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Stability analysis in COVID-19 model

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ملخص

اهتماما بالفيروس الجديد, قمنا بدراسة النموذج **SEIRV** الذي يصف سلوكه. في البداية نقدم لمحة عامة عن بعض نماذج هذا المرض التي تم العمل عليها مسبقا. بعد ذلك نقوم بايجاد نقاط استقرارالنظام. ثم نعطي نتائج الاستقرار المحلى والعالمي. اخيرا, نقدم بعض النتائج العددية بناءً على احصائيات مدينة ووهان الصينية [2] .

Abstract

Interesting to the new virus, we studied the SEIRV model which describe the behavior of the virus. At first, we give a literature overview of some disease models. Next, we analyse the equilibrium points of our system. After that, we give the local and global stability results. Finaly, we present some numerical results based on the statistics of the Chinese city, Wuhan [2].

Résumé

Suite à notre préoccupation du nouveau virus, nous avons étudié le modèle SEIRV qui décrit le comportement du virus. Dans un premier temps, nous donnons un aperçu de la littérature sur certains modèles de maladies. Ensuite, nous analysons les points d'équilibre de notre système. Après cela, nous donnons les résultats de stabilité locale et globale. Enfin, nous présentons quelques résultats numériques basés sur les statistiques de la ville chinoise, Wuhan [2].

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CONTENTS

INTRODUCTION

Coronaviruses are a large family that causes respiratory infections that include colds, high fever, coughing and other symptoms, it is the third zoonotic human coronavirus emerging in the current century, after the severe acute respiratory syndrome coronavirus (SARS-CoV) in 2002 that spread to 37 countries and the Middle East respiratory syndrome coronavirus (MERS-CoV) in 2012 that spread to 27 countries [2]. The first appeared in Wuhan, China in march 2019 [3], then it spread to most countries of the world and claimed many lives, as the number of deaths so far reached more than three millions deaths in the world.

It was more interest in this disease and how to control it to limit its spread and reducing the rates of injuries and deaths. The transmission of the disease from a person to other is through shaking hands, touching the eyes, nose or mouth, touching the infected surfaces and objects ... clinical evidence shows that the incubation period of this disease ranges from 2 to 14 days. During this period of time, infected individuals may not develop any symptoms and may not be aware of their infection, yet they are capable of transmitting the disease to other people [4], we can reduce the spread of the disease by following the preventive measures (wearing a medical mask, quarantine, cleaning hands, social distancing ect $\cdots$).

Mathematical models play an increasingly important role in our understanding of the transmission of infectious diseases. Epidemiological models are studied by Mathematics are constructive in comprising, proposing, planning, implementing, testing theories, prevention, evaluating a variety of detection. On the other hand, to study, examine, analyze, predict and capture the behaviour of viruses, diseases, threads and others, the mathematics is the only tool that can help us to better understand disease behavior, To detect and cure those diseases properly , it is called differential and integral operators are used to model real world problems in all fields of sciences as they are able to repilcate some behaviors observed in real world [5], we need an effective method to solve these models. For the solution of the system of linear and nonlinear differential equations. Recently, several mathematical, computational, clinical and examination studies have been put forward for modeling, prediction, treatment and fight disease.

Statement of the problem

In this work, we relied on the proposed model in [2] which describes the transmission of the disease and its focus in environment, as well as the role of preventive measures in controlling the virus and its spread.

The authors divided the total human population into four compartments, the susceptible (denoted by S), the exposed (denoted by E), the infected (denoted by I), and the recovered (denoted by R). Individuals in the infected class have fully developed disease symptoms and can infect other people. Individuals in the exposed class are in the incubation period, they do not show symptoms but are still capable of infecting others. Thus, another interpretation of the E and I compartments in our model is that they contain asymptomatic infected and symptomatic infected individuals, respectively, see (e.g. [4]).

The previous hypothesis are based on the following shema

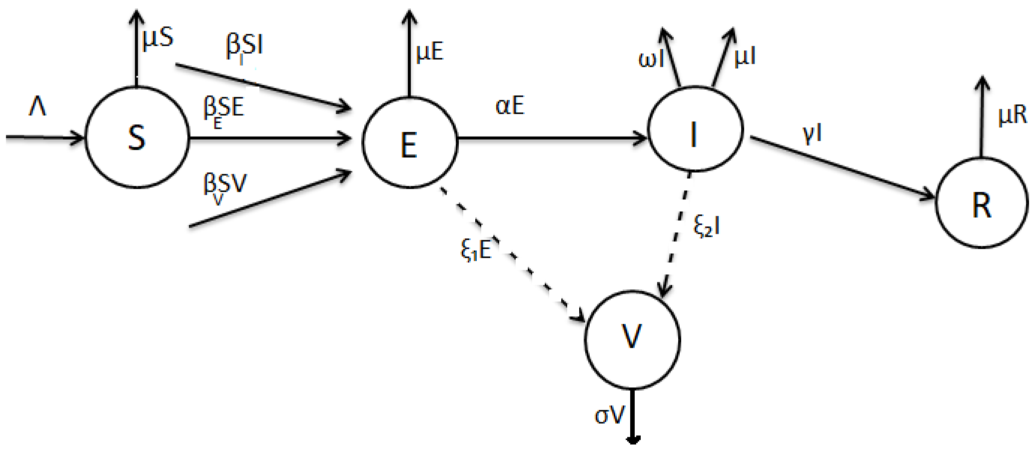

Figure 1: SEIRV Diagramm.

The scheme (1) can be translated into a set of differential equations:

$$\begin{cases} \frac{dS}{dt}=\Lambda-\beta_E(E)SE-\beta_I(I)SI-\beta_V(V)SV-\mu S, \quad t>0,\\ \frac{dE}{dt}=\beta_E(E)SE+\beta_I(I)SI+\beta_V(V)SV-(\alpha+\mu)E, \quad t>0,\\ \frac{dI}{dt}=\alpha E-(\omega+\gamma+\mu)I, \quad t>0,\\ \frac{dR}{dt}=\gamma I\text{-}\mu R, \quad t>0,\\ \frac{dV}{dt}=\xi_1 E+\xi_2 I-\sigma V, \quad t>0. \end{cases} \tag{1}$$

Here. Λ represent the population influx, μ is the natural death rate of human hosts, α^{-1} is the incubation period between the infection and the onset of symptoms, w is the disease-induced death rate, γ the rate of recovery from infection, ξ_1 is the rates of the exposed individuals contributing the coronavirus to the environmental reservoir, ξ_2 is the rates of the infected individuals contributing the coronavirus to the environmental reservoir and σ is the removal rate of the virus from the environment. The system (1), associated with the following initial conditions

$$S(0)=S_0, \quad E(0)=E_0, \quad I(0)=I_0, \quad R(0)=R_0, \quad V(0)=V_0. \tag{2}$$

The functions $\beta_E(E)$ and $\beta_I(I)$ represents the direct, human-to-human transmission rates between the exposed and susceptible individuals, and between the infected and susceptible individuals, respectively. The function $\beta_V(V)$ is the indirect, environment-tohuman transmission rate.

Our objective in this work is to study the local and global stability of the equilibrium points of the system (1) and to give a numerical results. This work is organized as follows

- In chapter (1), we give some mathematical tools.
- In chapter (2), we present the literature review of some Covid-19 models.
- In chapter (3), we analyze the equilibrium of the SEIRV model.
- In chapter (4), we study the local and global stability of our problem.
- In chapter (5), we give a numerical results of SEIRV model and we present the conclusion.

CHAPTER 1

PRELIMINARIES

The mathematical analysis of dynamic systems resulting from epidemiological modeling uses matrices of a very particular type. The differential systems studied in this thesis are nonlinear. We will present most of the results that were used in this work. These results are classic. The stability of a dynamic system, will be revisited. Finally we will present the method of Lyaponov and La Salle theoreme and the calculation of the basic reproduction number $\mathcal{R}_0$. In the following we will need the following definitions and notations.

We introduce some notations that are used in our work and we give the definition of the parameters that are used in the numerical results.

1.1 Notations

- The determinant of real and complex martices, is denoted by $\det(A)$.
- The trace of real and complex matrices, is denoted by $tr(A)$.
- The spectral radius of A, is denoted by $\rho(A)$.
- The disease-free equilibrium, is denoted by (**DFE)**.
- The ordinary differential equations, is denoted by (**EDO**).

1.2 Parameters

- Λ : Influx rate.
- β_{E0} : Transmission constant between S and E.
- β_{I0} : Transmission constant between S and I.
- β_{V0} : Transmission constant between S and V.
- c : Transmission adjustment coefficient.
- μ : Natural death rate.
- $1/\alpha$: Incubation period.
- w : Disease-induced death rate.
- γ : Recovery rate.
- σ : Removal rate of virus.
- ξ_1 : Virus shedding rate by exposed people.
- ξ_2 : Virus shedding rate by infected people.

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1.3 Theoretical framework

The following are some general notions we use in our work.

Definition 1. *(Equilibrium point) We say that X^E an equilibrium point of a system*

$$\begin{cases} \frac{dX(t)}{dt} = f(X(t)), \\ X(0) = X_0, \end{cases} \tag{1.1}$$

if X^E verify the equation $f(X^E) = 0$.

Definition 2. *[35] The equilibrium X^E is said to be stable if for everything $\epsilon > 0$, it exists $\eta > 0$, as for all solution $X(t)$ of (1.1), we have*

$$||X(0) - X^E|| < \eta \Longrightarrow ||X(t) - X^E|| < \epsilon. \tag{1.2}$$

Definition 3. *[35](Stability of an equilibrium point) Let $x_0 \in \Omega$ be an equilibrium point of the system*

$$\dot{x} = X(x).$$

We say that x_0 is a stable equilibrium point for $\dot{x} = X(x)$ or that the system $\dot{x} = X(x)$ is stable in x_0, if for any positive, there exists a positive real number δ such that for any $x \in \Omega$ with $\| x(0) - x_0 \| < \delta$, the solution $X_t(x(0)) = x(t)$. If moreover there exists δ_0 such that $0 < \delta_0 < \delta$ and

$$\| x(0) - x_0 \| < \delta_0 \Rightarrow lim_{t\to+\infty} x(t) = x_0,$$

x_0 is said to be asymptotically stable. The system is said to be unstable at x_0 if it is not stable at x_0.

Definition 4. *[35](Attractive point of equilibrium) - The equilibrium point x_0 is said to be attractive (we will also say that the system*

$$\dot{x} = X(x)$$

is attractive in x_0 if there exists a neighborhood $D \subset \Omega$ of x_0 such that for any initial condition x starting in D, the corresponding solution $X_t(x)$ of the system $\dot{x} = X(x)$ is defined for all $t \geq 0$ and tends to x_0 as t tends to infinity. In other words,

$$lim_{t\to\infty} X_t(x) = x_0$$

for any initial condition $x_0 \in D$,

$$\dot{x} = X(x)$$

.

Definition 5. *[35] The point* x_0 *is said to be globally attractive if*

$$lim_{t\to\infty} X_t(x) = x_0,$$

for any initial condition $x \in \Omega$.

Definition 6. *[35](globally asymptotically stable equilibrium) Let* $x_0 \in \Omega$ *be an equilibrium point of the system* $\dot{x} = X(x)$. *This system is said to be globally asymptotically stable at* x_0 *in* Ω *if it is both stable, attractive and its basin of attraction is* Ω *as a whole.*

Definition 7. *[35](Locally asymptotically stable) Let*

$$J(X^E) = \frac{\partial f}{\partial X}(X^E),$$

the Jacobian matrix of f *evaluates at point* X^E. *Consider the following linear system*

$$\frac{dX}{dt} = AX, \tag{1.3}$$

where $A = J(X^E)$ *is say the linearized or the linear approximation of the non-linear system* (1.1) *in* X^E. *The study of the stability of the origin for the linearized allows in certain cases to characterize the stability of the* (1.1). *More precisely, we have,*

- *If all the eigenvalues of the matrix* A *are of strictly negative real part, then the system* (1.1) *is stable.*

- *If there is at least one eigenvalue of the matrix* A *of strictly positive real part then, the system* (1.1) *is unstable.*

Definition 8. *[35](Globally asymptotically stable) The equilibrium point* X^E *is say to be globally asymptotically stable if it is stable, and for any* $X(t)$ *solution for* (1.1), *we have*

$$\lim_{t\to\infty} ||X(t) - X^E|| = 0. \tag{1.4}$$

Definition 9. *[6] The basic reproduction number* R *is the spectral radius of the next generation matrix, namely*

$$R = \rho\left(FV^{-1}\right). \tag{1.5}$$

The following interpretation is given to the matrix FV^{-1}: Let us consider an infected individual introduced into a compartment FV^{-1} of a population without disease. The entry (i, k) of the matrix V^{-1} is the average time that the individual will spend in compartment i during his life, assuming that the infection has been blocked. The entry (j, i) of matrix F is the speed at which an infected person in compartment i produces infections in compartment j. Thus the entry (j, k) of FV^{-1} is the expected number of new infections in compartment j produced by an infected individual originally introduced into compartment k. The spectral radius of the matrix FV^{-1} is the basic reproduction number. That is to say $R = \rho(FV^{-1})$.

Definition 10. *[35](Lyapunov function) A function $V : \Omega \to R$ is a Lyapunov function for the system $\dot{x} = X(x)$ (X continue) if it decreasing along the trajectories of the system. If V is of class C^1, this amounts to saying that his derivative $\dot{V}$ with respect to the system $\dot{x} = X(x)$ is negative on Ω, i.e., $\dot{V}(x) \leq 0$ for all $x \in \Omega$.*

Theorem 11. *[32](LaSalle's invariance principle) Let Ω be a subset of $\mathbb{R}^n$; suppose that Ω is an open positively invariant for the system (1.1) at x_0. Let $V : \Omega \to R$ be a class C^1 function for the system (1.1) in x_0 such than :*

- $\dot{V} \leq 0$ *on Ω;*
- *Let $E = \{x \in \Omega \mid V(x) = 0\}$ and L the largest set invariant by X and $L \subset E$.*

Then, any bounded solution starting in Ω tends towards the set L as $t \longrightarrow \infty$.

This theorem is a very important tool for the analysis of systems; unlike Lyapunov,it requires neither of the function V to be positive definite, nor of its derivative $\dot{V}$ to be negative. However, heonly provides information on the attractiveness of the considered system at the equilibrium point x_0 .

Corollary 12. *[32] Suppose $\Omega \subset \mathbb{R}^n$ is a connected open such that $x_0 \in \Omega$. Let $V : \Omega \to R$ be a positive definite function of class C^1 such that $\dot{V} \leq 0$ on U. Let $E = \{x \in \Omega \mid V(x) = 0\}$, suppose that the largest positively invariant set containedin E is reduced to the point x_0 . Then, x_0 is an asymptotically stable equilibrium point for the system (1.1). If these conditions are satisfied for $U = \Omega$ if moreover V is proper on Ω, i.e., if*

$$limV(x) = +\infty$$

when

$$d(x, \frac{\partial}{\partial x}\Omega) + \|x\| \to +\infty,$$

then all trajectories are bounded for all $t \geq 0$ and x_0 is a globally stable equilibrium point for the system $\dot{x} = X(x)$.

Corollary 13. *[35] Under the assumptions of the previous theorem, if the set L is reduced to the point $x_0 \in \Omega$, then x_0 is globally asymptotic stable for the system $\dot{x} = X(x)$ defined in Ω.*

Definition 14. *[36](Routh-Hurwitz Stability Criterion) The method depend upon inequalities involving the so-called Hurwitz determinants.*

$$D_k \equiv \begin{vmatrix} a_1 & a_0 & 0 & \cdots & \cdots & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2k-1} & a_{2k-2} & \cdots & \cdots & \cdots & \cdots & a_k \end{vmatrix}, \quad k = 1, 2, \cdots, n, \tag{1.6}$$

where $a_j = 0$ for $j > n$ associated with the coefficients of $Q(s)$. In its most general and perhaps most efficient from the Routh–Hurwitz criterion may be stated as

Theorem 15. *[36] If the polynomial*

$$Q(s) = a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n.$$

has real coefficients, with $a_0 > 0$, then any one of the following conditions is necessary and sufficient for every zero of $Q(s)$ to have negative real part:

(i) $a_n > 0, a_{n-2} > 0, a_{n-4} > 0, \cdots, D_1 > 0, D_3 > 0, \cdots$.

(ii) $a_n > 0, a_{n-2} > 0, a_{n-4} > 0, \cdots, D_2 > 0, D_4 > 0, \cdots$.

(iii) $a_n > 0, a_{n-1} > 0, a_{n-3} > 0, \cdots, D_1 > 0, D_3 > 0, \cdots$.

