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Theme

Study of some fractional elliptic problems

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Dedication

I am dedicating this modest thesis to my parents, to all my family, friends, and peers.

ملخص

العمل المقدم في هذه الأطروحة مخصص لدراسة الوجود وتعددية الحلول الموجبة غير التافهة لمسألة p ـ لابلاس الكسرية.

تم الحصول على النتائج باستخدام بعض تقنيات التنويع. النتيجة الأولى هي وجود حلين موجبين غير تافهين ، باستخدام منوعة نيهاري وطريقة الألياف ، في الحالات الحرجة ودون الحرجة.

النتيجة الثانية تحتوي على وجود ثلاثة حلول مختلفة ، فقط في الحالة الحرجة ، باستخدام مبدأ إيكيلاند المتغير، في ظل فرضيات مختلفة.

الكلمات المفتاحية : المعامل الكسري ، منوعة نهاري ، طريقة الألياف ، مبدأ إيكيلاند المتغير.

Résumé

Le travail présenté dans cette thèse est consacré à l´étude de l´existence et la multiplicité de solutions positives non triviales d´un problème p-laplacien fractionnaire. Les résultats sont obtenus en utilisant certaines techniques variationnelle. La première résultat est celle de l´existence de deux solutions positives non-triviales, en utilisant la variété de Nehari et la méthode de fibering, dans les cas critique et sous-critique. La deuxième résultat contient l´existence de trois solutions distincts, seulement dans le cas critique, en utilisans le principe variationnelle d´Ekeland, sous différentes hypothèses.
Mots-clés: Opérateur fractionnaire, la variété de Nehari, méthode des fibering, principe d'Ekeland.

Abstract

The work presented in this thesis is devoted to the study of the existence and multiplicity positive non-trivial solutions of a fractional p-Laplacian problem. The results are obtained using some variational techniques. The first result is that of the existence of two non-trivial positive solutions, using the Nehari manifold and the fibering method, in the critical and subcritical cases. The second result contains the existence of three distinct solutions, only in the critical case, using Ekeland's variational principle, under different assumptions.
Keywords: Fractional operator, Nehari Manifold, Fibering method, Ekeland's variational principle.

Introduction

The topic of differential fractional equations is one of the most important subjects since the concept of fractional calculus appeared in the correspondence of Leibniz and Lopital in 1695. this topic has several important applications in the real world. Physics, financial, mechanics, chemestry. And other various phenomena in diverse fields that can be studied as a fractional equation such as nuclear reactor dynamics, thermoelasticity, mecanical vebrations, bilogical tissues, fractional entropy, fractional diffusion [29, 31], phase transitions [4, 36], materials science [12], water waves [20, 23], conservation laws [13]. For more details on this subject, we refer to [9, 19].

Our interest in the work is related to the fractional Laplacian (the Ries fractional derevative), which appears in physical sciences, describing an unusual diffusion process as result of the random jumpers that are able to move between nearby sites, or distant sites by means of Lèvey flights, in the general case, fracional Laplacian describes the contribution to a conservation law of a non-local process affected by the global state of intrest at given time. This study is concerned with fractional Laplacian equations with regular nonlinearities. There are many works on the existence of solutions for this kind of elliptic equations such as

$$\begin{cases} (-\Delta)^s u = \lambda u^p + u^q \text{ in } \Omega, u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $N > 2s, 0 < s < 1, p, q > 0$, and $\lambda > 0$, see for example [1, 11]. In [33, 34], Servadei and Valdinoci, studied the following more general problem

$$\begin{cases} (-\Delta)^s u = f(x, u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2}$$

Under appropriate conditions on the nonlinearity f, and by using variational method, the authors proved the existence and multiplicity of non-negative solutions to the subcritical

growth problems (2). Critical exponent problems like (2), are studied in [11,28,35].
Problems similar to (1) have been also studied in the local setting with different elliptic operators; see [5,14,25]. Very recently, Saoudi et al. [33], considered an extension of (1), more precisely

$$\begin{cases} \mathcal{L}u = \lambda a(x)u^p + b(x)u^q \text{ in } \Omega, u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where $\mathcal{L}$ is a nonlocal operator which is a generalization of the fractional laplacian equation. By constrained minimization on suitable subsets of Nehari manifold combined with fibering maps, the authors proved that for $\lambda > 0$, small enough, problem (3), has at least two non-negative solutions.
As far as we know, in this direction, the first example for the p-Laplacian operator, was given in [27]. After that, problems involving fully nonlinear operators has been studied in [22]. Ghanmi [27], considered the following elliptic problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{q-2} u + f(x,u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4}$$

where $(-\Delta)_p^s$ is the p-fractional Laplacian operator. Using the decomposition of the Nehari manifold, the auther, proved that the non-local elliptic problem (4), has at least two nontrivial solutions. We also refer to [7,8] where the author obtained a multiplicity result for a more general problem.
In this thesis, we consider the following p-fractional Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{E}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > ps$, $s \in (0,1)$, λ and μ are positive parameters and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_+$, are continuous functions.
In the first chapter, we start by giving some basic notions, that concern the functional framework necessary to obtain the results of the existence of solutions for the considered problem.

In the second chapter, we present the variational methods, which contains the critical point theory, Nehari manifold, fibering map, palais-smale condition, and Ekeland's variational principal.
In the third chapter, we use the Nehari Manifold and fibering maps to obtain the existence of two positive solutions.
In the fourth chapter, under different assumptions, we obtain the existence of three different solutions, by using Ekeland's variational principal.

CHAPTER 1

Preliminaries

Contents

In this chapter, we briefly recall the essential definitions and notions which we will use in the later chapters such as space of continuous functions, $L^p(\Omega)$ spaces, Sobolev and fractional Sobolev spaces, and some basic theorems.

1.1 Functional spaces

1.1.1 Space of continuous functions

Definition 1.1 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}$ a function. We say that u is continuous if*

$$\forall x_0 \in \Omega, \forall \varepsilon > 0, \exists \delta > 0,$$

such that

$$x \in E, \|x - x_0\| < \delta \Longrightarrow |u(x) - u(y)| < \varepsilon,$$

where the norm in $\mathbb{R}^N$ is the Euclidean norm.

Definition 1.2 *Let Ω be an open set in $\mathbb{R}$. We define :*

$$C(\Omega) := \{u : \Omega \to \mathbb{R} \; u \textit{ is continuous}\},$$

$$C\left(\overline{\Omega}\right) := \left\{u : \Omega \to \mathbb{R} \; u \textit{ is continous and extends continuously to } \overline{\Omega}\right\}.$$

Let

$$\|.\|_C : C\left(\overline{\Omega}\right) \to \mathbb{R},$$

$$u \longmapsto \sup_{x \in \Omega} |u(x)| \;\; \textit{is a norm.}$$

1.1.2 $L^p(\Omega)$ Spaces

[15]

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$, we set

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \backslash \; f \text{ is measurable and } \int |f|^p \, d\mu < \infty \right\},$$

we define the L^p norm of f by

$$\|f\|_{L^p} = \|f\|_p = \left(= \int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

If $p = \infty$, the space $L^{\infty}(\Omega)$ satisfy

$$L^{\infty}(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} / \ f \text{ is measurable and } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ a.e on } \Omega\},$$

we define the L^{∞} norm of f by

$$\|f\|_{L^{\infty}} = \|f\|_{\infty} = \inf\{C; |f| \leq C \text{ a.e on } \Omega\},$$

$L^{\infty}(\Omega)$ is a Banach space.

If $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for scalar product

$$(f, g) = \int_{\Omega} f(x)g(x)dx.$$

We denote by $L^1_{loc}(\Omega)$ the set of locally integrable functions on Ω and we write

$$L^1_{loc}(\Omega) = \left\{u : u \in L^1(K) \text{ for all compact } K \text{ of } \Omega\right\}.$$

Remark 1.1 *If $f \in L^{\infty}(\Omega)$ then we have*

- $|f| \leq \|f\|_{L^{\infty}}$ *a.e. on* Ω.
- $L^p(\Omega) \subset L^1_{loc}(\Omega)$ *for all* $1 \leq p \leq \infty$.
- $\left(L^p(\Omega), \|.\|_p\right)$ *is Banach space for* $1 \leq p \leq \infty$ *separable for* $1 \leq p < \infty$ *and reflexive for* $1 < p < \infty$

Theorem 1.1 *[15]* ***(Hölder's inequality)***

Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

1.1.3 Sobolev Space $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.3 *[15] The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \exists g_1, ..., g_N \in L^p(\Omega) \text{ such that} \int\limits_{\Omega} u \frac{\partial \varphi}{\partial x_i} = -\int\limits_{\Omega} g_i \varphi \; \forall \varphi \in C_c^\infty(\Omega),\ \forall i = \overline{1, N} \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ we define $\frac{\partial u}{\partial x_i} = g_i$,and we write

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}.$$

Proposition 1.1 *[15] $W^{1,p}(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and it is separable for $1 \leq p < \infty$.*

Corollary 1.1 *[24] Let $1 \leq p \leq \infty$. We have*

- *$W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, if $p < N$,*
- *$W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [p, +\infty)$, if $p = N$,*
- *$W^{1,p}(\Omega) \subset L^\infty(\Omega)$, if $p > N$,*

and all these injections are continuous. Moreover, if $p > N$ we have, for all $u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}} |x - y|^\alpha \text{ a.e. } x, y \in \Omega,$$

with $\alpha = 1 - (N/p)$ and C is a constant depends only on Ω, p, and N. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Theorem 1.2 *[24] **(Rellich–Kondrachov)**. Suppose that Ω is bounded and of class C^1. Then we have the following compact injections:*

- *$W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [1, p^*) W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [1, p^*)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, if $p < N$,*
- *$W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [p, +\infty)$, if $p = N$,*
- *$W^{1,p}(\Omega) \subset C(\overline{\Omega})$, $\forall q \in [p, +\infty)$, if $p > N$.*

In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$, with compact injection for all p (and all N).

1.1.4 Fractional Sobolev spaces $W^{s,p}(\Omega)$

Let Ω be a smooth bounded set in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, we introduce fractional Sobolev space

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

We consider the space

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R}, u \in L^p(\Omega) \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Sigma) \right\},$$

with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_{\Sigma} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \right)^{\frac{1}{p}}.$$

Proposition 1.2 *[24] The space $W^{s,p}(\Omega)$ is of local type, that is, for every u in $W^{s,p}(\Omega)$ and for every $\varphi \in D(\Omega)$, the product φu belongs to $W^{s,p}(\Omega)$.*

Proposition 1.3 *[24] The space $D(\mathbb{R}^N)$ is dense in $W^{s,p}(\Omega)$.*

Theorem 1.3 *[24] [28]Let $s \in]0,1[$ and let $p \in]1,\infty[$. We have:*

- *If $sp < N$, then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for every $q \le Np/(N-sp)$.*
- *If $N = sp$, then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for every $q < \infty$.*
- *If $sp > N$, then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and, more precisely,*

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow C_b^{0,s-N/p}(\mathbb{R}^N).$$

Proposition 1.4 *[24] Let $s \in [0,1[$ and let $p > 1$. Let Ω be an open set that admits an (s,p)-extension; then $D(\overline{\Omega})$, the space of restrictions to Ω of functions in $D(\mathbb{R}^N)$, is dense in $W^{s,p}(\mathbb{R}^N)$.*

Corollary 1.2 *[24] Let $s \in]0,1[$ and let $p \in]1,\infty[$. Let Ω be a Lipschitz open set. We then have:*

- *If $sp < N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \le Np/(N-sp)$.*
- *If $N = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.*

- *If $sp > N$, then $W^{s,p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ and, more precisely,*

$$W^{s,p}(\Omega) \hookrightarrow C_b^{0,s-N/p}(\Omega).$$

Theorem 1.4 *[24] Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^N$. Let $s \in [0,1[$, let $p > 1$, and let $N \geq 1$. We then have:*

- *If $sp < N$, then the embedding of $W^{s,p}(\Omega)$ into L^k is compact for every*

$$k < Np/(N - sp).$$

- *If $sp = N$, then the embedding of $W^{s,p}(\Omega)$ into L^q is compact for every $q < \infty$.*

- *If $sp > N$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ is compact for $\lambda < s - N/p$.*

1.2 Notions on operators

Let $(X, \|.\|)$ be a real Banach space and let X' be topological dual.

