Democratic People's Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Larbi Tebessi – Tebessa -

Faculty of Exact Sciences and Science of Nature and life

Mathematics and Informatics Department

Thesis

For obtaining a doctorate degree on science
in : MATHEMATICS
Option : APPLIED MATHEMATICS

By : **Bahi Mohammed Cherif**

Entitled :

The Maximum Norm Analysis of a Nonmatching Grids Method for Nonlinear Parabolic PDES

In front of the committee :

Mr.Rebiai belgacem	Prof	**Larbi Tebessi University- Tebessa**	President
Mr. Boulaaras Salah	Prof	**Qassim University -Saudi Arabia**	Supervisor
Mr. Zarai Abderrahmane	MCA	**Larbi Tebessi University- Tebessa**	Co-Supervisor
Ms. Messaoudene Hadia	MCA	**Larbi Tebessi University- Tebessa**	Examiner
Mr. Guefaifia Rafique	MCA	**Larbi Tebessi University- Tebessa**	Examiner
Ms. Degaichia Hakima	MCA	**Larbi Tebessi University- Tebessa**	Examiner
Mr. Bencheick le Hocine Mohamed Amine	MCA	**University Center of Tamanrasset**	Examiner

Defense Date : June30, 2019

Contents

Résumé

Dans cette thèse, une analyse de norme maximale d'une méthode de grilles non appariées associée à un schéma d'éléments finis temporel ainsi que la méthode spatiale de Galerkin pour l'équation parabolique avec des termes sources linéaires et des termes sources non linéaires. En outre, une estimation a posteriori de l'erreur pour la méthode de Schwarz généralisée avec des conditions aux limites de Dirichlet sur l'équation HJB évolutive des interfaces avec des problèmes de valeur aux limites du second ordre est obtenue en utilisant la même méthode mentionnée précédemment. En outre, l'utilisation de l'algorithme de Benssoussan-Lions permet de déduire le comportement asymptotique de tous les problèmes précédents selon une norme uniforme. Dans les travaux suivants, la convergence géométrique de l'estimation d'erreur des algorithmes de Schwarz correspondants, continue et discrète, d'une nouvelle classe d'EDP elliptiques non linéaires sera démontrée et les résultats de certaines expériences numériques seront présentés pour appuyer la théorie.

Mots clés : Méthode des grilles non matching, EDPs non linéaire, méthode de Schwarz.

Abstract

In this thesis, a maximum norm analysis of a nonmatching grids method combined with a finite element time scheme as well as Galerkin spatial method for parabolic equation with linear source term and with nonlinear source terms is considered. Also, an a posteriori error estimates for the generalized Schwarz method with Dirichlet boundary conditions on the interfaces evolutionary HJB equation with second order boundary value problems are derived using the same previous mentioned method. Furthermore, a result of asymptotic behaviors for all previous problems on uniform norm are deduced by using Benssoussan-Lions' algorithm. In the next works. The geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate of a new class of non linear elliptic PDEs will be proved and the results of some numerical experiments will be presented to support the theory.

key words: Nonmatching Grids Method, Nonlinear PDEs, Schwarz method.

ملخص

في هذه الأطروحة ،قمنا بمحاكاة عددية لأصناف مختلفة من المتباينات المتغيرة التطورية (المكافئية) باستخدام مخطط الفروق المنتهية والعناصر المنتهية (المحددة) ، حيث درسنا السلوك التقاربي بواسطة المعيار المنتظم للمتباينات المتغيرة والشبه المتغيرة ، أين يكون الطرف الثاني للمتباينة خطي ، وغير خطي في حالة أخرى ، بالإضافة الى تقدير الخطأ لمعادلة (آميلتون ، جاكوبي ، بيلمان) والتي تنبثق بدورها الى متباينات متغيرة أو شبه متغيرة.

وكذلك طريقة قالاركين للمعادلات المكافئية وفق الشروط الابتدائية الخطية وغير الخطية .

علاوة على ذلك ، فإن استخدام خوارزمية Benssoussan-Lions يجعل من الممكن استنتاج السلوك المقارب لجميع المشكلات السابقة وفقًا لقاعدة موحدة.

في العمل التالي ، سيتم عرض التقارب الهندسي لتقدير الخطأ لخوارزميات شوارتز المستمرة وغير المستمرة لنوع جديد من المعادلات التفاضلية الجزئية الاهليجية الغير خطية وسيتم تقديم نتائج بعض التجارب العددية لدعم النظرية.

الكلمات المفتاحية : طريقة الشبكات غير المتطابقة ، المعادلات التفاضلية الجزئية غير الخطية ، طريقة شوارتز.

List of publication

1-S. Boulaaras, **M. C. Bahi,** M. Alnegga and A. Zarai: A posteriori error estimates for generalized Schwarz method for HJB equation related to management of energy production with mixed boundary condition, Applied Sciences, Vol.19, 2017, 22-30. Its publication link is: http://www.mathem.pub.ro/apps/v19/A19-bo-E15.pdf

The link of the journal of the first paper in official SCOPUS website is:
https://www.scopus.com/sourceid/25193?origin=recordpage

2-S. Boulaaras, **M. C. Bahi ,** The maximum norm analysis of a nonmatching grids method for a class of parabolic equation, Boletim da Sociedade Paranaense de Matemática, in press, Jan. 2018. doi:10.5269/bspm.v38i4.40272, Its publication link is:
http://www.spm.uem.br/bspm/pdf/next/25.pdf

The link of the journal of second paper in official SCOPUS website is:
https://www.scopus.com/sourceid/19700201433?origin=resultslist

3-**MC. Bahi,** S. Boulaaras, M. Haiour, and A. Zarai, The maximum norm analysis of a nonmatching grids method for a class of parabolic equation with linear source terms, Applied Science APPS, Volume 20 (2018), pp. 1-17. Its link publication:
http://www.mathem.pub.ro/apps/v20/A20-ba-ZE92.pdf

The link of the journal of the third paper in official SCOPUS website is:
https://www.scopus.com/sourceid/25193?origin=recordpage

Acknowledgement

In the first place at all, my gratitude goes to Prof. Salah Boulaaras, my supervisor and Dr. Abderrahmane Zarai my co-supervisor. they always listens, advices, encourages and guides me, they were behind this work. I shall always be impressed by them availability and the care with which he comes to read and correct my manuscript. They were able to motivate each step of my work by relevant remarks and let me able to advance in my research in this area which is a new for me. I would like to thank them very sincerely for their availability even at a distance, with them I have a lot of fun to work.
I wish to thank Pr.Rebiai Belgacem , who makes me the honor of chairing the jury of my defense.
I am very grateful to the dissertation committee:Dr. Hadia Messaoudene,Dr Rafik Guefaifia, Dr.Hakima Degaichia and Dr.Mohamed Amine Bencheick le Hocine which they have agreed to rescind this thesis and to be part of my defense panel. Of course, this thesis is the culmination of many years of study. I am indebted to all those who guided me on the road of mathematics. I thank from my heart, my spouse, my sons, my children and daughters. I think for my parents, whose the work could not succeed without their endless support and encouragement since I was young. They find in the realization of this work, the culmination of their efforts and express my most affectionate gratitude.

Notations

Ω : Bounded domain in $\mathbb{R}^2$.

Γ : Topological boundary of Ω.

$x = (x_1, x_2)$: Generic point of $\mathbb{R}^2$.

$dx = dx_1 dx_2$: Lebesgue measuring on Ω.

∇u : Gradient of u.

Δu : Laplacien of u.

$D(\Omega)$: Space of differentiable functions with compact support in Ω.

$D'(\Omega)$: Distribution space.

$C^k(\Omega)$: Space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: Space of functions p-th power integrated on with measure of dx.

$\|f\|_p = \left(\int\limits_{\Omega} \left(|f|^P \right) \right)^{\frac{1}{p}}.$

$W^{1,p}(\Omega) = \{u \in L^p(\Omega),\ \nabla u \in L^p(\Omega)\}.$

H : Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}$.

<u>If X is a Banach space</u>

$L^p(0,T;X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable;} \int\limits_0^T \|f(t)\|_X^p \ dt < \infty \right\}.$

$L^\infty(0,\ T;\ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } ess - \sup\limits_{t\ \in[0,\ T]} \|f(t)\|_X^p \ < \infty \right\}.$

$C^k([0,\ T];\ X)$:Space of functions $k-$times continuously differentiable for $[0,\ T] \longrightarrow X$.

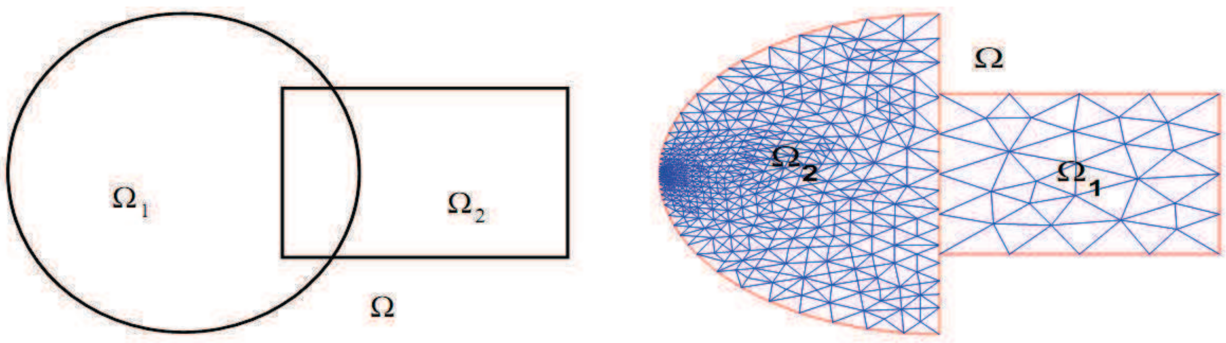
$D([0,\ T];\ X)$: Space of functions continuously differentiable with compact support in $[0,\ T]$.

$B_X = \{x \in X;\ \|x\| \leq 1\}$: Unit ball.

Introduction

Overlapping domain decomposition methods include the Original Schwarz alternating method and the additive Schwarz method. Even without conjugate gradient acceleration, the multiplicative method can take many fewer iterations than the additive version. However, the multiplicative version is not as parallelizable. We consider in fourth chapter the tow methods: the overlapping domain decomposition method, more precisely the additive Schwarz method, and the non-overlapping method. The local problems are linked together by suitable coupling terms or transmission conditions. Moreover, Hermann Schwarz was a German analyst of the 19th century. He was interested in proving the existence and uniqueness of the Poisson problem. At his time, there were no Sobolev spaces nor Lax-Milgram theorem. The only available tool was the Fourier transform, limited by its very nature to simple geometries. H.A. Schwarz in 1870, in order to consider more general situations, devised an iterative algorithm for solving Poisson problem set on a union of simple geometries : this is the alternating Schwarz method. (See figure 1) The alternating Schwarz method, introduced by was probably the first example of a domain decomposition method. Starting with a decomposition into two overlapping subdomains decomposition into two overlapping subdomains and the equations are solved iteratively on the subdomains using Dirichlet values of the neighbor domains computed in the previous step. In this way H. Schwarz could show the existence of a solution of the Poisson problem for a domain with non smooth boundary.

Also in our thesis, we have studied an a posteriori error estimates for the generalized overlapping domain decomposition method (DDM) with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with linear source terms using the theta time scheme combined with a finite element spatial approximation, similar to that in our published papers in (Boulaaras et al. [9]), (Boulaaras and Haiour [10]), (Boulaaras and Haiour [11]),(Haiour and Boulaaras [20]) which investigated Laplace operator i.e.,a posteriori error estimates for the generalized Schwarz method (GSM), for evolutionary Hamilton-Jacobi-Belmann (HJB) equation with linear source terms related to management of energy production with mixed boundary condition (MBC) are established using a theta scheme with a Galerkin spatial approximation and the techniques of the residual a posteriori error analysis.



(a) The original example of H.A. SCHWARZ (b) Decomposition into simple domains

Figure 1: *The figure shows two simple decompositions. (a) is an overlapping decomposition. In (b) the meshes of Ω_1 and Ω_2 are nonmatching at the interface.*

Figure 1:

The DDM has been used to solve the stationary and evolutionary boundary value problems on domains which consists of two or more overlapping sub-domains (see Badea [3], Bensoussan and Lions [5], Nataf [25], Boulaaras and Haiour [11], Otto and Lube [29]). The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the sub-domain. The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomains. Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in Douglas and Huang [16], Engquist and Zhao [17], Chan et al. [14] and the references therein. Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See proceedings of the annual domain decomposition conference beginning with Engquist and Zhao [17] . Moreover, The a priory estimate of the error for stationary problem is given in several papers, see for instance Lions Bensoussan and Lions [5] in which a variational formulation of the classical Schwarz method is derived. In Chan et al. Chan et al. [14] a geometry related convergence results are obtained. Douglas and Huang Douglas and Huang [16] studied the accelerated version of the GODDM, Engquist and Zhao Engquist and Zhao [17] studied the convergence for simple rectangular or circular geometries; however, these authors did not give a criterion to stop the iterative process. All these results can also be found in the recent books on domain decomposition methods of Quarteroni and Valli Quarteroni and Valli [32], Toselli and Widlund Toselli and Widlund [34] .

Recently Maday and Magoul'es Maday and Magoules [23], Maday and Magoules [24] presented an improved version of the Schwarz method for highly heterogeneous media. This method uses new optimized interface conditions specially designed to take into account the heterogeneity between the subdomains on the interfaces. A recent overview of the current state of the art on domain decomposition methods can be found in two special issues of the computer methods in applied mechanics and engineering journal, edited by Farhat and Le Tallec Farhat and Lesoinne [18], Magoul'es and Rixen Rixen and Magoulès [33] and in Nataf Nataf [25] . In general, the a priory estimate for stationary problems is not suitable for assessing the quality of the approximate solution on subdomains since it depends mainly on the exact solution itself which is unknown. The alternative approach is to use the approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the nineties of the last century with finite element methods, see the monographs Ainsworth and Oden [1], Verfürth [35] and the references therein. In their paper Otto and Lube Otto and Lube [29] gave an a posteriori estimate for a nonoverlapping domain decomposition algorithm that said that "the better the local solutions fit together at the interface the better the errors of the subdomain solutions will be." This error estimate enables us to know with certainty when one must stop the iterative process as soon as the required global precision is reached. A posteriori error analysis for the elliptic case was also used by Bernardi et al. Bernardi et al. [6] to determine an optimal value of the penalty parameter for penalty domain decomposition methods to construct fast solvers. In recent research, in Boulbrachene and Al Farei [13] the authors proved the error analysis in the maximum norm for a class of linear elliptic problems in the context of overlapping nonmatching grids and they established the optimal L^∞ error estimate between the discrete Schwarz sequence and the exact solution of the PDE. H. Benlarbi and A.-S. Chibi and Boulaaras and Haiour [11] derived a posteriori error estimates for the generalized overlapping domain decomposition method GODDM i.e., with Robin boundary conditions on the interfaces, for second order boundary value problems. They shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interfaces. After discretization of the domain by finite elements, they use the techniques of the residual a posteriori error analysis to get an posteriori error estimate for the discrete solutions on subdomains.

Our thesis is organized as follows:

In chapter 1: We lay down some fundamental definitions and theorems on functional analysis, which will be needed some them in the body of the thesis, however we give some definitions

Sobolev spaces of fractional order and trace theorems.

In chapter2 : In this chapter, we will introduce the domain decomposition method (DDM, in short). In numerical partial differential equations, domain decomposition methods solve a boundary value problem by splitting it into smaller boundary value problems on subdomains and iterating to coordinate the solution between adjacent subdomains. The basic idea behind DD methods consists in subdividing the computational domain Ω, on which a boundary-value problem is set, into two or more subdomains on which discretized problems of smaller dimension are to be solved, with the further potential advantage of using parallel solution algorithms. There are two ways of subdividing the computational domain into subdomains: one with disjoint subdomains, the others with overlapping subdomains. In non-overlapping methods, the closure of subdomains intersect only on their interface.

In chapter3 : Motivated by the idea which has been introduced by M. Haiour and S.Boulaaras (Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121,No. 4, November 2011,pp.481–493), we provide a maximum norm analysis of a theta scheme combined with finite element Schwarz alternating method for a class of parabolic equation on two overlapping subdomains with non-matching grids (Bahi et al. [4]). We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a stability analysis on the theta scheme which given by our work in (App. Math. Comp., 217, 6443–6450 (2011)), we establish, on each subdomain, an optimal asymptotic behavior between the discrete Schwarz sequence and the asymptotic solution of parabolic differential equations.

