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Common fixed points for generalized α-implicit contractions in partial metric space and an application

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بسم الله الرحمن الرحيم

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ملخص

في هذه المذكرة ، قمنا بدراسة بعض نظريات النقطة الثابتة المشتركة باستخدام مفهوم alpha_admissible الزوجي المعممة بواسطة Aydi et al. التي تعمم تعريف alpha_admissible. المؤلفون استنادًا إلى النتائج السابقة، عرفوا تقلص alpha_$implicit$ المعممة في الفضاءات المترية الجزئية ، ثم برهنوا بعض نتائج النقاط الثابتة المشتركة لمثل هذه التقلصات. كما أعطوا بعض النتائج والمبرهنات التي تحقق وتأكد مدى صحة النتائج التي تم الحصول عليها. في النهاية، قدمنا ملخص عن البرمجة الديناميكية قبل عرض تطبيق يحقق نظرية المقال المدروس، كما طرحنا بعض الأمثلة التي تجعل المفاهيم والنتائج الجديدة واضحة.

Abstract

In this memoir, we study some common fixed point theorems using the concept of α-admissible pair of mappings generalized by Aydi et al. [3] which are generalizing the definition of α-admissible mappings. Authors Based on the previous results, defined generalized α-implicit contractions in the framework of partial metric spaces, then they contribute some common fixed point results for such contractions. Also gave some consequences and corollaries from their obtained results. In the end, we present a synthesis of dynamic programming before show an application involving their theorem, and some examples are presented making effective the new concepts and results.

Résumé

Dans ce mémoire, nous étudions quelques théorèmes de point fixe populaires en utilisant le concept de paire α-admissible pour les applications généralisées par Aydi et al. [3] qui généralisent la définition des appellations α-admissibles. Sur la base des résultats précédents, les auteurs ont défini des contractions α-implicites généralisées sous des espaces métriques partiels, puis ont contribué à certains résultats de point fixe communs obtenus à partir de la contraction précédente. Il a également donné quelques résultats et les résultats obtenus à partir de ses résultats. la fin, nous présentons une synthèse de la programmation dynamique avant de montrer une application qui inclut sa théorie, et quelques exemples sont donnés qui rendent ses nouveaux concepts et résultats efficaces

Dedication

This work is firstly dedicated to my beloved father who has given me invaluable educational opportunities and has always supported me to be the person I am today.
I also would like to dedicate this thesis to my family members, my mother,My brothers Soltan, Djamel, Ismail whose support, guidance and encouragement have enriched my soul and inspired me to complete this research, to my sisters: Louiza, Nacira, Djamila, Salma, Sana.
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Introduction

Banach contraction principle is the most important utile in nonlinear analysis which is proved by Stefan Banach [4] in 1922. His valuable work developed in two ways generalizing the metric spaces or the contractions conditions by imposing fewer conditions for example. As a result of those generalizations, so many metric spaces have been introduced which are uniformly convex Banach spaces, strictly convex Banach spaces, cone metric spaces, pseudo metric spaces, B-metric spaces, fuzzy metric spaces, etc.

In this memoir, we consider another generalization of a metric space, called partial metric space. This notion was introduced by Matthews (see e.g. [21]) to solve some difficulties in theory field for computer science. After Matthews (1994) enormous works were done in partial metric spaces. Several authors have demonstrated the existence and uniqueness of fixed points which also provide applications, see for example Bucatin et al (2009), Aydi, Karapinar and Shantanawi (2011), Oltra and Valero (2004), Altun, Sola, and Simsek (2010), Ciric, Samet, Aydi, and Vetro (2011), Pragadeeswarar and Marudai (2014), Altun and Erduran (2011) and Romaguera (2009).In Pragadeeswarar and Marudai (2014) they established the fixed point theory of the contraction map which satisfies the logical expression in partial metric spaces. Common fixed point theory generally includes conditions on permutation, continuity, completeness, Popa [24, 26] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions, from the application of common fixed point theorem we discuss here dynamic programming which introduced by H.Aydi.

Our memoir consists of three chapters, in the first one, we present some concepts and preliminaries that we needed in the following chapter.

In the second chapter, we provide the essentials of our memoir where we study and detail Aydi's paper.

In the last chapter, we introduce some concepts of dynamic programming and show an application of the main results studies.

Chapter 1

Notions and preliminaries

1.1 Metric spaces

In this chapter we present some concepts and preliminaries that we need in the following chapter.

Definition 1.1.1 *[32] A metric on a set* X *is a function* $d : X \times X \to \mathbb{R}$ *such that*

$M0 : 0 \leq d(x, y)$ for all $x, y \in X$ (nonnegativity),

$M1$: if $x = y$ then $d(x, y) = 0$ (equality implies indistancy),

$M2$: if $d(x, y) = 0$ then $x = y$ (indistancy implies equality),

$M3$: $d(x, y) = d(y, x)$ (symmetry), and

$M4$: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.(triangularity).

Example 1.1.1 *[32] Let* $d(x, y) = |x - y|, (\mathbb{R}, d)$ *is a metric space. The first two conditions are obviously satisfied, and the third follows from the ordinary triangle inequality for real numbers:*

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

Example 1.1.2 *[32] Let* $d(x, y) = |x - y|, (\mathbb{R}^n, d)$ *is a metric space. The first two conditions are obviously satisfied, and the third follows from the triangle inequality for vectors the same way as above :*

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

1.1.1 Sequences and Cauchy sequences

A sequence in a metric space X is an infinite list

$$x_1, x_2, x_3, \ldots$$

of points in X. (More formally, a sequence in X can be defined as a function $f : \mathbb{N} \to X$ but we will not need this point of view here.) Note that the terms in the sequence do not have to be distinct; in particular, for any $x \in X$ we can talk about the constant sequence

$$x, x, x, \ldots$$

We will use the notation (x_n) for the sequence whose n-th term is x_n.

Definition 1.1.2 *A sequence (x_n) in a metric space (X, d) is said to converge to $x \in X$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(x_n, x) < \epsilon \; for \; n \geq N.$$

Example 1.1.3 *In any metric space, a constant sequence $x, x, x, \ldots$ converges to x.*

Example 1.1.4 *This is a standard elementary example. Consider the sequence $\left(\frac{1}{n}\right)$ in $\mathbb{R}$ with the standard metric. Intuitively the terms of this sequence are getting closer and closer to 0^-let us prove that the sequence indeed converges to 0.*

Let $\epsilon > 0$. We want to find an index N where for $n \geq N$ we have

$$\frac{1}{n} < \epsilon.$$

It is the Archimedian Property that tells us what N to choose: pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$ we have

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

We conclude that the sequence $\left(\frac{1}{n}\right)$ converges to 0.

Definition 1.1.3 *A sequence$\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Let E be a nonempty subset of a metric space (X, d), and let $S = \{d(p, q) : p, q \in E\}$. The diameter of E is sup S.

If $\{x_n\}$ is a sequence in X and if E_N consists of the points $x_N, x_{N+1}, \ldots$, it is clear that $\{x_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} diam E_N = 0$$

there is many types of metric spaces for exempes: cone metric spaces, pseudo metric spaces, B-metric spaces, fuzzy metric spaces,partial metric spaces etc.

1.2 Partial metric spaces

Partial metric spaces, introduced by Matthews $[1, 2]$,are a generalization of the notion of the metric space in which the definition of metric, the condition $d(x, x) = 0$ is replaced by the condition $d(x, x) \leq d(x, y)$.

