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Theme

The existence of weak positive solutions of the elliptic problem via the Sub-supersolution method

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Dedication

To everyone who helped me in this search.

To my dear parents.

To my large family,especially the small family,represented by the wife,children,brothers and sisters.

ملخص

الهدف من هذه الأطروحة هو دراسة وجود حلول موجبة ضعيفة لمسائل كيرشوف الإهليجية بواسطة طريقة الحلول العلوية والسفلية , تسمى الإشكالية التي درسناها غير محلية نظرا لوجود مؤثر كيرشوف $M\left(\int_\Omega \frac{1}{P(x)}|\nabla u|^{P(x)}dx\right)$, حيث M هي دالة مستمرة ومتزايدة على $\mathbb{R}^+$ وقيمها موجبة تماما , مما يعني أن المعادلة لم تعد مطابقة نقطيا وهذا مما يسبب بعض الصعوبات الرياضياتية التي تجعل دراسة مثل هذه الإشكالية مثيرة للاهتمام , ولهذا السبب فقد أجرينا عملية حصر الحل بواسطة اثنين من الحلول الموجبة الضعيفة باستخدام طريقة الحلول العلوية والسفلية.

الكلمات المفتاحية :حل ضعيف ,حل علوي ,حل سفلي ,مؤثر كيرشوف

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Abstract

The object of this thesis is to study the existence of weak positive solutions to Kirchhoff's elliptic problems by using the Sub-supersolution method.

The problem that studied is called nonlocal due to the presence of the Kirchhoff operator $M(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)$,where M is a continuous and increasing function on $\mathbb{R}^+$ and its values are completely positive.This means that the equation is no longer point-identical.This causes some mathematical difficulties that make studying such a problem interesting.

It is for this reason that we have performed confining the solution with two weak positive solutions using the Sub-supersolution method.

Keywords: weak solution,subsolution,supersolution,the Kirchhoff operator.

Résumé

L'objet de cette thèse est d'étudier l'existence de solutions faiblement positives aux problèmes elliptiques de Kirchhoff par la méthode de sous-supersolution.

Le problème que nous avons étudié est dit non local du fait de la présence de l'opérateur de Kirchhoff $M(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx), où\ M$ est une fonction continue et croissante sur $\mathbb{R}^+$ et ses valeurs sont complètement positives.Cela signifie que l'équation n'est plus la même en un point.Cela conduit à des difficultés mathématiques qui rendent intéressante l'étude d'un tel problème.

C'est pour cette raison que nous avons réalisé le confinement de la solution avec deux solutions faiblement positives en utilisant la méthode de sous-supersolution.

Mots clés: solution faible,sous-solution,supersolution,l'opérateur de Kirchhoff.

CONTENTS

General introduction

The study of nonlinear elliptic equations with quasilinear homogeneous type operators such as the p-Laplace operator can be performed basing on the theory of standard Sobolev spaces $W^{m,p}$,and thus weak solutions can be found.These spaces are made up of functions which have weak derivatives and satisfy some integrability conditions.Hence,in the case of nonhomogeneous $p(x)$-Laplace operators,the normal framework for this approach is the use of the so-called variable exponent Sobolev spaces.The general idea consists of replacing the Lebesgue spaces $L^{p}(\Omega)$ by more general spaces $L^{p(x)}(\Omega)$,called variable exponent Lebesgue spaces.The resulting space will be denoted by $W^{m,p}(\Omega)$ and called a variable exponent Sobolev space if the role played by $L^{p}(\Omega)$ in the definition of the Sobolev spaces $W^{m,p(x)}(\Omega)$ is assigned rather to a variable Lebesgue space $L^{p(x)}(\Omega)$.Lot of properties of Sobolev spaces were extended to Orlicz–Sobolev spaces,especially by O'Neill [46] (excellent account of those works can be found with Adams [3]).The spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ have been closely studied in the works published by Edmunds et al ([19];[18]) and the paper of Musielak [44],as well as in Kovacik and Rkosnk,Mihailescu and Radulescu ([38];[43]],and Samko and Vakulov [50].In the last decades,Variable Sobolev spaces have been used in several domains to model various phenomena.As major application,Chen,Levine and Rao [13] have presented a framework for image restoration,which is based on a variable exponent Laplacian.Moreover,modelling of electrorheological fluids (also referred to as smart fluids) is highly considered as an important application which adopts nonhomogeneous Laplace operators.In fact,since the middle of the last century,several experimental studies for different materials relying on such an advanced theory have been carried out.We may consider the works of WillisWinslow in 1949 as the most important discovery in electrorheological fluids,showing that their viscosity depends on the electric field in the fluid,which is an interesting property.Thus,they were able to increase the viscosity by up to five orders of magnitude,and the phenomenon has been known as the Winslow effect.Some more technical applications are showed by Pfeiffer et al.[49] and a general account of the underlying physics can be found with Halsey[32].

Our approach to this thesis is based on the method of sub and super-solutions.The concepts of sub- and super-solution were introduced by Nagumo (Proc $Phys-$Math $Soc\ Jpnl9 : 861 - 866, 1937$) in 1937 who proved,using also the shooting method,the existence of at least one solution for a class of nonlinear Sturm-Liouville problems.In fact,the premises of the sub and super-solution method can be traced back to Picard.He applied,in the early $1880s$,the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems.This is the starting point of the use of sub-and super-solutions in connection with monotone methods.Picard's techniques were applied later by Poincaré (J Math Pures Appl 4:137230,1898) in connection with problems arising in astrophysics.

Since the structure of the $p(x)-$Laplace is more complicated than that of the $p-$Laplace operator,such as it is nonhomogeneous,the extension from p-Laplace operator to $p(x)-$Laplace operator will not be well-worn.Furthermore,many methods for $p-$Laplacian are not true for the $p(x)-$Laplacian; for instance,if Ω is bounded,then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)/\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx} \tag{0.1}$$

is zero generally,and as it is shown in [21],$\lambda_{p(x)}$ would be positive only under some special conditions.In spite of the fact that the first eigenvalue and the first eigenfunction of the $p(x)-$Laplacian may not be existing,having a positive first eigenvalue λ_p and getting the first eigenfunction are very interesting in the study of $p-$Laplacian problem.Hence,discussing the existence of solutions of variable exponent problems has more problems.The existence of positive weak solutions for the following p-Laplacian problem is considered in [31]

$$\begin{cases} -\Delta_p u = \lambda f\ (v) & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where the first eigenfunction has been used to construct the subsolution of $p-$Laplacian problem.Under the condition that

$$\lim_{u\to+\infty}\frac{f\left(M(g(u))^{\frac{1}{p-1}}\right)}{u^{p-1}}=0, \text{for all } M>0, \tag{0.3}$$

the authors gave the existence of positive solutions for problem (0.2) provided that λ is large enough.In [11],the existence and nonexistence of positive weak solutions to the following quasilinear elliptic system

$$\begin{cases} -\Delta_p u=\lambda u^{\alpha}v^{\gamma} & \text{in } \Omega, \\ -\Delta_q v=\lambda u^{\delta}v^{\beta} & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

has been considered where the first eigenfunction has been used to construct the subsolution of problem (0.4) and he obtained the following results:

(i) If $\alpha,\beta\geq 0$,$\gamma,\delta>0$,$\theta=(p-1-\alpha)(q-1-\beta)-\gamma\delta>0$,then the problem (0.4) has a positive weak solution for each $\lambda>0$.

(ii) If $\theta=0$ and $p\gamma=q(p-1-\alpha)$,then there exists $\lambda_0>0$ such that for $0<\lambda<\lambda_0$,then problem (0.4) has no nontrivial nonnegative weak solution.For further generalizations of system (0.4) we refer to [9] and [27].As described previously,among the $p(x)-$Laplacian problems,the first eigenvalue and the first eigenfunction of the $p(x)-$Laplacian may not be existing even if there is a first eigenfunction of the $p(x)-$Laplacian.Owing to the nonhomogeneous of the $p(x)-$Laplacian,the first eigenfunction would not be used in the construction of the subsolutions of $p(x)-$Laplacian problems.Furthermore,some symmetry conditions are imposed in [4],[53],[54],in order to study the existence of solutions for the problem (0.2).Moreover,the existence of positive solutions of the system is investigated in [55]

$$\begin{cases} -\Delta_{p(x)} u=\lambda^{p(x)}f(v) & \text{in } \Omega, \\ -\Delta_{p(x)} v=\lambda^{p(x)}g(u) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{0.5}$$

without any symmetry conditions.Motivated by the ideas introduced in [55] and [30],where the authors in [55] proved the existence of a positive solution when λ is large enough and satisfies the condition (0.3) and they did not assume any symmetric condition,and did not assume any sign condition on $f(0)$ and $g(0)$.Also the authors proved the existence of positive solutions with multiparameter.

in this thesis,we extend this given system of differential equations,

$$\begin{cases} -M\left(\frac{1}{p(x)}\int_{\Omega}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda^{p(x)}\left[\lambda_1 a(x) f(v)+\mu_1 c(x) h(u)\right] & \text{in } \Omega, \\ -M\left(\frac{1}{p(x)}\int_{\Omega}|\nabla v|^{p(x)}dx\right)\Delta_{p(x)}v=\lambda^{p(x)}\left[\lambda_2 b(x) g(u)+\mu_2 d(x) \tau(v)\right] & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega, 1 < p \in C^1\left(\overline{\Omega}\right)$ is a functions with $1 < p^- := \inf_{\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < \infty$, and $\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called $p(x)-$Laplacian,and $M\left(\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)$ is called Kirchhoff operator where M is a continuous and increasing function on $\mathbb{R}^+$ and its values are completely positive.

$\lambda, \lambda_1, \lambda_2, \mu_1$,and μ_2 are positive parameters,and f,g,h,τ are monotone functions in $[0, +\infty[$ such that

$$\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty,$$

and satisfying some natural growth condition at $u=\infty$.

An extension of the previous studies and with the same method used in modeling physical phenomena,we generalized the following Kirchhoff equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{0.6}$$

presented by Kirchhoff in 1883,see [37] This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during the vibrations.The parameters in (0.6) ;have the following meanings: L is the length of the string,h is the area of the cross-section,E is the Young modulus of the material,ρ is the mass density,and P_0 is the initial tension.

In this thesis we have divided it into three chapters as :

Chapter 1 : We present the concepts and theories that where used in the remaining chapters of the thesis

Chapter 2 : We studied the following system of differential equations

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u|^2 dx\right)\Delta u=\lambda_1 \alpha(x) f(v)+\mu_1 \beta(x) h(u) & \text{in } \Omega, \\ -B\left(\int_{\Omega}|\nabla v|^2 dx\right)\Delta v=\lambda_2 \gamma(x) g(u)+\mu_2 \eta(x) \tau(v) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{0.7}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$,and A,B :$\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions,$\alpha, \beta, \gamma, \eta \in C(\overline{\Omega})$,$\lambda_1, \lambda_2, \mu_1$,and μ_2 are nonnegative parameters.

Since the first equation in (0.7) contains an integral over Ω,it is no longer a pointwise identity; therefore it is often called nonlocal problem.This problem models several physical and biological systems,where u describes a process which depends on the average of itself,such as the population density,

By imposing five conditions on the case data,

(A_1) A, B:$\mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exists $a_i, b_i > 0$,$i = 1, 2$,such that $a_1 \leq A(t) \leq a_2$,$b_1 \leq B(t) \leq b_2$ for all $t \in \mathbb{R}^+$.

(A_2) $\alpha, \beta, \gamma, \eta \in C(\overline{\Omega})$ *and for all* $x \in \Omega$ $\alpha(x) \geq \alpha_0 > 0$,$\beta(x) \geq \beta_0 > 0$,$\gamma(x) \geq \gamma_0 > 0$,$\eta(x) \geq \eta_0 > 0$.

(A_3) f,g,h,and τ are continuous on $[0, +\infty[$,C^1 on $(0, +\infty)$,and increasing functions such that $\begin{cases} \lim_{t\to+\infty} f(t) = +\infty, \lim_{t\to+\infty} g(t) = +\infty, \\ \lim_{t\to+\infty} h(t) = +\infty, \lim_{t\to+\infty} \tau(t) = +\infty. \end{cases}$

(A_4) *It holds that* $\lim_{t\to+\infty} \frac{f(K(g(t)))}{t} = 0$, for all $K > 0$.

(A_5) $\lim_{t\to+\infty} \frac{h(t)}{t} = \lim_{t\to+\infty} \frac{\tau(t)}{t} = 0$.

we have reached the following main conclusion :

Theorem 0.1 .*Assume that the conditions* $(A_1)-(A_5)$ *hold,and* M *is a nonincreasing function atisfying (2.3).Then for* $\lambda_1\alpha_0 + \mu_1\beta_0$ *and* $\lambda_2\gamma_0 + \mu_2\eta_0$ *are large then problem (0.7) has a large positive weak solution.*

Finally,in Chapter Three,we examined the following Kirchhoff elliptic system of differential equations

$$\begin{cases} -M\left(\frac{1}{p(x)} \int_\Omega |\nabla u|^{p(x)} dx\right) \Delta_{p(x)} u = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] & \text{in } \Omega, \\ -M\left(\frac{1}{p(x)} \int_\Omega |\nabla v|^{p(x)} dx\right) \Delta_{p(x)} v = \lambda^{p(x)} \left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega, 1 < p \in C^1\left(\overline{\Omega}\right)$ is a functions with $1 < p^- := \inf_\Omega p(x) \leq p^+ := \sup_\Omega p(x) < \infty$,and $\Delta_{p(x)} u = div\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is called $p(x)-$Laplacian,and $M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right)$ is called Kirchhoff operator where M is a continuous and increasing function on $\mathbb{R}^+$ and its values are completely positive.

$\lambda, \lambda_1, \lambda_2, \mu_1$,and μ_2 are positive parameters,and f,g,h,τ are monotone functions in $[0, +\infty[$ such that

$$\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty,$$

and satisfying some natural growth condition at $u=\infty$.

When certain conditions are met about the data of the problem,

(H_1) M: $[0, +\infty) \to [m_0, \infty]$ is a continuous and increasing function with $m_0 > 0$.

(H_2) $p \in C^1\left(\overline{\Omega}\right)$ and $1 < p^- \leq p^+$.

(H_3) f, g, h, τ: $[0, +\infty[\to \mathbb{R}$ are C^1,monotone functions such that $\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty$.

(H_4) $\lim_{u\to+\infty} \frac{f\left(L(g(u))^{\frac{1}{p^- -1}}\right)}{u^{p^- -1}} = 0$, for all $L > 0$.

(H_5) $\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = 0$,$\lim_{u\to+\infty} \frac{\tau(u)}{u^{p^- -1}} = 0$.

(H_6) $a, b, c, d : \overline{\Omega} \to (0, +\infty)$ are continuous functions,such that
$a_1 = \min_{x\in\overline{\Omega}} a(x), b_1 = \min_{x\in\overline{\Omega}} b(x), c_1 = \min_{x\in\overline{\Omega}} c(x), d_1 = \min_{x\in\overline{\Omega}} d(x)$,
$a_2 = \max_{x\in\overline{\Omega}} a(x), b_2 = \max_{x\in\overline{\Omega}} b(x), c_2 = \max_{x\in\overline{\Omega}} c(x), d_2 = \max_{x\in\overline{\Omega}} d(x)$.

We arrive at the main conclusion

Theorem 0.2 *Assume that the conditions $(H_1) - (H_6)$ are satisfied.Then problem (0.8) has a positive solution when λ is large enough.*

At the end of the thesis,we presented some prospects that we aspire to generalize our search results to wider spaces.

CHAPTER 1

Concepts about Lebesgue and Sobolev spaces with variable exponents.

1.1 *History of function spaces with variable exponents.*

One of the reasons for the huge development of the theory of classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ (where $1 \leq p \leq \infty$) is the description of many phenomena arising in applied sciences.For instance,many materials can be modeled with sufficient accuracy using the function spaces L^p and $W^{1,p}$ where p is a fixed constant.For some nonhomogeneous materials,for instance electrorheological fluids (sometimes referred to as "smart fluids"),this approach is not adequate,but rather the exponent p should be allowed to vary.This leads us to the study of variable exponent Lebesgue and Sobolev spaces,$L^{p(x)}$ and $W^{1,p(x)}$,where p is a real–valued function.

Variable exponent Lebesgue spaces appeared in the literature in 1931 in the paper by Orlicz [47].He was interested in the study of function spaces that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\rho(\lambda u) = \int_{\Omega} \varphi(\lambda \left|u\left(x\right)\right|)dx \tag{1.1}$$

for some $\lambda > 0$ and φ satisfying some natural assumptions,where Ω is an open set in $\mathbb{R}^N$.This space is denoted by L^{φ} and it is now called Orlicz space.

