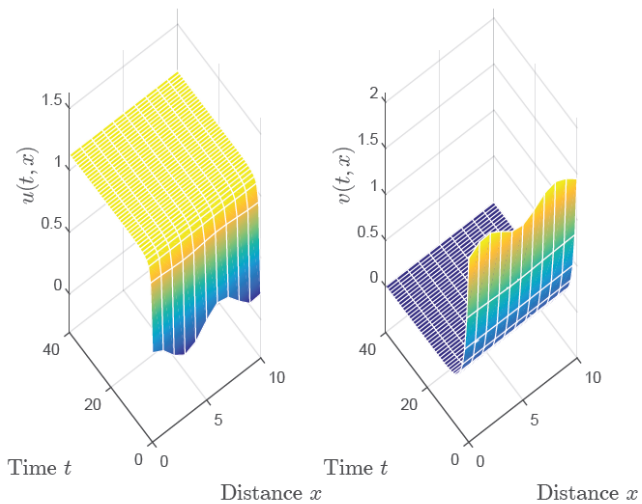

Figure 3.10: Numerical solutions of system (3.5) subject to the fourth set of parameters.

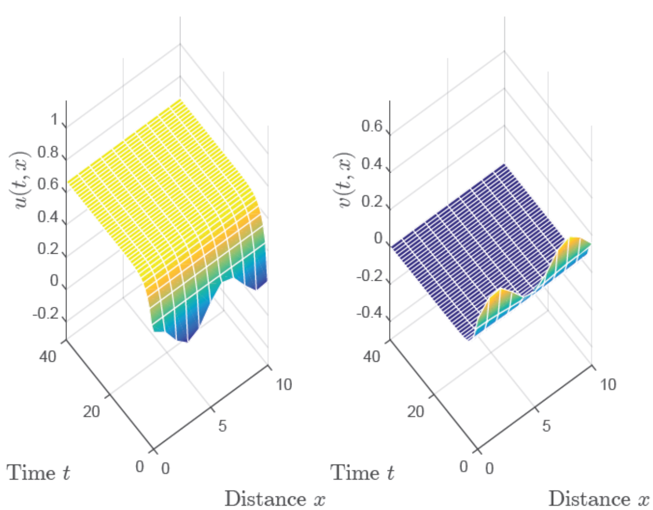

Figure 3.11: Numerical solutions of system (3.5) subject to the fifth set of parameters.

Conclusion

We have arrived here with you until the end of the mathematical scientific research in which we addressed the stability of the epidemic reaction-diffusion system.

Where we proceeded from the beginning of the research with preliminary concepts and theories related to global and local asymptotic stability, presenting the most general form of an reaction-diffusion system, and then dedicating the study to a model of the reaction-diffusion of an epidemiological (susceptible, infectious) with the non-linear incidence under the conditions $(2.4)-(2.5)$ and determining R_0 the basic reproduction number by which the discussion takes place.

In the case of ODEs, the disease-free equilibrium is asymptotic stable if R_0 is less than unity, while the endemic equilibrium is asymptotic stable if R_0 is greater than unity, and by applying the Lyapunov function we determine the state of global stability PDEs, we confirming this results numericaly.

Hence, we suggest some aspects for future research, which are:

• When R_0 is equal to the unit. Are the disease-free and endemic equilibrium asymptotically stable?

• When condition (2.5) is like this:

$$\varphi(v) \leq v\varphi'(v) \text{ for all } v > 0.$$

Will the results of this study change and how will that be ?

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