Definition 1.2.1 *[23]A partial metric or p-metric is a function* $p : U^{\ 2} \to \mathbb{R}$ *such that,*

$(P1)\ \forall x, y \in U, x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$
$(P2)\ \forall x, y \in U, p(x, x) \leq p(x, y)$
$(P3)\ \forall x, y \in U, p(x, y) = p(y, x)$
$(P4)\ \forall x, y, z \in U, p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$

Example 1.2.1 *[25] Let a function* $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by* $p(x, y) = max\{x, y\}$ *for any* $x, y \in \mathbb{R}^+$*. Then,* $(\mathbb{R}^+, p)$*is a partial metric space where the self-distance for any point* $x \in \mathbb{R}^+$ *is its value itself.*

Example 1.2.2 *[25] Consider a function* $p : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^+$ *defined by* $p(x, y) = -min(x, y)$*for any* $x, y \in \mathbb{R}^-$*. The pair* $(\mathbb{R}^-, p)$ *is a partial metric space for which* p *is called theusual partial metric on* $\mathbb{R}^-$ *and where the self-distance for any point* $x \in \mathbb{R}^-$ *is its absolute value.*

Definition 1.2.2 *[3] Each partial metric* p *on* X *generates a* T_0 *topology* τ_p *on* X *which has as a base the family of open p-balls* $\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\}$*, where* $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ *for all* $x \in X$ *and* $\varepsilon > 0$*.*

Remark 1.2.1 *If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by*

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad (1)$$

is a metric on X

1.2.1 Cauchy sequences and convergent sequences

Definition 1.2.3 *A sequence $x = \{x_n\}$ of points in a partial metric space (X, p) is Cauchy if there exists $\alpha \geq 0$ such that for each $\epsilon > 0$ there exists k such that for all $n, m > k, |p(x_n, x_m) - a| < \epsilon$.*

In other words, x is Cauchy if the numbers $p(x_n, x_m)$ converge to some α as n and m approach infinity, that is, if $\lim_{n,m\to\infty} p(x_n, x_m) = \alpha$. Note that then $\lim_{n\to\infty} p(x_n, x_n) = \alpha$, and so if (X, p) is a metric space then $\alpha = 0$.

Lemma 1.2.1 *[3] A sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is Cauchy in the metric space (X, d_p).*

Remark 1.2.2 *A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover*

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 1.2.2 *[3] Let (X, p) be a partial metric space. Then,*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p)if and only if it is a Cauchy sequence in the metric space (X, p^s),*

(b) *X is complete if and only if the metric space $(X, p^s)$is complete. Furthermore, $\lim_{n-\to+\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$$

Definition 1.2.4 *[3] A partial metric space (X, d) is said to be complete if every Cauchy sequences $\{x_n\}$ in X converges to $x \in X$, such that*

$$p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$$

1.3 Contraction's concepts

1.3.1 Definition and examples

Definition 1.3.1 *[27] (Contraction). Let (X,d) be a complete metric space. A function $f : X \to X$ is called a contraction if there exists $k < 1$ such that for any $x, y \in X$,*

$$d(f(x), f(y) \leq kd(x,y).$$

Example 1.3.1 *[27] Consider the metric space $(\mathbb{R}, d)$ where d is the Euclidean distance metric, i.e. $d(x,y) = |x-y|$ The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{x}{a} + b$ is a contraction if $a > 1$. In this specific case we can find a fixed point. Since a fixed means that $f(x) = x$, we want $x = \frac{x}{a} + b$. Solving for x gives us $x = \frac{ab}{a-1}$.*

1.3.2 Applying contractions multiple times

From these examples, we have the reason for assumption that contractions in general have fixed point. To show that any contraction has a fixed point we will find a point that should be fixed and prove that point is indeed a fixed point. Let $f : X \to X$ be any contraction. If, for a moment, we believe that all contractions have fixed points, then $f^2(x)$ should have a fixed point since it's a contraction.

Proposition 1.3.1 *[27] Suppose f is contraction. Then f^n is also a contraction. Furthermore, if k is the constant for f, k^n is the constant for $f^{\ n}$.*

1.3.3 Some types of contractions

Several authors have defined contractual type mappings on a complete metric space X, which are generalizations of the Banach's contraction, which have the property that each such mapping has a unique fixed point. Now we come up with multiple definitions corresponding to the contractual type.

Definition 1.3.2 *[28] Let (X,d) be a complete metric space and $T : X \to X$ a mapping. we say that T is a Rakotch contraction if there is a monotonic and decreasing function*

$\alpha :]0, +1[\rightarrow [0, 1[$ such that, for all $x, y \in X, x \neq y$;

$$d(Tx, Ty) \leq \alpha(d(x, y))d\,(x, y)$$

Definition 1.3.3 *[18] Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. We say that T is a Kannan contraction if there is a number a, $0 < a < \frac{1}{2}$, such as, for all $x, y \in X$, $x \neq y$;*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

Definition 1.3.4 *[9] Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. we say that T is a Bianchini contraction if there is a number $h, 0 < h < 1$, such that, for all $x, y \in X$*

$$d(Tx, Ty) \leq hmax\,\{d(x, Tx), d(y, Ty)\}$$

Definition 1.3.5 *[27] Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. We say that T is a Reich contraction if there are positive numbers a, b, c, satisfy $a + b + c < 1$,*

such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y)$$

Definition 1.3.6 *[31] Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. We say that T is a Sehgal contraction if for all $x, y \in X$,$x \neq y$,*

$$d(Tx, Ty) < max\,\{d(x; Tx), d(y, Ty), d(x, y)\}$$

Definition 1.3.7 *[13] Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. We say that T is a contraction of Zamfirescu if for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) < max\left\{d(x, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Definition 1.3.8 *[12] Let A be a self-mapping on a metric space (M,d) and $\{f_i\}_{i=1}^{5}$ be a set of Wong (auxiliary) functions. We say that A is a Wong type contraction if the following inequality holds:*

$$d(Ap, Aq) \leq a_1 d(p, q) + a_2 d(p, Ap) + a_3 d(q, Aq) + a_4 d(p, Aq) + a_5 d(Ap, q)$$

for any $p, q \in M$ with $p \neq q$ where $a_i = f_i(d(p, q))/d(p, q)$.

1.3.4 Fixed point theorem

We now examine how a familiar theorem from the theory of metric spaces can be transferred to partial metric spaces.

Definition 1.3.9 *[16] For each partial metric space* (X, p) *a contraction is a function* $f : X \to X$ *for which there exists a* $c \in [0, 1)$ *such that for all* x, y *in* X, $p(f(x), f(y)) \leq c.p(x, y)$.

Theorem 1.3.1 (Matthews [11]) *For each contraction* f *over a complete partial metric space* (X, p) *there exists a unique* x *in* X *such that* $x = f(x)$*.Also,* $p(x, x) = 0$*. Thus the contraction fixed point theorem is extended to partial metric spaces.*

Chapter 2

Common fixed point theorem

In this chapter, [3]we examine the work of H. Aydi et al who have provided some common fixed point theorems regarding generalized α-implicit contractions over partial metric spaces. As consequences of their obtained ,we show their results that prove five theorems on partial metric spaces, and give some examples to illustarte their results and concepts.

Recently, Samet et al. [30] introduced the concept of α-admissible maps.

2.0.5 Implicit relations

Popa [24,26] proved several fixed point theorems that satisfy appropriate implicit relations. To prove these results, Popa considered ψ to be the set of all continuous functions $\psi(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfyies the following conditions:

(ψ_1) is non-increasing in variables t_5 and t_6,

(ψ_2) there exists $k \in (0, 1)$ such that for$u, v \geq 0$ with

$(\psi_{2a})(u, v, v, u, u + v, 0) \leq 0$ or

(ψ_{2b}) $(u, v, u, v, 0, u + v) \leq 0$ implies $u \leq kv$,

$(\psi_3)(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Lemma 2.0.1 *[3] Let (X, p) be a partial metric space, $F : X \to X$ be a given mapping. Suppose that F is continuous at $x_0 \in X$. Then, for all sequence $\{x_n\}$ in X such that $x_n \to x_0$, we have $Fx_n \to Fx_0$.*

Popa considered an implicit contraction type condition rather than the usual explicit contractive conditions.This direction of research has produced a consistent literature on fixed point, common fixed point and coincidence point theorems in various ambient spaces.For more details, see [2,7,14,23,25,30]. Now, denote N the set of positive integers and the set of functions

$\psi[0,\infty) \to [0,\infty)$ satisfying: $(\psi_1)\psi$ is nondecreasing, $(\psi_2)\sum_{n=1}^{\infty}\psi^n(t) < \infty$ for each $t \in R^+$, where $\psi^{\ n}$ isthe nth iterate of ψ .