In chapter 4: This chapter deals with the maximum norm analysis of a nonmatching grids method for a class of parabolic equation with nonlinear source terms using Euler time scheme combined with a finite element spatial methods with respect to the same boundary conditions (Boulaaras et al. [8]) which presented in the forth chapter.

In chapter 5: Finally, an a posteriori error estimates for the generalized Schwarz method with Dirichlet boundary conditions (Boulaaras et al. [7]) on the interfaces evolutionary HJB equation with second order boundary value problems are derived using the same previous mentioned method.

Chapter 1

Preliminary and functional analysis

In this chapter we will introduce and state some necessary materials needed in the proof of our results, and shortly the basic results which concerning the Banach spaces, Hilbert space, the L^p space, Sobolev spaces and other theorems. The knowledge of all this notations and results are important for our study.

1.1 Banach spaces - definition and properties

We first review some basic facts from calculus in the most important class of linear spaces "Banach spaces".

Definition 1.1.1 *A Banach space is a complete normed linear space X. Its dual space $X^{'}$ is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.*

Proposition 1.1.1 *(Yosida [36]) $X^{'}$ equipped with the norm*

$$\|f\|_{X'} = \sup \left\{ |f(u)| : \|u\|_X \leq 1 \right\},$$

is also a Banach space.

Definition 1.1.2 *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim_{n\longrightarrow\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \longrightarrow u$, or $\lim\limits_{n\longrightarrow\infty} u_n = u$

Definition 1.1.3 *A sequence* (u_n) *in* X *is weakly convergent to* u *if and only if*

$$\lim_{n\longrightarrow\infty} f(u_n) = f(u),$$

for every $f \in X'$ *and this is denoted by* $\lim\limits_{n\longrightarrow\infty} u_n = u$.

1.1.1 Banach fixed-point theorem

Definition 1.1.4 *Let* (X, d) *be a metric space. Then a map* $T : X \to X$ *is called a contraction mapping on* X *if there exists* $q \in [0, 1)$ *such that*

$$d(T(x), T(y)) \leqslant qd(x, y),$$

for all x, y *in* X.

Theorem 1.1.1 *(Yosida [36]) Let* (X, d) *be a non-empty complete metric space with a contraction mapping* $T : X \to X$. *Then* T *admits a unique fixed-point* x^* *in* X *(i.e.* $T(x^*) = x^*$*). Furthermore,* x^* *can be found as follows: start with an arbitrary element* x^0 *in* X *and define a sequence* $\{x_n\}$ *by* $x_n = T(x_{n-1})$. *Then* $x_n \longrightarrow x^*$.

1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. Then, we must give some important results on these spaces here.

Definition 1.2.1 *A Hilbert space* H *is a vectorial space supplied with inner product* (u, v) *such that* $\|u\| = \sqrt{(u,\ u)}$ *is the norm which let* H *complete.*

(**The Cauchy-Schwarz inequality**) Every inner product satisfies the Cauchy-Schwarz inequality

$$|(x_1,\ x_2)| \leq \|x_1\| \, \|x_2\| \, .$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Corollary 1.2.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence which converges to* u, *in the weak topology and* $(v_n)_{n\in\mathbb{N}}$ *is an other sequence which converge weakly to* v, *then*

$$\lim_{n\longrightarrow\infty} (v_n, u_n) = (v, u) \, .$$

Theorem 1.2.1 (Lax-Milgram) *Let V be a real Hilbert space, $L(.)$ a continuous linear form on V, $a(\cdot,\cdot)$ a continuous coercive bilinear form on V. Then the problem*

$$\begin{cases} \text{find } u \in V \text{ such hat} \\ \\ a(u,v) = L(v) \text{ for every } v \in V. \end{cases}$$

has a unique solution. Further, this solution depends continuously on the linear form L.

1.3 Functional spaces

1.3.1 The $L^p(\Omega)$ spaces

Now we define Lebesgue spaces and collect some properties of these spaces. We consider $\mathbb{R}^2$ with the Lebesgue-measure μ. If $\Omega \subset \mathbb{R}^2$ is a measurable set, two measurable functions $f, g : \Omega \longrightarrow \mathbb{R}$ are called equivalent, if $f = g$ a.e. (almost everywhere) in Ω. An element of a Lebesgue space is an equivalence class.

Definition 1.3.1 *Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$. Define the standard Lebesgue space $L^p(\Omega)$, by*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ is measurable; } d \underset{\Omega}{\in} t \left| f(x) \right|^p \, dx < \infty \right\}.$$

Notation 1.3.1 *For $p \in \mathbb{R}$, and $1 \leq p < \infty$ denote by*

$$\|f\|_p = \left(\int_{\Omega} \left| f(t) \right|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ \begin{array}{l} f : \Omega \longrightarrow \mathbb{R} \text{ is measurable and there exist a constant } C, \\ \\ \text{such that, ; } \left| f(t) \right| < C \text{ a.e in } \Omega. \end{array} \right\}$$

Also, we denote by

$$\|f\|_\infty = \inf \left\{ C, \left| f(t) \right| < C \text{ a.e in } \Omega \right\}.$$

Theorem 1.3.1 *(Yosida [36])* $\left(L^p(\Omega), \|.\|_p\right), (L^\infty(\Omega), \|.\|_\infty)$ *are a Banach spaces.*

Remark 1.3.1 *In particularly, when* $p = 2,\ L^2(\Omega)$ *equipped with the inner product*

$$(f,\ g)_\Omega = \int_\Omega f(x).g(x)\,dx,$$

is a Hilbert space.

1.3.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.3.2 *(Yosida [36]) (Hölder's inequality)*
Let $1 \le p < \infty$*. Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then,* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f.g|\ \ dx \le \|f\|_p\ \ \|g\|_q\ .$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

Lemma 1.3.1 *(Minkowski inequality)*
For $1 \le p < \infty$*, we have*

$$\|u+v\|_p \le \|u\|_p + \|v\|_p\,.$$

1.4 The Sobolev space $W^{m,\ p}(\Omega)$

Proposition 1.4.1 *Let* Ω *be an open domain in* $\mathbb{R}^N$*, Then the distribution* $T \in D'(\Omega)$ *is in* $L^p(\Omega)$ *if there exists a function* $f \in L^p(\Omega)$ *such that*

$$(T,\varphi) = \int_\Omega f(x)\,g(x)\ \ dx,\ \textit{for all } \varphi \in D(\Omega)\,.$$

where $1 \le p < \infty$*, and it is well-known that* f *is unique.*

Definition 1.4.1 *Let $m \in \mathbb{N}^*$ and $p \in [0,\ \infty[$. The $W^{m,\ p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,\ p}(\Omega) = \left\{ \begin{array}{l} f \in L^p(\Omega),\ \textit{such that } \partial^\alpha f \in L^p(\Omega) \textit{ for all } \alpha \in \mathbb{N}^m \textit{ such that,} \\ \\ |\alpha| = \sum\limits_{j=1}^{n} \alpha_j \leq m,\ \textit{where, } \partial^\alpha = \partial_1^{\alpha_1}\ \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}. \end{array} \right\}$$

Theorem 1.4.1 *$W^{m,\ p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,\ p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p \ \textit{for all } f \in W^{m,\ p}(\Omega).$$

Definition 1.4.2 *When $p = 2$, we prefer to denote by $W^{m,\ 2}(\Omega) = H^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \left(\|\partial^\alpha f\|_{L^2}\right)^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$(u,\ v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Definition 1.4.3 *$H_0^m(\Omega)$ is given by the completion of $D(\Omega)$ with respect to the norm $\|.\|_{H^m(\Omega)}$.*

Remark 1.4.1 *Clearly $H_0^m(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^m(\Omega)}$. The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega) := [H_0^m(\Omega)]^*$.*

Lemma 1.4.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$D(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow D'(\Omega).$$

Now the smoothness of the boundary $\partial\Omega := \overline{\Omega} - \Omega$ can be described:

Definition 1.4.4 *Let Ω be an open subset of $\mathbb{R}^d$, $0 \leqslant \lambda \leqslant 1$, $m \in \mathbb{N}$. We say that its boundary that its boundary $\partial\Omega$ is of class $C^{m;\lambda}$ if the following conditions are satisfied:*
For every $x \in \partial\Omega$ there exist a neighborhood V of x in $\mathbb{R}^d$ and new orthogonal coordinates $\{y_1, ..., y_d\}$ such that V is a hypercube in the new coordinates:

$$V = \{(y_1, ..., y_d) : -a_i < y_i < a_i, i = 1, ...d\}$$

and there exists a function $\varphi \in C^{m;\lambda}(V^{'})$ *with*

$$V^{'} = \{(y_1, ..., y_{d-1}) : -a_i < y_i < a_i, i = 1, ...d-1\}$$

and such that

$$\begin{aligned} \left|\varphi(y^{'})\right| &\leqslant \frac{1}{2}a_d, \quad \forall y^{'} := (y_1, ..., y_{d-1}) \in V^{'}, \\ \Omega \cap V &= \left\{(y^{'}, y_d) \in V : y_d < \varphi(y^{'})\right\}, \\ \partial\Omega \cap V &= \left\{(y^{'}, y_d) \in V : y_d = \varphi(y^{'})\right\}. \end{aligned}$$

A boundary of class $C^{0;1}$ *is called Lipschitz boundary.*

1.5 The $L^p(0,\ T;\ X)$ spaces

Definition 1.5.1 *Let* X *be a Banach space, denote by* $L^p(0,\ T; X)$ *the space of measurable functions*

$$f :]0,\ T[\longrightarrow X$$

$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}}\, dt = \|f\|_{L^p(0,\ T,\ X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,\ T,\ X)} = \sup_{t\ \in]0,\ T[} ess\, \|f(t)\|_X.$$

Theorem 1.5.1 *The space* $L^p(0,\ T,\ X)$ *is a Banach space.*

Lemma 1.5.1 *Let* $f \in L^p(0,\ T,\ X)$ *and* $\dfrac{\partial f}{\partial t} \in L^p(0,\ T,\ X)$, $(1 \leq p \leq \infty)$, *then, the function* f *is continuous from* $[0,\ T]$ *to* X. *i. e.* $f \in C^1(0, T, X)$. *Since our study based on some known algebraic inequalities, we want to recall few of them here.*

1.6 Sobolev spaces of fractional order and trace theorems

In this section let $\Omega \subset \mathbb{R}^d$ is a measurable set with Lipschitz boundary $\partial\Omega$. The boundary $\partial\Omega$ of Ω will be denoted by $\Gamma := \partial\Omega$. On the $(d-1)$ -dimensional set it is also possible to define Sobolev spaces :

Definition 1.6.1 *$H^{\frac{1}{2}}(\Gamma)$ is defined by*

$$H^{\frac{1}{2}}(\Gamma) := \left\{ u \in L^2(\Gamma) : |u|_{\frac{1}{2},\Gamma} < \infty \right\}$$

where the seminorme $|.|_{\frac{1}{2},\Gamma}$ is given by

$$|u|_{\frac{1}{2},\Gamma} := \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{|u(x)-u(y)|}{|x-y|^d} ds(x)ds(y), \quad u \in H^{\frac{1}{2}}(\Gamma).$$

Theorem 1.6.1 ***(Allaire [2])*** *$H^{\frac{1}{2}}(\Gamma)$ with the scalar product*

$$(u,v)_{\frac{1}{2},\Gamma} := \int\limits_{\Gamma} uvds + \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^d} ds(x)ds(y),$$

is a Hilbert space.

Definition 1.6.2 *Let $\Gamma_1 \subset \Gamma$ be a proper, connected $(d-1)$-dimensional relative open subset. Then we define*

$$H^{\frac{1}{2}}(\Gamma_1) := \left\{ u \in L^2(\Gamma_1) : \exists \widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \text{ with } u = \widetilde{u}\,|_{\Gamma 1} \right\},$$

with norm

$$\|u\|_{\frac{1}{2},\Gamma_1} := \inf_{\substack{\widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \\ \widetilde{u}\,|_{\Gamma 1} = u}} \|\widetilde{u}\|_{\frac{1}{2},\Gamma}, \qquad u \in H^{\frac{1}{2}}(\Gamma_1).$$

Now we construct a particular subspace of $H^{\frac{1}{2}}(\Gamma_1)$. For $v \in H^{\frac{1}{2}}(\Gamma_1)$ the zero extension of v into $\Gamma - \Gamma_1$ will be denoted by $\widetilde{v}$. So we can define:

Definition 1.6.3 *$H_{00}^{\frac{1}{2}}(\Gamma_1)$ is defined by*

$$H_{00}^{\frac{1}{2}}(\Gamma_1) := \left\{ v \in L^2(\Gamma_1) : \widetilde{v} \in H^{\frac{1}{2}}(\Gamma) \right\}.$$

Notice that

$$(u,v)_{H_{00}^{\frac{1}{2}}(\Gamma_1)} := (u,v)_{\frac{1}{2},\Gamma_1} + \int_{\Gamma_1} \frac{uv}{\rho(x,\partial\Gamma_1)} ds(x),$$

where $\rho(x,\partial\Gamma_1)$ is a positive function which behaves like the distance between x and $\partial\Gamma_1$, defines a scalar product in $H_{00}^{\frac{1}{2}}(\Gamma_1)$.

Remark 1.6.1 *Grisvard [19] By a direct calculation, for all $v \in L^2(\Gamma_1)$ we obtain tow positive constants c_1, c_2 such that:*

$$c_1 \|v\|_{\frac{1}{2},\Gamma_1} \leqslant \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} \leqslant c_2 \|v\|_{\frac{1}{2},\Gamma_1}.$$

Therefore $H_{00}^{\frac{1}{2}}(\Gamma_1)$ is a Hilbert space. The dual of these spaces are denoted by

$$H^{-\frac{1}{2}}(\Gamma_1) := \left[H_{00}^{\frac{1}{2}}(\Gamma_1) \right]^*, \quad H_{00}^{-\frac{1}{2}}(\Gamma_1) := \left[H^{\frac{1}{2}}(\Gamma_1) \right]^*.$$

Next we present some trace theorems.
Let be $u \in C(\overline{\Omega})$. Then we can define the trace of u on $\partial\Omega$:

$$\gamma_0(u) := u\,|_{\partial\Omega}\,.$$

This trace operator can be extended:

Theorem 1.6.2 *Grisvard [19] Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with boundary $\partial\Omega \in C^{0;1}$. Then the trace mapping γ_0 defined on $C^{0;1}(\overline{\Omega})$ extends uniquely to a bounded, surjective linear map:*

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega).$$

Moreover the right inverse of the trace operator exists:

Theorem 1.6.3 *Grisvard [19] Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary $\partial\Omega$. Then there exists a linear bounded operator*

$$\begin{aligned} E \quad &: \quad H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega), \quad \textit{such hat} \\ \gamma_0(E(\varphi)) \quad &= \quad \varphi, \quad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Note that the preceding theorems allow the definition of the following equivalent norm on $H^{\frac{1}{2}}(\partial\Omega)$:

$$\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)} := \inf_{\substack{u \in H^1(\Omega) \\ \gamma_0(u) = u}} \|u\|_{H^1(\Omega)}, \qquad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega).$$

Sometimes the simpler notation $u\,|_{\partial\Omega} = \gamma_0(u)$ is used for functions$u \in H^1(\Omega)$. With the trace operator γ_0 we can characterize the space $H_0^1(\Omega)$:

Theorem 1.6.4 *Grisvard [19] Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with boundary $\partial\Omega \in C^{0;1}$. Then $H_0^1(\Omega)$ is the kernel of trace operator γ_0, i.e,*

$$\begin{aligned} H_0^1(\Omega) &= N(\gamma_0) = \left\{u \in H^1(\Omega) : \gamma_0(u) = 0\right\} \\ &= \left\{u \in H^1(\Omega) : u\,|_{\partial\Omega} = 0\right\}. \end{aligned}$$

Definition 1.6.4 *Let Ω is an open smooth domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Gamma_D \subsetneq \partial\Omega$ such that $\mathrm{mes}(\Gamma_D) > 0$. We set*

$$\begin{aligned} H^1_{\Gamma_D}(\Omega) &= \left\{u \in H^1(\Omega) : \gamma_0(u) = 0 \text{ on } \Gamma_D\right\} \\ & \left\{u \in H^1(\Omega) : u\,|_{\Gamma_D} = 0\right\}. \end{aligned}$$

Lemma 1.6.1 *$H^1_{\Gamma_D}(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^1(\Omega)}$.*

1.6.1 Inequality of Poincaré

Now we cite a variant of the inequality of Poincaré. It allows to estimate the function values of functions $u \in H^1(\Omega)$ by the first derivatives of functions $u \in H^1(\Omega)$.

