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Option: Evolution Problems

Theme

Global Existence and Asymptotic Behavior for some Classes of Partial Differential Equations with Delay

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KAMACHE HOURIA

Dedication

This modest work of study is dedicated to :

My dear parent KAMACHE Mouhamed and Kamache Rebiha, who never stopped supporting me and encouraging me during my years of study.

My dear husband YESSAAD Aissa , who has encouraged me all the way and taken with me charge of this study, and my beloved daughter ALA.

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And I also dedicate it to :

All the teachers of Department of Mathematics.

All my friends.

KAMACHE Houria

الملخص

الهدف الرئيسي من هذه الأطروحة هو دراسة الوجود الكلي ، الاضمحلال العام ونتائج الإنفجار لحلول بعض المعادلات التطورية غير الخطية مع أنواع مختلفة من الشروط الحدية وحدود زمن التأخير. يتكون هذا العمل من ثلاثة فصول.

في الفصل 1، نقدم بعض الرموز ، الفرضيات ، الملاحظات، والنتائج الأساسية وبعض النظريات الأساسية في التحليل الدالي.

الفصل 2 ، مخصص لدراسة مسألة ذات قيمة ابتدائية وحدية لمعادلة من نوع كيرشوف مع حدي التأخر و المصدر الغير خطيين.

فى الفصل 3 ، قمنا بدراسة معادلة اللزوجة ذات الألواح من نوع كيرشوف مع شروط ديناميكية حدية و حدي التأخير و المصدر المطبقة على الحافة.

تحصلنا على الوجود الكلي للحلول بنظرية بئر الكمون . وقد تم إثبات نتيجة الإضمحلال للطاقة من خلال توظيف معادلة الطاقة و توابع ليابونوف ، و نتيجة إنفجار الحلول بالإعتماد على طريقة جورجييف و تودوروفا.

الكلمات المفتاحية : معادلة من نوع كيرشوف ، شروط حدية غير خطية ، شروط ديناميكية حدية ، حد التأخير ، الوجود الكلي ، الإضمحلال العام ، الإنفجار.

RÉSUMÉ

L'objectif principal de cette thèse est d'étudier l'existence globale, la décroissance générale et le résultat d'explosion de solutions à certaines équations d'évolutions non linéaires avec différents types de conditions aux limites et de termes de retard. Ce travail se compose de trois chapitres:

Dans le chapitre 1, nous donnons quelques notations, présentons nos hypothèses et principaux résultats et quelques théorèmes principaux en analyse fonctionnelle.

Chapitre 2 est consacré à l'étude d'un problème aux limites initial pour une équation de type Kirchhoff avec retard aux limites non linéaire et termes sources.

Dans le chapitre 3, nous étudions une équation de plaque viscoélastique de Kirchhoff avec des conditions aux limites dynamiques, un retard et des termes sources agissant sur la frontière.

L'existence globale de solutions a été obtenue par la théorie des puits de potentiel, le résultat de décroissance général de l'énergie a été établi en introduisant une énergie appropriée et des fonctionnelles de Lyapunov et le résultat d'explosion de solutions basées sur la méthode de Georgiev et Todorova.

Mots clés

Équation de type Kirchhoff, conditions aux limites non linéaires, conditions aux limites dynamiques, terme de retard, existence globale, décroissance générale, explosion.

ABSTRACT

The main goal of this thesis is to study the global existence, general decay, and blow-up results of solutions for some nonlinear evolutions equations with different types of boundary conditions and delay terms. This work consists of three chapters:

In chapter 1, we give some notations, present our assumptions and main results and some main theorems in functional analysis.

Chapter 2 is devoted to study an initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms.

In chapter 3, we study a viscoelastic Kirchhoff plate equation with dynamic boundary conditions, delay and source terms acting on the boundary.

The global existence of solutions has been obtained by potential well theory, the general decay result of energy has been established by introducing suitable energy and Lyapunov functionals, and the blow up result of solutions based on the method of Georgiev and Todorova.

Key words

Kirchhoff type-equation, nonlinear boundary conditions, dynamic boundary conditions, delay term, global existence, general decay, blow up.

CONTENTS

INTRODUCTION

A great number of processes of the applied sciences can be modeled by means of evolution equations or systems. Nonlinear partial differential equations and systems exhibit a number of properties which are absent from the linear theories ; these nonlinear properties are often related to important features of the real world phenomena which the mathematical model is supposed to describe; at the same time these new properties are closely connected with essential new difficulties of the mathematical treatment. The study of nonlinear processes has been a continuous source of new problems and it has motivated the introduction of new methods in the areas of mathematical analysis, partial differential equations and other disciplines, becoming a most active area of mathematical research.

The Kirchhoff-type equation was introduced by Kirchhoff [36] in order to study nonlinear vibrations of an elastic string. Kirchhoff was the first one who study the oscillations of stretched strings and plates. The existence, decay, and blow-up of solutions in this case have been discussed by many authors. For example, the following Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u). \tag{0.1}$$

Equation (0.1) with $M \equiv 1$, is reduces to a nonlinear wave equation, which has been extensively studied, see for instance [37, 38, 21, 13] and the references therein. When $M \neq 1$. Matsuyama and Ikehata [41] studied (0.1) for $g(u_t) = \delta|u_t|^p u_t$ and $f(u) = \xi|u|^p u$. They proved existence of the global solutions by using Faedo-Galerkin's method and the decay of energy based on the method of Nakao [49]. Ono [52] studied (0.1) with $M(s) = bs$, $g(u_t) = -\Delta u$, and $f(u) = \xi|u|^p u$. They showed that the solutions blow up in finite time with negative initial energy. Later, Wu and Tsai [59] studied (0.1) with different damping terms (u_t, Δu_t, and $|u_t|^{m-2}u_t$), they obtained unique local solution and finite time blow up of solutions, we also refer the interested reader to [6, 53, 62] and the references therein.

In the matter of the study of plates, plate equations have been investigated for many years due to their importance in various physical areas such as vibration and elasticity theories of solid mechanics. The viscoelastic plate equations have been studied by many authors and several stability results have been established. Rivera et al. [48] studied an initial-boundary problem for the following viscoelastic plate equation,

$$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0, \tag{0.2}$$

together with initial and dynamical boundary conditions and proved that the sum of the first and second energies decays exponentially (respectively polynomially) if the kernel g decays exponentially (respectively polynomially). Alabau-Boussouira et al. [62] studied (0.2) with $\gamma = 0$ and semilinear source terms $f(u)$. They established exponential and polynomial decay results for sufficiently small initial data. Recently, Messaoudi and Mukiawa [45] [62] studied (0.2) in the bounded domain $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$ with non traditional boundary conditions and $\gamma = 0$, they established the well-posedness and the decay rate of the energy. Yang et al. [61] studied the following plate equation with velocity-dependent density and memory term effects

$$|u_t|^\rho u_{tt} - \Delta u_{tt} + \Delta^2 u - M\left(\int_\Omega |\Delta u|^2 dx\right)\Delta u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0, \tag{0.3}$$

and established the decay rate of energy by exploring only the memory. For more results related to the plate equation, we refer the reader to [47, 3, 31, 39] and the references therein.

The boundary condition describes the reflection of sound at surfaces of some materials with memory of interest in engineering practice. It is quite general and covers a fairly large variety of physical configurations. Such types of boundary conditions are usually called dynamic boundary conditions. The dynamic boundary conditions represent the Newton's law for the attached mass, (for more details [4, 9, 14, 27, 28]).

There are so many results concerning the wave equation with boundary conditions. Vitillaro [58] considered the following wave equation with nonlinear boundary damping and source terms

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in}\,\Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on}\,\Gamma_0 \times (0,\infty), \\ u_\nu = -|u_t|^{m-2}u_t + |u|^{p-2}u, & \text{on}\,\Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{0.4}$$

He proved local existence of the solutions, global existence when $p \leq m$ or the initial data was chosen suitably. Zhang and Hu [63] proved the asymptotic behavior of the solution for problem (0.4) when the initial data is inside a stable set, and the nonexistence of the solution when $p > m$ and the initial data is inside an unstable set. nonlinear source and boundary damping terms. Gerbi and Said-Houair [22] studied the following problem

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ u_{tt}(x,t) = -\Big[\dfrac{\partial u}{\partial \nu}(x,t) + \alpha \dfrac{\partial u_t}{\partial \nu}(x,t) + \gamma |u_t|^{m-2}\Big], & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{0.5}$$

and proved the local existence by using the Faedo-Galerkin approximations combined with a contraction mapping theorem and showed the exponential growth of the energy. Later Gerbi and Said-Houair [23] established the global existence and asymptotic stability of solutions starting in a stable set by combining the potential well method and the energy method. A blow-up result for the case $m = 2$ with initial data in the unstable set was also obtained, we also refer other works [1, 11, 12, 26] and the references therein.

Recently, many mathematical researchers have been studying partial differential equations with time delay effects, and established so many results concerning the global well-posedness of these systems. It is well known that time delay effects often appear in many chemical, physical, and economical phenomena because these phenomena depend not only on the present state but also on the past history of the system. The presence of delay can be a source of instability and an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms are added. Nicaise and Pignotti [50] considered the following wave equation with a delay term

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0. \tag{0.6}$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. Then, they extended the result to the time-dependent delay case in the work of Nicaise and Pignotti[51]. Kafini et al. [32] investigated the following nonlinear wave equation with delay

$$u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + \mu_1 u_t + \mu_1 u_t(t-\tau) = b|u|^{p-2}u. \tag{0.7}$$

They proved the blow-up result of solutions with negative initial energy and $p \geq m$. Later, Kafini et al. [33] considered the blow up of solutions with negative initial energy for the second-order abstract evolution system with delay. For the plate equation with time delay term, Park [54] considered a weak viscoelastic plate equation with a time-varying delay

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \sigma(t)\int_0^t g(t-s)\Delta u(s)ds + a_0 u_t + a_1 u_t(t-\tau(t)) = 0, \tag{0.8}$$

and proved a general decay result of energy under the assumption $|a_1| < \sqrt{1-d}a_0$. Feng [17] considered the following plate equation with a memory term and a time delay term in the internal feedback:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \sigma(t)\int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau(t)), \tag{0.9}$$

and obtained the global well-posedness with $|\mu_2| \leq u_1$ and decay rate of energy under the assumption $|\mu_2| < \mu_1$. Yang [60] studied (0.9) with $M \equiv 1$ and proved the existence of global solution under suitable assumptions on the relaxation function g. Moreover, under some restrictions on μ_1 and u_2, exponential decay results of the energy for the concerned problem are obtained via an appropriate Lyapunov function. For the related works of equations with delay term, we also refer other works [7, 25, 15, 34, 35, 42] and the references therein.

In recent years, the study of partial differential equations with delay term acting in the boundary have has been considered by numerous authors, see for example [8, 24, 18, 19, 40]. Gerbi and Said-Houari [24] considered the damped wave equation with dynamic boundary conditions and a delay boundary term

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ u_{tt} = -\Big(\dfrac{\partial u}{\partial \nu}(x,t) + \alpha \dfrac{\partial u_t}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\Big), & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x-\tau), & x \in \Gamma_1,\ t \in [0,\tau), \end{cases} \tag{0.10}$$

and proved the global existence of the solutions and the exponential stability of the system. Ferhat and Hakem [18] considered the following wave equation with dynamic boundary conditions and nonlinear delay term

$$\begin{cases} u_{tt} - \Delta u + \delta \Delta u_t - \sigma(t) \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, & in \Omega \times (0,\infty), \\ u(x,t) = 0, & on \Gamma_0 \times (0,\infty), \\ u_{tt} = -a\Big[\dfrac{\partial u}{\partial \nu}(x,t) + \delta \dfrac{\partial u_t}{\partial \nu}(x,t) - \sigma(t) \displaystyle\int_0^t g(t-s)\Delta u(s)ds \dfrac{\partial u}{\partial \nu}(x,t) & \\ +\mu_1 |u_t|^{m-1} u_t(x,t) + \mu_2 |u_t(x,t-\tau)|^{m-1} u_t(x,t-\tau)\Big], & on \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau(0)) = f_0(x-\tau(0)), & on \Gamma_1 \times (0,\infty). \end{cases} \tag{0.11}$$

By using the Faedo-Galerkin approximation combined with a contraction mapping theorem and the concept of stable sets, they showed the local and global existence of solutions. Also, they proved the general decay results by exploiting the perturbed Lyapunov functionals. For the related works of equations with delay boundary term.

This thesis is composed of three chapters divided as follows:

Chapter 1 : Preliminaries

This Chapter contends to present some notations and preliminaries, especially we recall some basic knowledge in functional analysis.

Chapter 2 : Global existence, asymptotic behavior and blow up of solutions for a Kirchhoff-type equation with nonlinear boundary delay and source terms

In this party, we are concerned with the global existence, decay, and blow up of solutions to the initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + u_t = 0.$$

Chapter 3 : General Decay and blow up of Solutions for the Kirchhoff plate equation with dynamic boundary conditions, delay and source terms

In this party, we investigate the global existence, decay, and blow up of solutions to the following initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)ds = 0.$$

CHAPTER 1

PRELIMINARIES

1.1 Function analysis

1.1.1 Banach spaces

Definition 1.1. *Let X be a vector space over $\mathbb{R}$, a real-valued function $\|.\|$ defined on X and satisfying the following conditions is called a norm:*

- $\|u\| \geq 0$, $\|u\| = 0$ *if and only if* $u = 0$.
- $\|\lambda u\| = |\lambda|\|u\|$ *; for all* $u \in X$ *and* $\lambda \in \mathbb{R}$.
- $\|u + v\| \leq \|u\| + \|v\|$, $\forall\ v, u \in X$.

