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Study of some identification problems by using the Sentinel method

Presented by:

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بسم الله الرحمن الرحيم

Dedication

Praise be to God until praise reaches its end

To my honorable parents

To my dear sisters

To my dear friends, family and relatives

Billal Elhamza

Acknowledgment

In the name of Allah, the Most Gracious and the Most Merciful Alhamdulillah, all praises to Allah for giving me the courage and the patience finish this thesis.

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Abstract

This thesis aims to study of various kinds of some identification problems by the use of the sentinel method which is introduced by Jacques Louis Lions. We apply this method for the first time in order to identify parameters and coefficients in some identification problems. Actually, We identify the potential coefficient in the wave equation with incomplete data, the bulk modulus coefficient in the acoustic equation with incomplete data, the potential coefficient in the Schrodinger equation with an incomplete initial condition, and the identification of the diffusion coefficient in the diffusion equation with incomplete data. This method is based on three considerations, a state equation, a state observation, and a functional stationary to certain parameters. The existence of this functional is related to solving an optimal control problem. The determination of these coefficients is crucial for understanding and predicting the behavior of many physical and biological systems.

Keywords: Sentinel method, Identification problems, Potential coefficient identification, Bulk modulus coefficient identification, Diffusion coefficient identification, Controllability, HUM, Penalization method.

Résumé

Cette thèse vise à étudier divers types de problèmes d'identification à l'aide de la méthode sentinelles, introduite par Jacques Louis Lions. Nous appliquons cette méthode pour la première fois afin d'identifier des paramètres et des coefficients dans certains problèmes d'identification. En particulier, nous identifions le coefficient de potentiel dans l'équation des ondes avec des données incomplètes, le coefficient de module de compression dans l'équation acoustique avec des données incomplètes, le coefficient de potentiel dans l'équation de Schrödinger avec une condition initiale incomplète, et l'identification du coefficient de diffusion dans l'équation de diffusion avec des données incomplètes. Cette méthode repose sur trois considérations : une équation d'état, une observation d'état, et une fonctionnelle stationnaire pour certains paramètres. L'existence de cette fonctionnelle est liée à la résolution d'un problème de contrôle optimal. La détermination de ces coefficients est cruciale pour comprendre et prédire le comportement de nombreux systèmes physiques et biologiques.

Mots-clés: Méthode sentinelles; Problèmes d'identification; Identification de coefficient de potentiel; Identification de coefficient de module de compression, Identification de coefficient de diffusion; Contrôlabilité; HUM; Méthode de pénalisation.

الملخص

تهدف هذه الأطروحة إلى دراسة أنواع مختلفة من مسائل تحديد الهوية باستخدام طريقة الحارس التي قدمها جاك لويس ليونز. نطبق هذه الطريقة لأول مرة من أجل تحديد المعلمات والمعاملات في بعض مشاكل تحديد الهوية. على وجه الخصوص ، نحدد المعامل المحتمل في معادلة الموجة ببيانات غير كاملة ، ومعامل معامل الضغط في المعادلة الصوتية مع بيانات غير كاملة ، والمعامل المحتمل في معادلة شرودنغر مع حالة أولية غير مكتملة ، وتحديد معامل الانتشار في معادلة الانتشار ببيانات غير كاملة. تعتمد هذه الطريقة على ثلاثة اعتبارات: معادلة الحالة ، ومراقبة الحالة ، والوظيفة الثابتة لمعلمات معينة. يرتبط وجود هذه الوظيفة بحل مشكلة التحكم الأمثل. يعد تحديد هذه المعاملات أمرًا بالغ الأهمية لفهم سلوك العديد من الأنظمة الفيزيائية والبيولوجية والتنبؤ به.

الكلمات الرئيسية: طريقة الحراس ؛ قضايا تحديد الهوية ؛ تحديد المعامل المحتمل ؛ تحديد معامل الانضغاط ، تحديد معامل الانتشار ؛ طريقة هيلبرت ؛ طريقة الجزاء.