Theorem 1.6.5 *Quarteroni and Valli [31] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Furthermore let $\Gamma_D \subset \Omega$ be a connected part of the boundary of Ω with $mes_{d-1}(\Gamma_D) > 0$. Then the inequality*

$$\|u\|_{0,\Omega} \leqslant C(\Omega, \Gamma_D)\ \ |u|_{1,\Omega}$$

is true for all $u \in H^1(\Omega)$ with $\gamma_0(u)\,|_{\Gamma_D} = 0$. The constant $C(\Omega, \Gamma_D)$ depend only on Ω and Γ_D and is bounded by the diameter of Ω.

Remark 1.6.2 *By the Inequality of Poincaré we deduce that the semi-norm $|.|_{1,\Omega}$ is an equivalent norm to $\|.\|_{1,\Omega}$ in $H^1_{\Gamma_D}(\Omega)$.*

1.6.2 Green's formula

Proposition 1.6.1 *(Necas [26]) Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary. Then for all $u, v \in H^1(\Omega)$ we have*

$$\int_{\Omega} (\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u)\ dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d,$$

where η_i is the i-th component of the outward normal vector η.

Chapter 2

Domain decomposition methods

In this chapter we will introduce the domain decomposition method (DDM, in short). In numerical partial differential equations, domain decomposition methods solve a boundary value problem by splitting it into smaller boundary value problems on subdomains and iterating to coordinate the solution between adjacent subdomains. The basic idea behind DD methods consists in subdividing the computational domain Ω, on which a boundary-value problem is set, into two or more subdomains on which discretized problems of smaller dimension are to be solved, with the further potential advantage of using parallel solution algorithms. There are two ways of subdividing the computational domain into subdomains: one with disjoint subdomains, the others with overlapping subdomains. In non-overlapping methods, the closure of subdomains intersect only on their interface. Let the domain Ω be the union of a disk and a rectangle. Consider the Poisson problem which consists in finding $u : \Omega \longrightarrow \mathbb{R}$ such that:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 2.0.5 *(Original Schwarz algorithm, cf. Dolean et al. [15]) The Schwarz algorithm is an iterative method based on solving alternatively sub-problems in domains Ω_1 and Ω_2.*

It updates $(u_1^m, u_2^m) \longrightarrow (u_1^{m+1}, u_2^{m+1})$ *by*

$$\begin{cases} -\Delta u_1^{m+1} = f, \ \ in\ \Omega_1, \\ \\ u_1^{m+1} = u_2^m \quad on \quad \partial\Omega_1 \cap \overline{\Omega_2}, \\ \\ u_1 = 0 \qquad\quad on \ \ \partial\Omega_1 \cap \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\Delta u_2^{m+1} = f, \quad in\ \Omega_2, \\ \\ u_2^{m+1} = u_1^{m+1} \quad on \ \ \partial\Omega_2 \cap \overline{\Omega_1}, \\ \\ u_2 = 0 \qquad\quad on \ \ \partial\Omega_2 \cap \partial\Omega. \end{cases}$$

H. Schwarz proved the convergence of the algorithm and thus the wellposedness of the Poisson problem in complex geometries. With the advent of digital computers, this method also acquired a practical interest as an iterative linear solver. Subsequently, parallel computers became available and a small modification of the algorithm (Boulaaras and Haiour [12]) makes it suited to these architectures. We present this method in a general case : Let given a model problem : find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Lu = f \ \ \text{in } \Omega, \\ \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

L being a generic second order elliptic operator, whose weak formulation reads

$$\begin{aligned} \text{find } u \ &\in \ V = H_0^1(\Omega) \ \text{ such that:} \\ a(u, v) \ &= \ (f, v), \quad \forall v \in V \end{aligned}$$

being $a(\cdot,\cdot)$ the bilinear form associated with L. Consider a decomposition of the domain Ω in two subdomains Ω_1 and Ω_2 such that

$$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}, \quad \Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset, \quad \partial\Omega_i \cap \Omega_j = \Gamma_i, \ i \neq j \ \text{ and } i, j = 1, 2.$$

Consider the following iterative method. Given u_2^0 on Γ_1, solve the following problems for $m \in \mathbb{N}^*$

$$\begin{cases} Lu_1^m = f \ \text{ in } \Omega_1, \\ \\ u_1^m = u_2^{m-1} \text{ on } \Gamma_1 \\ \\ u_1^m = 0 \quad \text{on } \partial\Omega_1 - \Gamma_1, \end{cases}$$

and

$$\begin{cases} Lu_2^m = f \ \text{ in } \Omega_2, \\ \\ u_2^m = \begin{cases} u_1^{m-1} \\ \\ u_1^m \end{cases} \quad \text{on } \Gamma_2 \\ \\ u_2^m = 0 \quad \text{on } \partial\Omega_2 - \Gamma_2, \end{cases} \tag{2.2}$$

In the case in which one chooses u_1^m on Γ_2 in (2.2) the method is named *multiplicative Schwarz* (MSM), it's algorithm is sequential. Whereas that in which we choose u_1^{m-1}, is named *additive Schwarz* (ASM), problems in domains Ω_1 and Ω_2 may be solved concurrently. The reason of this appointment is clarified in (Quarteroni and Valli [32]). Denoting the solution of iteration step i in subdomain Ω_j by u_j^i for the two-domain case the multiplicative variant can be described as follows : Starting with an initial guess, first a new solution in Ω_1 is computed. Then, already using this solution, the solution in Ω_2 is solved, and so on. In contrast the additive algorithm uses the solution of the previous step instead of the current solution (cf. Figure 2). The second method has got the advantage that the solution of all subdomain problems can be completely done in parallel. In the multi-domain case the multiplicative variant requires a coloring of the subdomains. We have thus two elliptic boundary-value problems with Dirichlet conditions for the two subdomains Ω_1 and Ω_2, and we would like the two sequences $(u_1^m)_{m\in\mathbb{N}^*}$ and $(u_2^m)_{m\in\mathbb{N}^*}$ to converge to the restrictions of the solution u of problem (2.1), that is

$$\lim_{m\to+\infty} u_1^m = u^m\,|_{\Omega_1} \quad \text{ and } \lim_{m\to+\infty} u_2^m = u^m\,|_{\Omega_2}\ .$$

It can be proven that the Schwarz method applied to problem (2.1) always converges, with a rate that increases as the measure $|\Omega_1 2|$ of the overlapping region Ω_{12} increases. It is easy to see that if the algorithm converges, the solutions $u_i^{\infty}, i = 1, 2$, in the intersection of the subdomains take the same values. The original algorithms ASM and MSM are very slow. Another weakness is the need of overlapping subdomains. Indeed, only the continuity of the solution is imposed and nothing is imposed on the matching of the fluxes. When there is no overlap convergence is thus impossible. n order to remedy the drawbacks of the original Schwarz method, Modify the original Schwarz method by replacing the Dirichlet interface conditions on $\partial\Omega_i \cap \partial\Omega$, $i = 1, 2$, by Robin interface conditions ($\partial\eta_i + \alpha$, where η_i is the outward normal to subdomain Ω_i, see Ortiz [28]).

2.1 The generalized overlapping domain decomposition method

During the last decades, more sophisticated Schwarz methods were designed, namely the *optimized Schwarz methods or generalized overlapping domain decomposition method.* These methods are based on a classical domain decomposition, but they use more effective transmission conditions than the classical Dirichlet conditions at the interfaces between subdomains. The first more effective transmission conditions were introduced by P.L. Lions (cf. Boulaaras and Haiour [12]). For elliptic problems, we have seen that Schwarz algorithms work only for overlapping domain decompositions and their performance in terms of iterations counts depends on the width of the overlap. The algorithm introduced by P.L. Lions (cf. Boulaaras and Haiour [12]) can be applied to both overlapping and non overlapping subdomains. It is based on improving Schwarz methods by replacing the Dirichlet interface conditions by Robin interface conditions. Let α be a positive number, the modified algorithm reads

$$\begin{cases} -\triangle u_1^m = f \quad \text{in } \Omega_1, \\ \\ \dfrac{\partial u_1^{m+1}}{\partial \eta_1} + \alpha_1 u_1^{m+1} = \dfrac{\partial u_2^m}{\partial \eta_1} + \alpha_1 u_2^m, \ \text{on } \Gamma_1 \\ \\ u_1^m = 0 \quad \text{on } \partial\Omega_1 - \Gamma_1, \end{cases}$$

and

$$
\begin{cases}
-\triangle u_2^m = f \quad \text{in } \Omega_2, \\[2ex]
\dfrac{\partial u_2^{m+1}}{\partial \eta_2} + \alpha_2 u_1^{m+1} = \dfrac{\partial u_1^m}{\partial \eta_2} + \alpha_2 u_1^m, \ \text{on } \Gamma_2 \\[2ex]
u_2^m = 0 \quad \text{on } \partial\Omega_2 - \Gamma_2,
\end{cases}
\tag{2.3}
$$

where η_1 and η_2 are the outward normals on the boundary of the subdomains.

It is also possible to consider other interface conditions than Robin conditions and optimize their choice with respect to the convergence factor.

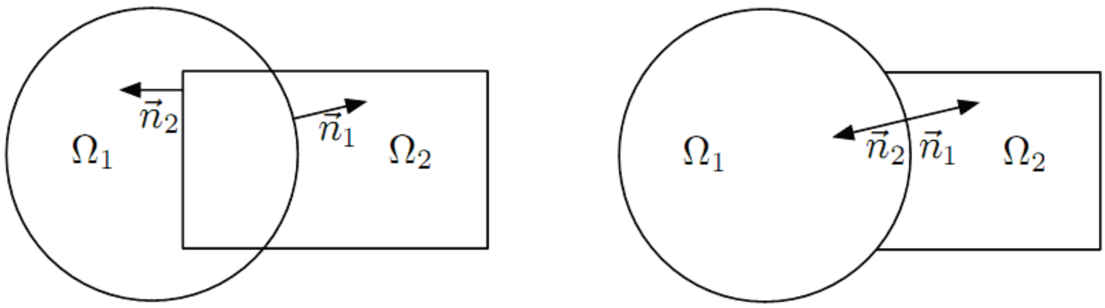


Figure 2.1: Outward normals for overlapping and non overlapping subdomains for P.L. Lions' algorithm.

Figure 2.1:

Chapter 3

Maximum norm analysis of a nonmatching grids method combined with a theta scheme for parabolic equation with linear source terms

This chapter deals with the error analysis in the maximum norm, in the context of the non-matching grids method, of the following evolutionary equation: find $u \in L^2(0, T, H_0^1(\Omega)) \cap C^2(0, T, H^{-1}(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, \quad \text{in } \Sigma, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \, u(., 0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0, T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ. The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative satisfies

$$\alpha < \beta, \quad \beta > 0. \tag{3.2}$$

f is a regular function such that

$$f \in L^2\left(0, T, L^2(\Omega)\right) \cap C^1\left(0, T, H^{-1}(\Omega)\right).$$

Let $(.,.)_\Omega$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall\xi \ > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

3.1 The discrete parabolic equation

The problem (3.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ solution of

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right)_\Omega + a(u, v) = (f, v) + (\varphi, v)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u(x, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{3.3}$$

where $a(.,.)$ is the bilinear form defined as:

$$u, v \in H_0^1(\Omega) : a(u, u) = (\nabla u, \nabla u) + (\alpha u, u) \tag{3.4}$$

3.1.1 The space discretization

Let Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$V_h^{(\varphi)} = \left\{ \begin{array}{l} v \in \left(L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \text{such that } v_h\mid_K = P_1,\ k \in \tau_h, \\ \\ v_h\left(.,0\right) = v_{h0} \text{ in } \Omega,\ \dfrac{\partial v_h}{\partial \eta} = \pi_h \varphi \text{ in } \Gamma_0, \\ \\ v_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{array} \right\} \tag{3.5}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1 and π_h is an interpolation operator on Γ_0. We consider r_h be the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v\left(M_i\right) \varphi_i\left(x\right).$$

The discrete maximum principle assumption (DMP)

We assume the matrices whose coefficients $a\left(\varphi_i, \varphi_j\right)$ are M-matrix (Maday and Magoules [23] and Boulaaras et al. [9]). For convenience in all the sequels, C will be a generic constant independent on h.It can be approximated the problem (3.1) by a weakly coupled system of the following parabolic equation $v \in H^1\left(\Omega\right)$

$$\left(\frac{\partial u}{\partial t}, v\right)_\Omega + a\left(u,v\right) = \left(f,v\right)_\Omega + \left(\varphi, v\right)_{\Gamma_0}. \tag{3.6}$$

We discretize in space, i.e., we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset \left(L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right)$, we get the following semi-discrete system of parabolic equation

$$\left(\frac{\partial u_h}{\partial t}, v_h\right)_\Omega + a\left(u_h,v_h\right) = \left(f,v_h\right)_\Omega + \left(\varphi, v_h\right)_{\Gamma_0}. \tag{3.7}$$

3.1.2 The time discretization

Now we apply the θ-scheme in the semi-discrete approximation (3.7). Thus we have, for any $\theta \in]0,1]$ and $k = 1,...,p$

$$\begin{aligned}&\left(u_h^k - u_h^{k-1}, v_h\right)_\Omega + (\Delta t)\, a\left(u_h^{\theta,k}, v_h\right) = \\ &(\Delta t)\left[\left(f^{\theta,k}, v_h\right)_\Omega + \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0}\right],\end{aligned} \tag{3.8}$$

where

$$\begin{aligned} u_h^{\theta,k} &= \theta u_h^k + (1-\theta)\, u_h^{k-1}, \\ f^{\ \theta,k} &= \theta f^k + (1-\theta)\, f^{k-1} \end{aligned} \tag{3.9}$$

and

$$\varphi^{\ \theta,k} = \theta \varphi^k + (1-\theta)\, \varphi^{k-1}. \tag{3.10}$$

By multiplying and dividing by θ and by adding $\left(\dfrac{u_h^{k-1}}{\theta\Delta t}, v_h\right)$ to both parties of the inequalities (3.1), we get

$$\begin{aligned}&\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right) = \left(f^{\ \theta,k} + \frac{u_h^{\theta,k-1}}{\theta\Delta t}, v_h\right)_\Omega + \\ &+ \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_h \in V_h^{(\varphi)}.\end{aligned} \tag{3.11}$$

Then, the problem (3.11) can be reformulated into the following coercive discrete system of parabolic variational equation

$$b\left(u_h^{\theta,k}, v_h\right) = \left(f^{\theta,k} + \mu u_h^{k-1}, v_h\right)_\Omega + \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0},\ v_h,\ u_h^{\theta,k} \in V_h^{(\varphi)}, \tag{3.12}$$

where

$$\left\{\begin{aligned} &b\left(u_h^{\theta,k}, v_h\right) = \mu\left(u_h^{\theta,k}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right),\ v_h \in V_h^{(\varphi)}, \\ &\mu = \frac{1}{\theta\Delta t} = \frac{p}{\theta T}.. \end{aligned}\right. . \tag{3.13}$$

Theorem 3.1.1 *(see Haiour and Boulaaras [20]). Under suitable regularity of the solution of problem (3.1), there exists a constant C independent of h such that*

$$\left\|\zeta_h^\infty - \zeta\right\| \le Ch^2 \left|\log h\right|^2. \tag{3.14}$$

Lemma 3.1.1 *(see Lui [22]) Let $w \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies $a(w,\phi) + \lambda(w,\phi) \geq 0$ or all nonnegative $\phi \in H^1(\Omega)$ and $w \geq 0$ on Γ, then $w \geq 0$ on $\bar{\Omega}$.*

Notation 3.1.1 *$(F^{\theta,k}, \varphi^{\theta,k})$; $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ be a pair of data and $\zeta^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k})$; $\widetilde{\zeta}^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ the corresponding solutions to (3.12) .*

Proposition 3.1.1 *Under the previous notation, we have*

$$\left\|\zeta_h^{\theta,k} - \zeta^{\theta,k}\right\|_{L\infty(\Omega)} \leq \max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k} - \widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Omega)}\}. \tag{3.15}$$

Proof. First, putting

$$\mu^{\theta,k} = \max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k} - \widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Gamma)}\}, \tag{3.16}$$

then

$$\begin{cases} \widetilde{F}^{\theta,k} \leq F^{\theta,k} + \left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \left(\frac{\lambda}{\beta}\right)\left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \lambda\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}, \left\|\varphi^{\theta,k} - \widetilde{\varphi}^{\theta,k}\right\|_{L\infty(\Gamma)}\} \\ \leq F^{\theta,k} + \lambda\mu^{\theta,k}. \end{cases}$$