However,we point out that in [47] the case $|u|^{p(x)}$ corresponding to variable exponents was not included.In the 1950's these problems were systematically studied by Nakano [45],who developed the theory of modular function spaces.Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered,see Nakano [45, p 284].Later,Polish mathematicians investigated the modular function spaces,see Musielak [44].Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers.In that context,we refer to the work of Tsenov [52] and Sharapudinov [51].They were interested in the minimization of functionals like

$$\int_a^b \left|u\left(x\right) - v(x)\right|^{p(x)} dx \tag{1.2}$$

where u is a given function and v varies over a finite dimensional subspace of $L^{p(x)}$[a, b].Zhikov [57] started a new direction of investigation,which created the relationship between spaces with variable exponent and variational integrals with nonstandard growth conditions.We also point out the contributions of Marcellini [42],who studied minimization problems with (p;q)-

growth,namely

$$inf \int_{\Omega} F(x, |\nabla u|)dx \tag{1.3}$$

where $t^p \leq F(x,t) \leq t^q + 1$ for all $t \geq 0$.The case corresponding to the variable exponent corresponds to $F(x,t) = t^{p(x)}$,where p: $\Omega \to (1, \infty)$ is a bounded function.

In 1991,Kovacik and Rakosnik [38] established several basic properties of spaces $L^p(\Omega)$ and,$W^{1,p}(\Omega)$ with variable exponents.Their results were extended by Fan and Zhao [26] in the framework of Sobolev spaces$W^{\mathrm{m},p}(\Omega)$.Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [1].Density of smooth functions in $W^{\mathrm{k},p}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rakosnik [18].

We also point out the important contributions of the Finnish research group on variable exponent spaces and image processing,whose main goal was to study nonlinear potential theory in variable exponent Sobolev spaces.The abstract theory of Lebesgue and Sobolev spaces with variable exponents was developed in the monograph by Diening,Harjulehto,*Hästö*,and Ruzicka [17].The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications.In 1920 Bingham was surprised to discover that some paints do not run like honey.He studied such a behavior and described a strange phenomenon.There are fluids that first flow,then stop spontaneously (Bingham fluids).Inside them,the forces that create the flows reach a threshold.As this threshold is not reached,the fluid flow deforms as a solid.Invented in the 17th century,the "Flemish medium" makes painting oil thixotropic: it flows under pressure of the brush,but freezes as soon as you leave it to rest.While the exact composition of the Flemish medium remains unknown,it is known that the bonds form gradually between its components,which is why the picture freezes in a few minutes.Thanks to this wonderful medium,Rubens was able to paint La Kermesse in only 24 hours.

Recent systematic study of partial differential equations with variable exponents was motivated by the description of several relevant models in electrorheological and thermorheological fluids,image processing,or robotics.In what follows,we give two relevant examples that justify the mathematical study of models involving variable exponents.The first example is due to Chen,Levine,Rao [12] and it concerns applications to image restoration.Let us consider an input I that corresponds to shades of gray in a domain $\subseteq \mathbb{R}^2$.

1.2 *Lebesgue spaces with variable exponents.*

We write $E = \{u : u \; is \; a \; measurable \; function \; in \; \Omega\}$.suth that $\Omega \subseteq \mathbb{R}^n$ be a measurable subset and meas $\Omega > 0$

Elements in E that are equal to each other almost everywhere are considered as one element. Let $p \in E$.In the following discussion we always assume that $u \in E$ and write

$$\phi(x,s) = s^{p(x)}, \forall x \in \Omega, s \geq 0 \tag{1.4}$$

$$\rho(u) = \rho_{p(x)}(u) = \int_{\Omega} \phi(x,|u|)dx = \int_{\Omega} |u(x)|^{p(x)}dx \tag{1.5}$$

$$L^{p(x)}(\Omega) = \{u \in E : \lim_{\lambda \to 0^+} \rho(\lambda u) = 0\} \tag{1.6}$$

$$L_0^{p(x)}(\Omega) = \{u \in E : \rho(u) < \infty\} \tag{1.7}$$

$$L_1^{p(x)}(\Omega) = \{u \in E : \forall \lambda > 0, \rho(\lambda u) < \infty\} \tag{1.8}$$

$$L_+^{\infty}(\Omega) = \{u \in L^{\infty}(\Omega) : ess \; \inf \; u \geq 1\} \tag{1.9}$$

It is easy to see that the function ϕ defined above belongs to the class Φ,which is defined in [22,p.33],i.e.,ϕ satisfies the following two conditions:

1) For all $x \in \Omega$,$\phi(x,\cdot) : [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function with $\phi(x,0) = 0$ and $\phi(x,s) > 0$ whenever $s > 0$; $\phi(x,s) \to \infty$ when $s \to \infty$.

2) For every $s \geq 0$,$\phi(.,s) \in E$.Obviously,ϕ is convex in s.

In view of the definition in [44],ρ is a convex modular over E,i.e.,$\rho : E \to [0,\infty]$ verifies the following properties $(a) - (c)$

(a) $\rho(u) = 0 \Leftrightarrow u = 0$,

(b) $\rho(-u) = \rho(u)$,

(c) $\rho(\alpha u + \beta v) \leq \alpha\rho(u) + \beta\rho(v)$,$\forall u,v \in E$,$\forall \alpha,\beta \geq 0$,$\alpha + \beta = 1$.

and thus by [44],$L^{p(x)}(\Omega)$ is a Nakano space,which is a special kind of Musielak-Orlicz space.$L_0^{p(x)}(\Omega)$ is a kind of generalized Orlicz class.It is easy to see that $L^{p(x)}(\Omega)$ is a linear subspace of E,and $L_0^{p(x)}(\Omega)$ is a convex subset of $L^{p(x)}(\Omega)$.In general we have

$$L_1^{p(x)}(\Omega) \subset L_0^{p(x)}(\Omega) \subset L^{p(x)}(\Omega)$$

By the properties of $\phi(x,s)$ we also have

$$L^{p(x)}(\Omega) = \{u \in E : \exists\lambda > 0, \rho(\lambda u) < \infty\}.$$

Theorem 1.1 *([26])The following two conditions are equivalent:*

$$\begin{aligned} &1) \quad p \in L_+^{\infty}(\Omega), \\ &2) \quad L_1^{p(x)}(\Omega) = L^{p(x)}(\Omega). \end{aligned}$$

Proof

1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1).If 1) is not true,then we can take a sequence $\{I_m\}$ of disjoint subsets of Ω with positive measure such that $p(x) > m$ for $x \in I_m$.Choosing an increasing sequence $\{u_m\} \subset (0,\infty)$ such that $u_m \to \infty$ as $m \to \infty$,we can find k_m satisfying the inequality

$$\int_{I_m} u_{k_m}^{p(x)} dx \geq \frac{1}{2^m}$$

By the absolute continuity of integral,we can shrink I_m to Ω_m such that

$$\int_{\Omega_m} u_{k_m}^{p(x)} dx = 2^m$$

Denote by $\chi_{\Omega_m}(x)$ the characteristic function of Ω_m,i.e

$$\chi_{\Omega_m}(x) = \begin{cases} 1 & if \quad x \in \Omega_m \\ 0 & if \quad x \notin \Omega_m \end{cases}$$

if we write

$$u_0(x) = \int_{m=1}^{\infty} u_{k_m} \chi_{\Omega_m}(x),$$

then we have

$$\int_{\Omega} |u_0(x)|^{p(x)} dx = \int_{n=1}^{\infty} \int_{\Omega_n} u_{k_n}^{p(x)} dx = \int_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$$\int_{\Omega}|2u_0(x)|^{p(x)}dx=\int_{n=1}^{\infty}\int_{\Omega_n}2^{p(x)}u_{k_n}^{p(x)}dx>\int_{n=1}^{\infty}2^n\int_{\Omega_n}u_{k_n}^{p(x)}dx=\infty$$

thus we have $u_0\in L^{p(x)}(\Omega)$,but $u_0\notin L_1^{p(x)}(\Omega)$.This contradicts condition (2),and we complete the proof. ∎

From now on we only consider the case where $p\in L_+^{\infty}(\Omega)$,i.e.,

$$1\leq p^-=:ess\ \inf\ p(x)\leq ess\ \sup\ p(x)=:p^+<\infty \tag{1.10}$$

For simplicity we write $E_\rho=L^{p(x)}(\Omega)=L_0^{p(x)}(\Omega)=L_1^{p(x)}(\Omega)$,and we call $L^{p(x)}(\Omega)$ generalized Lebesgue spaces.By [44],we can introduce the norm $\parallel u\parallel_{L^{p(x)}(\Omega)}$ on E_ρ (denoted by $\parallel u\parallel_\rho$) as

$$\parallel u\parallel_\rho=\ \inf\ \{\lambda>0:\rho\left(\frac{u}{\lambda}\right)\leq 1\}$$

and $(E_\rho,\parallel\ .\ \parallel_\rho)$ becomes a Banach space.It is not hard to see that under condition (1.10),p satisfies

(d) $\rho(u+u)\leq 2^{p^+}(\rho(u)+\rho(u));\forall\ u\in E_\rho$.

(e) For $u\in E_\rho$,if $\lambda>1$,we have

$$\rho\left(u\right)\leq\lambda\rho\left(u\right)(\leq\lambda^{p^-}\rho\left(u\right)(\leq\rho\left(\lambda u\right)\leq\lambda^{p^+}\rho\left(u\right)$$

and if $0<\lambda<1$,we have

$$\lambda^{p^+}\rho\left(u\right)\leq\rho\left(\lambda u\right)\leq\lambda^{p^-}\rho\left(u\right)\leq\lambda\rho\left(u\right)\leq\rho\left(u\right)$$

(f) For every fixed $u\in E_\rho\backslash\{0\}$,$\rho(\lambda u)$ is a continuous convex even function in λ,and it increases strictly when $\lambda\in[0,\infty)$

By property (f) and the definition of $\parallel u\parallel_\rho$,we have

Theorem 1.2 *([26]) Let $u\in E_\rho\backslash\{0\}$; then*

$$\parallel u\parallel_\rho=a\ if\ and\ only\ if\ \rho\left(\frac{u}{a}\right)=1$$

The norm $\parallel u\parallel_\rho$ is in close relation with the modular $\rho(u)$.We have

Theorem 1.3 *([26]) Let $u \in E_\rho$; then*

1) $\| u \|_\rho < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$,

2) $If \| u \|_\rho > 1, then \| u \|_\rho^{p^-} \leq \rho(u) \leq \| u \|_\rho^{p^+}$,

3) $If \| u \|_\rho < 1, then \| u \|_\rho^{p^+} \leq \rho(u) \leq \| u \|_\rho^{p^-}$.

Proof

From (f) and Theorem 1.2 we can obtain 1).We only prove 2) below,as the proof of 3) is similar.Assume that $\| u \|_\rho = a > 1$,by Theorem 1.2,$\rho(\frac{u}{a}) = 1$.Notice that $\frac{1}{a} < 1$,by (e).We have

$$\frac{1}{a^{p^+}}\rho(u) \leq \rho(\frac{u}{a}) = 1 \leq \frac{1}{a^{p^-}}\rho(u)$$

so we obtain 2). ■

Theorem 1.4 *([26]) Let u,$u_k \in E_p$,$k = 1, 2, \ldots$ Then the following statements are equivalent to each other*

1) $lim_{k\to\infty} \| u_k - u \|_\rho = 0$,

2) $lim_{k\to\infty} \rho(u_k - u) = 0$,

3) u_k *converges to* u in Ω *in measure and* $lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Proof

The equivalence of 1) and 2) can be obtained from Theorem 1.6 in [44] and the property e) of ρ stated above.Now we prove the equiva lence of 2) and 3).If 2) holds,i.e.,

$$\lim_{k\to\infty} \int_\Omega | u_k - u|^{p(x)} dx = 0$$

then it is easy to see that u_k converges to u in Ω in measure; thus $|u_k|^{p(x)}$ converges to $|u|^{p(x)}$ in measure.Using the inequality

$$|u_k|^{p(x)} \leq 2^{p^+-1}(|u_k - u|^{p(x)} + |u|^{p(x)})$$

and using the Vitali convergence theorem of integral we deduce that $\rho(u_k) \to \rho(u)$,so 3) holds.On the other hand,if 3) holds,we can deduce that $|u_k - u|^{p(x)}$ converges to 0 in Ω in measure.By the inequality

$$|u_k - u|^{p(x)} \leq 2^{p^+-1}(|u_k|^{p(x)} + |u|^{p(x)})$$

and condition $\rho(u_k) \to \rho(u)$,we get $\lim_{k\to\infty}\rho(u_k - u) = 0$.For arbitrary $u \in L^{p(x)}(\Omega)$,let

$$u_n(x) = \begin{cases} u(x), & if\ |u(x)| \leq n, \\ 0, & if\ |u(x)| > n. \end{cases}$$

It is easy to see that $\lim_{n\to\infty}\rho(u_n(x) - u(x)) = 0$. ■

so by Theorem 1.4 we get

Theorem 1.5 *([26])*
The set of all bounded measurable functions over Ω is dense in $(L^{p(x)}(\Omega), \| \, . \, \|_\rho)$.

For every fixed $s \geq 0$,under condition (1.10),the function $\phi\ (.,s)$ is local integral in Ω; thus by Theorem 7.7 and 7.10 in [44],we get

Theorem 1.6 *([44])The space $(L^{p(x)}(\Omega), \| \, . \, \|_\rho)$ is separable.*

By Theorem 7.6 in [44] we have

Theorem 1.7 *([44])The set S consisting of all simple integral functions over Ω is dense in the space $(L^{p(x)}(\Omega), \| \, . \, \|_\rho)$.*

When $\Omega \subseteq \mathbb{R}^n$ is an open subset,for every element in S,we can approximate it in the means of norm $\| \cdot \|_\rho$ by the elements in $C_0^\infty(\Omega)$ through the standard method of mollifiers,so we have

Theorem 1.8 *([44])*
If $\Omega \subseteq \mathbb{R}^n$ is an open subset,then $C_0^\infty(\Omega)$ is dense in the space $(L^{p(x)}(\Omega), \| \, . \, \|_\rho)$.

We now discuss the uniform convexity of $L^{p(x)}(\Omega)$.First we give the following conclusion:

Lemma 1.1 *([26]).Let $p(x) > 1$ be bounded.Then $\phi(x,s) = s^{p(x)}$ is strongly convex with respect to s ; i.e.,for arbitrary $a \in (0,1)$,there is $\delta(a) \in (0,1)$ such that for all $s \geq 0$ and $b \in [0,a]$,the inequality holds.*

$$\phi\left(x, \frac{1+b}{2}s\right) \leq (1 - \delta\,(a))\,\frac{\phi\,(a,s) + \phi\,(x,bs)}{2} \tag{1.11}$$

Proof

We rewrite (1.11) as

$$\left(\frac{1+b}{2}\right)^{p(x)} \leq (1 - \delta(a))\frac{1 + b^{p(x)}}{2}$$

It is easy to see that for almost all $x \in \Omega$ and $b \in [0,1)$,we always have

$$\left(\frac{1-b}{2}\right)^{p(x)} < (1+b^{p(x)})/2 \tag{1.12}$$

Let

$$\theta_x(t) = \left(\frac{1+t}{2}\right)^{p(x)} / \left(1+t^{p(x)}\right)/2$$

It is not hard to prove that for almost all $x \in \Omega$,$\theta(t)$ increases strictly in $[0,1)$.We only need to prove that the inequality

$$\theta_x(a) \leq 1 - \delta(a)$$

holds.If this is not so,then we can find a sequence $\{x_n\}$ of points in Ω such that

$$\lim_{n\to\infty} \theta_{x_n}(a) = 1$$

thus we can choose a convergence subsequence $p(x_{n_j})$ of $p(x_n)$ that still verifies

$$\lim_{n_j\to\infty} \theta_{x_{n_j}}(a) = 1$$

Setting

$$p^* = \lim_{n_j\to\infty} p(x_{n_j}) \in [p^-, p^+]$$

we get

$$\left(\frac{1+a}{2}\right)^{p^*} = (1+a^{p^*})/2n_j$$

which is a contradiction.Thus we must have

$$\sup_{x\in\Omega}\theta(a) < 1$$

i.e.there is $\delta(a) \in (0,1)$ such that for almost all $x \in \Omega$,we have

$$\theta(a) \leq 1 - \delta(a)$$

This completes the proof. ■

By Lemma 1.1 and Theorem 11.6 in [44],we can get immediately

Theorem 1.9 *([44]).If $p^- > 1$,$p^+ < \infty$,then $L^{p(x)}(\Omega)$ is uniform convex and thus is reflexive.*

Now we give an imbedding result.