(iv) $a_n > 0, a_{n-1} > 0, a_{n-3} > 0, \cdots, D_2 > 0, D_4 > 0, \cdots$,

where the determinants are given by (1.6).

CHAPTER 2

LITERATURE REVIEW OF SOME COVID-19 MODELS

In this chapter, we present some models that have already been worked on since the emergence of virus.

In [5], the author developed the following three models to understand the nature of covid-19:
The SIR model

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta IS}{N}, \\ \frac{dI}{dt} = \frac{\beta IS}{N} - \alpha \upsilon I, \\ \frac{dR}{dt} = \alpha \upsilon I, \end{cases} \tag{2.1}$$

the $SEIQR$ model

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta(\omega E+I+Q)S}{N}, \\ \frac{dE}{dt} = \frac{\beta(\omega E+I+Q)S}{N} - \varepsilon\tau E, \\ \frac{dI}{dt} = \varepsilon\tau E - \alpha\left(1-\upsilon\right)I - \upsilon\varphi I, \\ \frac{dQ}{dt} = \upsilon\varphi I - \alpha Q, \\ \frac{dR}{dt} = \alpha Q + \alpha\left(1-\upsilon\right)I, \end{cases} \tag{2.2}$$

and the $SEIQLR$ model

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta(\omega E+I+Q+\lambda L)S}{N}, \\ \frac{dE}{dt} = \frac{\beta(\omega E+I+Q+\lambda L)S}{N} - \tau E, \\ \frac{dI}{dt} = \varepsilon\tau E - \alpha\left(1-\upsilon\right)I - \upsilon\varphi I, \\ \frac{dQ}{dt} = \upsilon\varphi I - \alpha Q, \\ \frac{dL}{dt} = \left(1-\varepsilon\right)\tau E - \eta L, \\ \frac{dR}{dt} = \alpha Q + \alpha\left(1-\upsilon\right)I + \eta L, \end{cases} \tag{2.3}$$

where

$$S(t) + R(t) + I(t) = N.$$

The functions in the previous systems are

S	Susceptible,
E	Number of exposed cases,
I	Number of diagnosed cases,
H	Number of cured cases,
D	Number of dead cases,
N	Total population of Hubei Province,

and the parameters are given by

β	contagion rate
α	Removal rate for quarantine
η	Removal rate for the latent
S	Proportion of people with dominant infection
v	incubation period
v	Transfer rate of diagnosed cases
$1/\varphi$	Average delay edreporting period
λ	Infection-reducing factor sinthelatent infections
α	Curerate
μ	Antibody failure rate.

By an appropriate stability analysis, He spread from whitch it is possible to predict similar and sudden diseases in the future and to optimally control them by taking the necessary measures. In paper [13], the authors introduced the following $SEIARW$ model

$$\begin{cases} \frac{dS}{dt} = \Lambda - mS - \beta_p S\left(I + kA\right) - \beta_w SW, \\ \frac{dE}{dt} = \beta_p S\left(I + kA\right) + \beta_w SW - \left(1 - \delta\right)\omega E - \delta\omega' E - mE, \\ \frac{dI}{dt} = \left(1 - \delta\right)\omega E - \left(\gamma + m\right) I, \\ \frac{dA}{dt} = \delta\omega_p' E - \left(\gamma' + m\right) A, \\ \frac{dR}{dt} = \gamma I + \gamma' A - mR, \\ \frac{dW}{dt} = \mu I + \gamma' A - \varepsilon W, \end{cases} \tag{2.4}$$

associated with the initial conditions

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, A(0) = A_0, W(0) = W_0.$$

In system (2.4), the functions are

S	susceptible people
E	exposed people
I	symptomatic infected people
A	asymptomatic infected people
R	removed people
W	dead people.

and the parameters are

$\lambda = n \times N$	N refer to the total number of people
m	the death rate of people
β_p	the transmission rate from I to S
k	the multiple of the transmissible of A to that of I
β_w	the transmission rate from W to S,
δ	the proportion of asymptomatic infection rate of people
$\frac{1}{\omega}$	the incubation period of people
$\frac{1}{\omega}$	the latent period of people
$\frac{1}{\gamma}$	the infectious period of symptomatic infection of people
$\frac{1}{\gamma'}$	the infectious period of asymptomatic infection of people
μ	the shedding coefficients from I to$W$,
μ'	the shedding coefficients from A to$W$
$\frac{1}{\varepsilon}$	the life time of the virus in W.

They incorporates the homotopy analysis method the laplace transform,and the uniqueness of the solution and the stability of iteration approach using fixed point theory then compare the results to the results of the caputo derivative. In the work [14], khoshnaw et al., studied the following model

$$\begin{cases} \frac{dS}{dt} = \upsilon_4 - (\upsilon_1 + \upsilon_2 + \upsilon_3), \\ \frac{dE}{dt} = \upsilon_1 - (\upsilon_5 + \upsilon_6), \\ \frac{dI}{dt} = \upsilon_6 - (\upsilon_9 + \upsilon_{10} + \upsilon_{11}), \\ \frac{dA}{dt} = \upsilon_5 - \upsilon_8, \\ \frac{dS_q}{dt} = \upsilon_3 - \upsilon_4, \\ \frac{dE_q}{dt} = \upsilon_2 - \upsilon_7, \\ \frac{dH}{dt} = \upsilon_7 + \upsilon_{10} - (\upsilon_{12} + \upsilon_{13}), \\ \frac{dR}{dt} = \upsilon_8 + \upsilon_9 + \upsilon_{12}, \end{cases} \tag{2.5}$$

in system (2.5), the new functions define

S_q	Quarantined Susceptible Individuals
E_q	Quarantined Exposed Individuals
H	Quarantined Infected (Hospitalized) Individuals
R	Recovered Individuals Symbolsand Biological definitions,

and the parameters are

k_1	Contactrate
k_2	Probability of transmission percontact
k_3	Quarantined rate of exposed individuals
k_4	Transition rate of exposed individuals to the infected class
k_5	The multiple of the transmissibility of A to$I$
k_6	Rate at which the quarantined uninfected contacts
k_7	Probability of having symptomsamong infected individuals
k_8	Transition rate of symptomatic infected individuals tothe quarantined infected class
k_9	Transition rate of quarantined exposed individuals
k_{10}	Recovery rate of symptomatic infected individuals
k_{11}	Recovery rate of asymptomatic infected individuals
k_{12}	Recovery rate of quarantined infected individuals
k_{13}	Disease-induced death rate.

The most important in the previous work is the new parameter considered, called reaction rate, which is defined by

$$\begin{aligned}
&\upsilon_1 = k_1k_2\left(1-k_3\right)S\left(I+k_5A\right), \quad \upsilon_2 = k_1k_2k_3S\left(I+k_5A\right)\\
&\upsilon_3 = k_1k_3\left(1-k_2\right)S\left(I+k_5A\right), \quad \upsilon_4 = k_6S_q\\
&\upsilon_5 = k_5\left(1-k_7\right)E, \quad \upsilon_6 = k_4k_7E, \quad \upsilon_7 = k_9E_q, \quad \upsilon_8 = k_{11}A\\
&\upsilon_9 = k_{10}I, \quad \upsilon_{10} = k_8I, \quad \upsilon_{11} = k_{13}I, \quad \upsilon_{12} = k_{12}H, \quad \upsilon_{13} = k_{13}H.
\end{aligned}$$

The authors in [14] developed two models to this following model its idea is based on clinical progress, epidemiological personnel and intervention measures, he study the sensitivity analysis of the model. In [15], Mahrouf et al., developed the SRI model to the following model of forecasting Spreading of Covid-19

$$\begin{cases} \frac{dS(t)}{dt} = -\beta\left(1-u\right)\frac{S(t)I_s(t)}{N}, \\ \frac{dE(t)}{dt} = \beta\epsilon\left(1-u\right)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \alpha I_s\left(t\right) - \left(1-\alpha\right)\left(\mu_s+\eta_s\right)I_s\left(t\right), \\ \frac{dI(t)}{dt} = \beta\left(1-\epsilon\right)\left(1-u\right)\frac{S(t-\tau_1)I_s(t-\tau_1)}{N} - \eta_a I_a\left(t\right), \\ \frac{dA(t)}{dt} = \alpha\gamma_b I_s\left(t-\tau_2\right)\left(\mu_b+r_b\right)F_b\left(t\right), \\ \frac{dS_q(t)}{dt} = \alpha\gamma_b I_s\left(t-\tau_2\right)\left(\mu_g+r_g\right)F_g\left(t\right), \\ \frac{dE_q(t)}{dt} = \alpha\gamma_c I_s\left(t-\tau_2\right)\left(\mu_c+r_c\right)F_c\left(t\right), \\ \frac{dH(t)}{dt} = \eta_s\left(1-\alpha\right)I_s\left(t-\tau_3\right) + r_b F_b\left(t-\tau_4\right) + r_g F_g\left(t-\tau_4\right) + r_c F_c\left(t-\tau_4\right), \\ \frac{dR(t)}{dt} = \mu_s\left(1-\alpha\right)I_s\left(t-\tau_3\right) + \mu_b F_b\left(t-\tau_4\right) + \mu_g F_g\left(t-\tau_4\right) + \mu_c F_c\left(t-\tau_4\right), \end{cases} \tag{2.6}$$

in model (2.6), the parameters β, u and ϵ are the transmission rate, level of the preventive strateg and the proportion for the symptomatic individual respectevely. Note that $u, \epsilon \in [0,1]$. α is the proportion of the diagnosed symptomatic infected populattion that moves to the three forms: F_b, F_g and F_c, by the rates γ_b, γ_g and γ_c, respectively. The mean recovery period of these forms are denoted by $1/r_b$, $1/r_g$ and $1/r_c$, respectivy. The latter forms die also with the rates μ_b, μ_g and μ_c, respectively. Asymptomatic infected population, recover with rate η_a and the symptomatic infected ones recover or die with rates η_s and μ_s, respectively. τ_1, τ_2, τ_3 and τ_4 are the incubation period, the period of time , the time required before the death of individuals coming from the compartments Is, F_b, F_g, and F_c, respectively. The authors in [15] divided the population into eight groups : S :susceptible (symptomatic), I:infected but not transmittinng disease , F_b, F_g, and F_c refer to patients diagnosed , (under quarantine), and is divided into three categories: benign,critical and critical forms , D and M are the categories of cure and mortality. As a conclusion, the work extend the well-known SIR compartmental model to deterministic and stochastic time-delayed models in order to predict the epidemiological trend of COVID-19 in Morocco and to assess the potential role of multiple preventive measures and strategies imposed by Moroccan authorities. In [16], a novel forecast deterministic model for the Covid-19 was introduced as follows

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta IS}{N}, \\ \frac{dI}{dt} = \frac{\beta IS}{N} - I\gamma, \\ \frac{dr}{dt} = I\gamma, \end{cases} \tag{2.7}$$

where $N = S + I + r$ is the total number of population and

$$R(t) = \frac{\beta}{\gamma}s(t) = R_0 s(t).$$

The parameters of system (2.7) are

$\beta = 1/T_c$	contact frequency
T_c	the eaverage time between contacts
$1/\gamma = T_r$	the mean time between infection and removal
$r(t)$	the removed individuals
T_r	can be consequently interpreted as the time until removal from the infection process
$g(t)$	typical of the disease
$R(t)$	generalized effective reproduction number,

the authors focus on data for two exemplary countries , italy and germany predict the course of the $Covid-19$ for a period of four to five weeks with reasonable numerical stability. Bahloul [18] introduced the following SEIRAW model

$$\begin{cases} \frac{dS_B}{dt} = \Lambda_B - m_B S_B - \beta_B S_B I_B, \\ \frac{dE_B}{dt} = \beta_B S_B I_B - \omega_B E_B - m_B E_B, \\ \frac{dI_B}{dt} = \omega_B E_B - (\gamma_B + m_B) I_B, \\ \frac{dR_B}{dt} = \gamma_B I_B - m_B R_B, \\ \frac{dS_H}{dt} = \Lambda_H - m_H S_H - \beta_{BH} S_B I_B - \beta_H S_H I_H, \\ \frac{dE_H}{dt} = \beta_{BH} S_H I_B + \beta_H S_H I_H - \omega_H E_H - m_H E_H, \\ \frac{dI_H}{dt} = \omega_H E_H - (\gamma_B + m_B) I_B, \\ \frac{dR_H}{dt} = \gamma_H I_H - m_H R_H, \\ \frac{dS_P}{dt} = \Lambda_P - m_P S_P - \beta_P S_P (I_P + \kappa A_P) - \beta_W S_P W, \\ \frac{dE_P}{dt} = \beta_P S_P (I_P + \kappa A_P) + \beta_W S_P W - (1 - \delta_P) \omega_P E_P - \delta_P \omega'_P E_P - m_p E_P, \\ \frac{dI_P}{dt} = (1 - \delta_P) \omega_P E_P - (\gamma_P + m_P) I_P, \\ \frac{dA_P}{dt} = \delta_P \omega'_P E_P - (\gamma'_P + m_P) A_P, \\ \frac{dR_P}{dt} = \gamma_P I_P + \gamma'_P A_P - m_P R_P, \\ \frac{dW}{dt} = aW \frac{I_H}{N_H} + \mu_P I_P + \mu'_P A_P - \varepsilon W. \end{cases} \tag{2.8}$$

The functions in system (2.8) represents

S_B	Susceptible bats
E_B	exposed bats
I_B	infected bats
R_B	removed bats
S_H	susceptible hosts
E_H	exposed hosts
I_H	infected hosts
R_H	removed hosts

For the parameters, we have

n_B	birth rate
m_B	death rate
$\lambda_B = n_B \times N_B$	the number of the new born bats
N_B	the total number of bats
$\frac{1}{\omega_B}$	incubation period of bat infection
$\frac{1}{\gamma_B}$	infectious period of bat infection
$\Lambda_H = n_H \times N_H$	the new born bats
N_H	number of hosts
$\frac{1}{\omega_H}$	The incubation period
$\frac{1}{\gamma_H}$	the infectious period
β_H	the transmission rate.

The authors developed $BRHP$ model, to eastimating transmissibility and dynamics of $Covid-19$ transmission, they proved the new reproduction number R_0. In the work [19], the authors studied the following $SEIQRDP$ model

$$\begin{cases} \frac{dS(t)}{dt} = -\alpha S(t) - \beta \frac{S(t)I(t)}{N}, \\ \frac{dE(t)}{dt} = -\gamma E(t) + \beta \frac{S(t)I(t)}{N}, \\ \frac{dI(t)}{dt} = \gamma E(t) - \delta I(t), \\ \frac{dQ(t)}{dt} = \delta I(t) - \lambda(t) Q(t) - \kappa(t) Q(t), \\ \frac{dR(t)}{dt} = \lambda(t) Q(t), \\ \frac{dD(t)}{dt} = \kappa(t) Q(t), \\ \frac{dP(t)}{dt} = \alpha S(t), \end{cases} \tag{2.9}$$

where the functions are

$S(t)$	susceptible people
$E(t)$	exposed people
$I(t)$	infected people
$Q(t)$	quarantined
$R(t)$	removed people
$D(t)$	dead people.
$P(t)$	protected from COVID-19

and the parameters are

α	theprotectionrate
β	theinfectionrate
γ	theinverseoftheaveragelatenttime
δ	therateinquarantine,
$\lambda(t)$	thecurerate
$\frac{1}{\omega}$	the incubation period of people
$\frac{1}{\omega}$	the latent period of people
$\frac{1}{\gamma}$	the infectious period of symptomatic infection of people
$\frac{1}{\gamma'}$	the infectious period of asymptomatic infection of people
μ	the shedding coefficients from I to$W$,
μ'	the shedding coefficients from A to$W$
$\frac{1}{\varepsilon}$	the life time of the virus in W

This fractional-order $SEIQRDP$ model for Simulating the Dynamics of COVID-19 Epidemic, the author developed the $SEIQRDP$ model by using the fractional analysis, because its flexibility and accuracy of its description of complex physical systems also rovides new standards for virus control which predict the end of the virus August 12,2020. In [20], Khoshnaw et al., developed the following *seiarm* model

$$\begin{cases} \frac{ds}{dt} = b - \gamma s(t) - \frac{\delta s(t)(i(t)+\beta a(t))}{N} - \varepsilon s(t) m(t), \\ \frac{de}{dt} = \frac{\delta s(t)(i(t)+\beta a(t))}{N} + \varepsilon s(t) m(t) - (1-\vartheta)\theta e(t) - \vartheta \alpha e(t) - \gamma e(t), \\ \frac{di}{dt} = (1-\vartheta)\theta e(t) - (\rho+\gamma) i(t), \\ \frac{da}{dt} = \vartheta \alpha e(t) - (\sigma+\gamma) a(t), \\ \frac{dr}{dt} = \rho i(t) + \sigma a(t) - \gamma r(t), \\ \frac{dm}{dt} = \tau i(t) + \kappa a(t) - \omega m(t). \end{cases} \tag{2.10}$$

In system (2.10) the parameters represent

b	the rate of birth
γ	rate of death of infected population
σ	the transmission coefficient
β	transmissibility multiple
α	the transmission rate become sinfected
θ	the incubation period
ϑ	the amount of asymptomatic infection
ε	the disease transmission coefficient
ρ	recovery rate
σ	asymptotically infected population
κ	the influence of virus to m by i
ω	the rate of virus removing from m.