Definition 1.4 *Let $A : X \to X'$, we say that :*

- ***Continuous** if $\|Ax_n - Ax\|_{X'} \to 0$ when $\|x_n - x\|_X \to 0$.*

- ***Compact** if $A(\overline{B}_X)$ is relatively compact in X', where B_X denotes the ball unit in X.*

- ***Coercive** if*

$$\lim_{\|x\| \to +\infty} \frac{\langle A(x), x\rangle}{\|x\|} = +\infty.$$

- ***Monotonous** if*

$$\langle Au - Av, u - v\rangle \geq 0, \forall u, v \in X \text{ with } u \neq v.$$

- ***Strictly monotonous** if*

$$\langle Au - Av, u - v\rangle > 0, \forall u, v \in X \text{ with } u \neq v.$$

- ***Bounded** if the image by A of any bounded subset of X is a bounded subset of X'.*

- ***Semi-continuous***

$$\text{if } u_n \to u \text{ when } n \to \infty \text{ implies } Au_n \rightharpoonup Au \text{ when } n \to \infty.$$

- ***Strongly continuous***

$$\text{if } u_n \rightharpoonup u \text{ when } n \to \infty \text{ implies } Au_n \to Au \text{ when } n \to \infty.$$

1.3 Weak derivative

Definition 1.5 *[30] **(Directional derivative)***

Let w be a part of a Banach space X and $F : w \to \mathbb{R}$ a real valued function. If $u \in w$ and $z \in X$ we have $u + tz \in w$, we say that F admits (at the point u) a derivative in the direction z if the limit

$$\lim_{t \to \overset{+}{0}} \frac{F(u+tz) - F(u)}{t}, \text{ for all } t > 0 \text{ small enough}$$

exists. We will denote this limit $F'_z(u)$. The Gateaux differential generalizes the idea of a directional derivative.

Definition 1.6 *[30] **(Gateaux derivative)** Let w be a part of a Banach space X and $F : w \to \mathbb{R}$. If $u \in w$, we say that F is Gateaux differentiable in u, if there exists $l \in X'$ or $F(u+tz)$ for $t > 0$ small enough. The Gateaux differential is defined*

$$\langle l, z \rangle = \lim_{t \to \overset{+}{0}} \frac{F(u+tz) - F(u)}{t}.$$

Where $F'(u) := l$.

Definition 1.7 *[30] **(Frechet derivative)** Let X be a Banach space, W an open space in X and F a function. If $u \in w$, we say that F is differentiable (or derivable) in u (in the sense of Frechet) if there exists $l \in X'$, such that:*

$$\textit{for all} \quad v \in W \quad \textit{we have} \text{ , } F(v) - F(u) = \langle l, v-u \rangle + \sigma(v-u).$$

If F is differentiable, l is unique and we denote by $F'(u) := l$. The set of differentiable functions $w \to \mathbb{R}$ will be denoted by $C^1(w, \mathbb{R})$.

1.4 Convergence criteria

Theorem 1.5 *[15] **(Lebesgue's dominated convergence)** Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*

- *$f_n(x) \longrightarrow f$ a.e, on Ω,*

- *There is a function $g \in L^1(\Omega)$ such that for all n,*

$$|f_n(x)| \leq g(x), \quad a.e. \text{ on } \Omega.$$

Then

$$f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1} \longrightarrow 0.$$

Theorem 1.6 [30] ***(Vitali's convergence theorem)*** *Let $f_1, f_2, ...$be L^p-integrable functions on some measure space, for $1 \leq p < \infty$. The sequence $\{f_n\}$ converges in L^p to a measurable function f if and and only if*

- *The sequence $\{f_n\}$ converges to f in measure,*
- *The functions $\{|f_n|^p\}$ are uniformly integrable*
- *For every $\epsilon > 0$, there exists a set E of finite measure, such that $\int\limits_{E^c} |f_n|^p < \epsilon$ for all n.*

Theorem 1.7 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that*

$$\|f_n - f\|_p \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Then, there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a function $h \in L^p(\Omega)$ such that

- *$f_{n_k}(x) \longrightarrow f(x)$ $a.e$ on Ω,*
- *$|f_{n_k}(x)| \leq h(x)\ \forall k$, $a.e.$ on Ω.*

Lemma 1.1 [15] ***(Fatou's Lemma)***

Let (f_n) a sequence of functions in $L^1(\Omega)$ that satisfy

- *For all n, $f_n \geq 0$,*
- *$\sup\limits_n \int f_n < \infty$,*

For almost all $x \in \Omega$ we set $f(x) = \liminf\limits_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f(x)dx \leq \lim_{n\to\infty} \inf \int_\Omega f_n(x)dx.$$

Lemma 1.2 [16] ***(Brezis–Lieb).***[14] *Let Ω be an open bounded set in $\mathbb{R}^n$ and $1 < p < +\infty$, $(f_n)_n$ is sequence of measurable functions such that $f_n \to f$ a.e. in $L^p(\Omega)$, then*

$$f \in L^p(\Omega) \text{ and } \|f\|_p^p = \|f_n\|_p^p - \|f_n - f\|_p^p + \sigma(1).$$

Lemma 1.3 *[34] Let $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ satisfy the assumptions*

- $\gamma K \in L^1(\mathbb{R}^n)$, *where* $\gamma(x) = \min\{\|x\|^2, 1\}$,
- $K(x) = K(-x)$ *for any* $x \in \mathbb{R}^n \backslash \{0\}$.

And let v_j be a bounded sequence in X_0. Then, there exists $v_\infty \in L^v(\mathbb{R}^n)$ such that, up to a subsequence,

$$v_j \to v_\infty \text{ in } L^v(\mathbb{R}^n) \text{ as} j \to \infty, \text{ for any } v \in [1, 2^*).$$

Theorem 1.8 (Bolzano's Theorem) *Let a and b two real numbers with $a < b$ and let*

$$g : [a, b] \to \mathbb{R}$$

a continuous application where

$$g(a)g(b) \leq 0.$$

Then g admits at least one zero in $[a, b]$

Definition 1.8 *[15] Let $f : D \to R$ and let $x_0 \in D$. We say that f is lower semicontinuous function (l.s.c) at x_0 if for every $\epsilon > 0$, there exist $\delta > 0$ such that*

$$f(x_0) - \epsilon < f(x) \quad \text{for all} \quad x \in B(x_0; \delta) \cap D.$$

Or equivalently

$$\lim_{x \to x_0} \inf f(x) \geq f(x_0).$$

CHAPTER 2

Variational methods

Contents

This chapter contains some fundamental definitions and theorems, as well as the different variational techniques which, will be used to obtain the main results in this thesis.

2.1 Critical point theory

2.1.1 Critical point

Definition 2.1 (Homogeneous function) *Let f be a function of n variables defined on a set S for which $(tx_1, ..., tx_n) \in S$ whenever $t > 0$ and $(tx_1, ..., tx_n) \in S$. Then f is homogeneous of degree k if*

$$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n) \text{ for all } (x_1, ..., x_n) \in S \text{ and for all } \quad t > 0.$$

Definition 2.2 (Coercivity) *f is a coercive function if*

$$\lim_{\|x\| \to \infty} f(x) = \infty$$

Definition 2.3 (Critical point) *[30] A point $(u, v) \in E$ is critical for J_λ if $J'_\lambda(u, v) = 0$, otherwise (u, v) is regular. If $J_\lambda(u, v) = c$ for some critical point $(u, v) \in E$ of J_λ, the value c is critical, otherwise c is regular.*

Let E be a Banach space, $\Phi \in C^1(E, \mathbb{R})$and $\mathcal{N}$ is a set of constraints where:

$$\mathcal{N} = \{v \in E : \Phi(v) = 0\}.$$

Definition 2.4 (Lagrange multiplier) *[30] we suppose that for all $u \in \mathcal{N}$, we have $\Phi'(u) \neq 0$. If $J \in C^1(E, \mathbb{R})$ we say that $c \in \mathbb{R}$ is critical value of J on $\mathcal{N}$, if there exists $u \in \mathcal{N}$, and $\lambda \in \mathbb{R}$ such that*

$$J(u) = c \text{ and } J'(u) = \lambda \Phi'(u).$$

The point u is a critical point of J on $\mathcal{N}$ and the real λ is called the Lagrange multiplier for the critical value c (or the critical point u).

When X is a functional space and the equation $J'(u) = \lambda\Phi'(u)$ corresponds to a partial derivative equation, we say that $J'(u) = \lambda\Phi'(u)$ is the Euler-Lagrange equation (or the Euler's equation) satisfied by the critical point u on the constraint $\mathcal{N}$.

Theorem 2.1 *[30] Let $(E, \|.\|)$ be a Banach space, Ω an open in E and $J : \Omega \to \mathbb{R}$ a differentiable function on Ω and $\Phi \in C^1(\Omega, \mathbb{R}^n)$ of components $\Phi_1, ..., \Phi_n$. Given a point in $\mathbb{R}^n$, we set $K = \Phi^{-1}(a)$ which we assume not empty, if at a point $u_0 \in K$*

$$J(u_0) = \inf_{x \in K} J(u),$$

and if moreover the differential $\Phi'(u_0) \in L(E, \mathbb{R}^n)$ is surjective then there exist real numbers $\lambda_1, ..., \lambda_n$ for which

$$J'(u_0) = \sum_{i=1}^{n} \lambda_i \Phi_i'(u_0).$$

2.1.2 Palais-Smale condition

Definition 2.5 *A Plais-Smale sequence for the funcational I is a sequence $(x_n)_{n \in N}$ satisfying*

- *$I_{X_0}(x_n)_{n \in N}$ is Bounded.*
- *$I'_{X_0}(x_n)$ goes to zero in X'.*

Definition 2.6 *[32] the Palais-Smale condition is a compactness property related to functional defined on a Banach space. It states as follows: Let $I : E \to F$ be a C^1 functional defined on the Banach space E. and $c \in \mathbb{R}$.*
If for any given sequence $(x_n)_n$ in E such that $I(x_n) \to c$ and $I'(x_n) \to 0$ there exist a converging subsequence of $(x_n)_n$, we say that I satifies the Palais-Smale at level c.

Remark 2.1 *If $I(x_n)$ is bounded, $I'(x_n) \to 0$ in E' and $\|x_n\|_F$ is bounded we say that I satisfies a weak Palais-Smale condition.*

2.2 The Nehari Manifold

Nehari has introduced a variational method very useful in critical point theory and eventually came to bear his name. He considered a boundary value problem for a certain nonlinear second-order ordinary differential equation in an interval $[a, b]$ and proved that it has a nontrivial solution which may be obtained by constrained minimization. To describe Nehari'S method in an abstract setting, let E be a Banach space and $J \in C^1(E, \mathbb{R})$ a functional. The

Frechet derivative of J at u, $J'(u)$, is an element of the dual space E'. Suppose $u \neq 0$ is a critical point of J, i.e., $J^{'}(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} = \left\{ u \in E \backslash \{0\} : \left\langle J^{'}(u), u \right\rangle = 0 \right\}.$$

So $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of $J(u)$ by minimizing the energy functional J on the constraint $\mathcal{N}$ is called the Nehari manifold. Set

$$c := \inf_{u \in \mathcal{N}} J(u).$$

Under appropriate conditions on J one hopes that c is attained at some $u_0 \in \mathcal{N}$ and that u_0 is a critical point.

2.3 Fibering method

At the end of the 1990s, the fibering method or the decomposition method introduced by Pohozaev for investigating some variational problems, and its applications to nonlinear elliptic equations. Let X and Y be Banach spaces, and let A be a nonlinear operator acting from X to Y. We consider the equation

$$A(u) = h. \tag{2.1}$$

The fibering method is based on the representation of the solutions of the equation in the form

$$u = tv.$$

Where t is a real parameter, $t \neq 0$ in some open $J \subseteq \mathbb{R}$. Now, we give a complete description of the fibering method, we begin by defining the fibre map of the following

$$\phi(t) : \mathbb{R}^+ \to \mathbb{R} \text{ such that } \phi(t) = J(tu),$$

then, we calculate $\phi'(t), \phi''(t)$ the first and second derivative of $\phi(t)$. We decompose $\mathcal{N}$ into three parts $\mathcal{N}^+, \mathcal{N}^-$, and $\mathcal{N}^0$ corresponding respectively, to local minima, local maxima and points of inflection of ϕ defined as follows

$$\mathcal{N}^+ = \{u \in \mathcal{N} : \phi''(1) > 0\},$$

$$\mathcal{N}^- = \{u \in \mathcal{N} : \phi''(1) < 1\},$$

$$\mathcal{N}^0 = \{u \in \mathcal{N} : \phi''(1) = 0\},$$

and it is $\phi''(1)$ which is used for these definitions, since it is clear that if u is a local minimum for J, then u has a local minimum at $t = 1$. The method of decomposition $(F.M)$ makes it possible to find solutions to the non-coercive problems and in the absence of the continuity of the operator A.