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Notations and abbreviations

$\mathbb{R}^+$	Set of real positive numbers.
$\Vert.\Vert_H$	Norm of the linear space H.
$(.,.)_H$	Scalar product in H.
$\langle .,. \rangle_{H',H}$	The duality product between H and $H^{'}$
$\mathcal{M}_n(\mathbb{R})$	The space of square matrices of order n in $\mathbb{R}$
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$	The laplacian.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)^T$	The gradient.
div	The divergence.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(\mathcal{Y}, \mathcal{Z})$	The space of linear bounded operators from $\mathcal{Y}$ to $\mathcal{Z}$.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support tin Q.
$L^2(0,T;E)$	Space of square $E-$ norm summable functions

s.t.	such that.
iff	if and only if.
a.e.	almost everywhere.
PDEs	Partial differential equation.
HUM	Hilbert Uniqueness method.
UCP	Unique Continuation property.

Introduction

The use of PDEs to represent physical systems has been a long-standing practice in applied mathematics. To build a complete model, one needs to have certain state inputs in the form of initial or boundary data, along with structure inputs such as coefficients or source terms that reflect the physical characteristics of the system. Solving the direct problem involves obtaining a unique solution to a well-posed problem, which allows for the computation of various physical outputs of interest. However, if some of the necessary inputs are missing, we can attempt to deduce them from measured outputs through an inverse problem. When one or more unknown coefficients in a partial differential equation are the missing inputs, we refer to the problem as a coefficient identification problem. A classic examples of such a problems are the identification of a diffusion coefficient in a semilinear diffusion equation, the identification of the bulk modulus coefficient in acoustic equation and the potential coefficient in a wave equation and a schrödinger equation that have been dealt with in different ways.

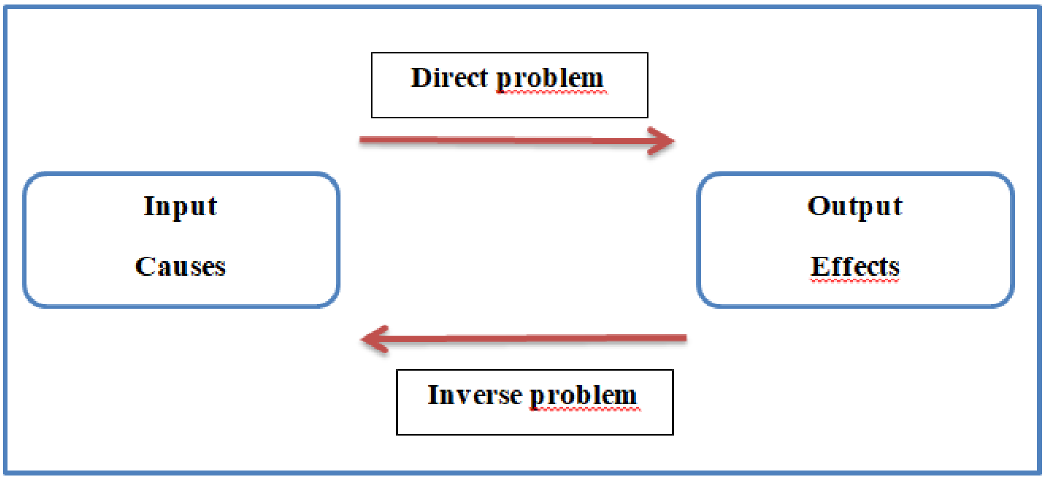

Figure 1: Direct and inverse problem.

To solve an identification problem, a widely used technique is the least squares method, she was introduced in 1795 by Gauss and Legendre for solving inverse problems. As early as 1805, Legendre presented his article "nouvelles méthodes pour la détermination des orbites des comètes" [35]. Since then, this method has remained the most popular parameter identification technique for both ordinary differential equations (ODEs) and partial differential equations (PDEs). The least squares technique consists of minimizing the squared distance between observed values and calculated values. In the least squares method, all unknown parameters play the same role. One does not make a difference between the parameters (with the source terms and with the initial terms). There are therefore risks of not being able to clearly separate the roles of each. Moreover, the available data may be insufficient in relation to the number of parameters sought, which leads to an infinity of possible solutions. We have, in this case, a problem of uniqueness of the solution, also for a set of data taken from the same domain, the resolution can lead to a strong disturbance of the solution, it is about a problem of stability . Faced with all these eventualities, it is generally said that the problem of least squares is badly posed, it is necessary, in this case, to introduce regularizing or stabilizing terms which reduce additional approximation errors.