They apply the Differential Transformation Method (DTM) to analyze and obtain the solution for the mathematical model previous above. In [21], Ahmad Naim introduce the new $SIUWR$ model as follows

$$\begin{cases} \frac{dS}{dt} = -\beta\left(I+U\right), \\ \frac{dI}{dt} = \beta\left(I+U\right) - \left(\gamma+\delta\right)I, \\ \frac{dU}{dt} = \delta I - \left(\eta+\alpha_1\right)U, \\ \frac{dW}{dt} = \gamma I - \left(\eta+\alpha_2\right)W, \\ \frac{dR}{dt} = \eta W + \eta U. \end{cases} \tag{2.11}$$

System (2.11) is complemented with the following initial conditions

$S(0)$	initial susceptible individuals
$I(0)$	initiala symptomatic infected individuals
$U(0)$	initial unreported symptomatic infected individuals
$W(0)$	initial reported symptomatic infected individuals
$R(0)$	initially recovered individuals

The parameters are given by

β	Transmission rate between susceptible individuals
γ	Transition rate between asymptomatic infected
δ	Transition rate between asymptomatic infected
$\frac{1}{\eta}$	Average time symptomatic infectious have symptoms
α_1	The unreported symptomatic death rate
α_2	The reported symptomatic death rate.

The work of [21] represent a quantitative and qualitative analysis of the COVID-19 pandemic model,he suggest an updated model that includes a system of differential equations with transmission parameters. Some key computational simulations and sensitivity analysis are investigated. Also, the local sensitivities for each model state concerning the model parameters are computed using three different techniques: non-normalizations, half normalizations, and full normalizations. The $SEIR$ developed model in the work [22]

$$\begin{cases} \frac{dS}{dt} = -\beta.I.\frac{S}{N}, \\ \frac{dE}{dt} = \beta.I.\frac{S}{N} - \alpha.E, \\ \frac{dI}{dt} = \alpha.E - \gamma I, \\ \frac{dR}{dt} = \gamma I, \end{cases} \tag{2.12}$$

where the parameters are

β	average contact rate
$\frac{1}{\alpha}$	measures of incubation period
$\frac{1}{\gamma}$	infectious period.

and the reproduction number is

$$R_0 = \frac{\beta}{\gamma}.$$

The system (2.12) of $COVID-19$ modeling in Saudi Arabia, by using the modified Susceptible-Exposed-Infectious-Recovered ($SEIR$), They calculated the reproduction number and simulation results. In [23], the SEI model is introduced as follows

$$\begin{cases} dS\left(t\right) = -E_f \beta S\left(t\right) C\left(t\right) dt, \\ dE\left(t\right) = \left(-E_f \beta S\left(t\right) C\left(t\right) - \gamma C\left(t\right)\right) dt + \sigma C\left(t\right) dW_t, \\ dI\left(t\right) = \gamma C\left(t\right) dt - \sigma C\left(t\right) dW_t, \end{cases} \tag{2.13}$$

where the functions C, S and R represents the infections, the susceptible and the recoveries. The Stochastic SIR model for $COVID-19$ Infection Dynamics for Karnataka after interventions – Learning from European Trends, this is a continuous work in which we are trying to find the model parameters everyday and project the possible scenarios, by varying the exposure factor for the rate of infection, as a result of evolving levels of quarantining. In paper [24], the $SEAIHRem$ model is given by

$$\begin{cases} \frac{dS(t)}{dt} = -l * \beta(t) . \left(\frac{S(t)A(t)}{N} + \frac{S(t)I(t)}{N} + \frac{S(t)H(t)}{N} \right), \\ \frac{dE(t)}{dt} = l * \beta(t) . \left(\frac{S(t)A(t)}{N} + \frac{S(t)I(t)}{N} + \frac{S(t)H(t)}{N} \right) - (\sigma + d) E(t), \\ \frac{dA(t)}{dt} = (1-\gamma) \delta E(t) - (k+d) A(t), \\ \frac{dI(t)}{dt} = \gamma \delta E(t) - 0.13 \lambda I(t) - 0.87 (\kappa + d) I(t), \\ \frac{dH(t)}{dt} = 0.13 \lambda I(t) - kH(t) - (\delta + d) H(t), \\ \frac{dRem(t)}{dt} = k [A(t) + 0.87 I(t) + H(t)] + d [A(t) + I(t) + H(t) + E(t)] + \delta H(t), \end{cases} \tag{2.14}$$

where

$\frac{1}{\sigma}$	incubation period
$\frac{1}{\lambda}$	the mean time between symptomon set to hospitalization
$\frac{1}{k}$	the mean infectious/recovery period
$\frac{1}{\delta}$	the mean time from hospitalization to death
γ	the clinical out break rate
l	the self-protective measures taken by individuals
d	the mitigation measurements taken by the government of the symptomatic .

The system with time-dependent (2.14) for the analyse the evolution of the $SARS-covid-2$ epidemic outbreak in Portugal, a time-dependent dynamic. The SIR model inspired in a model previously used during the MERS outbreak in South Korea was used to analyse the time trajectories of active and hospitalized cases in Portugal. In the work

[25], the $SEIHD$ model is presented by

$$\begin{cases} \frac{dS(t)}{dt} = \mu N(t) - \frac{\beta S(t)I(t)}{N} - \mu S(t), \\ \frac{dE(t)}{dt} = \frac{\beta S(t)I(t)}{N(t)} - \frac{\beta S(t-\tau)I(t-\tau)}{N(t)} e^{-\mu\tau} - \mu E(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t-\tau)I(t-\tau)}{N(t)} e^{-\mu\tau} - \gamma H(t) - \mu I(t) - (1-\omega) H_N(t), \\ \frac{dH(t)}{dt} = \alpha I(t) - \gamma H(t) - \mu H(t), \\ \frac{dH_N(t)}{dt} = (1-\alpha) I(t) - (1-\omega) H_N(t) - \mu H_N(t), \\ \frac{dR(t)}{dt} = \gamma H(t) + (1-\omega) H_N(t) - \mu R(t), \\ \frac{dD(t)}{dt} = \omega H_N(t) + (1-\gamma) H(t). \end{cases} \tag{2.15}$$

In system (2.15), the functions are

S	susceptible
I	infected
R	recuperated or deceased
H	hospitalised
H_N	infected people but not hospitalised(undetected)
D	infected people deceased due to the disease,

and the parameters are

β	Rate of contact of infected people with the population
μ	Recruitment and natural death rate
α	Rate of infected people hospitalised
γ	Recovery rate of infected people those whoare hospitalised
ω	Death rate due to the disease.

Daniel et al., studied the stability analysis of the epidemic model $COVID-19$ (2.15) in case of delay presented in the system. In the work [26], the authors introduced the following $SEIQCRW$ model

$$\begin{cases} \frac{dS}{dt} = \Lambda - b\left(t\right) - Q - \left(b_{qs} + \lambda\right) S, \\ \frac{dE}{dt} = b\left(t\right) - \left(\beta + b_{qs} + \lambda\right) E, \\ \frac{dI}{dt} = \beta E - \left(\delta + \lambda + b_{qs}\right) I, \\ \frac{dQ}{dt} = b_{qs} S + b_{qs} E - \left(b_{qs} + b_{qc} + \lambda\right) Q, \\ \frac{dC}{dt} = b_{qs} Q + b_{ic} I - \left(\delta + b_{cr} + \lambda\right) C, \\ \frac{dR}{dt} = b_{cr} C - \lambda R, \\ \frac{dW}{dt} = k_1 E + k_2 I - \lambda_w W. \end{cases} \tag{2.16}$$

Here, the parameters are

Λ	Birth rate
N	Total population
$\frac{1}{\beta}$	Incubation period
b_{cr}	Recovery rate
λ_w	Rate of removal of the virus from the environmental reservoir
λ	Death rate
b_{eq}	Rate at which exposed are quarantined
b_{ic}	Rate at which highly infectious individuals are confirmed
α_E	Transmission rate from the exposed to the susceptibleα_E
α_I	Transmission rate from the highly infected to the susceptible
α_W	Transmission rate from the environment to the susceptible
ν	Coefficient providing adjustment to the transmission rate
k_1	Rate at which the exposed are contributing
k_2	Rate at which the infected are contributing
b_{sq}	Rate at which susceptible are quarantined
b_{qs}	Rate at which quarantined move back to the susceptible class
b_{qc}	Rate at which quarantine individuals are confirmed
δ	Covid-19 induced death rate.

The Mathematical Model given by (2.16) tak account the transmission of Covid-19 with nonlinear Forces of Infection and the Need for prevention Measure in Nigeria, the authors touched in this paper on the boundedness of the solution, equilibrium point, stability of the free disease equilibrium point, the basic reproduction number , the existence of endemic equilibrium point, numerical results and discussion. Alanazi et al., [27], present the $SEIRP$ model in the following forme

$$\begin{cases} \frac{dS}{dt} = b - \frac{\beta_1 SP}{1+\alpha_1 P} - \frac{\beta_2 S(I_A+I_S)}{1+\alpha_2(I_A+I_S)} + \psi E - \mu S, \\ \frac{dE}{dt} = \frac{\beta_1 SP}{1+\alpha_1 P} + \frac{\beta_2 S(I_A+I_S)}{1+\alpha_2(I_A+I_S)} - \psi E - \mu E - \omega E, \\ \frac{dI_A}{dt} = (1-\delta)\,\omega E - (\mu+\sigma)\, I_A - \gamma_A I_A, \\ \frac{dI_S}{dt} = \delta\omega E - (\mu+\sigma)\, I_s - \gamma_S I_S, \\ \frac{dR}{dt} = \gamma_S I_S + \gamma_A I_A - \mu R, \\ \frac{dP}{dt} = \eta_A I_A + \eta_S I_S - \mu_P P. \end{cases} \tag{2.17}$$

The parameters are

b	Birth rate
μ	death rate
$\frac{1}{\mu}$	life expectancy
μ_P	Natural death rate
$\frac{1}{\mu_P}$	Life expectancy of pathogens in the environment
α_1	Proportion of interaction with an infectious environment
α_2	Proportion of interaction with an infectious individual
β_1	Rate of transmission from S to E due to contactwithP
β_2	Rate of transmission from S to E due to contact with$I_A$and/orI_S
δ	Proportion of symptomatic infectious people
ψ	Progression rate from E back to S due to robust immune system
ω	Progression rate from E to either$I_A$orI_S
σ	Death rate duetothe coronavirus
γ_S	Rate of recovery of the symptomatic population
γ_A	Rate of recovery of the asymptomatic human population
η_S	Rate of virus spread to environment by symptomatic infectious individual sor1
η_A	Rate of virusspread to environment by asymptomatic infectious individuals.

System (2.17) for $COVID-19$ dynamics incorporating the environment and social distancing. In paper [28] , the SIR and $SIR-F$ developed models are

$$\begin{cases} \frac{dS}{dT} = -N^{-1}, \\ \frac{dI}{dT} = N^{-1}\beta SI - \gamma I, \\ \frac{dR}{dT} = I\gamma, \end{cases} \tag{2.18}$$

and

$$\begin{cases} \frac{dS}{dT} = -N^{-1}\beta S, \\ \frac{dI}{dT} = -N^{-1}(1-\alpha_1)\beta SI - (\gamma+\alpha_2)I, \\ \frac{dR}{dT} = I\gamma, \\ \frac{dF}{dT} = N^{-1}\alpha_1\beta SI + \alpha_2 I. \end{cases} \tag{2.19}$$

For system (2.18), we have $N = S + I + R, T$ is the elapsed time from the start date, the authors studied the stability of the system and also they calculated the new reproductive rate(contact rate)R_0, For system (2.19), we have $N = S + I + R + F$ such that the parameters T, β and γ are the elapsed time from the start date, optimized contact rate and mortality rate. In work [28], it is concerned with societal behavior towards the disease and its translation into electronic data. They developed two models , SIR and $SIR - F$, to predict the epidemiological trend of $Covid - 19$ and monitor infection rates, deaths and recoveries.

CHAPTER 3

EQUILIBRIUM'S ANALYSIS OF THE SEIRV MODEL

In this chapter, we analyze the equilibrium points for the following problem

$$\begin{cases} \frac{dS}{dt}=\Lambda - \beta_E(E)SE - \beta_I(I)SI - \beta_V(V)SV - \mu S, \quad t>0, \\ \frac{dE}{dt}=\beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV - (\alpha+\mu)E, \quad t>0, \\ \frac{dI}{dt}= \alpha E - (\omega+\gamma+\mu)I, \quad t>0, \\ \frac{dR}{dt}= \gamma I\text{-}\mu R, \quad t>0, \\ \frac{dV}{dt} = \xi_1 E + \xi_2 I - \sigma V, \quad t>0. \end{cases} \tag{3.1}$$

Noting that the total population N is defined by

$$N(t) = S(t) + E(t) + I(t) + R(t). \tag{3.2}$$

The **SEIRV** model is an transmission system and for the epidemiologically meaningful, it is important to prove that all solutions with non-negative initial data will remain non-negative for all time see, (e.g. [2]).

3.1 Positivity and boundedness of solutions

For the Positivity of solutions, we introduce the following result

Theorem 16. *If $S(0), E(0), I(0), R(0)$ and $V(0)$ are non-negative. Then, the functions*

$S(t), E(t), I(t), R(t)$ and $V(t)$ are non-negative for all time $t > 0$. Moreover, we have

$$\limsup_{t\to\infty} \left(S(t) + E(t) + I(t) + R(t)\right) \leq \frac{\Lambda}{\mu},$$

and also

$$\limsup_{t\to\infty} V(t) \leq \frac{(\xi_1 + \xi_2)\,\frac{\Lambda}{\mu}}{\sigma}.$$

Furthermore, if

$$S(0) + E(0) + I(0) + R(0) \leq \frac{\Lambda}{\mu}.$$

Then, we have

$$S(t) + E(t) + I(t) + R(t) \leq \frac{\Lambda}{\mu},$$

and also

$$0 \leq V(t) \leq \frac{(\xi_1 + \xi_2)\,\frac{\Lambda}{\mu}}{\sigma}.$$

Proof. Let $S(t), E(t), I(t), R(t)$ and $V(t)$ be any solution with positive initial conditions. We have

$$N(t) = S(t) + E(t) + I(t) + R(t),$$

the time derivative of $N(t)$ along the solution of (3.1) is

$$\begin{aligned} \frac{d}{dt}N(t) &= \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt}, \\ &\leq \Lambda - \mu N(t). \end{aligned}$$

By using the theory of differential equations, we obtain the following homogene solution

$$\frac{d}{dt}N(t) = -\mu N(t) \Rightarrow N(t) = N_0 e^{-\mu t},$$

and the non-homogene solution given by

$$\frac{dN(t)}{dt} = \left(\frac{dN_0(t)}{dt} - \mu N_0(t)\right) e^{-\mu t}.$$

Hence, we have

$$N(t) \leq \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right) + N_0 e^{-\mu t},$$

and for $t \to \infty$, we have

$$\overline{\lim_{t\to\infty}} N(t) = \limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}. \tag{3.3}$$

From equation $(3.1)_4$, we have

$$\begin{aligned}\frac{d}{dt}V(t) &= \xi_1 E + \xi_2 I - \sigma V(t),\\ &\leq (\xi_1 + \xi_2)\,\frac{\Lambda}{\mu} - \sigma V(t).\end{aligned}$$

By using the theory of differential equations, we obtain the following homogene solution given by

$$\frac{dV(t)}{dt} = -\sigma V(t) \Rightarrow V(t) = V_0 e^{-\sigma t},$$

and the non-homogene solution given by

$$\frac{dV(t)}{dt} = \left(\frac{dV_0(t)}{dt} - \sigma V_0(t)\right) e^{-\sigma t}.$$