Definition 2.0.10 *[3] Let be the set of all continuous functions* $F(t_1,...,t_6) : R^6+ \to R$ *such that (F1): F is nondecreasing in variable* t_1 *and nonincreasing in variables* t_5 *and* t_6*, (F2): There exists* $h_1 \in \Psi$ *such that for all* $u, v \geq 0, F(u, v, v, u, u+v, u) \leq 0$ *implies* $u \leq h_1(v)$*,and* $F(u, v, u, v, u, u+v) \leq 0$ *implies* $u \leq h_1(v)$*.*

We give the following examples.

Example 2.0.2 $F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$*, where* $a, b, c, d, e \geq 0$ *such that* $a+b+c+2d+e < 1$ *and* $a+b+c+d+2e < 1$*.*

Example 2.0.3 $F(t_1,...,t_6) = t_1 - k\max\{t_2,...,t_6\}$*, where* $k \in [0, \frac{1}{2})$*.*

Example 2.0.4 $F(t_1,...,t_6) = t_1 - k\max\{t_2, \frac{t_3+t_4}{2} \frac{t_5+t_6}{2}\}$*, where* $k \in [0, \frac{2}{3})$*.*

Example 2.0.5 $F(t_1,...,t_6) = t_1 - a\max\{t_2,t_3,t_4\} - (1-a)[bt_5 + ct_6]$*, where* $b+2c$ *and* $c+2b$ *are in* $[0,1)$*.*

Definition 2.0.11 *For a nonempty set* X*, let* $T : X \to X$ *and* $\alpha : X \times X \to [0,\infty)$ *be mappings.We say that the self-mapping* T *on* X *is* α*-admissible if for all* $x, y \in X$*, we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1. \quad (2)$$

Many papers distributed with the above concept have been considered to prove some (common) fixed point results, for example see [1,16,17,19,21]. H. Aydi et al generalized definition 2.0.13 by introducing a pair of mappings defined in the following.

Definition 2.0.12 *For a nonempty set X, let $A, B : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings.We say that (A, B) is a generalized α-admissible pair if for all $x, y \in X$, we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Ax, By) \geq 1 \text{ and } \alpha(BAx, ABy) \geq 1. \tag{3}$$

Remark 2.0.1 *If the operator A is invertible such that $A = A^{-1}$ with $B = A$ in definition 2.0.14,we get definition 2.0.13. So the above class of mappings given in Definition 2.0.14 is not empty.*

Remark 2.0.2 *If A is α-admissible, it is obvious that (A, A) is a generalized α-admissible pair.*

We present the following examples.

Example 2.0.6 *Take $X = \{1, 2, \frac{1}{2}\}$. Consider $A = B : X \to X$ given by*

$$Ax = Bx = \frac{1}{x}.$$

The mappings A and B are well defined and for all $x, y \in X$,

$$\alpha(Ax, By) = \alpha(Ax, Ay) \text{ and } \alpha(BAx, ABy) = \alpha(x, y). \tag{4}$$

Take $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(1,1) = \alpha(1,2) = \alpha(2,1) = \alpha\left(1, \frac{1}{2}\right) = \alpha\left(\frac{1}{2}, 1\right) = 1,$$

and 0 otherwise. It is clear that if $\alpha(x, y) = 1$, we have $\alpha(Ax, Ay) = 1$. Thus, by (4), we can say that (A, B) is a generalized α-admissible pair.

Example 2.0.7 *Take $X = [0, \infty)$. Consider the mappings $A, B : X \to X$ given by*

$$Ax = \begin{cases} 1 \text{ if } x \in [0,3], \\ \frac{1}{2} \text{if } x > 3 \end{cases} \quad \text{and } Bx = \frac{3}{4}.$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 4 \text{ if } x, y \in [0,3], \\ \frac{1}{3} \text{ otherwise} \end{cases}$$

.

First, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x, y \in [0,3]$. Thus, for $x, y \in [0,3]$, we have

$$\alpha(Ax, By) = \alpha\left(1, \frac{3}{4}\right) = 4 \geq 1,$$

and

$$\alpha(BAx, ABy) = \alpha\left(B(1), A\left(\frac{3}{4}\right)\right) = \alpha\left(\frac{3}{4}, 1\right) = 4 \geq 1.$$

Therefore, (A, B) is a generalized α-admissible pair.

Example 2.0.8 *Take $X = [0,2]$ and the mappings $A, B : X \to X$:*

$$Ax = \begin{cases} \frac{x}{x+1} \textit{ if } x \in [0,1], \\ x - \frac{1}{2} \textit{ if } x \in (1,2] \end{cases} \quad \textit{and } Bx = \begin{cases} \frac{x}{2} \textit{ if } x \in [0,1], \\ 2x - 2 \textit{ if } x \in (1,2]. \end{cases}$$

Consider the mapping $\alpha : X \times X \to [0, \infty)$ given by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x, y \in [0,1], \\ 0 \text{ otherwise} \end{cases}.$$

Let $x, y \in X$ such that $\alpha(x,y) \geq 1$. By definition of α, this implies that $x, y \in [0,1]$. Thus,for $x, y \in [0,1]$, we have

$$\alpha(Ax, By) = \alpha\left(\frac{x}{x+1}, \frac{y}{2}\right) = 1,$$

and

$$\alpha(BAx, ABy) = \alpha\left(B(\frac{x}{x+1}), A\left(\frac{y}{2}\right)\right) = \alpha\left(\frac{x}{2(x+1)}, \frac{y}{y+2}\right) = 1.$$

Then, (A, B) is a generalized α-admissible pair.

Now, we introduce the concept of generalized α-implicit contractive mappings in the setting of partial metric spaces.

Definition 2.0.13 *Let (X,p) be a partial metric space and $A, B : X \to X$ be given mappings.*

We say that (A, B) is a generalized α-implicit contractive pair of mappings if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $F \in \Gamma$ such that

$$F(\alpha(x,y)p(Ax,By), p(x,y), p(x,Ax), p(y,By), p(x,By), p(y,Ax)) \leq 0, \tag{5}$$

for all $x, y \in X$. If we take $A = B$ in (5),

$$F(\alpha(x,y)p(Ax,Ay), p(x,y), p(x,Ax), p(y,Ay), p(x,Ay), p(y,Ax)) \leq 0, \tag{6}$$

then we say that A is a generalized α-implicit contractive mapping.

Theorem 2.0.2 *Let (X,p) be a complete partial metric space and $A, B : X \to X$ be generalized α-implicit contractive pair of mappings. Suppose that*

(i) (A, B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) A and B are continuous on (X, p).

Then there exists $u \in X$ such that

$$p(u, Au) = p(Au, Au), p(u, Bu) = p(Bu, Bu) \text{ and } p(u,u) = 0 \tag{7}$$

Assume in addition that

(v) $\alpha(z,z) \geq 1$ for all z verifying $p(z, Az) = p(Az, Az), p(z, Bz) = p(Bz, Bz)$ and $p(z,z) = 0$;

(vi) F satisfies

($F\gamma$): if $F(u,0,v,w,u,u) \leq 0$ for all $u,v,w \geq 0$, there exists $\gamma \in [0,1)$ such that $u \leq \gamma max\{v,w\}$.

Then, u is a common fixed point of A and B, that is,$u = Au = Bu$.