Lemma 1.1. $(X, \|.\|)$, *vector space* X *equipped with* $\|.\|$ *is called a normed space.*

Definition 1.2. *(Equivalent norms). We say that two norms* $\|.\|_1$ *and* $\|.\|_2$ *of a normed space* X *are equivalent, if there exist constants* α, $\beta > 0$ *such that*

$$\alpha\|.\|_1 \leq \|.\|_2 \leq \beta\|.\|_1, \ \ \forall x \in X.$$

Definition 1.3. *(Cauchy sequences). Let X be a normed space, and let $(x_n)_{n\in\mathbf{N}}$ be a sequence of elements of X. $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence if*

$$\forall\varepsilon > 0;\ \exists N > 0, \forall n, m \geq N,\ \ \|u_n - u_m\| < \varepsilon.$$

Definition 1.4. *(complet spaces). Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces.*

Definition 1.5. *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n **converges strongly** to u in X if and only if*

$$\lim \|u_n - u\|_X = 0$$

and this is denoted by $u_n \to u$ or $\lim_{n\to\infty} u_n = u$

1.1.2 Hilbert spaces

Definition 1.6. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.1. *(Riesz [10]). If $(H; \langle .,.\rangle)$ is a Hilbert space,$\langle .,.\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|_{H'} = \|x\|_H$*

Theorem 1.2. *([10]). Let $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in the Hilbert space H, it posses a subsequence which converges in the weak topology of H.*

Theorem 1.3. *([10]). In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 1.4. *([10]). Let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges to u, in the weak topology and $(u_n)_{n\in\mathbb{N}}$ is an other sequence which converge weakly to v, then*

$$\lim_{n\to\infty} \langle u_n, v_n\rangle = \langle u, v\rangle$$

1.1.3 The $L^p(\Omega)$ spaces

Definition 1.7. *Let $1 \leq p < \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $(n \in \mathbb{N})$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \; is\, measurable\, and \int_\Omega \|f(x)\|^p dx < \infty \right\}.$$

Definition 1.8. *For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega \|f(x)\|^p dx \right)^{\frac{1}{p}}.$$

Definition 1.9. *If $p = \infty$, we have*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \; is\, measurable\, and\, there\, exists\, a\, constant\; C \; such\, that \, |f(x)| \leq C \; a.e \, in \, \Omega \},$$

and

$$\|f\|_{L^\infty(\Omega)} = \inf \{ C, \; |f(x)| \leq C \; a.e \, in \, \Omega \}.$$

Theorem 1.5. *([10]). L^p is a vectorial space and $\| \cdot \|_p$ is a norm for all $1 \leq p \leq \infty$.*

Theorem 1.6. *(Fischer-Riesz [10]). L^p is a Banach space for all $1 \leq p \leq \infty$.*

1.1.4 The $L^p(0, T; X)$ spaces

Definition 1.10. *Let X be a Banach space, $1 \leq p < \infty$, denote by $L^p(0, T; X)$ the space of measurable functions*

$$L^p(0, T; X) = \left\{ f :]0, T[\to X \;\; is \;\; mesurable \;\; and \;\; \int_0^T \|f\|_X^p dt < \infty \right\},$$

if $p = \infty$

$$L^\infty(0, T; X) = \left\{ f :]0, T[\to X, f \, is\, measurable\, and\, there\, is\, a\, constant\, C\, suchthat \sup_{t \in [0,T]} ess \|f\|_X \leq C \right\},$$

Lemma 1.2. *The space $L^p(0,T;X)$ equipped with the norm*

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(x)\|_X^p dx\right)^{\frac{1}{p}}, \quad \textit{for } p < \infty,$$

and

$$\|f\|_{L^p(0,T;X)} = \sup_{t\in[0,T]} ess\|f(x)\|_X, \quad \textit{for } p = \infty,$$

is a Banach space.

1.1.5 The $W^{m,p}(\Omega)$ spaces

Definition 1.11. *(Weak derivatives). Let Ω be an open set in $\mathbb{R}^n$. Let $f, g \in L^{loc}(\Omega)$ and $i \in \{1, n\}$. We say that g is the weak partial derivative of f in the direction i if*

$$\int_\Omega f\partial_i\varphi dx = -\int_\Omega g\varphi dx, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

and we write $\partial_i f := g$.
Let $\alpha \in \mathbb{N}^n$ be a multi-index. We say that g is the weak α-th partial derivative of f and we write $\partial^\alpha f = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n} f := g$ if

$$\int_\Omega f\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}\varphi dx = (-1)^{|\alpha|}\int_\Omega g\varphi dx, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

where $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

Lemma 1.3. *Let $f \in L^{loc}(\Omega)$. If f has a weak α-th partial derivative, then it is uniquely defined.*

Definition 1.12. *Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \left\{f \in L^p(\Omega),\ \textit{such that } \partial^\alpha f \in L^p(\Omega) \textit{ for all } \alpha \in \mathbb{N}^m \textit{ such that } |\alpha| = \sum_{j=1}^n \alpha_j \leq m\right\},$$

where $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}$.

Theorem 1.7. *$W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|<m} \|\partial^\alpha f\|_{L^p},\ 1 \leq p < \infty,\ for\ \ all\ f \in W^{m,p}(\Omega).$$

Definition 1.13. *We denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Remark 1.1. *If $p = 2$, we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega) \quad and \quad H_0^m(\Omega) = W_0^{m,2}(\Omega),$$

supplied with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha|<m} \|\partial^\alpha f\|_{L^2} \right)^{\frac{1}{2}},$$

with usual scalar product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|<m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Theorem 1.8.

1. *$H^m(\Omega)$ supplied with inner product $\langle ., .\rangle_{H^m(\Omega)}$ is Hilbert space.*
2. *If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.*

Theorem 1.9. *(Rellich-Kondrachov [46]). Assume that Ω is an open domain in $\mathbb{R}^n, (n \geq 1)$, with smooth boundary $\partial\Omega$. Then,*

1. *If $1 \leq p \leq n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p*]$, where $p* = \dfrac{np}{n-p}$.*
2. *If $p = n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.*
3. *If $p > n$, we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \dfrac{p-n}{p}$.*

Proposition 1.1. *(Green's formula [10]). For all $u \in H^2(\Omega)$, $v \in H^1(\Omega)$, we have*

$$-\int_\Omega \Delta u v dx = \int_\Omega \nabla u \nabla v dx - \int_{\partial x} \frac{\partial u}{\partial \eta} v d\sigma,$$

where $\dfrac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

1.2 Some inequalities and lemmas used

Notation. for $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.10. *(Algebrique Young's inequality). For all $u, v \in \mathbb{R}^+$, we have*

$$|uv| \leq \alpha |u|^2 + \frac{|v|^2}{4\alpha}.$$

where α is any positive constant.

Theorem 1.11. *(Young's inequality [10]) For $u, v \geq 0$, the following inequality holds*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Theorem 1.12. *(Hölder's inequality [10]). Let $1 \leq p \leq \infty$. Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_p \|v\|_q.$$

when $p = q = 2$ one finds the Cauchy-Schwarz inequality.

1.3 Existence and asymptotic behavior of solution

The study of local existence and uniqueness of solutions of partial differential equations is based on the existence theory for abstract semilinear differential equations (see [20, 30, 55]).

Definition 1.14. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|\cdot\|)$, $F : X \to X$. We consider the following nonhomogeneous Cauchy problem*

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = F(u(t)), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{P}$$

A local solution of (P) is a solution define in a interval $[0, T)$ for $t < T$.

Maximal existence time

Definition 1.15. *Let $u(t)$ be a weak solution of* (P). *We define the maximal existence time T of $u(x,t)$ as follows:*

1. *If $u(t)$ exists for all $0 \leq t < \infty$, then $T = \infty$.*
2. *If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \leq t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.*

Remark 1.2. *If* 1 *is satisfied, we say that the solution $u(t)$ is global.*

Stabilization

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation. More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limits null or not and if this limits null, to give an estimate of the decay rate of the energy to zero.

They are several type of stabilization:

- Strong stabilization: $E(t) \xrightarrow[t \to \infty]{} 0$.
- Uniform stabilization: $E(t) \leq K e^{-ct}, \quad k,\, c > 0,\ \forall\, t > 0$.
- Polynomial stabilization: $E(t) \leq K t^{-c}, \quad k,\, c > 0,\ \forall\, t > 0$.
- Logarithmic stabilization: $E(t) \leq K \left(\log(t)\right)^{-c}, \quad k,\, c > 0,\ \forall\, t > 0$.

Blow-up

Definition 1.16. *(Finite time blow-up). Let $u(t)$ be a weak solution of* (P). *We call $u(t)$ blows up in finite time if the maximal existence time T is finite and*

$$\lim_{t \to T} \|u(t)\| = +\infty.$$

CHAPTER 2

GLOBAL EXISTENCE, ASYMPTOTIC BEHAVIOR AND BLOW UP OF SOLUTIONS FOR A KIRCHHOFF-TYPE EQUATION WITH NONLINEAR BOUNDARY DELAY AND SOURCE TERMS

This chapter is devoted to prove the global existence, decay, and the blow up of solutions to a Kirchhoff-type equation with nonlinear boundary delay and source terms.

2.1 The problem statement

We consider the following initial boundary value problem for a Kirchhoff-type equation with nonlinear boundary delay and source terms

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + u_t = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ M\left(\|\nabla u\|_2^2\right)\dfrac{\partial u}{\partial \nu} + \mu_1 |u_t|^{m-2}u_t + \mu_2 |u_t(t-\tau)|^{m-2}u_t(t-\tau) = |u|^{p-2}u, & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau), & x \in \Gamma_1,\ t > 0, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\mathrm{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\frac{\partial u}{\partial \nu}$ denotes the unit outer normal derivative, $M(s)$ is a positive C^1-function satisfying $M(s) = a + bs^\gamma$, $\gamma > 0$, $a > 0$, $b \geq 0$, $s \geq 0$, p, $m > 2$, μ_1 are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay, and u_0, u_1, f_0 are given functions belonging to suitable spaces.

2.2 Preliminary results

In this section, we will give some notations, assumptions, and some preliminary lemmas needed in the proof of our main results.

Throughout this thesis, to simplify the notations, we denote

$$\|u\|_p := \|u\|_{L^p(\Omega)}, \qquad \|u\|_{p,\Gamma_1} := \|u\|_{L^p(\Gamma_1)}, \qquad \|\nabla u\|_2 := \|u\|_{H^1},$$

where

$$\|u\|_{L^p(\Gamma_1)} = \left(\int_{\Gamma_1} |u|^p dx\right)^{\frac{1}{p}}.$$

Next, assume the following assumptions:

(A_1) $p \geq 2\gamma + 2$, if $n = 1, 2$, $2\gamma + 2 \leq p \leq \dfrac{n+2}{n-2}$, if $n \geq 3$.

(A_2) $|\mu_2| < \mu_1$.

Lemma 2.1. *(General Poincaré's Inequality). If then there is an optimal constant c_p such that*

$$2 \le p \le \frac{2n}{n-2},\ n \ge 3, \quad or\ p \ge 2,\ n = 1, 2,$$

then

$$\|u\|_p \le c_p \|\nabla u\|_2,\ \ \forall\ \ u \in H_0^1(\Omega). \tag{2.2}$$

Moreover, using the trace theorem, we have

$$\|u\|_{p,\Gamma_1} \le c_* \|\nabla u\|_2, \quad \forall\, u \in\ H^1_{\Gamma_0}(\Omega), \tag{2.3}$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) | u_{|\Gamma_0} = 0\}.$$

Proof. The proof can be found in [2, 16] □

To deal with the time delay term, motivated by Nicaise and Pignotti [50], we introduce a new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho),\ \ x \in \Gamma_1,\ \ \rho \in (0, 1),\ t > 0, \tag{2.4}$$

which gives us

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0,\ \ in\ \Gamma_1 \times (0, 1) \times (0, \infty). \tag{2.5}$$

Then, problem (2.1) is equivalent to

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u + u_t = 0, & x \in \Omega,\ t > 0, \\ u(x, t) = 0, & x \in \Gamma_0,\ t > 0, \\ M\left(\|\nabla u\|_2^2\right) \dfrac{\partial u}{\partial \nu} + \mu_1 |u_t|^{m-2} u_t + \mu_2 |z(1, t)|^{m-2} z(1, t) = |u|^{p-2} u, & x \in \Gamma_1,\ t > 0, \\ \tau z_t(\rho, t) + z_\rho(\rho, t) = 0, & x \in \Gamma_1,\ \rho \in (0, 1),\ t > 0, \\ z(\rho, 0) = f_0(-\tau\rho), & x \in \Gamma_1,\ \rho \in (0, 1), \\ u(x, 0) = u_0(x),\ \ u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \tag{2.6}$$

We first state a local existence theorem that can be established by Faedo-Galerkin Method, see for instance [5, 18, 29, 57].

Theorem 2.1. *(Local existence). Assume that* $(A_1)-(A_2)$ *hold. Then, for any* $(u_0, u_1, f_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \cap L^m(\Gamma_1) \times L^2(\Gamma_1 \times (0,1))$ *be given. Then, there exists a unique local solution* u *of problem* (2.1) *such that*

$$u \in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right), \ \ u_t \in L^\infty\left([0,T]; L^2(\Omega)\right) \cap L^m([0,T]\times\Gamma_1),$$

for some $T>0$.