Hence, we have

$$N(t) \leq \frac{\Lambda}{\mu}\left(1 - e^{-\sigma t}\right) + V_0 e^{-\sigma t},$$

and for $t \to \infty$, we have

$$\overline{\lim_{t\to\infty}} V(t) = \limsup_{t\to\infty} V(t) \leq \frac{(\xi_1 + \xi_2)\,\frac{\Lambda}{\mu}}{\sigma}. \tag{3.4}$$

Clearly, it has been proved that all the solutions of (3.1) which initiate in $\mathbb{R}^4_+$ confined in the region D defined by

$$D = \left\{(S, E, I, R, V) \in \mathbb{R}^5_+ : S(t) + E(t) + I(t) + R(t) \leq \frac{\Lambda}{\mu}, \quad 0 \leq V(t) \leq \frac{(\xi_1 + \xi_2)\,\frac{\Lambda}{\mu}}{\sigma}.\right\} \tag{3.5}$$

So, the solution are bounded in the interval $[0, \infty)$. ■

3.2 Existence of the equilibrium points

In the section, we find the quilibrium pionts of model (3.1). By solving the **SEIRV** model equation, we get

$$\begin{cases}\frac{dS}{dt} = 0,\\ \frac{dE}{dt} = 0,\\ \frac{dI}{dt} = 0,\\ \frac{dR}{dt} = 0,\\ \frac{dV}{dt} = 0,\end{cases}$$

Then, we have

$$\begin{cases} \Lambda - \beta_E ESE - \beta_I ISI - \beta_V VSV - \mu S = 0, \\ \beta_E ESE + \beta_I ISI + \beta_V VSV - (\alpha + \mu) E = 0, \\ \alpha E - (w + \gamma + \mu) I = 0, \\ \gamma I - \mu R = 0, \\ \xi_1 E + \xi_2 I - \sigma V = 0. \end{cases} \tag{3.6}$$

3.2.1 Existence of disease-free equilibrium

The disease-free equilibrium (**DFE**) of the **SEIRV** model (3.1) existe only when

$$E = I = R = V = 0,$$

it is given by

$$X_0 = (S_0, E_0, I_0, R_0, V_0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right). \tag{3.7}$$

The basic reproduction number of the model (3.1) is given based on the Defintion (9) as follows

$$\mathcal{R}_0 = \rho\left(F\mathcal{V}^{-1}\right).$$

We have

$$F(X_0) = \begin{bmatrix} \beta_E(0) S_0 & \beta_I(0) S_0 & \beta_V(0) S_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.8}$$

and

$$\mathcal{V}(X_0) = \begin{bmatrix} \alpha + \mu & 0 & 0 \\ -\alpha & w_1 & 0 \\ -\xi_1 & -\xi_2 & \sigma \end{bmatrix}, \tag{3.9}$$

where

$$w_1 = w + \gamma + \mu.$$

Since,

$$\det(\mathcal{V}) = (\alpha + \mu)(w_1)(\sigma) \neq 0.$$

Therfore, the matrix $\mathcal{V}$ is inverse and the inverse is given by

$$\mathcal{V}^{-1} = \frac{1}{\det(\mathcal{V})} \begin{bmatrix} w_1 \sigma & 0 & 0 \\ \alpha\sigma & \sigma(\alpha + \mu) & 0 \\ \alpha\xi_2 + \xi_1 w_1 & \xi_2(\alpha + \mu) & (\alpha + \mu) w_1 \end{bmatrix}. \tag{3.10}$$

After a simple calculation

$$\mathcal{V}^{-1} = \begin{bmatrix} \frac{1}{(\alpha+\mu)} & 0 & 0 \\ \frac{\alpha}{w_1(\alpha+\mu)} & \frac{1}{w_1} & 0 \\ \frac{\alpha\xi_2+\xi_1 w_1}{w_1\sigma(\alpha+\mu)} & \frac{\xi_2}{w_1\sigma} & \frac{1}{\sigma} \end{bmatrix}. \tag{3.11}$$

By a simple calculation, we have

$$F\mathcal{V}^{-1} = \begin{bmatrix} Z_1 & Z_2 & Z_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.12}$$

such that

$$\begin{aligned} Z_1 &= \frac{\beta_E(0) S_0}{(\alpha+\mu)} + \frac{\alpha\beta_I(0) S_0}{w_1(\alpha+\mu)} + \frac{\beta_V(0) S_0 (\alpha\xi_2 + \xi_1 w_1)}{w_1\sigma(\alpha+\mu)}, \\ Z_2 &= \frac{\beta_I(0) S_0}{w_1} + \frac{\beta_V(0) S_0}{w_1\sigma}, \\ Z_2 &= \frac{\beta_V(0) S_0}{\sigma}, \end{aligned}$$

Let's remember that

$$\rho\left(F\mathcal{V}^{-1}\right) = \max_{i=1,2,3} \lambda_i,$$

where λ_i are the eigenvalues of the matrix $F\mathcal{V}^{-1}$. We have

$$\begin{aligned} 0 = \left|F\mathcal{V}^{-1} - \lambda I\right| &= (Z_1 - \lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} \\ &- Z_2\begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix} + Z_3\begin{vmatrix} 0 & -\lambda \\ 0 & 0 \end{vmatrix} \\ &= (Z_1 - \lambda)\lambda^2. \end{aligned} \tag{3.13}$$

The resolution of the equation (3.13), give us

$$\lambda_1 = 0, \quad \lambda_2 = 0 \quad \text{and} \quad \lambda_3 = Z_1.$$

Therefore, the basic reproduction number is given based on the method used in [33] as follows

$$\mathcal{R}_0 = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \tag{3.14}$$

where

$$\mathcal{R}_1 = \frac{\beta_E(0)S_0}{\alpha+\mu},$$

$$\mathcal{R}_2 = \frac{\alpha\beta_I(0)S_0}{w_1(\alpha+\mu)},$$

$$\mathcal{R}_3 = \frac{(\alpha\xi_2+\xi_1 W_1)\beta_V(0)S_0}{\sigma w_1(\alpha+\mu)}.$$

Such that:

- $\mathcal{R}_1$: Mesures the contributions from human- to- human transmission fights "exposed to peoples sensitive".

- $\mathcal{R}_2$: Measure the contributions from the human-to-human transmission routes " infected-tosusceptible, respectively".

- $\mathcal{R}_3$: Represents the contribution from the environment-tohuman transmission route.

3.2.2 Existence of endemic equilibrium

By solving system (3.6), we calculate the equilibrium pionts, then we obtain

$$\begin{cases} S = \frac{1}{\mu}\left[\Lambda - (\beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV)\right], \\ (\alpha+\mu)E = \beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV, \\ E = \frac{W_1}{\alpha}I, \\ R = \frac{\gamma}{\mu}I, \\ V = \frac{\alpha\xi_2+\xi_1 I}{\sigma\alpha}. \end{cases} \tag{3.15}$$

Hence, we get

$$\begin{cases} S = \frac{1}{\mu}\left(\Lambda - (\alpha+\mu)E\right), \\ E = \frac{W_1}{\alpha}I, \\ R = \frac{\gamma}{\mu}I, \\ V = \frac{\alpha\xi_2+\xi_1 W_1}{\sigma\alpha}I. \end{cases} \tag{3.16}$$

Then, by using the second equation of (3.16), we obtain

$$\begin{cases} S = \frac{1}{\mu}\left(\Lambda - \frac{w_1(\alpha+\mu)}{\alpha} I\right), \\ E = \frac{w_1}{\alpha} I, \\ R = \frac{\gamma}{\mu} I, \\ V = \frac{\alpha\xi_2+\xi_1 W_1}{\sigma\alpha} I. \end{cases} \tag{3.17}$$

It follows from the first two equations of (3.17) that S can be denoted by a function of I, namely,

$$S = \phi(I). \tag{3.18}$$

By using the first equation in (3.6), then S can be exressed by the function ψ.

$$\begin{aligned} \psi(I) := S &= \frac{\alpha+\mu}{\beta_E(E)E + \beta_I(I)\underbrace{I}_{=\frac{\alpha}{w_1}E} + \beta_V(V)\underbrace{V}_{=\frac{w_1\xi_1+\alpha\xi_2}{\sigma\alpha}I}} \cdot E \\ &= (\alpha+\mu)\left[\beta_E(\frac{w_1}{\alpha}I) + \frac{\alpha}{w_1}\beta_I(I) + \frac{w_1\xi_1+\alpha\xi_2}{\sigma w_1}\beta_V(\frac{w_1\xi_1+\alpha\xi_2}{\sigma\alpha}I)\right]^{-1}, \end{aligned} \tag{3.19}$$

the function (3.19) give

$$\begin{aligned} \psi(0) &= (\alpha+\mu)\left[\beta_E(0) + \frac{\alpha}{w_1}\beta_I(0) + \frac{w_1\xi_1+\alpha\xi_2}{\sigma w_1}\beta_V(0)\right]^{-1} \\ &= \frac{S_0}{\mathcal{R}_0}. \end{aligned} \tag{3.20}$$

The intersection of curves $S = \phi(I)$ for $I \geq 0$ and $S = \psi(I)$ for $I \geq 0$ in $\mathbb{R}_+$ determine the equilibria non-DFE, i.e., from equation (3.18), we get

$$\frac{d}{dI}\phi(I) = -\frac{w_1(\alpha+\mu)}{\mu\alpha} < 0 \tag{3.21}$$

$$\Leftrightarrow \quad \phi \text{ is streactly decreasing,}$$

from equation (3.19), we get

$$\frac{d}{dI}\psi(I) = -\frac{Z_4}{Z_5} \geq 0 \tag{3.22}$$

$$\Leftrightarrow \quad \psi \text{ is increasing.}$$

Because

$$Z_4 = \frac{w_1}{\alpha}\overbrace{\frac{d}{dI}\beta_E(\frac{w_1}{\alpha}I)}^{\leq 0} + \frac{\alpha}{w_1}\overbrace{\frac{d}{dI}\beta_I(I)}^{\leq 0} + \frac{(w_1\xi_1+\alpha\xi_2)^2}{\sigma^2\alpha w_1}\overbrace{\frac{d}{dI}\beta_V(\frac{w_1\xi_1+\alpha\xi_2}{\sigma\alpha}I)}^{\leq 0}.$$

and

$$Z_5 = \left[\beta_E(\frac{w_1}{\alpha}I) + \frac{\alpha}{w_1}\beta_I(I) + \frac{w_1\xi_1+\alpha\xi_2}{\sigma w_1}\beta_V(\frac{w_1\xi_1+\alpha\xi_2}{\sigma\alpha}I)\right]^2.$$

Furthermore we have :

$$\phi(0) = S_0 \quad \text{and} \quad \phi(I_1) = 0 \quad \text{with} \quad I_1 = \frac{\Lambda\alpha}{w_1(\alpha+\mu)}$$

and also from the equation (3.14) and (3.19) we have

$$\psi(0) = \frac{S_0}{\mathcal{R}_0}. \tag{3.23}$$

The equation (3.23) we have the two following cases

- Intersection insidde of $\mathbb{R}^2$ if

$$\mathcal{R}_0 > 1 \quad \Leftrightarrow \quad \frac{\phi(0)}{\psi(0)} > 1 \quad \Leftrightarrow \quad \phi(0) > \psi(0).$$

  This gives a unique endemic equilibrium (EE) X.

- No intersection inside $\mathbb{R}^2$ if

$$\mathcal{R}_0 \leq 1 \quad \Leftrightarrow \quad \frac{\phi(0)}{\psi(0)} \leq 1 \quad \Leftrightarrow \quad \phi(0) \leq \psi(0).$$

  This does not give an endemic equilibrium (EE) X_* but gives an equilibrium (DFE) X_0.

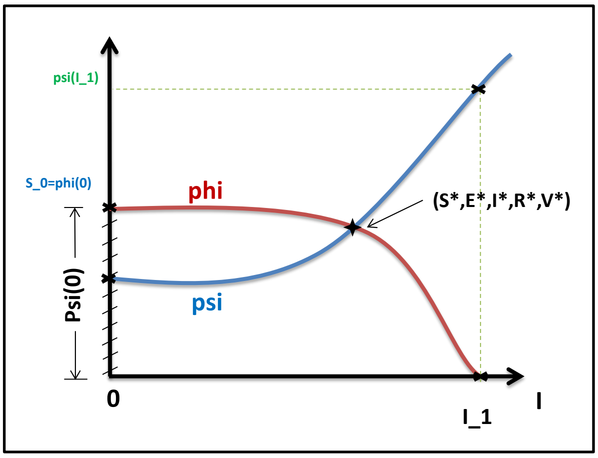

Figure 3.1: Illustration of ψ and ϕ functions

In the following, we carry out a study on the overall stability of DFE. By a simple principle of comparison, we find that

$$0 \leq S + E + I + R \leq S_0, \tag{3.24}$$

$$0 \leq V \leq \frac{(\xi_1 + \xi_2)S_0}{\sigma}, \tag{3.25}$$

By using (3.24) and (3.25), we can get

$$\Omega = \left\{ (S, E, I, R, V) \in \mathbb{R}^5_+ : 0 \leq S + E + I + R \leq S_0, \quad 0 \leq V \leq \frac{(\xi_1 + \xi_2)S_0}{\sigma} \right\}. \tag{3.26}$$

Proposition 17. *If $R_0 > 1$, the model has a two equilibria, the **DFE** X_0 and the **EE** X_*. If $R_0 \leq 1$, the system* (3.1) *admits a unique equilibrium X_0.*

We conclude that the system (3.1) has two possible non-negative equilibria namely the disease-free equilibrium (**DFE**) X_0 and the (**EE**) X_*.

CHAPTER 4

STABILITY ANALYSIS OF THE SEIRV MODEL

4.1 Local stability

In this section we study the local stability of equilibre points of the model (3.1).

4.1.1 Local stability of the disease-free equilibrium

Let examine the local stability of the disease-free equilibriumis $X_0 = (S_0, 0, 0, 0, 0)$. In order to simplify the notations, we adopt the abbreviations

$$\beta_E(E) = \beta_E, \quad \beta_I(I) = \beta_I, \quad \beta_V(V) = \beta_V$$
$$\beta_E(0) = \beta_{E0}, \quad \beta_I(0) = \beta_{I0}, \quad \beta_V(0) = \beta_{V0}.$$

Proposition 18. *Let $\mathcal{R}_0 < 1$. Then, the disease-free equilibrium (**DFE**) of the system* (3.6) *is locally asymptotically stable.*

Proof. The Jacobian matrix for the system (3.1) is given by

$$J(X) = \begin{bmatrix} -\beta_E E - \beta_I I - \beta_V V - \mu & -\beta_E' SE - \beta_E S & -\beta_I' SI - \beta_I S & 0 & -\beta_V' SV - \beta_V S \\ \beta_E E + \beta_I I + \beta_V V & \beta_E' SE + \beta_E S - (\alpha + \mu) & \beta_I' SI + \beta_I S & 0 & \beta_V' SV + \beta_V S \\ 0 & \alpha & -w_1 & 0 & 0 \\ 0 & 0 & \gamma & -\mu & 0 \\ 0 & \xi_1 & \xi_2 & 0 & -\sigma. \end{bmatrix} \tag{4.1}$$

The evaluation of (4.27) at $X_0 = (S_0, 0, 0, 0, 0)$ gives us

$$J(X_0) = \begin{bmatrix} -\mu & -\beta_{E0}S_0 & -\beta_{I0}S_0 & 0 & -\beta_{V0}S_0 \\ 0 & \beta_{E0}S_0 - (\alpha+\mu) & \beta_{I0}S_0 & 0 & \beta_{V0}S_0 \\ 0 & \alpha & -w_1 & 0 & 0 \\ 0 & 0 & \gamma & -\mu & 0 \\ 0 & \xi_1 & \xi_2 & 0 & -\sigma \end{bmatrix}.$$

It is clear that $-\mu$ is a double eigenvalue, so by deleting the first and fourth columns and likewise the first and fourth rows, the Jacobian matrix will reduce to

$$J(X_0) = \begin{bmatrix} \beta_{E0}S_0 - \alpha_\mu & \beta_{I0}S_0 & \beta_{I0}S_0 \\ \alpha & -w_1 & 0 \\ \xi_1 & \xi_2 & -\sigma \end{bmatrix}, \tag{4.2}$$

where

$$\alpha_\mu = \alpha + \mu.$$

By recalling that

$$\begin{aligned} \mathcal{R}_0 &= \frac{\beta_{E0}S_0}{\alpha_\mu} + \frac{\alpha\beta_{I0}S_0}{w_1\alpha_\mu} + \frac{\beta_{V0}S_0(\alpha\xi_2 + w_1\xi_1)}{\sigma w_1\alpha_\mu} \\ &:= \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \end{aligned} \tag{4.3}$$

we get

$$J(X_0) = \begin{bmatrix} \mathcal{R}_{01} & \mathcal{R}_{02} & \mathcal{R}_{03} \\ \alpha & -w_1 & 0 \\ \xi_1 & \xi_2 & -\sigma \end{bmatrix}, \tag{4.4}$$

where

$$\mathcal{R}_{01} := \beta_{E0}S_0 - \alpha_\mu = \alpha_\mu(\mathcal{R}_1 - 1), \tag{4.5}$$

$$\mathcal{R}_{02} := \beta_{I0}S_0 = \frac{w_1\alpha_\mu\mathcal{R}_2}{\alpha}, \tag{4.6}$$

$$\mathcal{R}_{03} := \beta_{V0}S_0 = \frac{\mathcal{R}_3\sigma w_1\alpha_\mu}{w_1\xi_1 + \alpha\xi_2}. \tag{4.7}$$