Theorem 1.10 *([26]).Let meas $\Omega < \infty$,$p_1(x)$,$p_2(x) \in E$,and let condition (1.10) be satisfied.Then the necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ we haue $p_1(x) \le p_2(x)$,and in this case,the imbedding is continuous.*

The norm $\| \, . \, \|_\rho$ of $L^{p(x)}(\Omega)$ defined before is usually called the Luxem-bury norm.We can introduce another norm ρ as

$$\| u \|_\rho^* = \inf \left\{ \lambda \left(1 + \rho\left(\frac{u}{\lambda}\right)\right), \lambda > 0 \right\} \tag{1.13}$$

This is called the Amemiya norm.The above two norms are equivalent; they satisfy

$$\| u \|_\rho \le \| u \|_\rho^* \le 2 \| u \|_\rho, \forall u \in L^{p(x)}(\Omega)$$

A simple calculation shows that if $p(x) = p$ is a constant and we write

$$\| u \|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{1/p}$$

then we have

$$\| u \|_\rho = \| u \|_{L^p(\Omega)}, \| u \|_\rho^* = 2 \| u \|_{L^p(\Omega)}$$

If $p^- > 1$,we can also introduce the so-called Orlicz norm as

$$\| u \|_\rho' = \| u \|_{L^p(\Omega)}' = \sup\{ \left| \int_\Omega u\,(x)\, v\,(x)\, dx \right| : \rho_{q(x)}\,(v) \le 1, v\,(x) \in L^{q(x)}\,(\Omega)\}$$

and we have

$$\| u \|_\rho \le \| u \|_\rho' \le 2 \| u \|_\rho, \forall u \in L^{p(x)}(\Omega)$$

so $\| u \|_\rho'$ is equivalent to $\| u \|_\rho$ and u p.For the norm $\| u \|_\rho$,we have the Hölder inequality

$$| \int_\Omega u(x)v(x)dx| \le \| u \|_{\rho_{p(x)}} \| v \|_{\rho_{q(x)}}', \forall u(x) \in L^{p(x)}\,(\Omega)\,, v(x) \in L^{q(x)}(\Omega)$$

and therefore we have

$$| \int_\Omega u(x)v(x)dx| \le 2 \| u \|_{\rho_{p(x)}} \| v \|_{\rho_{q(x)}}, \forall u(x) \in L^{p(x)}(\Omega), v(x) \in L^{q(x)}(\Omega)$$

Where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$$

Definition 1.1 *Let $u \in L^{p(x)}(\Omega)$,let $D \subset \Omega$ be a measurable subset,and let χ_D be the characteristic function of E.If $lim_{meas\ D \to 0} \parallel u(x)\ \chi_D(X) \parallel_\rho = 0$ then we say that u is absolutely continuous with respect to norm $\parallel . \parallel_\rho$.*

Theorem 1.11 *([26]) $u \in L^{p(x)}(\Omega)$ is absolutely continuous with respect to norm $\parallel . \parallel_\rho$.*

As $L^{p(x)}(\Omega) = \{u \in E : \forall \lambda > 0, \rho(\lambda u) < \infty\}$ for arbitrary $s > 0$,we have $\rho(\frac{u}{\varepsilon}) < \infty$.Let

$$u_n(x) = \begin{cases} u(x), & if\ |u(x)| \leq n, \\ 0, & if\ |u(x)| > n. \end{cases}$$

Then by Theorem 1.5,we can take N such that

$$\parallel u - u_N \parallel \leq \frac{\varepsilon}{2}$$

Because $u_N(x)$ is bounded,we can find $\delta > 0$ such that when meas $D < \delta$,we have

$$\parallel u_N(x)\chi_D(x) \parallel_\rho < \frac{\varepsilon}{2},$$

and thus we get

$$\parallel u(x)\chi_D(x) \parallel_\rho \leq \parallel (u - u_N(x))\chi_D(x) \parallel_\rho + \parallel u_N(x)\chi_D(x) \parallel_\rho < \varepsilon$$

Let $\alpha \in E$ and $0 < a \leq \alpha(x) \leq b < \infty$,where a and b are positive constants.Setting φ_α : $\Omega \times R^+ \to R^+$ as

$$\varphi_\alpha(x, s) = \alpha(x)\varphi(x, s) = \alpha(x)s^{p(x)}.$$

Similar to the definition of ρ and E_ρ,let

$$\rho_\alpha(u) = \int_\Omega \varphi_\alpha(x, |u(x)|)dx,$$

and

$$E_{\rho_\alpha} = \{u \in E\ :\ \lim_{\lambda \to 0^+} \rho_\alpha(\lambda u) = 0\}.$$

By

$$a\ \varphi(x,s) \le \varphi_\alpha(x,s) \le b\varphi(x,s)$$

And

$$a\ \rho(u) \le \rho_\alpha(u) \le b\rho(u)$$

We have $E_{\rho_\alpha} = E_\rho = L^{p(x)}(\Omega)$.If we define the norm $\|\ .\ \|_{\rho_\alpha}$ of E_ρ as before,

$$\|\ u\ \|_{\rho_\alpha} = \inf\ \{\lambda > 0 : \rho_\alpha(\frac{u}{\lambda}) \le 1\} \tag{1.14}$$

it is easy to see that $\|\ .\ \|_{\rho_\alpha}$ and $\|\ .\ \|_\rho$ are equivalent norms on E_ρ.

Let us begin to discuss the conjugate space of $L^{p(x)}(\Omega)$,i.e.,the space $(L^{p(x)}(\Omega))^*$ consisting of all continuous linear functionals over $L^{p(x)}(\Omega)$.We suppose that $p(x)$ satisfies condition (1.10) and $p^- > 1$.By the definition in [18,p.33] $\varphi(x,s) = s^{p(x)}$ belongs to the class Φ,and for $x \in \Omega$,φ is convex in s and satisfies

$$(0) : \lim_{s\to 0^+} \frac{\varphi\,(x,s)}{s} = 0$$

$$(\infty) : \lim_{s\to\infty} \frac{\varphi(x,s)}{s} = \infty$$

Let $\varphi_p(x,s) = \frac{1}{p(x)}s^{p(x)}$.Then φ_p also belongs to the class Φ.Writing

$$\rho_p(u) = \int_\Omega \varphi_p(x,|u(x)|)dx$$

$$\|\ u\ \|_{\rho_p} = \inf\ \{\lambda > 0 : \rho_p(\frac{u}{\lambda}) \le 1\}$$

$\|\ u\ \|_{\rho_p}$ is an equivalent norm on $L^{p(x)}(\Omega)$.Obviously,the Young's conjuga tive function of φ_p is

$$\varphi_p^*(x,s) = \frac{1}{q(x)}s^{q(x)}$$

where $q(x)$ is the conjugative function of $p(x)$,i.e.,$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.It is obvious that $(\varphi_p^*)^* = \varphi_p$,and q^-,q^+ are conjugative numbers of p^+,p^- respectively.In particular,we have $q^- > 1$ and $q^+ < \infty$.Writing

$$\rho_p^*(v) = \int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}dx = \int_\Omega \varphi_p^*(x,|v(x)|)dx$$

$E^*_{\rho_p} = \{v \in E \lim_{\lambda \to 0^+} \rho^*_p(\lambda v) = 0\}$,we have

$$E^*_{\rho_p} = L^{q(x)}(\Omega) = L^{q(x)}_0(\Omega) = \{v \in E : \int_\Omega |v(x)|^{q(x)} dx < \infty\}$$

By Corollary 13.14 and Theorem 13.17 in [44].we have

Theorem 1.12 *([44]).*$(L^{p(x)}(\Omega))^* = L^{q(x)}(\Omega)$,*i.e*

1^o) *For evey* $v \in L^{q(x)}(\Omega)$,f *defined by*

$$f(u) = \int_\Omega u(x)v(x)dx, \forall u \in L^{p(x)}(\Omega) \tag{1.15}$$

is a continuous linear functional over $L^{p(x)}(\Omega)$

2^o) *For every continuous linear functional* f *on* $L^{p(x)}(\Omega)$,*there is a unique element* $u \in L^{q(x)}(\Omega)$ *such that* f *is exactly defined by (1.15)*

From.Theorem 1.12 we can also deduce that when $p^- > 1$,$p^+ < \infty$,the space $L^{p(x)}(\Omega)$ is reflexive.We know that for Banach space $(X, \| \,.\, \|)$ the norm $\| \,.\, \|'$ on its conjugate space X^* is usually defined by the formulation

$$\| x^* \|' = \sup\{\prec x^*, x \succ : \| x \| \leq 1\} \tag{1.16}$$

where $x^* \in X^*$,$\prec x^*, x \succ = x^*(x)$,and the inequality holds.

$$| \prec x^*, x \succ | \leq \| x^* \|' \| x \|, \forall x \in X, x^* \in X^* \tag{1.17}$$

It is obvious that the norm $\| \,.\, \|'$.on X^* depends on the norm $\| x \|$ on X.Now we take $X = L^{p(x)}(\Omega)$,then $X^* = L^{q(x)}(\Omega)$.For $v \in X^*$ and $u \in X$,

$$\prec u, v \succ = \int_\Omega u(x)v(x)dx \tag{1.18}$$

If we use the norm $\| \,.\, \|_{\rho_p}$ on X,then according to Theorem 13.11 in [44],we have

$$\| v \|_{\rho^*_p} \leq \| v \|'_{\rho^*_p}, \forall v \in X^* \tag{1.19}$$

An interesting question we are concerned with is the relation between the prime norm $\| \cdot \|_{L^{q(x)}(\Omega)}$ of X^* and the norm $\| \cdot \|'_\rho$ of X^* when X is equipped with norm $\| \cdot \|_\rho$.It is well known that when $p(x)$ is a constant $p \in (1,\infty)$,the two norms defined above are exactly the same.Here we give

Theorem 1.13 *([44]) Under the aboue assumptions,for arbitrary $v \in L^{q(x)}(\Omega)$,we have*

$$\| v \|_{L^{q(x)}(\Omega)} \leq \| v \|'_\rho \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right) \| v \|_{L^{q(x)}(\Omega)} \tag{1.20}$$

Proof

For $v \in L^{q(x)}(\Omega)$,$u \in L^{p(x)}(\Omega)$,setting $\| v \|_{L^{q(x)}(\Omega)}= a$,$\| u \|_{L^{p(x)}(\Omega)}= b \leq 1$,

$$\begin{aligned}\int_\Omega \tfrac{u(x)}{b}\tfrac{v(x)}{a}dx &\leq \int_\Omega \tfrac{1}{p(x)}|\tfrac{u(x)}{b}|^{p(x)}dx + \int_\Omega \tfrac{1}{q(x)}|\tfrac{v(x)}{a}|^{q(x)}dx \\ &\leq \tfrac{1}{p^-}\int_\Omega |\tfrac{u(x)}{b}|^{p(x)}dx + \tfrac{1}{q^-}\int_\Omega |\tfrac{v(x)}{a}|^{q(x)}dx = \tfrac{1}{p^-}+\tfrac{1}{q^-}\end{aligned}$$

So we get

$$\int_\Omega u(x)v(x)dx \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right) ab \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right) a,$$

and then

$$\| v \|'_\rho \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right) \| v \|_{L^{q(x)}(\Omega)} .$$

On the other hand,for $v \in L^{q(x)}(\Omega)$ with

$$\| v \|_{L^{q(x)}(\Omega)}= a, u(x) = |\frac{v(x)}{a}|^{q(x)-1}\text{sgn } u(x).$$

Then$|u(x)|^{p(x)} = |\frac{v(x)}{a}|^{q(x)}$Thus $u(x) \in L^{p(x)}(\Omega)$.And $\| u \|_{L^{p(x)}(\Omega)}= 1$.So

$$\int_\Omega u(x)v(x)dx = \int_\Omega a|\frac{v(x)}{a}|^{q(x)}dx = a = \| v \|_{L^{q(x)}(\Omega)} .$$

This equality means that $\| v \|'_\rho \geq \| v \|_{L^{q(x)}(\Omega)}$.The proof is completed. ■

This theorem can be regarded as a generalization of conclusion (1.19).The importance of Nemytsky operators from $L^{p_1}(\Omega)$ to $L^{p_2}(\Omega)$ is well known.Here we give the basic properties of Nemytsky operators from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$.Let p_1,$p_2 \in L^\infty_+(\Omega)$.We denote by p_1,p_2 the modular corresponding to p_1 and p_2,respectively.Let $g(x,u)(x \in \Omega, u \in R)$ be a Caratheodory function,and G is the Nemytsky operator defined by g,i.e.,$(Gu)(x) = g(x,u(x))$.We have

Theorem 1.14 *([26]).If G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$,then G is continuous and bounded,and there is a constant $b \geq 0$ and a nonnegative function $a \in L^{p_2(x)}(\Omega)$ such that for $x \in \Omega$ and $u \in R$,the following inequality holds*

$$g(x,u) \leq a(x) + b|u|^{p_1(x)/p_2(x)} \tag{1.21}$$

On the other hand,if g satisfies (1.21),then G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$,and thus G is continuous and bounded.

As an application,we give an example.

Example 1.1 *Let Ω be a measurable set in $\mathbb{R}^n$ and meas$(\Omega) < \infty$,$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the condition*

$$f(x,u) \leq a(x) + b|u|^{p(x)},$$

where $p(x) \in L^{\infty}_{+}(\Omega)$,$a(x) \in L^1(\Omega)$,$a(x) \geq 0$,$b \geq 0$ is a constant.Then the functional

$$J(u) = \int_{\Omega} f(x,u(x))dx$$

defined on $L^{p(x)}(\Omega)$ is continuous and J is uniformly bounded on a bounded set in $L^{p(x)}(\Omega)$.

1.3 *Sobolev spaces with variable exponents.*

In this section we will give some basic results on the generalized Lebesgue-Sobolev space $W^{m,p(x)}(\Omega)$,where Ω is a bounded domain of $\mathbb{R}^n$,m is a positive integer and $p \in L^{\infty}_{+}(\Omega)$.$W^{m,p(x)}(\Omega)$ is defined as

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \ : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq m\}$$

$W^{m,p(x)}(\Omega)$ is a special class of socalled generalized Orlicz-Sobolev spaces.

For $p(x) = 2$,we have

$$H^m(\Omega) = W^{m,2}(\Omega).$$

For $m = 0$,we have

$$W^{0,p(x)}(\Omega) = L^{p(x)}(\Omega).$$

We define the subspace $W_0^{m,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$:

$$W_0^{m,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p(x)}(\Omega)}$$

We call $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$,which we also note :

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} = W_0^{1,2}(\Omega)$$

From [35].we know that $W^{m,p(x)}(\Omega)$ can be equipped with the norm $\|u\|_{W^{m,p(x)}(\Omega)}$ as Banach spaces,where

$$\|u\|_{W^{m,p(x)}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}$$

According to [36] and Theorem 1.9 in Section 2,we already have

Theorem 1.15 *([26]).$W^{m,p(x)}(\Omega)$ is separable and reflexive.*

Theorem 1.16 *([44]) When $p^- > 1$,the function spaces $W_0^{1,p(x)}(\Omega)$ is reflexive uniformly convex Banach spaces.Moreover,for any measurable bounded exponent p,the spaces $W_0^{1,p(x)}(\Omega)$ is separable*

An immediate consequence of Theorem 1.7

Theorem 1.17 *([26]).Assume that $p_1(x)$,$p_2(x) \in L_+^\infty(\Omega)$.If $p_1(x) \leq p_2(x)$,then $W^{m,p_2(x)}(\Omega)$ can be imbedded into $W^{m,p_1(x)}(\Omega)$ continuously.*

Now let us generalize the well known Sobolev imbedding theorem of $W^{m,p}(\Omega)$ to $W^{m,p(x)}(\Omega)$.We have

Theorem 1.18 *([26]).Let $p,q \in C(\overline{\Omega})$ and $p,q \in L_+^\infty(\Omega)$.Assume that*

$$mp\,(x) < n, q(x) < \frac{np(x)}{n - mp(x)}, \forall x \in \overline{\Omega}.$$

Then there is a continuous and compact imbedding $W^{m,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

Remark 1.1 *We do not known whether we have the imbedding*

$$W^{\mathbf{m},\mathbf{p}(\mathbf{x})}(\Omega) \to L^{\mathbf{p}^*(\mathbf{x})}(\Omega)$$

but if the assumption on $p(x)$ is not satisfied,we cannot have it.

Example 1.2 *Let* $\Omega = \{x = (x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\} \subseteq \mathbb{R}^2$,$p(x) = 1 + x_2$,$u(x) = (2 + x_2)^{1/(1+x_2)}$*; then we have* $u(x) \in W^{1,p(x)}(\Omega)$ *and* $p^*(x) = 2(1 - x_2)/(1 - x_2)$.*It is easy to test that* $u \notin L^{p^*(x)}(\Omega)$.