2.3.1 Example of application

We consider the following problem:

$$\begin{cases} -\Delta u(x) = f(x, u(x)) \text{ in } \Omega, \\ \\ u(x) = 0 \text{ on } x \in \partial\, \Omega. \end{cases} \tag{P}$$

Let $E = W_0^{1,2}(\Omega)$ be the Banach space. The energy functional $J : E \to \mathbb{R}$ corresponding to the problem (P) defined as follows

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u(x)|^2\, dx - \int_{\Omega} F(x, u(x))dx.$$

Where $F(x, u(x)) = \int_0^u f(x, s)dx$. Obviously, the functional J may not be bounded on all the space but can be on some parts of E (called the Nehari manifold $\mathcal{N}$) defined as follows

$$\mathcal{N} = \left\{ u \in E : \left\langle J^{'}(u), u \right\rangle = 0 \right\}.$$

Theorem 2.2 *Let $u \in E\setminus\{0\}$ and $t > 0$. Then $tu \in \mathcal{N}$ if and only if $\phi_u'(t) = 0$ where*

$$\phi_u(t) = J(tu).$$

Proof By definition, one has

$$\phi_u(t) = J(tu).$$

Therefore

$$\phi_u'(t) = \langle J'(u), u \rangle = \frac{1}{t}\langle J'(tu), tu \rangle.$$

If $\phi_u'(t) = 0$, then $\langle J'(tu), tu \rangle = 0$ i.e $tu \in \mathcal{N}$. In other terms, the points of the manifold $\mathcal{N}$ correspond to the stationary points of the maps $\phi_u(t)$.

On the other hand, we decompose $\mathcal{N}$ into three parts $\mathcal{N}^+, \mathcal{N}^-, \mathcal{N}^0$ corresponding to local minima, local maxima and points of inflection of $\phi_u(t)$. For that, we calculate the second derivative of $\phi_u(t)$

$$\phi'_u(t) = \langle J'(tu), u\rangle$$

$$= \int_\Omega |\nabla(tu)|\,|\nabla u|\,dx - \lambda \int_\Omega f(x,tu)udx$$

$$= t\int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega f(x,tu)udx.$$

So

$$\phi''_u(t) = \int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega (f'_u(x,tu)u)udx$$

$$= \int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega f'_u(x,tu)u^2dx.$$

Thus, we conclude $\mathcal{N}^+, \mathcal{N}^-$,and $\mathcal{N}^0$defined as follows

$$\mathcal{N}^0 = \left\{u \in \mathcal{N}, \phi''_u(1) = 0\right\},$$

$$\mathcal{N}^+ = \left\{u \in \mathcal{N}, \phi''_u(1) > 0\right\},$$

$$\mathcal{N}^- = \left\{u \in \mathcal{N}, \phi''_u(1) < 0\right\}.$$

Since it is clear that if u is a local minimum for J, then u has a local minimum at $t = 1$.

Theorem 2.3 *Let $u \in \mathcal{N}$. Then*

- (i) $\phi'_u(1) = 0.$,
- (ii)

$$\begin{cases} u \in \mathcal{N}^+ \textit{ if } \phi''_u(1) > 0, \\ u \in \mathcal{N}^- \textit{ if } \phi''_u(1) < 0, \\ u \in \mathcal{N}^0 \textit{ if } \phi''_u(1) = 0. \end{cases}$$

Proof Let $u \in \mathcal{N}$ if and only if

$$\left\langle J^{'}(u), u \right\rangle = 0,$$

which is equivalent to : $\phi'_u(1) = 0$ hence (i).

For (ii), there are three cases:

case $1 : u \in \mathcal{N}^{+}$, then

$$\int_{\Omega} (|\nabla u|^2 - \lambda f'_u(x,u)u^2)dx > 0$$

which is equivalent to $\phi''_u(1) > 0$.

case $2 : u \in \mathcal{N}^{-}$, then

$$\int_{\Omega} (|\nabla u|^2 - \lambda f'_u(x,u)u^2)dx < 0$$

which is equivalent to $\phi''_u(1) < 0$.

case $3 : u \in \mathcal{N}^{0}$, then

$$\int_{\Omega} (|\nabla u|^2 - \lambda f'_u(x,u)u^2)dx = 0$$

which is equivalent to $\phi''_u(1) = 0$.

The following theorem attests that the minimizers of J on the manifold $\mathcal{N}$ are true, in general, critical points of J.

Theorem 2.4 *Suppose u_0 is a local minimizer for J on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$.*
Then

$$J'(u_0) = 0.$$

Proof According to Lagrange's multiplier theorem

$$\exists\, \eta \in \mathbb{R} : J'(u_0) = \eta \xi'(u_0),$$

so

$$\langle J'(u_0), u_0 \rangle = \eta \, \langle \xi'(u_0), u_0 \rangle \, .$$

The constraint ξ defined as follows

$$\xi(u) = \langle J'(u), u \rangle = \int_{\Omega} (|\nabla u|^2 - \lambda f(x,u)u)dx.$$

For all $u_0 \in \mathcal{N}$, we have

$$\langle J'(u_0), u_0 \rangle = \eta \, \langle \xi'(u_0), u_0 \rangle = 0.$$

Therefore

$$\int_\Omega (|\nabla u_0|^2 - \lambda f(x,u_0)u_0)dx = 0,$$

then

$$\int_\Omega (|\nabla u_0|^2 \, dx = \lambda \int_\Omega f(x,u_0)u_0 dx,$$

thus

$$\begin{aligned}\langle \xi'(u_0), u_0\rangle &= \int_\Omega (2\,|\nabla u_0|^2 - \lambda f'_u(x,u_0)u_0^2)dx - \lambda \int_\Omega f(x,u_0)u_0 dx \\ &= \int_\Omega (|\nabla u_0|^2 - \lambda f'_u(x,u_0)u_0^2)dx \\ &= \phi''_{u_0}(1) \neq 0.\end{aligned}$$

Which implies that $\eta = 0$, then $J'(u_0) = 0$.

2.4 Ekeland's variational principle

In general, it is not clear that a bounded and lower semi-continuous functional E actually attains its infimum. The analytic function $f(x) = \arctan x$, for example, neither attains its infimum nor its supremum on the real line.
A variant due to Ekeland of Dirichlet's principle, however, permits one to construct minimizing sequences for such functionals E whose elements u_m each minimize a functional E_m, for a sequence of functionals $\{E_m\}$ converging locally uniformly to E.

Theorem 2.5 *[26] Let E be a reflexive Banach space with norm $\|.\|$, and $J : E \to \mathbb{R}$ is coercive and weakly lower semi-continuous on E , that is, suppose the following conditions are fullfilled:*

- $J(u,v) \to \infty$ as $\|(u,v)\| \to \infty$, $(u,v) \in E$.
- For any $(u,v) \in E$, any sequence (u_n, v_n) in E such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in E there holds$J(u,v) \leq \liminf_{n\to\infty} J(u_n, v_n)$. Then J is bounded from below on E and attains its infimum in E such that

$$J(u_0, v_0) = \inf_E J.$$

Theorem 2.6 *[21] Let M be a complete metric space with metric d, and let $J : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, bounded from below, and $\neq \infty$.Then for any $\epsilon, \delta > 0$, any $u \in M$ with*

$$J(u) \leq \inf_M J(u) + \epsilon,$$

there is an element $v \in M$ strictly minimizing the functional

$$J_v(w) \leq J(w) + \frac{\epsilon}{\delta} d(v, w),$$

Moreover, we have

$$J(v) \leq J(u), \; d(u, v) \leq \delta.$$

CHAPTER 3

Existence result for sub-critical and critical p-fractional elliptic equations via Nehari manifold method

Contents

3.1 Introduction

In this chapter we study the fractional p-Laplacian problem (E), using fibering maps and Nehari manifold, we obtain existence result for either, subcritical and critical cases see [3]. This chapter is organized as follows : in first and second sections, we introduce our problem, and we give some preliminaries (spaces, definitions, fibering maps...). In the third section we give a first result of existence, and in the fourth section we establish the second existence result.

We consider the p-fractional Laplacian problem (E), where $\Omega \subset \mathbb{R}^n (n > ps)$, is a bounded smooth domain, $s \in (0,1)$, λ, μ are positive parameters, the functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_+$, are continuous and positively homogeneous of degrees q and r respectively, that is, for all $t > 0$ and $(x,u) \in \Omega \times \mathbb{R}$, we have

$$\begin{cases} f(x,tu) = t^q f(x,u), \\ \\ g(x,tu) = t^r g(x,u), \end{cases} \tag{3.1}$$

for some constants q, r satisfying

$$1 < r+1 < p < q+1 \leq p_s^* := \frac{np}{n-sp}. \tag{3.2}$$

Note that, the primitive functions

$$\begin{cases} F(x,u) = \int\limits_0^u f(x,s)ds, \\ G(x,u) = \int\limits_0^u g(x,s)ds, \end{cases}$$

are in $C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and they are positively homogeneous of degrees $q+1$ and $r+1$ respectively. Moreover, the so-called Euler identities hold, that is

$$\begin{cases} (q+1)\, F(x,u) = uf(x,u), \\ \\ (r+1)\, G(x,u) = uG(x,u). \end{cases} \tag{3.3}$$

We can easily prove the existance of two positive constants γ_1, γ_2, such that, for all $(x,u) \in \Omega \times \mathbb{R}$, we have

$$F(x,u) \leq \gamma_1 \left|u\right|^{q+1}, \text{ and } G(x,u) \leq \gamma_2 \left|u\right|^{r+1}. \tag{3.4}$$

Put

$$\Omega^c = \mathbb{R}^n \setminus \Omega, \text{ and } Q = \mathbb{R}^{2n} \diagdown (\Omega^c \times \Omega^c).$$

We introduce the functional space

$$X = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ measurable: } u \in L^p(\Omega) \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(Q) \right\},$$

Endowed with the norm

$$\|u\|_X = \left(\|u\|^p_{L^p(\Omega)} + \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy. \right)^{\frac{1}{p}}$$

Also, we define the space

$$X_0 = \{ u \in X : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}, \tag{3.5}$$

Equipped with the norm

$$\|u\|_{X_0} = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}. \tag{3.6}$$

It is well known that X_0 is a separable Banach space. Moreover, for all $u, v \in X_0$, we have the duality product

$$\mathcal{T}(u,v) = \langle (-\Delta)^s_p u, v \rangle_{X_0} = \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+ps}} dxdy. \tag{3.7}$$

Definition 3.1 *We say that $u \in X_0$, is a weak solution of problem (E), if for all $v \in X_0$, we have the following weak formulation*

$$\mathcal{T}(u,v) = \lambda \|u\|^p_p + \int_\Omega f(x,u)v(x)dx + \mu \int_\Omega g(x,u)v(x)dx.$$

Associated to the problem (E), we define the functional $J_{\lambda,\mu} : X_0 \to \mathbb{R}$, as

$$J_{\lambda,\mu}(u) = \frac{1}{p} A(u) - B(u) - \mu C(u), \tag{3.8}$$

where

$$A(u) = \|u\|^p_{X_0} - \lambda \|u\|^p_p,$$

$$B(u) = \int_\Omega F(x,u)dx,$$

and

$$C(u) = \int_\Omega G(x,u)dx.$$

Note that, $J_{\lambda,\mu} \in C^1(X_0, \mathbb{R})$, and $J'_{\lambda,\mu} : X_0 \to X'_0$ is given by

$$\langle J'_{\lambda,\mu}(u), u \rangle = A(u) - (q+1)B(u) - \mu(r+1)C(u), \tag{3.9}$$

where X'_0, is the dual space of X_0.

Let λ_1 be the first eigenvalue of the fractional p-Laplacian equation subject to homogeneous Dirichlet boundary conditions. Then, our first result (about the sub-critical and concave case) is the following.