So

$$b\left(\widetilde{\zeta}^{\theta,k}, \phi\right) \leq b\left(\zeta^{\theta,k}, \phi\right) + \lambda\left(\mu^{\theta,k}, \phi\right), \text{ for all } \phi \geq 0,\ \phi \in H_0^1(\Omega) \tag{3.17}$$

and thus

$$b\left(\widetilde{\zeta}^{\theta,k}, \phi\right) \leq b\left(\zeta^{\theta,k} + \mu^{\theta,k}, \phi\right) = \left(F^{\theta,k} + \lambda\mu^{\theta,k}, \phi\right).$$

On the other hand,we have

$$\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k} \geq 0 \text{ on } \Gamma_0. \tag{3.18}$$

So

$$b(\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k} \geq 0. \tag{3.19}$$

By using the result of lemma 1, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega} \tag{3.20}$$

Similarly, interchanging the roles of the couples $(F^{\theta,k}, \varphi^{\theta,k})$ and $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega}, \tag{3.21}$$

which completes the proof. ∎

Remark 3.1.1 *Proposition 1 stays true for the discrete case.*

Lemma 3.1.2 *(Lui [22]) Let $w \in V_h$ satisfy $b(w^{\theta,k}, \phi_s) > 0$ for $s = 1, 2...m(h)$and $w^{\theta,k} \geq 0$ on Γ_0.then $w^{\theta,k} \geq 0$ on $(\overline{\Omega})$.*

Notation 3.1.2 *$(F^{\theta,k}, \varphi^{\theta,k})$; $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ be a pair of data and $\zeta_h^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k}); \widetilde{\zeta}_h^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ the corresponding solutions to (3.12) .*

Proposition 3.1.2 *Let DMP hold, we have*

$$\left\| \zeta_h^{\theta,k} - \widetilde{\zeta}_h^{\theta,k} \right\|_{L\infty(\Omega)} \leq \max\{ \left(\frac{1}{\beta}\right) \left\| F^{\theta,k} - \widetilde{F}^{\theta,k} \right\|_{L\infty(\Omega)}, \left\| \varphi^{\theta,k} - \widetilde{\varphi}^{\theta,k} \right\|_{L\infty(\Gamma_0)} \} \tag{3.22}$$

Proof. The proof is similar to that of the continuous case. ∎

3.2 Schwarz alternating methods for parabolic equation

We decompose (Ω) in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$;$i \neq j$ is empty. Let

$$V_i^{(w_j^{\theta,k})} = \begin{cases} v \in \left(L^2\left(0, T, H_0^1\left(\Omega\right)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v = w_j \text{ on } \Gamma_i. \end{cases}$$

We associate with problem (3.12) the following system: find $(u_1^{\theta,k}, u_2^{\theta,k}) \in V_1^{\theta,k} \times V_2^{\theta,k}$ solution to

$$\begin{cases} b_1(u_1^{\theta,k}, v) = (F^{\theta,k}, v)_{\Omega 1} + (\varphi^{\theta,k}, v)_{\Gamma_{01}}, \\ \\ b_2(u_2^{\theta,k}, v) = (F^{\theta,k}, v)_{\Omega 2} + (\varphi^{\theta,k}, v)_{\Gamma_{02}}, \end{cases} \tag{3.23}$$

where

$$b_i(u_i^{\theta,k}, v) = \int_{\Omega_i} (\bigtriangledown u^{\theta,k}.\nabla v^{\theta,k} + \alpha u^{\theta,k}.v^{\theta,k})dx \tag{3.24}$$

and

$$u_i^{\theta,k} = u^{\theta,k}/\Omega_i; i = 1, 2$$

3.2.1 Continuous Schwarz sequences

Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$b(u_0, v) = (F^{\theta,k}, v). \tag{3.25}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences$\left(u_1^{n+1}\right)$on Ω_1 such that
$u_1^{\theta,k,n+1} \in V_1^{\left(u_2^{\theta,k,n}\right)}$ solves of

$$b_1(u_1^{\theta,k,n+1}, v) = (F_1^{\theta,k}, v), \tag{3.26}$$

where

$$F_1^{\theta,k} = f^{\theta,k} + \lambda u_1^{\theta,k-1,n+1}$$

and $(u_2^{\theta,k,n+1})$on Ω_2 such that $u_2^{\theta,k,n+1} \in V_2^{\left(\theta,k,u_1^{\theta,k,n+1}\right)}$ solves

$$b_2(u_2^{\theta,k,n+1}, v) = (F_1^{\theta,k}, v), \tag{3.27}$$

where

$$F_1^{\theta,k} = f^{\theta,k} + \lambda u_2^{\theta,k-1,n+1}$$

Theorem 3.2.1 *Haiour and Boulaaras [20] The sequences $(u_h^{n+1}); (u_h^{n+1})$, $n \geq 0$ produced by the Schwarz alternating method converge geometrically to a solution u of the elliptic obstacle problem. More precisely, there exist $k_1, k_2 \in (0, 1)$ which depend on (Ω_1, γ_2) and $(\Omega_2, \gamma 1)$ such that for all $n \geq 0$,*

$$\sup_{\overline{\Omega}_1} \left|u_h - u^{2n+1}\right| \leq \delta_1^n \delta_2^n \sup_{\gamma_1} \left|u_h - u_h^0\right| \tag{3.28}$$

and

$$\sup_{\overline{\Omega}_2} \left|u_h - u^{2n}\right| \leq \delta_1^n \delta_2^{n-1} \sup_{\gamma_2} \left|u_h - u_h^0\right|. \tag{3.29}$$

3.2.2 The discrete Schwarz sequences

As we have defined before, for $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C(\Omega_i)$, we set

$$V_{hi}^{(w_j^{\theta,k})} = \begin{cases} v \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v = \phi \text{ on } \Gamma_{01} \cap \Gamma_{02};\ v = \pi_{h_i}(w) \text{ on } \Gamma_{0i}, \end{cases}$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.
Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.26) and (3.27) .
Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.25), we respectively, define by $u_{1h}^{\theta,k,n+1} \in V_{h1}^{\left(u_{2h}^{\theta,k,n}\right)}$ such that

$$b_1(u_{1h}^{\theta,k,n+1}, v) = (F^{\theta,k}(u_{1h}^{\theta,k,n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0 \tag{3.30}$$

and $u_{2h}^{\theta,k,n+1} \in V_{h2}^{\left(u_{1h}^{\theta,k,n+1}\right)}$ such that

$$b_2(u_{2h}^{\theta,k,n+1}, v) = (F^{\theta,k}(u_{2h}^{\theta,k,n+1}), v), \forall v \in V_h^{(\varphi)};\ n \geq 0. \tag{3.31}$$

3.3 Maximum norm analysis of asymptotic behavior

3.3.1 Error Analysis for the stationary case

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

Two auxiliary Schwarz sequences

For $w_{2h}^0 = u_{2h}^0$, we define the sequences $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ such that $u_{1h}^{\theta,\infty,n+1} \in V_{h1}^{\left(u_2^{\theta,\infty,n}\right)}$ solves

$$b_1(w_{1h}^{\theta,\infty,n+1}, v) = (F^{\theta,k}(u_{1h}^{\theta,\infty,n+1}), v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{3.32}$$

and $w_{2h}^{\theta,\infty,n+1} \in V_{2h}^{\left(u_{1h}^{\theta,\infty,n+1}\right)}$ solves

$$b_2(w_{2h}^{\theta,\infty,n+1}, v) = (F^{\theta,k}(u_{2h}^{\theta,\infty,n+1}), v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{3.33}$$

respectively. It is then clear that $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ are the finite element approximation of $u_1^{\theta,\infty,n+1}$ and $u_2^{\theta,\infty,n+1}$ defined in (3.32), (3.33), respectively. Then, as $F^{\theta,k}(.)$ is continuous,$\left\| F^{\theta,k}\left(u_i^{\theta,k,n+1}\right)\right\|_\infty \leq \lambda \left\| u_i^{\theta,k,n+1}\right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear parabolic problems, we have

$$\left\| u_i^{\theta,k,n} - u_{ih}^{\theta,k,n}\right\|_{L\infty(\Omega_i)} \leq Ch^2 \left|\log h\right| \tag{3.34}$$

where C is a constant independent of both h and n.

Notation 3.3.1 *From now on, we shall adopt the following notations:* $|.|_1 = |.|_{L\infty(\Gamma_1)}$, $|.|_2 = |.|_{L\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L\infty(\Gamma_2)}$, *and we set* $\pi_{h_1} = \pi_{h_2} = \pi_h$.

3.3.2 Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{\theta,k,n+1} = T_h u_{ih}^{k-1,n+1}, k = 1, ..., p,\ u_{ih}^{\theta,k,n+1} \in V_{hi}^{\left(u_2^{\theta,k,n}\right)} \tag{3.35}$$

where $u_h^{\theta,k}$ is the solution of the problem (3.35) and the first iteration u_h^0 is solution of (3.25).

Proposition 3.3.1 *Boulaaras et al. [9]Under the previous hypotheses and notations, we have the following estimate of convergence if* $\theta \geq \dfrac{1}{2}$

$$\left\| u_h^{\theta,k,n+1} - u_h^\infty \right\|_\infty \leq \left(\frac{1}{1+\theta\Delta t}\right)^k \left\| u_h^\infty - u_{h_0}\right\|_\infty, \tag{3.36}$$

if $0 \leq \theta < \dfrac{1}{2}$, *we have*

$$\left\| u_h^{\theta,k,2n+1} - u_h^\infty \right\|_\infty \leq \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^k \left\| u_h^\infty - u_{h_0}\right\|_\infty, \tag{3.37}$$

where $\rho\left(A\right)$ *is the spectral radius of the elliptic operator.*

Lemma 3.3.1 *Let $\rho = \dfrac{\alpha}{\beta}$. Then, under assumption (3.2), there exists a constant C independent of both h and n such that*

$$\left\| u_i^{\theta,\infty,n+1} - u_{ih}^{\theta,\infty,n+1} \right\|_i \leq \frac{ch^2 \left|\log h\right|}{1-\rho}, \quad i = 1, 2. \tag{3.38}$$

Proof. We know from standard error estimate on uniform norm for linear problem Nitsche [27] that there exists a constant C independent of h such that

$$\left\| u^0 - u_h^0 \right\|_{L\infty(\Omega)} \leq ch^2 \left|\log h\right|. \tag{3.39}$$

Since $\dfrac{1}{2} < \rho < 1$, then $1 < \rho/(1-\rho)$ and

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq ch^2 \left|\log h\right| \leq \frac{\rho ch^2 \left|\log h\right|}{1-\rho}. \tag{3.40}$$

Let us now prove (3.41) by induction. Indeed for $n = 1$, using the result of Propsition1, we have in Ω_1

$$\begin{aligned}
\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 &\leq \left\| u_1^{\theta,k,1} - w_{1h}^{\theta,k,1} \right\|_1 + \left\| w_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \\
&\leq ch^2 \left|\log h\right| + \left\| w_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \\
&\leq ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\| F^{\theta,k}\left(u_1^{\theta,k,1}\right) - F^{\theta,k}\left(u_{1h}^{\theta,k,1}\right) \right\|_1, \left| u_2^0 - u_{2h}^0 \right|_1 \\
&\leq ch^2 \left|\log h\right| + \max\{\left(\frac{1}{\beta}\right) \left\| F^{\theta,k}\left(u_1^{\theta,k,1}\right) - F^{\theta,k}\left(u_{1h}^{\theta,k,1}\right) \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2 \\
&\leq ch^2 \left|\log h\right| + \max\{\rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2.
\end{aligned}$$

We then have to distinguish between two cases

$$\max\{\rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2\} = \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \tag{3.41}$$

or

$$\max\{\rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2\} = \left\| u_2^0 - u_{2h}^0 \right\|_2. \tag{3.42}$$

(3.41) implies

$$\begin{cases} \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k1} \right\|_1 \leq ch^2 |\log h| + \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1, \\ \\ \|u_2^0 - u_{2h}^0\|_2 \leq \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1, \end{cases}$$

then

$$\begin{cases} \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \dfrac{ch^2 |\log h|}{1-\rho}. \\ \\ \|u_2^0 - u_{2h}^0\|_2 \leq \rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \frac{\rho Ch^2 |\log h|}{1-\rho}. \end{cases}$$

(29) implies

$$\begin{cases} \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq ch^2 |\log h| + \|u_2^0 - u_{2h}^0\|_2 \\ \\ \qquad\qquad \leq \|u_2^0 - u_{2h}^0\|_2, \end{cases}$$

so, by multiplying (3.42) by ρ we get

$$\rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \leq \rho ch^2 |\log h| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2. \tag{3.43}$$

So,$\rho \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1$ is bounded by both $\rho ch^2 |\log h| + \rho \|u_2^0 - u_{2h}^0\|_2$and $\|u_2^0 - u_{2h}^0\|_2$, this implies that

$$\rho \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \rho ch^2 |\log h| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2, \tag{3.44}$$

or

$$\rho ch^2 |\log h| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \left\| u_2^0 - u_{2h}^0 \right\|_2, \tag{3.45}$$

that is (3.44) implies

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \frac{\rho ch^2 |\log h|}{1-\rho} \tag{3.46}$$

and (3.45) implies

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \geq \frac{\rho ch^2 |\log h|}{1-\rho}. \tag{3.47}$$

It follows that only the case (3.44) is true, that is,

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \frac{\rho ch^2 |\log h|}{1-\rho}, \tag{3.48}$$

then

$$\begin{aligned}\rho\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1 &\leq ch^2\left|\log h\right|+\left\|u_2^0-u_{2h}^0\right\|_2\\ &\leq ch^2\left|\log h\right|+\frac{\rho Ch^2\left|\log h\right|}{1-\rho}\\ &\leq \frac{ch^2\left|\log h\right|}{1-\rho}.\end{aligned}$$

So, in both cases (3.41) and (3.42), we have

$$\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\leq\frac{ch^2\left|\log h\right|}{1-\rho}. \tag{3.49}$$

Similarly, we have in Ω_2

$$\begin{aligned}\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2 &\leq ch^2\left|\log h\right|+\left\|w_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2\\ &\leq ch^2\left|\log h\right|+\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}\left(u_2^{\theta,k,1}\right)-F^{\theta,k}\left(u_{2h}^{\theta,k,1}\right)\right\|_2,\left|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right|_2\}\\ &\leq ch^2\left|\log h\right|+\max\{\left(\frac{1}{\beta}\right)\left\|F^{\theta,k}\left(u_2^{\theta,k,1}\right)-F^{\theta,k}\left(u_{2h}^{\theta,k,1}\right)\right\|_2,\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\}\\ &\leq ch^2\left|\log h\right|+\max\{\rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2,\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\}.\end{aligned}$$

So

$$\max\{\rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2,\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\}=\rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2 \tag{3.50}$$

or

$$\max\{\rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2,\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1\}=\left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1. \tag{3.51}$$

cases (3.50) implies

$$\begin{aligned}\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2 &\leq ch^2\left|\log h\right|+\rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2,\\ \left\|u_1^{\theta,k,1}-u_{1h}^{\theta,k,1}\right\|_1 &\leq \rho\left\|u_2^{\theta,k,1}-u_{2h}^{\theta,k,1}\right\|_2\end{aligned}$$

so

$$\begin{cases} \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \leq \dfrac{ch^2 \left|\log h\right|}{1-\rho}, \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \\ \leq \rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \\ \leq \dfrac{\rho ch^2 \left|\log h\right|}{1-\rho} \leq \dfrac{ch^2 \left|\log h\right|}{1-\rho}, \end{cases}$$

while case (3.51) implies

$$\begin{cases} \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \leq ch^2 \left|\log h\right| + \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \\ \rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \leq \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 . \end{cases}, \tag{3.52}$$

So, by multiplying (3.52) by ρ we get

$$\rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2 \leq \rho ch^2 \left|\log h\right| + \rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 . \tag{3.53}$$

Hence $\rho \left\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\right\|_2$ is bounded by both $\rho ch^2|logh|+\rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1$ and $\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k1}\right\|_1$, then

$$\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \leq \rho ch^2 \left|\log h\right| + \rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \tag{3.54}$$

or

$$ch^2 \left|\log h\right| + \rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \leq \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 , \tag{3.55}$$

which (3.54) implies

$$\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 \leq \frac{\rho ch^2 \left|\log h\right|}{1-\rho} < \frac{ch^2 \left|\log h\right|}{1-\rho} \tag{3.56}$$

or (3.55) implies

$$\frac{\rho ch^2 \left|\log h\right|}{1-\rho} \leq \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 < \frac{ch^2 \left|\log h\right|}{1-\rho}. \tag{3.57}$$