1.4 *Maximum principle.*

The maximum principle is one of the most useful and best known tools employed in the study of partial differential equations.The maximum principle enables us to obtain information about the uniqueness,approximation,boundedness and symmetry of the solution,the bounds for the first eigenvalue,for quantities of physical interest (maximum stress,the torsional stiffness,electrostatic capacity,charge density etc),the necessary conditions of solvability for some boundary value problems,etc.

The first subsection specializes the maximum principle for partial differential equations to the one variable case.We present the one dimensional classical maximum principle and a new extension.In subsection two,we present the classical maximum principle of Hopf for elliptic operators and some possible extensions

1.4.1 *The one dimensional case*

The one dimensional maximum principle represents a generalization of the following simple result: *Let the smooth function u satisfy the inequality* $u'' \geq 0$ *in* $\Omega = (\alpha, \beta)$.*Then the maximum of* u *in* Ω *occurs on* $\partial\Omega = \{\alpha, \beta\}$ *(on the boundary of* Ω*)*,i.e.,

$$\max_{\overline{\Omega}} u = \max\{u(\alpha), u(\beta)\}.$$

Theorem 1.19 (*one dimensional weak maximum principle*) *([16]) Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *be a nonconstant function satisfying* $Lu \equiv u'' + b(x)u' \geq 0$ *in* Ω,*with* b *bounded in closed subintervals of* Ω.*Then,*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Drawing the graph of a function u satisfying $u'' \geq 0$ ($u'' \neq 0$) reveals us the interesting fact that at a point on $\partial\Omega$ (where u attains its maximum),the slope of u is nonzero.More precisely,$\frac{du}{dn} > 0$ at such a point.Here $\frac{d}{dn}$ denotes the outward derivative on $\partial\Omega$,i.e.,

$$\frac{du}{dn}(\alpha) = -u^{'}(\alpha), \frac{du}{dn}(\beta) = u^{'}(\beta).$$

The next theorem is an extension of this result:

Theorem 1.20 (***one dimensional strong maximum principle***) *([16]).Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a nonconstant function satisfying $Lu \equiv u^{''} + b(x)u^{'} + c(x)u \geq 0$ in Ω,with b and c bounded in closed subintervals of Ω and $c \leq 0$ in Ω.Then a nonnegative maximum can occur only on $\partial\Omega$,and $du/dn > 0$ there.If $c \equiv 0$ in Ω then,u takes its maximum on $\partial\Omega$ and $du/dn > 0$ there.*

The following simple counterexample shows that we have to impose some restrictions to c: The function $u(x) = e^{-x} \sin x$ satisfies
$Lu \equiv u^{''} + 2u^{'} + 3u \geq 0$ in $\Omega = (0, \pi)$.

1.4.2 *The* n *dimensional case*

In this subection,we treat the n dimensional variants of results presented in section 1,some possible extensions for nonlinear equations and for equations for higher order as well as their applications.We consider the linear operator (summation convention is assumed,i.e.,summation from 1 to n is understood on repeated indices)

$$Lu = a^{ij}(x)u_{ij}u + b^i(x)u_i + c(x)u, a^{ij}(x) = a^{ji}(x),$$

where $x = (x_1, x_n) \in \Omega$,$\Omega$ is a bounded domain (unless otherwise stated) of B^n,$n \geq 1$ and

$$u_i = \frac{\partial u}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

The operator L is called elliptic at a point $x \in \Omega$ if the matrix $[a^{ij}(x)]$ is positive,i.e.,if $\lambda(x)$ and $\Lambda(x)$ denote respectively the minimum and maximum eigenvalues of $[a^{ij}(x)]$,then

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2,$$

for all $\xi = (\xi_1, \xi_n) \in B^n - \{0\}$.If $\lambda \geq 0$,then L is elliptic in Ω,If Λ/λ is bounded in Ω,then L is called uniformly elliptic in Ω.

Theorem 1.21 (*weak maximum principle*) *([29],Theorem 3.1).Let L be elliptic in* Ω*.Suppose that* $|b^i|/\lambda < +\infty$ *in* Ω*,*$i = 1,n$*.If* $Lu \geq 0$ *in* Ω*,*$c = 0$ *in* Ω *and* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$*,then the maximum of* u *in* $\overline{\Omega}$ *is achieved on* $\partial\Omega$*,that is:*

$$\max_{\overline{\Omega}} u \ = \max_{\partial\Omega} u$$

Remark 1.2 *Theorem 1.20 holds under the weaker hypothesis: the matrix* $[[a^{\mathbf{i}j}]$ *is nonnegative and the ratio* $|b^{\mathbf{k}}|/a^{\mathbf{k}k}$ *is locally bounded for some* $k \in \{1, n\}$*.*

Theorem 1.22 (*the strong maximum principle of E. Hopf*) *([34])Let L be uniformly elliptic,*$c = 0$ *and* $Lu \geq 0$ *in* Ω *(not necessarily bounded),where* $u \in C^2$ *Then,if* u *attains its maximum in the interior of* Ω*,then* u *is constant.If* $c \leq 0$ *and* c/λ *is bounded then* u *cannot attain a nonnegative maximum in the interior of* Ω*,unless* u *is constant.*

1.5 *Eigenvalue problem.*

Definition 1.2 (an eigenvalue) *We say that* $u \in W_0^{1,p}(\Omega)$, $u \neq 0$,*is an eigenfunction of the operator* $-\triangle_p u$ *if*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx = \lambda \int_\Omega |u|^{p-2} u.\varphi dx \tag{1.22}$$

for all $\varphi \in C_0^\infty(\Omega)$*.The corresponding real number* λ *is called eigenvalue.*

Let λ_1 defined by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{1.23}$$

Equivalent

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p dx; \int_\Omega |u|^p dx = 1, u \in W_0^{1,p}(\Omega), u \neq 0 \right\}$$

λ_1 is the first eigenvalue of the p-Laplacian operator with zero Dirichlet conditions at the boundary.

Lemma 1.2 *λ_1 is isolated,then there exists $\delta > 0$ such that in an interval $(\lambda_1, \lambda_1 + \delta)$,there does not exist another eigenvalue of (1.22)*

Lemma 1.3 *a) Let $p \geq 2$,then for all $\xi_1, \xi_2 \in \mathbb{R}^n$*

$$|\xi_2|^p \geq |\xi_1|^p + p\,|\xi_1|^{p-2} \langle \xi_1, \xi_2 - \xi_1 \rangle + C\,(p)\,|\xi_1 - \xi_2|^p\,,$$

b) Let $p < 2$,then for all $\xi_1, \xi_2 \in \mathbb{R}^n$

$$|\xi_2|^p \geq |\xi_1|^p + p\,|\xi_1|^{p-2} \langle \xi_1, \xi_2 - \xi_1 \rangle + C\,(p)\,\frac{|\xi_1 - \xi_2|^p}{(|\xi_2| + |\xi_1|)^{2-p}},$$

where $C(p)$ is a component dependent only on p.

Lemma 1.4 *The first eigenvalue $\lambda_1$is simple.i.e if $u.v$ are two eigenfunctions associated with λ_1,then,there exists k such that: $u = kv$*

Lemma 1.5 *Let u be an eigenfunction associated with the eigenvalue λ_1,then u does not change sign on Ω,moreover if $u \in C^{1,\alpha}$,then it does not vanish on $\overline{\Omega}$.*

CHAPTER 2

Existence of positive weak solutions for a class of Kirrchoff elliptic systems with multiple parameters.

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2.1 *Introduction.*

In this chapter,we consider the following system of differential equations

$$\begin{cases} -A\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u), & \text{in } \Omega, \\ -B\left(\int_\Omega |\nabla v|^2 dx\right)\Delta v = \lambda_2 \gamma(x) g(u) + \mu_2 \eta(x) \tau(v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$.
A,B : $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions,$\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$,$\lambda_1, \lambda_2, \mu_1$,and μ_2 are nonnegative parameters.

Since the first equation in (2.1) contains an integral over Ω,it is no longer a pointwise identity; therefore it is often called nonlocal problem.This problem models several physical and biological systems,where u describes a process which depends on the average of itself,such as the population density,see [56].

In recent years,problems involving Kirchhoff type operators have been studied in many papers,we refer to ([4 - 9],in which the authors have used different methods to get the existence of solutions for (2.1) in the single equation case.In the papers ([48]; [56]),Z.Zhang et al.studied the existence of nontrivial singn-changing solutions for system (2.1) where $A(t) = B(t) = 1$ via sub-supersolution method.Our work is motivated by the recent results in ([6],[7],[28],[37]).In the papers [7] (Theorem 2),Azzouz and Bensedik studied the existence of a positive solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^{p-2}u + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $p > 1$,*i.e.*the nonlinear term at infinity.f is a sign-changing function.Using the sub-supersolution method combining a comparison principle introduced in [6],the authors established the existence of a positive solution for (2.2) where the parameter $\lambda > 0$ is small enough.In the present chapter,we consider system (2.1) in the case when the nonlinearities are " sublinear" at infinity,see the condition (H_3).We are inspired by the ideas in the interesting paper [28],in which the authors considered system (2.1) in the case $A(t) = B(t) = 1$.More precisely,under suitable conditions on f,g,we shall show that system (2.1) has a positive solution for $\lambda > \lambda^*$ large enough.To our best knowledge,this is a new research topic for nonlocal problems,see [37].In current chapter,motivated by previous works in ([7] and [28]) and by using sub-super solutions method,we study the existence of weak positive solution for a class of Kirrchoff elliptic systems in bounded domains with multiple parameters.

2.2 *Definitions and theories*

Lemma 2.1 (Comparison principle) *[6]Assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function satisfying*

$$M(s)>m_0, for\ all\ s\geq s_0. \tag{2.3}$$

Assume that u,v are two non-negative functions such that

$$\begin{cases} -\mathrm{M}\left(\int_\Omega |\nabla \mathrm{u}|^2 \mathrm{dx}\right)\Delta \mathrm{u} \geq -\mathrm{M}\left(\int_\Omega |\nabla \mathrm{v}|^2 \mathrm{dx}\right)\Delta \mathrm{v}, & \text{in } \Omega,\\ \mathrm{u}=\mathrm{v}=0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then

$$u\geq v\ ,in\ \ \overline{\Omega}$$

Proof

Suppose further that the function $H(t)=tM(t^2)$,$t\geq 0$ is an increasing on $\mathbb{R}^+$.We follow along the lines of Alves' work in [6].Multiplying both sides of the inequality by v and u and integrating,we get

$$\frac{M(\parallel u\parallel^2)\parallel u\parallel^2}{M(\parallel v\parallel^2)}\geq (u,v)\geq \frac{M(\parallel v\parallel^2)\parallel v\parallel^2}{M(\parallel u\parallel^2)}$$

and so

$$M(\parallel u\parallel^2)\parallel u\parallel\geq M(\parallel v\parallel^2)\parallel v\parallel$$

i.e

$$H(\parallel u\parallel)\geq H(\parallel v\parallel).$$

Since H is increasing,we obtain

$$\parallel u\parallel\geq\parallel v\parallel$$

Then

$$M(\parallel u\parallel^2)\leq M(\parallel v\parallel^2) \tag{2.5}$$

Because M is nonincreasing.On the other hand,by application of the maximum principle to (2.2),we get

$$M(\| u \|^2)u \geq M(\| v \|^2)v.$$

This with (2.5),yield $u \geq v$.This ends the proof ■

We give the following two definitions before we give our main result.

Definition 2.1 (weak solution) *Let $(u,v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$,$(u,v)$ is said a weak solution of (2.1) if it satisfies for all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$*

$$\begin{cases} A\left(\int_\Omega |\nabla u|^2 dx\right) \int\limits_\Omega \nabla u \nabla \phi dx = \lambda_1 \int\limits_\Omega \alpha(x) f(v) \phi dx + \mu_1 \int\limits_\Omega \beta(x) h(u) \phi dx \ in\ \Omega, \\ B\left(\int_\Omega |\nabla v|^2 dx\right) \int\limits_\Omega \nabla v \nabla \psi dx = \lambda_2 \int\limits_\Omega \gamma(x) g(u) \psi dx + \mu_2 \int\limits_\Omega \eta(x) \tau(v) \psi dx \ in\ \Omega. \end{cases}$$

Definition 2.2 (weak subsolution and supersolution) *A pair of nonnegative functions $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ in $(H_0^1(\Omega) \times H_0^1(\Omega))$ are called a weak subsolution and supersolution of (2.1) if they satisfy $(\underline{u},\underline{v}), (\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$.For all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.*

$$\begin{cases} A\left(\int\limits_\Omega |\nabla \underline{u}|^2 dx\right) \int\limits_\Omega \nabla \underline{u} \nabla \phi dx \leq \lambda_1 \int\limits_\Omega \alpha(x) f(\underline{v}) \phi dx + \mu_1 \int\limits_\Omega \beta(x) h(\underline{u}) \phi dx \quad in\ \Omega, \\ B\left(\int\limits_\Omega |\nabla \underline{v}|^2 dx\right) \int\limits_\Omega \nabla \underline{v} \nabla \psi dx \leq \lambda_2 \int\limits_\Omega \gamma(x) g(\underline{u}) \psi dx + \mu_2 \int\limits_\Omega \eta(x) \tau(\underline{v}) \psi dx \quad in\ \Omega, \end{cases}$$

and

$$\begin{cases} A\left(\int\limits_\Omega |\nabla \overline{u}|^2 dx\right) \int\limits_\Omega \nabla \overline{u} \nabla \phi dx \geq \lambda_1 \int\limits_\Omega \alpha(x) f(\overline{v}) \phi dx + \mu_1 \int\limits_\Omega \beta(x) h(\overline{u}) \phi dx \ in\ \Omega, \\ B\left(\int\limits_\Omega |\nabla \overline{v}|^2 dx\right) \int\limits_\Omega \nabla \overline{v} \nabla \psi dx \geq \lambda_2 \int\limits_\Omega \gamma(x) g(\overline{u}) \psi dx + \mu_2 \int\limits_\Omega \eta(x) \tau(\overline{v}) \psi dx \ in\ \Omega. \end{cases}$$

2.3 *Existence of positive weak solutions.*

In this section,we shall state and prove the main result of this chapter.Let us assume the following assumptions

(H_1) $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exists $a_i, b_i > 0$, $i = 1, 2$,such that $a_1 \leq A(t) \leq a_2$,$b_1 \leq B(t) \leq b_2$ for all $t \in \mathbb{R}^+$.

(H_2) $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$ *and for all* $x \in \Omega$
$\alpha(x) \geq \alpha_0 > 0$,$\beta(x) \geq \beta_0 > 0$,$\gamma(x) \geq \gamma_0 > 0$,$\eta(x) \geq \eta_0 > 0$.

(H_3) f, g, h, *and* τ *are continuous on* $[0, +\infty[$, C^1 *on* $(0, +\infty)$, *and increasing functions such that*
$$\begin{cases} \lim_{t\to+\infty} f(t) = +\infty, \lim_{t\to+\infty} g(t) = +\infty \\ \lim_{t\to+\infty} h(t) = +\infty, \lim_{t\to+\infty} \tau(t) = +\infty \end{cases}$$

(H_4) *It holds that* $\lim_{t\to+\infty} \frac{f(K(g(t)))}{t} = 0$, for all $K > 0$.