Now, we define the energy associated with problem (2.6) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2} + \frac{\xi}{m}\int_0^1 \|z(\rho,t)\|^m_{m,\Gamma_1} d\rho - \frac{1}{p}\|u\|^p_{p,\Gamma_1}, \tag{2.7}$$

where ξ be a positive constant satisfying

$$\tau(m-1)|\mu_2| \le \xi \le \tau(m_1 - |\mu_2|). \tag{2.8}$$

Lemma 2.2. *Let* u *be a solution of problem* (2.1)*. Then,*

$$E'(t) \le -\|u_t\|_2^2 - m_0\left(\|u_t\|^m_{m,\Gamma_1} + \|z(1,t)\|^m_{m,\Gamma_1}\right) \le 0. \tag{2.9}$$

Proof. Multiplying the first equation in (2.6) by u_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2} - \frac{1}{p}\|u\|^p_{p,\Gamma_1}\right] \\ &= -\|u_t\|_2^2 - \mu_1\|u_t\|^m_{m,\Gamma_1} - \mu_2\int_{\Gamma_1}|z(1,t)|^{m-2}z(1,t)u_t dx. \end{aligned} \tag{2.10}$$

Multiplying the second equation in (2.6) by ξz^{m-1} and integrating over $\Gamma_1 \times (0,1)$, we obtain

$$\begin{aligned} \frac{\xi}{m}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 |z(\rho,t)|^m d\rho dx &= -\frac{\xi}{m\tau}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial\rho}|z(\rho,t)|^m d\rho dx \\ &= \frac{\xi}{m\tau}\left(\|u_t\|^m_{m,\Gamma_1} - \|z(1,t)\|^m_{m,\Gamma_1}\right). \end{aligned} \tag{2.11}$$

Using Young's inequality, we have

$$-\mu_2 \int_{\Gamma_1} |z(1,t)|^{m-2} z(1,t) u_t dx \leq \frac{(m-1)|\mu_2|}{m} \|z(1,t)\|_{m,\Gamma_1}^m + \frac{|\mu_2|}{m} \|u_t\|_{m,\Gamma_1}^m. \tag{2.12}$$

Combining (2.10), (2.11), and (2.12), we obtain

$$E'(t) \leq -\|u_t\|_2^2 - m_0 \left(\|u_t\|_{m,\Gamma_1}^m + \|z(1,t)\|_{m,\Gamma_1}^m \right), \tag{2.13}$$

where $m_0 = \min \left\{ \mu_1 - \frac{\xi}{m\tau} - \frac{|\mu_2|}{m}, \frac{\xi}{m\tau} - \frac{(m-1)|\mu_2|}{m} \right\}$, which is positive by (2.8). □

Similar to the result in [43], we can prove the following lemma.

Lemma 2.3. *There exists a positive constant $C_* > 1$ depending on Γ_1 only such that*

$$\|u\|_{p,\Gamma_1}^s \leq C_* \left(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p \right),$$

for any $u \in H_{\Gamma_1}^1(\Omega)$, $2 \leq s \leq p$.

2.3 Global Existence

In this section, we will prove that the solutions established in Theorem 2.1 are global in time. For this purpose, we define the functionals

$$I(t) = I(u(t)) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^{2\gamma+2} - \|u\|_{p,\Gamma_1}^p, \tag{2.14}$$

and

$$J(t) = J(u(t)) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2\gamma+2}\|\nabla u\|_2^{2\gamma+2} + \frac{\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho - \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{2.15}$$

Then, it is obvious that

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t). \tag{2.16}$$

In order to show our result, we first establish the following lemma.

Lemma 2.4. *Assume that* $(A_1)-(A_2)$ *hold and for any* $(u_0, u_1, f_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \cap L^m(\Gamma_1) \times L^2(\Gamma_1 \times (0,1))$*, such that*

$$I(0) > 0 \quad and \quad \alpha = \frac{c_*^p}{a}\left[\frac{2p}{a(p-2)}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{2.17}$$

then,

$$I(t) > 0, \ \forall t > 0. \tag{2.18}$$

Proof. Since $I(0) > 0$, then by continuity of u, there exist a time $T_* < T$ such that

$$I(t) \geq 0, \ \forall t \in [0, T_*]. \tag{2.19}$$

Using (2.14), (2.15), (2.16), and (2.9), we see that

$$\begin{aligned} J(t) &= \frac{1}{p}I(t) + \frac{a(p-2)}{2p}\|\nabla u\|_2^2 + \frac{b(p-2\gamma-2)}{p(2\gamma+2)}\|\nabla u\|_2^{2\gamma+2} + \frac{\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho \\ &\geq \frac{a(p-2)}{2p}\|\nabla u\|_2^2 + \frac{b(p-2\gamma-2)}{p(2\gamma+2)}\|\nabla u\|_2^{2\gamma+2}, \end{aligned} \tag{2.20}$$

and

$$\|\nabla u\|_2^2 \leq \frac{2p}{a(p-2)}J(t) \leq \frac{2p}{a(p-2)}E(t) \leq \frac{2p}{a(p-2)}E(0). \tag{2.21}$$

Exploiting (2.3), (2.17), and (2.21), we get

$$\|u\|_{p,\Gamma_1}^p \leq c_*^p\|\nabla u\|_2^p \leq \frac{c_*^p}{a}\left[\frac{2p}{a(p-2)}E(0)\right]^{\frac{p-2}{2}} a\|\nabla u\|_2^2 = \alpha a\|\nabla u\|_2^2 < a\|\nabla u\|_2^2, \ \forall t \in [0, T_*]. \tag{2.22}$$

Therefore, we have

$$I(t) > 0, \ \forall t \in [0, T_*]. \tag{2.23}$$

By repeating the procedure, T_* is extended to T. The proof is completed. □

Theorem 2.2. *Assume that the conditions of Lemma 2.4 hold, then the solution of problem* (2.1) *is global and bounded.*

Proof. It is sufficient to show that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2,$$

is bounded independently of t. By using (2.9), (2.16), and (2.20), we have

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{a(p-2)}{2p}\|\nabla u\|_2^2, \tag{2.24}$$

which means,

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(0), \tag{2.25}$$

where C is a positive constant. □

2.4 General decay

In this section, we state and prove the decay result of solution to problem (2.1). For this goal, we set

$$F(t) := E(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon}{2}\|u\|_2^2, \tag{2.26}$$

where ε is a positive constant to be specified later.

Lemma 2.5. *Let u be a solution of problem (2.1). Then, there exist two positive constants α_1 and α_2 depending on ε such that*

$$\alpha_1 E(t) \leq F(t) \leq \alpha_2 E(t). \tag{2.27}$$

Theorem 2.3. *Let $(u_0, u_1, f_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \cap L^m(\Gamma_1) \times L^2(\Gamma_1 \times (0,1))$. Assume that $(A_1)-(A_2)$ hold. Then, there exist two positive constant K and k such that*

$$E(t) \leq K e^{-kt}, \quad t \geq 0.$$

Proof. Differentiate (2.26) with respect to t, using (2.6) and (2.9), we obtain

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon \int_\Omega u u_{tt} dx + \varepsilon \int_\Omega u u_t dx \\ &\leq -m_0\|u_t\|_{m,\Gamma_1}^m - m_0\|z(1,t)\|_{m,\Gamma_1}^m - (1-\varepsilon)\|u_t\|_2^2 - a\varepsilon\|\nabla u\|_2^2 - b\varepsilon\|\nabla u\|_2^{2\gamma+2} \\ &\quad + \varepsilon\|u\|_{p,\Gamma_1}^p - \varepsilon\mu_1 \int_{\Gamma_1} |u_t|^{m-2} u_t u d\Gamma - \varepsilon\mu_2 \int_{\Gamma_1} |z(1,t)|^{m-2} z(1,t) u d\Gamma. \end{aligned} \tag{2.28}$$

By using Young's inequality for $\eta > 0$, we get

$$\begin{aligned}\mu_1 \int_{\Gamma_1} |u_t|^{m-2} u_t u d\Gamma &\leq \mu_1^m \eta \|u\|_{m,\Gamma_1}^m + c(\eta)\|u_t\|_{m,\Gamma_1}^m \leq \mu_1^m \eta c_*^m \|\nabla u\|_2^m + c(\eta)\|u_t\|_{m,\Gamma_1}^m \\ &\leq \eta c_1 \|\nabla u\|_2^2 + c(\eta)\|u_t\|_{m,\Gamma_1}^m,\end{aligned} \tag{2.29}$$

and

$$\mu_2 \int_{\Omega} |z(1,t)|^{m-2} z(1,t) u dx \leq \eta c_2 \|\nabla u\|_2^2 + c(\eta)\|z(1,t)\|_{m,\Gamma_1}^m, \tag{2.30}$$

where c_1 and c_2 are positive constants which depend only on m and $E(0)$. Combining (2.29)-(2.30) with (2.28), we obtain

$$\begin{aligned}F'(t) &\leq -(m_0 - \varepsilon c(\eta))\|u_t\|_{m,\Gamma_1}^m - (m_0 - \varepsilon c(\eta))\|z(1,t)\|_{m,\Gamma_1}^m - (1-\varepsilon)\|u_t\|_2^2 \\ &\quad -\varepsilon(a - \eta(c_1 + c_2))\|\nabla u\|_2^2 - \varepsilon b \|\nabla u\|_2^{2\gamma+2} + \varepsilon \|u\|_{p,\Gamma_1}^p.\end{aligned} \tag{2.31}$$

First, we choose η so small satisfying

$$a - \eta(c_1 + c_2) > 0.$$

For any fixed η, we choose ε so small that (2.27) remain valid and

$$m_0 - \varepsilon c(\eta) > 0, \ \ 1 - \varepsilon > 0.$$

Consequently, inequality (2.31) becomes

$$F'(t) \leq -c_3 E(t), \ \ \forall t > 0. \tag{2.32}$$

Using (2.27), we obtain

$$F'(t) \leq -c_3 E(t) \leq \frac{c_3}{\alpha_2} F(t), \ \ \forall t > 0. \tag{2.33}$$

A simple integration of (2.33), leads to

$$F(t) \leq c_4 e^{-kt}, \ \ \forall t > 0. \tag{2.34}$$

Again (2.27), gives

$$E(t) \leq Ke^{-kt}, \ \ \forall t > 0. \tag{2.35}$$

□

2.5 Blows up

In this section, we state and prove the finite time blow up of solutions to problem (2.1) with $E(0) < 0$.

Theorem 2.4. *Let $(A_1) - (A_2)$ and $E(0) < 0$ holds. Then, the solution of problem (2.1) blows up in finite time T^* and*

$$T^* \leq \frac{1-\sigma}{\omega\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Proof. Set

$$H(t) = -E(t), \tag{2.36}$$

then (2.9) gives

$$H'(t) = -E'(t) \geq m_0 \left(\|u_t\|_{m,\Gamma_1}^m + \|z(1,t)\|_{m,\Gamma_1}^m\right) \geq 0, \tag{2.37}$$

and $H(t)$ is an increasing function. From (2.7) and (2.36), we see that

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{2.38}$$

Next, we define

$$\Psi(t) = H(t)^{1-\sigma} + \varepsilon \int_\Omega u_t u dx + \frac{\varepsilon}{2}\|u\|_2^2, \tag{2.39}$$

where ε is a positive constants to be specified later and

$$0 < \sigma \leq \frac{p-m}{p(m-1)}. \tag{2.40}$$

Taking a derivative of $\Psi(t)$ and using (2.6), we have

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H'(t)H(t)^{-\sigma} + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega u u_{tt}dx + \varepsilon\int_\Omega u u_t dx \\ &= (1-\sigma)H'(t)H(t)^{-\sigma} + \varepsilon\|u_t\|_2^2 - \varepsilon a\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^{2\gamma+2} + \varepsilon\|u\|_{p,\Gamma_1}^p \\ &\quad -\varepsilon\mu_1\int_{\Gamma_1}|u_t|^{m-2}u_t u d\Gamma - \varepsilon\mu_2\int_{\Gamma_1}|z(1,t)|^{m-2}z(1,t)u d\Gamma.\end{aligned} \tag{2.41}$$

Applying Young's inequality for $\eta > 0$, we have

$$\begin{aligned}\mu_1\int_{\Gamma_1}|u_t|^{m-2}u_t u d\Gamma &\le \frac{\mu_1^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{m}\eta^{-\frac{m}{m-1}}\|u_t\|_{m,\Gamma_1}^m \\ &\le \frac{\mu_1^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}H'(t).\end{aligned} \tag{2.42}$$

Similarly, we have

$$\mu_2\int_\Omega |z(1,t)|^{m-2}z(1,t)u d\Gamma \le \frac{|\mu_2|^m\eta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}H'(t). \tag{2.43}$$

A substitution of (2.42)-(2.43) into (2.41), we have

$$\begin{aligned}\Psi'(t) &\ge \left\{(1-\sigma)H(t)^{-\sigma} - \varepsilon\frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}\right\}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon a\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^{2\gamma+2} \\ &\quad +\varepsilon\|u\|_{p,\Gamma_1}^p - \frac{(\mu_1^m + |\mu_2|^m)\eta^m}{m}\|u\|_{m,\Gamma_1}^m.\end{aligned} \tag{2.44}$$

Using (2.7) and (2.36), for a constant $\mu > 0$, we see that

$$\begin{aligned}\Psi'(t) &\ge \left\{(1-\sigma)H(t)^{-\sigma} - \varepsilon\frac{m-1}{mm_0}\eta^{-\frac{m}{m-1}}\right\}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2 \\ &\quad +\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} + \varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p - \frac{(\mu_1^m+|\mu_2|^m)\eta^m}{m}\|u\|_{m,\Gamma_1}^m \\ &\quad +\frac{\mu\xi}{m}\int_0^1\|z(\rho,t)\|_{m,\Gamma_1}^m d\rho + \mu\varepsilon H(t).\end{aligned} \tag{2.45}$$

Therefore, by taking $\eta = (kH(t)^{-\sigma})^{-\frac{m-1}{m}}$ where $k > 0$ to be specified later, we see that

$$\begin{aligned}\Psi'(t) \;\geq&\; \left\{(1-\sigma) - \varepsilon k\frac{(m-1)}{mm_0}\right\} H(t)^{-\sigma}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2 \\ &+\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} + \varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p + \frac{\mu\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho \\ &-\frac{(\mu_1^m + |\mu_2|^m)}{m}k^{1-m}H(t)^{\sigma(m-1)}\|u\|_{m,\Gamma_1}^m + \mu\varepsilon H(t).\end{aligned} \tag{2.46}$$