Therefore, we calculate the eigenvalues of the reduced matrix by the following equation

$$\det(J(X_0) - \lambda I) = 0,$$

which leads to the following characteristic polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{4.8}$$

the coefficients are

$$a_1 = \sigma + w_1 - \mathcal{R}_{01}, \tag{4.9}$$

$$a_2 = -\alpha\mathcal{R}_{02} - (\sigma + w_1)\mathcal{R}_{01} - \xi_1\mathcal{R}_{03} + \sigma w_1, \tag{4.10}$$

$$a_3 = -\mathcal{R}_{01}\sigma w_1 - \mathcal{R}_{02}\sigma\alpha - \mathcal{R}_{03}(\alpha\xi_2 + w_1\xi_1). \tag{4.11}$$

For the application of the stability conditions to the equation (4.8). That result condition from Theorem (15) is

$$a_1 > 0, \qquad a_3 > 0, \qquad a_1a_2 - a_3 > 0 \tag{4.12}$$

Now, we consider the equation (4.9) and by using (4.5), we get

$$\begin{aligned} a_1 &= \sigma + w_1 - \mathcal{R}_{01} \\ &= \sigma + w_1 - \alpha_\mu(\mathcal{R}_1 - 1) \\ &> 0. \end{aligned} \tag{4.13}$$

Therefore, (4.13) will be checked if and only if $\mathcal{R}_0 < 1$. Next, we consider the equation (4.11) and by using (4.5)-(4.7) with (4.3), we obtain

$$\begin{aligned} a_3 &= -\mathcal{R}_{01}\sigma w_1 - \mathcal{R}_{02}\sigma\alpha - \mathcal{R}_{03}(\alpha\xi_2 + w_1\xi_1) \\ &= -\alpha_\mu\sigma w_1(\mathcal{R}_1 - 1) - \frac{\sigma\alpha w_1\alpha_\mu\mathcal{R}_2}{\alpha} - \mathcal{R}_3\sigma w_1\alpha_\mu \\ &= -\alpha_\mu\sigma w_1\mathcal{R}_1 + \alpha_\mu\sigma w_1 - \sigma w_1\alpha_\mu\mathcal{R}_2 - \mathcal{R}_3\sigma w_1\alpha_\mu \\ &= \alpha_\mu\sigma w_1(1 - \mathcal{R}_1 - \mathcal{R}_2 - \mathcal{R}_3) \\ &= \alpha_\mu\sigma w_1(1 - \mathcal{R}_0) \\ &> 0. \end{aligned} \tag{4.14}$$

Hence, (4.14), will be checked if and only if $\mathcal{R}_0 < 1$. Finally, we investigate the third stability condition, with some algebraic computations, we have

$$\begin{aligned} a_1a_2 - a_3 &= [(\sigma + w_1) - \mathcal{R}_{01}] \left[-\alpha\mathcal{R}_{02} - (\sigma + w_1)\mathcal{R}_{01} - \xi_1\mathcal{R}_{03} + \sigma w_1\right] \\ &\quad - \alpha_\mu \sigma w_1 (1 - \mathcal{R}_0) \\ &> [(\sigma + w_1) - \mathcal{R}_{01}] \left[-\alpha\mathcal{R}_{02} - (\sigma + w_1)\mathcal{R}_{01} - \xi_1\mathcal{R}_{03}\right] \\ &\quad - \alpha_\mu \sigma w_1 (1 - \mathcal{R}_0). \end{aligned} \tag{4.15}$$

Recalling that

$$\mathcal{R}_0 = \frac{\mathcal{R}_{01}}{\alpha_\mu} + 1 + \frac{\alpha\mathcal{R}_{02}}{w_1\alpha_\mu} + \frac{(w_1\xi_1 + \alpha\xi_2)\mathcal{R}_{03}}{\sigma w_1 \alpha_\mu}. \tag{4.16}$$

Then, by using (4.16), we get

$$\begin{aligned} -(\sigma + w_1)\mathcal{R}_{01} &= \alpha_\mu(\sigma + w_1)(1 - \mathcal{R}_0) + \frac{(\sigma + w_1)\alpha\mathcal{R}_{02}}{w_1} \\ &\quad + \frac{(\sigma + w_1)(w_1\xi_1 + \alpha\xi_2)\mathcal{R}_{03}}{\sigma w_1}. \\ &= \alpha_\mu(\sigma + w_1)(1 - \mathcal{R}_0) + \frac{\sigma\alpha\mathcal{R}_{02}}{w_1} + \alpha\mathcal{R}_{02} + \xi_1\mathcal{R}_{03} \\ &\quad + \frac{\alpha\xi_2\mathcal{R}_{03}}{w_1} + \frac{(w_1\xi_1 + \alpha\xi_2)\mathcal{R}_{03}}{\sigma}. \end{aligned} \tag{4.17}$$

Substituting (4.17) into (4.15), we obtain

$$\begin{aligned} a_1a_2 - a_3 &> [(\sigma + w_1) - \mathcal{R}_{01}] \left[-\alpha\mathcal{R}_{02} - (\sigma + w_1)\mathcal{R}_{01} - \xi_1\mathcal{R}_{03}\right] \\ &\quad - \alpha_\mu \sigma w_1 (1 - \mathcal{R}_0) \\ &> [(\sigma + w_1) - \mathcal{R}_{01}] \Bigg[-\alpha\overline{\mathcal{R}}_2 + \alpha_\mu(\sigma + w_1)(1 - \mathcal{R}_0) + \frac{\sigma\alpha\mathcal{R}_{02}}{w_1} + \alpha\overline{\mathcal{R}}_2 + \xi_1\overline{\mathcal{R}}_3 \\ &\quad + \frac{\alpha\xi_2\mathcal{R}_{03}}{w_1} + \frac{(w_1\xi_1 + \alpha\xi_2)\mathcal{R}_{03}}{\sigma} - \xi_1\overline{\mathcal{R}}_3\Bigg] - \alpha_\mu \sigma w_1 (1 - \mathcal{R}_0) \\ &> [(\sigma + w_1) - \mathcal{R}_{01}] \left[\alpha_\mu(\sigma + w_1)(1 - \mathcal{R}_0)\right] - \alpha_\mu \sigma w_1 (1 - \mathcal{R}_0) \\ &> \alpha_\mu(1 - \mathcal{R}_0)\left((\sigma + w_1)^2 - (\sigma + w_1)\mathcal{R}_{01} - \sigma w_1\right) \\ &> \alpha_\mu(1 - \mathcal{R}_0)\left((\sigma + w_1)^2 - \sigma w_1\right) \\ &> \alpha_\mu(1 - \mathcal{R}_0)\left(\sigma^2 + w_1^2\right) > 0. \end{aligned} \tag{4.18}$$

Inequality (4.18) will be checked if and only if $\mathcal{R}_0 < 1$. From the above relation all the stability conditions (4.12) are satisfied and the disease-free equilibrium X_0 is locally asymptotically stable. ■

4.1.2 Local stability of endemic equilibrium

We study the stability local of endemic equilibrium X_* in (3.16) in the system (3.1), and let

$$P_* < -\frac{\beta_P^*}{\beta_P^{*\prime}}, \tag{4.19}$$

where P_* can represent E_*, I_* or V_*. The stability result is given as follows

Theorem 19. *Let $\mathcal{R}_0 > 1$, and assume that the hypothesis (4.19) is verified. Then the endemic equilibrium X_* of the SEIRV model is locally asymptotically stable.*

Proof. In order to simplify the notations, we adopt the abbreviations

$$\beta_E(E_*) = \beta_E^*, \quad \beta_I(I_*) = \beta_I^*, \quad \beta_V(V_*) = \beta_V^*.$$

The Jacobian matrix for the system (3.1) evaluated in $X_* = (S_*, E_*, I_*, V_*, R_*)$ is given by

$$J(X_*) = \begin{bmatrix} L_1 & -AS_* & -BS_* & 0 & -CS_* \\ L_2 & L_3 & BS_* & 0 & CS_* \\ 0 & \alpha & -w_1 & 0 & 0 \\ 0 & 0 & \gamma & -\mu & 0 \\ 0 & \xi_1 & \xi_2 & 0 & -\sigma \end{bmatrix}, \tag{4.20}$$

where

$$L_1 := -\frac{\Lambda}{S_*} = -\beta_E^* E_* - \beta_I^* I_* - \beta_V^* V_* - \mu, \tag{4.21}$$

$$L_2 := \frac{\alpha_\mu E_*}{S_*} = \beta_E^* E_* + \beta_I^* I_* + \beta_V^* V_*, \tag{4.22}$$

$$A = \beta_E^{*\prime} E_* + \beta_E^*, \tag{4.23}$$

$$L_3 := AS_* - \alpha_\mu, \tag{4.24}$$

$$B = \beta_I^{*\prime} I_* + \beta_I^*, \tag{4.25}$$

$$C = \beta_V^{*\prime} V_* + \beta_V^*. \tag{4.26}$$

It is clear that $-\mu$ is a eigenvalue, so by deleting the fourth columns and the fourth rows, the Jacobian matrix will reduce to

$$J(X_*) = \begin{bmatrix} L_1 & -AS_* & -BS_* & -CS_* \\ L_2 & L_3 & BS_* & CS_* \\ 0 & \alpha & -w_1 & 0 \\ 0 & \xi_1 & \xi_2 & -\sigma \end{bmatrix}. \tag{4.27}$$

The characteristic equation corresponding to $J(X_*)$ is given by

$$\det\left(J(X_*) - \lambda I\right) = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4, \tag{4.28}$$

where

$$\begin{aligned}
b_1 &= -(L_3 + L_1) + w_1 + \sigma, && (4.29)\\
b_2 &= L_1L_3 - (L_1 + L_3)(\sigma + w_1) - S_*(\alpha B + \xi_1 C) + \sigma w_1 + AS_*L_2, && (4.30)\\
b_3 &= L_1L_3(\sigma + w_1 + \sigma w_1) - (L_1 + L_3)\sigma w_1 + L_2S_*((\sigma + w_1)A + \xi_1 C + \alpha B) && (4.31)\\
&\quad + L_1S_*(\alpha B + \xi_1 C) - S_*(\alpha\xi_2 C + \alpha\sigma B + w_1\xi_1 C),\\
b_4 &= L_1L_3\sigma w_1 + L_1S_*(\alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) && (4.32)\\
&\quad + L_2S_*(\sigma w_1 A + \alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C).
\end{aligned}$$

Note that

$$\begin{aligned}
L_1 &= -\frac{\Lambda}{S_*} = -\frac{1}{S_*}(\mu S_* + \alpha_\mu E_*)\\
&= -\mu - \frac{\alpha_\mu E_*}{S_*}\\
&= -\mu - L_2. && (4.33)
\end{aligned}$$

Firstly, from equation (4.29), we have $b_1 > 0$. Then, from equation (4.30)and by using (4.33) with (4.19), we get

$$\begin{aligned}
b_2 &= L_1L_3 - (L_1 + L_3)(\sigma + w_1) - S_*(\alpha B + \xi_1 C) + \sigma w_1 + AS_*L_2\\
&= L_1L_3 - (L_1 + L_3)(\sigma + w_1) - S_*(\alpha B + \xi_1 C) + \sigma w_1 + AS_*(-L_1 - \mu)\\
&= L_1(L_3 - AS_*) - (L_1 + L_3)(\sigma + w_1) - S_*(\mu A + \alpha B + \xi_1 C) + \sigma w_1\\
&= L_1(-\alpha_\mu) - (L_1 + L_3)(\sigma + w_1) - S_*(\mu A + \alpha B + \xi_1 C) + \sigma w_1\\
&> 0.
\end{aligned}$$

Then, $b_2 > 0$. Next, from equation (4.31) and by using (4.33) with (4.19), we get

$$\begin{aligned}
b_3 &= L_1L_3(\sigma + w_1 + \sigma w_1) - (L_1 + L_3)\sigma w_1 + L_2S_*((\sigma + w_1)A + \xi_1 C + \alpha B) \\
&\quad + L_1S_*(\alpha B + \xi_1 C) - S_*(\alpha\xi_2 C + \alpha\sigma B + w_1\xi_1 C) \\
&= L_1L_3(\sigma + w_1 + \sigma w_1) - (L_1 + L_3)\sigma w_1 \\
&\quad + S_*(-L_1 - \mu)((\sigma + w_1)A + \xi_1 C + \alpha B)) \\
&\quad + L_1S_*(\alpha B + \xi_1 C) - S_*(\alpha\xi_2 C + \alpha\sigma B + w_1\xi_1 C) \\
&= L_1L_3(\sigma + w_1 + \sigma w_1) - (L_1 + L_3)\sigma w_1 - L_1S_*(\sigma + w_1)A \\
&\quad - L_1S_*\alpha B - \mu S_*(\xi_1 C + \alpha B) + L_1S_*(\alpha B + \xi_1 C) \\
&\quad - S_*(\alpha\xi_2 C + \alpha\sigma B + w_1\xi_1 C) - \mu S_*(\sigma + w_1)A - L_1S_*\xi_1 C \\
&= L_1(\sigma + w_1)(L_3 - AS_*) + L_1L_3(\sigma w_1) - (L_1 + L_3)\sigma w_1 \\
&\quad - S_*(\mu(\sigma + w_1)A + (\alpha\sigma + \mu\alpha)B + (\alpha\xi_2 + w_1\xi_1 + \mu\xi_1)C) \\
&= L_1(\sigma + w_1)(-\alpha_\mu) + L_1L_3(\sigma w_1) - (L_1 + L_3)\sigma w_1 \qquad (4.34) \\
&\quad - S_*(\mu(\sigma + w_1)A + (\alpha\sigma + \mu\alpha)B + (\alpha\xi_2 + w_1\xi_1 + \mu\xi_1)C) \\
&> 0.
\end{aligned}$$

Finally, from (4.32). By using (4.33) and (4.19), we get

$$\begin{aligned}
b_4 &= L_1L_3\sigma w_1 + L_1S_*(\alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&\quad + L_2S_*(\sigma w_1 A + \alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&= L_1L_3\sigma w_1 + L_1S_*(\alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&\quad + S_*(-\mu - L_1)(\sigma w_1 A + \alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&= L_1L_3\sigma w_1 + L_1S_*(\alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&\quad - S_*(\mu\sigma w_1 A + \mu\alpha\sigma B + \mu(\alpha\xi_2 + w_1\xi_1)C) \\
&\quad - L_1S_*\sigma w_1 A - L_1S_*(\alpha\sigma B + (\alpha\xi_2 + w_1\xi_1)C) \\
&= -S_*(\mu\alpha\sigma B + \mu(\alpha\xi_2 + w_1\xi_1)C + \sigma w_1\mu A) + L_1\sigma w_1(L_3 - AS_*) \\
&= -S_*(\mu\alpha\sigma B + \mu(\alpha\xi_2 + w_1\xi_1)C + \sigma w_1\mu A) + L_1\sigma w_1(-\alpha_\mu) \\
&= \mu\sigma w_1 + K_0 + \frac{\alpha_\mu\Lambda\sigma w_1}{S_*} \\
&> 0.
\end{aligned}$$

where

$$K_0 = \left(-\frac{\alpha S_* B}{w_1} - \frac{CS_*(\alpha\xi_2 + w_1\xi_1)}{\sigma w_1} - AS_*\right) > 0.$$

Now, it's clear that

$$b_j > 0 \text{ for } j = 1, 2, 3, 4.$$

By using Theorem (15), X_* is locally asymptotically stable if the following conditions hold

$$\begin{aligned} &\text{(i) } b_1 b_2 - b_3 > 0, \\ &\text{(ii) } \ b_3(b_1 b_2 - b_3) - b_1^2 b_4 > 0. \end{aligned} \tag{4.35}$$