(H_5) $\lim_{t\to+\infty} \frac{h(t)}{t}, \lim_{t\to+\infty} \frac{\tau(t)}{t} = 0.$

Theorem 2.1 *([10])Assume that conditions* $(H_1) - (H_5)$ *hold,and* M *is a nonincreasing function atisfying (2.3).Then for* $\lambda_1\alpha_0 + \mu_1\beta_0$ *and* $\lambda_2\gamma_0 + \mu_2\eta_0$ *are large then problem (2.1) has a large positive weak solution.*

Proof

Let σ be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\| = 1$.Let $k_0, m_0, \delta > 0$ such that $f(t)$,$g(t)$,$h(t)$, $\tau(t) \geq -k_0$ for all $t \in \mathbb{R}^+$ and $|\nabla\phi_1|^2 - \sigma\phi_1^2 \geq m_0$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$.For each $\lambda_1\alpha_0 + \mu_1\beta_0$ and $\lambda_2\gamma_0 + \mu_2\eta_0$ large,let us define

$$\begin{cases} \underline{u} = \left(\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{2m_0a_1}\right)\phi_1^2 \\ \underline{v} = \left(\frac{(\lambda_2\gamma_0+\mu_2\eta_0)k_0}{2m_0b_1}\right)\phi_1^2 \end{cases},$$

where a_1, b_1 are given by the condition (H_1).We shall verify that $(\underline{u}, \underline{v})$ is a subsolution of problem (2.1) for $\lambda_1\alpha_0 + \mu_1\beta_0$ and $\lambda_2\gamma_0 + \mu_2\eta_0$ large enough.Indeed,let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$

in Ω.By $(H_1)-(H_3)$,a simple calculation shows that

$$\begin{aligned} A\left(\int_{\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\int_{\overline{\Omega}_\delta}\nabla \underline{u}.\nabla\phi dx &= A\left(\int_{\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}\int_{\overline{\Omega}_\delta}\phi_1\nabla\phi_1.\nabla\phi dx \\ &= \frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}A\left(\int_{\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right) \\ &\times\left\{\int_{\overline{\Omega}_\delta}\nabla\phi_1\nabla(\phi_1\phi)\,dx-\int_{\overline{\Omega}_\delta}|\nabla\phi_1|^2\phi dx\right\} \\ &= \frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0a_1}A\left(\int_{\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\int_{\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx. \end{aligned}$$

On $\overline{\Omega}_\delta$ we have $|\nabla\phi_1|^2-\sigma\phi_1^2\geq m_0$,then by $(H_3): f(\underline{v}), h(\underline{u}), g(\underline{u}), \tau(\underline{v})\geq\frac{k_0}{m_0}$ that

$$\begin{aligned} A\left(\int_{\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\int_{\overline{\Omega}_\delta}\nabla \underline{u}\nabla\phi dx &\leq \frac{(\lambda_1\alpha_0+\mu_1\beta_0)k_0}{m_0}\int_{\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx \\ &\leq \lambda_1\int_{\overline{\Omega}_\delta}\alpha(x)f(\underline{v})\phi dx+\mu_1\int_\Omega\beta(x)h(\underline{u})\phi dx. \end{aligned} \tag{2.6}$$

Next,on $\Omega\backslash\overline{\Omega}_\delta$ we have $\phi_1\geq r$ for some $r>0$.and therefor by the conditions $(H_1)-(H_3)$ and the definition of $\underline{v}$,it follows that for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough.

$$\begin{aligned} \lambda_1\int_\Omega\alpha(x)f(\underline{v})\phi dx+\mu_1\int_\Omega\beta(x)h(\underline{u})\phi dx &\geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0a_2}{m_0a_1}\sigma\int_{\Omega\backslash\overline{\Omega}_\delta}\phi dx \\ &\geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right).\sigma\int_{\Omega\backslash\overline{\Omega}_\delta}\phi dx \\ &\geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right) \\ &\times\int_{\Omega\backslash\overline{\Omega}_\delta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx \\ &= A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\nabla \underline{u}\nabla\phi dx \end{aligned}$$

So

$$\lambda_1\int_\Omega\alpha(x)f(\underline{v})\phi dx+\mu_1\int_\Omega\beta(x)h(\underline{u})\phi dx\geq A\left(\int_{\Omega\backslash\overline{\Omega}_\delta}|\nabla \underline{u}|^2dx\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\nabla \underline{u}\nabla\phi dx \tag{2.7}$$

Relation (2.6) and (2.7) imply that

$$A\left(\int_\Omega|\nabla \underline{u}|^2dx\right)\int_\Omega\nabla \underline{u}\nabla\phi dx\leq\lambda_1\int_\Omega\alpha(x)f(\underline{v})\phi dx+\mu_1\int_\Omega\beta(x)h(\underline{u})\phi dx \tag{2.8}$$

for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough and any $\phi\in H_0^1(\Omega)$ with $\phi\geq 0$ in Ω. Similarly,

$$B\left(\int_\Omega|\nabla \underline{v}|^2dx\right)\int_\Omega\nabla \underline{v}\nabla\psi dx\leq\lambda_2\int_\Omega\gamma(x)g(\underline{u})\psi dx+\mu_2\int_\Omega\eta(x)\tau(\underline{v})\psi dx \tag{2.9}$$

for $\lambda_2\gamma_0+\mu_2\eta_0>0$ large enough and any $\psi\in H_0^1(\Omega)$ with $\psi\geq 0$ in Ω.From (2.8) and (2.9),$(\underline{u},\underline{v})$ is a subsolution of problem (2.1).Moreover,we have $\underline{u}>0$ and $\underline{v}>0$ in Ω,$\underline{u}\to+\infty$ and $\underline{v}\to+\infty$

as $\lambda_1\alpha_0+\mu_1\beta_0 \rightarrow +\infty$ and $\lambda_2\gamma_0+\mu_2\eta_0 \rightarrow +\infty$. Next We shall construct a supersolution of problem (2.1).Let e be the solution of the following problem

$$\begin{cases} -\Delta e= 1 & \text{in } \Omega \\ e= 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

Let

$$\begin{cases} \overline{u}=Ce \\ \overline{v}=\left(\frac{\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty}{b_1}\right)\left[g\left(C\|e\|_\infty\right)\right]e \end{cases}$$

where e is given by (2.10) and $C>0$ is a large positive real number to be chosen later.We shall verify that $(\overline{u},\overline{v})$ is a supersolution of problem (2.1) Let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω.Then we obtain from (2.10) and the condition (H_1) that

$$\begin{aligned} A\left(\int_\Omega |\nabla\overline{u}|^2dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi dx &= A\left(\int_\Omega |\nabla\overline{u}|^2dx\right)C\int_\Omega \nabla\omega.\nabla\phi dx \\ &= A\left(\int_\Omega |\nabla\overline{u}|^2dx\right)C\int_\Omega \phi dx \\ &\geq a_1C\int_\Omega \phi dx. \end{aligned}$$

By (H_4) and (H_5),we can choose C large enough so that

$$a_1C \geq \lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty}{b_1}g\left(C\|e\|_\infty\right)\|e\|_\infty\right)+\mu_1\|\beta\|_\infty h\left(C\|e\|_\infty\right).$$

Therefore

$$\begin{aligned} A\left(\int_\Omega |\nabla\overline{u}|^2dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi dx &\geq \left[\lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty}{b_1}g\left(C\|e\|_\infty\right)\|e\|_\infty\right)+\mu_1\|\beta\|_\infty h\left(C\|e\|_\infty\right)\right] \\ &\times\int_\Omega \phi dx \\ &\geq \lambda_1\|\alpha\|_\infty\int_\Omega f\left(\left[\frac{\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty}{b_1}\right]g\left(C\|e\|_\infty\right)\|e\|_\infty\right)\phi dx \\ &+\mu_1\int_\Omega h\left(C\|e\|_\infty\right)\phi dx \\ &\geq \lambda_1\int_\Omega \alpha(x)f(\overline{v})\phi dx+\mu_1\int_\Omega \beta(x)h(\overline{u})\phi dx. \end{aligned}$$

So

$$A\left(\int_\Omega |\nabla\overline{u}|^2dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi dx \geq \lambda_1\int_\Omega \alpha(x)f(\overline{v})\phi dx+\mu_1\int_\Omega \beta(x)h(\overline{u})\phi dx. \tag{2.11}$$

Also

$$B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx \geq (\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty)\int_\Omega g\left(C\|e\|_\infty\right)\psi dx \\ =\lambda_2\int_\Omega \gamma\left(x\right)g\left(\overline{u}\right)\psi dx+\mu_2\int_\Omega \eta\left(x\right)g\left(C\|e\|_\infty\right)\psi dx \tag{2.12}$$

Again by (H_4) and (H_5) for C large enough we have

$$g\left(C\|e\|_\infty\right)\geq \tau\left[\frac{(\lambda_2\|\gamma\|_\infty+\mu_2\|\eta\|_\infty)}{b_1}g\left(C\|e\|_\infty\right)\|e\|_\infty\right]\geq \tau\left(\overline{v}\right). \tag{2.13}$$

From (2.12) and (2.13),we have

$$B\left(\int_\Omega |\nabla\overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx\geq \lambda_2\int_\Omega \gamma\left(x\right)g\left(\overline{u}\right)\psi dx+\mu_2\int_\Omega \eta\left(x\right)\tau\left(\overline{v}\right)\psi dx. \tag{2.14}$$

From (2.11) and (2.14) we have$(\overline{u},\overline{v})$ is a subsolution of problem (2.1) with $u\leq\overline{u}$ and $v\leq\overline{v}$ for C large.To obtain a weak solution of problem (2.1) we shall use the arguments by Azzouz and Bensedik [7] (observe that f,g,h,and τ does not depend on x).For this purpose,we define a sequence $(u_n\ ,v_n)\in(H_0^1(\Omega)\times H_0^1(\Omega))$ as follows: $u_0:=\overline{u}, v_0=\overline{v}$ and (u_n,v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2 dx\right)\Delta u_n=\lambda_1\alpha\left(x\right)f\left(v_{n-1}\right)+\mu_1\beta\left(x\right)h\left(u_{n-1}\right) & \text{in } \Omega,\\ -B\left(\int_\Omega |\nabla v_n|^2 dx\right)\Delta v_n=\lambda_2\gamma\left(x\right)g\left(u_{n-1}\right)+\mu_2\eta\left(x\right)\tau\left(v_{n-1}\right) & \text{in } \Omega,\\ u_n=v_n=0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Problem (2.15) is $(A,B)-$linear in the sense that,if $(u_{n-1},v_{n-1})\in(H_0^1(\Omega)\times H_0^1(\Omega))$ is a given,the right hand sides of (2.15) is independent of u_n,v_n.Set $A(t)=tA(t^2)$,$B(t)=tB(t^2)$.Then since $A(\mathbb{R})=\mathbb{R}$, $B(\mathbb{R})=\mathbb{R}$,$f(v_{n-1})$,$h(u_{n-1})$, $g(u_{n-1})$,and $\tau(v_{n-1})\in L^2(\Omega)$,we deduce from a result in [6] that system (2.15) has a unique solution $(u_n,v_n)\in(H_0^1(\Omega)\times H_0^1(\Omega))$.By using (2.15) and the fact that (u_0,v_0) is a supersolution of (2.1),we have

$$\begin{cases} -A\left(\int_\Omega |\nabla u_0|^2 dx\right)\Delta u_0\geq\lambda_1\alpha\left(x\right)f\left(v_0\right)+\mu_1\beta\left(x\right)h\left(u_0\right)=-A\left(\int_\Omega |\nabla u_1|^2 dx\right)\Delta u_1,\\ -B\left(\int_\Omega |\nabla v_0|^2 dx\right)\Delta v_0\geq\lambda_2\gamma\left(x\right)g\left(u_0\right)+\mu_2\eta\left(x\right)\tau\left(v_0\right)=-B\left(\int_\Omega |\nabla v_1|\, dx\right)\Delta v_1, \end{cases}$$

and by Lemma 2.1,$u_0 \geq u_1$ and $v_0 \geq v_1$.Also,since $u_0 \geq \underline{u}$,$v_0 \geq \underline{v}$ and the monotonicity of f,h,g,and τ one has

$$\begin{aligned}
-A\left(\int_\Omega |\nabla u_1|^2 dx\right)\Delta u_1 &= \lambda_1 \alpha(x) f(v_0) + \mu_1 \beta(x) h(u_0)\\
&\geq \lambda_1 \alpha(x) f(\underline{v}) + \mu_1 \beta(x) h(\underline{u})\\
&\geq -A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\Delta \underline{u},
\end{aligned}$$

$$\begin{aligned}
-B\left(\int_\Omega |\nabla v_1|^2 dx\right)\Delta v_1 &= \lambda_2 \gamma(x) g(u_0) + \mu_2 \eta(x) \tau(v_0)\\
&\geq \lambda_2 \gamma(x) g(\underline{u}) + \mu_2 \eta(x) \tau(\underline{v})\\
&\geq -B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\Delta \underline{v},
\end{aligned}$$

from which,according to Lemma 2.1,$u_1 \geq \underline{u}$,$v_1 \geq \underline{v}$.for u_2, v_2 we write

$$\begin{aligned}
-A\left(\int_\Omega |\nabla u_1|^2 dx\right)\Delta u_1 &= \lambda_1 \alpha(x) f(v_0) + \mu_1 \beta(x) h(u_0)\\
&\geq \lambda_1 \alpha(x) f(v_1) + \mu_1 \beta(x) h(u_0)\\
&= -A\left(\int_\Omega |\nabla u_2|^2 dx\right)\Delta u_2,
\end{aligned}$$

and

$$\begin{aligned}
-B\left(\int_\Omega |\nabla v_1| dx\right)\Delta v_1 &= \lambda_2 \gamma(x) g(u_0) + \mu_2 \eta(x) \tau(v_0)\\
&\geq \lambda_1 \alpha(x) g(u_1) + \mu_2 \beta(x) \tau(v_1)\\
&= -B\left(\int_\Omega |\nabla v_2|^2 dx\right)\Delta v_2,
\end{aligned}$$

and then $u_1 \geq u_2$,$v_1 \geq v_2$.Similarly,$u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned}
-A\left(\int_\Omega |\nabla u_2|^2 dx\right)\Delta u_2 &= \lambda_1 \alpha(x) f(v_1) + \mu_1 \beta(x) h(u_1)\\
&\geq \lambda_1 \alpha(x) f(\underline{v}) + \mu_1 \beta(x) h(\underline{u})\\
&\geq -A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\Delta \underline{u}
\end{aligned}$$

$$\begin{aligned}
-B\left(\int_\Omega |\nabla v_2|^2 dx\right)\Delta v_2 &= \lambda_2 \gamma(x) g(u_1) + \mu_2 \eta(x) \tau(v_1)\\
&\geq \lambda_2 \gamma(x) g(\underline{u}) + \mu_2 \eta(x) \tau(\underline{v})\\
&\geq -B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\Delta \underline{v}
\end{aligned}$$

Repeating this argument we get a bounded monotone sequence $(u_n\ , v_n) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots \geq \underline{u} > 0 \tag{2.16}$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq \ldots \geq v_n \geq \ldots \geq \underline{v} > 0 \tag{2.17}$$

Using the continuity of the functions f, h,g,and τ and the definition of the sequences u_n , v_n ,there exist constants C_i> 0,i= 1, ..., 4 independent of n such that

$$|f(v_{n-1})| \leq C_1, |h(u_{n-1})| \leq C_2, |g(u_{n-1})| \leq C_3 \tag{2.18}$$

and $|\tau(u_{n-1})| \leq C_4$ for all n.From (2.18),multiplying the first equation of (2.15) by u_n,integrating,using the Hölder inequality we can show that

$$\begin{aligned} a_1 \int_\Omega |\nabla u_n|^2 dx &\leq A\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega |\nabla u_n|^2 dx \\ &= \lambda_1 \int_\Omega \alpha(x) f(v_{n-1}) u_n dx + \mu_1 \int_\Omega \beta(x) h(u_{n-1}) u_n dx \\ &\leq \lambda_1 \|\alpha\|_\infty \int_\Omega |f(v_{n-1})| |u_n| dx + \mu_1 \|\beta\|_\infty \int_\Omega |h(u_{n-1})| |u_n| dx \\ &\leq C_1 \lambda_1 \int_\Omega |u_n| dx + C_2 \mu_1 \int_\Omega |u_n| dx \leq C_5 \|u_n\|_{H_0^1(\Omega)} \end{aligned}$$

or

$$\|u_n\|_{H_0^1(\Omega)} \leq C_5, \forall n \in \mathbb{N} \tag{2.19}$$

where C_5> 0 is a constant independent of n.Similarly,there exist C_6> 0 independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_6, \forall n \in \mathbb{N}. \tag{2.20}$$

From (2.19) and (2.20),we infer that (u_n, v_n) has a subsequence which weakly converges in $H_0^1(\Omega, \mathbb{R}^2)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$.Being monotone and also using a standard regularity argument,(u_n, v_n) converges itself to (u, v).Now,letting $n \to +\infty$ in (2.17),we deduce that (u, v) is a positive solution of system (2.1).The proof of theorem is now completed. ■

CHAPTER 3

Existence of positive solutions for nonlocal elliptic systems.

3.1 *Introduction*

The study of differential equations and variational problems with nonstandard $p(x)-$growth conditions is a new and interesting topic.It arises from nonlinear elasticity theory,electrorheological fluids,etc.(see [2],[8],[12]).Many existence results have been obtained on this kind of problems.In [14],[20],[24],[22],[23],X.L.Fan et al.studied the regularity of solutions for differential equations with nonstandard $p(x)$-growth conditions.

In this chapter,we are interested in the $p(x)$-Kirchhoff systems of the form

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda^{p(x)}\left[\lambda_1 a\left(x\right)f\left(v\right)+\mu_1 c\left(x\right)h\left(u\right)\right] & \text{in } \Omega, \\ -M\left(\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\Delta_{p(x)}v=\lambda^{p(x)}\left[\lambda_2 b\left(x\right)g\left(u\right)+\mu_2 d\left(x\right)\tau\left(v\right)\right] & \text{in } \Omega, \\ u=v=0 & \text{on}\partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega\subseteq\mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega, 1<p\in C^1\left(\overline{\Omega}\right)$ is a functions with $1<p^-:=\inf_\Omega p\left(x\right)\leq p^+:=\sup_\Omega p\left(x\right)<\infty$,and $\Delta_{p(x)}u=div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called $p(x)-$Laplacian,and $M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)$ is called the Kirchhoff operator where it satisfies the condition

(H_1) $M:[0,+\infty)\to[m_0,\infty]$ is a continuous and increasing function with $m_0>0$.