Theorem 3.1 *Let $s \in (0,1)$. Assume that the nonlinearities f, g are continuous satisfying (3.1). If*

$$1 < r+1 < p < q+1 < p_s^*, \text{ and } n > ps.$$

Then, for all $\lambda \in (0, \lambda_1)$, there exists $\mu_*(\lambda) > 0$, such that, for all $\mu \in (0, \mu_*(\lambda))$, problem (E) has at least two nontrivial solutions.

The second main result of this chapter is devoted to the critical case $(q = p_s^* - 1)$. Since the embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$, is not compact, then the energy functional does not satisfy the Palais-Smale condition globally, but it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant, that we can defined by the following expression

$$S_p = \inf_{v \in X_0 \setminus \{0\}} \frac{\|v\|^p_{X_0}}{\|v\|^p_{L^{p_s^*}}}. \tag{3.10}$$

Theorem 3.2 *Assume that $s \in (0,1), n > ps$ and $0 < r < 1 < p < q = p_s^* - 1$. If there exist $t_0 > 0$ and $u_0 \in X_0 \backslash \{0\}$, with $u_0 > 0$ in $\mathbb{R}^n$, such that*

$$\frac{1}{p} A(u_0) t_0^p - t_0^{p_s^*} B(u_0) = \frac{s}{n} \left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{sp}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{sp}}, \tag{3.11}$$

Then, for all $\lambda \in (0, \lambda_1)$, there exists $\mu^*(\lambda) > 0$, such that, for all $\mu \in (0, \mu^*(\lambda))$, problem (E) has at least two nontrivial solutions.

Remark 3.1 *The condition (3.11), can be guaranteed by the following Lamma.*

Lemma 3.1 *If $s \in (0,1), n > ps$ and $0 < r < 1 < p < q = p_s^* - 1$. Then, there exist $t_0 > 0$ and $u_0 \in X_0 \backslash \{0\}$, with $u_0 > 0$ in $\mathbb{R}^n$, such that*

$$\left(\frac{1}{p} A(u_0) t_0^p - t_0^{p_s^*} B(u_0)\right) < \frac{s}{n} \left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{sp}}.$$

Proof For all $u \in X_0 \setminus \{0\}$, we define the function $\zeta_u : (0,\infty) \to \mathbb{R}$, as follows:

$$\zeta_u(t) := \frac{1}{p} A(tu) - B(tu) = \frac{A(u)}{p} t^p - B(u) t^{p_s^*}.$$

It is easy to see that ζ is of class C^1, moreover, for all $t > 0$, we have

$$\zeta_u'(t) = t^{p-1}\left(A(u) - p_s^* B(u) t^{p_s^*-p}\right).$$

Since $\lim_{t\to 0} \zeta_u(t) = 0$ and $\lim_{t\to\infty} \zeta_u(t) = -\infty$. Then, ζ atteinds its global maximum at

$$t_* = \left(\frac{A(u)}{p_s^* B(u)}\right)^{\frac{1}{p_s^*-p}}.$$

Moreover, from (2.11) (2.12) and the fact that $q = p_s^* - 1$, we obtain

$$\begin{aligned}
\sup_{t>0} \zeta(t) &= \zeta(t_*) \\
&= \frac{A(u)}{p}\left(\frac{A(u)}{p_s^* B(u)}\right)^{\frac{p}{p_s^*-p}} - B(u)\left(\frac{A(u)}{p_s^* B(u)}\right)^{\frac{p_s^*}{p_s^*-p}} \\
&= (p_s^*)^{-\frac{p}{p_s^*-p}}\left(\frac{1}{p} - \frac{1}{p_s^*}\right) A(u)^{\frac{p_s^*}{p_s^*-p}} B(u)^{-\frac{p}{p_s^*-p}} \\
&= \frac{s}{n}(p_s^*)^{-\frac{n}{sp_s^*}} A(u)^{\frac{n}{sp}} B(u)^{-\frac{n}{sp_s^*}} \\
&\geq \frac{s}{n}(\gamma_1 p_s^*)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{sp}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{sp}} > 0.
\end{aligned}$$

Therefore, from the variations of the function ζ, we deduce the existence of $0 < \tilde{t}_1 < t_* < \tilde{t}_2$, such that

$$\zeta\left(\tilde{t}_1\right) = \zeta\left(\tilde{t}_2\right) = \frac{s}{n}(\gamma_1 p_s^*)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{sp}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{-\frac{n}{sp}}.$$

This ends the proof.

3.2 Nehari manifold and fibering maps analysis

In this section, we collect some basic results that will be used in the forthcoming sections. As the energy functional $J_{\lambda,\mu}$ is not bounded below on X_0, it is useful to show that $J_{\lambda,\mu}$ is bounded on some suitable subset of X_0. A good candidate is the so-called Nehari manifold defined by

$$\mathcal{N}_{\lambda,\mu} = \{u \in X_0 \backslash \{0\}, \langle J_{\lambda,\mu}'(u), u\rangle_{X_0} = 0\}.$$

It is easy to see that $u \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$A(u) - (q+1)B(u) - \mu(r+1)C(u) = 0. \tag{3.12}$$

Hence, from (3.9), we see that $\mathcal{N}_{\lambda,\mu}$ contain all nontrivial critical points which are solutions of problem (E). It is useful to understand $\mathcal{N}_{\lambda,\mu}$ in terms of the stationary points of mapping $\varphi_u : (0,\infty) \to \mathbb{R}$, much known as fiber maps, as

$$\varphi_u(t) = J_{\lambda,\mu}(tu).$$

For more details and properties about these maps, we refer the reader to [10,17,18]. Taking derivative with respect to the variable$t$, we get

$$\varphi_u'(t) = \langle J_{\lambda,\mu}'(tu), u\rangle_{X_0} = \frac{1}{t}\langle J_{\lambda,\mu}'(tu), tu\rangle_{X_0}.$$

So $tu \in \mathcal{N}_{\lambda,\mu}$, if and only if $\varphi_u'(t) = 0$, in particular, $u \in \mathcal{N}_{\lambda,\mu}$, if and only if $\varphi_u'(1) = 0$. In order to obtain multiplicity of solutions, we split $\mathcal{N}_{\lambda,\mu}$ into the following three parts

$$\mathcal{N}_{\lambda,\mu}^{+} = \{u \in \mathcal{N}_{\lambda,\mu} : \varphi_u''(1) > 0\} = \{u \in X_0 : \varphi_u'(1) = 0 \text{ and } \varphi_u''(1) > 0\},$$

$$\mathcal{N}_{\lambda,\mu}^{-} = \{u \in \mathcal{N}_{\lambda,\mu} : \varphi_u''(1) < 0\} = \{u \in X_0 : \varphi_u'(1) = 0 \text{ and } \varphi_u''(1) < 0\},$$

$$\mathcal{N}_{\lambda,\mu}^{0} = \{u \in \mathcal{N}_{\lambda,\mu} : \varphi_u''(1) = 0\} = \{u \in X_0 : \varphi_u'(1) = 0 \text{ and } \varphi_u''(1) = 0\}.$$

Note that, from (3.12), we obtain

$$\begin{aligned}
\varphi_u''(1) &= (p-1)\,A(u) - q(q+1)B(u) - \mu r(r+1)C(u)\\
&= (p-q-1)(q+1)B(u) + \mu(r+1)(p-r-1)C(u)\\
&= (p-q-1)A(u) + \mu(r+1)(q-r)C(u)\\
&= (p-r-1)A(u) - (q+1)(r-q)B(u). \qquad (3.13)
\end{aligned}$$

Lemma 3.2 *If u_0 is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, such that $u_0 \notin \mathcal{N}_{\lambda,\mu}^{0}$, then u_0 is a critical point of $J_{\lambda,\mu}$.*

Proof Let u_0 be a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then u_0 is a solution of the minimization problem

$$\begin{cases} \min\limits_{u\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u) = J_{\lambda,\mu}(u_0), \\ \\ \beta(u_0) = 0, \end{cases}$$

where

$$\beta(u) = A(u) - (q+1)B(u) - \mu(r+1)C(u).$$

From Lagrangian multipliers theorem, there exists $\delta \in \mathbb{R}$, such that

$$J'_{\lambda,\mu}(u_0) = \delta\beta'(u_0). \tag{3.14}$$

Since $u_0 \in \mathcal{N}_{\lambda,\mu}$, then, we have

$$\delta\langle\beta'(u_0), u_0\rangle_{X_0} = \langle J'_{\lambda,\mu}(u_0), u_0\rangle_{X_0} = 0. \tag{3.15}$$

On the other hand, from (3.12) and the contraint $\beta(u_0) = 0$, we get

$$\langle\beta'(u_0), u_0\rangle_{X_0} = (p-1)A(u_0) - q(q+1)B(u_0) - \mu r(r+1)C(u_0) = \varphi''_{u_0}(1). \tag{3.16}$$

By combining equations (3.15), (3.16) with the fact that $u_0 \notin \mathcal{N}^0_{\lambda,\mu}$, we get $\delta = 0$. Finally, by substitution of δ in equation (3.14), we obtain $J'_{\lambda,\mu}(u_0) = 0$.

In order to understand the Nehari manifold and fibering maps, let us consider the function $\psi_u : (0,\infty) \to \mathbb{R}$, defined by

$$\psi_u(t) = t^{p-r-1}A(u) - (q+1)t^{q-r}B(u). \tag{3.17}$$

From (3.12), we see that $tu \in \mathcal{N}_{\lambda,\mu}$, if and only if

$$\psi_u(t) = \mu(r+1)C. \tag{3.18}$$

Moreover, from the fact that

$$\psi'_u(t) = (p-r-1)t^{p-r-2}A(u) - (q+1)(q-r)t^{q-r-1}B(u), \tag{3.19}$$

we see that, if $tu \in \mathcal{N}_{\lambda,\mu}$, then

$$t^r\psi'_u(t) = \varphi''_u(t). \tag{3.20}$$

Therefore, $tu \in \mathcal{N}^+_{\lambda,\mu}$, $\left(\text{respectively}, tu \in \mathcal{N}^-_{\lambda,\mu}\right)$ if and only if $\psi'_u(t) > 0$ $\left(\text{respectively}, \psi'_u(t) < 0\right)$. By simple calculation we can prove the following result.

Lemma 3.3 *Let $u \in X_0$ such that, $u \neq 0$. Then, we have*

(i) ψ_u *has a unique critical point at*

$$t_{\max}(u) = \left[\frac{(p-r-1)}{(q+1)(q-r)}\frac{A(u)}{B(u)}\right]^{\frac{1}{q+1-p}}.$$

Moreover

$$\psi_u(t_{\max}) = \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q+1-p}}\left(\frac{q+1-p}{q-r}\right)A(u)^{\frac{q-r}{q+1-p}}B(u)^{-\frac{p-r-1}{q+1-p}}. \tag{3.21}$$

(ii) $\lim_{t\to\infty}\psi_u(t) = -\infty.$

(iii) ψ_u *is strictly increasing on* $(0, t_{\max}(u))$ *and strictly decreasing on* $(t_{\max}(u), +\infty)$.