Exploiting (2.38), we have

$$H(t)^{\sigma(m-1)}\|u\|_{m,\Gamma_1}^m \leq C_p^m H(t)^{\sigma(m-1)}\|u\|_{p,\Gamma_1}^m \leq \frac{C_p^m}{p^\sigma}\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m}. \tag{2.47}$$

Combining (2.46) and (2.47), we get

$$\begin{aligned}\Psi'(t) \;\geq&\; \left\{(1-\sigma) - \varepsilon k\frac{(m-1)}{mm_0}\right\} H(t)^{-\sigma}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{\mu}{2}-1\right)\|\nabla u\|_2^2 \\ &+\varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} + \varepsilon\left(1-\frac{\mu}{p}\right)\|u\|_{p,\Gamma_1}^p + \frac{\mu\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho \\ &-\varepsilon\frac{(\mu_1^m + |\mu_2|^m)}{m}\frac{C_p^m k^{1-m}}{p^\sigma}\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m} + \mu\varepsilon H(t).\end{aligned} \tag{2.48}$$

Applying Lemma 2.3 for $s = \sigma p(m-1) + m < p$, we get

$$\|u\|_{p,\Gamma_1}^{\sigma p(m-1)+m} \leq C_*\left(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p\right). \tag{2.49}$$

Combining (2.49) with (2.48), we obtain

$$\begin{aligned}\Psi'(t) \;\geq&\; \left\{(1-\sigma) - \varepsilon\frac{(m-1)k}{mm_0}\right\} H(t)^{-\sigma}H'(t) + \varepsilon\left(1+\frac{\mu}{2}\right)\|u_t\|_2^2 \\ &+\varepsilon\left(a\left(\frac{\mu}{2}-1\right) - c_\sigma k^{1-m}\right)\|\nabla u\|_2^2 + \varepsilon b\left(\frac{\mu}{2\gamma+2}-1\right)\|\nabla u\|_2^{2\gamma+2} \\ &+\varepsilon\left(\left(1-\frac{\mu}{p}\right) - c_\sigma k^{1-m}\right)\|u\|_{p,\Gamma_1}^p + \frac{\mu\xi}{m}\int_0^1 \|z(\rho,t)\|_{m,\Gamma_1}^m d\rho + \mu\varepsilon H(t),\end{aligned} \tag{2.50}$$

where $c_\sigma = \dfrac{C_*(\mu_1^m + |\mu_2|^m)}{m}\dfrac{C_p^m}{p^\sigma}$.

At this point, we choose $2\gamma+2<\mu<p$ such that

$$\frac{\mu}{2}-1>0,\ \ \frac{\mu}{2\gamma+2}-1>0,\ \ 1-\frac{\mu}{p}>0.$$

When μ is fixed, we choose k large enough such that

$$a\left(\frac{\mu}{2}-1\right)-c_\sigma k^{1-m}>0,\ \ \left(1-\frac{\mu}{p}\right)-c_\sigma k^{1-m}>0.$$

Once k and μ are fixed, we select $\varepsilon>0$ small enough so that

$$(1-\sigma)-\varepsilon k\frac{(m-1)}{mm_0}>0,\ \ \Psi(0)=H(0)^{1-\sigma}+\varepsilon\int_\Omega u_1u_0dx+\frac{\varepsilon}{2}\|u_0\|_2^2>0.$$

Then inequality (2.50) becomes

$$\Psi'(t)\geq K\left(\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2\gamma+2}+\|u\|_{p,\Gamma_1}^p+H(t)\right), \tag{2.51}$$

where K is a positive constant.

On the other hand, we will estimate $\Psi^{\frac{1}{1-\sigma}}(t)$. Applying Hölder and Youngs inequalities, we have

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\sigma}}\leq C\|u\|_p^{\frac{1}{1-\sigma}}\|u_t\|_2^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_p^{\frac{\mu}{1-\sigma}}+\|u_t\|_2^{\frac{\theta}{1-\sigma}}\right), \tag{2.52}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. Take $\theta=2(1-\sigma)$ which gives $\frac{\mu}{1-\sigma}=\frac{2}{1-2\sigma}$. Then, (2.52) becomes

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_p^{\frac{2}{1-2\sigma}}+\|u_t\|_2^2\right), \tag{2.53}$$

It follows from (2.25) and (2.38), we have

$$\|u\|_p^{\frac{2}{1-2\sigma}}\leq c_p^{\frac{2}{1-2\sigma}}\|\nabla u\|_2^{\frac{2}{1-2\sigma}}\leq c_p^{\frac{2}{1-2\sigma}}(CE(0))^{\frac{1}{1-2\sigma}}\leq c_p^{\frac{2}{1-2\sigma}}(CE(0))^{\frac{2}{1-2\sigma}}\frac{H(t)}{H(0)}. \tag{2.54}$$

Similar to (2.54), we have

$$\|u\|_2^{\frac{2}{1-\sigma}}\leq c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}}\leq c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}}\frac{H(t)}{H(0)}\leq c_2^{\frac{2}{1-\sigma}}(CE(0))^{\frac{1}{1-\sigma}}\frac{\|u\|_{p,\Gamma_1}^p}{pH(0)}. \tag{2.55}$$

Combining (2.54)-(2.55) and (2.39), we get

$$\Psi^{\frac{1}{1-\sigma}}(t) \leq \widetilde{K}\left(\|u_t\|_2^2 + \|u\|_{p,\Gamma_1}^p + H(t)\right), \tag{2.56}$$

where $\widetilde{K}$ is a positive constant.

It follows from (2.51) and (2.56), we find that

$$\Psi'(t) \geq \omega \Psi^{\frac{1}{1-\sigma}}(t), \;\; \forall t > 0, \tag{2.57}$$

where κ is a positive constant.

A simple integration of (2.57) over $(0, t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\omega\sigma t}{1-\sigma}}.$$

Consequently, the solution of problem (2.1) blows up in finite time T^*. □

CHAPTER 3

GENERAL DECAY AND BLOW UP OF SOLUTIONS FOR THE KIRCHHOFF PLATE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS, DELAY AND SOURCE TERMS

This chapter is devoted to prove the global existence, decay, and the blow up of solutions for a viscoelastic Kirchhoff plate equation with dynamic boundary conditions, delay and source terms acting on the boundary. The results of this chapter have been the subject of the international publication: H. Kamache, N. Boumaza, B. Gheraibia, General decay and blow up of solutions for the Kirchhoff plate equation with dynamic boundary conditions, delay and source terms. Zeitschrift für angewandte Mathematik und Physik 73 (2) (2022): 76.

3.1 Formulation of the problem

we investigate the following viscoelastic Kirchhoff plate equation with dynamic boundary conditions, delay and source terms

$$\begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta^2 u(s)ds = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = \dfrac{\partial u}{\partial v}(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ u_{tt}(x,t) = \dfrac{\partial \Delta u}{\partial v}(x,t) - M(\|\nabla u\|^2)\dfrac{\partial u}{\partial v}(x,t) & \\ +\sigma(t)\displaystyle\int_0^t g(t-s)\dfrac{\partial \Delta u}{\partial v}ds - \mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau) + |u|^{p-2}u, & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x-\rho\tau), & x \in \Gamma_1,\ t \in [0,\tau), \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\text{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\frac{\partial u}{\partial \nu}$ denotes the unit outer normal derivative, $p > 2$, μ_1, μ_2 are positive functions, $\tau > 0$ represents the time delay, M, σ and g are functions satisfy some conditions to be specified later, and u_0, u_1, f_0 are given functions belonging to suitable spaces.

3.2 Assumptions preliminary lemmas

In this section, we will give some notations, sufficient conditions, assumptions, and some preliminary lemmas needed in the proof of our main results.

Throughout this chapter, we denote

$$\|\Delta u\|_2 := \|u\|_{H^2(\Omega)},$$

and

$$H^2_{\Gamma_0}(\Omega) = \{u \in H^2(\Omega) | u_{|\Gamma_0} = 0\}.$$

Let c_ϱ be the smallest positive number satisfying

$$\|\nabla u\|_2 \leq c_\varrho \|\Delta u\|_2, \quad \forall\, u \in\ H^2(\Omega). \tag{3.2}$$

Next, assume the following assumptions:
(A_1) : $g, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$g(s) \geq 0,\ l_0 = \int_0^\infty g(s)ds < \infty,\ \sigma(t) > 0,\ 1 - 2\sigma(t)\int_0^t g(s)ds \geq l > 0, \tag{3.3}$$

with $l = 1 - l_0$.
(A_2) There exists a positive differentiable functions ξ satisfying

$$g'(t) \leq 0, \quad \sigma'(t) \leq 0, \quad g'(t) \leq -\xi(t)g(t),\ \text{for}\, t \geq 0,\ \lim_{t\to\infty} \frac{\sigma'(t)}{\xi(t)\sigma(t)} = 0. \tag{3.4}$$

(A_3) : M is a C^1-function and satisfying

$$M(s) \geq -m_0, \quad \text{and} \quad M(s)s \geq \hat{M}(s),\ s > 0, \tag{3.5}$$

where $\hat{M}(s) = \displaystyle\int_0^s M(\tau)d\tau$.
(A_4): The constant p satisfies

$$p \geq 2, \quad \text{if}\, n = 1, 2, \quad 2 < p \leq \frac{n+2}{n-2}, \ \text{if}\, n \geq 3. \tag{3.6}$$

(A_5): The constants μ_1 and μ_2 satisfy

$$\mu_2 < \mu_1. \tag{3.7}$$

Assume further that g satisfies

$$\sigma(t)\int_0^\infty g(s)ds < \frac{(N - 2 - 2M_0)}{(N - 2 - (1/2\eta))}. \tag{3.8}$$

By using the direct calculations, we have

$$\begin{aligned} &\sigma(t)\int_0^t g(t-s)\int_\Omega u(s)dsu_t(t)dx \\ &= -\frac{d}{dt}\left[\frac{\sigma(t)}{2}(g\circ u)(t) - \frac{\sigma(t)}{2}\|u(t)\|_2^2\int_0^t g(s)ds\right] - \frac{\sigma(t)}{2}g(t)\|u(t)\|_2^2 \\ &+\frac{\sigma(t)}{2}(g'\circ u)(t) + \frac{\sigma'(t)}{2}(g\circ u)(t) - \frac{\sigma'(t)}{2}\|u(t)\|_2^2\int_0^t g(s)ds. \end{aligned} \tag{3.9}$$

where

$$(g \circ u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|_2^2 ds.$$

Now, we introduce, as in [50], the new variable

$$z(x, \rho, t) = u_t(x, t - \rho\tau), \ \ x \in \Omega, \ \ \rho \in (0, 1), \ \ t > 0 \tag{3.10}$$

which gives us

$$\tau z_t(x, \rho, \tau) + z_\rho(x, \rho, \tau) = 0, \ \ \text{in}\, \Omega \times (0, 1) \times (0; \infty) \tag{3.11}$$

Then, problem (3.1) is equivalent to

$$\begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \sigma(t) \displaystyle\int_0^t g(t-s)\Delta^2 u ds = 0, x \in \Omega, t > 0, \\ u(x, t) = \dfrac{\partial u}{\partial v}(x, t) = 0, \quad x \in \Gamma_0, t > 0, \\ u_{tt}(x, t) = \dfrac{\partial \Delta u}{\partial v}(x, t) - M(\|\nabla u\|_2^2)\dfrac{\partial u}{\partial v}(x, t) + \sigma(t) \displaystyle\int_0^t g(t-s)\dfrac{\partial \Delta u}{\partial v} ds \\ -\mu_1 u_t(x, t) - \mu_2 z_\rho(x, \rho, t) + |u|^{p-2}u, \ \ x \in \Gamma_1, t > 0, \\ \tau z_t(x, \rho, \tau) + z_\rho(x, \rho, \tau) = 0, \ \ in \Gamma_1 \times (0; 1) \times (0; \infty), \\ z(0, t) = u_t(t), \quad x \in \Gamma_1, t > 0, \\ z(x, \rho, 0) = f_0(x - \rho\tau) \ \ x \in \Gamma_1, t \in [0; \tau). \end{cases} \tag{3.12}$$

We now state, without a proof, local existence result, which can be established by using the Fadeo-Galerkin method (see [5, 18, 29]).