We can estimate $b_1 b_2 - b_3$ as follows

$$\begin{aligned} b_1 b_2 - b_3 = {} & [-(L_3 + L_1) + (w_1 + \sigma)] \times \\ & \quad [-\alpha_\mu L_1 - (L_1 + L_3)(\sigma + w_1) - S_*(\mu A + \alpha B + \xi_1 C) + \sigma w_1] \\ & - [-\alpha_\mu L_1(\sigma + w_1) + L_1 L_3(\sigma w_1) - (L_1 + L_3)\sigma w_1 \\ & \qquad -S_*(\mu(\sigma + w_1)A + (\alpha\sigma + \mu\alpha)B + (\alpha\xi_2 + w_1\xi_1 + \mu\xi_1)C)] \\ = {} & \alpha_\mu L_1(L_3 + L_1) + (L_1 + L_3)^2(\sigma + w_1) \\ & + (L_3 + L_1)(S_*(\mu A + \alpha B + \xi_1 C)) - (w_1 + \sigma)\alpha_\mu L_1 - (\sigma + w_1)^2(L_1 + L_3) \\ & - (\sigma + w_1)(S_*(\mu A + \alpha B + \xi_1 C) - \sigma w_1) \\ & + \alpha_\mu L_1(\sigma + w_1) - L_1 L_3(\sigma w_1) + (L_1 + L_3)\sigma w_1 - (L_1 + L_3)\sigma w_1 \\ & + S_*(\mu(\sigma + w_1)A + (\alpha\sigma + \mu\alpha)B + (\alpha\xi_2 + w_1\xi_1 + \mu\xi_1)C) \\ = {} & \alpha_\mu L_1(L_3 + L_1) + (L_1 + L_3)^2(\sigma + w_1) \\ & + (L_3 + L_1)(S_*(\mu A + \alpha B + \xi_1 C)) - (\sigma + w_1)^2(L_1 + L_3) - L_1 L_3(\sigma w_1) \\ & + S_*(\mu\alpha B + \alpha\xi_2 C + \mu\xi_1 C) + (\sigma w_1)(\sigma + w_1) \\ = {} & \alpha_\mu L_1(L_3 + L_1) + (L_1 + L_3)^2(\sigma + w_1) + L_1(S_*(\mu A + \alpha B + \xi_1 C)) \\ & + (AS_* - \alpha_\mu)(S_*(\mu A + \alpha B + \xi_1 C)) - (\sigma + w_1)^2(L_1 + L_3) - L_1 L_3(\sigma w_1) \\ & + S_*(\mu\alpha B + \alpha\xi_2 C + \mu\xi_1 C) - S_*(w_1\alpha B + \sigma\xi_1 C) + (\sigma w_1)(\sigma + w_1) \\ = {} & \alpha_\mu L_1(L_3 + L_1) + (L_1 + L_3)^2(\sigma + w_1) + L_1(S_*(\mu A + \alpha B + \xi_1 C)) \\ & + AS_*^2(\mu A + \alpha B + \xi_1 C) - S_*((\alpha\mu + \mu^2)A + \alpha^2 B + \alpha(\xi_1 - \xi_2)C) \\ & - (\sigma + w_1)^2(L_1 + L_3) - L_1 L_3(\sigma w_1) - S_*(w_1\alpha B + \sigma\xi_1 C) + (\sigma w_1)(\sigma + w_1) \\ = {} & (L_1 S_* + AS_*^2)(\mu A + \alpha B + \xi_1 C) \\ & - S_*((\alpha\mu + \mu^2)A + (\alpha^2 + w_1\alpha)B + (\alpha\xi_1 + \sigma\xi_1 - \alpha\xi_2)C) \\ & + \alpha_\mu L_1(L_3 + L_1) + (L_1 + L_3)^2(\sigma + w_1) - (\sigma + w_1)^2(L_1 + L_3) \\ & - L_1 L_3(\sigma w_1) + (\sigma w_1)(\sigma + w_1) \\ = {} & K_3 + K_2 L_1 L_3 + K_1 C S_*, \end{aligned}$$

where

$$K_1 = L_1 + AS_* - \alpha\xi_1 - \sigma\xi_1 + \alpha\xi_2 < 0, \tag{4.36}$$

$$K_2 = \alpha_\mu + 2(\sigma + w_1) - \sigma w_1 > 0, \tag{4.37}$$

$$\begin{aligned} K_3 = {} & (L_1 S_* + AS_*^2)(\mu A + \alpha B) - S_*((\alpha\mu + \mu^2)A + (\alpha^2 + w_1\alpha)B) + (\sigma w_1)(\sigma + w_1) \\ & - (\sigma + w_1)^2(L_1 + L_3) + (\alpha_\mu + \sigma + w_1)L_1^2 + (\sigma + w_1)L_3^2 > 0 \end{aligned} \tag{4.38}$$

Then, we get

$$b_1 b_2 - b_3 > 0.$$

For the last inequality in (4.35), we have

$$\begin{aligned} b_3(b_1b_2 - b_3) - b_1^2 b_4 &= b_3\left[K_3 + K_2L_1L_3 + K_1CS_*\right] \\ &\quad - \left[-(L_3 + L_1) + (w_1 + \sigma)\right]^2 \left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right] \\ &= b_3\left[K_3 + K_2L_1L_3 + K_1CS_*\right] \\ &\quad + \left[2(w_1 + \sigma)(L_3 + L_1) - (\sigma + w_1)^2 - (L_3 + L_1)^2\right] \times \\ &\qquad \left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right] \\ &= b_3\left[K_3 + K_2L_1L_3 + K_1CS_*\right] \\ &\quad + \left[2(w_1 + \sigma)(L_3 + L_1)\right]\left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right] \\ &\quad - \left[(\sigma + w_1)^2 + (L_3 + L_1)^2\right]\left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right]. \end{aligned}$$

From inequalities (4.34) and (4.38), we have

$$\begin{aligned} b_3K_3 = {} & b_3\left[(L_1S_* + AS_*^2)(\mu A + \alpha B) - S_*((\alpha\mu + \mu^2)A + (\alpha^2 + w_1\alpha)B) + (\sigma w_1)(\sigma + w_1) \right. \\ & \left. - (\sigma + w_1)^2(L_1 + L_3) + (\alpha_\mu + \sigma + w_1)L_1^2 + (\sigma + w_1)L_3^2\right] \\ > {} & \left[(\sigma + w_1)^2 + (L_3 + L_1)^2\right]\left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right]. \end{aligned} \tag{4.39}$$

From inequalities (4.34), (4.37) and (4.36), we have

$$b_3\left[K_2L_1L_3 + K_1CS_*\right] > -\left[2(w_1 + \sigma)(L_3 + L_1)\right]\left[\mu\sigma w_1 K_0 + L_1\sigma w_1(-\alpha_\mu)\right]. \tag{4.40}$$

By using (4.40) and (4.39), we conclude that

$$b_3(b_1b_2 - b_3) - b_1^2 b_4 > 0.$$

Thus, by Routh-Hurwitz stability criterion (15), X_* is locally asymptotically stable. ∎

4.2 Global stability

In this section we study the global stability of the equilibrium points of the model SEIRV (3.1).

4.2.1 Global stability of the disease-free eqiulibrium

Assume that the following assumptions hold

- The functions
$$\beta_E(E), \quad \beta_I(I) \quad \text{and} \quad \beta_V(V) \quad \text{are decreasing.} \tag{4.41}$$

- The functions
$$\beta_E(E), \quad \beta_I(I) \quad \text{and} \quad \beta_V(V) \quad \text{are positive.} \tag{4.42}$$

- The functions satisfy
$$\beta_E'(E) \leq 0, \quad \beta_I'(I) \leq 0 \quad \text{and} \quad \beta_V'(V) \leq 0. \tag{4.43}$$

Then, the result of the global stability of the disease-free eqiulibrium (**DFE**) of system (3.9) is given by

Theorem 20. *If $\mathcal{R}_0 \leq 1$, the disease-free equilibrium (**DFE**) X_0 is globally asymptotically stable in Ω. If $\mathcal{R}_0 > 1$, the equilibrium (**DFE**) X_0 is unstable and there is a unique endemic equilibrium (**EE**) X_*. In addition, the disease is uniformly persistent inside Ω, denoted by $\mathring{\Omega}$ such that*

$$\liminf_{t\to\infty}(E, I, V) > (\varepsilon, \varepsilon, \varepsilon), \quad \text{with} \quad \varepsilon > 0.$$

Proof. Let $X = (E, I, V)^T$. Then, by using the system (3.1) we obtain

$$\begin{aligned}
\frac{dE}{dt} &= \beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV - (\alpha + \mu)E, \\
\frac{dI}{dt} &= \alpha E - (w + \gamma + \mu)I, \\
\frac{dV}{dt} &= \xi_1 E + \xi_2 I - \sigma V.
\end{aligned} \tag{4.44}$$

On the one hand, by the system (4.44), the derivative of X is given by

$$\frac{dX}{dt} = \begin{bmatrix} \beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV - (\alpha+\mu)E \\ \alpha E - (w+\gamma+\mu)I \\ \xi_1 E + \xi_2 I - \sigma V \end{bmatrix}.$$

On the other hand, we have

$$F - \mathcal{V} = \begin{bmatrix} \beta_E(0)S_0 - \alpha - \mu & \beta_I(0)S_0 & \beta_V(0)S_0 \\ \alpha & -w_1 & 0 \\ \xi_1 & \xi_2 & -\sigma \end{bmatrix}.$$

Hence, we get

$$(F - \mathcal{V})X = \begin{bmatrix} \beta_I(0)S_0 I + \beta_V(0)S_0 V - E(\alpha+\mu-\beta_E(0)S_0) \\ \alpha E - I w_1 \\ \xi_1 E + \xi_2 I - \sigma V \end{bmatrix}.$$

With assumptions (4.41)-(4.43) and Domain (3.26), we have

$$\frac{dX}{dt} \leq (F - \mathcal{V})X. \tag{4.45}$$

By simple calculation,

$$\mathcal{V}^{-1}F = \frac{1}{\alpha+\mu}\begin{bmatrix} \beta_E(0)S_0 & \beta_I(0)S_0 & \beta_V(0)S_0 \\ \frac{\alpha}{w_1}\beta_E(0)S_0 & \frac{\alpha}{w_1}\beta_I(0)S_0 & \frac{\alpha}{w_1}\beta_V(0)S_0 \\ \frac{\beta_E(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} & \frac{\beta_I(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} & \frac{\beta_V(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} \end{bmatrix}.$$

The eigenvalues of $\mathcal{V}^{-1}F$ are

$$\lambda_i = \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha\beta_I(0)S_0\sigma + \alpha\beta_V(0)S_0\xi_2 + \beta_E(0)S_0\sigma w_1 + \beta_V(0)S_0 w_1\xi_1}{\alpha\sigma w_1 + \mu\sigma w_1} \end{bmatrix}.$$

Therefore, we obtain

$$\begin{aligned}\rho(\mathcal{V}^{-1}F) = \max_{i=1,2,3} \lambda_i &= \frac{\alpha\beta_I(0)S_0\sigma + \alpha\beta_V(0)S_0\xi_2 + \beta_E(0)S_0\sigma w_1}{\alpha\sigma w_1 + \mu\sigma w_1} \\ &+ \frac{\beta_V(0)S_0w_1\xi_1}{\alpha\sigma w_1 + \mu\sigma w_1} \\ &= \rho(F\mathcal{V}^{-1}).\end{aligned}$$

Let $U = (\beta_E(0), \beta_I(0), \beta_V(0))^T$, so, we have

$$U(\mathcal{V}^{-1}F) = \frac{1}{\alpha+\mu}\times \begin{bmatrix} \beta_E(0)S_0\beta_E(0) + \frac{\alpha}{w_1}\beta_E(0)S_0\beta_I(0) + \frac{\beta_E(0)\beta_V(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} \\ \beta_I(0)S_0\beta_E(0) + \frac{\alpha}{w_1}\beta_I(0)S_0\beta_I(0) + \frac{\beta_I(0)\beta_V(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} \\ \beta_V(0)S_0\beta_E(0) + \frac{\alpha}{w_1}\beta_V(0)S_0\beta_I(0) + \frac{\beta_V(0)\beta_V(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1} \end{bmatrix}. \tag{4.46}$$

On the other hand, we have

$$\mathcal{R}_0U = \begin{bmatrix} \left(\frac{\alpha\beta_I(0)S_0\sigma+\alpha\beta_V(0)S_0\xi_2+\beta_E(0)S_0\sigma w_1+\beta_V(0)S_0w_1\xi_1}{\alpha\sigma w_1+\mu\sigma w_1}\right)\beta_E(0) \\ \left(\frac{\alpha\beta_I(0)S_0\sigma+\alpha\beta_V(0)S_0\xi_2+\beta_E(0)S_0\sigma w_1+\beta_V(0)S_0w_1\xi_1}{\alpha\sigma w_1+\mu\sigma w_1}\right)\beta_I(0) \\ \left(\frac{\alpha\beta_I(0)S_0\sigma+\alpha\beta_V(0)S_0\xi_2+\beta_E(0)S_0\sigma w_1+\beta_V(0)S_0w_1\xi_1}{\alpha\sigma w_1+\mu\sigma w_1}\right)\beta_V(0) \end{bmatrix} \tag{4.47}$$

Then, we conclude from (4.46) and (4.47), the following relation

$$U(\mathcal{V}^{-1}F) = \mathcal{R}_0U. \tag{4.48}$$

Now, consider the following Lyapunov function:

$$\mathcal{L}_0 = U\mathcal{V}^{-1}X.$$

The derivative is given as follows

$$\begin{aligned}\frac{d\mathcal{L}_0}{dt} &= U\mathcal{V}^{-1}\frac{dX}{dt} \\ &\leq U\mathcal{V}^{-1}(F-\mathcal{V})X \qquad \text{with equality (4.45)} \\ &= U\mathcal{V}^{-1}FX - U\mathcal{V}^{-1}\mathcal{V}X \\ &= U\mathcal{R}_0X - UX \qquad \text{with equality (4.48)} \\ &= U(\mathcal{R}_0-1)X.\end{aligned}$$

In conclusion, we have

$$\frac{d\mathcal{L}_0}{dt} \leq U(\mathcal{R}_0 - 1)X. \tag{4.49}$$

From the inequality (4.49), we distinguish the following cases

- For $\mathcal{R}_0 < 1$. We fix $\zeta = \mathcal{R}_0 - 1 < 0$, then from the inequality (4.49) we have

$$\begin{aligned}
\frac{d\mathcal{L}_0}{dt} = 0 &\Rightarrow U(\mathcal{R}_0 - 1)X \geq 0 \\
&\Rightarrow \begin{cases} U\zeta X = 0 \\ \quad\vee \\ U\zeta X > 0 \end{cases} \\
&\Rightarrow \begin{cases} U\zeta X = 0 \Rightarrow UX = 0 \quad \text{since} \quad \zeta < 0 \\ \quad\vee \\ U\zeta X > 0 \quad \underbrace{\text{Impossible since}}_{\Downarrow} \\ \qquad\qquad \overbrace{UX \geq 0 \quad \text{and} \quad \zeta < 0.} \end{cases} \\
&\Rightarrow UX = 0 \Rightarrow X = 0 \quad \text{since} \quad U \neq 0 \\
&\Rightarrow (E, I, V) = (0, 0, 0).
\end{aligned} \tag{4.50}$$

With the equations of system (3.6) and the equality (4.50) we obtain

$$(S, E, I, R, V) = (S_0, 0, 0, 0, 0).$$

So the invariant set on which $\frac{d\mathcal{L}_0}{dt} = 0$ contains the single point X_0.