In the rest of this chapter, we assume that $\lambda \in (0, \lambda_1)$ and $\mu \in (0, \mu_*(\lambda))$, where

$$\mu_*(\lambda) = \frac{1}{\gamma_2}\left(\frac{q+1-p}{(q-r)(r+1)}\right)\left(\frac{p-r-1}{(q+1)(q-r)\gamma_1}\right)^{\frac{p-r-1}{q+1-p}}\left(S_p\,|\Omega|^{\frac{p-p_s^*}{p_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)\right)^{\frac{q-r}{q+1-p}}.$$

Lemma 3.4 *For all $u \in \mathcal{N}_{\lambda,\mu}$, there exist unique $0 < t_1 < t_{max}(u) < t_2$, such that $t_1 u \in \mathcal{N}^+_{\lambda,\mu}$ and $t_2 u \in \mathcal{N}^-_{\lambda,\mu}$.*

Proof Let $u \in \mathcal{N}_{\lambda,\mu}$, then, from (3.4), (3.10) and the Hõlder inequality, we get

$$\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|^p_{X_0} \le A(u) \le \|u\|^p_{X_0}, \tag{3.22}$$

$$B(u) \le \gamma_1\,|\Omega|^{\frac{p_s^*-q-1}{p_s^*}}\,\|u\|^{q+1}_{L^{p_s^*}} \le \gamma_1 S_p^{-\frac{q+1}{p}}\,|\Omega|^{\frac{p_s^*-q-1}{p_s^*}}\,\|u\|^{q+1}_{X_0}, \tag{3.23}$$

and

$$C(u) \le \gamma_2 S_p^{-\frac{r+1}{p}}\,|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\,\|u\|^{r+1}_{X_0}. \tag{3.24}$$

By combining (3.22), (3.23) and (3.24) with (3.21), we obtain

$$\psi_u(t_{\max}) - \mu(r+1)C(u)$$
$$= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q+1-p}}\left(\frac{q+1-p}{q-r}\right)A^{\frac{q-r}{q+1-p}}B^{-\frac{p-r-1}{q+1-p}} - \mu(r+1)C(u) \tag{3.25}$$

$$\ge \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q+1-p}}\left(\frac{q+1-p}{q-r}\right)\gamma_1^{-\frac{p-r-1}{q+1-p}}S_p^{\frac{(q+1)(p-r-1)}{p(q+1-p)}}\,|\Omega|^{-\frac{(p-r-1)(p_s^*-q-1)}{p_s^*(q+1-p)}}$$
$$\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{q-r}{q+1-p}}\|u\|^{r+1}_{X_0} - \mu(r+1)\gamma_2 S_p^{-\frac{r+1}{p}}\,|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\,\|u\|^{r+1}_{X_0}$$

$$\ge (\mu_*(\lambda)-\mu)(r+1)\gamma_2 S_p^{-\frac{r+1}{p}}\,|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\,\|u\|^{r+1}_{X_0} > 0. \tag{3.26}$$

Hence, from the variation of ψ, there exist unique $0 < t_1 < t_{max}(u) < t_2$, such that $\psi_u'(t_1) > 0$, $\psi_u'(t_2) < 0$, moreover

$$\psi_u(t_1) = \mu(r+1)C(u) = \psi_u(t_2).$$

Finally, equations (3.18) and (3.19), implies that $t_1 u \in \mathcal{N}^+_{\lambda,\mu}$ and $t_2 u \in \mathcal{N}^-_{\lambda,\mu}$.

Lemma 3.5 *For all* $(\lambda,\mu) \in (0,\lambda_1) \times (0,\mu_*(\lambda))$, *we have* $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Proof Suppose otherwise. Let $u_0 \in \mathcal{N}^0_{\lambda,\mu}$. Since $\varphi''_{u_0}(1) = 0$, then, from (3.13) we have

$$(p-r-1)A(u_0) - (q+1)(q-r)B(u_0) = 0.$$

Therefore

$$B(u_0) = \frac{(p-r-1)}{(q+1)(q-r)}A(u_0). \tag{3.27}$$

On the other hand

$$\begin{aligned} 0 = \varphi'_{u_0}(1) &= A(u_0) - (q+1)B(u_0) - \mu(r+1)C(u_0) \\ &= A(u_0) - \frac{p-r-1}{q-r}A(u_0) - \mu(r+1)C(u_0) \\ &= \frac{q-p+1}{q-r}A(u_0) - \mu(r+1)C(u_0), \end{aligned}$$

which implies that

$$C(u_0) = \frac{(q-p+1)}{\mu\,(q-r)\,(r+1)}A(u_0). \tag{3.28}$$

Consequently, from (3.21) and (3.27), we get

$$\begin{aligned} &\psi_{u_0}(t_{\max}) - \mu(r+1)C(u_0) = \\ &= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q-p+1}} \left(\frac{q-p+1}{q-r}\right)\left(\frac{A(u_0)^{q-r}}{B(u_0)^{p-r-1}}\right)^{\frac{1}{q-p+1}} - \mu(r+1)C(u_0) \\ &= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q-p+1}} \left(\frac{q-p+1}{q-r}\right)\left(\frac{p-r-1}{(q+1)(q-r)}\right)^{-\frac{p-r-1}{q-p+1}} A(u_0) - \left(\frac{p-q-1}{r-q}\right)A(u_0) = 0. \end{aligned} \tag{3.29}$$

So $\psi_{u_0}(t_{\max}) - \mu(r+1)C(u_0) = 0$ which is a contradiction with (3.25). So $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Lemma 3.6 $J_{\lambda,\mu}$ *is coercive and bounded from below on* $\mathcal{N}_{\lambda,\mu}$.

Proof Let $u \in \mathcal{N}_{\lambda,\mu}$. Then, from (3.12), we get

$$B(u) = \frac{1}{q+1}\left(A(u) - \mu(r+1)C(u)\right),$$

which implies that

$$J_{\lambda,\mu}(u) = \left(\frac{1}{p} - \frac{1}{q+1}\right) A(u) - \mu \left(\frac{q-r}{q+1}\right) C(u).$$

So using (3.12), we obtain

$$J_{\lambda,\mu}(u) \geq \left(\frac{1}{p} - \frac{1}{q+1}\right)\left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^p_{X_0} - \gamma_2 S_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^*-r-1}{p_s^*}} \|u\|^{r+1}_{X_0}.$$

Since $\lambda < \lambda_1$ and $r+1 < p < q$, then, we conclude that the functional $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.

Note that by Lemma 3.5, we can write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$, and by Lemma 3.6, we can define

$$\alpha^-_{\lambda,\mu} = \inf_{u \in \mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u) \text{ and } \alpha^+_{\lambda,\mu} = \inf_{u \in \mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u).$$

3.3 Proof of Theorem (3.1)

In order to prove Theorem 3.1, we need to present several results.

Proposition 3.1 *There exists a minimizer $u_{\lambda,\mu}$ in $\mathcal{N}^+_{\lambda,\mu}$ for $J_{\lambda,\mu}$ satisfying:*

(1) $J_{\lambda,\mu}(u_{\lambda,\mu}) = \alpha^+_{\lambda,\mu} < 0.$

(2) $u_{\lambda,\mu}$ *is a solution of problem (E).*

Proof Since $J_{\lambda,\mu}$ is bounded from below on $\mathcal{N}^+_{\lambda,\mu}$, then, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}^+_{\lambda,\mu}$. That is

$$\lim_{k\to\infty} J_{\lambda,\mu}(u_k) = \inf_{u \in \mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u). \tag{3.30}$$

From (3.30) and Lemma (3.6), the sequence $\{u_k\}$ is bounded in X_0, So, up to a subsequence, there exists $u_{\lambda,\mu} \in X_0$, such that

$$u_k \rightharpoonup u_{\lambda,\mu}, \text{ weakly in } X_0.$$

On the other hand, by Lemma 1.3 , up to a subsequence still denoted $\{u_k\}$, we have

$$u_k \to u_{\lambda,\mu} \text{ in } L^\sigma(\mathbb{R}^n), u_k \to u_{\lambda,\mu} \text{ a.e. in } \mathbb{R}^n \text{ as } k \to \infty.$$

By [15] [Theorem IV-9], there exists $l \in L^\sigma(\mathbb{R}^n)$, such that

$$|u_k(x)| \leq l(x) \text{ in } \mathbb{R}^n.$$

A simple calculation shows that

$$B(u_k) < \gamma_1 \|u_k\|_{L^{q+1}}^{q+1} \text{ and } C(u_k) < \gamma_2 \|u_k\|_{L^{r+1}}^{r+1}.$$

Therefore, by the dominated convergence theorem, as k tends to infinity, we have

$$B(u_k) \to B(u_{\lambda,\mu}) \text{ and } C(u_k) \to C(u_{\lambda,\mu}), \tag{3.31}$$

From Lemma (3.4) there exists $t_1 > 0$, such that

$$t_1 u_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^{+} \text{ and } J_{\lambda,\mu}(t_1 u_{\lambda,\mu}) < 0.$$

Hence, we get

$$\alpha_{\lambda,\mu}^{+} = \inf_{u\in\mathcal{N}_{\lambda,\mu}^{+}} J_{\lambda,\mu}(u) < 0.$$

Next, we show that $u_k \to u_{\lambda,\mu}$ strongly in X_0. We proceed by contradiction and we assume that $\|u_{\lambda,\mu}\|_{X_0} < \liminf_{k\to\infty} \|u_k\|_{X_0}$. This implies that

$$\begin{aligned}\lim_{k\to\infty} \varphi'_{u_k}(t_1) &= \lim_{k\to\infty} \left[t_1^{p-1}A(u_k) - (q+1)t_1^q B(u_k) - \mu(r+1)t_1^r C(u_k)\right]\\ &> t_1^{p-1}A(u_{\lambda,\mu}) - (q+1)t_1^q B(u_{\lambda,\mu}) - \mu(r+1)t_1^r C(u_{\lambda,\mu})\\ &= \varphi'_{u_{\lambda,\mu}}(t_1) = 0.\end{aligned}$$

So $\varphi'_{u_k}(t_1) > 0$ for k large enough. Since $u_k \in \mathcal{N}_{\lambda,\mu}^{+}$, then, $\varphi'_{u_k}(t) < 0$, for $t \in (0, t_1)$ and $\varphi'_{u_k}(1) = 0$. This yields to $t_1 > 1$. Now, the fact that $\varphi_{u_{\lambda,\mu}}$ is decreasing on $(0, t_1)$, implies that

$$J_{\lambda,\mu}(t_1 u_{\lambda,\mu}) \le J_{\lambda,\mu}(u_{\lambda,\mu}) < \lim_{k\to\infty} J_{\lambda,\mu}(u_k) = \inf_{u\in\mathcal{N}_{\lambda,\mu}^{+}} J_{\lambda,\mu}(u),$$

which is a contradiction. Hence, $u_k \to u_{\lambda,\mu}$ strongly in X_0. Moreover, as k tends to infinity, we have

$$J_{\lambda,\mu}(u_k) \to J_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{u\in\mathcal{N}_{\lambda,\mu}^{+}} J_{\lambda,\mu}(u).$$

Namely, $u_{\lambda,\mu}$ is a minimizer of $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^{+}$. Finally, from Lemma 3.3, we see that $u_{\lambda,\mu}$ is a solution of ((E)).

Proposition 3.2 *If* $0 < r < 1 < q < p_s^* - 1$. *Then,* $J_{\lambda,\mu}$ *has a minimizer* $v_{\lambda,\mu}$ *in* $\mathcal{N}_{\lambda,\mu}^{-}$ *satisfying*

(1) $J_{\lambda,\mu}(v_{\lambda,\mu}) = \alpha_{\lambda,\mu}^{-} > 0.$

(2) $v_{\lambda,\mu}$ is a solution of problem (E).

Proof Since $J_{\lambda,\mu}$ is bounded from below on $\mathcal{N}^-_{\lambda,\mu}$, then, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}^-_{\lambda,\mu}$ satisfying

$$\lim_{k\to\infty} J_{\lambda,\mu}(u_k) = \inf_{u\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u).$$

By the same argument given in the proof of Proposition 3.1, there exists $v_{\lambda,\mu} \in X_0$ such that, up to a subsequence,

$$A(u_k) \to A(v_{\lambda,\mu}), B(u_k) \to B(v_{\lambda,\mu}) \text{ and } C(u_k) \to C(v_{\lambda,\mu}), \text{ as } k \to \infty.$$

Moreover, from the analysis of the fibering maps φ_u , we know that there exists $t_2 > t_{\max}(u)$ such that $t_2 v_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$. Now, we prove that $u_k \to v_{\lambda,\mu}$, strongly in $\mathcal{N}^-_{\lambda,\mu}$. If not, then, we have

$$\|v_{\lambda,\mu}\|_{X_0} < \liminf_{k\to\infty} \|u_k\|_{X_0}.$$

Since $\{u_k\} \subset \mathcal{N}^-_{\lambda,\mu}$, then, we get $J_{\lambda,\mu}(u_k) > J_{\lambda,\mu}(tu_k)$ for all $t > t_{\max}$.
On the other hand, using the fact that $t_2 v_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$, we obtain

$$\begin{aligned} J_{\lambda,\mu}(t_2 v_{\lambda,\mu}) &= \frac{t_2^p}{p} A(v_{\lambda,\mu}) - t_2^{q+1} B(v_{\lambda,\mu}) - \mu t_2^{r+1} C(v_{\lambda,\mu}) \\ &< \lim_{k\to\infty}\inf \left(\frac{t_2^p}{p} A(u_k) - t_2^{q+1} B(u_k) - \mu t_2^{r+1} C(u_k)\right) \\ &= \lim_{k\to\infty}\inf J_{\lambda,\mu}(t_2 u_k) \\ &\leq \liminf J_{\lambda,\mu}(u_k) = \alpha^-_{\lambda,\mu}, \end{aligned}$$

which is a contradiction. We conclude that $u_k \to v_{\lambda,\mu}$ strongly in X_0. So

$$J_{\lambda,\mu}(u_k) \to J_{\lambda,\mu}(v_{\lambda,\mu}) = \inf_{u\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u), k \to \infty.$$

Namely, $v_{\lambda,\mu}$ is a minimizer of $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$. Finally, from Lemma 2, we get that $v_{\lambda,\mu}$ is a solution of ((E)).