Proof. In the first step of the proof we will prove that (x_n) is a cauchy sequence in (X,p)

By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$. Take $x_1 = Ax_0$ and $x_2 = Bx_1$. By induction, we construct a sequence (x_n) such that

$$x_{2n} = Bx_{2n-1} \text{and } x_{2n+1} = Ax_{2n} \forall n = 1,2,... \tag{8}$$

We have $\alpha(x_0,x_1) \geq 1$ and since (A,B) is a generalized α-admissible pair, so

$$\alpha(x_1,x_2) = \alpha(Ax_0,Bx_1) \geq 1 \text{and } \alpha(x_2,x_3) = \alpha(Bx_1,Ax_2) = \alpha(BAx_0,ABx_1) \geq 1.$$

Similar to above, we obtain

$$\alpha(x_n,x_{n+1}) \geq 1, \text{for all } n = 0,1,.... \tag{9}$$

On the other hand, by (iii), we have

$$\alpha(x_2,x_1) = \alpha(BAx_0,Ax_0) \geq 1.$$

Applying again (iii)

$$\alpha(x_4,x_3) = \alpha(BAx_2,Ax_2) \geq 1.$$

Continuing the same process, we obtain

$$\alpha(x_{2n},x_{2n-1}) \geq 1 \text{for all } n = 1,2, \tag{10}$$

We claim that (x_n) is a Cauchy sequence in (X,p).From (5), we have

In (5) we put $x_{2n-2} = x$ and $y = x_{2n-1}$, with using (8), we have

$$F(\alpha(x_{2n-2},x_{2n-1})p(Ax_{2n}-2,Bx_{2n-1}),p(x_{2n-2},x_{2n-1}),p(x_{2n-2},Ax_{2n-2}),$$

$$p(x_{2n-1},Bx_{2n-1}),p(x_{2n-2},Bx_{2n-1}),p(x_{2n-1},Ax_{2n-2})) \leq 0,$$

that is,

$$F(\alpha(x_{2n-2},x_{2n-1})p(x_{2n-1},x_{2n}),p(x_{2n-2},x_{2n-1}),p(x_{2n-2},x_{2n-1})$$

$$, p(x_{2n-1}, x_{2n}), p(x_{2n-2}, x_{2n}), p(x_{2n-1}, x_{2n-1})) \leq 0.$$

Using (9) and (p4) in the fifth variable, we get due to (F1) in the first and fifth variables

$$F(p(x_{2n-1}, x_{2n}), p(x_{2n-2}, x_{2n-1}), p(x_{2n-2}, x_{2n-1}), p(x_{2n-1}, x_{2n}), \tag{11}$$

$$p(x_{2n-2}, x_{2n-1}) + p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n})) \leq 0.$$

By (F2), we obtain

$$p(x_{2n-1}, x_{2n}) \leq h_1(p(x_{2n-2}, x_{2n-1})). \tag{12}$$

Similarly, from (5),puting $x_{2n} = x$ and $x_{2n-1} = y$ with using (8), we have

$$F(\alpha(x_{2n}, x_{2n-1})p(Ax_{2n}, Bx_{2n-1}), p(x_{2n}, x_{2n-1}), p(x_{2n}, Ax_{2n}),$$

$$p(x_{2n-1}, Bx_{2n-1}), p(x_{2n}, Bx_{2n-1}), p(x_{2n-1}, Ax_{2n})) \leq 0,$$

that is,

$$F(\alpha(x_{2n}, x_{2n-1})p(x_{2n+1}, x_{2n}), p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), \tag{13}$$

$$p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n}), p(x_{2n-1}, x_{2n+1})) \leq 0,$$

By (10) and (F1) in the first variable, we get

$$F(p(x_{2n+1}, x_{2n}), p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n}), p(x_{2n-1}, x_{2n+1})) \leq 0, \tag{14}$$

By (p2),

$p(x_{2n}, x_{2n}) \leq p(x_{2n}, x_{2n+1})$ and by (p4), $p(x_{2n-1}, x_{2n+1}) \leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})$.

So applying (F1) in the fifth and sixth variables, we find that

$$F(p(x_{2n+1}, x_{2n}), p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \tag{15}$$

$$p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})) \leq 0,$$

By (F2), we obtain

$$p(x_{2n}, x_{2n+1}) \leq h_1(p(x_{2n-1}, x_{2n})). \tag{16}$$

Combining (16) and (12), we obtain

$$p(x_n, x_{n+1}) \leq h_1(p(x_{n-1}, x_n)) \text{ for all } n = 1, 2, ... \tag{17}$$

We deduce

$$p(x_n, x_{n+1}) \leq h_1^n(p(x_0, x_1)) \text{ for all } n = 0, 1, 2, ... \tag{18}$$

Since $h_1 \in \Psi$, we get

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{19}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p)

$$\begin{aligned} p(x_n, x_{n+k}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+k-1}, x_{n+k}) \\ &\leq (h_1^n + h_1^{n+1} + \cdots + h_1^{n+k-1})(p(x_0, x_1)) \\ &\leq \sum_{m=n}^{\infty} h_1^m(p(x_0, x_1)). \end{aligned} \tag{20}$$

Since $h_1 \in \Psi$, the above implies that

$$p(x_n, x_{n+k}) \to 0 \text{ as } n \to \infty, \text{for all } k. \tag{21}$$

Since $p^s(x, y) \leq 2p(x, y)$ for all $x, y \in X$, so

$$\lim_{n\to\infty} p^s(x_n, x_{n+k}) = 0.$$

It follows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s).Since (X, p) is complete, then from Lemma 1.2.2, (X, p^s) is a complete metric space.Therefore, the sequence $\{x_n\}$ converges to some $u \in X$, that is, $\lim_{n\to\infty} p^s(x_n, u) = 0$.

From the properties (b) in Lemma 1.2.2, we have

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{m\geq n\to\infty} p(x_n, x_m)$$

By (21), we get

$$p(u, u) = \lim_{n-\to\infty} p(x_n, u) = \lim_{m\geq n-\to\infty} p(x_n, x_m) = 0. \tag{22}$$

This implies that

$$\lim_{n\to\infty} p(x_{2n+1}, u) = \lim_{n\to\infty} p(x_{2n+2}, u) = 0. \tag{23}$$

By (8), it means that

$$\lim_{n\to\infty} p(Ax_{2n}, u) = \lim_{n\to\infty} p(Bx_{2n+1}, u) = 0. \tag{24}$$

We shall prove that $u = Au = Bu$, that is, u is a common fixed point of A and B.

Using $x_{2n+1} \to u$ in (X, p) and $p(u, u) = 0$ in **Lemma 1.2**, we have

$$\lim_{n\to\infty} p(x_{2n+1}, Au) = p(u, Au). \tag{25}$$

By continuity of A and $x_{2n} \to u$ in (X, p), we have

$$\lim_{n\to\infty} p(x_{2n+1}, Au) = \lim_{n\to\infty} p(Ax_{2n}, Au) = p(Au, Au). \tag{26}$$

From (25) and (26), we obtain

$$p(u, Au) = p(Au, Au). \tag{27}$$

Using continuity of B and similarly to (27), we get

$$p(u, Bu) = p(Bu, Bu). \tag{28}$$

Thus, from (22), (27) and (28), (7) holds. So by condition (v), we have

$$\alpha(u, u) \geq 1.$$

Applying (5) for $x = y = u$

$$F(\alpha(u, u)p(Au, Bu), p(u, u), p(u, Au), p(u, Bu), p(u, Bu), p(u, Au)) \leq 0,$$

i.e,

$$F(\alpha(u, u)p(Au, Bu), 0, p(u, Au), p(u, Bu), p(u, Bu), p(u, Au)) \leq 0.$$

Due to the fact that $\alpha(u, u) \geq 1$ and by (F1) in the first variable, we get

$$F(p(Au, Bu), 0, p(u, Au), p(u, Bu), p(u, Bu), p(u, Au)) \leq 0.$$

Remember that $p(u, Bu) = p(Bu, Bu) \leq p(Au, Bu)$ and $p(u, Au) = p(Au, Au) \leq p(Au, Bu)$, so applying (F1) in the fifth and sixth variables, we obtain

$$F(p(Au, Bu), 0, p(u, Au), p(u, Bu), p(Au, Bu), p(Au, Bu)) \leq 0.$$

Since F satisfies property $(F\gamma)$, so

$$p(Au, Bu) \leq \gamma max\{p(u, Au), p(u, Bu)\} \leq \gamma p(Au, Bu),$$

which holds unless $p(Au, Bu) = 0$. Thus, $Au = Bu$. We deduce from (27) and (28) that $p(u, Au) = 0 = p(u, Bu)$ so

$$u = Au = Bu$$

.This completes the proof.