$\lambda,\lambda_1,\lambda_2,\mu_1$,and μ_2 are positive parameters,and f,g,h,τ are monotone functions in $[0,+\infty[$ such that

$$\lim_{u\to+\infty}f\left(u\right)=\lim_{u\to+\infty}g\left(u\right)=\lim_{u\to+\infty}h\left(u\right)=\lim_{u\to+\infty}\tau\left(u\right)=+\infty$$

and satisfying some natural growth condition at $u=\infty$.

In this chapter,motivated by the ideas introduced in ([5]) and the properties of Kirchhoff type operators in [33],we study the existence of positive solutions for system (3.1) by using the sub- and super solutions techniques.To our best knowledge,this is a new research topic for nonlocal problems.The remainder of this chapter is organized as follows.In Section 2,we present properties of $p(x)-$Kirchhoff$-$Laplace operator.In Section 3 is devoted to state and prove the main result.

3.2 *Properties of* p(x)$-$*Kirchhoff*$-$*Laplace operator*

In this section,we discuss the $p(x)-$Kirchhoff$-$Laplace operator

Definition 3.1 (differentiable in the Gateaux sense) *Let X and Y be two normalized vector spaces and let f be a map of an open U of E with values in F. We say that f is differentiable in the Gateaux sense at point a of U if there exists an continuous linear application L: $E \longrightarrow F$ such that,for all v of E*

$$\lim_{t\to 0^+} \frac{1}{t}(f(a+tv) - f(a)) = L(v)$$

L is then called the Gateaux differential of f at a.

This notion is weaker than the usual notion of differentiability,also called differentiability in the sense of Fréchet.Indeed,if f is differentiable in a in the sense of Fréchet and if L is its differential,then

$$\exists \varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \varepsilon(x) \overset{X\longrightarrow 0}{\longrightarrow} 0, \forall v \in E, \left|\frac{1}{t}(f(a+tv) - f(a)) - L(v)\right| \leq \varepsilon(|t|)$$

If f is differentiable in a in the sense of Gateaux,and if L is its differential,then

$$\forall v \in E, \exists \varepsilon_v : \mathbb{R} \longrightarrow \mathbb{R}, \varepsilon_v(x) \overset{X\longrightarrow 0}{\longrightarrow} 0, \left|\frac{1}{t}(f(a+tv) - f(a)) - L(v)\right| \leq \varepsilon_v(|t|)$$

This notion of differentiability was introduced by Gateaux in 1913 in order to establish a theory of integration in infinite dimension.For each $u \in X$,define

$$\phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$.For simplicity we write $X = W^{1p(x)}(\Omega)$,denote by $u_n \rightharpoonup u$ and $u_n \to u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X,respectively.It is obvious that the functional ϕ is a Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $\phi'(u) \in X^*$,given by

$$\langle \phi'(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v(x)dx.$$

$\langle .,. \rangle$ is the duality pairing between X and X^*.

Therefore,the $p(x)$ Kirchhoff Laplace operator is the derivative operator of ϕ in the weak sense.We have the following properties about the derivative operator of ϕ.

Lemma 3.1 *([15])*

(i) $\phi' : X \to X^*$ *is a continuous,bounded and strictly monotone operator.*

(ii) ϕ' *is a mapping of type* (S_+)*,i.e.if* $u_n \rightharpoonup u$ *in* X *and*
$\overline{lim}_{n\longrightarrow\infty}\langle\phi'(u_n)- \phi'(u),u_n - u\rangle \leq 0$*,then* $u_n \to u$ *in* X.

(iii) $\phi'(u) : X \to X^*$ *is a homeomorphism.*

Proof

(i) It is obvious that ϕ' is continuous and bounded since M is continuous.For any $u,v \in X$ with $u \neq v$,without loss of generality,we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \geq \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx$$

(otherwise,changing the role of u and v in the following proof).Therefore,we have

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) \geq M\left(\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx\right) \tag{3.2}$$

since $M(t)$ is a monotone function.Using Cauchy's inequality,we have

$$\nabla u\nabla v \leq |\nabla u||\nabla v| \leq \frac{|\nabla u|^2 + |\nabla v|^2}{2} \tag{3.3}$$

Using (3.3),we can easily obtain that

$$\int_\Omega |\nabla u|^{p(x)}dx - \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx \geq \int_\Omega \frac{|\nabla u|^{p(x)-2}}{2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \tag{3.4}$$

and

$$\int_\Omega |\nabla v|^{p(x)}dx - \int_\Omega |\nabla v|^{p(x)-2}\nabla u\nabla v dx \geq \int_\Omega \frac{|\nabla v|^{p(x)-2}}{2}\left(|\nabla v|^2 - |\nabla u|^2\right)dx \tag{3.5}$$

Moreover,by Young's inequality,we obtain

$$\int_\Omega |\nabla u|^{p(x)-2}|\nabla v|^2 dx \leq \int_\Omega \left(\frac{p(x)-2}{p(x)}|\nabla u|^{p(x)} + \frac{2}{p(x)}|\nabla v|^{p(x)}\right)dx \tag{3.6}$$

and

$$\int_\Omega |\nabla v|^{p(x)-2}|\nabla u|^2 dx \leq \int_\Omega \left(\frac{p(x)-2}{p(x)}|\nabla v|^{p(x)} + \frac{2}{p(x)}|\nabla u|^{p(x)}\right)dx \tag{3.7}$$

From (3.6) and (3.7),we can see that

$$\int_\Omega |\nabla u|^{p(x)-2}|\nabla v|^2dx + \int_\Omega |\nabla v|^{p(x)-2}|\nabla u|^2dx \le \int_\Omega \left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right)dx \tag{3.8}$$

Therefore,using (3.2),(3.4),(3.5) and (3.8),we obtain

$$\begin{aligned}
\langle \phi^{'}(u) - \phi^{'}(v), u-v\rangle &= \langle \phi^{'}(u), u\rangle - \langle \phi^{'}(u), v\rangle - \langle \phi^{'}(v), u\rangle + \langle \phi^{'}(v), v\rangle \\
&= M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx \\
&-M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx \\
&-M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\int_\Omega |\nabla v|^{p(x)-2}\nabla u\nabla v dx \\
&+M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\int_\Omega |\nabla v|^{p(x)}dx \\
&\ge M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega \tfrac{|\nabla u|^{p(x)-2}}{2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&-M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v^{p(x)}dx\right)\int_\Omega \tfrac{|\nabla v|^{p(x)-2}}{2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&\ge M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\int_\Omega \tfrac{|\nabla u|^{p(x)-2}}{2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&-M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\int_\Omega \tfrac{|\nabla v|^{p(x)-2}}{2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&\ge M\left(\int_\Omega \tfrac{1}{p(x)}|\nabla v|^{p(x)}dx\right) \\
&\times \left[\int_\Omega \tfrac{1}{2}\left(|\nabla u|^{p(x)-2} - |\nabla v|^{p(x)-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx\right] \ge 0
\end{aligned} \tag{3.9}$$

i.e.$\phi^{'}$ is monotone.We claim that $\phi^{'}$ is strictly monotone.Indeed,if

$$\langle \phi^{'}(u) - \phi^{'}(v), u-v\rangle = 0$$

then we have

$$\int_\Omega \left(|\nabla u|^{p(x)-2} - |\nabla v|^{p(x)-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx = 0$$

so $|\nabla u| = |\nabla v|$.Thus,we obtain

$$\begin{aligned}
\langle \phi^{'}(u) - \phi^{'}(v), u-v\rangle &= \langle \phi^{'}(u), u-v\rangle - \langle \phi^{'}(v), u-v\rangle \\
&= M\left(\int_\Omega \tfrac{1}{p(x)}(|\nabla u|^{p(x)}dx\right)\left(\int_\Omega |\nabla u|^{p(x)-2}(\nabla u - \nabla v)^2 dx)\right) = 0
\end{aligned}$$

i.e.$u - v$ is a constant.In view of $u = v = 0$ on $\partial\Omega$,we have $u \equiv v$,which is contrary with $u \ne v$.Therefore,$\langle \phi^{'}(u) - \phi^{'}(v), u-v\rangle > 0$.It follows that Φ' is a strictly monotone operator in X.

(ii) From (i),if $u_n \to u$ and $\overline{\lim}_{n\to+\infty}\langle \phi^{'}(u_n) - \phi^{'}(u), u_n - u\rangle \leq 0$ then

$$\lim{}_{n\to+\infty}\langle \phi^{'}(u_n) - \phi^{'}(u), u_n - u\rangle = 0$$

In view of (3.9),∇u_n converges in measure to ∇u in Ω,so we get a subsequence (which we still denote by u_n) satisfying $\nabla u_n(x) \to u(x)$,$a.e.x \in \Omega$.By Fatou lemma we get

$$\underline{\lim}_{n\to+\infty}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \tag{3.10}$$

From $u_n \to u$ we have$\lim_{n\to+\infty}\langle \phi^{'}(u_n), u_n - u\rangle = \lim_{n\to+\infty}\langle \phi^{'}(u_n) - \phi^{'}(u), u_n - u\rangle =$ 0 On the other hand,we also have

$$\begin{aligned}
\langle \phi^{'}(u_n), u_n - u\rangle &= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla u dx\right)\\
&\geq M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx - \int_\Omega |\nabla u_n|^{p(x)-1}\nabla u dx\right)\\
&\geq M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}dx\\
&-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega \left(\frac{p(x)-1}{p(x)}|\nabla u_n|^{p(x)} + \frac{1}{p(x)}|\nabla u|^{p(x)}\right)dx\\
&\geq M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\\
&\geq m_0\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)
\end{aligned} \tag{3.11}$$

According to (3.10)-(3.11) we obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx.$$

Using the similar method in [25],we have

$$\lim_{n\to+\infty}\int_\Omega |\ \nabla u_n - \nabla u|^{p(x)}dx = 0 \tag{3.12}$$

From Theorem 1.4 (See Chapter 1) and (3.12) $u_n \to u$,i.e.$\phi^{'}$ is of type (S_+).

(iii) It is clear that $\phi^{'}$ is an injection since $\phi^{'}$ is a strictly monotone operator in X.Since

$$\lim_{\|u\|\to+\infty}\frac{\langle \phi^{'}(u), u\rangle}{\|\ u\ \|} \geq \lim_{\|u\|\to+\infty}\frac{m_0\int_\Omega |\nabla u|^{p(x)}dx}{\|u\ \|} = +\infty$$

$\phi^{'}$ is coercive,thus $\phi^{'}$ is a surjection in view of Minty-Browder theorem [49,Theorem 26A].Hence $\phi^{'}$ has an inverse mapping $\Psi := (\phi^{'})^{-1} : X^* \to X$.Therefore,the continuity of Ψ is sufficient to ensure $\phi^{'}$ to be a homeomorphism.If f_n,$f \in X^*$.$f_n \to f$,let $u_n = \Psi(f_n)$,$u = \Psi(f)$,then $\phi^{'}(u_n) = f_n$,$\phi^{'}(u) = f$.So $\{u_n\}$ is bounded in X.Without loss of generality,we can assume that $u_n \rightharpoonup u$.Since $f_n \to f$,then

$$\lim_{n\to+\infty} \langle \phi^{'}(u_n), u_n - u\rangle = \lim_{n\to+\infty} \langle f_n, u_n - u\rangle = 0$$

Since $\phi^{'}$ is of type (S_+),$u_n \to u$,so Ψ is continuous

Now we give a useful definition of the principle of comparison ■

Definition 3.2 *If $u, v \in W_0^{1,p(x)}(\Omega)$, We say that*

$$-M(I_0(u))\Delta_{p(x)}u \le -M(I_0(v))\Delta_{p(x)}v$$

if for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \ge 0$

$$M(I_0(u)) \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi dx \le M(I_0(v)) \int_\Omega |\nabla v|^{p(x)-2}\nabla v.\nabla\varphi dx$$

Where $I_0(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx$

we give a general principle of sub-supersolution method for the problem (3.1) based on the regularity results and the comparison principle

Lemma 3.2 (Comparison principle) *Let $u, v \in W^{1,p(x)}(\Omega)$ and (H_1) holds.If*

$$-\mathrm{M}(\mathrm{I}_0(\mathrm{u}))\Delta_{\mathrm{p(x)}}\mathrm{u} \le -\mathrm{M}(\mathrm{I}_0(\mathrm{v}))\Delta_{\mathrm{p(x)}}\mathrm{v} \quad (3.13)$$

and $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$ then $u \le v$ in Ω

Proof

Taking $\lambda = 0$ in the proof of Theorem 3.2 of [40]. ■

Lemma 3.3 *([33]).Let (H_1) hold.$M > 0$ and let u be the unique solution of the problem*

$$\begin{cases} -\mathrm{M}(\mathrm{t})\,\mathrm{div}\left(|\nabla \mathrm{u}|^{\mathrm{p(x)}-2}\nabla \mathrm{u}\right) = \mathcal{M} & \text{in } \Omega \\ \mathrm{u} = 0 & \text{on } \partial\Omega \end{cases} \quad (3.14)$$

Set $h = \frac{m_0 p^-}{2|\Omega|^{\frac{1}{N}} C_0}$*.when* $M \geq h$ *then* $|u|_\infty \leq C^* M^{\frac{1}{p^- -1}}$ *and when* $M < h$ *then* $|u|_\infty \leq C_* M^{\frac{1}{p^+ -1}}$*,where* C^* *and* C_**are positive constants depending* $p^+, p^-, N, |\Omega|, C_0$ *and* m_0.

Proof

Let u be the solution of (3.14),Lemma 3.2 implies $u \geq 0$.For $k \geq 0$,set $A_k = \{x \in \Omega : u(x) > k\}$.Taking $(u-k)^+$ as a test function in (3.14) and using the Young inequality,we have

$$\begin{aligned} \int_{A_k} |\nabla u|^{p(x)} dx = \tfrac{\mathcal{M}}{M(t)} \int_{A_k} (u-k)dx \quad & \leq \tfrac{\mathcal{M}|\Omega|^{\frac{1}{N}} C_0}{m_0 p^-} \int_{A_k} \varepsilon^{p(x)} |\nabla u|^{p(x)} dx \\ & + \tfrac{\mathcal{M}|A_k|^{1/N} C_0}{m_0 (p^+)'} \int_{A_k} \varepsilon^{-p'(x)} dx \end{aligned} \tag{3.15}$$

When $\mathcal{M} \geq h$,taking

$$\varepsilon = \left(\frac{m_0 p^-}{2\mathcal{M}|\Omega|^{1/N} C_0} \right)^{1/p^-} = \left(\frac{h}{\mathcal{M}} \right)^{1/p^-},$$

then $\varepsilon \leq 1$ and

$$\frac{\mathcal{M}\,|\Omega|^{1/N} C_0}{m_0 p^-} \int_{A_k} \varepsilon^{p(x)} |\nabla u|^{p(x)} dx \leq \frac{\mathcal{M}\,|\Omega|^{1/N} C_0}{m_0 p^-} \varepsilon^{p^-} \int_{A_k} |\nabla u|^{p(x)} dx = \frac{1}{2} \int_{A_k} |\nabla u|^{p(x)} dx.$$

Consequently,from this and (3.15),it follows that

$$\int_{A_k} |\nabla u|^{p(x)} dx \leq \frac{2\mathcal{M}\,|A_k|^{1/N} C_0}{m_0 (p^+)} \int_{A_k} \varepsilon^{-p'(x)} dx \leq \frac{2\mathcal{M} C_0 \varepsilon^{-(p^-)'}}{m_0 (p^+)} |A_k|^{1+1/N} \tag{3.16}$$

From (3.15) and (3.16),we have

$$\int_{A_k} (u-k)dx = \frac{M(t)}{\mathcal{M}} \int_{A_k} |\nabla u|^{p(x)} dx \leq M \left(\frac{2MMC_0 \varepsilon^{-(p^-)'}}{p^- m_0 (p^+)'} |\Omega|^{1+1/N} \right) \frac{2C_0 \varepsilon^{-(p^-)'}}{m_0 (p^+)'} |A_k|^{1+1/N} \tag{3.17}$$

By Lemma 5.1 in([39],Chapter 2) and (3.17) implies that

$$|u|_\infty \leq \gamma\,(N+1)\,|\Omega|^{1/N} \tag{3.18}$$

Where

$$\gamma = M \left(\frac{2\mathcal{M} C_0 \varepsilon^{-(p^-)'}}{p^- m_0 (p^+)} |\Omega|^{1+1/N} \right) \frac{2C_0 \varepsilon^{-(p^-)'}}{m_0 (p^+)}$$