Hence, (3.54) and (3.55) are true because they both coincide with (3.49). So, there is either a contradiction and thus case (3.50) is impossible or case (3.51) is possible only if

$$\left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 = \rho ch^2 \left|\log h\right| + \rho \left\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\right\|_1 , \tag{3.58}$$

that is

$$\left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 = \frac{\rho c h^2 \left| \log h \right|}{1-\rho}, \tag{3.59}$$

thus

$$\begin{aligned} \left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 &\leq c h^2 \left| \log h \right| + \left\| u_1^{\theta,k,1} - u_{1h}^{\theta,k,1} \right\|_1 \\ &\leq c h^2 \left| \log h \right| + \frac{\rho c h^2 \left| \log h \right|}{1-\rho} \\ &\leq \frac{c h^2 \left| \log h \right|}{1-\rho}, \end{aligned}$$

that is, both cases (3.50) and (3.51) imply

$$\left\| u_2^{\theta,k,1} - u_{2h}^{\theta,k,1} \right\|_2 \leq \frac{c h^2 \left| \log h \right|}{1-\rho}. \tag{3.60}$$

Now, let us assume that

$$\left\| u_2^{\theta,k,n} - u_{2h}^{\theta,k,n} \right\|_2 \leq \frac{c h^2 \left| \log h \right|}{1-\rho} \tag{3.61}$$

and prove that

$$\begin{cases} \left\| u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1} \right\|_1 \leq \dfrac{c h^2 \left| \log h \right|}{1-\rho} \\[2ex] \left\| u_2^{\theta,k,n+1} - u_{2h}^{\theta,k,n+1} \right\|_2 \leq \dfrac{c h^2 \left| \log h \right|}{1-\rho} \end{cases}$$

■

Theorem 3.3.1 *Let $h = max\,(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\| u_i^{\theta,k,n+1} - u_{ih}^{\theta,k,n+1} \right\|_1 \leq \frac{c h^2 \left| \log h \right|}{1-\rho}, \quad \forall i = 1, 2. \tag{3.62}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\delta = \delta_1 \delta_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned} \left\| u_1^{\theta,k} - u_{1h}^{\theta,k,n+1} \right\|_1 &\leq \left\| u_1^{\theta,k} - u_1^{\theta,k,n+1} \right\|_1 + \left\| u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1} \right\|_1 \\ &\leq \delta_1^n \delta_2^n \left| u^0 - u \right|_1 + \frac{c h^2 \left| \log h \right|}{1-\rho} \\ &\leq \delta^{2n} \left| u^0 - u \right|_1 + \frac{c h^2 \left| \log h \right|}{1-\rho}. \end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{3.63}$$

and thus

$$\begin{aligned}\left\| u_1^{\theta,k} - u_{1h}^{\theta,k,n+1} \right\|_1 &\leq ch^2 + ch^2 \left|\log h\right| \\ &\leq ch^2 \left|\log h\right|,\end{aligned}$$

which is the desired result. ■

3.3.3 Asymptotic behavior

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h(T)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution of (3.13)

Theorem 3.3.2 *According to the results of the proposition 3 and the theorem 3, we have for the first case* $\theta \geq \dfrac{1}{2}$

$$\left\| u_{1h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{1}{1+\beta\theta\Delta t} \right)^p \right], \tag{3.64}$$

and

$$\left\| u_{2h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{1}{1+\beta\theta\Delta t} \right)^p \right], \tag{3.65}$$

and for the second case $0 \leq \theta < \dfrac{1}{2}$

$$\left\| u_{1h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right] \tag{3.66}$$

and

$$\left\| u_{2h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right| + \left(\frac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right], \tag{3.67}$$

where C is a constant independent of h and k.

Proof. We have

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq \left\| u_h^{\theta,p,2n+1} - u_h^\infty \right\|_\infty + \left\| u_h^\infty - u^\infty \right\|_\infty .$$

Using the proposition 4.3.1 and the theorem 4.3.1, we have for $\theta \geq \dfrac{1}{2}$

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right|^3 + \left(\frac{1}{1+\beta\theta\Delta t} \right)^p \right],$$

and for $0 \leq \theta < \dfrac{1}{2}$ we have

$$\left\| u_h^{\theta,p,2n+1} - u^\infty \right\|_\infty \leq C \left[h^2 \left|\log h\right|^3 + \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)} \right)^p \right]$$

The proof for (3.66) and (3.67) case is similar. ■

Remark 3.3.1 *It can be seen in the previous estimates (3.64) up to (3.67),* $\left(\dfrac{1}{1+\beta\theta\Delta t} \right)^p$, $\left(\dfrac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)} \right)^p$, *goes to* 0 *when* p *tend to infinity.*
Therefore, the estimation order for both the coercive and noncoercive problems is

$$\left\| u^\infty - u_{1h}^{\infty,n+1} \right\|_{L^\infty\left(\bar{\Omega}_1\right)} \leq C h^2 \left|\log h\right|^3$$

and

$$\left\| u^\infty - u_{2h}^{\infty,n+1} \right\|_{L^\infty\left(\bar{\Omega}_2\right)} \leq C h^2 \left|\log h\right|^3 .$$

Chapter 4

Maximum norm analysis of a nonmatching grids method for a class of parabolic equation

This chapter deals with the error analysis in the maximum norm, in the context of the non-matching grids method, of the following evolutionary equation:
find $u \in L^2(0,T;H_0^1(\Omega)) \cap C^2(0,T,H^{-1}(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f(u), \quad \text{in } \Sigma, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0,\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.
The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative satisfies

$$\alpha < \beta, \quad \beta > 0. \tag{4.2}$$

$f(.)$ is a nonlinear and Lipschitz functions with Lipschitz constant c and satisfying the following condition

$$\begin{cases} f \in L^2\left(0,T,L^2\left(\Omega\right)\right) \cap C^1\left(0,T,H^{-1}\left(\Omega\right)\right) \\ \\ c < \beta. \end{cases} . \tag{4.3}$$

Let $(.,.)_\Omega$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined in Perthame [30] as impulse control problem:

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall\xi \ > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

4.1 The discrete parabolic equation

The problem (4.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0,T,H_0^1\left(\Omega\right)\right)$ solution of

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right)_\Omega + a\left(u,v\right) = \left(f\left(u\right),v\right)_\Omega + \left(\varphi,v\right)_{\Gamma_0}, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u\left(x,0\right) = u_0 \text{ in } \Omega, \end{cases} \tag{4.4}$$

where $a(.,.)$ is the bilinear form defined as:

$$a\left(u,v\right) = \left(\nabla u, \nabla v\right)_\Omega - \left(\alpha u, v\right)_\Omega \tag{4.5}$$

4.1.1 The spatial discretization

We discretize the problem (4.4) with respect to time by using Euler scheme. Therefore, we search a sequence of elements $u^k \in H_0^1(\Omega)$ which approaches $u(t_k)$, $t_k = k\Delta t$, with initial data $u^0 = u_0$.

Thus, we have for $k = 1, ..., n$

$$\begin{cases} \left(\dfrac{u^k - u^{k-1}}{\Delta t}, v\right) + a\left(u^k, v\right) = \left(f\left(u^k\right), v\right) + (\varphi, v)_{\Gamma_0}, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u\,(x, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{4.6}$$

4.1.2 The spatial discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_l, $l = \{1, ..., m\,(h)\}$ defined by $\varphi_l\,(M_s) = \delta_{ls}$ where M_s is a vertex of the considered triangulation. We introduce the following discrete spaces V^h of finite element

$$V^h = \left\{ \begin{array}{l} v \in L^2\left(0, T, H_0^1\,(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right), \text{ such that} \\ \\ v\mid_K \in P_1,\ K \in \tau_h, \text{ and } u\,(.,0) = u_0 \text{ in } \Omega, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0,\ u\,(x,0) = u_0 \text{ in } \Omega. \end{array} \right\} \tag{4.7}$$

where r_h is the usual interpolation operator defined by

$$v \in L^2\left(0, T, H_0^1\,(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right),\ r_h v = \sum_{i=1}^{m(h)} v\,(M_i)\,\varphi_i\,(x) \tag{4.8}$$

and P_1 denotes the space of polynomials with degree at most 1. In the sequel of the paper, we shall make use of the discrete maximum principle assumption (dmp). In other words, we shall assume that the matrices $(A)_{ps} = a\,(\varphi_p, \varphi_s)$ is M-matrices (Toselli and Widlund [34]). We discretize in space the problem (4.6), i.e. that we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset H_0^1$, we get the following discrete PQVIs.

$$\begin{cases} \left(\dfrac{u_h^k - u_h^{k-1}}{\Delta t}, v_h\right) + a\left(u_h^k, v_h\right) \geq \left(f\left(u_h^k\right), v_h\right) + (\varphi, v)_{\Gamma_0}, \\ \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u_h^0(x) = u_{h0} \text{ in } \Omega, \end{cases} \tag{4.9}$$

which implies

$$\begin{cases} \left(\dfrac{u_h^k}{\Delta t}, v_h\right) + a\left(u_h^k, v_h\right) \geq \left(f\left(u_h^k\right) + \dfrac{u_h^{k-1}}{\Delta t}, v_h\right) + (\varphi, v)_{\Gamma_0}, \\ \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u_h^0(x) = u_{h0} \text{ in } \Omega. \end{cases} \tag{4.10}$$

Then, the problem (4.10) can be reformulated into the following coercive discrete system of elliptic quasi-variational equations (EQVIs)

$$\begin{cases} b\left(u_h^k, v_h\right) = \left(f\left(u_h^k\right) + \lambda u_h^{k-1}, v_h\right) + (\varphi, v)_{\Gamma_0}, \ u_h^k \in V^h, \\ \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u_h^0(x) = u_{h0} \text{ in } \Omega, \end{cases} \tag{4.11}$$

such that

$$\begin{cases} b\left(u_h^k, v_h\right) = \lambda\left(u_h^k, v_h\right) + a\left(u_h^k, v_h\right),\ u_h^k \in V^h, \\ \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n},\ k = 1, ..., n. \end{cases} \tag{4.12}$$

4.1.3 An iterative discrete algorithm

As we have chosen before in the iterative semi-discrete algorithm $u_h^0 = u_{h0}$ the solution of the following full-discrete equation

$$b\left(u_h^0, v_h\right) = \left(g^0, v_h\right),\ v_h \in V^h, \tag{4.13}$$

where g^0 is a linear and a regular function. Now we give the full following discrete algorithm

$$u_h^k = T_h u^{k-1}, k = 1, .., n, \tag{4.14}$$

where u_h^k is the solution of the problem (4.11). Let $F^{k-1}(w) = f\left(u_h^k\right) + \lambda w$, $\tilde{F}^{k-1}(\tilde{w}) = f\left(\tilde{u}\ _h^k\right) + \lambda\tilde{w} \in L^\infty(\Omega)$ be the corresponding right-hand sides to the EQVIs.

Lemma 4.1.1 *[(Boulaaras and Haiour [12]) (Boulaaras and Haiour [10]) Under the previous assumption and the dmp we have, if*

$$F^{k-1}(w) \geqq F^{k-1}(\tilde{w}),$$

then

$$u_h^k = \partial\left(F^{k-1}(w)\right) \geqq \tilde{u}_h^k = \partial\left(F^{k-1}(\tilde{w})\right).$$

We shall first recall some results related to coercive quasi variational inequalities that are necessarily in proving some useful qualitative properties.

Proof. The proof of the Lemma is very similar to that in (Douglas and Huang [16] and Lions [21]) for free boundary problem. ■

Definition 4.1.1 *ζ_h^k is said to be a subsolution for the system of EQVIs (25) if*

$$\begin{cases} b\left(\zeta_h^k, \varphi_s\right) \leq \left(f\left(\zeta_h^k\right) + \lambda \zeta_h^{k-1}, \varphi_s\right) + \left(\varphi, \varphi_s\right)_{\Gamma_0}, \ \forall \varphi_s, \quad s = 1, ..., m\left(h\right), \\ u_h = 0 \ in \ \Gamma/\Gamma_0, \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \ in \ \Gamma_0, \\ u_h^0\left(x\right) = u_{h0} \ in \ \Omega, \end{cases}$$

Notation 4.1.1 *Let X_h be the set of discrete subsolutions. Then, we have the following theorem.*

Theorem 4.1.1 *Under the discrete maximum principle, the solution of the system of EQVI (25) is the maximum element of X_h.*

Proof. We denote by $\varphi^+ = \max(\varphi, 0)$, $\varphi^- = \max(-\varphi, 0)$.
Let $w_h \in V_h$ be a solution of the following of the full discrete system of parabolic quai variational inequalities using Euler time scheme combined with a finite element spatial approximation (Boulaaras et al. [9])

$$\begin{cases} b\left(w_h, \breve{v}_h\right) = \left(f\left(w_h\right) + \lambda w_h, \tilde{v}_h\right) + \left(\varphi, \tilde{v}_h\right)_{\Gamma_0}, \ \forall \tilde{v}_h \in V_h, \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u_h^0\left(x\right) = u_{h0} \text{ in } \Omega, \end{cases} \tag{4.15}$$

where $\breve{v}_h = \sum_{s=1}^{m(h)} \tilde{v}_s \varphi_s$. Since $\tilde{v}$ is a trial function, we choose $\tilde{v}_h = w_h - v_h$ and $v_h > 0$. Thus

$$b\left(w_h, \varphi_s\right) \leq \left(f\left(z_h\right) + \lambda w_h, \varphi_s\right), \tag{4.16}$$

that is to say $w_h \in X_h$.On the other hand; let z_h be a subsolution, such that

$$w_h \leq z_h. \tag{4.17}$$

Then we have

$$b\left(z_h, \varphi_s\right) \leq \left(f\left(w_h\right) + \lambda w_h, \varphi_s\right).$$

Setting $v_h = \left(z_h - w_h\right)^+ \geq 0$ as a trial function. Yields

$$b\left(z_h, \left(z_h - w_h\right)^+\right) \leq \left(f\left(z_h\right) + \lambda w_h, \left(z_h - w_h\right)^+\right)$$

and since w_h is a subsolution too, we have

$$b\left(w_h, \left(z_h - w_h\right)^+\right) \leq \left(f\left(z_h\right) + \lambda w_h, \left(z_h - w_h\right)^+\right).$$

Thus, we deduce

$$-b\left(\left(z_h - w_h\right)^+, \left(z_h - w_h\right)^+\right) \geq 0.$$

Under the coerciveness of the bilinear, we can get

$$\left(z_h - w_h\right)^+ = 0.$$

Therefore

$$z_h \leq w_h. \tag{4.18}$$

Thus, from (4.17) and (4.18) we obtain $z_h = w_h$. ■

Theorem 4.1.2 *see Haiour and Boulaaras [20] . Under suitable regularity of the solution of problem (4.1), there exists a constant C independent of h such that*

$$\left\|\zeta_h^\infty - \zeta\right\| \leq Ch^2 \left|\log h\right|. \tag{4.19}$$

Lemma 4.1.2 *(see Lui [22]) Let $w \in H^1(\Omega) \cap C\left(\bar{\Omega}\right)$ satisfies $a(w, \phi) + \lambda(w, \phi) \geq 0$ or all nonnegative $\phi \in H^1(\Omega)$ and $w \geq 0$ on Γ, then $w \geq 0$ on $\bar{\Omega}$.*

Notation 4.1.2 $(F^{k-1}, \varphi); (\widetilde{F}^{k-1}, \widetilde{\varphi})$ *be a pair of data and $\zeta = \partial(F^{k-1}, \varphi); \widetilde{\zeta} = \partial(\widetilde{F}^{k-1}, \widetilde{\varphi})$ the corresponding solutions to (4.6) .*

Proposition 4.1.1 *Under the previous notation, we have*

$$\left\|\zeta_h - \zeta\right\|_{L\infty(\Omega)} \leq \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \left\|\varphi - \widetilde{\varphi}\right\|_{L\infty(\Gamma)}\}. \tag{4.20}$$

Proof. First, putting

$$\mu^k = \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \left\|\varphi - \widetilde{\varphi}\right\|_{L\infty(\Gamma)}\}, \tag{4.21}$$

then

$$\widetilde{F}^k \leq F^k + \left\|F^k - \widetilde{F}^k\right\|_{L\infty(\Omega)}$$

$$\leq F^k + \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \left\|\varphi - \widetilde{\varphi}\right\|_{L\infty(\Gamma)}\}$$

$$\leq F^k + \lambda\mu^k.$$

So

$$b\left(\widetilde{\zeta}^k, \phi\right) \leq b\left(\zeta^k, \phi\right) + \lambda\left(\mu^k, \phi\right), \text{ for all } \phi \geq 0,\, \phi \in H_0^1(\Omega) \tag{4.22}$$

and thus

$$b\left(\widetilde{\zeta}^k, \phi\right) \leq b\left(\zeta^k + \mu^k, \phi\right) = \left(F^k + \lambda\mu^k, \phi\right).$$