If we take $A = B$ in Theorem 2.0.3, we get the following result. ■

Corollary 2.0.1 *Let (X, p) be a complete partial metric space and $A : X \to X$ a given mapping. Assume there exists $F \in \Gamma$ such that*

$$F(\alpha(x, y)p(Ax, Ay), p(x, y), p(x, Ax), p(y, Ay), p(x, Ay), p(y, Ax)) \leq 0$$

for all $x, y \in X$. Suppose that

(i) A is an α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) A is continuous on (X, p).

Then there exists $u \in X$ such that

$$p(u, Au) = p(Au, Au), \text{and } p(u, u) = 0. \tag{29}$$

Assume in addition that

(iv) $\alpha(z, z) \geq 1$ for all z verifying $p(z, Az) = p(Az, Az)$ and $p(z, z) = 0$;

(v) F satisfies

$(F\gamma)$ if $F(u, 0, v, w, u, u) \leq 0$ for all $u, v, w \geq 0$, there exists $\gamma \in [0, 1)$ such that $u \leq \gamma max\{v, w\}$.

Then, u is a fixed point of A, that is, $u = Au$.

Proof. The proof follows from the lines in the proof of Theorem 2.0.3, except that we do need hypothesis (iii) of Theorem 2.0.3.Considering the metric case in Theorem 2.0.3, we have ■

Corollary 2.0.2 *Let (X,d) be a complete metric space and $A,B : X \to X$ be given mappings.*

Suppose there exists $F \in \Gamma$ such that

$$F(\alpha(x,y)d(Ax,By), d(x,y), d(x,Ax), d(y,By), d(x,By), d(y,Ax)) \leq 0,$$

for all $x,y \in X$. Suppose that

(i) (A,B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) A and B are continuous on (X,d).

Then there exists $u \in X$ such that $u = Au = Bu$.

Proof. Due to conditions (i)–(iv), there exists $u \in X$ such that from Corollary 2.0.1, (7) becomes

$$u = Au = Bu,$$

that is, u is a common fixed point of A and B. Here, we do not need conditions (v) and (vi) given in Theorem 2.0.3. ■

Corollary 2.0.3 *Let (X,d) be a complete metric space and $A : X \to X$ a given mapping.*

Assume there exists $F \in \Gamma$ such that

$$F(\alpha(x,y)d(Ax,Ay), d(x,y), d(x,Ax), d(y,Ay), d(x,Ay), d(y,Ax)) \leq 0$$

for all $x,y \in X$. Suppose that

(i) A is an α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) A is continuous on (X,d).

Then there exists $u \in X$ such that $u = Au$.

Proof. Due to conditions (i)–(iii), there exists $u \in X$ such that from Theorem 2.0.3, (29) becomes

$$Au = u$$

■

When replacing the continuity of T on (X, p) by the continuity of T on (X, p^s), the conditions (v) and (vi) of Theorem 2.0.3 are omitted and we naturally state the following result.

Theorem 2.0.3 *Let (X, p) be a complete partial metric space and $A, B : X \to X$ be a generalized α-implicit contractive pair of mappings. Suppose that*

(i) (A, B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) A and B are continuous on (X, p^s).

Then there exists $u \in X$ such that u is a common fixed point of A and B, that is, $u = Au = Bu$.

Analogously, we can derive the following result by letting $A = B$ in Theorem 2.0.4.

Corollary 2.0.4 *Let (X, p) be a complete partial metric space and $A : X \to X$ a given*

mapping. Assume there exists $F \in \Gamma$ such that

$$F(\alpha(x, y)p(Ax, Ay), p(x, y), p(x, Ax), p(y, Ay), p(x, Ay), p(y, Ax)) \leq 0$$

for all $x, y \in X$. Suppose that

(i) A is a α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) A is continuous on (X, p^s).

Then there exists $u \in X$ such that u is a fixed point of A, that is, $u = Au$.

Taking the operator F presented by Example 2.0.4 in Theorem 2.0.4, we have

Corollary 2.0.5 *Let (X, p) be a complete partial metric space and $A, B : X \to X$ satisfying*

$$\alpha(x,y)p(Ax,By) \leq k \max\{p(x,y), p(x,Ax), p(y,By), p(x,By), p(y,Ax)\} \qquad (30)$$

where $k \in [0, \frac{1}{2})$. Suppose that

(i) (A, B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) A and B are continuous on (X, p^s).

Then there exists $u \in X$ such that u is a common fixed point of A and B, that is, $u = Au = Bu$.

For $A = B$, we have the following result.

Corollary 2.0.6 *Let (X, p) be a complete partial metric space and $A : X \to X$ be a mapping such that*

$$\alpha(x,y)p(Ax,Ay) \leq k \max\{p(x,y), p(x,Ax), p(y,Ay), p(x,Ay), p(y,Ax)\} \qquad (31)$$

where $k \in [0, \frac{1}{2})$. Suppose that

(i) A is a α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) A is continuous on (X, p^s).

Then there exists $u \in X$ such that u is a fixed point of A and B, that is, $u = Au$.

Note that in Theorem 2.0.3, the continuity hypothesis of F is not required. But this hypothesis is essential for Theorem 2.0.5. In the next result, we drop the continuity hypothesis of A and B and we replace it by the following:

(H) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and

$\alpha(x, x_{n(k)}) \geq 1$ for all k.

Theorem 2.0.4 *Let (X, p) be a complete partial metric space and $A, B : X \to X$ be generalized α-implicit contractive pair of mappings. Suppose that*

(i) (A, B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) (H) is satisfied.

Then there exists a $u \in X$ such that $u = Au = Bu$. We also have $p(u, u) = 0$.

Proof. Following the proof of Theorem 2.0.3, the sequence $\{x_n\}$defined by (8) is Cauchy and converges to some $u \in X$ in (X, p). Remember that (9) and (10) hold, so from condition (iv), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{2n(k)}, u) \geq 1$ and $\alpha(u, x_{2n(k)-1}) \geq 1$ for all k. We shall show that $u = Au = Bu$.

Taking $x = x_{2n(k)}$ and $y = u$ in (5)

$$F(\alpha(x_{2n(k)}, u)p(Ax_{2n(k)}, Bu), p(x_{2n(k)}, u), p(x_{2n(k)}, Ax_{2n(k)}),$$

$$p(u, Bu), p(x_{2n(k)}, Bu), p(u, Ax_{2n(k)})) \leq 0.$$

Having $\alpha(x_{2n(k)}, u) \geq 1$, so by (F1) in the first variable, we have

$$F(p(x_{2n(k)+1}, Bu), p(x_{2n(k)}, u), p(x_{2n(k)}, x_{2n(k)+1}),$$

$$p(u, Bu), p(x_{2n(k)}, Bu), p(u, x_{2n(k)+1})) \leq 0.$$

Letting k tend to infinity and using continuity of F, we have

$$F(p(u, Bu), 0, 0, p(u, Bu), p(u, Bu), 0) \leq 0.$$

Using (F1) in the sixth variable, we can write

$$F(p(u, Bu), 0, 0, p(u, Bu), p(u, Bu), p(u, Bu)) \leq 0.$$

By (F2), it follows that $p(u, Bu) \leq h_1(0) = 0$, which implies that $u = Bu$.