From (3.17) and (3.18),we obtain

$$|u|_\infty \leq C^* \mathcal{M}^{1/(p^- -1)}$$

Where

$$C^{*}=\frac{(N+1)(2C_0)^{(p^-)'}}{(p^+)'m_0^{(p^-)\prime-})^{(p^-)/p^-}}|\Omega|^{(p^-)'/N}M\left(\frac{(2\mathcal{M}C_0)^{(p^-)'}}{p^-(p^+)m_0^{(p^-)\prime-})^{(p^-)'/p^-}}|\Omega|^{(p^-)'/N}\right)$$

When $\mathcal{M}<h$,taking

$$\varepsilon=\left(\frac{m_0p^-}{2\mathcal{M}|\Omega|^{1/N}C_0}\right)^{1/p^+}=\left(\frac{h}{\mathcal{M}}\right)^{1/p^+}$$

(noting that in this case $\varepsilon>1$) and using arguments similar to those above,we can obtain

$$|u|_\infty\le C_*\mathcal{M}^{1/(p^+-1)}$$

Where

$$C_{*}=\frac{(N+1)(2C_0)^{(p^+)'}}{(p^+)'m_0^{(p^+)\prime-})^{(p^+)/p^+}}|\Omega|^{(p^+)'/N}M\left(\frac{(2\mathcal{M}C_0)^{(p^+)'}}{p^-(p^+)m_0^{(p^+)\prime-})^{(p^+)'/p^+}}|\Omega|^{(p^+)'/N}\right).$$

The proof is complete. ■

Before going to the next lemma,we will use the notation $d\,(x,\partial\Omega)$ to denote the distance of $x\in\Omega$ to the boundary of Ω.Denote $d\,(x)=d\,(x,\partial\Omega)$ and

$$\partial\Omega_\varepsilon=\{x\in\Omega{:}d\,(x,\partial\Omega)<\varepsilon.\}$$

From Lemma 14.16 in [29],Since $\partial\Omega$ is C^2 regularly,there exists a constant $\delta\in(0,1)$ such that $d\,(x)\in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$ and $|\nabla d\,(x)|=1$.Denote

$$v_1\,(x)=\begin{cases}\gamma d\,(x) & \text{if } d\,(x)<\delta,\\ \gamma\delta+\int_\delta^{d(x)}\gamma\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p^--1}}(\lambda_1a_1+\mu_1c_1)^{\frac{2}{p^--1}}dt & \text{if } \delta\le d\,(x)<2\delta,\\ \gamma\delta+\int_\delta^{2\delta}\gamma\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p^--1}}(\lambda_1b_1+\mu_1d_1)^{\frac{2}{p^--1}}dt & \text{if } 2\delta\le d\,(x).\end{cases}$$

And

$$v_2\,(x)=\begin{cases}\gamma d\,(x) & \text{if } d\,(x)<\delta,\\ \gamma\delta+\int_\delta^{d(x)}\gamma\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p^--1}}(\lambda_2a_2+\mu_2c_2)^{\frac{2}{p^--1}}dt & \text{if } \delta\le d\,(x)<2\delta,\\ \gamma\delta+\int_\delta^{2\delta}\gamma\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p^--1}}(\lambda_2b_2+\mu_2d_2)^{\frac{2}{p^--1}}dt & \text{if } 2\delta\le d\,(x).\end{cases}$$

Obviously,$0 \leq v_1(x), v_2(x) \in C^1(\overline{\Omega})$.Considering

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}\omega(x)=\eta & \text{in } \Omega, \\ \omega=0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

we have the following result

Lemma 3.4 *([15]).If positive parameter η is large enough and ω is the unique solution of (3.19).then we have*

(i) For any $\theta \in (0,1)$ there exists a positive constant C_1 such that $C_1\eta^{\frac{1}{p^+-1+\theta}} \leq max_{x\in\overline{\Omega}}\,\omega(x)$

(ii) There exists a positive constant C_2 such that $max_{x\in\overline{\Omega}}\,\omega(x) \leq C_2\eta^{\frac{1}{p^--1}}$.

3.3 *Main Result*

Throughout the section,we will assume that:

(H_1) $M:[0,+\infty) \to [m_0,\infty]$ is a continuous and increasing function with $m_0 > 0$.

(H_2) $p \in C^1(\overline{\Omega})$ *and* $1 < p^- \leq p^+$.

(H_3) f, g, h, τ: $[0,+\infty[\to \mathbb{R}$ are C^1,monotone functions such that $\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty$.

(H_4) $\lim_{u\to+\infty}\dfrac{f\left(L(g(u))^{\frac{1}{p^--1}}\right)}{u^{p^--1}} = 0$, for all $L > 0$.

(H_5) $\lim_{u\to+\infty}\frac{h(u)}{u^{p^--1}} = 0$,*and* $\lim_{u\to+\infty}\frac{\tau(u)}{u^{p^--1}} = 0$.

(H_6) $a, b, c, d : \overline{\Omega} \to (0,+\infty)$ are continuous functions,such that
$a_1 = \min_{x\in\overline{\Omega}} a(x), b_1 = \min_{x\in\overline{\Omega}} b(x), c_1 = \min_{x\in\overline{\Omega}} c(x), d_1 = \min_{x\in\overline{\Omega}} d(x)$,
$a_2 = \max_{x\in\overline{\Omega}} a(x), b_2 = \max_{x\in\overline{\Omega}} b(x), c_2 = \max_{x\in\overline{\Omega}} c(x), d_2 = \max_{x\in\overline{\Omega}} d(x)$.

Before we give the main result,we provide some basic definitions as follows

Definition 3.3 (a weak solution) *If $u, v \in W_0^{1,p(x)}(\Omega)$,$(u,v)$ is called a weak solution of* (3.1) *if it satisfies for all $\varphi \in W_0^{1,p(x)}(\Omega)$.$\varphi \geq 0$.*

$$\begin{cases} M(I_0(u))\displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi dx = \int_\Omega \lambda^{p(x)}\left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right]\varphi dx \\ M(I_0(v))\displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla v.\nabla\varphi dx = \int_\Omega \lambda^{p(x)}\left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right]\varphi dx \end{cases}$$

Definition 3.4 (a subsolution and supersolution) *We say that (u,v) is called a sub solution (respectively a super solution) of problem (3.1) if for all $\varphi \in W_0^{1,p(x)}(\Omega).\varphi \geq 0$.*

$$\begin{cases} M(I_0(u))\int\limits_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi dx \leq (resp \geq) \int\limits_\Omega \lambda^{p(x)}\left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right]\varphi dx \\ M(I_0(v))\int\limits_\Omega |\nabla u|^{p(x)-2}\nabla v.\nabla\varphi dx \leq (resp \geq) \int\limits_\Omega \lambda^{p(x)}\left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right]\varphi dx \end{cases}$$

Theorem 3.1 *([41]) Assume that the conditions $(H_1)-(H_6)$ are satisfied.Then problem (3.1) has a positive solution when λ is large enough.*

Proof

We shall establish Theorem 3.1 by constructing a positive subsolution (ϕ_1,ϕ_2) and supersolution (z_1,z_2) of (3.1).such that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$.that is,(ϕ_1,ϕ_2) and (z_1,z_2) satisfies

$$\begin{cases} M(I_0(\phi_1))\int\limits_\Omega |\nabla\phi_1|^{p(x)-2}\nabla\phi_1.\nabla q dx \leq \int\limits_\Omega \lambda^{p(x)}\left[\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)\right] q dx, \\ M(I_0(\phi_2))\int\limits_\Omega |\nabla\phi_2|^{p(x)-2}\nabla\phi_2.\nabla q dx \leq \int\limits_\Omega \lambda^{p(x)}\left[\lambda_2 b(x) g(\phi_1) + \mu_2 d(x) \tau(\phi_2)\right] q dx \end{cases}$$

and

$$\begin{cases} M(I_0(z_1))\int\limits_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1.\nabla q dx \geq \int\limits_\Omega \lambda^{p(x)}\left[\lambda_1 a(x) f(z_2) + \mu_1 c(x) h(z_1)\right] q dx, \\ M(I_0(z_2))\int\limits_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2.\nabla q dx \geq \int\limits_\Omega \lambda^{p(x)}\left[\lambda_2 b(x) g(z_1) + \mu_2 d(x) \tau(z_2)\right] q dx, \end{cases}$$

for all $q \in W_0^{1,p(x)}(\Omega)$ with $q \geq 0$.According to the sub-super solution method for $p(x)$-Kirchhoff type equations (see [33]),then problem (3.1) has a positive solution.

Step 1.We will construct a subsolution of (3.1).Let $\sigma \in (0,\delta)$ is small enough.Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1 & ,d(x) < \sigma \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma}\left(\frac{2\delta - t}{2\delta-\sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt & ,\sigma \leq d(x) < 2\delta \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma}\left(\frac{2\delta - t}{2\delta-\sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt & , 2\delta \leq d(x) \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)}-1 & ,d(x)<\sigma \\ e^{k\sigma}-1+\int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}(\lambda_2 b_1+\mu_2 d_1)^{\frac{2}{p^--1}}dt & ,\sigma\le d(x)<2\delta \\ e^{k\sigma}-1+\int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}(\lambda_2 b_1+\mu_2 d_1)^{\frac{2}{p^--1}}dt & ,2\delta\le d(x) \end{cases}$$

It is easy to see that $\phi_1,\phi_2\in C^1\left(\overline{\Omega}\right)$,Denote

$$\alpha=\min\left\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)|+1)},1\right\}$$
$$\zeta=\min\{\lambda_1 f(0)+\mu_1 h(0),\lambda_2 g(0)+\mu_2\tau(0),-1\}$$

By some simple computations we can obtain

$$-\Delta_{p(x)}\phi_1 \begin{cases} -k\left(e^{kd(x)}\right)^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\Delta d}{k}\right] & ,d(x)<\sigma \\ \begin{cases} \left[\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^--1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}\nabla p\nabla d+\Delta d\right]\right] \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^--1}-1}(\lambda_1 a_1+\mu_1 c_1) \end{cases} & ,\sigma\le d(x)<2\delta \\ 0 & ,2\delta\le d(x) \end{cases}$$

$$-\Delta_{p(x)}\phi_2 \begin{cases} -k\left(e^{kd(x)}\right)^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\Delta d}{k}\right] & ,d(x)<\sigma \\ \begin{cases} \left[\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^--1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}\nabla p\nabla d+\Delta d\right]\right] \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^--1}-1}(\lambda_2 b_1+\mu_2 d_1) \end{cases} & ,\sigma\le d(x)<2\delta \\ 0 & ,2\delta\le d(x) \end{cases}$$

from (H_4) there exists a positive constant $L>1$ such that

$$f(L-1)\ge 1,g(L-1)\ge 1,h(L-1)\ge 1,\tau(L-1)\ge 1.$$

Let $\sigma=\frac{1}{k}\ln L$,then

$$\sigma k=\ln L \tag{3.20}$$

If k is sufficiently large,from (3.20),we have

$$-\Delta_{p(x)}\phi_1\le -k^{p(x)}\alpha,d(x)<\sigma \tag{3.21}$$

Let $\frac{\lambda\zeta}{m_0}=k\alpha$,then

$$-k^{p(x)}\alpha\ge -\lambda^{p(x)}\frac{\zeta}{m_0}.$$

From(3.21),we have

$$\begin{aligned}-M\left(I_0\left(\phi_1\right)\right)\Delta_{p(x)}\phi &\leq M\left(I_0\left(\phi_1\right)\right)\lambda^{p(x)}\frac{\zeta}{m_0}\\ &\leq \lambda^{p(x)}\zeta \leq \lambda^{p(x)}\left(\lambda_1 a_1 f\left(0\right)+\mu_1 c_1 h\left(0\right)\right) \quad ,d\left(x\right)<\sigma\\ &\leq \lambda^{p(x)}\left(\lambda_1 a\left(x\right)f\left(\phi_2\right)+\mu_1 c\left(x\right)h\left(\phi_1\right)\right)\end{aligned} \tag{3.22}$$

Since $d\left(x\right)\in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$,there exists a positive constant C_3 such that

$$\begin{aligned}-M\left(I_0\left(\phi_1\right)\right)\Delta_{p(x)}\phi_1 &\leq m_0\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^--1}-1}\left(\lambda_1+\mu_1\right)\\ &\times\left(\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^--1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}\nabla p\nabla d+\Delta d\right]\right) \quad ,\sigma\leq d\left(x\right)<2\delta\\ &\leq C_3 m_0\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\lambda_1 a_1+\mu_1 c_1\right)\ln k\end{aligned}$$

If k is sufficiently large,let $\frac{\lambda\zeta}{m_0}=k\alpha$,then we have

$$\begin{aligned}C_3 m_0\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\lambda_1 a_1+\mu_1 c_1\right)\ln k &= C_3 m_0(KL)^{p(x)-1}\left(\lambda_1 a_1+\mu_1 c_1\right)\ln k\\ &\leq \lambda^{p(x)}\left(\lambda_1 a_1+\mu_1 c_1\right)\end{aligned}$$

Then

$$-M\left(I_0\left(\phi_1\right)\right)\Delta_{p(x)}\phi_1\leq\lambda^{p(x)}\left(\lambda_1 a_1+\mu_1 c_1\right) \quad ,\sigma\leq d\left(x\right)<2\delta \tag{3.23}$$

Since $\phi_1\left(x\right),\phi_2\left(x\right)$ and f,h are monotone,when λ is large enough we have

$$-M\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}\phi_1\leq\lambda^{p(x)}\left(\lambda_1 a\left(x\right)f\left(\phi_2\right)+\mu_1 c\left(x\right)h\left(\phi_1\right)\right) \quad ,\sigma\leq d\left(x\right)<2\delta$$

$$\begin{aligned}-M\left(I_0\left(\phi_1\right)\right)\Delta_{p(x)}\phi_1=0 &\leq \lambda^{p(x)}\left(\lambda_1 a_1+\mu_1 c_1\right)\\ &\leq \lambda^{p(x)}\left(\lambda_1 a\left(x\right)f\left(\phi_2\right)+\mu_1 c\left(x\right)h\left(\phi_1\right)\right)\end{aligned} \quad ,2\delta\leq d\left(x\right) \tag{3.24}$$

Combining (3.22),(3.23) and (3.24),we can conclude that

$$-M\left(I_0\left(\phi_1\right)\right)\Delta_{p(x)}\phi_1\leq\lambda^{p(x)}\left(\lambda_1 a\left(x\right)f\left(\phi_2\right)+\mu_1 c\left(x\right)h\left(\phi_1\right)\right) \quad ,\text{a.e.on }\Omega \tag{3.25}$$

Similarly

$$-M\left(I_0\left(\phi_2\right)\right)\Delta_{p(x)}\phi_2\leq\lambda^{p(x)}\left(\lambda_2 b\left(x\right)g\left(\phi_1\right)+\mu_2 d\left(x\right)\tau\left(\phi_2\right)\right) \quad ,\text{a.e.on }\Omega \tag{3.26}$$

From (3.25) and (3.26),we can see that (ϕ_1,ϕ_2) is a subsolution of problem (3.1)

Step 2.We will construct a supersolution of problem (3.1).We consider

$$\begin{cases} -M\left(I_0\left(z_1\right)\right)\Delta_{p(x)}z_1{=}\frac{\lambda^{p^+}}{m_0}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu & \text{in } \Omega \\ -M\left(I_0\left(z_2\right)\right)\Delta_{p(x)}z_2{=}\frac{\lambda^{p^+}}{m_0}\left(\lambda_2 b_2{+}\mu_2 d_2\right)g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu\right)\right) & \text{in } \Omega \\ z_1{=}z_2{=}\ 0 & \text{on } \partial\Omega \end{cases}$$

where

$$\beta{=}\beta\left(\lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu\right) = \max_{x\in\overline{\Omega}} z_1\left(x\right).$$

We shall prove that (z_1, z_2) is a supersolution of problem (3.1).