On the other hand,we have

$$\zeta^k + \phi - \widetilde{\zeta}^k \geq 0 \text{ on } \Gamma_0. \tag{4.23}$$

So

$$b(\zeta^k + \phi - \widetilde{\zeta}^k \geq 0. \tag{4.24}$$

By using the result of lemma 1, we get

$$\widetilde{\zeta}^k + \phi - \zeta^k \geq 0 \text{ on } \overline{\Omega} \tag{4.25}$$

Similarly, interchanging the roles of the couples (F^k, φ) and $(\widetilde{F}^k, \widetilde{\varphi}^k)$, we get

$$\widetilde{\zeta}^k + \phi - \zeta^k \geq 0 \text{ on } \overline{\Omega}, \tag{4.26}$$

which completes the proof. ■

Remark 4.1.1 *Proposition 1 stays true for the discrete case.*

Lemma 4.1.3 *(Lui [22]) Let $w \in V_h$ satisfy $b(w^k, \phi_s) > 0$ for $s = 1, 2..., m(h)$and $w^{\theta,k} \geq 0$ on Γ_0.then $w^{\theta,k} \geq 0$ on $(\overline{\Omega})$.*

Notation 4.1.3 *(F^k, φ); $(\widetilde{F}^k, \widetilde{\varphi}^k)$ be a pair of data and $\zeta_h^k = \partial(F^k, \varphi); \widetilde{\zeta}_h^k = \partial(\widetilde{F}^k, \widetilde{\varphi})$ the corresponding solutions to (4.6) .*

Proposition 4.1.2 *Let DMP hold, we have*

$$\left\|\zeta_h^k - \widetilde{\zeta}_h^k\right\|_{L\infty(\Omega)} \leq \max\{c\left\|u_h^k - \widetilde{u}_h^k\right\|_{L\infty(\Omega)} + \lambda\left\|u_h^{k-1} - \widetilde{u}_h^{k-1}\right\|_{L\infty(\Omega)}, \left\|\varphi - \widetilde{\varphi}\right\|_{L\infty(\Gamma)}\} \tag{4.27}$$

Proof. The proof is similar to that of the continuous case. ■

4.2 Schwarz alternating methods for parabolic equation

We decompose (Ω) in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$;$i \neq j$ is empty. Let

$$V_i^{(w_j^k)} = \begin{cases} v \in \left(L^2\left(0, T, H_0^1\left(\Omega\right)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v = w_j \text{ on } \Gamma_i. \end{cases}$$

We associate with problem (4.11) the following system: find $(u_1^k, u_2^k) \in V_1^k \times V_2^k$ solution to

$$\begin{cases} b_1(u_1^k, v) = (F^k, v)_{\Omega 1} + (\varphi^k, v)_{\Gamma_{01}}, \\ \\ b_2(u_2^k, v) = (F^k, v)_{\Omega 2} + (\varphi^k, v)_{\Gamma_{02}}, \end{cases} \tag{4.28}$$

where

$$b_i(u_i^k, v) = \int\limits_{\Omega_i} (\bigtriangledown u^k.\nabla v^k + \alpha u^k.v^k)dx \tag{4.29}$$

and

$$u_i^k = u^k/\Omega_i; i = 1, 2$$

4.2.1 Continuous Schwarz sequences

Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$b(u_0, v) = (F^k, v). \tag{4.30}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences$\left(u_1^{n+1}\right)$on Ω_1 such that $u_1^{k,n+1} \in V_1^{\left(u_2^{k,n}\right)}$ solves of

$$b_1(u_1^{k,n+1}, v) = (F_1^k, v), \tag{4.31}$$

where

$$F_1^k = f^k\left(u_1^{k,n+1}\right) + \lambda u_1^{k-1,n+1}$$

and $(u_2^{k,n+1})$on Ω_2 such that $u_2^{k,n+1} \in V_2^{\left(k,u_1^{k,n+1}\right)}$ solves

$$b_2(u_2^{k,n+1}, v) = (F_1^k, v), \tag{4.32}$$

where

$$F_2^k = f^k\left(u_2^{k,n+1}\right) + \lambda u_2^{k-1,n+1}$$

Theorem 4.2.1 *Boulaaras and Haiour [11] The sequences* $(u_h^{n+1}); (u_h^{n+1})$, $n \geq 0$ *produced by the Schwarz alternating method converge geometrically to a solution u of the elliptic obstacle problem. More precisely, there exist* $k_1, k_2 \in (0, 1)$ *which depend on* (Ω_1, γ_2) *and* $(\Omega_2, \gamma 1)$ *such that for all* $n \geq 0$,

$$\sup_{\overline{\Omega}_1} \left|u_h - u^{2n+1}\right| \leq \delta_1^n \delta_2^n \sup_{\gamma_1} \left|u_h - u_h^0\right| \tag{4.33}$$

and

$$\sup_{\overline{\Omega}_2} \left|u_h - u^{2n}\right| \leq \delta_1^n \delta_2^{n-1} \sup_{\gamma_2} \left|u_h - u_h^0\right|. \tag{4.34}$$

4.2.2 The discrete Schwarz sequences

As we have defined before, for $i = 1, 2$, let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C\left(\Omega_i\right)$, we set

$$V_{hi}^{(w_j^k)} = \left\{ \begin{array}{l} v \in \left(L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v = \phi \text{ on } \Gamma_{01} \cap \Gamma_{02};\ v = \pi_{h_i}\left(w\right) \text{ on } \Gamma_{0i}, \end{array} \right\}$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (4.31) and (4.32) . Indeed, let u_{0h} be the discrete analog of u_0, defined in (4.30), we respectively, define by $u_{1h}^{k,n+1} \in V_{h1}^{\left(u_{2h}^{k,n}\right)}$ such that

$$b_1(u_{1h}^{k,n+1}, v) = (F_1^k, v), \forall v \in V_h^{(\varphi)};\ n \geq 0 \tag{4.35}$$

and $u_{2h}^{k,n+1} \in V_{h2}^{\left(u_{1h}^{k,n+1}\right)}$ such that

$$b_2(u_{2h}^{k,n+1}, v) = (F_2^k, v), \forall v \in V_h^{(\varphi)};\ n \geq 0. \tag{4.36}$$

4.3 Maximum norm analysis of asymptotic behavior

4.3.1 Error analysis for the stationary case

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

Two auxiliary Schwarz sequences

For $w_{2h}^0 = u_{2h}^0$, we define the sequences $w_{1h}^{\infty,n+1}$ and $w_{2h}^{\infty,n+1}$ such that $u_{1h}^{\infty,n+1} \in V_{h1}^{\left(u_2^{\infty,n}\right)}$ solves

$$b_1(w_{1h}^{\infty,n+1}, v) = (F_1^k, v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{4.37}$$

and $w_{2h}^{\infty,n+1} \in V_{2h}^{\left(u_{1h}^{\infty,n+1}\right)}$ solves

$$b_2(w_{2h}^{\infty,n+1}, v) = (F_2^k, v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{4.38}$$

respectively. It is then clear that $w_{1h}^{\infty,n+1}$ and $w_{2h}^{\infty,n+1}$ are the finite element approximation of $u_1^{\infty,n+1}$ and $u_2^{\infty,n+1}$ defined in (4.37), (4.38), respectively. Then, as $F^k\left(.\right)$ is continuous, $\left\| F^k\left(u_i^{k,n+1}\right)\right\|_\infty \leq \lambda \left\| u_i^{k,n+1}\right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear parabolic problems, we have

$$\left\| u_i^{k,n} - u_{ih}^{k,n} \right\|_{L\infty(\Omega_i)} \leq Ch^2 \left| \log h \right| \tag{4.39}$$

where C is a constant independent of both h and n.

Notation 4.3.1 *From now on, we shall adopt the following notations:* $|.|_1 = |.|_{L\infty(\Gamma_1)}$, $|.|_2 = |.|_{L\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L\infty(\Gamma_2)}$, *and we set* $\pi_{h_1} = \pi_{h_2} = \pi_h$.

4.3.2 Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{k,n+1} = T_h u_{ih}^{k-1,n+1}, k = 1, ..., p, \ u_{ih}^{k,n+1} \in V_{hi}^{\left(u_2^{k,n}\right)} \tag{4.40}$$

where u_h^k is the solution of the problem (4.1) and the first iteration u_h^0 is solution of (4.30).

Proposition 4.3.1 *(Boulaaras and Haiour [10]). Under the previous hypotheses and notations, we have the following estimate of convergence*

$$\left\| u_h^{k,n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{\lambda + c}{\beta + \lambda} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}. \tag{4.41}$$

Lemma 4.3.1 *Let* $\rho = \dfrac{\lambda + c}{\beta + \lambda}$. *Then, under assumption (4.2), there exists a constant* C *independent of both* h *and* n *such that*

$$\left\| u_i^{\infty,n+1} - u_{ih}^{\infty,n+1} \right\|_i \leq \frac{Ch^2 \left| \log h \right|}{1 - \rho}, \quad i = 1, 2. \tag{4.42}$$

Proof. We know from standard error estimate on uniform norm for linear problem Nitsche [27] that there exists a constant C independent of h such that

$$\left\| u^0 - u_h^0 \right\|_{L=(\Omega)} \leq Ch^2 \left| \log h \right|. \tag{4.43}$$

Since $\dfrac{1}{2} < \rho < 1$, then $1 < \rho / (1 - \rho)$ and

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq Ch^2 \left| \log h \right| \leq \frac{\rho C h^2 \left| \log h \right|}{1 - \rho}. \tag{4.44}$$

Let us now prove (4.42) by induction. Indeed for $n = 1$, using the result of Propsition1, we have in Ω_1

$$\begin{aligned}
\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 &\leq \left\|u_1^{k,1} - w_{1h}^{k,1}\right\|_1 + \left\|w_1^{k,1} - u_{1h}^{k,1}\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \left\|w_1^{k,1} - u_{1h}^{k,1}\right\|_1 \\
&\leq Ch^2 \left|\log h\right| + \max\{\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\}.
\end{aligned}$$

We then have to distinguish between two cases

$$\max\{\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \tag{4.45}$$

or

$$\max\{\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\} = \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{4.46}$$

(4.45) implies

$$\begin{cases}
\left\|u_1^{k,1} - u_{1h}^{k1}\right\|_1 \leq Ch^2 \left|\log h\right| + \rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1, \\
\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1,
\end{cases}$$

then

$$\begin{cases}
\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}. \\
\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho}.
\end{cases}$$

(4.46) implies

$$\begin{cases}
\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq Ch^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\
\qquad\qquad\qquad\quad \leq \left\|u_2^0 - u_{2h}^0\right\|_2,
\end{cases}$$

so, by multiplying (4.46) by ρ we get

$$\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2. \tag{4.47}$$

So,$\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1$ is bounded by both $\rho Ch^2 |\log h| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2$and $\left\|u_2^0 - u_{2h}^0\right\|_2$, this implies that

$$\rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \rho C h^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{4.48}$$

or

$$\rho C h^2 \left|\log h\right| + \rho \left\|u_2^0 - u_{2h}^0\right\|_2 \leq \left\|u_2^0 - u_{2h}^0\right\|_2, \tag{4.49}$$

that is (4.48) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \tag{4.50}$$

and (4.49) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}. \tag{4.51}$$

It follows that only the case (4.48) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 \left|\log h\right|}{1-\rho}, \tag{4.52}$$

then

$$\begin{aligned} \rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 &\leq C h^2 \left|\log h\right| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \frac{\rho C h^2 \left|\log h\right|}{1-\rho} \\ &\leq \frac{C h^2 \left|\log h\right|}{1-\rho}. \end{aligned}$$

So, in both cases (4.45) and (4.46), we have

$$\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq \frac{C h^2 \left|\log h\right|}{1-\rho}. \tag{4.53}$$

Similarly, we have in Ω_2

$$\begin{aligned} \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2 &\leq C h^2 \left|\log h\right| + \left\|w_2^{k,1} - u_{2h}^{k,1}\right\|_2 \\ &\leq C h^2 \left|\log h\right| + \max\{\rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1\}. \end{aligned}$$

So

$$\max\{\rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1\} = \rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2 \tag{4.54}$$

or

$$\max\{\rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2, \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1\} = \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1. \tag{4.55}$$

cases (4.54) implies

$$\begin{aligned} \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 &\leq Ch^2 \left|\log h\right| + \rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2, \\ \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 &\leq \rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \end{aligned}$$

so

$$\begin{cases} \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \\ \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \\ \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho} \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}, \end{cases}$$

while case (4.55) implies

$$\begin{cases} \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \leq Ch^2 \left|\log h\right| + \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \\ \rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \leq \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1. \end{cases}, \tag{4.56}$$

So, by multiplying (4.56) by ρ we get

$$\rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1. \tag{4.57}$$

Hence $\rho \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2$ is bounded by both $\rho Ch^2|logh|+\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1$ and $\left\| u_1^{k,1} - u_{1h}^{k1} \right\|_1$, then

$$\left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \tag{4.58}$$

or

$$Ch^2 \left|\log h\right| + \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \tag{4.59}$$

which (4.58) implies

$$\left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} < \frac{Ch^2 \left|\log h\right|}{1-\rho} \tag{4.60}$$

or (4.59) implies

$$\frac{\rho C h^2 |\log h|}{1-\rho} \leq \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 < \frac{C h^2 |\log h|}{1-\rho}. \tag{4.61}$$

Hence, (4.58) and (4.59) are true because they both coincide with (4.53). So, there is either a contradiction and thus case (4.54) is impossible or case (4.55) is possible only if

$$\left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 = \rho C h^2 |\log h| + \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \tag{4.62}$$

that is

$$\left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 = \frac{\rho C h^2 |\log h|}{1-\rho}, \tag{4.63}$$

thus

$$\begin{aligned} \left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 &\leq C h^2 |\log h| + \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \\ &\leq C h^2 |\log h| + \frac{\rho C h^2 |\log h|}{1-\rho} \\ &\leq \frac{C h^2 |\log h|}{1-\rho}, \end{aligned}$$

that is, both cases (4.54) and (4.55) imply

$$\left\| u_2^{k,1} - u_{2h}^{k,1} \right\|_2 \leq \frac{C h^2 |\log h|}{1-\rho}. \tag{4.64}$$

Now, let us assume that

$$\left\| u_2^{k,n} - u_{2h}^{k,n} \right\|_2 \leq \frac{C h^2 |\log h|}{1-\rho} \tag{4.65}$$

and prove that

$$\begin{cases} \left\| u_1^{k,n+1} - u_{1h}^{k,n+1} \right\|_1 \leq \dfrac{C h^2 |\log h|}{1-\rho} \\ \left\| u_2^{k,n+1} - u_{2h}^{k,n+1} \right\|_2 \leq \dfrac{C h^2 |\log h|}{1-\rho} \end{cases}$$

■

Theorem 4.3.1 *Let $h = max\,(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\| u_i^{k,n+1} - u_{ih}^{k,n+1} \right\|_1 \leq \frac{ch^2 \left|\log h\right|}{1-\rho}, \quad \forall i = 1, 2. \tag{4.66}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\delta = \delta_1\delta_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned}
\left\| u_1^k - u_{1h}^{k,n+1} \right\|_1 &\leq \left\| u_1^k - u_1^{k,n+1} \right\|_1 + \left\| u_1^{k,n+1} - u_{1h}^{k,n+1} \right\|_1 \\
&\leq \delta_1^n \delta_2^n \left|u^0 - u\right|_1 + \frac{ch^2 \left|\log h\right|}{1-\rho} \\
&\leq \delta^{2n} \left|u^0 - u\right|_1 + \frac{ch^2 \left|\log h\right|}{1-\rho}.
\end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{4.67}$$

and thus

$$\begin{aligned}
\left\| u_1^k - u_{1h}^{k,n+1} \right\|_1 &\leq ch^2 + ch^2 \left|\log h\right| \\
&\leq ch^2 \left|\log h\right|,
\end{aligned}$$

which is the desired result. ■

4.3.3 Asymptotic behavior

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h(T)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution of (4.4)

Theorem 4.3.2 *According to the results of the proposition 3 and the theorem 3, we have*

$$\left\| u_{1h}^{p,n+1} - u^\infty \right\|_\infty \leq C\left[h^2 \left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right], \tag{4.68}$$

and

$$\left\| u_{2h}^{p,n+1} - u^\infty \right\|_\infty \leq C\left[h^2 \left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right], \tag{4.69}$$

where C is a constant independent of h and k.