Similarly, by taking $x = u$ and $y = x_{2n(k)-1}$ in (5), we have

$$F(\alpha(u, x_{2n(k)-1})p(Au, Bx_{2n(k)-1}), p(u, x_{2n(k)-1}), p(u, Au),$$

$$p(x_{2n(k)-1}, Bx_{2n(k)-1}), p(u, Bx_{2n(k)-1}), p(x_{2n(k)-1}, Au)) \leq 0.$$

By (F1) and having $\alpha(u, x_{2n(k)-1}) \geq 1$, we get

$$F(p(Au, x_{2n(k)}), p(u, x_{2n(k)-1}), p(u, Au), p(x_{2n(k)-1}, x_{2n(k)}),$$

$$p(u, x_{2n(k)}), p(x_{2n(k)-1}, Au)) \leq 0.$$

Letting $k \to \infty$ and using continuity of F, we have

$$F(p(Au, u), 0, p(u, Au), 0, 0, p(u, Au)) \leq 0.$$

Using (F1) in the fifth variable, we can write

$$F(p(Au, u), 0, p(u, Au), 0, p(u, Au), p(u, Au)) \leq 0.$$

By (F2), it follows that $p(u, Au) \leq h_1(0) = 0$, which implies that $u = Au$.

Now, consider (H1) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k. ■

We state the following corollaries.

Corollary 2.0.7 *Let (X, p) be a complete partial metric space and $A : X \to X$ a given mapping. Assume there exists $F \in \Gamma$ such that*

$$F(\alpha(x, y)p(Ax, Ay), p(x, y), p(x, Ax), p(y, Ay), p(x, Ay), p(y, Ax)) \leq 0$$

for all $x, y \in X$. Suppose that

(i) A is an α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iv) (H1) is satisfied.

Then there exists a $u \in X$ such that $u = Au$. We also have $p(u, u) = 0$.

Proof. Taking $B = A$ in Theorem 2.0.5, we get the result. ■

Corollary 2.0.8 *Let (X, p) be a complete partial metric space and $A, B : X \to X$ satisfying*

$$\alpha(x, y)p(Ax, By) \leq k \max\{p(x, y), p(x, Ax), p(y, By), p(x, By), p(y, Ax)\},$$

where $k \in [0, \frac{1}{2})$. Suppose that

(i) (A,B) is a generalized α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) $\alpha(BAx, Ax) \geq 1$ for all $x \in X$;

(iv) (H) is satisfied.

Then there exists a $u \in X$ such that $u = Au = Bu$.

Proof. It suffices to consider in Theorem 2.0.5 the operator F given by Example 2.0.4. ■

Corollary 2.0.9 *Let (X, p) be a complete partial metric space and $A : X \to X$ be a mapping such that*

$$\alpha(x, y)p(Ax, Ay) \leq k \max\{p(x, y), p(x, Ax), p(y, Ay), p(x, Ay), p(y, Ax)\},$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Suppose that

(i) A is an α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$;

(iii) (H1) is satisfied.

Then there exists a $u \in X$ such that $u = Au$.

Proof. It suffices to consider in Corollary 2.0.7 the operator F given by Example 2.0.4 ■

The following two corollaries are ´ Ciri´c [10] type results in the setting of partial metric spaces.

Corollary 2.0.10 *Let (X, p) be a complete partialmetric space and $A, B : X \to X$ satisfying*

$$p(Ax, By) \leq k \max\{p(x, y), p(x, Ax), p(y, By), p(x, By), p(y, Ax)\}$$

where $k \in [0, \frac{1}{2})$. Then there exists a $u \in X$ such that $u = Au = Bu$.

Proof. It suffices to take $\alpha(x, y) = 1$ in Corollary 2.0.8 ■

Corollary 2.0.11 *Let (X, p) be a complete partialmetric space and $A : X \to X$ be a mapping such that*

$$p(Ax, Ay) \leq k \max\{p(x, y), p(x, Ax), p(y, Ay), p(x, Ay), p(y, Ax)\}$$

where $k \in [0, \frac{1}{2})$. Then there exists a $u \in X$ such that $u = Au$.

Proof. The proof follows easily when taking $\alpha(x, y) = 1$ in Corollary 2.0.9 ■

To prove uniqueness of the common fixed point given in Theorem 2.0.3 (resp. Theorem 2.0.5),

we need to take the following additional hypotheses.

(U) For all $x, y \in CF(A, B)$, we have $\alpha(x, y) \geq 1$, where $CF(A, B)$ denotes the set of common fixed points of A and B,

(F3) : For all $t > 0, F(t, t, 0, 0, t, t) > 0$.

Theorem 2.0.5 *Adding conditions (U) and (F3) to the hypotheses of Theorem 2.0.3 (resp.Theorem 2.0.5), we obtain that u is the unique common fixed point of A and B.*

Proof. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Au = Bu$ and $v = Av = Bv$ with $u = v$. Of course from Theorem 2.0.3 (resp. Theorem 2.0.5), such u and v satisfy$p(u,u) = p(v,v) = 0$. By (5), we get

$$F(\alpha(u,v)p(Au,Av), p(u,v), p(u,Au), p(v,Bv), p(u,Bv), p(v,Au)) \leq 0,$$

i.e,

$$F(\alpha(u,v)p(u,v), p(u,v), p(u,u), p(v,v), p(u,v), p(v,u)) \leq 0.$$

Due to the fact that $\alpha(u,v) \geq 1$, so by (F1) in the first variable, we get

$$F(p(u,v), p(u,v), 0, 0, p(u,v), p(u,v)) \leq 0.$$

Since F satisfies property (F3), so it is a contradiction. Hence $u = v$. ■

Theorem 2.0.6 *Adding conditions (U) and (F3) to the hypotheses of Corollary 2.0.1 (resp.Corollary 2.0.7), we obtain that u is the unique fixed point of A.*

The following two examples illustrate Theorem 2.0.6 where A and B have a unique common fixed point.

Example 2.0.9 *Take* $X = [0, \frac{4}{3}]$ *endowed with the complete standard partial metric* $p(x,y) = max\{x,y\}$. *Consider the mappings* $A, B : X \to X$ *given by*

$$Ax = \begin{cases} \frac{x}{3} \text{ if } x \in [0,1] \\ 2x - 53 \text{ if } x \in [1, 43] \end{cases}, \text{and } Bx = \begin{cases} \frac{x}{3} \text{ if } x \in [0,1] \\ x - \frac{2}{3} \text{ if} x \in \left[1, \frac{2}{4}\right] \end{cases}.$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x, y \in [0,1], \\ 0 otherwise. \end{cases}$$

First, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. By definition of α, this implies that $x, y \in [0,1]$.

Thus,

$$\alpha(Ax, By) = \alpha\left(\frac{x}{3}, \frac{y}{3}\right) = 1,$$

and

$$\alpha(BAx, ABy) = \alpha\left(\frac{x}{9}, \frac{y}{9}\right) = 1.$$

Then, (A, B) is a generalized α-admissible pair. For all $x \in [0, 1]$, we have

$$\alpha(BAx, Ax) = \alpha\left(\frac{x}{9}, \frac{x}{3}\right) = 1.$$

On the other hand, for all $x \in [1, \frac{4}{3}]$, we have $(2x - \frac{5}{3}) \in [\frac{1}{3}, 1] \subset [0, 1]$, so

$$\alpha(BAx, Ax) = \alpha\left(B\left(2x - \frac{5}{3}\right), 2x - \frac{5}{3}\right) = \alpha\left(\frac{2x - \frac{5}{3}}{3}, 2x - \frac{5}{3}\right) = 1.$$

From the two above identities, we get

$$\alpha(BAx, Ax) = 1 \text{ for all } x \in X.$$

If x or y is not in $[0, 1], \alpha(x, y) = 0$, so (30) holds. Now, we restraint to the case where $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned} \alpha(x, y)p(Ax, By) &= \max\left\{\frac{x}{3}, \frac{y}{3}\right\} \\ &= \frac{1}{3}\max\{x, y\} = kp(x, y) \\ &\leq k\max\{p(x, y), p(x, Ax), p(y, By), p(x, By), p(y, Ax)\}, \end{aligned}$$

where $k = \frac{1}{3}$. Then, (30) is satisfied. Moreover, the mappings A and B are continuous on (X, p^s) and there exists $x_0 = 0$ such that $\alpha(x_0, Ax_0) = \alpha(0, 0) = 1$. Thus, all hypotheses of Corollary 2.0.5 are verified, so there exists a common fixed point of A and B. But, since the hypothesis (U) is satisfied, so applying Theorem 2.0.6, the above common fixed point is unique and it is $u = 0$.