Proof of Theorem (3.1) By Propositions 3.1, 3.2 and Lemma 3.3, we get that problem ((E)) has two solutions $u_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ and $v_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$on X_0. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, then, $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$ are distinct. This completes the proof of Theorem 1.1.

3.4 Proof of Theorem (3.2)

Put

$$M = \left(\frac{p-r-1}{p}\right)\left(\frac{n(r+1)}{sp_s^*}\right)^{\frac{r+1}{p-r-1}}\left(\frac{p_s^*-r-1}{p}\right)^{\frac{p}{p-r-1}}\left(\gamma_2 S_p^{-\frac{r+1}{p}}|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\right)^{\frac{p}{p-r-1}}. \quad (3.32)$$

Proposition 3.3 *Assume that* $0 < r < 1 < q = p_s^* - 1$. *Then, every Palais-Smale sequence* $\{u_k\} \subset X_0$ *for* $J_{\lambda,\mu}$ *at level* c, *with*

$$c < \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}}\mu^{\frac{p}{p-r-1}}, \quad (3.33)$$

has a convergent subsequence, where S_p *is given by equation (3.10) .*

Proof From Lemma 3.6, we see that $\{u_k\}$ is bounded in X_0. So up to a sequence, still denoted by $\{u_k\}$, there exists $u_* \in X_0$ such that $u_k \rightharpoonup u_*$ weakly in X_0. Therefore

$$A(u_k) \to A(u_*), \text{ as } k \to \infty.$$

Moreover, by [34], [lemma 8], we have that

$$u_k \rightharpoonup u_* \text{ weakly in } L^{p_s^*}(\mathbb{R}^n), u_k \to u_* \text{ in } L^{r+1}(\mathbb{R}^n), u_k \to u_* \text{ in } \mathbb{R}^n.$$

Since $1 \le r+1 < p_s^*$. then, from [15] Theorem IV-9, there exists $l \in L^{r+1}(\mathbb{R}^n)$ such that:

$$|u_k(x)| \le l(x) \text{ in } \mathbb{R}^n.$$

So the dominated convergence theorem, implies that

$$C(u_k) \longrightarrow C(u_*), \text{ as } k \to \infty.$$

On the other hand, from Brezis-Lieb Lemma 1.2, we get

$$A(u_k) = A(u_k - u_*) + A(u_*) + o(1),$$

and

$$B(u_k) = B(u_k - u_*) + B(u_*) + o(1).$$

Consequently,

$$\begin{aligned}\langle J'_{\lambda,\mu}(u_k), u_k\rangle_{X_0} &= A(u_k) - p_s^* B(u_k) - \mu(r+1)C(u_k)\\ &= A(u_k-u_*) + A(u_*) - p_s^*\left[B\left(u_k-u_*\right) + B(u_*)\right] - \mu(r+1)C(u_k) + o(1)\\ &= \langle J'_{\lambda,\mu}(u_*), u_*\rangle_{X_0} + A(u_k-u_*) - p_s^* B\left(u_k-u_*\right).\end{aligned}$$

Since

$$\langle J'_{\lambda,\mu}(u_*), u_*\rangle_{X_0} = 0 \text{ and } \lim_{k\to\infty}\langle J'_{\lambda,\mu}(u_k), u_k\rangle_{X_0} \longrightarrow 0,$$

then, we obtain

$$\lim_{k\to\infty} A(u_k-u_*) = \lim_{k\to\infty} p_s^* B\left(u_k-u_*\right). \tag{3.34}$$

We aim to prove that $b := \lim_{k\to\infty} A(u_k-u_*) = 0$. By contradiction, we assume that $b>0$. So from (3.22), we get

$$p_s^* B\left(u_k-u_*\right) \le p_s^*\gamma_1 S_p^{-\frac{p_s^*}{p}}\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{p_s^*}{p}}\left(A(u_k-u_*)\right)^{\frac{p_s^*}{p}},$$

which yields to

$$b \ge \left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{sp}}. \tag{3.35}$$

On the other hand, we have

$$\begin{aligned}c &= \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A(u_k) - B(u_k) - \mu C(u_k)\right)\\ &= \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A(u_k-u_*) - B\left(u_k-u_*\right) - \frac{1}{p}A(u_*) - B(u_*) - \mu C(u_k)\right) + o(1)\\ &= J_{\lambda,\mu}\left(u_*\right) + b\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\\ &\ge J_{\lambda,\mu}(u_*) + \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{sp}}\\ &> J_{\lambda,\mu}\left(u_*\right) + c.\end{aligned}$$

Therefore, $J_{\lambda,\mu}(u_*) < 0$. In particular, $u_* \ne 0$, and

$$B(u_*) > \frac{1}{p}A(u_*) - \mu C(u_*). \tag{3.36}$$

So from (3.35), we obtain

$$\begin{aligned}c = \lim_{k\longrightarrow\infty} J_{\lambda,\mu}(u_k) &= \lim_{k\longrightarrow\infty}\left(J_{\lambda,\mu}(u_k) - \frac{1}{p}\langle J'_{\lambda,\mu}(u_k), u_k\rangle_{X_0}\right)\\ &= \lim_{k\longrightarrow\infty}\left[\left(\frac{p_s^*}{p}-1\right)\left(B\left(u_k-u_*\right)\right) + B(u_*) - \mu\left(\frac{p-r-1}{p}\right)C(u_k)\right]\\ &= \frac{s}{n}b + \frac{sp_s^*}{n}B(u_*) - \mu\left(\frac{p-r-1}{p}\right)C(u_*)\\ &\ge \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} + \frac{sp_s^*}{n}B(u_*) - \mu\left(\frac{p-r-1}{p}\right)C(u_*).\end{aligned}$$

Using (3.22), (3.24) and (3.36), we obtain

$$\begin{aligned} c &> \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} + \frac{sp_s^*}{n}\left(\frac{1}{p}A(u_*)-\mu C(u_*)\right) - \mu\left(\frac{p-r-1}{p}\right)C(u_*). \\ &= \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} + \frac{sp_s^*}{np}A(u_*) - \mu\left(\frac{p-r-1}{p}+\frac{sp_s^*}{n}\right)C(u_*) \\ &= \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} + \frac{sp_s^*}{np}A(u_*) - \mu\left(\frac{p_s^*-r-1}{p}\right)C(u_*) \\ &> \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} + \frac{sp_s^*}{np}A(u_*) \\ &\quad -\mu\gamma_2 S_p^{-\frac{r+1}{p}}|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\left(\frac{p_s^*-r-1}{p}\right)\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p}}\left(A(u_*)\right)^{\frac{r+1}{p}} \\ &= \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - h((A(u_*))^{\frac{1}{p}}), \end{aligned} \tag{3.37}$$

where h is defined by

$$h(\xi) = \mu\gamma_2 S_p^{-\frac{r+1}{p}}|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\left(\frac{p_s^*-r-1}{p}\right)\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p}}\xi^{r+1} - \frac{sp_s^*}{np}\xi^p.$$

A simple calculation shows that h attaints its maximum at

$$\xi_0 = \left(\mu n\,(r+1)\,\gamma_2 S_p^{-\frac{r+1}{p}}|\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\left(\frac{p_s^*-r-1}{spp_s^*}\right)\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p}}\right)^{\frac{1}{p-r-1}},$$

and

$$\sup_{\xi>0} h(\xi) = h(\xi_0) = M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}}\mu^{\frac{p}{p-r-1}}, \tag{3.38}$$

where M is defined in (3.32)

By combing equations (3.37) and (3.38), we obtain

$$c \geq \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}}\mu^{\frac{p}{p-r-1}}.$$

Which is a contradiction. Hence, $b=0$. So $u_k \to u_*$ strongly in X_0. This completes the proof.

Proposition 3.4 *There exist $\mu^*(\lambda)>0$, $t_0>0$ and $u_0\in X_0$, such that, for all $(\lambda,\mu)\in(0,\lambda_1)\times(0,\mu^*(\lambda))$, we have*

$$J_{\lambda,\mu}(t_0u_0) \leq \frac{s}{n}\left(p_s^*\gamma_1\right)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}}\mu^{\frac{p}{p-r-1}}. \tag{3.39}$$

In particular,

$$\alpha^-_{\lambda,\mu} < \frac{s}{n}\left(p_s^*\gamma_1\right)^{-\frac{n}{sp_s^*}} S_p^{\frac{n}{ps}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}}\mu^{\frac{p}{p-r-1}}. \tag{3.40}$$

Proof Put

$$\mu_1(\lambda) = \left(\frac{s}{nM}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\right)^{\frac{p-r-1}{p}} \left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{p_s^*-r-1}{p_s^*-p}}.$$

Then, for $0 < \mu < \mu_1(\lambda)$, we have

$$\frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}} > 0. \tag{3.41}$$

By condition (3.11), there exists t_0 and $u_0 \in X_0 \setminus \{0\}$ such that

$$\begin{aligned} J_{\lambda,\mu}(t_0u_0) &= \frac{1}{p}A(u_0)t_0^p - t_0^q B(u_0) - \mu t_0^{r+1}C(u_0) \\ &= \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - \mu t_0^{r+1}C(u_0). \end{aligned}$$

Put

$$\mu_2(\lambda) = \left(\frac{t_0^{r+1}C(u_0)}{M}\right)^{\frac{p-r-1}{r+1}} \left(1-\frac{\lambda}{\lambda_1}\right).$$

Then, for all $\mu \in (0, \mu_2(\lambda))$ we have

$$-\mu t_0^{r+1}C(u_0) < -M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}.$$

So from (3.42), we get

$$J_{\lambda,\mu}(t_0u_0) < \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}.$$

Therefore,(3.39) hold true.

Finally, if we put

$$\mu^*(\lambda) = \min_{0<\lambda<\lambda_1} \left(\mu_*(\lambda), \mu_1(\lambda), \mu_2(\lambda)\right).$$

Then, for all $0 < \mu < \mu^*(\lambda)$ and using the analysis of fibering maps $\varphi_u(t) = J_{\lambda,\mu}(tu)$, we get

$$\alpha_{\lambda,\mu}^- < \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} - M\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{r+1}{p-r-1}} \mu^{\frac{p}{p-r-1}}.$$

This completes the proof.

Proof of Theorem 3.2 By Propositions 3.3 and 3.4, there exists two sequences $\{u_k^+\}$ and $\{u_k^-\}$ in X_0, such that

$$J_{\lambda,\mu}(u_k^+) \longrightarrow \alpha_{\lambda,\mu}^+,\ J'_{\lambda,\mu}(u_k^+) \longrightarrow 0,$$

and

$$J_{\lambda,\mu}(u_k^-) \longrightarrow \alpha_{\lambda,\mu}^-,\ J'_{\lambda,\mu}(u_k^-) \longrightarrow 0.$$

as $k \longrightarrow \infty$. We observe that from the analysis of fibering maps $\varphi_u(t)$, we have $\alpha^+_{\lambda,\mu} < 0$. Similar to the proof of Propositions 3.1 and 3.2 and Theorem 3.1, problem ((E)) has two solutions $u_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ and $v_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$ in X_0.Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, then these two solutions are distinct. This finishes the proof.

CHAPTER 4

Multiple solutions for the p-fracional laplacian problem with critical growth

Contents

4.1 Introduction

In this chapter we study the same fractional p-Laplacian problem (E) with different assumptions on the non-linearities for the critical case, using Ekeland's variational principal with the mountain pass theorem see [6]. The first section is devoted to some basic notions, in the second section, we prove several lemmas to be used in the third section for the purpose of obtaining our main existence result [2].