Instead of using output least squares to identify coefficients, another approach is the equation error method which is a mathematical technique used in system identification to estimate the parameters of a mathematical model of a system. the equation error method has been described in literature [21], [22], [25] and [41]. In this method, the measured over-specification is used as input to the differential equation in the direct problem, which is then transformed into an equation for the unknown coefficient. This equation establishes a direct relationship between the unknown coefficient values and the measured data values. However, since the relationship is often complex, it may be difficult to determine the properties of the input-to-output mapping. The effectiveness of the equation error method varies depending on the problem at hand.

While not neglecting the fundamental "least squares" method, which remains by far the most important for such problems, it can be useful to attempt what is called the "Sentinels Method". A sentinel is a linear form acting on observations that must satisfy sensitivity conditions to certain parameters of the system and insensitivity conditions to others. Thus, the idea of sentinels seems a bit different. The idea is that with a suitable set of sentinels, one can identify the interesting unknowns and free oneself from the others. For example, suppose that the equation of the system describes the kinetics of a pollutant in a river or lake and that the source is potential polluters. What is interesting in this case is obviously to know what the polluters have dumped into the river and not the state of the lake at the initial time.

The method of sentinels allows us to reconstruct a parameter or an approximation of it independently of other data that we do not want to identify, so sentinels are a "parameter identification method." Identification problems have many motivations related to important physical problems, and the application field of parameter identification methods is extremely vast, with abundant literature on the subject.

Sentinels were introduced by J. L. Lions in notes to the CRAS [34]. He later published a book on this topic [33], where he studied the existence of sentinels insensitive to disturbances without constraints of sensitivity to relevant data. The study of their existence leads to the resolution of the optimal problem of distributed systems.

There are numerous theoretical and numerical results as well as many applications to real physical problems motivated by researchers and industry. For example, the works of G. Chavent [13], who is also the author of a paper on sentinels, specifically dealing with the relationship between sentinels and least squares, and the works of O. Nakoulima [43] and [44]. One can also refer to the works of J.P. Kernevez's team [1], [2] and [15], for the numerical treatment of pollution identification problems in distributed systems, pollution detection in an aquifer [**?**] , determination of missing parameters in a lake, and pollution search in a river [5]. Since then, several authors have focused on application this method for various problem, see for instance [3], [12], [37], [38], [50].

This thesis focuses on the study of coefficient inverse problems, which consist of identifying coefficients in hyperbolic and parabolic equations. In practice, this type of equation can cover a wide range of applications ranging from biology to environmental studies, chemistry, medicine, and so on. Furthermore, a motivation for this thesis is an application of the sentinel method for the first time to identify the potential coefficient in an inverse problem of wave equation see [17], and to identify the bulk modulus coefficient in an inverse problem of acoustic equation see [18] and to identify the potential coefficient in an inverse problem of Schrödinger equation see [19].

Content of the thesis

This thesis is organized as follows:

In the first chapter, we have two section. In the first section, we provide some definitions and properties of the controllability and the observability and the optimal control. In the second section, we introduce the basis of sentinel theory. We introduce the concepts of the sentinel and we giving an exemple of the application of this method.

In the second chapter, we have two sections, the first of which is devoted to the application of the sentinel method to a wave equation with incomplete data to identifying the potential coefficient, and the second of which is devoted to the application of the sentinel method to an acoustic equation to identifying the bulk modulus coefficient.

The third chapter is devoted to two section. In the first, we applied the sentinel method to a schrödinger equation with incomplete data for the determination of the potential coefficient. In the second, we applied the sentinel method for the identification of the diffusion coefficient in a diffusion equation with incomplete data.

Finally, we end the thesis with a conclusion and perspectives describing the main obtained results and perspectives for further research projects on the topic.

Chapter 1

Preliminaries and basics tools

In this first chapter, we recall some basic tools and we present some preliminary results essential for this work. We give in particular some fundamental results on the controllability of distributed systems, and the sentinel method.