Example 2.0.10 *Take* $X = \{0, 1, 2\}$ *and* $A = B : X \to X$ *such that*

$$A0 = B0 = 0, A1 = B1 = 0 \text{ and } A2 = B2 = 1. \tag{32}$$

Take $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(0, 0) = \alpha(0, 1) = \alpha(0, 2) = \alpha(1, 0) = \alpha(2, 0) = 1,$$

and 0 otherwise. Clearly, if $\alpha(x,y) \geq 1$, so x or y is equal to 0, and then by (32), in this case we have $\alpha(Ax, By) = \alpha(Ax, Ay) = 1$ and

$$\alpha(BAx, ABy) = 1.$$

So (A, B) is a generalized α-admissible pair. Moreover,

$$\alpha(BA0, A0) = \alpha(BA1, A1) = \alpha(BA2, A2) = 1.$$

We have $\alpha(0, A0) = 1$. Now, we define the partial metric p by

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}.$$

Notice that $p(1,1) = \frac{1}{2}$, so p is not a metric on X. Since $p^s(x,y) = |x-y|$, so (X,p) is a complete partial metric space.

Going back to Example 1.1, take

$$F(t_1, ..., t_6) = t_1 - ct_4,$$

where $0 \leq c < 1$. Therefore the inequality (5) we want to prove becomes

$$\alpha(x,y)p(Ax, By) \leq cp(y, By). \tag{33}$$

Consider $c = \frac{3}{5}$

If x and y are different to 0, we have $\alpha(x,y) = 0$, so (33) holds. Now, we restraint to the case where x or y is equal to 0. By symmetry of α and p, it suffices to take the cases

$$(x = 0, y = 1), (x = 0, y = 2) \text{ and } (x = y = 0).$$

Case 1: $(x = 0, y = 1)$. We have

$$\alpha(0,1)p(A0, B1) = p(0,0) = 0 \leq cp(1, B1)$$

Case 2: $(x = 0, y = 2)$. We have

$$\alpha(0,2)p(A0, B2) = p(0,1) = \frac{3}{4} = \frac{5}{4}c = cp(2, B2).$$

Case 3: $(x = 0, y = 0)$. We have

$$\alpha(0,0)p(A0, B0) = p(0,0) = 0 = cp(0, B0).$$

Then, (33) is satisfied.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. By definition of α, this implies that $x_n = 0$ or $x_{n+1} = 0$ for all n, so there exists a subsequence $\{x_{n(k)}\}$ such that

$$\alpha(x_{n(k)}, x) = 1 \text{ and } \alpha(x, x_{n(k)}) = 1,$$

that is, the hypothesis (H) is satisfied. Thus, applying Theorem 2.0.5, the mappings A and B have a common fixed point. Here, the hypothesis (U) also holds. So applying Theorem 2.0.6,

0 is the unique common fixed point of A and B.

Chapter 3

Application to dynamic programming

Dynamic programming [6] is a collection of methods for solving sequential decision problems. The methods are based on decomposing a multistage problem into a sequence of interrelated one-stage problems. Fundamental to this decomposition is the principle of optimality, which was developed by Richard Bellman in the 1950s. Its importance is that an optimal solution for a multistage problem can be found by solving a functional equation relating the optimal value for a (t + 1)-stage problem to the optimal value for a t-stage problem.

3.0.6 Structure of dynamic programming problems

Dynamic programming (DP for short) [8] is the principal method for analysing a large and diverse class of sequential decision problems. Examples include deterministic and stochastic optimal control problems with continuous state space, Markov and semi-Markov decision problems with discrete state space, minimax problems, and sequential zero-sum games. While the nature of these problems may vary widely, their underlying structures is very similar. In all cases, there is an underlying mapping that based on a dynamic system associated with it and the corresponding cost for each stage. This mapping, the DP operator, provides a compact "mathematical signature" of the problem. It defines the cost function of policies and the optimal cost function, and it provides a convenient shorthand notation for algorithmic description and analysis.

3.0.7 A deterministic optimal control example

To clarify our viewpoint, let us consider the discrete time-specific optimum control problem described by the system equation

$$x_{k+1} = f(x_k, u_k), \qquad k = 0, 1, ... \tag{1.1}$$

Here x_k is the state of the system which takes the values in the set X (the state space), and u_k is the control element which takes the values in the set U (the control space).In stage k, there is a cost

$$\alpha^k g(x_k, u_k)$$

incurred when u_k is applied in the case x_k , where α is a scalar in $(0, 1]$ that has the interpretation of the discount factor at $\alpha < 1$. The controls are chosen as a function of the current state, subject to a constraint that depends on that state. In particular, at state x the control is constrained to take values in a given set $U(x) \subset U$. Thus we are interested in optimizing the (nonstationary) set of policies

$$\Pi = \{\mu_0, \mu_1, ...\} | \mu_k \in M, k = 0, 1, ...$$

where M is the set of functions $\mu : X \to U$ defined by

$$M = \{\mu | \mu(x) \in U(x), \forall x \in X.\}$$

The total cost of a policy $\pi = \{\mu_0, \mu_1, ...\}$ over an infinite number of stages (an infinite horizon) and starting at an initial state x_0 is the limit superior of the N-step costs

$$J_\pi(x_0) = \limsup_{N \to \infty} \sum_{k=0}^{N-1} \alpha^k g\left(x_k, \mu_k\left(x_k\right)\right), \tag{1.2}$$

where the state sequence $\{x_k\}$ is generated by the deterministic system (1.1) under the policy π :

$$x_{k+1} = f\left(x_k, \mu_k(x_k)\right), \qquad k = 0, 1, ...$$

(We use limit superior rather than limit to cover the case where the limit does not exist.) The optimal cost function is

$$J^*(x) = \inf_{\pi \in J} \pi(x), x \in X.$$

For any policy $\pi = \{\mu_0, \mu_1, ...\}$, consider the policy $\pi_1 = \{\mu_1, \mu_2, ...\}$ and write by using Eq. (1.2),

$$J_\pi(x) = g\left(x, \mu_0(x) + \alpha J_{\pi_1}\right) f(x, \mu_0(x)).$$

We have for all $x \in X$

$$\begin{aligned} J^*(x) &= \inf_{\pi=\{\mu_0,\pi_1\}\in\Pi} \left\{g\left(x, \mu_0(x)\right) + \alpha J_{\pi 1}\left(f(x, \mu_0(x))\right)\right\} \\ &= \inf_{\mu_0 \in M} \left\{g\left(x, \mu_0(x)\right) + \alpha \inf_{\pi_1 \in \Pi} J_{\pi_1}\left(f(x, \mu_0(x))\right)\right\} \\ &= \inf_{\mu_0 \in M} \left\{g\left(x, \mu_0(x)\right) + \alpha J^*\left(f(x, \mu_0(x)\right)\right\}. \end{aligned}$$

The minimization over $\mu_0 \in M$ can be written as minimization over all $u \in U(x)$, so we can write the preceding equation as

$$J^*(x) = \inf_{u \in U(x)} \left\{g(x, u) + \alpha J^*\left(f(x, u)\right)\right\}, \forall x \in X. \tag{1.3}$$

This equation is an example of Bellman's equation, which plays a central role in DP analysis and algorithms. If it can be solved for J^*,then a perfect constant policy $\{\mu^*, \mu^*, ...\}$ can be obtained usually by minimizing the right-hand side of each x, i.e.,

$$\mu^*(x) \in \arg\min_{u \in U(x)} \left\{g(x, u) + \alpha\left(J^* f(x, u)\right)\right\}, \forall x \in X. \tag{1.4}$$

We now note that both Eqs. (1.3) and (1.4) can be stated in terms of the expression

$$H(x, u, J) = g(x, u) + \alpha J f(x, u), x \in X, u \in U(x).$$

Defining

$$(T\mu J)(x) = Hx, \mu(x), J, x \in X,$$

and

$$(TJ)(x) = \inf_{u \in U(x)} H(x, u, J) = \inf_{\mu \in M} (T\mu J)(x), x \in X,$$

we see that Bellman's equation (1.3) can be written compactly as

$$J^* = TJ^*,$$

i.e., J^* is the fixed point of T , viewed as a mapping from the set of functions on X into itself. Moreover, it can be similarly seen that J_μ, the cost function of the stationary policy $\{\mu, \mu, ...\}$, is a fixed point of T_μ. In addition, the optimality condition (1.4) can be stated compactly as

$$T_{\mu^*} J^* = TJ^*.$$

We will see later that additional properties, as well as a variety of algorithms for finding J^* can be stated and analyzed using the mappings T and T_μ.