We consider the problem (E), where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > ps$, $s \in (0,1)$, λ and μ are positive parameters, the functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_+$, are positive continuous differentiable with respect to the second argument where $f(x,0) = 0$, $g(x,0) = 0$, and satisifying the following conditions:

There exist positive constants α_i and β_i for $i = 1,2,3,4$ such that

$$\min(\alpha_1, \beta_1) \leq \max(\alpha_1, \beta_1) < \frac{1}{p-1} < p < \min(\alpha_2, \beta_2) \leq \max(\alpha_2, \beta_2) < \min(p_s^*, \alpha_4, \beta_4).$$

Moreover, for any $u \in L^{p_s^*}(\Omega)$, we have

$$\alpha_3 \|u\|_{L^{p_s^*}(\Omega)}^{p_s^*} \leq \alpha_2 \int_\Omega F(x,u)dx \leq \int_\Omega f(x,u)udx \leq \alpha_1 \int_\Omega f_u(x,u)u^2 dx \leq \alpha_4 \|u\|_{L^{p_s^*}(\Omega)}^{p_s^*} \tag{4.1}$$

and

$$\beta_3 \|u\|_{L^q(\Omega)}^q \leq \beta_2 \int_\Omega G(x,u)dx \leq \int_\Omega g(x,u)udx \leq \beta_1 \int_\Omega g_u(x,u)u^2 dx \leq \beta_4 \|u\|_{L^q(\Omega)}^q, \tag{4.2}$$

for some q with $p < q < p_s^*$. Where

$$p_s^* = \frac{np}{n - sp}.$$

and F, G are defined by

$$\begin{cases} F(x,u) = \int_0^u f(x,s)ds, \\ G(x,u) = \int_0^u g(x,s)ds. \end{cases}$$

Our main result of this paper is the following theorem.

Theorem 4.1 *If Equations (4.1), and (4.2) hold, then there exists $\mu^* > 0$, such that for every $\lambda \in (0, \lambda_1)$ and $\mu > \mu^*$, problem (1) admits three different nontrivial solutions. Moreover, these solutions are, one negative, one positive and the other has non-constant sign.*

4.2 Functional settings

we will use in this proof the same approach as in [37]. That is, we will construct three disjoint sets K_1, K_2 and K_3 not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of C^1- manifolds $M_i \in X$ that will be constructed by imposing a sign restriction and a normalizing condition. Let,

$$M_1 = \left\{ u \in X_0 : \int_\Omega u_+ > 0 \text{ and } A(u_+) - \int_\Omega f(x,u)u_+ - \mu \int_\Omega g(x,u)u_+ = 0 \right\},$$

$$M_2 = \left\{ u \in X_0 : \int_\Omega u_- > 0 \text{ and } A(u_-) - \int_\Omega f(x,u)u_- - \mu \int_\Omega g(x,u)u_- = 0 \right\},$$

$$M_3 = M_1 \cap M_2,$$

where $u_+ = \max\{u, o\}, u_- = \max\{-u, 0\}$ are the negative and positive part of u,
Finally we define

$$K_1 = \{u \in M_1 : u \geq 0\},$$

$$K_2 = \{u \in M_2 : u \leq 0\},$$

$$K_3 = M_3.$$

Lemma 4.1 *For every* $w_0 \in X_0, w_0 > 0, \quad (w_0 < 0)$, *there exists* $t_\mu > 0$ *such that* $t_\mu w_0 \in M_1, (t_\mu w_0 \in M_2)$. *Moreover,* $\lim_{\mu \to \infty} t_\mu = 0$.

As consequences of this lemma, if $w_0, w_1 \in X_0$, where $w_0 > 0$ and $w_1 < 0$, with disjoint supports, there exist $t'_\mu, t_\mu > 0$ such that $t'_\mu w_0 + t_\mu w_1 \in M_3$. Moreover $t'_\mu, t_\mu \to 0$ as $\mu \to \infty$.

Proof For $w \in X_0, w \geq 0$ we consider the functional

$$\phi(w) = A(w) - \int_\Omega f(x,w)w dx - \mu \int_\Omega g(x,w)w dx.$$

Given $w_0 \geq 0$, we will prove that $\phi(t_\mu w_0) = 0$ for some $t_\mu > 0$. Using conditions 4.1,4.2 we get

$$\begin{aligned} \phi(tw_0) &= A(tw_0) - \int_\Omega f(x,tw_0)tw_0 dx - \mu \int_\Omega g(x,tw_0)tw_0 dx \\ &\geqslant t^p A(w_0) - \alpha_4 t^{p_s^*} \|w_0\|^{p_s^*}_{L^{p_s^*}(\Omega)} - \mu \beta_4 t^q \|w_0\|^q_{L^q(\Omega)}, \end{aligned}$$

and

$$\phi(tw_0) \leqslant t^p A(w_0) - \alpha_3 t^{p_s^*} \|w_0\|^{p_s^*}_{L^{p_s^*}(\Omega)} - \mu \beta_3 t^q \|w_0\|^q_{L^q(\Omega)}.$$

Since $p < q < p_s^*$ we have that $\phi(tw_0)$ is negative for t large enough, and positive for t small enough . we can explicitly give an upper bound t_μ, We note that

$$\phi(tw_0) \leqslant t^p A(w_0) - \mu\beta_3 t^q \|w_0\|^q_{L^q(\Omega)},$$

so its enough to choose t_1 such that

$$t_1^p A(w_0) - \mu\beta_3 t_1^q \|w_0\|^q_{L^q(\Omega)} = 0.$$

i.e.,

$$t_1 = \left(\frac{A(v_0)}{\mu\beta_3 \|w_0\|^q_{L^q(\Omega)}} \right)^{\frac{1}{q-p}}.$$

Hence by Bolzano's theorem 1.8, we can choose $t_\mu \in [0, t_1]$, and we can see that $t_\mu \to 0$ as $\mu \to +\infty$.

Lemma 4.2 *For every $u \in K_i, i = 1, 2, 3,$ we have*

$$\begin{aligned} \|u\|^p_{X_0} &\leqslant \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1} \left(\int_\Omega f(x,u)udx + \mu \int_\Omega g(x,u)udx \right) \\ &\leqslant \left(\frac{1}{p} - \frac{1}{\min{(\alpha_2, \beta_2)}} \right) \Phi(u) \leqslant \left(\frac{1}{p} + \frac{1}{\min{(\alpha_2, \beta_2)}} \right) \|u\|^p_{X_0} \end{aligned}$$

Proof Let $u \in K_i$, we have that

$$A(u) = \int_\Omega f(x,u)udx + \mu \int_\Omega g(x,u)udx, \tag{4.3}$$

and by 3.22 we get

$$\|u\|^p_{X_0} \leq \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1} \left(\int_\Omega f(x,u)udx + \mu \int_\Omega g(x,u)udx \right).$$

This establishes the first inequality.

Further more we have from conditions 4.1, 4.2 and 4.3

$$(B(u) + \mu C(u)) \leqslant \frac{1}{\min{(\alpha_2, \beta_2)}} \left(\int_\Omega f(x,u)udx + \mu \int_\Omega g(x,u)udx \right),$$

and therefore

$$\begin{aligned} \Phi(u) &= \frac{1}{p} A(u) - (B(u) + \mu C(u)) \\ &= \left(\frac{1}{p} - \frac{1}{\min{(\alpha_2, \beta_2)}} \right) \left(\int_\Omega f(x,u)udx + \mu \int_\Omega g(x,u)udx \right). \end{aligned}$$

This proves the middle inequality. Now, we will prove the third inequality as follows

$$|\Phi(u)| \leq \frac{1}{p}A(u) + B(u) + \mu C(u),$$

by 4.1, 4.2 we get

$$\begin{aligned}\Phi(u) &\leq \frac{1}{p}A(u) + \frac{1}{\alpha_2}\int_\Omega f(x,u)udx + \frac{\mu}{\beta_2}\int_\Omega g(x,u)udx,\\ &\leq \frac{1}{p}A(u) + \max\left(\frac{1}{\alpha_2},\frac{1}{\beta_2}\right)\left(\int_\Omega f(x,u)udx + \mu\int_\Omega g(x,u)udx\right),\\ &= \frac{1}{p}A(u) + \frac{1}{\min(\alpha_2,\beta_2)}\left(\int_\Omega f(x,u)udx + \mu\int_\Omega g(x,u)udx\right),\end{aligned}$$

and by 4.3, 3.22 we find

$$\Phi(u) \leq \left(\frac{1}{p} + \frac{1}{\min(\alpha_2,\beta_2)}\right)A(u) \leq \left(\frac{1}{p} + \frac{1}{\min(\alpha_2,\beta_2)}\right)\|u\|^p_{X_0}.$$

This finishes the proof

Lemma 4.3 *There exists $c > 0$ such that,*

$$\begin{aligned}\|u_-\|_{X_0} &\geq c \textit{ for all } u \in K_2,\\ \|u_+\|_{X_0} &\geq c \textit{ for all } u \in K_1,\\ \|u_-\|_{X_0}, \|u_+\|_{X_0} &\geq c \textit{ for all } u \in K_3.\end{aligned}$$

Proof Using the fact that X_0 injects in $L^r(\Omega)$, for $r \in]0, p_s^*]$ and the conditions 4.1, 4.2 and by the definition of K_i. we have that

$$\begin{aligned}A(u_\pm) &= \int_\Omega f(x,u_\pm)u_\pm dx + \mu\int_\Omega g(x,u_\pm)u_\pm dx \leq \alpha_4\|u_\pm\|^{p_s^*}_{L^{p_s^*}(\Omega)} + \beta_4\mu\|u_\pm\|^q_{L^q(\Omega)},\\ &\leq \alpha_4 c_1\|u_\pm\|^{p_s^*}_{X_0} + \beta_4 c_2\|u_\pm\|^q_{X_0},\end{aligned}$$

and by 3.22 we get for some positive constantes c_1 and c_2

$$\|u_\pm\|^p_{X_0} \leq \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1}\left(\alpha_4 c_1\|u_\pm\|^{p_s^*}_{X_0} + \beta_4 c_2\|u_\pm\|^q_{X_0}\right).$$

As $p < q < p_s^*$, this finishes the proof.

Lemma 4.4 *There exists $l > 0$ such that $\Phi(u) \geq l\|u\|^p_X$ for every $u \in X_0$ if $\|u\|_{X_0}$ is small enough.*

Proof By 4.1, 4.2 and 3.22 we have for some positive constantes l_1 and l_2

$$\begin{aligned}\Phi(u) &= \frac{1}{p}A(u) - B(u) - \mu C(u)\\ &\geq \frac{1}{p}\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|_X^p - \left(\frac{\alpha_4}{\alpha_2}\|u\|^{p_s^*}_{L^{p_s^*}(\Omega)} + \frac{\beta_4}{\beta_2}\|u\|^q_{L^q(\Omega)}\right)\\ &\geq l_1\|u\|^p_{X_0} - l_2\left(\|u\|^{p_s^*}_{X_0} + \|u\|^q_{X_0}\right)\end{aligned}$$

As conseqences if $\|u\|_{X_0}$ is small enough, as $p < q < p_s^*$ we get

$$\Phi(u) \geq l\|u\|^p_{X_0}.$$

Now we introduce lemma for describing the properties of the manifolds M_i

Lemma 4.5 *M_i, is a C^1 sub-manifold of X_0 of co-dimension 1 $(i = 1, 2)$ and of co-dimension 2 for $i = 3$. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_uX_0 = T_uM_i \bigoplus \texttt{span}\left\langle u_-, u_+\right\rangle.$$

where T_uM is the tangent space at u of the banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof Let us denote

$$\bar{M}_1 = \left\{u \in X_0 : \int_\Omega u_+ dx > 0\right\},$$
$$\bar{M}_2 = \left\{u \in X_0 : \int_\Omega u_- dx > 0\right\},$$
$$\bar{M}_3 = \bar{M}_1 \cap \bar{M}_2.$$

We see that $M_i \subset \bar{M}_i$.