Theorem 3.1. *Suppose that* $(A_1) - (A_5)$ *hold. Then, for any* $(u_0, u_1) \in H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *and* $f_0 \in L^2(\Omega \times (0, 1))$*, there exists a unique weak solution of problem* (3.1) *satisfying*

$$\begin{aligned} & u \in L^\infty\left((0, T); H^2_{\Gamma_0}(\Omega)\right), \\ & u_t \in L^\infty\left((0, T); L^2(\Gamma_1)\right), \\ & u_{tt} \in L^\infty\left((0, T); L^2(\Omega)\right) \bigcap L^\infty\left((0, T); L^2(\Gamma_1)\right). \end{aligned} \tag{3.13}$$

Now, we define the energy associated with problem (3.12) as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{1}{2}\left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\sigma(t)(g \circ \Delta u) + \frac{1}{2}\hat{M}\left(\|\nabla u\|_2^2\right) \\ &+ \frac{\xi}{2}\int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho - \frac{1}{p}\|u\|_{p,\Gamma_1}^p, \end{aligned} \tag{3.14}$$

where ξ be a positive constant satisfying

$$\tau\mu_2 \leq \xi \leq \tau(\mu_1 - \mu_2). \tag{3.15}$$

Lemma 3.1. *Let u be a solution of problem* (3.1). *Then,*

$$E'(t) \leq -c_0\left(\|u_t\|_{2,\Gamma_1}^2 + \|z(1,t)\|_{2,\Gamma_1}^2\right) + \frac{\sigma(t)}{2}\left(g' \circ \Delta u\right)(t) - \frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2. \tag{3.16}$$

Proof. Multiplying the first equation in (3.12) by u_t, integrating over Ω, and using (3.9), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{1}{2}\left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\sigma(t)\left(g \circ \Delta u\right)(t) \\ &+ \frac{1}{2}\hat{M}\left(\|\nabla u\|_2^2\right) - \frac{1}{p}\|u\|_{p,\Gamma_1}^p\Big] \\ =& -\frac{\sigma(t)}{2}g(t)\|\Delta u\|_2^2 + \frac{\sigma(t)}{2}\left(g' \circ \Delta u\right)(t) + \frac{\sigma'(t)}{2}\left(g \circ \Delta u\right)(t) \\ &- \frac{\sigma'(t)}{2}\Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 - \mu_1\|u_t\|_{2,\Gamma_1}^2 - \mu_2\int_{\Gamma_1} z(1,t)u_t dx. \end{aligned} \tag{3.17}$$

Multiplying the second equation in (3.12) by ξz and integrating over $\Gamma_1 \times (0,1)$, we obtain

$$\begin{aligned} \frac{\xi}{2\tau}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 |z(\rho,t)|^2 d\rho dx &= -\frac{\xi}{2\tau}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial\rho}|z(\rho,t)|^2 d\rho dx \\ &= \frac{\xi}{2\tau}\left(\|u_t\|_{2,\Gamma_1}^2 - \|z(1,t)\|_{2,\Gamma_1}^2\right). \end{aligned} \tag{3.18}$$

Using Young's inequality, we have

$$-\mu_2\int_{\Gamma_1} z(1,t)u_t dx \leq \frac{\mu_2}{2}\|z(1,t)\|_{2,\Gamma_1}^2 + \frac{\mu_2}{2}\|u_t\|_{2,\Gamma_1}^2. \tag{3.19}$$

Combining (3.17), (3.18), and (3.19), we obtain

$$E'(t) \leq -c_0 \left(\|u_t\|_{2,\Gamma_1}^2 + \|z(1,t)\|_{2,\Gamma_1}^2 \right) + \frac{\sigma(t)}{2} \left(g' \circ \Delta u \right)(t) - \frac{\sigma'(t)}{2} \left(\int_0^t g(s)ds \right) \|\Delta u\|_2^2, \tag{3.20}$$

where $c_0 = \min \left\{ \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}, \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right\}$, which is positive by (3.15). □

Lemma 3.2. *([44]). Suppose that $p \leq 2\frac{n-1}{n-2}$ holds. Then, there exists a positive constant $C > 1$ depending only on Ω such that*

$$\|u\|_p^s \leq C \left(\|\nabla u\|_2^2 + \|u\|_p^p \right),$$

for any $u \in H_0^1(\Omega)$, $2 \leq s \leq p$.

3.3 Global Existence

In this section, we will prove that the solutions established in Theorem 3.1 are global in time. For this purpose, we define the functionals:

$$\begin{aligned} I(t) &= \hat{M}\left(\|\nabla u\|_2^2\right) + \left(1 - \sigma(t) \int_0^t g(s)ds\right) \|\Delta u\|_2^2 + \sigma(t)(g \circ \Delta u) \\ &+ \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho - \|u\|_{p,\Gamma_1}^p, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} J(t) &= \frac{1}{2}\left(1 - \sigma(t) \int_0^t g(s)ds\right) \|\Delta u\|_2^2 + \frac{1}{2}\sigma(t)(g \circ \Delta u) + \frac{1}{2}\hat{M}\left(\|\nabla u\|_2^2\right) \\ &+ \frac{\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho - \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \end{aligned} \tag{3.22}$$

Then, it is obvious that

$$E(t) = J(t) + \frac{1}{2}\left[\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right]. \tag{3.23}$$

Lemma 3.3. *Assume that $(A_1) - (A_5)$ hold, and for any $(u_0, u_1) \in H_{\Gamma_1}^2(\Omega) \times L^2(\Omega)$, such that*

$$I(0) > 0 \quad \text{and} \quad \alpha = \frac{c_*^p c_\varrho^p}{l} \left[\frac{2p}{l(p-2)} E(0) \right]^{\frac{p-2}{2}} < 1, \tag{3.24}$$

then

$$I(t) > 0, \quad \textit{for all } t > 0. \tag{3.25}$$

Proof. Since $I(0) > 0$, then by continuity of u, there exist a time $t_1 > 0$ such that

$$I(t) \geq 0, \quad \forall t \in (0, t_1). \tag{3.26}$$

We assume that

$$\{I(t_0) = 0 \,\text{and}\, I(t) > 0, \text{ for all}\, 0 \leq t < t_0\}. \tag{3.27}$$

This, together with (3.21), (3.22), (3.26), and (A_1), give that

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\Big[\hat{M}\left(\|\nabla u\|_2^2\right) + \left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \sigma(t)\left(g \circ \Delta u\right)(t) \\ &\quad + \xi \int_0^1 \|z(\rho, t)\|_2^2 d\rho\Big] + \frac{1}{p} I(t) \\ &\geq \frac{p-2}{2p}\Big[\hat{M}\left(\|\nabla u\|_2^2\right) + \left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \sigma(t)\left(g \circ \Delta u\right)(t) \\ &\quad + \xi \int_0^1 \|z(\rho, t)\|_2^2 d\rho\Big]. \end{aligned} \tag{3.28}$$

Using (A_1), (3.16), and (3.28), we obtain

$$\begin{aligned} l\|\Delta u\|_2^2 &\leq \left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \leq \frac{2p}{p-2} J(t) \\ &\leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \quad \forall\, t \in [0, t_0]. \end{aligned} \tag{3.29}$$

Exploiting Lemma 3.1, (3.29) and (3.24), we obtain

$$\begin{aligned} \|u(t_0)\|_{p,\Gamma_1}^p &\leq c_*^p c_\varrho^p \|\Delta u(t_0)\|_2^p \leq \frac{c_*^p c_\varrho^p}{l}\left(\frac{2p}{l(p-2)} E(0)\right)^{\frac{p-2}{2}} l\|\Delta u(t_0)\|_2^2 \\ &= \alpha l \|\Delta u(t_0)\|_2^2 < \left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta u(t_0)\|_2^2. \end{aligned} \tag{3.30}$$

Therefore,

$$I(t_0) > 0.$$

which contradicts to (3.27). Thus, $I(t) > 0$ on $[0, T]$. □

Theorem 3.2. *Assume that the conditions of Lemma 3.3 hold, then the solution of* (3.1) *is global and bounded.*

Proof. It suffices to show that

$$\|u_t\|_{2,\Gamma_1}^2 + \|u_t\|_2^2 + \|\Delta u\|_2^2$$

is bounded independently of t. Using (3.16), (3.23) and (3.29), we have

$$\begin{aligned} E(0) &\geq E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 \\ &\geq \frac{p-2}{2p}\left(l\|\Delta u\|_2^2\right) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2. \end{aligned} \tag{3.31}$$

Therefore,

$$\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2 \leq CE(0), \tag{3.32}$$

where C is a positive constant. This infers that the solution of (3.1) is bounded and global in time. □

3.4 General decay

In this section, we prove the energy decay result by constructing a suitable Lyapunov functionals.

Theorem 3.3. *Let $(u_0, u_1) \in H_{\Gamma_0}^2 \times L^2(\Omega)$ be given. Assume that $(A_1) - (A_5)$ hold. Then, for each $t > 0$, there exist two positive constants K and k such that, for any solution of problem* (3.1) *, the energy satisfies*

$$E(t) \leq Ke^{-k\int_{t_0}^{t}\xi(s)\sigma(s)ds}. \tag{3.33}$$

For this goal, we define the following functionals

$$F(t) := E(t) + \varepsilon_1\sigma(t)\varphi(t) + \varepsilon_2\sigma(t)\psi(t), \tag{3.34}$$

where ε_1 and ε_2 are some positive constants to be specified later and

$$\varphi(t) = \int_\Omega u u_t dx + \int_{\Gamma_1} u u_t dx. \tag{3.35}$$

$$\psi(t) = -\left(\int_\Omega u_t dx + \int_{\Gamma_1} u u_t dx\right)\int_0^t g(t-s)(u(t)-u(s))ds. \tag{3.36}$$

In order to show our stability result, we need the following lemmas:

Lemma 3.4. *Let (u, z) be a solution of problem (3.12). Then there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \le F(t) \le \alpha_2 E(t), \;\; t > 0. \tag{3.37}$$

Proof. By Hölder's, Young's, Sobolev-Poincare inequalities, (3.2), and (2.3), we obtain

$$\begin{aligned}
|F(t)-E(t)| &\le (\varepsilon_1+\varepsilon_2)\frac{|\sigma(t)|}{2}\|u_t\|_2^2 + \le (\varepsilon_1+\varepsilon_2)\frac{|\sigma(t)|}{2}\|u_t\|_{2,\Gamma_1}^2 + \varepsilon_1\frac{|\sigma(t)|}{2}\left(\|u\|_2^2+\|u\|_{2,\Gamma_1}^2\right)\\
&\quad +\varepsilon_2\frac{|\sigma(t)|}{2}(1-l)\int_0^t g(t-s)\left(\|u(t)-u(s)\|_2^2+\|u(t)-u(s)\|_{2,\Gamma_1}^2\right)ds\\
&\le (\varepsilon_1+\varepsilon_2)\frac{\sigma(0)}{2}\|u_t\|_2^2 + (\varepsilon_1+\varepsilon_2)\frac{\sigma(0)}{2}\|u_t\|_{2,\Gamma_1}^2 + \varepsilon_1\frac{\sigma(0)}{2}\left(c_p^2+c_*^2\right)c_\varrho^2\|\Delta u\|_2^2\\
&\quad +\varepsilon_2\frac{|\sigma(t)|}{2}(1-l)\left(c_p^2+c_*^2\right)c_\varrho^2\,(g\circ\Delta u)(t))\\
&\le c(\varepsilon_1+\varepsilon_2)E(t).
\end{aligned} \tag{3.38}$$

If we take ε_1 and ε_2 to be sufficiently small, then (3.37) follows from (3.38). This completes the proof. □

Lemma 3.5. *The functional $\varphi(t)$ defined in (3.35) satisfies*

$$\begin{aligned}
\varphi'(t) &\le \|u_t\|_2^2 + (1+\varepsilon)\|u_t\|_{2,\Gamma_1}^2 - c_\varepsilon\|\Delta u\|_2^2 + \frac{1}{4}\sigma(t)(g\circ\Delta u)(t)\\
&\quad +\varepsilon\|z(1,t)\|_{2,\Gamma_1}^2 + \|u\|_{p,\Gamma_1}^p.
\end{aligned} \tag{3.39}$$

Proof. Differentiate (3.35) with respect to t and using first equation of (3.12), we get

$$\begin{aligned}\varphi'(t) &= \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \int_\Omega u u_{tt} dx + \int_{\Gamma_1} u u_{tt} dx \\ &= \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 - \|\Delta u\|_2^2 - M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p \\ &\quad + \sigma(t)\int_\Omega\int_0^t g(t-s)\Delta u(s)ds\Delta u(t)dx - \mu_1\int_{\Gamma_1} u_t u dx - \mu_2\int_{\Gamma_1} z(1,t)u dx.\end{aligned} \tag{3.40}$$

Applying Hölder's inequality, Young's inequality and (3.2), we obtain

$$\begin{aligned}I_1 &= \int_\Omega \Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx + \Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\ &\le 2\Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + \frac{1}{4}(g\circ\Delta u)(t),\end{aligned} \tag{3.41}$$

and

$$I_2 = \mu_1\int_{\Gamma_1} u_t u dx \le \varepsilon\|u_t\|_{2,\Gamma_1}^2 + \frac{\mu_1^2}{4\varepsilon}\|u\|_{2,\Gamma_1}^2 \le \varepsilon\|u_t\|_{2,\Gamma_1}^2 + \frac{\mu_1^2 c_*^2 c_\varrho^2}{4\varepsilon}\|\Delta u\|_2^2. \tag{3.42}$$

Similarly

$$I_3 = \mu_2\int_{\Gamma_1} z(1,t)u dx \le \varepsilon\|z(1,t)\|_{2,\Gamma_1}^2 + \frac{\mu_2^2}{4\varepsilon}\|u\|_{2,\Gamma_1}^2 \le \varepsilon\|z(1,t)\|_{2,\Gamma_1}^2 + \frac{\mu_2^2 c_*^2 c_\varrho^2}{4\varepsilon}\|\Delta u\|_2^2. \tag{3.43}$$

A substitution of (3.41)-(3.43) into (3.40), we obtain

$$\begin{aligned}\varphi'(t) &\le \|u_t\|_2^2 + (1+\varepsilon)\|u_t\|_{2,\Gamma_1}^2 - c_\varepsilon\|\Delta u\|_2^2 + \frac{1}{4}\sigma(t)(g\circ\Delta u)(t) \\ &\quad + \varepsilon\|z(1,t)\|_{2,\Gamma_1}^2 + \|u\|_{p,\Gamma_1}^p.\end{aligned} \tag{3.44}$$

where $c_\varepsilon = \{l - m_0 c_\varrho^2 - ((\mu_1^2+\mu_2^2)c_*^2 c_\varrho^2/4\varepsilon)\} > 0.$ □

Lemma 3.6. *The functional $\psi(t)$ defined in* (3.36) *satisfies*

$$
\begin{aligned}
\psi'(t) \ &\leq -\left\{\int_0^t g(s)ds - \eta\right\}\|u_t\|_2^2 + \eta\left\{1 + 2(1-l)^2\sigma(t) + M_0 c_\varrho^2\right\}\|\Delta u\|_2^2 + \eta\|u\|_{p,\Gamma_1}^p \\
&\quad +(1-l)\left\{\frac{1}{4\eta}(1 + M_0 c_\varrho^2 + (\mu_1^2+\mu_2^2)c_*^2 c_\varrho^2 + c_\eta \widetilde{c}) + \left(2\eta + \frac{1}{4\eta}\right)\sigma(t)\right\}(g\circ\Delta u)(t) \\
&\quad -\left\{\int_0^t g(s)ds - 2\eta\right\}\|u_t\|_{2,\Gamma_1}^2 + \eta\|z(1,t)\|_{2,\Gamma_1}^2 - \frac{g(0)\left(c_p^2 + c_*^2\right)c_\varrho^2}{4\eta}(g'\circ\Delta u)(t).
\end{aligned}
\tag{3.45}
$$