- For $\mathcal{R}_0 = 1$, we have

$$\frac{d\mathcal{L}_0}{dt} = U\mathcal{V}^{-1}\frac{dX}{dt}, \tag{4.51}$$

such that

$$U^T = \begin{bmatrix} \beta_E(0) & \beta_I(0) & \beta_V(0) \end{bmatrix},$$

and

$$\mathcal{V}^{-1} = \begin{bmatrix} \frac{1}{\alpha+\mu} & 0 & 0 \\ \frac{\alpha}{w_1(\alpha+\mu)} & \frac{1}{w_1} & 0 \\ \frac{\alpha\xi_2+w_1\xi_1}{\sigma w_1(\alpha+\mu)} & \frac{\xi_2}{\sigma w_1} & \frac{1}{\sigma} \end{bmatrix}$$

$$\frac{dX}{dt} = \begin{bmatrix} \beta_E(E)SE + \beta_I(I)SI + \beta_V(V)SV - (\alpha+\mu)E \\ \alpha E - (w+\gamma+\mu)I \\ \xi_1 E + \xi_2 I - \sigma V \end{bmatrix}.$$

By using equation (4.51), we get

$$U\mathcal{V}^{-1} = \begin{bmatrix} \frac{\beta_E(0)}{\alpha+\mu} + \frac{\alpha\beta_I(0)}{w_1(\alpha+\mu)} + \frac{\beta_V(0)(\alpha\xi_2+w_1\xi_1)}{\sigma w_1(\alpha+\mu)} & \frac{\beta_I(0)}{w_1} + \frac{\beta_V(0)\xi_2}{\sigma w_1} & \frac{\beta_V(0)}{\sigma} \end{bmatrix}.$$

So, by using the following notation

$$Y = U\mathcal{V}^{-1}\frac{dX}{dt},$$

we can find

$$\begin{aligned}
Y =& \frac{\beta_V(0)\left(\xi_1 E + \xi_2 I - \sigma V\right)}{\sigma} \\
&+ \frac{\left(\alpha E - I w_1\right)\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1} \\
&+ \left[\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right] \\
&\times \left[\frac{\beta_E(E)ES - (\alpha+\mu)E + \beta_I(I)IS + S\beta_V(V)V}{\alpha+\mu}\right] \\
=& E\left[\frac{\beta_V(0)\xi_1}{\sigma} + \frac{\alpha\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1}\right. \\
&\quad - \left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right) \\
&\quad + \frac{\beta_E(E)S}{\alpha+\mu}\times \\
&\quad \left.\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right)\right] \\
&+ I\left[\frac{\beta_V(0)\xi_2}{\sigma} - \beta_I(0) - \frac{\beta_V(0)\xi_2}{\sigma} + \frac{\beta_I(I)S}{\alpha+\mu}\times\right. \\
&\quad \left.\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right)\right] \\
&+ V\left[-\beta_V(0) + \frac{S\beta_V(V)}{\alpha+\mu}\times\right. \\
&\quad \left.\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right)\right].
\end{aligned}$$

by using (3.20) with $\mathcal{R}_0 = 1$, we can get

$$S_0 = (\alpha+\mu)\left[\beta_E(0) + \frac{\alpha}{w_1}\beta_I(0) + \frac{w_1\xi_1 + \alpha\xi_2}{\sigma w_1}\beta_V(0)\right]^{-1}. \tag{4.52}$$

By making some simplification using (4.52), we have

$$\begin{aligned}
Y = E &\left[\frac{\beta_V(0)\xi_1}{\sigma} + \frac{\alpha\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1} - \underbrace{\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{\sigma w_1}\right)}_{\frac{\alpha+\mu}{S_0}}\right.\\
&\left. + \beta_E(E)S\underbrace{\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{(\alpha+\mu)\sigma w_1}\right)}_{\frac{1}{S_0}}\right]\\
&+ I\left[\frac{\beta_V(0)\xi_2}{\sigma} - \beta_I(0) - \frac{\beta_V(0)\xi_2}{\sigma} + \beta_I(I)S\underbrace{\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{(\alpha+\mu)\sigma w_1}\right)}_{\frac{1}{S_0}}\right]\\
&+ V\left[-\beta_V(0) + S\beta_V(V)\underbrace{\left(\frac{\alpha\beta_I(0)\sigma + \beta_E(0)\sigma w_1 + \beta_V(0)\alpha\xi_2 + \beta_V(0)w_1\xi_1}{(\alpha+\mu)\sigma w_1}\right)}_{\frac{1}{S_0}}\right]
\end{aligned}$$

Then, we get

$$\begin{aligned}
Y &= E\left(\frac{\beta_V(0)\xi_1}{\sigma} + \frac{\alpha\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1} - \frac{\alpha+\mu}{S_0} + \frac{\overbrace{S\beta_E(E)}^{\leq \beta_E(0)S_0}}{S_0}\right)\\
&\quad + I\underbrace{\left(-\beta_I(0) + \frac{\beta_I(I)S}{S_0}\right)}_{\leq 0} + V\underbrace{\left(-\beta_V(0) + \frac{S\beta_V(V)}{S_0}\right)}_{\leq 0}\\
&\leq E\left(\frac{\beta_V(0)\xi_1}{\sigma} + \frac{\alpha\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1} - \frac{\alpha+\mu}{S_0} + \beta_E(0)\right)\\
&\leq E\frac{\alpha+\mu}{S_0}\left[-1 + \frac{S_0}{\alpha+\mu}\left(\frac{\beta_V(0)\xi_1}{\sigma} + \frac{\alpha\left(\beta_I(0)\sigma + \beta_V(0)\xi_2\right)}{\sigma w_1} + \frac{\beta_E(0)S_0}{\alpha+\mu}\right)\right]\\
&= E\frac{\alpha+\mu}{S_0}\left[-1 + \frac{\alpha\beta_I(0)S_0\sigma + \alpha\beta_V(0)S_0\xi_2 + \beta_E(0)S_0\sigma w_1}{\alpha\sigma w_1 + \mu\sigma w_1} + \frac{\beta_V(0)S_0 w_1\xi_1}{\alpha\sigma w_1 + \mu\sigma w_1}\right]\\
&= E\frac{\alpha+\mu}{S_0}\left(-1 + \mathcal{R}_0\right)\\
&= 0.
\end{aligned}$$

As a conclusion for the proof of Theorem (4.2.1), we obtain the following two cases

- $E = I = V = 0$.
- $\beta_E(E) = \beta_E(0), \beta_I(I) = \beta_I(0), \beta_V(V) = \beta_V(0)$ and $S = S_0$.

Each of the cases would indicate that X_0(DEF) is the only set invariant on

$$\left\{ X \in \Omega : \frac{d\mathcal{L}_0}{dt} = 0 \right\}.$$

Therefore, when $\mathcal{R}_0 \leq 0$, the largest invariant set over which $\frac{\mathcal{L}_0}{dt}$ always consists of a singleton X_0. By the LaSalle invariance principle [32], the DFE is globally asymptotically stable in Ω. In on the other hand, if $\mathcal{R}_0 > 1$, then it follows from the continuity of the vector fields that $\frac{\mathcal{L}_0}{dt} > 0$ in a neighborhood of DFE in $\mathring{\Omega}$. Thus, the DFE is unstable by Lyapunov's theory of stability. To prove the next limit

$$\liminf_{t\to\infty}(E, I, V) > (\varepsilon, \varepsilon, \varepsilon), \quad \text{pour} \quad \varepsilon > 0,$$

we most follow the proof of Theorem 2.5 in paper [30]. ■

4.2.2 Global stability of the endemic equilibrium

Theorem 21. *If $\mathcal{R}_0 > 1$, then the unique endimic equilibrium X_* of the system* (3.1) *is globally asymptotically stable in $\mathring{\Omega}$.*

Proof. Either the following functional

$$L(y(t)) = \int_{y_*}^{y} \frac{x - y_*}{x} dt,$$

for $y > 0$ and with $y_* > 0$. We Calculate the derivative of L(y(t)) as follows

$$\begin{aligned}
\frac{d}{dx}L(y(x)) = \frac{d}{dx}(L \circ y) &= \frac{d}{dx}L(y(x))(\frac{dy}{dx}) \\
&= \frac{d}{dx}\int_{y_*}^{y} \frac{x - y_*}{x} dx(\frac{dy}{dx}) \\
&= \left[\frac{x - y_*}{x}\right]_{y_*}^{y} (\frac{dy}{dx}) \\
&= \frac{y - y_*}{y}(\frac{dy}{dx}).
\end{aligned}$$

For the function S, we have

$$\begin{aligned}
\frac{dL(S)}{dt} &= \frac{S-S_*}{S}\,(\beta_E(E_*)S_*E_* - \beta_E(E)SE + \beta_I(I_*)S_*I_* \\
&\quad -\beta_I(I)SI + \beta_V(V_*)S_*V_* - \beta_V(V)VS) - \underbrace{\frac{(S-S_*)^2\mu}{S}}_{\geq 0} \\
&\leq \frac{S-S_*}{S}\,(\beta_E(E_*)S_*E_* - \beta_E(E)SE + \beta_I(I_*)S_*I_* \\
&\quad -\beta_I(I)SI + \beta_V(V_*)S_*V_* - \beta_V(V)VS) \\
&\quad = (1-\frac{S_*}{S})\,[\beta_E(E_*)S_*E_* - \beta_E(E)SE + \beta_I(I_*)S_*I_* \\
&\quad -\beta_I(I)SI + \beta_V(V_*)S_*V_* - \beta_V(V)VS] \\
&\quad = (\beta_E(E_*)S_*E_* - \beta_E(E)SE + \beta_I(I_*)S_*I_* \\
&\qquad -\beta_I(I)SI + \beta_V(V_*)S_*V_* - \beta_V(V)VS) \\
&\qquad - \frac{\beta_E(E_*)S_*E_*S_*}{S} + \frac{\beta_E(E)SES_*}{S} - \frac{\beta_I(I_*)S_*IS_*}{S} \\
&\qquad + \frac{\beta_I(I)SIS_*}{S} - \frac{\beta_V(V_*)V_*S_*}{S} + \frac{\beta_V(V)SVS_*}{S} \\
&\quad = \beta_E(E_*)S_*E_*\left[\left(1-\frac{S_*}{S}\right) - \frac{\beta_E(E)SE}{\beta_E(E_*)S_*E_*} + \frac{\beta_E(E)E}{\beta_E(E_*)E_*}\right] \\
&\qquad + \beta_I(I_*)S_*I_*\left[\left(1-\frac{S_*}{S}\right) - \frac{\beta_I(I)IS}{\beta_I(I_*)I_*S_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*}\right] \\
&\qquad + \beta_V(V_*)S_*V_*\left[\left(1-\frac{S_*}{S}\right) - \frac{\beta_V(V)VS}{\beta_V(V_*)V_*S_*} + \frac{\beta_V(V)V}{\beta_V(V_*)V_*}\right].
\end{aligned}$$

For the function E, we have

$$\begin{aligned}
\frac{dL(E)}{dt} &= \frac{E-E_*}{E}\,[\beta_E(E)SE - \beta_E(E_*)S_*E + \beta_I(I)SI \\
&\qquad - \beta_I(I_*)S_*I_* + \beta_V(V)SV - \beta_V(V_*)S_*V_* - (\alpha+\mu)\frac{E_*E}{E_*}\Bigg] \\
&= \frac{E-E_*}{E}\,[\beta_E(E)SE - \beta_E(E_*)S_*E + \beta_I(I)SI - \beta_I(I_*)S_*I_* + \beta_V(V)SV \\
&\qquad -\beta_V(V_*)S*V*] \underbrace{-(1-\frac{E_*}{E})(\alpha+\mu)E_*\frac{E}{E_*}}_{=\gamma},
\end{aligned}$$

such that

$$\begin{aligned}\gamma &= (1-\frac{E_*}{E})\,(-(\alpha+\mu)E_*E_*) \\ &= \left(-(\alpha+\mu)E_*\frac{E}{E_*}+(\alpha+\mu)\frac{E_*}{E}E_*\frac{E}{E_*}\right) \\ &= \frac{(\alpha+\mu)E}{E_*}\left(-E_*+\frac{E_*}{E}E_*\right) \\ &= \frac{(\alpha+\mu)EE_*}{E_*}\left(-1+\frac{E_*}{E}\right) \qquad (4.53) \\ &= \frac{(\alpha+\mu)E_*}{E_*}\,(-E+E_*) \\ &= \frac{(\alpha+\mu)E_*E_*}{E_*}\left(1-\frac{E}{E_*}\right) = -\gamma. \qquad (4.54)\end{aligned}$$

In addition, we have

$$\begin{aligned}E_* \text{ a point of equilibrium} &\Rightarrow E = E_* \\ &\Rightarrow \quad (4.53) \quad \text{and} \quad (4.54) \quad \Rightarrow \gamma = -\gamma \\ &\Rightarrow \gamma = 0.\end{aligned}$$

Then, by using simplifications, we get

$$\begin{aligned}\frac{dL(E)}{dt} &= (1-\frac{E_*}{E})\,[\beta_E(E)SE-\beta_E(E_*)S_*E+\beta_I(I)SI-\beta_I(I_*)S_*I_* \\ &\quad +\beta_V(V)SV-\beta_V(V_*)S_*V_*] \\ &= \beta_E(E)SE-\beta_E(E_*)S_*E+\beta_I(I)SI-\beta_I I_*)S*I_*+\beta_V(V)SV \\ &\quad -\beta_V(V_*)S_*V_*-\frac{\beta_E(E)SEE_*}{E}+\frac{\beta_E(E_*)S_*EE_*}{E}-\frac{\beta_I(I)SIE_*}{E} \\ &\quad +\frac{\beta_I(I_*)S_*I_*E_*}{E}-\frac{\beta_V(V)SVE_*}{E}+\frac{\beta_V(V_*)S_*V_*E_*}{E} \\ &= \beta_E(E_*)S_*E_*\left[1-\frac{E}{E_*}+\frac{\beta_E(E)SE}{\beta_E(E_*)S_*E_*}-\frac{\beta_E(E)S}{\beta_E(E_*)S_*}\right] \\ &\quad +\beta_I(I_*)I_*S_*\left[\underbrace{-1+\frac{E_*}{E}}_{=1-\frac{E}{E*}}-\frac{\beta_I(I)ISE_*}{\beta_I(I_*)I_*S_*E}+\frac{\beta_I(I)IS}{\beta_I(I_*)I_*S_*}\right] \\ &\quad +\beta_V(V_*)V_*S_*\left[\underbrace{-1+\frac{E_*}{E}}_{=1-\frac{E}{E*}}-\frac{\beta_V(V)VSE_*}{\beta_V(V_*)V_*S_*E}+\frac{\beta_V(V)VS}{\beta_V(V_*)V_*S_*}\right].\end{aligned}$$

Adding up the previous estimates, we obtain

$$
\begin{aligned}
\frac{dL(S)}{dt}+\frac{dL(E)}{dt} &= \beta_E(E_*)S_*E_*\left[\left(1-\frac{S_*}{S}\right)-\frac{\beta_E(E)SE}{\beta_E(E_*)S_*E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}\right]\\
&+\beta_I(I_*)S_*I_*\left[\left(1-\frac{S_*}{S}\right)-\frac{\beta_I(I)IS}{\beta_I(I_*)I_*S_*}+\frac{\beta_I(I)I}{\beta_I(I_*)I_*}\right]\\
&+\beta_V(V_*)S_*V_*\left[\left(1-\frac{S_*}{S}\right)-\frac{\beta_V(V)VS}{\beta_V(V_*)V_*S_*}+\frac{\beta_V(V)V}{\beta_V(V_*)V_*}\right]\\
&+\beta_E(E_*)S_*E_*\left[1-\frac{E}{E_*}+\frac{\beta_E(E)SE}{\beta_E(E_*)S_*E_*}-\frac{\beta_E(E)S}{\beta_E(E_*)S_*}\right]\\
&+\beta_I(I_*)I_*S_*\left[1-\frac{E}{E_*}-\frac{\beta_I(I)ISE_*}{\beta_I(I_*)I_*S_*E}+\frac{\beta_I(I)IS}{\beta_I(I_*)I_*S_*}\right]\\
&+\beta_V(V_*)V_*S_*\left[1-\frac{E}{E_*}-\frac{\beta_V(V)VSE_*}{\beta_V(V_*)V_*S_*E}+\frac{\beta_V(V)VS}{\beta_V(V_*)V_*S_*}\right]\\
&\leq \beta_E(E_*)S_*E_*\left[2-\frac{S_*}{S}-\frac{E}{E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}-\frac{\beta_E(E)S}{\beta_E(E_*)S_*}\right]:=A\\
&+\beta_I(I_*)S_*I_*\left[2-\frac{S_*}{S}-\frac{E}{E_*}+\frac{\beta_I(I)I}{\beta_I(I_*)I_*}-\frac{\beta_I(I)ISE_*}{\beta_I(I_*)I_*S_*E}\right]:=B\\
&+\beta_V(V_*)S_*V_*\left[2-\frac{S_*}{S}-\frac{E}{E_*}+\frac{\beta_V(V)V}{\beta_V(V_*)V_*}-\frac{\beta_V(V)VSE_*}{\beta_V(V_*)V_*S_*E}\right]:=C
\end{aligned}
$$