The set $\bar{M}_i$ is open in X_0, than it will be enough to prove that M_i is C^1 sub-manifold of $\bar{M}_i$. In order to do this, we have to construct a C^1-functions $\phi_i : \bar{M}_i \to R^d$ with $d = 1$ for $i = 1, 2$ and $d = 2$ for $i = 3$ and we will get $M_i = \phi_i^{-1}(0)$, where 0 is regular value of ϕ_i. First we define

$$\begin{aligned}\phi_1(u) &= A(u_+) - \int_\Omega f(x,u)u_+dx - \mu\int_\Omega g(x,u)u_+dx \texttt{ for } u \in \bar{M}_1,\\ \phi_2(u) &= A(u_-) - \int_\Omega f(x,u)u_-dx - \mu\int_\Omega g(x,u)u_-dx \texttt{ for } u \in \bar{M}_2,\\ \phi_3(u) &= \left(\phi_1\left(u\right), \phi_2\left(u\right)\right) \texttt{ for } u \in \bar{M}_3.\end{aligned}$$

We can easly see that $M_i = \phi_i^{-1}(0)$. From standard arguments see [15], ϕ_i is of class C^1. Therefore, we just need to prove that 0 is a regular value for ϕ_i. To do this we calculate for $u \in M_1$,

$$\begin{aligned}
\langle \phi_1'(u), u_+\rangle &= pA(u_+) - \int_\Omega f(x,u)u_+dx - \int_\Omega f_u(x,u)u_+^2dx - \mu\int_\Omega g(x,u)u_+dx - \mu\int_\Omega g_u(x,u)u_+^2dx,\\
&\le pA(u_+) - \int_\Omega \left(1+\frac{1}{\alpha_1}\right) f(x,u)u_+dx + \mu\left(1+\frac{1}{\beta_1}\right)\int_\Omega g(x,u)u_+dx,\\
&\le pA(u_+) - \left(1+\frac{1}{\max(\alpha_1,\beta_1)}\right)\left(\int_\Omega f(x,u)u_+dx + \mu\int_\Omega g(x,u)u_+dx\right),\\
&= \left(p-1-\frac{1}{\max(\alpha_1,\beta_1)}\right)A(u_+).
\end{aligned}$$

We know that $\alpha_1, \beta_1 < \frac{1}{p-1}$. Hence the last term is strictly negative by lemma 4.3. Therefore, M_1 is a C^1 sub-manifold of X. we can argue the same way for M_2 and M_3 Since we have

$$\langle \phi_1'(u), u_+\rangle = \langle \phi_2'(u), u_-\rangle = 0.$$

Now, we will prove that K_i is complete,

Let u_k be a Cauchy sequence in K_i, then $u_k \to u$ in X. Moreover $(u_k)_\mp \to (u)_\mp$ in X. and we can deduce by 4.3 and by continuity that $u \in K_i$. Finally, we have the deomposition

$$T_uX = T_uM_1 \bigoplus \text{span}\,\langle u_+\rangle\,.$$

Where $M_1 = \{u : \phi_1(u) = 0\}$ and $T_uM_1 = \{v : \langle \phi_1'(u), v\rangle = 0\}$. Let $v \in T_uX_0$ be unit tangential vector, then $v = v_1 + v_2$ where $v_2 = \gamma u_+$ and $v_1 = v - v_2$. Let us take γ as

$$\gamma = \frac{\langle \phi_1'(u), v\rangle}{\langle \phi_1'(u), u_+\rangle}.$$

With this choice, we have that $v_1 \in T_uM_1$. then $\langle \phi_1'(u), v_1\rangle = 0$. We use the same argument to show that $T_uX = T_uM_2 \bigoplus \text{span}\,\langle u_-\rangle$, and $T_uX = T_uM_3 \bigoplus \text{span}\,\langle u_-, u_+\rangle$. This estabishes the uniform continuity of the projections onto T_uM_i.

Lemma 4.6 *The fuctional Φ verifies the palais-Smale condition for energy level*

$$c < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}. \tag{4.4}$$

where S_p is the Sobolev constant given by 3.10.

Let $\{u_k\} \subset X_0$ be a $(PS)_c$ sequence for $\Phi_{\lambda,\mu}$. Then, there exists a subsequence of $\{u_k\}$, which converges strongly in X_0.

Proof From Lemma 4.2, we see that $\{u_k\}$ is bounded in X_0. Then, up to a sequence, still denoted by $\{u_k\}$, there exists $u_* \in X_0$ such that $u_k \to u_*$ weakly in X_0, that is

$$A\left(u_k\right) \to A\left(u_*\right), \text{ as } k \to \infty.$$

Moreover, by [34], [lemma 8], we have that

$$u_k \to u_* \text{ weakly in } L^{p_s^*}\left(\mathbb{R}^n\right), u_k \to u_* \text{ in } L^{r+1}(\mathbb{R}^n), u_k \to u_* \text{ in } \mathbb{R}^n.$$

As $k \to \infty$, and by [15], [theorem IV-9] , there exists $l \in L^{r+1}(\mathbb{R}^n)$ such that:

$$|u_k(x)| \leq l(x) \text{ in } \mathbb{R}^n,$$

for any $1 \leq q < p_s^*$. Therefore, by dominated convergence theorem, we have that

$$C\left(u_k\right) \longrightarrow C\left(u_*\right), \text{ as } k \to \infty.$$

By Brezis-Lieb [38], [Lemma 1.32], we get

$$\begin{aligned} A\left(u_k\right) &= A(u_k - u_*) + A\left(u_*\right) + o(1), \\ B\left(u_k\right) &= B\left(u_k - u_*\right) + B\left(u_*\right) + o(1). \end{aligned}$$

Then,

$$\begin{aligned} \left\langle \Phi'_{\lambda,\mu}(u_k), u_k \right\rangle_{X_0} &= A\left(u_k\right) - p_s^* B\left(u_k\right) - \mu q C\left(u_k\right) \\ &= A(u_k - u_*) + A\left(u_*\right) - p_s^*(B\left(u_k - u_*\right) + B\left(u_*\right)) - \mu q C\left(u_k\right) + o(1) \\ &= \left\langle J'_{\lambda,\mu}(u_*), u_* \right\rangle_{X_0} + A(u_k - u_*) - p_s^* B\left(u_k - u_*\right). \end{aligned}$$

By $\left\langle \Phi'_{\lambda,\mu}(u_*), u_* \right\rangle_{X_0} = 0$ and $\left\langle \Phi'_{\lambda,\mu}(u_k), u_k \right\rangle_{X_0} \longrightarrow 0$ as $k \longrightarrow \infty$., we know that

$$A(u_k - u_*) \longrightarrow b \text{ and } p_s^* B\left(u_k - u_*\right) \longrightarrow b. \tag{4.5}$$

If $b = 0$, the proof is complete. Assuming $b > 0$, by 3.22, we get

$$p_s^* B\left(u_k - u_*\right) \leq \frac{\alpha_4}{\alpha_2} p_s^* S_p^{-\frac{p_s^*}{p}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{-\frac{p_s^*}{p}} \left(A(u_k - u_*)\right)^{\frac{p_s^*}{p}}.$$

Then

$$b \geq \left(\frac{\alpha_4}{\alpha_2} p_s^*\right)^{\frac{-n}{sp_s^*}} \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

On the other hand, we have

$$c = \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A\left(u_k\right) - B\left(u_k\right) - \mu C\left(u_k\right)\right)$$

$$= \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A(u_k - u_*) - B\left(u_k - u_*\right) - \frac{1}{p}A\left(u_*\right) - B\left(u_*\right) - \mu C(u_k)\right) + o(1)$$

$$= \Phi_{\lambda,\mu}\left(u_*\right) + b(\frac{1}{p} - \frac{1}{p_s^*}) \geq \Phi_{\lambda,\mu}(u_*) + \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

By the assumption that $c < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}$, we obtain $\Phi_{\lambda,\mu}(u_*) < 0$. In particular, $u_* \neq 0$, and

$$B\left(u_*\right) > \tfrac{1}{p}A\left(u_*\right) - \mu C\left(u_*\right). \tag{4.6}$$

Then,

$$c = \lim_{k\longrightarrow\infty}\Phi_{\lambda,\mu}(u_k) = \lim_{k\longrightarrow\infty}\left(\Phi_{\lambda,\mu}(u_k) - \frac{1}{p}\left\langle\Phi'_{\lambda,\mu}(u_k), u_k\right\rangle_{X_0}\right)$$

$$= \lim_{k\longrightarrow\infty}(\frac{p_s^*}{p} - 1)(B\left(u_k - u_*\right)) + B\left(u_*\right) - \mu(\frac{p-q}{p})C(u_k)$$

$$= \frac{sp_s^*}{n}\left(B\left(u_k - u_*\right) + B\left(u_*\right)\right) - \mu(\frac{p-q}{p})C\left(u_*\right)$$

$$\geq \frac{s}{n}\left(\frac{p_s^*\alpha_4}{\alpha_2}\right)^{\frac{-n}{sp_s^*}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}} + \frac{sp_s^*}{n}B\left(u_*\right) + \mu\left(\frac{q-p}{p}\right)C\left(u_*\right).$$

Then,

$$c \geq \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

Then, we get a contradiction with our hypothesis. Hence, $b = 0$ and, we conclude that $u_k \to u_*$ strongly in X_0. This completes the proof.

4.3 Proof of the main result

In this section, we will prove the main result (Theorem4.1). First of all, we begin by remark that if $u \in K_i$ is a critical point of the restricted functional $\Phi_{\lambda,\mu}|_{K_i}$. Then u is also a critical point of the unrestricted functional $\Phi_{\lambda,\mu}$. Which implies that u is a weak solution for problem (1).

Lemma 4.7 *If c satisfies (4.4), then the functional $\Phi_{\lambda,\mu}$ defined on K_i satifies the Plais-Smale condition at level c.*

Proof Let $(u_k) \in K_i$ be a sequence such that $\Phi_{\lambda,\mu}(u_k)$ is uniformly bounded and $\Phi'_{\lambda,\mu}(u_k) \to 0$. Let $v_j \in T_{u_j}X_0$, be a unit tangenttial vector such that

$$\langle \Phi'_{\lambda,\mu}(u_j), v_j \rangle = \|\Phi'_{\lambda,\mu}(u_j)\|_{X'}.$$

By lemma 4.5, we have that $v_j = w_j + y_j$, for some $w_j \in T_{u_j}M_i$ and $y_j \in span \left\langle (u_j)_+, (u_j)_- \right\rangle$. Since $\Phi_{\lambda,\mu}(u_j)$ is uniformly bounded then, by lemma 4.2, u_j is also uniformly bounded in X_0. So, w_j is uniformly bounded in X_0. Therefore, as j tends to infinity, we get

$$\|\Phi'_{\lambda,\mu}(u_j)\|_{X'} = \langle \Phi'_{\lambda,\mu}(u_j), v_j \rangle = \langle \Phi'_{\lambda,\mu}|_{K_i}(u_j), v_j \rangle \to 0.$$

As a consequences we get

$$\Phi'_{\lambda,\mu}|_{K_i}(u_k) \to 0.$$

Finally, the result follows immediately from Lemma 4.6.

Now, we need to show that the fuctional $\Phi|_{K_i}$. satifies the hypothesis of the Ekeland's Variational Principle [26]. We have as a direct consequence of the construction of the manifold K_i that Φ is bounded below over K_i.
Hence, by Ekeland's Variational Principle, there existe $v_k \in K_i$ such that

$$\Phi(v_k) \to c_i := \inf_{K_i} \Phi \text{ and } (\Phi|_{K_i})'(v_k) \to 0.$$

We have frome lemma 4.1 if we choose μ large, then we get

$$c_i < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

For instance, for c_1, we get the choosing $w_0 \geq 0$,

$$c_1 \leq \Phi(t_\mu w_0) \leq \frac{1}{p} t_\mu^p A(w_0).$$

Therefore $c_1 \to 0$ as $\mu \to +\infty$. Moreover, it follows from lemma 4.1 that

$$c_i < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}} \text{ for } \mu > \mu^*(p, q, n, \alpha_3, \beta_3).$$

frome lemma 4.6, there exists a convergent subsequence extracted from v_k still denoted v_k. Therefore the funcational Φ has a critical point in K_i, $i = 1, 2, 3$.

Conclusion

In this thesis we studied a non-local elliptic fractional Laplacian problem with regular non-linearity using two different variational techniques under different conditions.
In the first work the non-linearities are two continuous functions satisfying homogenous conditions. We proved the existence of two non-trivial positive solutions for the subcritical case by applying fibering maps, the Nehari manifold, and some basic calculations. The second main result obtained concerns the critical case we proceed in the same way depending on some additional convergence criteria with a little more complicated calculations, due to the lack of compacity of the embedding, and as a result of that the energy functional does not satisfy the Palais-Smale condition globally, except in an appropriate condition due to the best critical Sobolev constant.
In the second work the non-linearities functions satisfying different conditions, we proved the existence of three distinct solutions for the critical case, using Ekeland's variational principle.
Finally, the results obtained in this thesis concern a fractional p-Laplacian problem using different variational techniques under different assumptions for the critical case, and it can be generalized with $p(x, y)$-fractional operators and other types of nonlinearities.

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