Proof. Differentiating (3.36) with respect to t and using equation (3.12), we get

$$
\begin{aligned}
\psi'(t) \ &= -\left(\int_\Omega u_{tt} + \int_{\Gamma_1} u_{tt}\right)\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\quad -\left(\int_\Omega u_t + \int_{\Gamma_1} u_t\right)\int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&\quad -\left(\int_0^t g(s)ds\right)\left(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right) \\
&= J_1 + \cdots + J_7 - \left(\int_0^t g(s)ds\right)\left(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right).
\end{aligned}
\tag{3.46}
$$

By using Hölder's inequality, Young's inequality, and (A_1), we obtain

$$
J_1 = \int_\Omega \Delta u \int_0^t g(t-s)(\Delta u(t) - \Delta u(s))dsdx \leq \eta\|\Delta u\|_2^2 + \frac{(1-l)}{4\eta}(g\circ\Delta u)(t), \tag{3.47}
$$

and

$$
\begin{aligned}
J_2 \ &= -\sigma(t)\int_\Omega\left(\int_0^t g(t-s)\Delta u(s)ds\right)\left(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\right)dx \\
&\leq 2\eta(1-l)^2\sigma(t)\|\Delta u\|_2^2 + \left(2\eta + \frac{1}{4\eta}\right)(1-l)\sigma(t)(g\circ\Delta u)(t).
\end{aligned}
\tag{3.48}
$$

Now, since $\|\nabla u\|_2^2 \leq 2c_\varrho^2 E(0)/l$, for all $t \geq 0$, taking

$$
M_0 = \sup\{M(s),\, 0 < s \leq 2c_\varrho^2 E(0)/l\},
$$

we see that $M\left(\|\nabla u\|_2^2\right) \leq M_0 < \infty$. Then, we have

$$
\begin{aligned}
J_3 &= M\left(\|\nabla u\|_2^2\right)\int_\Omega \nabla u \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\leq M_0 c_\varrho^2 \eta \|\Delta u\|_2^2 + \frac{M_0(1-l)c_\varrho^2}{4\eta}\left(g\circ\Delta u\right)(t).
\end{aligned} \tag{3.49}
$$

By using Hölder's inequality, Young's inequality, and (A_1), we obtain

$$
\begin{aligned}
J_4 &= \int_{\Gamma_1} u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq \eta\|u_t\|_{2,\Gamma_1}^2 + \frac{(1-l)}{4\eta}\int_0^t g(t-s)\|u(t)-u(s)\|_{2,\Gamma_1}^2 ds \\
&\leq \eta\|u_t\|_{2,\Gamma_1}^2 + \frac{(1-l)c_*^2 c_\varrho^2}{4\eta}(g\circ\Delta u)(t).
\end{aligned} \tag{3.50}
$$

Similar to (3.50), we have

$$
\begin{aligned}
J_5 &= \int_{\Gamma_1} z(1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq \eta\|z(1,t)\|_{2,\Gamma_1}^2 + \frac{(1-l)c_*^2 c_\varrho^2}{4\eta}(g\circ\Delta u)(t).
\end{aligned} \tag{3.51}
$$

Applying Hölder's inequality, Young's inequality, (A_1), (2.3), and (3.2), we have

$$
\begin{aligned}
J_6 &= \int_{\Gamma_1} |u|^{p-2}u \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq \eta\|u\|_{p,\Gamma_1}^p + c_\eta \int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^p dx \\
&\leq \eta\|u\|_{p,\Gamma_1}^p + c_\eta(1-l)^{p-1}\int_0^t g(t-s)\|u(t)-u(s))\|_{p,\Gamma_1}^p ds \\
&\leq \eta\|u\|_{p,\Gamma_1}^p + c_\eta(1-l)^{p-1}c_*^p c_\varrho^p \int_0^t g(t-s)\|\Delta u(t)-\Delta u(s))\|_2^p ds \\
&\leq \eta\|u\|_{p,\Gamma_1}^p + c_\eta(1-l)\widetilde{c}\left(g\circ\Delta u\right)(t).
\end{aligned} \tag{3.52}
$$

Finally,

$$\begin{aligned} J_7 &= \left(\int_\Omega u_t + \int_{\Gamma_1} u_t\right)\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &\leq \eta\|u_t\|_2^2 + \eta\|u_t\|_{2,\Gamma_1}^2 - \frac{g(0)\left(c_p^2+c_*^2\right)c_\varrho^2}{4\eta}(g'\circ\Delta u)(t). \end{aligned} \tag{3.53}$$

A substitution of (3.47)-(3.53) into (3.46), we obtain

$$\begin{aligned} \psi'(t) \leq & -\left\{\int_0^t g(s)ds - \eta\right\}\|u_t\|_2^2 + \eta\left\{1+2(1-l)^2\sigma(t)+M_0c_\varrho^2\right\}\|\Delta u\|_2^2 + \eta\|u\|_{p,\Gamma_1}^p \\ & +(1-l)\left\{\frac{1}{4\eta}(1+M_0c_\varrho^2+(\mu_1^2+\mu_2^2)c_*^2c_\varrho^2+c_\eta\widetilde{c}) + \left(2\eta+\frac{1}{4\eta}\right)\sigma(t)\right\}(g\circ\Delta u)(t) \\ & -\left\{\int_0^t g(s)ds - 2\eta\right\}\|u_t\|_{2,\Gamma_1}^2 + \eta\|z(1,t)\|_{2,\Gamma_1}^2 - \frac{g(0)\left(c_p^2+c_*^2\right)c_\varrho^2}{4\eta}(g'\circ\Delta u)(t). \end{aligned} \tag{3.54}$$

□

Lemma 3.7. *The functional $F(t)$ defined in (3.34) satisfies*

$$F'(t) \leq -k_0\sigma(t)E(t) + k_1\sigma(t)(g\circ\Delta u)(t), \quad t > t_0, \tag{3.55}$$

where k_0 and k_1 are some positive constants.

Proof. First, since the function g is a positive and continuous function, for any $t_0 > 0$ we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0, \quad \text{for all } t \geq t_0.$$

Differentiating (3.34) and using Lemmas 3.5, 3.6, we get

$$
\begin{aligned}
F'(t) &= E'(t) + \varepsilon_1\sigma'(t)\varphi(t) + \varepsilon_1\sigma(t)\varphi'(t) + \varepsilon_2\sigma'(t)\psi(t) + \varepsilon_2\sigma(t)\psi'(t) \\
&\leq -\sigma(t)\left\{\varepsilon_2(g_0-\eta)-\varepsilon_1\right\}\|u_t\|_2^2 - \sigma(t)\left\{\frac{c_0}{\sigma(0)} - \varepsilon_1(1+\varepsilon) + \varepsilon_2(g_0-2\eta)\right\}\|u_t\|_{2,\Gamma_1}^2 \\
&\quad -\sigma(t)\left\{\varepsilon_1 c_\varepsilon - \varepsilon_2\eta\left(1+2(1-l)^2\sigma(t)+M_0c_\varrho^2\right) + \frac{\sigma'(t)}{2\sigma(t)}\left(\int_0^t g(s)ds\right)\right\}\|\Delta u\|_2^2 \\
&\quad +\sigma(t)\left\{\varepsilon_1\frac{\sigma(t)}{4} + \varepsilon_2 c_\eta + \varepsilon_2\left(2\eta+\frac{1}{4\eta}\right)(1-l)\sigma(t)\right\}(g\circ\Delta u)(t) \\
&\quad +\sigma(t)\left\{\frac{1}{2} - \varepsilon_2\frac{g(0)\left(c_p^2+c_*^2\right)c_\varrho^2}{4\eta}\right\}(g'\circ\Delta u)(t) + \sigma(t)\left\{\varepsilon_1+\varepsilon_2\eta\right\}\|u\|_{p,\Gamma_1}^p \\
&\quad -\sigma(t)\left\{\frac{c_0}{\sigma(0)} - \varepsilon_1\varepsilon - \varepsilon_2\eta\right\}\|z(1,t)\|_{2,\Gamma_1}^2 + \varepsilon_1\sigma'(t)\left(\int_\Omega uu_t dx + \int_{\Gamma_1} uu_t dx\right) \\
&\quad +\varepsilon_2\sigma'(t)\left(\int_\Omega u_t + \int_{\Gamma_1} u_t\right)\int_0^t g(t-s)(u(t)-u(s))dsdx.
\end{aligned}
\tag{3.56}
$$

On the other hand, we have

$$
\sigma'(t)\int_\Omega uu_t dx + \sigma'(t)\int_{\Gamma_1} uu_t dx \leq -\frac{\sigma'(t)}{2}\|u_t\|_2^2 - \frac{\sigma'(t)}{2}\|u_t\|_{2,\Gamma_1}^2 - \sigma'(t)(c_p^2+c_*^2)c_\varrho^2\|\Delta u\|_2^2,
$$

and

$$
\begin{aligned}
&\sigma'(t)\left(\int_\Omega u_t + \int_{\Gamma_1} u_t\right)\int_0^t g(t-s)(u(t)-u(s))dsdx. \\
&\leq -\frac{\sigma'(t)}{2}\left(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right) - \frac{\sigma'(t)(c_p^2+c_*^2)c_\varrho^2}{2}\left(\int_0^t g(s)ds\right)(g\circ\Delta u)(t).
\end{aligned}
$$

Consequently, inequality (3.56) becomes

$$
\begin{aligned}
F'(t) \ \leq & -\sigma(t)\left\{\varepsilon_2(g_0-\eta)-\varepsilon_1+\frac{\sigma'(t)}{\sigma(t)}\right\}\|u_t\|_2^2 \\
& -\sigma(t)\left\{\frac{c_0}{\sigma(0)}-\varepsilon_1(1+\varepsilon)+\varepsilon_2(g_0-2\eta)+\frac{\sigma'(t)}{2\sigma(t)}\right\}\|u_t\|_{2,\Gamma_1}^2 \\
& -\sigma(t)\Big\{\varepsilon_1 c_\varepsilon-\varepsilon_2\eta\left(1+2(1-l)^2\sigma(t)+M_0c_\varrho^2\right) \\
& +\frac{\sigma'(t)}{2\sigma(t)}\Big(\int_0^t g(s)ds\Big)+\frac{(c_p^2+c_*^2)c_\varrho^2}{2}\frac{\sigma'(t)}{\sigma(t)}\Big\}\|\Delta u\|_2^2 \\
& +\sigma(t)\Big\{\varepsilon_1\frac{\sigma(t)}{4}+\varepsilon_2c_\eta+\varepsilon_2\left(2\eta+\frac{1}{4\eta}\right)(1-l)\sigma(t) \\
& -\frac{(c_p^2+c_*^2)c_\varrho^2}{2}\frac{\sigma'(t)}{\sigma(t)}\int_0^t g(s)ds\Big\}(g\circ\Delta u)(t) \\
& +\sigma(t)\left\{\frac{1}{2}-\varepsilon_2\frac{g(0)(c_p^2+c_*^2)c_\varrho^2}{4\eta}\right\}(g'\circ\Delta u)(t) \\
& +\sigma(t)\left\{\varepsilon_1+\varepsilon_2\eta\right\}\|u\|_{p,\Gamma_1}^p-\sigma(t)\left\{\frac{c_0}{\sigma(0)}-\varepsilon_1\varepsilon-\varepsilon_2\eta\right\}\|z(1,t)\|_{2,\Gamma_1}^2.
\end{aligned}
\tag{3.57}
$$

First, we choose η so small such that

$$
g_0-\eta>\frac{1}{2}g_0 \quad \text{and} \quad \frac{\eta}{c_\varepsilon}(1+2(1-l)^2+M_0c_\varrho^2)<\frac{1}{4}g_0.
$$

For any fixed $\eta>0$, we choose ε_1 and ε_2 small enough to satisfy

$$
\frac{g_0}{4}\varepsilon_2<\varepsilon_1<\varepsilon_2\frac{g_0}{2}, \tag{3.58}
$$

and

$$
\begin{aligned}
c_1 &= \varepsilon_2(g_0-\delta)-\varepsilon_1>0, \\
c_2 &= \varepsilon_1c_\varepsilon-\varepsilon_2\eta\left(1+2(1-l)^2+M_0c_\varrho^2\right)>0, \\
c_3 &= \varepsilon_2(g_0-2\eta)-\varepsilon_1(1+\varepsilon)>0.
\end{aligned}
$$

Further, we pick ε_1 and ε_2 so small that (3.37) and (3.59) remain valid and

$$\begin{aligned} c_4 &= \frac{1}{2} - \varepsilon_2 \frac{g(0)(c_p^2 + c_*^2)c_\varrho^2}{4\eta} > 0, \\ c_5 &= \frac{c_0}{\sigma(0)} - \varepsilon_1\varepsilon - \varepsilon_2\eta > 0. \end{aligned}$$

Therefore, (3.57) becomes

$$\begin{aligned} F'(t) &\leq -\sigma(t)\left\{c_1 + \frac{\sigma'(t)}{\sigma(t)}\right\}\|u_t\|_2^2 - \sigma(t)\left\{c_3 + \frac{\sigma'(t)}{2\sigma(t)}\right\}\|u_t\|_{2,\Gamma_1}^2 \\ &\quad -\sigma(t)\left\{c_2 + \frac{\sigma'(t)}{2\sigma(t)}\Big(\int_0^t g(s)ds\Big) + \frac{(c_p^2 + c_*^2)c_\varrho^2}{2}\frac{\sigma'(t)}{\sigma(t)}\right\}\|\Delta u\|_2^2 \\ &\quad +\sigma(t)\left\{c_4 - \frac{(c_p^2 + c_*^2)c_\varrho^2}{2}\frac{\sigma'(t)}{\sigma(t)}\Big(\int_0^t g(s)ds\Big)\right\}(g \circ \Delta u)(t) + c_6\sigma(t)\|u\|_{p,\Gamma_1}^p. \end{aligned} \tag{3.59}$$