We have S_* equilibrium point $\Leftrightarrow S=S_*$. Then, A,B and C are calculated as follows

$$
\begin{aligned}
A &= \beta_E(E_*)S_*E_*\left[\underbrace{2-\frac{S_*}{S}}_{=1}-\frac{E}{E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}-\frac{\beta_E(E)S}{\beta_E(E_*)S*}\right]\\
&= \beta_E(E_*)S_*E_*\left[1-\frac{E}{E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}-\frac{\beta_E(E)}{\beta_E(E_*)}\right]\\
&= \beta_E(E_*)S_*E_*\left[\frac{\beta_E(E)}{\beta_E(E_*)}\left(\frac{\beta_E(E_*)}{\beta_E(E)}-1\right)-\frac{E}{E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}\right], \qquad \frac{\beta_E(E)}{\beta_E(E_*)}\leq 1 \quad (4.55)\\
&\leq \beta_E(E_*)S_*E_*\left[\frac{\beta_E(E_*)}{\beta_E(E)}-1-\frac{E}{E_*}+\frac{\beta_E(E)E}{\beta_E(E_*)E_*}\right]\\
&\leq \beta_E(E_*)S_*E_*\left(\frac{\beta_E(E)E}{\beta_E(E_*)E_*}-1\right)\left(1-\frac{\beta_E(E_*)}{\beta_E(E)}\right).
\end{aligned}
$$

We can use the following inequalities

$$
1-\frac{\beta_E(E_*)}{\beta_E(E)}\leq 0 \;\;\Leftrightarrow\;\; E_*\leq E \;\;\Leftrightarrow\;\; \frac{\beta_E(E)E}{\beta_E(E_*)E_*}-1\geq 0. \tag{4.56}
$$

By using (4.56) and (4.55) we have

$$A \leq \beta_E(E_*)S_*E_* \left(\underbrace{\frac{\beta_E(E)E}{\beta_E(E_*)E_*} - 1}_{\leq 0} \right) \left(\underbrace{1 - \frac{\beta_E(E_*)}{\beta_E(E)}}_{\geq 0} \right)$$
$$\leq 0. \tag{4.57}$$

Now, we have

$$\begin{aligned}
B &= \beta_I(I_*)S_*I_* \left[2 - \frac{S_*}{S} - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} - \frac{\beta_I(I)ISE_*}{\beta_I(I_*)I_*S*E}\right] \\
&= \beta_I(I_*)S_*I_* \left[1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} - \frac{\beta_I(I)IE_*}{\beta_I(I_*)I_*E}\right] \\
&\leq \beta_I(I_*)S_*I_* \left[1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} + \frac{\beta_I(I)}{\beta_I(I_*)}\left(1 - \frac{E_*I}{EI_*}\right)\right], \qquad 1 \geq \frac{\beta_I(I)}{\beta_I(I_*)} \geq 0. \\
&\leq \beta_I(I_*)S_*I_* \left[1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} + 1 - \frac{E_*I}{EI_*}\right], \qquad 1 \leq \frac{\beta_I(I_*)}{\beta_I(I)} - 1. \\
&\leq \beta_I(I_*)S_*I_* \left[\frac{\beta_I(I_*)}{\beta_I(I)} - 1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} + 1 - \frac{E_*I}{EI_*}\right] \\
&\leq \beta_I(I_*)S_*I_* \left[-\frac{I}{I_*} + \frac{I}{I_*} + \frac{\beta_I(I_*)}{\beta_I(I)} - 1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} + 1 - \frac{E_*I}{EI_*}\right] \qquad (4.58) \\
&\leq \beta_I(I_*)S_*I_* \left[-\frac{I}{I_*} + \frac{I}{I_*} + \frac{\beta_I(I_*)}{\beta_I(I)} - 1 - \frac{E}{E_*} + \frac{\beta_I(I)I}{\beta_I(I_*)I_*} + \ln\frac{EI_*}{E_*I}\right].
\end{aligned}$$

we have

$$1 - \frac{1}{x} \leq \ln x, \quad x = \frac{E_*I}{EI_*} \quad \text{(see Fig (4.1))}$$

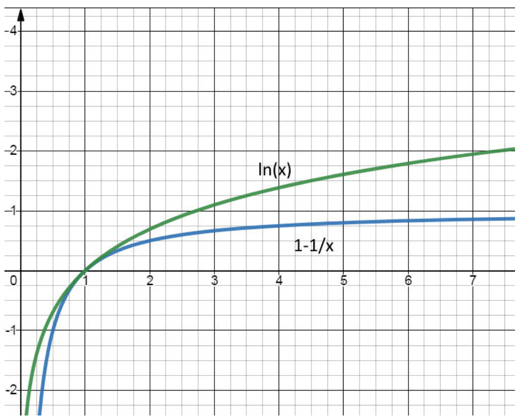

Figure 4.1: Behavior of $1 - \frac{1}{x}$ and $\ln x$

Hence, we get

$$\ln \frac{EI_*}{E_*I} = \ln \frac{\frac{E}{E_*}}{\frac{I}{I_*}} = \ln \frac{E}{E_*} - \ln \frac{I}{I_*}. \tag{4.59}$$

We apply (4.59)

$$\begin{aligned}
B &\leq \beta_I(I_*)S_*I_* \left[\frac{\beta_I(I)I}{\beta_I(I_*)I_*} - \frac{I}{I_*} - 1 + \frac{\beta_I(I_*)}{\beta_I(I)} + \frac{I}{I_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{I}{I_*}\right] \\
&\leq \beta_I(I_*)S_*I_* \left[\left(\underbrace{\frac{\beta_I(I)I}{\beta_I(I_*)I_*} - 1}_{\leq 0}\right)\left(\underbrace{1 - \frac{\beta_I(I_*)}{\beta_I(I)}}_{\geq 0}\right) + \frac{I}{I_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{I}{I_*}\right] \\
&\leq \beta_I(I_*)S_*I_* \left[\frac{I}{I_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{I}{I_*}\right]
\end{aligned} \tag{4.60}$$

For C, we apply the same reasoning as for B. We then obtain

$$C \leq \beta_V(V_*)S_*V_* \left[\frac{V}{V_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{V}{V_*}\right]. \tag{4.61}$$

Let's use (4.57),(4.61) and (4.60), so

$$\begin{aligned}
\frac{dL(S)}{dt} + \frac{dL(E)}{dt} &\leq \beta_I(I_*)S_*I_* \left[\frac{I}{I_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{I}{I_*}\right] \\
&\quad + \beta_V(V_*)S_*V_* \left[\frac{V}{V_*} - \frac{E}{E_*} + \ln \frac{E}{E_*} - \ln \frac{V}{V_*}\right].
\end{aligned}$$

For I, we have

$$\begin{aligned}
\frac{dL(I)}{dt} &= \frac{I - I_*}{I}\frac{dI}{dt} \\
&= \frac{I - I_*}{I}\left[\alpha\,(E - E_*) - w_1\underbrace{(I - I_*)}_{=0}\right], \ I_* \text{ point of equilibrium} \\
&= \alpha E - \alpha E_* - \frac{\alpha E I_*}{I} + \frac{\alpha E_* I_*}{I} \\
&= \alpha E - \frac{\alpha E I_*}{I} + \alpha E_*\left(-1 + \frac{I_*}{I}\right) \quad \text{see (4.53) and (4.54)} \\
&= \alpha E - \frac{\alpha E I_*}{I} + \alpha E_*\left(1 - \frac{I}{I_*}\right) \\
&= \alpha E_*\left(\frac{E}{E_*} - \frac{I}{I_*} - \frac{I_* E}{I E_*} + 1\right) \quad \text{see (4.58) and (4.59).} \\
&\leq \alpha E_*\left(\frac{E}{E_*} - \frac{I}{I_*} + \ln \frac{I}{I_*} - \ln \frac{E}{E_*}\right).
\end{aligned}$$

For V, we have

$$\begin{aligned}
\frac{dL(V)}{dt} &= \frac{V-V_*}{V}\frac{dV}{dt} \\
&= \frac{V-V_*}{V}\left[\xi_1\left(E-E_*\right)+\xi_2\left(I-I_*\right)-\sigma\underbrace{\left(V-V_*\right)}_{=0}\right], \ V_* \text{ point of equilibrium} \\
&= \frac{V-V_*}{V}\left[\xi_1\left(E-E_*\right)+\xi_2\left(I-I_*\right)\right] \\
&= \xi_1 E - \frac{\xi_1 E V_*}{V} + \xi_1 E_*\left(-1+\frac{V_*}{V}\right) \quad \text{see (4.53) and (4.54)} \\
&\quad + \xi_2 I - \frac{\xi_2 I V_*}{V} + \xi_2 I_*\left(-1+\frac{V_*}{V}\right) \quad \text{see (4.53) and (4.54)} \\
&= \xi_1 E - \frac{\xi_1 E V_*}{V} + \xi_1 E_*\left(1-\frac{V}{V_*}\right) \\
&\quad + \xi_2 I - \frac{\xi_2 I V_*}{V} + \xi_2 I_*\left(1-\frac{V}{V_*}\right) \\
&= \xi_1 E_*\left(\frac{E}{E_*}-\frac{V}{V_*}-\frac{V_*E}{VE_*}+1\right) \quad \text{see (4.58) and (4.59).} \\
&\quad + \xi_2 I_*\left(\frac{I}{I_*}-\frac{V}{V_*}-\frac{V_*I}{VI_*}+1\right) \quad \text{see (4.58) and (4.59).} \\
&\leq \xi_1 E_*\left(\frac{E}{E_*}-\frac{V}{V_*}+\ln\frac{V}{V_*}-\ln\frac{E}{E_*}\right) \\
&\quad + \xi_2 I*\left(\frac{I}{I_*}-\frac{V}{V_*}+\ln\frac{V}{V_*}-\ln\frac{I}{I_*}\right).
\end{aligned}$$

Define the following Lyapunov functional $\mathcal{L}_1(t) = L(S) + L(E) + c_1 L(I) + c_2 L(V)$. By using the previous estimates, we get

$$\begin{aligned}
\frac{d\mathcal{L}_1}{dt} &\leq \beta_I(I_*)S_* I_*\left[\frac{I}{I_*}-\frac{E}{E_*}+\ln\frac{E}{E_*}-\ln\frac{I}{I_*}\right] \\
&\quad + \beta_V(V_*)S_*V_*\left[\frac{V}{V_*}-\frac{E}{E_*}+\ln\frac{E}{E_*}-\ln\frac{V}{V_*}\right] \\
&\quad + c_1\alpha E_*\left(\frac{E}{E_*}-\frac{I}{I_*}+\ln\frac{I}{I_*}-\ln\frac{E}{E_*}\right) \\
&\quad + c_2\xi_1 E_*\left(\frac{E}{E_*}-\frac{V}{V_*}+\ln\frac{V}{V_*}-\ln\frac{E}{E_*}\right) \\
&\quad + c_2\xi_2 I_*\left(\frac{I}{I_*}-\frac{V}{V_*}+\ln\frac{V}{V_*}-\ln\frac{I}{I_*}\right) \\
&= \left(\ln\frac{E}{E_*}-\frac{E}{E_*}\right)\left(\beta_I(I_*)S_*I_* + \beta_V(V_*)S_*V_* - c_1\alpha E_* - c_2\xi_1\right) \qquad (4.62) \\
&\quad + \left(\frac{I}{I_*}-\ln\frac{I}{I_*}\right)\left(\beta_I(I_*)S_*I_* - c_1\alpha E_* + c_2\xi_2 I_*\right) \qquad (4.63) \\
&\quad \left(\frac{V}{V_*}-\ln\frac{V}{V_*}\right)\left(\beta_V(V_*)S_*V_* - c_2\xi_1 E_* - c_2\xi_2 I_*\right) \qquad (4.64)
\end{aligned}$$

We must choose $c_2 > 0$ such that

$$\beta_V(V_*)S_*V_* - c_2\xi_1 E_* - c_2\xi_2 I_* = 0, \tag{4.65}$$

then, solving equation (4.65) gives us

$$c_2 = \frac{\beta_V(V_*)S_*V_*}{\xi_1 E_* + \xi_2 I_*} = \frac{w_1\beta_V(V_*)S_*V_*}{w_1(\xi_1 E_* + \xi_2 I_*)}.$$

Note that $w_1 I_* = \alpha E_*$. Therefore

$$c_2 = \frac{w_1\beta_V(V_*)S_*V_*}{(w_1\xi_1 + \alpha\xi_2)E_*}.$$

By replacing c_2 in (4.63) we find c_1. Equality being valid if and only if $(S, E, I, V) = (S_*, E_*, I_*, V_*)$. So, X_* is globally asymptotically stable inside of Ω. ■

CHAPTER 5

NUMERICAL RESULTS

In this chapter, we present the numerical results. At first, we consider the following functions for the three transmission rates

$$\beta_E(E) = \frac{\beta_{E0}}{1+cE}, \quad \beta_I(I) = \frac{\beta_{I0}}{1+cI}, \quad \beta_V(V) = \frac{\beta_{V0}}{1+cV}, \tag{5.1}$$

where β_{E0}, β_{I0} and β_{V0} (all positive constants) denote the maximum values of these transmission rates, and c is a positive coefficient providing adjustment to the (otherwise constant) transmission rates.

Next, we fix the parameters defined in (1.2) as in the work [2].

$$\begin{aligned}
\Lambda &= 271.23 \\
\beta_{E0} &= 3.11 \times 10^{-8}. \\
\beta_{I0} &= 0.62 \times 10^{-8}. \\
\beta_{V0} &= 1.03 \times 10^{-8}. \\
c &= 1.01 \times 10^{-4}. \\
\mu &= 3.01 \times 10^{-5}. \\
\alpha &= 1/7. \\
w &= 0.01. \\
\gamma &= 1/15. \\
\sigma &= 1. \\
\xi_1 &= 2.3. \\
\xi_2 &= 0.
\end{aligned}$$

Then, based on [1], the initial condition is set as

$$(S(0), E(0), I(0), R(0), V(0)) = (8999015, 500, 475, 10)\,.$$

Now, we are able to evaluate the basic reproduction number

$$\mathcal{R}_0 = 4.1780.$$

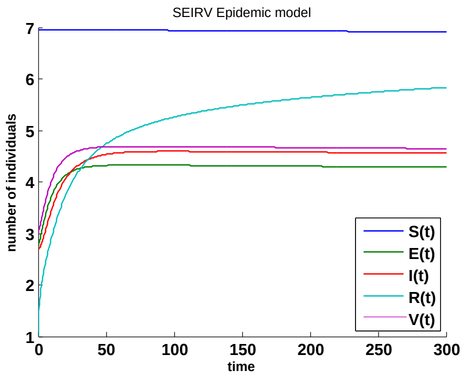

Figure 5.1: A simulation result for SEIRV model

By reading Figure (5.1), we conclude: The incorporation of the environmental reservoir in the transmission dynamics of the disease, and with non-constant transmission rates. We observe the change in epidemiological status and environmental conditions and reflect the impact of the implemented disease control measures.

In addition, we have performed a numerical test using simple, constant transmission rates in our model

$$\beta_E(E) = \beta_{E0}, \quad \beta_I(I) = \beta_{I0}, \quad \beta_V(V) = \beta_{V0}, \tag{5.2}$$

equivalent to setting $c = 0$ in (5.1). The Figure (5.2) shows a prediction of the outbreak

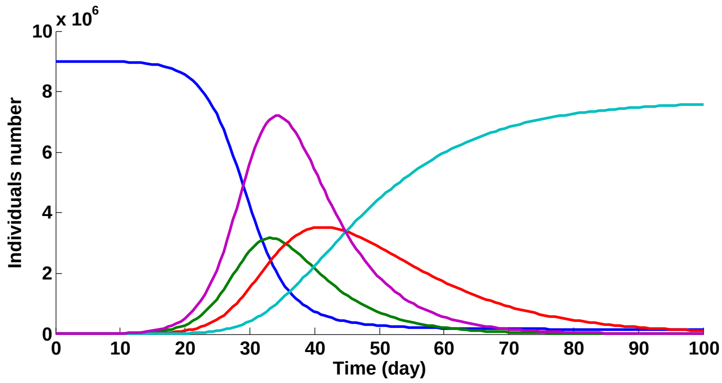

Figure 5.2: A simulation result for SEIRV model with constant transmission rates

size in this setting. Compared to Figure (5.1), we now clearly observe a significantly higher level of infection.

CONCLUSION

In this work, we collected some of the covid-19 models and mentioned the studies related to each model, we studied local and global asymptotic stability analysis. Many aspects of the epidemiology of COVID-19 are still unknown, which adds challenges to mathematical modeling. The numerical results demonstrates that using fixed transmission rates, which do not take into account the strong disease control measures currently on-going, may overestimate the epidemic severity and generate misguided information.

5.1 perspectives

In studying the local stability of the state of $\mathcal{R}_0 = 1$. You need a detailed (paradoxes). Exposure detailed analysis of this model in the PDE status and explains its application by declared data, in order to eliminate the disease eventually among the interference strategy: Optimal control of the epidemic by determining (vaccination rate, quarantine, treatment cost, locking countries).

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