Proof. We have

$$\left\|u_h^{p,2n+1} - u^\infty\right\|_\infty \leq \left\|u_h^{p,2n+1} - u_h^\infty\right\|_\infty + \left\|u_h^\infty - u^\infty\right\|_\infty .$$

Using the proposition 3 and the theorem 3, we have for $\theta \geq \dfrac{1}{2}$

$$\left\|u_h^{p,2n+1} - u^\infty\right\|_\infty \leq C\left[h^2\left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right].$$

■

Remark 4.3.1 *It can be seen in the previous estimates (4.68) and (4.69),* $\left(\dfrac{\lambda + c}{\beta + \lambda}\right)^p$ *goes to 0 when p tend to infinity. Therefore, the estimation order for both the coercive and noncoercive problems is*

$$\left\|u^\infty - u_{1h}^{\infty,n+1}\right\|_{L^\infty\left(\bar{\Omega}_1\right)} \leq Ch^2\left|\log h\right|$$

and

$$\left\|u^\infty - u_{2h}^{\infty,n+1}\right\|_{L^\infty\left(\bar{\Omega}_2\right)} \leq Ch^2\left|\log h\right|.$$

Chapter 5

A posteriori error estimates for generalized Schwarz method for HjB equation related to management of energy production with mixed boundary condition

In this chapter, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with linear source terms using the theta time scheme combined with a finite element spatial approximation, similar to that in our published papers in (Boulaaras et al. [9]), (Boulaaras and Haiour [10]), (Boulaaras and Haiour [11]),(Haiour and Boulaaras [20]) which investigated Laplace operator i.e.,a posteriori error estimates for the generalized Schwarz method (GSM), for evolutionary Hamilton-Jacobi-Belmann equation with linear source terms related to management of energy production with mixed boundary condition (MBC) are established using a theta scheme with a Galerkin spatial approximation and the techniques of the residual a posteriori error analysis.

We consider the following evolutionary inequalities: find $u(x,t)$ such that $u \in L^2(0,T;K(u))$,

$u_t \in L^2(0,T;L^2(\Omega))$ and

$$\frac{\partial u^i}{\partial t} + \max_{i=1,\ldots,M}\left(-\Delta u^i + a_0^i u^i - f^i\right) = 0, \quad \text{in } K, \tag{5.1}$$

where K is an implicit convex set defined as follows:

$$K = \left\{ \begin{array}{l} u^i \in L^2(0,T,H_0^1(\Omega)) \cap C^{\ 2}(0,T,H^{-1}(\Omega)), \\ u^i(x) \leq l + u^{i+1}, \ u^i = 0 \text{ in } \Gamma, \quad u^i = 0 \text{ in } \Gamma/\Gamma_0, \\ u^i(x,0) = u_0^i \text{ in } \Omega, i = 1,...,M. \end{array} \right\}$$

Ω is a bounded smooth domain in $\mathbb{R}^d$, $d \geq 1$ and Σ is a set in $\mathbb{R}\times\mathbb{R}^d$ defined as $\Sigma = [0,\ T]\times\Omega$ with $T < +\infty$. and $a_0^i \in L^2(0,T,L^\infty(\Omega)) \cap C^{\ 0}(0,T,H^{-1}(\Omega))$, $i \leq 1,...,M$ and the right hand side $f^i \in \left(L^2(0,T,L^2(\Omega)) \cap C^{\ 1}(0,T,H^{-1}(\Omega))\right)^M$. The problem (5.1) can be approximated by the following system of the continuous parabolic inequalities: find $\left(u^1,u^2,...,u_h^M\right) \in \left(L^2(0,T,H_0^1(\Omega))\right)^M$ solution to

$$\frac{\partial u^i}{\partial t} + A^i u^i \leq f^{\ i} \text{ in } K,$$

which is similar to that in (Boulaaras and Haiour [10]) and Boulaaras and Haiour [11] which investigated the evolutionary free boundary problems. The problem (5.1) can be transformed into the following system of evolutionary quasi variational inequalities: find $u^i \in L^2(0,T,H_0^1(\Omega))$ solution of

$$\left\{ \begin{array}{l} \left(\frac{\partial u^i}{\partial t}, v - u^i\right)_\Omega + a^i(u^i, v-u^i) \geq (f^i, v-u^i)_\Omega + (\varphi^i, v-u^i)_{\Gamma_0}, \\ u^i = 0 \text{ in } \Gamma/\Gamma_0, \ u^i(x,0) = u_0^i \text{ in } \Omega, \\ \dfrac{\partial u^i}{\partial \eta} = \varphi^i \text{ in } \Gamma_0, \ i = 1,...,M, \end{array} \right. \tag{5.2}$$

where $a^i(.,.)$ is the bilinear form defined as: for $u,v \in H_0^1(\Omega)$:
$a^i(u^i,u^i) = (\nabla u^i, \nabla u^i) - (a_0^i u^i, u^i)$ and $a_0 \in L^2(0,T,L^\infty(\Omega)) \cap C^{\ 0}(0,T,H^{-1}(\Omega))$ is sufficiently smooth functions and satisfy the following condition: $a_0(t,x) \geq \beta > 0$, β is a constant. M is an operator given by $Mu^i = k + \inf_{i\neq\mu} u^\mu$ where $k > 0$ and $\mu > 0$ and Γ_0 is the part of the boundary defined by: $\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall\xi > 0,\ x + \xi \notin \bar{\Omega}\right\}$ where $\overrightarrow{\eta}_i$ is the normal vector, the symbol $(.,.)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$ and in (Perthame [30]) M is satisfying the following proprieties: for all $u,v \in C(\Omega)$

$$\left\{ \begin{array}{l} M(\delta u + (1-\delta)v) \geq \delta M(u) + (1-\delta)M(v), \\ \text{For all } \eta \in \mathbb{R}, M(u+\eta) = M(u) + \eta. \end{array} \right.$$

The system of evolutionary quasi-variational inequality (5.1) has arisen from many scientific, engineering and economic problems, for examples, heat control problem, Stefan problem, and American option problem (Bensoussan and Lions [5]).

5.1 The discrete system of parabolic quasi-variational inequalities

In (Boulaaras and Haiour [10]), the problem (5.2) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$\begin{cases} b^i\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right) \geq \left(f^{i,\ \theta,k} + \mu u_h^{i,k-1}, v_h - u_h^{i,\theta,k}\right)_\Omega \\ + \left(\varphi^{i,\theta,k}, \left(v_h - u_h^{i,\theta,k}\right)\right)_{\Gamma_0}, \\ v_h,\ u_h^{i,\theta,k} = \theta u_h^{i,k} + (1-\theta)\, u_h^{i,k-1} \in V_h^i,\ \theta \in [0,1], \\ f^{\ i,\theta,k} = \theta f^{i,k} + (1-\theta)\, f^{i,k-1},\ \varphi^{\ \theta,k} = \theta\varphi^k + (1-\theta)\,\varphi^{k-1}, \end{cases} \tag{5.3}$$

where

$$\begin{cases} b^i\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right) = \mu\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right)_\Omega + a\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right), \\ \mu = \dfrac{1}{\theta\Delta t} = \dfrac{p}{\theta T} \end{cases}. \tag{5.4}$$

and

$$V^{i,h} = \left\{ \begin{array}{l} v^i \in \left(L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right)^M, \\ \text{such that } v_h^i \mid_K = P_1,\ k \in \tau_h,\ v_h^i \leq r_h M v_h^i, \\ v_h^i\left(.,0\right) = v_{ih0} \text{ in } \Omega,\ \dfrac{\partial v_h^i}{\partial \eta} = \varphi^i \text{ in } \Gamma_0,\ v_h^i = 0 \text{ in } \Gamma\backslash\Gamma_0, \end{array} \right\} \tag{5.5}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1and r_h be the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v\left(M_i\right)\varphi_i\left(x\right).$$

5.2 The space continuous for the generalized Schwarz method

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$\Omega_1 \cap \Omega_2 = \Omega_{12}$, $\partial\Omega_s \cap \Omega_t = \Gamma_s$, $s \neq t$ and $s, t = 1, 2$.We need the spaces
$V_s = H^1(\Omega) \cap H^1(\Omega_s) = \{v \in H^1(\Omega_i) : \ v_{\partial\Omega_i \cap \partial\Omega} = 0\}$ and
$W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \{v_{\Gamma_s}, \ v \in V_s$ and $v = 0$ on $\partial\Omega_s \backslash \Gamma_s\}$, which is a subspace of
$H^{\frac{1}{2}}(\Gamma_s) = \{\psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s}$ for $\varphi \in V_s, \ s = 1, 2\}$, with its norm
$\|\varphi\|_{W_s} = \inf_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}$. We define the continuous counterparts of the continuous Schwarz sequences by $u_1^{i,k,m+1} \in \left(H_0^1(\Omega)\right)^M$, $m = 0, 1, 2, ..., i = 1, ..., M$ solution of

$$\begin{cases} c^i\left(u_1^{i,\theta,k,m+1}, v - u_1^{i,\theta,k,m+1}\right) \geq \\ \left(F^{i,\theta}\left(u_1^{i,\theta,k-1,m+1}\right), v - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} + \left(\varphi^i, v - u^{i,\theta,k,m+1}\right)_{\Gamma_0}, \\ u_1^{i,\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ \dfrac{\partial u_1^{i,\theta,k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{i,\theta,k,m+1} = \dfrac{\partial u_2^{i,\theta,k,m}}{\partial\eta_1} + \alpha_1 u_1^{i,\theta,k,m} \text{ on } \Gamma_1, \end{cases} \tag{5.6}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1, 2$. In the next sections, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions of generalized Schwarz alternating method, we get

$$\begin{aligned} &\left(-\Delta\, u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} = \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} \\ &- \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} - \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \end{aligned}$$

thus we can deduce

$$\begin{aligned}
&\left(-\Delta u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} = \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} \\
&-\left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} - \\
&\left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_2} + \alpha_1\, u_2^{i,\theta,k,m} - \alpha_1 u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,\theta,k,m+1}, v_1^i - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&-\left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1},
\end{aligned}$$

thus the problem (5.6) equivalent to; find $u_1^{i,\theta,k,m+1} \in V_1$ such that

$$\begin{aligned}
&c(u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}) + \left(\alpha_1 u_1^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&\geq \left(F^\theta(u_1^{i,\theta,k-1,m+1}), v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} + \left(\varphi^i, v - u_1^{i,\theta,k,m+1}\right)_{\Gamma_0} \\
&+\left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1
\end{aligned} \tag{5.7}$$

and we have $u_2^{i,\theta,k,m+1} \in V_2$

$$\begin{aligned}
&c^i(u_2^{i,\theta,k,m+1}, v_2 - u_2^{i,\theta,k,m+1}) + \left(\alpha_2 u_2^{i,\theta,k,m+1}, v_2 - u_2^{i,\theta,k,m+1}\right)_{\Gamma_2} \\
&\geq \left(F^i(u_2^{i,\theta,k-1,m+1}), v_2 - u_2^{i,\theta,k,m+1}\right)_{\Omega_2} + \left(\varphi^i, v - u^{i,\theta,k,m+1}\right)_{\Gamma_0} \\
&\left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_1^{i,\theta,k,m}, v_2 - u_2^{i,\theta,k,m+1}\right)_{\Gamma_2}.
\end{aligned} \tag{5.8}$$

5.3 A posteriori error estimate

To define the auxiliary inequalities, we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that $\Omega = \Omega_1 \cup \Omega_3$, with $\Omega_1 \cap \Omega_3 = \varnothing \quad \Omega = \Omega_2 \cup \Omega_4$ and $\Omega_2 \cap \Omega_4 = \phi$. Let $(u_1^{i,k,m}, u_2^{i,k,m})$ be the solution of problems (5.6), we define the couple $(u_1^{i,k,m}, u_3^{i,k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping inequalities

$$\begin{cases} \dfrac{u_1^{i,k,m+1} - u_1^{i,k-1,m+1}}{\Delta t} - \Delta\ u_1^{i,\theta,k,m+1} + a_0^{i,k} u_1^{i,\theta,k,m+1} \geq F^{\ i,\theta}\ \left(u_1^{i,\theta,k-1,m+1}\right)\ \text{in}\ \Omega_1, \\ u_1^{i,\theta,k,m+1} = 0, \quad \text{on}\ \partial\Omega_1 \cap \partial\Omega,\ k = 1, ..., n, \\ \dfrac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha u_1^{i,\theta,k,m} = \dfrac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m},\ \text{on}\ \Gamma_1 \end{cases} \tag{5.9}$$

and

$$\begin{cases} \dfrac{u_3^{i,k,m+1} - u_3^{i,k-1,m+1}}{\Delta t} - \Delta\ u_3^{i,\theta,k,m+1} + a_0^{i,k} u_3^{\theta,k,m+1} \geq F^{\theta}\left(u_3^{i,\theta,k-1,m+1}\right) \text{in}\ \Omega_3, \\ u_3^{i,\theta,k,m+1} = 0, \quad \text{on}\ \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_3^{i,\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{i,\theta,k,m} \text{on}\ \Gamma_2 = \dfrac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m},\ \text{on}\ \Gamma_1. \end{cases} \tag{5.10}$$

It can be taken $\epsilon_1^{i,\theta,k,m} = u_2^{i,\theta,k,m+1} - u_3^{i,\theta,k,m+1}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{i,\theta,k,m+1}$ and $u_3^{i,\theta,k,m+1}$ of the problem (5.6) and (resp.,(5.9) and (5.10)) in Ω_3. Because both overlapping and the nonoverlapping problems converge see (Otto and Lube [29]) that is, $u_2^{i,\theta,k,m+1}$ and $u_3^{i,\theta,k,m+1}$ tend to $u_3^{i,\theta,k}$ (resp. $u_3^{i,\theta,k}$), then $\epsilon_1^{i,\theta,k,m}$ should tend to naught when m tends to infinity in V_2 . By taking

$$\begin{aligned} \Lambda_3^{i,k,m} &= \frac{\partial u_2^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m},\ \Lambda_1^{i,k,m} = \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m}, \\ \Lambda_3^{i,k,m} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial \epsilon_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,\theta,k,m}, \\ \Lambda_1^{i,k,m} &= \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m}. \end{aligned} \tag{5.11}$$

Using Green formula, (5.9) and (5.10) can be reformulated to the following system of elliptic variational equations

$$\begin{aligned} & c(u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}) + \left(\alpha_1 u_1^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\ & \geq \left(F^{i,\theta}(u_1^{i,\theta,k-1,m+1}), v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} + \left(\varphi^i, v - u_1^{i,\theta,k,m+1}\right)_{\Gamma_0} \\ & + \left(\Lambda_3^{k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} & c(u_3^{i,\theta,k,m+1}, v_3 - u_3^{i,\theta,k,m+1}) + \left(\alpha_3 u_3^{i,\theta,k,m+1}, v_3 - u_3^{i,\theta,k,m+1}\right)_{\Gamma_1} \\ & \geq \left(F^{i,\theta}(u_3^{i,\theta,k-1,m+1}), v_3 - u_3^{i,\theta,k,m+1}\right)_{\Omega 3} + \left(\varphi^i, v - u_3^{i,\theta,k,m+1}\right)_{\Gamma_0} \\ & + \left(\Lambda_1^{i,k,m}, v_3 - u_3^{i,\theta,k,m+1}\right)_{\Gamma_1}, \forall v_3 \in V_3. \end{aligned} \tag{5.13}$$

On the other hand by taking

$$\theta_1^{i,k,m} = \frac{\partial \epsilon_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,\theta,k,m}, \tag{5.14}$$

we get

$$\begin{aligned}
\Lambda_3^{i,\theta,k,m} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial (u_2^{i,\theta,k,m} - u_3^{i,\theta,k,m})}{\partial \eta_1} + \alpha_1 (u_2^{i,\theta,k,m} - u_3^{i,\theta,k,m}) \\
&= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial \epsilon_1^{i,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m} \\
&= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m}.
\end{aligned} \tag{5.15}$$

Using (5.14) we have

$$\begin{aligned}
\Lambda_3^{i,k,m+1} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} = -\frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_3} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} \\
&= \alpha_3 u_3^{i,\theta,k,m} - \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} - \alpha_3 u_1^{i,\theta,k,m} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} \\
&= (\alpha_1 + \alpha_3) u_3^{i,\theta,k,m} - \Lambda_1^{i,k,m} + \theta_1^{i,k,m+1}
\end{aligned} \tag{5.16}$$

and the last equation in (5.16), we have

$$\begin{aligned}
&\Lambda_1^{i,k,m+1} = -\frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_3 u_1^{i,\theta,k,m} = \alpha_1 u_1^{i,\theta,k,m} - \frac{\partial u_2^{i,\theta,k,m}}{\partial \eta_1} - \alpha_1 u_2^{i,\theta,k,m} + \\
&\alpha_3 u_1^{i,\theta,k,m} + \alpha_3 u_1^{i,\theta,k,m} = (\alpha_1 + \alpha_3) u_1^{i,\theta,k,m} - \Lambda_3^{i,k,m} + \theta_3^{i,k,m+1}.
\end{aligned} \tag{5.17}$$