Since $\lim_{t\to\infty} \dfrac{-\sigma'(t)}{\sigma(t)} = 0$, we can choose $t_1 > t_0$ so that (3.59) takes the form

$$\begin{aligned} F'(t) &\leq -\sigma(t)\left\{c_1\|u_t\|_2^2 + c_2\|\Delta u\|_2^2 + c_3\|u_t\|_{2,\Gamma_1}^2 - c_6\|u\|_{p,\Gamma_1}^p\right\} + c_4\sigma(t)(g \circ \Delta u)(t) \\ &\leq -k_0\sigma(t)E(t) + k_1\sigma(t)(g \circ \Delta u)(t), \quad \forall\, t_1 > t_0. \end{aligned} \tag{3.60}$$

where k_i, $i = 0, 1$ are some positive constants. This completes the proof. □

Proof of Theorem 3.3: Multiplying (3.55) by $\xi(t)$, we have

$$\xi(t)F'(t) \leq -k_0\sigma(t)\xi(t)E(t) + k_1\sigma(t)\xi(t)(g \circ \Delta u)(t) \tag{3.61}$$

Since (A_2) and using the fact that $-\sigma(t)(g' \circ \Delta u)(t) \leq -2E(t)$ by Lemma 3.1, we obtain

$$\begin{aligned} \xi(t)F'(t) &\leq -k_0\sigma(t)\xi(t)E(t) - k_1\sigma(t)(g' \circ \Delta u)(t) \\ &\leq -k_0\sigma(t)\xi(t)E(t) - k_1\Big(2E'(t) + \sigma'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2\Big). \end{aligned} \tag{3.62}$$

Since $\zeta(t)$ is nonincreasing, we have

$$\frac{d}{dt}\left(\zeta(t)F(t)+2k_5E(t)\right)\leq -k_4\sigma(t)\zeta(t)E(t)-k_1\sigma'(t)\left(\int_0^s g(s)ds\right)\|\Delta u\|_2^2. \tag{3.63}$$

Observing from definition of $E(t)$ and assumption (A_2) that $l\|\nabla u\|_2^2\leq E(t)$, we get

$$\begin{aligned}\frac{d}{dt}\left(\zeta(t)F(t)+2k_5E(t)\right) &\leq -k_4\sigma(t)\zeta(t)E(t)-k_5\sigma'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2\\ &\leq -k_4\sigma(t)\zeta(t)E(t)-\frac{2k_5E(t)}{l}\sigma'(t)\int_0^t g(s)ds \\ &\leq -\sigma(t)\zeta(t)\left(k_4+\frac{2k_5l_0\sigma'(t)}{l\sigma(t)\zeta(t)}\right)E(t).\end{aligned} \tag{3.64}$$

Since $\lim_{t\to\infty}\dfrac{-\sigma'(t)}{\sigma(t)\zeta(t)}=0$, we can choose $t_1\geq t_0$ such that $k_4+\frac{2k_5l_0\sigma'(t)}{l\sigma(t)\zeta(t)}$ for $t\geq t_1$. Finally, let

$$L(t)=\zeta(t)F(t)+2k_5E(t),$$

then we can easily see that $L(t)$ is equivalent to $E(t)$. Thus, we arrive at

$$L'(t)\leq -k\zeta(t)\sigma(t)L(t)\quad for\, t\geq t_1. \tag{3.65}$$

Integrating (3.65) over (t_1,t) with respect to t, we get

$$L(t)\leq L(t_1)e^{-k\int_{t_1}^t\zeta(s)\sigma(s)ds}\quad t\geq t_0.$$

Consequently, the equivalent relations of $L(t),F(t)$ and $E(t)$ yield the desired result.

3.5 Blow up

In this section, we state and prove the blow-up result of problem (3.1) with $E(0)<0$.

Theorem 3.4. *Let $(A_1)-(A_5)$, (3.8) and the initial energy $E(0)<0$ hold. Then, the solution of problem (3.1) blows up in finite time.*

Proof. Let

$$H(t) = -E(t), \tag{3.66}$$

then $E(0) < 0$ and (3.16), gives

$$H'(t) = -E'(t) \geq c_0 \left(\|u_t\|_{2,\Gamma_1}^2 + \|z(1,t)\|_{2,\Gamma_1}^2 \right) \geq 0, \tag{3.67}$$

and $H(t)$ is an increasing function. From (3.14) and (3.66), we obtain

$$0 \leq H(0) \leq H(t) \leq \frac{1}{p} \|u\|_{p,\Gamma_1}^p. \tag{3.68}$$

Next, we define

$$G(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx + \varepsilon \int_{\Gamma_1} u u_t dx, \tag{3.69}$$

where $\varepsilon > 0$ is a small constant and will be chosen later, and

$$0 < \sigma \leq \min \left\{ \frac{p-2}{2p}, \frac{p-2}{p} \right\}. \tag{3.70}$$

Taking a derivative of (3.69) and using first equation in (3.12), we have

$$\begin{aligned}
G'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2,\Gamma_1}^2 + \varepsilon \int_\Omega u u_{tt} dx + \varepsilon \int_{\Gamma_1} u u_{tt} dx \\
&= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2,\Gamma_1}^2 - \varepsilon \|\Delta u\|_2^2 - \varepsilon M \left(\|\nabla u\|_2^2 \right) \|\nabla u\|_2^2 \\
&\quad + \varepsilon \|u\|_{p,\Gamma_1}^p + \varepsilon \sigma(t) \int_\Omega \int_0^t g(t-s) \Delta u(s) ds \Delta u(t) dx - \varepsilon \mu_1 \int_{\Gamma_1} u_t u dx \\
&\quad - \varepsilon \mu_2 \int_{\Gamma_1} z(1,t) u dx.
\end{aligned} \tag{3.71}$$

Applying Hölder's and Young's inequalities, for $\eta, \delta > 0$, we have

$$\int_\Omega \Delta u \int_0^t g(t-s) \Delta u(s) ds dx \geq \left(1 - \frac{1}{4\eta} \right) \left(\int_0^t g(s) ds \right) \|\Delta u\|_2^2 - \eta (g \circ \Delta u)(t), \tag{3.72}$$

and

$$\mu_1 \int_{\Gamma_1} u_t u dx \leq \delta \mu_1^2 \|u\|_{2,\Gamma_1}^2 + \frac{1}{4\delta} \|u_t\|_{2,\Gamma_1}^2 \leq \delta \mu_1^2 \|u\|_{2,\Gamma_1}^2 + \frac{1}{4 c_0 \delta} H'(t). \tag{3.73}$$

Similarly, we have

$$\mu_2 \int_{\Gamma_1} z(1,t) u dx \leq \delta \mu_2^2 \|u\|_{2,\Gamma_1}^2 + \frac{1}{4\delta} \|z(1,t)\|_{2,\Gamma_1}^2 \leq \delta \mu_2^2 \|u\|_{2,\Gamma_1}^2 + \frac{1}{4c_0\delta} H'(t). \tag{3.74}$$

Combining these estimates (3.72)-(3.74) and (3.71), we get

$$\begin{aligned} G'(t) &\geq \left\{ (1-\sigma) H^{-\sigma}(t) - \frac{\varepsilon}{4c_0\delta} \right\} H'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2,\Gamma_1}^2 \\ &\quad - \varepsilon \left\{ 1 + M_0 - \sigma(t) \left(1 - \frac{1}{4\eta} \right) \left(\int_0^t g(s) ds \right) \right\} \|\Delta u\|_2^2 \\ &\quad - \varepsilon \eta \sigma(t) (g \circ \Delta u)(t) - \varepsilon (\mu_1^2 + \mu_2^2) \delta \|u\|_{2,\Gamma_1}^2 + \varepsilon \|u\|_{p,\Gamma_1}^p . \end{aligned} \tag{3.75}$$

It follows from (3.14) and (3.66), for constant $N > 0$, we see that

$$\begin{aligned} G'(t) &\geq \left\{ (1-\sigma) H^{-\sigma}(t) - \varepsilon \frac{\varepsilon}{4c_0\delta} \right\} H'(t) + \varepsilon \left\{ \frac{N}{2} + 1 \right\} \|u_t\|_2^2 + \varepsilon \left\{ \frac{N}{2} + 1 \right\} \|u_t\|_{2,\Gamma_1}^2 \\ &\quad + \varepsilon \left\{ \left(\frac{N}{2} - 1 - M_0 \right) - \sigma(t) \left(\frac{N}{2} - 1 - \frac{1}{4\eta} \right) \left(\int_0^t g(s) ds \right) \right\} \|\Delta u\|_2^2 \\ &\quad + \varepsilon \left\{ \frac{N}{2} - \eta \right\} \sigma(t) (g \circ \Delta u)(t) + \varepsilon \left\{ \frac{N}{p} - 1 \right\} \|u\|_{p,\Gamma_1}^p - \varepsilon (\mu_1^2 + \mu_2^2) \delta \|u\|_{2,\Gamma_1}^2 \\ &\quad + \varepsilon \frac{N\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho + N \varepsilon H(t). \end{aligned} \tag{3.76}$$

Using (3.8), we find that, for some number η with $0 < \eta < N/2$,

$$\begin{aligned} G'(t) &\geq \left\{ (1-\sigma) H^{-\sigma}(t) - \frac{\varepsilon}{4c_0\delta} \right\} H'(t) + \varepsilon \left\{ \frac{N}{2} + 1 \right\} \|u_t\|_2^2 + \varepsilon \left\{ \frac{N}{2} + 1 \right\} \|u_t\|_{2,\Gamma_1}^2 \\ &\quad + \varepsilon a_1 \|\Delta u\|_2^2 + \varepsilon a_2 \sigma(t) (g \circ \Delta u)(t) + \varepsilon a_3 \|u\|_{p,\Gamma_1}^p + \varepsilon \frac{N\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho \\ &\quad - \varepsilon (\mu_1^2 + \mu_2^2) \delta \|u\|_{2,\Gamma_1}^2 . \end{aligned} \tag{3.77}$$

where

$$a_1 = \left(\frac{N}{2} - 1 - M_0 \right) - \sigma(t) \left(\frac{N}{2} - 1 - \frac{1}{4\eta} \right) \left(\int_0^t g(s) ds \right) > 0,$$

$$a_2 = \frac{N}{2} - \eta > 0, \; a_3 = \frac{N}{p} - 1 > 0.$$

Therefore, by taking $\delta = H(t)^\sigma/4c_0k$, where $k > 0$ to be specified later, we have

$$\begin{aligned} G'(t) &\geq \{(1-\sigma) - \varepsilon k\} H^{-\sigma}(t)H'(t) + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_2^2 + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_{2,\Gamma_1}^2 \\ &+\varepsilon a_1 \|\Delta u\|_2^2 + \varepsilon a_2 \sigma(t)(g \circ \Delta u)(t) + \varepsilon a_3 \|u\|_{p,\Gamma_1}^p + \varepsilon \frac{N\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho \\ &-\varepsilon \frac{(\mu_1^2+\mu_2^2)}{4c_0k} H(t)^\sigma \|u\|_{2,\Gamma_1}^2. \end{aligned} \tag{3.78}$$

Exploiting (3.68), we have

$$H(t)^\sigma \|u\|_{2,\Gamma_1}^2 \leq C_p^2 H(t)^\sigma \|u\|_{p,\Gamma_1}^2 \leq \frac{C_p^2}{p^\sigma} \|u\|_{p,\Gamma_1}^{\sigma p+2}. \tag{3.79}$$

Substituting (3.79) into (3.78), we get

$$\begin{aligned} G'(t) &\geq \{(1-\sigma) - \varepsilon k\} H^{-\sigma}(t)H'(t) + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_2^2 + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_{2,\Gamma_1}^2 \\ &+\varepsilon a_1 \|\Delta u\|_2^2 + \varepsilon a_2 \sigma(t)(g \circ \Delta u)(t) + \varepsilon a_3 \|u\|_{p,\Gamma_1}^p + \varepsilon \frac{N\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho \\ &-\varepsilon \frac{C_p^2(\mu_1^2+\mu_2^2)}{4c_0 p^\sigma k} \|u\|_{p,\Gamma_1}^{\sigma p+2}. \end{aligned} \tag{3.80}$$

From (3.70), and Lemma 2.3, for $s = \sigma p + 2 \leq p$, we deduce

$$\|u\|_{p,\Gamma_1}^{\sigma p+2} \leq C_* \left(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p\right) \leq C_* c_\varrho^2 \left(\|\Delta u\|_2^2 + \|u\|_{p,\Gamma_1}^p\right). \tag{3.81}$$

Combining (3.81) with (3.80), we get

$$\begin{aligned} G'(t) &\geq \{(1-\sigma) - \varepsilon k\} H^{-\sigma}(t)H'(t) + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_2^2 + \varepsilon \left\{\frac{N}{2}+1\right\} \|u_t\|_{2,\Gamma_1}^2 \\ &+\varepsilon \left\{a_1 - \frac{a_4}{k}\right\} \|\Delta u\|_2^2 + \varepsilon a_2 \sigma(t)(g \circ \Delta u)(t) + \varepsilon \left\{a_3 - \frac{a_4}{k}\right\} \|u\|_{p,\Gamma_1}^p \\ &+\varepsilon \frac{N\xi}{2} \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho, \end{aligned} \tag{3.82}$$

where $a_4 = \left(C_* c_\varrho^2 C_p^2(\mu_1^2+\mu_2^2)\right)/(4c_0 p^\sigma)$.