For $q \in W_0^{1,p(x)}(\Omega)$ with $q \geq 0$,it is easy to see that

$$\begin{aligned} M(I_0\left(z_2\right))_\Omega|\nabla z_2|^{p(x)-2}\nabla z_2.\nabla q dx &= \frac{1}{m_0}M(I_0\left(z_2\right))\int_\Omega \lambda^{p^+}\left(\lambda_2 b_2{+}\mu_2 d_2\right) \\ &\times \quad g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu\right)\right) \\ &\geq \int_\Omega \lambda^{p^+}\lambda_2 b\left(x\right)g\left(z_1\right)q dx \\ &+ \int_\Omega \lambda^{p^+}\mu_2 d\left(x\right)g\left(\beta\left(\lambda^{\ p^+}\left(\lambda_1{+}\mu_1\right)\mu\right)\right)q dx \end{aligned} \tag{3.27}$$

By (H_6),for μ large enough,using Lemma 3.4,we have

$$\begin{aligned} g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu\right)\right) &\geq \tau\left(C_2\left[\lambda^{p^+}\left(\lambda_2 b_2{+}\mu_2 d_2\right)g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu\right)\right)\right]^{\frac{1}{p^- -1}}\right) \\ &\geq \tau\left(z_2\right) \end{aligned} \tag{3.28}$$

Hence

$$M(I_0\left(z_2\right))\int_\Omega|\nabla z_2|^{p(x)-2}\nabla z_2.\nabla q dx{\geq}\int_\Omega \lambda^{p^+}\lambda_2 b\left(x\right)g\left(z_1\right)q dx{+}\int_\Omega \lambda^{p^+}\mu_2 d\left(x\right)\tau\left(z_2\right)q dx \tag{3.29}$$

Also

$$\begin{aligned} M(I_0\left(z_1\right))\int_\Omega|\nabla z_1|^{p(x)-2}\nabla z_1.\nabla q dx &= \frac{1}{m_0}M(I_0\left(z_1\right))\int_\Omega \lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu q dx \\ &\geq \int_\Omega \lambda^{p^+}\left(\lambda_1 a_2{+}\mu_1 c_2\right)\mu q dx. \end{aligned}$$

By $(H_4),(H_5)$ and Lemma 3.4,when μ is sufficiently large,we have

$$\begin{aligned}(\lambda_1 a_2+\mu_1 c_2)\,\mu &\geq \frac{1}{\lambda^{p^+}}\left[\frac{1}{C_2}\beta\left(\lambda^{p^+}(\lambda_1 a_2+\mu_1 c_2)\,\mu\right)\right]^{p^--1}\\ &\geq \mu_1 h\left(\beta\left(\lambda^{p^+}(\lambda_1 a_2+\mu_1 c_2)\,\mu\right)\right)\\ &+\lambda_1 f\left(C_2\left[\lambda^{p^+}(\lambda_2 b_2+\mu_2 d_2)\,g\left(\beta\left(\lambda^{p^+}(\lambda_1 a_2+\mu_1 c_2)\,\mu\right)\right)\right]^{\frac{1}{p^--1}}\right).\end{aligned}$$

Then

$$M(I_0(z_1))\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1.\nabla q dx\geq\int_\Omega \lambda^{p^+}\lambda_1 a(x)\,f(z_2)\,q dx+\int_\Omega \lambda^{p^+}\mu_1 c(x)\,h(z_1)\,q dx \tag{3.30}$$

According to (3.29) and (3.30),we can conclude that (z_1,z_2) is a supersolution of problem (3.1) It only remains to prove that $\phi_1\leq z_1$ and $\phi_2\leq z_2$.
In the definition of $v_1(x)$,let

$$\gamma=\frac{2}{\delta}\left(\max_{\overline{\Omega}}\phi_1(x)+\max_{\overline{\Omega}}|\nabla\phi_1|(x)\right).$$

We claim that

$$\phi_1(x)\leq v_1(x)\,,\forall x\in\Omega. \tag{3.31}$$

From the definition of v_1,it is easy to see that

$$\phi_1(x)\leq 2\max_{\overline{\Omega}}\phi_1(x)\leq v_1(x)\,,\text{when } d(x)=\delta$$

and

$$\phi_1(x)\leq 2\max_{\overline{\Omega}}\phi_1(x)\leq v_1(x)\,,\text{when } d(x)\geq\delta.$$

$$\phi_1(x)\leq v_1(x)\,,\text{when } d(x)<\delta.$$

Since $v_1-\phi_1\in C^1\left(\overline{\partial\Omega_\delta}\right)$,there exists a point $x_0\in\overline{\partial\Omega_\delta}$ such that

$$v_1(x_0)-\phi_1(x_0)=\min_{x_0\in\overline{\partial\Omega_\delta}}(v_1(x_0)-\phi_1(x_0)).$$

If $v_1(x_0)-\phi_1(x_0)<0$,it is easy to see that $0<d(x)<\delta$ and then

$$\nabla v_1(x_0)-\nabla\phi_1(x_0)=0.$$

From the definition of v_1,we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}}\phi_1(x_0) + \max_{\overline{\Omega}}|\nabla\phi_1|(x_0)\right) > |\nabla\phi_1|(x_0).$$

It is a contradiction to

$$\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0.$$

Thus (3.31) is valid.

Obviously,there exists a positive constant C_3 such that

$$\gamma \leq C_3\lambda.$$

Since $d(x) \in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$,according to the proof of Lemma 3.4,there exists a positive constant C_4 such that

$$M(I_0(v_1)) - \Delta_{p(x)}v_1(x) \leq C_*\gamma^{p(x)-1+\theta} \leq C_4\lambda^{p(x)-1+\theta}.a.e \text{ in } \Omega, \text{where } \theta \in (0,1).$$

When $\eta \geq \lambda^{p^+}$ is large enough,we have

$$-\Delta_{p(x)}v_1(x) \leq \eta.$$

According to the comparison principle,we have

$$v_1(x) \leq \omega(x), \forall x \in \Omega. \tag{3.32}$$

From (3.31) and (3.32) when $\eta \geq \lambda^{p^+}$and $\lambda \geq 1$ is sufficiently large,we have

$$\phi_1(x) \leq v_1(x) \leq \omega(x), \forall x \in \Omega. \tag{3.33}$$

According to the comparison principle,when μ is large enough,we have

$$v_1(x) \leq \omega(x) \leq z_1(x), \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (3.33),it is easy to see that

$$\phi_1(x) \leq v_1(x) \leq \omega(x) \leq z_1(x), \forall x \in \Omega,$$

when $\mu \geq 1$ and λ is large enough.

from Lemma 3.4 we can see that $\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu)$ is large enough,then

$$\frac{\lambda^{p^+}}{m_0}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu))$$

is large enough.Similarly,we have $\phi_2 \leq z_2$.This completes the proof of Theorem 3.1 ■

Conclusion

The sub-supersolution method has allowed us to prove that there is at least one weak solution,but the uniqueness of the solution remains an open problem

Prospect

Fractional Sobolev spaces are well known since the beginning of the last century,especially in the framework of harmonic analysis.More recently,a large amount of papers were written on problems involving the fractional diffusion $(-\triangle)^s$,$0 < s < 1$

the authors tried to see which results "survive" when the Laplacian is replaced by the fractional Laplacian.Then,they introduced a suitable functional space to study an equation in which a fractional variable exponent operator is present

In the future,we will,in our turn,generalize the results obtained in our research into Sobolev Fractional Spaces,especially depending on the reference

[1] A.Bahrouni,"Comparison and sub-supersolution principles for the fractional p(x)-Laplacian," *J.Math.Anal.Appl.*,vol.458,no.2,pp.1363–1372,2018.

BIBLIOGRAPHY

[1] Acerbi, E., and Mingione, G. Regularity results for a class of quasiconvex functionals with nonstandard growth. *Ann. della Sc. Norm. Super. di Pisa-Classe di Sci. 30*, 2 (2001), 311–339.

[2] Acerbi, E., and Mingione, G. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal. 164*, 3 (2002), 213–259.

[3] Adams, R. Sobolev Spaces, vol. 65 of Pure and Appl. Math, 1975.

[4] Afrouzi, G. A., and Ghorbani, H. Positive solutions for a class of p (x)-Laplacian problems. *Glas. Math. J. 51*, 3 (2009), 571–578.

[5] Afrouzi, G. A., Shakeri, S., and Chung, N. T. Existence of positive solutions for variable exponent elliptic systems with multiple parameters. *Afrika Mat. 26*, 1-2 (2015), 159–168.

[6] Alves, C. O., and Corrêa, F. On existence of solutions for a class of problem involving a nonlinear operator. *Comm. Appl. Nonlinear Anal 8*, 2 (2001), 43–56.

[7] Azzouz, N., and Bensedik, A. Existence Results for an Elliptic Equation of Kirchhoff-Type with Changing Sign Data. *Funkc. Ekvacioj 55*, 1 (2012), 55–66.

[8] Bensedik, A., and Bouchekif, M. On an elliptic equation of Kirchhoff-type with a potential asymptotically linear at infinity. *Math. Comput. Model. 49*, 5-6 (2009), 1089–1096.

[9] Bidaut-Véron, M.-F., and Pohozaev, S. Nonexistence results and estimates for some nonlinear elliptic problems. *J. d'analyse mathématique 84*, 1 (2001), 1–49.

[10] Boulaaras, S., and Guefaifia, R. Existence of positive weak solutions for a class of Kirrchoff elliptic systems with multiple parameters. *Math. Methods Appl. Sci. 41*, 13 (2018), 5203–5210.

[11] Chen, C. On positive weak solutions for a class of quasilinear elliptic systems. *Nonlinear Anal. Theory, Methods Appl. 62*, 4 (2005), 751–756.

[12] Chen, Y., Levine, S., and Rao, M. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math. 66*, 4 (2006), 1383–1406.

[13] Chen, Y., Levine, S., and Rao, R. Functionals with p (x)-growth in image processing. *Preprint* (2004).

[14] Chung, N. T. Multiple solutions for ap (x)-Kirchhoff-type equation with sign-changing nonlinearities. *Complex Var. Elliptic Equations 58*, 12 (2013), 1637–1646.

[15] Dai, G. Three solutions for a nonlocal Dirichlet boundary value problem involving the p (x)-Laplacian. *Appl. Anal. 92*, 1 (2013), 191–210.

[16] Danet, C.-P. The classical maximum principle, some of its extensions and applications. *Ann. Acad. Rom. Sci. Ser. Math. its Appl. 3* (2011), 273–299.

[17] Diening, L., Harjulehto, P., Hästö, P., and Ruzicka, M. *Lebesgue and Sobolev spaces with variable exponents.* Springer, 2011.

[18] Edmunds, D., and Rákosník, J. Sobolev embeddings with variable exponent. *Stud. Math. 3*, 143 (2000), 267–293.

[19] Edmunds, D. E., Lang, J., and Nekvinda, A. On L p (x) norms. *Proc. R. Soc. London. Ser. A Math. Phys. Eng. Sci. 455*, 1981 (1999), 219–225.

[20] Fan, X. On the sub-supersolution method for p (x)-Laplacian equations. *J. Math. Anal. Appl. 330*, 1 (2007), 665–682.

[21] Fan, X., Zhang, Q., and Zhao, D. Eigenvalues of p (x)-Laplacian Dirichlet problem. *J. Math. Anal. Appl. 302*, 2 (2005), 306–317.

[22] Fan, X., and Zhao, D. A class of De Giorgi type and Hölder continuity. *Nonlinear Anal. Theory, Methods Appl. 36*, 3 (1999), 295–318.

[23] Fan, X., and Zhao, D. The quasi-minimizer of integral functionals with m (x) growth conditions. *Nonlinear Anal. Theory, Methods Appl. 39*, 7 (2000), 807–816.

[24] Fan, X., and Zhao, D. On the spaces Lp (x)(Ω) and Wm, p (x)(Ω). *J. Math. Anal. Appl. 263*, 2 (2001), 424–446.

[25] Fan, X.-L., and Zhang, Q.-H. Existence of solutions for p (x)-Laplacian Dirichlet problem. *Nonlinear Anal. Theory, Methods Appl. 52*, 8 (2003), 1843–1852.

[26] Fan, X. L., and Zhao, D. On the space Lp (x) and Wm, p (x). *J. Math. Anal. Appl 263* (2001), 424–446.

[27] Filippucci, R. Quasilinear elliptic systems in RN with multipower forcing terms depending on the gradient. *J. Differ. Equ. 255*, 7 (2013), 1839–1866.

[28] García-Melián, J., and Iturriaga, L. Some counter examples related to the stationary Kirchhoff equations. In *Proc. Amer. Math. Soc* (2016), vol. 144, pp. 3405–3411.

[29] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* springer, 2015.

[30] Guefaifia, R., and Boulaaras, S. Existence of positive solutions for a class of $$\left (p\left (x\right), q\left (x\right)\right) $$-Laplacian systems. *Rend. del Circ. Mat. di Palermo Ser. 2 67*, 1 (2018), 93–103.

[31] Hai, D. D., and Shivaji, R. An existence result on positive solutions for a class of p-Laplacian systems. *Nonlinear Anal. Theory, Methods Appl. 56*, 7 (2004), 1007–1010.

[32] Halsey, T. C. Electrorheological fluids. *Science (80-.). 258*, 5083 (1992), 761–766.

[33] Han, X., and Dai, G. On the sub-supersolution method for p (x)-Kirchhoff type equations. *J. Inequalities Appl. 2012*, 1 (2012), 283.

[34] Hopf, E. *Elementare Bemerkungen über die Lösungen partieller Differentialgleichngen zweiter Ordnung vom elliptischen Typus.* 1927.

[35] Hudzik, H. On generalized Orlicz-Sobolev space. *Funct. Approx. Comment. Math 4* (1976), 37–51.

[36] Hudzik, H. The problem of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces\(W_Mˆ k (\Omega)\). *Comment. Math. 21*, 2 (1979).

[37] Kirchhoff, G. Vorlesungen\Huber. *Mech. Leipzig, Teubner* (1883).

[38] Kovácik, O. On spaces Lˆ< p (x)> and Wˆ< k, p (x). *Czechoslov. Math. J. 41* (1991), 592–618.

[39] Ladyzhenskaya, O. A., and Ural'tseva, N. N. Linear and quasilinear elliptic equations, 1968. *Leon Ehrenpr. Acad. Press. New York* (1968).

[40] Ma, R., Dai, G., and Gao, C. Existence and multiplicity of positive solutions for a class of p(x)-Kirchhoff type equations. *Bound. Value Probl. 2012*, 1 (2012), 16.

[41] Mairi, B., Guefaifia, R., Boulaaras, S., and Bouali, T. Existence of positive solutions for a new class of nonlocal p (x)-Kirchhoff elliptic systems via sub-super solutions concept. *Appl. Sci. APPS 20* (2018), 117–128.

[42] Marcellini, P. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differ. Equ. 90*, 1 (1991), 1–30.

[43] MihĂČilescu, M., and RĂČdulescu, V. A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. *Proc. R. Soc. A Math. Phys. Eng. Sci. 462*, 2073 (2006), 2625–2641.

[44] Musielak, J. Orlicz spaces and modular spaces. *Lect. notes Math. 1034* (1983), 1–216.

[45] Nakano, H. Modulared semi-ordered linear spaces, Tokyo Math. *B. Ser. 1* (1950).

[46] O'Neil, R. Fractional integration in Orlicz spaces. I. *Trans. Am. Math. Soc. 115* (1965), 300–328.

[47] Orlicz, W. Über konjugierte exponentenfolgen. *Stud. Math. 3*, 1 (1931), 200–211.

[48] Perera, K., and Zhang, Z. Nontrivial solutions of Kirchhoff-type problems via the Yang index. *J. Differ. Equ. 221*, 1 (2006), 246–255.

[49] PFEIFFER, C., MAVROIDIS, C., BAR-COHEN, Y., AND DOLGIN, B. P. Electrorheological-fluid-based force feedback device. In *Telemanipulator Telepresence Technol. VI* (1999), vol. 3840, International Society for Optics and Photonics, pp. 88–99.

[50] SAMKO, S., AND VAKULOV, B. Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. *J. Math. Anal. Appl. 310*, 1 (2005), 229–246.

[51] SHARAPUDINOV, I. I. Topology of the space âĎŠ p (t)([0, 1]). *Math. notes Acad. Sci. USSR 26*, 4 (1979), 796–806.

[52] TSENOV, I. Generalization of the problem of best approximation of a function in the space Ls, Uch. *Zap. Dagestan Gos. Univ 7* (1961), 25–37.

[53] ZHANG, Q. Existence of positive solutions for a class of p (x)-Laplacian systems. *J. Math. Anal. Appl. 333*, 2 (2007), 591–603.

[54] ZHANG, Q. Existence of positive solutions for elliptic systems with nonstandard p (x)-growth conditions via sub-supersolution method. *Nonlinear Anal. Theory, Methods Appl. 67*, 4 (2007), 1055–1067.

[55] ZHANG, Q. Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems. *Nonlinear Anal. Theory, Methods Appl. 70*, 1 (2009), 305–316.

[56] ZHANG, Z., AND PERERA, K. Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. *J. Math. Anal. Appl. 317*, 2 (2006), 456–463.

[57] ZHIKOV, V. V. Averaging of functionals of the calculus of variations and elasticity theory. *Math. USSR-Izvestiya 29*, 1 (1987), 33.