Lemma 5.3.1 *[(Perthame [30]) Let $u_s^k = u_{\Omega s}^k$, $e_s^{\theta,k,m+1} = u_s^{\theta,k,m+1} - u_s^k$ and $\eta_s^{k,m+1} = \Lambda_s^{k,m+1} - \Lambda_s^k$. Then for $s, t = 1, 3$, $s \neq t$, we have*

$$\begin{aligned}
&c_s^i(e_s^{i,\theta,k,m+1}, v_s - e_s^{i,\theta,k,m+1}) + \left(\alpha_s e_s^{i,\theta,k,m+1}, v_s - e_s^{i,k,m+1}\right)_{\Gamma s} \\
&= \left(\eta_t^{i,k,m}, v_s - e_s^{i,k,m+1}\right)_{\Gamma_s}, \forall v_s \in V_s
\end{aligned} \tag{5.18}$$

and

$$\left(\eta_s^{i,k,m+1}, \psi^i\right)_{\Gamma_s} = \left((\alpha_s + \alpha_t) e_s^{i,k,m+1}, v_s\right)_{\Gamma_s} - \left(\eta_t^{i,k,m}, \psi^i\right)_{\Gamma_s} + \left(\theta_t^{i,k,m+1}, \psi^i\right)_{\Gamma s}, \forall \psi \in V_s. \tag{5.19}$$

Theorem 5.3.1 *[(Perthame [30]) We have*

$$\left\| u_{1,h}^{i,\theta,k,m+1} - u_{1,h}^{i,\theta,k} \right\|_{1,\Omega_1} + \left\| u_{3,h}^{i,\theta,k,m+1} - u_{3,h}^{i,\theta,k} \right\|_{1,\Omega_3} \leqslant C \left\| u_{1,h}^{i,\theta,k,m+1} - u_{3,h}^{i,\theta,k,m} \right\|_{W_1},$$

$$\left\|u_{2,h}^{i,\theta,k,m+1}-u_{2,h}^{i,\theta,k}\right\|_{1,\Omega_2}+\left\|u_{4,h}^{i,\theta,k,m+1}-u_{4,h}^{i,\theta,k}\right\|_{1,\Omega_4}\leqslant C\left\|u_{2,h}^{i,\theta,k,m+1}-u_{4,h}^{i,\theta,k,m}\right\|_{W_2}.$$

and

$$\begin{aligned}&\left\|u_{1,h}^{i,\theta,k,m+1}-u_{1,h}^{i,\theta,k}\right\|_{1,\Omega_1}+\left\|u_{2,h}^{i,\theta,k,m}-u_{2,h}^{i,\theta,k}\right\|_{1,\Omega_2}\leqslant C\ (\left\|u_{1,h}^{i,\theta,k,m+1}-u_{2,h}^{i,\theta,k,m}\right\|_{W_1}\\&+\left\|u_{2,h}^{i,\theta,k,m}-u_{1,h}^{i,\theta,k,m}\right\|_{W_2}+\left\|e_{1,h}^{i,k+1,m}\right\|_{W_1}+\left\|e_{2,h}^{i,k+1,m}\right\|_{W_2}).\end{aligned}$$

Theorem 5.3.2 *Let* $u_s^{i,\theta,k}=u^{i,\theta,k}\mid_{\Omega_s}$ *where* u *is the solution of problem (5.1), the sequences* $\left(u_{1,h}^{i,\theta,k,m+1},u_{2,h}^{i,\theta,k,m}\right)_{m\in\mathbb{N}}$ *are solutions of the discrete problems (5.12) and (5.13). Then there exists a constant* C *independent of* h *such that*

$$\left\|u_{1,h}^{i,\theta,k,m+1}-u_1^{i,\theta,k}\right\|_{1,\Omega_1}+\left\|u_{2,h}^{i,\theta,k,m}-u_2^{i,\theta,k}\right\|_{1,\Omega_2}\leqslant C\left\{\sum_{i=1}^{2}\sum_{T\in\tau_h}\left(\eta_i^T\right)+\eta_{\Gamma_s}\right\},$$

where

$$\eta_{\Gamma_s}=\left\|u_{h,s}^{i,\theta,k,*}-u_{h,t}^{i,\theta,k,*-1}\right\|_{W_{h,s}}+\left\|\epsilon_{i,h}^{i,\theta,k,*}\right\|_{W_{h,s}}$$

and

$$\eta_s^T=h_T\left\|\begin{array}{c}F\left(u_{h,s}^{i,\theta,k-1,*}\right)+u_{h,s}^{i,\theta,k-1}+\\ \Delta\ u_{h,s}^{i,\theta,k,*}-\left(1+\lambda a_{h0}^k\right)u_{h,s}^{i,\theta,k}\end{array}\right\|_{0,T}+\sum_{E\in\varepsilon_h}h_E^{\frac{1}{2}}\left\|\left[\frac{\partial u_{h,s}^{i,\theta,k,*}}{\partial\eta_E}\right]\right\|_{0,E},$$

where C *is a constant independent of* h *and* k *and the symbol* $*$ *is corresponds to* $m+1$ *when* $s=1$ *and to* m *when* $s=2$.

Proof. We have by using the triangle inequality

$$\sum_{s=1}^{2}\left\|u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k,*}\right\|_{1,\Omega_s}\leqslant\sum_{s=1}^{2}\left\|u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k}\right\|_{1,\Omega_s}+\sum_{s=1}^{2}\left\|u_{h,s}^{i,\theta,k}-u_{s,h}^{i,*}\right\|_{1,\Omega_s}. \tag{5.20}$$

The second term on the right-hand side of (5.20) is bounded by

$$\sum_{s=1}^{2}\sum_{i=1}^{2}\left\|u_{h,s}^{i,\theta,k}-u_{s,h}^{i,*}\right\|_{1,\Omega_s}\leqslant C\sum_{s=1}^{2}\eta_{\Gamma_s}^{i}.$$

To bound the first term on the right-hand side of (5.20) we use the residual equation and the technique of the residual a posteriori error estimation (Otto and Lube [29]) , to obtain for

$v_h \in V^h$

$$\begin{cases} c(u_s^{i,\theta,k} - u_{h,s}^{\theta,k}, v_s) = c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s - v_{h,s}) \\ \leq \sum_{T\subset\Omega_s} \int_T \begin{pmatrix} F^{i,\theta}\left(u_{h,s}^{i,\theta,k-1}\right) + u_{h,s}^{i,\theta,k-1} + \mu\Delta\, u_{h,s}^{i,\theta,k} - \\ \left(1+\mu a_{h0}^{i,k}\right) u_{h,s}^{i,\theta k} \end{pmatrix} (v_s - v_{h,s})\, ds \\ - \sum_{E\subset\Omega_s} \int_E \left[\dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_E}\right] (v_s - v_{h,s})\, ds - \sum_{E\subset\Gamma_s} \int_E \dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_E} (v_s - v_{h,s})\, ds' \\ + \sum_{E\subset\Omega_s} \int_T \left(F^\theta\left(u_s^{i,\theta,k}\right) - F^\theta\left(u_{h,s}^{i,\theta k}\right)\right)(v_s - v_{h,s}) d\sigma + \left(\dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_s}, v_s - v_{h,s}\right)_{\Gamma_s}, \end{cases}$$

where $F^\theta\left(u_{h,s}^{i,\theta,k}\right)$ is any approximation of $F^\theta\left(u_s^{i,\theta,k}\right)$. Therefore,

$$\begin{aligned} &\sum_{s=1}^{2} c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s) \\ &\leq \sum_{s=1}^{2}\sum_{T\subset\Omega_s} \left\| \begin{matrix} F^\theta\left(u_{h,s}^{i,\theta,k}\right) + u_{h,s}^{i,\theta,k-1} + \mu\Delta\, u_{h,s}^{i,\theta,k} \\ -\left(1+\mu a_{h0}^{i,k}\right) u_{h,s}^{i,\theta,k} \end{matrix} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} \\ &+\sum_{s=1}^{2}\sum_{E\subset\Omega_s} \left\| \left[\frac{\partial u_{h,s}^{i,\theta,k}}{\partial \eta_E}\right] \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} + \sum_{s=1}^{2}\sum_{E\subset\Gamma_s} \left\| \frac{\partial u_{h,s}^{i,\theta,k}}{\partial \eta_E} \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} \\ &+\sum_{s=1}^{2}\sum_{T\subset\Omega_s} c \left\| u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} + \sum_{s=1}^{2}\sum_{T\subset\Omega_s} \left\| \frac{\partial u_{h,s}^{i,\theta,k}}{\partial \eta_s} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T}, \end{aligned} \tag{5.21}$$

Using the following fact

$$\left\| u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k} \right\|_{1,\Omega_s} \leqslant \sup_{v_s^i \in K} \frac{c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s + ch_s^{i,T})}{\|v_s^i + ch_s^T\|_{1,\Omega_i}},$$

we get

$$\sum_{s=1}^{2} c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s + ch_s^{i,T}) \leq \sum_{s=1}^{2}\left(\sum_{T\subset\Omega_s} \eta_s^{i,T}\right)\sum_{s=1}^{2} \|v_s\|_{1,\Omega_s}. \tag{5.22}$$

Finally, by combining (5.20) and (5.21) the required result follows. ■

Conclusion

In this thesis, a maximum norm analysis of a nonmatching grids method combined with a finite element time scheme as well as Galerkin spatial method for parabolic equation with linear source term and with nonlinear source terms. Also, an a posteriori error estimates for the generalized Schwarz method with Dirichlet boundary conditions on the interfaces evolutionary HJB equation with second order boundary value problems are derived using the same previous mentioned method. Furthermore, a result of asymptotic behaviors for all previous problems on uniform norm are deduced by using Benssoussan-Lions' algorithm. In the next works. The geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate of a new class of non linear elliptic PDEs will be proved and the results of some numerical experiments will be presented to support the theory.

Bibliography

[1] Ainsworth, M., Oden, J. T., 2011. A posteriori error estimation in finite element analysis. Vol. 37. John Wiley & Sons.

[2] Allaire, G., 2007. Numerical analysis and optimization: an introduction to mathematical modelling and numerical simulation. Oxford university press.

[3] Badea, L., 1991. On the schwarz alternating method with more than two subdomains for nonlinear monotone problems. SIAM Journal on Numerical Analysis 28 (1), 179–204.

[4] Bahi, M. C., Boulaaras, S., Haiour, M., Zarai, A., 2018. The maximum norm analysis of a nonmatching grids method for a class of parabolic equation with linear source terms. Applied Science APPS 20 (official SCOPUS).

[5] Bensoussan, A., Lions, J., 1982. Contrôle impulsionnel et inéquations quasi variationnelles. Paris.

[6] Bernardi, C., Chacón Rebollo, T., Chacón Vera, E., Franco Coronil, D., 2009. A posteriori error analysis for two non-overlapping domain decomposition techniques. Applied Numerical Mathematics 59 (6), 1214–1236.

[7] Boulaaras, S., Bahi, M., Alnegga, M., Zarai, A., 2017. A posteriori error estimates for generalized schwarz method for hjb equation related to management of energy production with mixed boundary condition. Applied Sciences 19, 22–30.

[8] Boulaaras, S., Bahi, M. C., Haiour, M., ???? The maximum norm analysis of a nonmatching grids method for a class of parabolic equation with nonlinear source terms. Boletim da Sociedade Paranaense de Matemática 38 (4), 157–174.

[9] Boulaaras, S., Habita, K., Haiour, M., 2015. Asymptotic behavior and a posteriori error estimates for the generalized overlapping domain decomposition method for parabolic equation. Boundary Value Problems 2015 (1), 124.

[10] Boulaaras, S., Haiour, M., 2013. The maximum norm analysis of an overlapping schwarz method for parabolic quasi-variational inequalities related to impulse control problem with the mixed boundary conditions. Appl. Math. Inf. Sci 7 (1), 343–353.

[11] Boulaaras, S., Haiour, M., 2015. A general case for the maximum norm analysis of an overlapping schwarz methods of evolutionary hjb equation with nonlinear source terms with the mixed boundary conditions. Applied Mathematics & Information Sciences 9 (3), 1247.

[12] Boulaaras, S., Haiour, M., 2015. A new proof for the existence and uniqueness of the discrete evolutionary hjb equations. Applied Mathematics and Computation 262, 42–55.

[13] Boulbrachene, M., Al Farei, Q., 2014. Maximum norm error analysis of a nonmatching grids finite element method for linear elliptic pdes. Applied Mathematics and Computation 238, 21–29.

[14] Chan, T. F., Hou, T. Y., Lions, P.-L., 1991. Geometry related convergence results for domain decomposition algorithms. SIAM Journal on Numerical Analysis 28 (2), 378–391.

[15] Dolean, V., Jolivet, P., Nataf, F., 2015. An introduction to domain decomposition methods: algorithms, theory, and parallel implementation. Vol. 144. SIAM.

[16] Douglas, J., Huang, C.-S., 1997. An accelerated domain decomposition procedure based on robin transmission conditions. BIT Numerical Mathematics 37 (3), 678–686.

[17] Engquist, B., Zhao, H.-K., 1998. Absorbing boundary conditions for domain decomposition. Applied numerical mathematics 27 (4), 341–365.

[18] Farhat, C., Lesoinne, M., 2000. Two efficient staggered algorithms for the serial and parallel solution of three-dimensional nonlinear transient aeroelastic problems. Computer methods in applied mechanics and engineering 182 (3-4), 499–515.

[19] Grisvard, P., 1985. Elliptic problems in nonsmooth domains, volume 24 of monographs and studies in mathematics. pitman.

[20] Haiour, M., Boulaaras, S., 2011. Overlapping domain decomposition methods for elliptic quasi-variational inequalities related to impulse control problem with mixed boundary conditions. Proceedings-Mathematical Sciences 121 (4), 481–493.

[21] Lions, P.-L., 1988. On the schwarz alternating method. i. In: First international symposium on domain decomposition methods for partial differential equations. Paris, France, pp. 1–42.

[22] Lui, S., 2002. On linear monotone iteration and schwarz methods for nonlinear elliptic pdes. Numerische Mathematik 93 (1), 109–129.

[23] Maday, Y., Magoules, F., 2006. Improved ad hoc interface conditions for schwarz solution procedure tuned to highly heterogeneous media. Applied mathematical modelling 30 (8), 731–743.

[24] Maday, Y., Magoules, F., 2006. A survey of various absorbing interface conditions for the schwarz algorithm tuned to highly heterogeneous media.

[25] Nataf, F., 2007. Recent developments on optimized schwarz methods. In: Domain Decomposition Methods in Science and Engineering XVI. Springer, pp. 115–125.

[26] Necas, J., 1967. Les méthodes directes en théorie des équations elliptiques.

[27] Nitsche, J., 1977. L-convergence of finite element approximations. In: Mathematical aspects of finite element methods. Springer, pp. 261–274.

[28] Ortiz, E. L., 1987. Numerical approximation of partial differential equations. Vol. 133. Elsevier.

[29] Otto, F.-C., Lube, G., 1999. A posteriori estimates for a non-overlapping domain decomposition method. Computing 62 (1), 27–43.

[30] Perthame, B., 1985. Some remarks on quasi-variational inequalities and the associated impulsive control problem. In: Annales de l'Institut Henri Poincare (C) Non Linear Analysis. Vol. 2. Elsevier, pp. 237–260.

[31] Quarteroni, A., Valli, A., 1994. Introduction. Springer.

[32] Quarteroni, A., Valli, A., 1999. Domain decomposition methods for partial differential equations numerical mathematics and scientific computation. Quarteroni, A. Valli–New York: Oxford University Press.–1999.

[33] Rixen, D., Magoulès, F., 2007. Domain decomposition methods: recent advances and new challenges in engineering. Computer Methods in Applied Mechanics and Engineering 8 (196), 1345–1346.

[34] Toselli, A., Widlund, O., 2006. Domain decomposition methods-algorithms and theory. Vol. 34. Springer Science & Business Media.

[35] Verfürth, R., 1996. A review of a posteriori error estimation and adaptive mesh-refinement techniques. John Wiley & Sons Inc.

[36] Yosida, K., 1995. Functional analysis. reprint of the sixth (1980) edition. classics in mathematics. Springer-Verlag, Berlin 11, 14.