First, we fix k such that

$$0 < k < \min\{a_4/a_1, a_4/a_3\}.$$

Once k is fixed, we select $\varepsilon > 0$ small enough such that

$$(1-\sigma) - \varepsilon k > 0, \quad and \quad G(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 dx + \varepsilon \int_{\Gamma_1} u_0 u_1 dx > 0.$$

Therefore, we obtain from (3.82) that

$$\begin{aligned} G'(t) \quad &\geq \omega \Big(H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\Delta u\|_2^2 + \sigma(t)(g \circ \Delta u)(t) \\ &+ \int_0^1 \|z(\rho,t)\|_{2,\Gamma_1}^2 d\rho + \|u\|_{p,\Gamma_1}^p \Big), \end{aligned} \tag{3.83}$$

where ω is a positive constant.

We now estimate $G(t)^{\frac{1}{1-\sigma}}$. By Hölder's inequality, we have

$$\left| \int_\Omega u u_t dx \right| \leq \|u\|_2 \|u_t\|_2 \leq c\|u\|_p \|u_t\|_2, \tag{3.84}$$

which implies

$$\left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\sigma}} \leq c\|u\|_p^{\frac{1}{1-\sigma}} \|u_t\|_2^{\frac{1}{1-\sigma}}. \tag{3.85}$$

Young's inequality yields

$$\left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\sigma}} \leq c \left(\|u\|_p^{\frac{\mu}{1-\sigma}} + \|u_t\|_2^{\frac{\theta}{1-\sigma}} \right), \tag{3.86}$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. To be able to use Lemma 3.2, we take $\theta = 2(1-\sigma)$ which gives $\frac{\mu}{1-\sigma} = \frac{2}{1-2\sigma} \leq p$. Therefore, (3.85) becomes

$$\left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\sigma}} \leq c\left(\|u\|_p^s + \|u_t\|_2^2 \right),$$

were $s = \frac{2}{1-2\sigma}$. Again Lemma 3.2 gives

$$\left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_p^p + \|\nabla u\|_2^2 + \|u_t\|_2^2 \right) \leq c_1 \left(\|\Delta u\|_2^2 + \|u_t\|_2^2 \right). \tag{3.87}$$

By applying Lemma 2.3 and similar to (3.87), we have

$$\left|\int_{\Gamma_1} uu_t dx\right|^{\frac{1}{1-\sigma}} \le C_* \left(\|u\|_{p,\Gamma_1}^p + \|\nabla u\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right) \le c_2 \left(\|u\|_{p,\Gamma_1}^p + \|\Delta u\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\right). \tag{3.88}$$

Combining these estimates (3.87)-(3.88) and (3.69), we get

$$\begin{aligned} G^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon \int_\Omega uu_t dx + \varepsilon \int_{\Gamma_1} uu_t dx\right)^{\frac{1}{1-\sigma}} \\ &\le c_3 \left(H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\Delta u\|_2^2 + \|u\|_{p,\Gamma_1}^p\right). \end{aligned} \tag{3.89}$$

Combining (3.89) with (3.83), we find that

$$G'(t) \ge \kappa G^{\frac{1}{1-\sigma}}(t),\ t \ge 0, \tag{3.90}$$

where κ is a positive constant. A simple integration of (3.90) over $(0,t)$ yields

$$G^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{G^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\kappa\sigma t}{1-\sigma}}.$$

Consequently, the solution of problem (3.1) blows up in finite time T^* and $T^* \le \frac{1-\sigma}{\kappa\sigma G^{\frac{\sigma}{1-\sigma}}(0)}$. □

CONCLUSION

In this thesis, we have studied the asymptotic behavior (Decay rate and Blow-up) of solutions for two Kirchhoff-type problems with nonlinear boundary conditions, delay and source terms.

First, we gave the results of local and global existence of solutions using Faedo-Galerkin and potential well methods respectively.

Then, we examined the general decay result of energy, by introducing suitable energy and perturbed Lyapunov Functional.

Finally, we thought about the finite time blow-up results of solutions with negative initial energy. The main tool used is based on the method of Georgiev and Todorova.

BIBLIOGRAPHY

[1] M. Aassila, M.M. Cavalcanti, V.N. Domingos Cavalcanti, Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term, Calc. Var. Partial Differential Equations 15 (2002) 155–183.

[2] R. A. Admas, Sobolev Space, New York: Academac press, 1975.

[3] D. Andrade, M. A. Jorge Silva, T. F. Ma, Exponential stability for a plate equation with p-Laplacian and memory terms. Math. Methods Appl. Sci. 35(2012) 417-426.

[4] K.T. Andrews, K.L. Kuttler, M. Shillor, Second order evolution equations with dynamic boundary conditions. J. Math. Anal. Appl. 197 (3) (1996) 781-795.

[5] N. Boumaza, B. Gheraibia, On the existence of a local solution for an integro-differential equation with an integral boundary condition. Bol. Soc. Mat. Mex. 26 (2020) 521-534.

[6] N. Boumaza, B. Gheraibia, General decay and blowup of solutions for a degenerate viscoelastic equation of Kirchhoff type with source term. J. Math. Anal. Appl. 489 (2)(2020) 124185.

[7] N. Boumaza, M. Saker, B. Gheraibia, Asymptotic behavior for a viscoelastic Kirchhoff-Type equation with delay and source terms. Acta Appl. Math. 171 (1), 18 (2021).

[8] N. Boumaza, B. Gheraibia, Global existence, nonexistence, and decay of solutions for a wave equation of p-Laplacian type with weak and p-Laplacian damping, nonlinear boundary delay and source terms. Asymptotic Analysis. (2021) 1–16.

[9] B.M. Budak, A.A. Samarskii, A.N. Tikhonov, A collection of problems on mathematical physics. Translated by A.R.M. Robson, The Macmillan Co. New York, 1964.

[10] H. Brezis, Functional Analysis, Sobolev Space and Partial Differential Equations. New york, Spring, 2011.

[11] M.M. Cavalcanti, V.N. Domingos Cavalcanti, P. Martinez, Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term, J. Diffe. Equa. 203 (1) (2004) 119-158.

[12] M.M. Cavalcanti, V.N. Domingos Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, J. Diffe. Equa. 236 (2007) 407-459.

[13] H. Chen, G. Liu, Global existence, uniform decay and exponential growth for a class of semilinear wave equation with strong damping, Acta. Math. Sci. Ser. B 33(1)(2013) 41-58.

[14] F. Conrad, O. Morgul, stabilization of a flexible beam with a tip mass. SIAM J. Control Optim. 36 (6) (1998) 1962–1986.

[15] QY. Dai, ZF. Yang, Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay. Z. Angew. Math. Phys. 65(5) (2014) 885-903.

[16] L. C. Evans, Partial Differential Equations(Second Edition), Rhode Island: American Mathematical Society Providence, 2010.

[17] B. Feng, Global well-posedness and stability for a viscoelastic plate equation with a time delay. Math. Probl. Eng. (2015) Article ID 585021 (2015).

[18] M. Ferhat, Ali Hakem, Global existence and energy decay result for a weak viscoelastic wave equations with a dynamic boundary and nonlinear delay term. Comput. Math. Appl. 71 (2016) 779–804.

[19] M. Ferhat, A. Hakem, Asymptotic behavior for a weak viscoelastic wave equations with a dynamic boundary and time varying delay term. J. Appl. Math. Comput. 51 (2016) 509–526.

[20] A. Friedman, Partial differential equations of parabolic type. Prentice–Hall, Englewood Cliffs, N.J., 1964.

[21] V. Georgiev, D. Todorova, Existence of solutions of the wave equations with nonlinear damping and source terms, J. Diffe. Equa. 109 (1994) 295-308.

[22] S. Gerbi, B. Said-Houari, Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions. Adv. Differ. Equ. 13 (2008) 1051-1074.

[23] S. Gerbi, B. Said-Houari, Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions. Nonlinear Anal. 74 (2011) 7137-7150.

[24] S. Gerbi, B. Said-Houari, Existence and exponential stability of a damped wave equation with dynamic boundary conditions and a delay term. Appl. Math. Comput. 218 (2012) 11900–11910.

[25] B. Gheraibia, N. Boumaza, General decay result of solutions for viscoelastic wave equation with Balakrishnan-Taylor damping and a delay term. Z. Angew. Math. Phys. 71, 198 (2020).

[26] P. J. Graber, B. Said-Houari, Existence and asymptotic behavior of the wave equation with dynamic boundary conditions. Appl. Math. Optim. 66 (2012) 81-122.

[27] M. Grobbelaar–Van Dalsen, On fractional powers of a closed pair of operators and a damped wave equation with dynamic boundary conditions. Appl. Anal. 53 (1–2) (1994) 41–54.

[28] M. Grobbelaar–Van Dalsen, On the initial–boundary-value problem for the extensible beam with attached load. Math. Methods Appl. Sci. 19 (12) (1996) 943–957.

[29] A. Guesmia, SA. Messaoudi, C.M. Webler, Well-posedness and optimal decay rates for the viscoelastic Kirchhoff equation. Bol. Soc. Parana. Mat. 35(3) (2017) 203–224.

[30] D. Henry, Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics. Vol. 840, Springer-Verlag, Berlin, 1981.

[31] M.A. Jorge Silva, J.E. MunÕz Rivera, R. Racke, On a classes of nonlinear viscoelastic Kirchhoff plates: well-posedness and general decay rates. Appl. Math. Optim. 73(2016) 165–194.

[32] M. Kafini, SA. Messaoudi, A blow-up result in a nonlinear wave equation with delay. Mediterr. J. Math. 13(1) (2016) 237-247.

[33] M. Kafini, SA. Messaoudi, S. Nicaise, A blow-up result in a nonlinear abstract evolution system with delay. Nonlinear Differ. Equ. Appl. 23(2) 2016 13.

[34] J-R. Kang, Global nonexistence of solutions for viscoelastic wave equation with delay. Math. Meth. Appl. Sci. 41 (16)(2018) 1–8.

[35] M. Kirane and B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay. Z. Angew. Math. Phys. 62 (2011) 1065-1082.

[36] G. Kirchhoff, Vorlesungen über Mechanik, Teubner, Leipzig, 1883.

[37] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J. Math. Anal. 5 (1974) 138-146.

[38] H.A. Levine, S. Ro Park, Global existence and global nonexistence of solutions of the Cauchy problems for a nonlinearly damped wave equation, J. Math. Anal. Appl. 228 (1998) 181-205.

[39] X. Lin, F. Li, Asymptotic energy estimates for nonlinear petrovsky plate model subject to viscoelastic damping. Abstr. Appl. Anal. 2012 (2012) 1-25.

[40] W. Liu, B. Zhu, G. Li, D. Wang, General decay for a viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term. Evol. Equ. Control The. 6 (2017) 239-260.

[41] T. Matsuyama, R. Ikehata, On global solutions and energy decay for the wave equation of Kirchhoff type with nonlinear damping terms. J. Math. Anal. Appl. 204(3) (1996) 729-753.

[42] F.Z. Mahdi, M. Ferhat, A. Hakem, Blow up and asymptotic behavior for a system of viscoelastic wave equations of Kirchhoff type with a delay term. Adv. Theory Nonlinear Anal. Appl. 2 (2018) 146-167.

[43] S.A. Messaoudi, Blow up and global existence in a nonlinear viscoelastic wave equation. Math. Nachr. 260 (2003) 58-66.

[44] S.A. Messaoudi, Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation. J. Math. Anal. Appl. 320(2006) 902–915

[45] S.A. Messaoudi, SE. Mukiawa, Existence and decay of solutions to a viscoelastic plate equation. Math. Nachr.Electron. J. Diff. Equ. Vol. 2016 (2016), No. 22, pp. 1-14.

[46] S. Migorski, A. Ochal, M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics. Nonlinear Analysis B: Real World Appl. 12 (2011) 3384-3396.

[47] S.E. Mukiawa, S.A. Messaoudi, Blow up Result for a viscoelastic plate equation with nonlinear source. Bol. Soc. Paran. Mat. (2020).

[48] J. Munõz Rivera, E. Lapa and R. Barreto, Decay rates for viscoelastic plates with memory. J. Elasticity 44 (1996) 61-87.

[49] M. Nakao, Existence of global smooth solutions to the initial-boundary value problem for the quasi-linear wave equation with a degenerate dissipative term. J. Diffe. Equa. 98(2) (1992) 299-327.

[50] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. 45(5) (2006) 1561-1585.

[51] S. Nicaise, C. Pignotti, Interior feedback stabilization of wave equations with time dependence delay. Electron J Differ Equ. 41 (2011) 1-20.

[52] K. Ono, Blowing up and global existence of solutions for some degenerate nonlinear wave equations with some dissipation. Nonlinear Anal. 30(7) (1997) 4449-4457.

[53] K. Ono, Global existence, decay and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings. J. Diffe. Equa. 137(2) (1997) 273-301.

[54] SH. Park, Decay rate estimates for a weak viscoelastic beam equation with time-varying delay. Appl. Math. Lett. 31 (2014) 46-51.

[55] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag New York, Inc, 1983.

[56] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Rational Mech. Anal. 30 (1968) 148–172.

[57] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equation. By the American Mathematical Society, (1997).

[58] E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, J. Diffe. Equa. 186 (2002) 259-298.

[59] S-T. Wu, L-Y. Tsai, Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation. Nonlinear Anal. 65 (2006) 243-246.

[60] Z. Yang, Existence and energy decay of solutions for the Euler-Bernoulli viscoelastic equation with a delay. Z. Angew. Math. Phys. 66 (2015) 727-745.

[61] X-G. Yang, R. O. Araujo, R. S.Teles, Exponential stability of a mildly dissipative viscoelastic plate equation with variable density. Math Meth Appl Sci. 42 (2019) 733-741.

[62] R. Zeng, C. L. Mu, S.M. Zhou, A blow-up result for Kirchhoff-type equations with high energy. Math. Methods Appl. Sci. 34 (4)(2011) 479-486.

[63] H. Zhang, Q. Hu, Asymptotic behavior and nonexistence of wave equation with nonlinear boundary condition, Commun. Pure Appl. Anal. 4 (2005) 861-869.