The mappings T_μ can also be used in the context of DP problems with a limited number of stages (a finite horizon). In particular, for a given policy $\pi = \{\mu_0, \mu_1, ...\}$ and the final cost $\alpha_N \bar{J}(x_N)$ for the state x_N at

the end of N stages, consider the $N-$stage cost function

$$J_{\pi,N}\left(x_0\right) = \alpha N \bar{J}(x_N) + \sum_{k=0}^{N-1} \alpha^k g\left(x_k, \mu_k(x_k)\right). \tag{1.5}$$

Then it can be verified by induction that for all initial states x_0, we have

$$J_{\pi,N}\left(x_0\right) = (T_{\mu0} T_{\mu1} \cdots T_{\mu N-1} \bar{J})(x_0). \tag{1.6}$$

Here $T_{\mu0} T_{\mu1} \ldots T_{\mu N-1}$ is the composition of the mappings $T_{\mu0}, T_{\mu1}, ...T_{\mu N-1}$,i.e., for all J,

$$(T_{\mu0} T_{\mu1} J)(x) = T_{\mu0}(T_{\mu1} J)(x), x \in X,$$

and more generally

$$(T_{\mu0} T_{\mu1} \cdots T_{\mu N-1} J)(x) = T_{\mu0}(T_{\mu1}(\cdots (T_{\mu N-1} J)))(x), x \in X,$$

The finite horizon cost functions $J_{\pi,N}$ of π can be defined in terms of the mappings $T\mu$[cf. Eq. (1.6)], and so can the infinite horizon cost function J_π :

$$J_\pi(x) = \limsup_{N \to \infty} (T_{\mu0} T_{\mu1} \cdots T_{\mu N-1} \bar{J})(x), x \in X, \tag{1.7}$$

where $\bar{J}$ is the zero function, $\bar{J}(x) = 0$ for all$x \in X$.

3.0.8 Connection with Fixed Point Methodology

Bellman's equation (1.3) and the optimality condition (1.4), mentioned in terms of the mappings $T\mu$ and T , highlight an important point, which is that DP theory is closely related to the theory of abstract mappings and their fixed points. Analogs of the Bellman equation, $J^* = TJ^*$, optimality conditions, results and other computational methods are applicable to a large variety of DP models, and can be stated compactly as described above in terms of the corresponding mappings $T\mu$ and T . The gains of this abstraction is greater generality and mathematical vision, as well as a more unified, economical, and streamlined analysis.

3.0.9 Abstract dynamic programming models

Problem Formulation

In this chapter,H.Aydi et all [3] present an application on dynamic programming. The existence of solutions of functional equations and system of functional equations has been studied in dynamic programming using various fixed point theorems. The reader can refer to [4,5,6] for a more detailed explanation of the background above. In this chapter, we present their demonstration of the existence of a common solution for classes of functional equations using Corollary 2.0.10.

Throughout this chapter, we assume that U and V are Banach spaces,$W \subseteq U$ is a state space and $D \subseteq V$ is a decision space. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well.

In particular, we are interested in solving the following two functional equations arising in dynamic programming:

$$r(x) = \sup_{y \in D}\{g(x,y) + G(x,y,r(\tau(x,y))) - b, x \in W, \tag{34}$$

$$q(x) = \sup_{y \in D}\{g(x,y) + Q(x,y,q(\tau(x,y))) - b, x \in W, \tag{35}$$

where $b > 0, \tau : W \times D \to W, g : W \times D \to \mathbb{R}$ and $G, Q : W \times D \times \mathbb{R} \to \mathbb{R}$.

Here,we study the existence of $h_* \in B(W)$ a common solution of the functional equations (34) and (35).

Let $B(W)$ denote the set of all the bounded real-valued functions on W. It is well known that $B(W)$ endowed with the partial metric

$$p_b(h,k) = b + \sup_{x\in W}|h(x) - k(x)|, h, k \in B(W). \tag{36}$$

is a complete partial metric space.

Now, take the mappings $A, B : B(W) \to B(W)$ defined by

$$A(h)(x) = \sup_{y\in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\} - b, x \in W, \tag{37}$$

and

$$B(h)(x) = \sup_{y\in D}\{g(x,y) + Q(x,y,h(\tau(x,y)))\} - b, x \in W. \tag{38}$$

Obviously, if the functions g, G and Q are bounded then A and B are well-defined. We will prove the following result.

Theorem 3.0.7 *Suppose that there exists $k \in [0, \frac{1}{2})$ such that for every $(x,y) \in W \times D$ and $h_1, h_2 \in B(W)$, the inequality*

$$|G(x,y,h_1(\tau(x,y))) - Q(x,y,h_2(\tau(x,y)))| \le kp_B(h_1,h_2)$$

,holds. Then, A and B have a common fixed point in $B(W)$.

Proof. Let $\lambda > 0$ be an arbitrary positive real number,$x \in W$ and $h_1, h_2 \in B(W)$. Then by (37) and (38), there exist $y_1, y_2 \in D$ such that

$$A(h_1)(x) < g(x,y_1) + G(x,y_1,h_1(\tau(x,y_1))) - b + \lambda \tag{39}$$

$$B(h_2)(x) < g(x,y_2) + Q(x,y_2,h_2(\tau(x,y_2))) - b + \lambda \tag{40}$$

$$A(h_1)(x) \ge g(x,y_2) + G(x,y_2,h_1(\tau(x,y_2))) \tag{41}$$

and

$$B(h_2)(x) \ge g(x,y_1) + Q(x,y_1,h_2(\tau(x,y_1))). \tag{42}$$

Then from (39) and (42), it follows easily that

$$\begin{aligned} A(h_1)(x) - B(h_2)(x) &\leq G(x, y_1, h_1(\tau(x, y_1))) - b + \lambda - Q(x, y_1, h_2(\tau(x, y_1))) \\ &\leq |G(x, y_1, h_1(\tau(x, y_1))) - Q(x, y_1, h_2(\tau(x, y_1)))| + \lambda - b \\ &\leq kp_b(h_1, h_2) + \lambda - b. \end{aligned}$$

Similarly, from (40) and (41), we get

$$B(h_2)(x) - A(h_1)(x) \leq kp_b(h_1, h_2) + \lambda - b.$$

We deduce from above inequalities that

$$|A(h_1)(x) - B(h_2)(x)| + b \leq kp_B(h_1, h_2) + \lambda. \tag{43}$$

Since the inequality (43) is true for any$x \in W$, then

$$p_B(A(h_1), B(h_2)) \leq kp_B(h_1, h_2) + \lambda. \tag{44}$$

Again $\lambda > 0$ is arbitrary, so

$$p_b(A(h_1), B(h_2)) \leq kp_b(h_1, h_2) \leq k \max\{p_b(h_1, h_2), p_b(h_1, Ah_1), p_b(h_2, Bh_2), p_b(h_1, Bh_2), p_b(h_2, Ah_1)\}. \tag{45}$$

So Corollary 2.0.10 is applicable. Consequently, the mappings A and B have a common fixed point, that is, the functional equations (34) and (35) has a common solution $h_* \in B(W)$. ■

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