Sentinel was introduced by J. L .Lions. Many types of systems are discussed and the author studies the existence of sentinels insensitive to disturbances without constraints of sensitivity to interesting data. The study of their existence leads to the resolution of the problem of controllability of distributed systems.

1.1 Controllability and Observability

1.1.1 Description of the systems

Let H, U be a functional space (Hilbert space); H is the state space, U is the control space. $(A, D(A))$ the infinitesimal generator of a semi-group $(S(t))_{t\geq 0}$ in H. And let $B \in \mathcal{L}(U, H)$. For all $u \in L^2(0, T, U)$ (control function), the Cauchy problem:

$$\begin{cases} y_t &= Ay + Bu \quad \text{in } (0, T), \\ y(0) &= y_0 \qquad\qquad , \end{cases} \tag{1.1}$$

admits for all $y_0 \in H$ a unique solution $y \in L^2((0, T); H)$. In addition $y \in C(0, T, H)$ and is given by:

$$y(t; u) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds. \tag{1.2}$$

We then assume that the system (1.1) is augmented by the output:

$$z(t) = Cy(t), \tag{1.3}$$

where C is an operator of $\mathcal{L}(H, Z)$ such that Z is the observation space.

1.1.2 Controllability: definitions

The standard problem of controllability for the equation (1.1) may be formulated roughly as follows:

Given a time $T > 0$, and a initial state y_0, the goal is to determine whether there exists a control driving the given initial data to the given final ones y_d in time T, where y_d is a desired state chosen a priori.

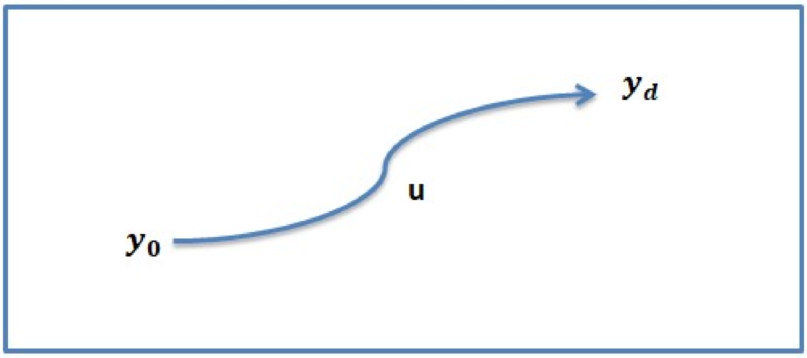

Figure 1.1: Notion of controllability

Now, let us introduce some notions of the typical controllability problems, associated to an infinite dimensional control system.

Exact controllability

Definition 1 *The system* (1.1) *is said to be exactly controllable in time* $T > 0$ *iff, for all initial state* $y_0 \in H$ *and desired state* $y_d \in H$, *there exists a control function* $u \in L^2(0, T, U)$ *such that,*

$$y(T) = y_d. \tag{1.4}$$

The definition (1) is equivalent to the following characterization properties

Proposition 1 *The system* (1.1) *is said to be exactly controllable in time* $T > 0$ *iff,*

$$\exists \gamma > 0, \text{ such that } \|y^*\|_{H^*} \leq \gamma \|B^* S^* (.) y^*\|_{L^2(0,T;U)}, \text{ for all } y^* \in H. \tag{1.5}$$

The adjoint A^* *of* A *generates the semigroup* $(S^*(t))_{t\geq 0}$ *adjoint* $(S(t))_{t\geq 0}$ *of which is also strongly continuous on the dual* H^* *of* H*, the operator* B^* *is the adjoint of* B*.*

Proof. See [9] ■

Remark 1

Let F_t *the linear operator defined by* $F_t = \int_0^t S(t-s) Bu(s) ds$*. Then, the definition* (1) *is equivalent to*

$$Im(F_t) = H. \tag{1.6}$$

Null controllability

Definition 2 *The system* (1.1) *is said to be zero-controllable in time* $T > 0$ *iff, for all initial state* $y_0 \in H$*, there exists a control function* $u \in L^2(0, T, U)$ *such that*

$$y(T) = 0. \tag{1.7}$$

Approximate controllability

Definition 3 *The system* (1.1) *is said to be approximately controllable in time* $T > 0$ *if and only if, for all initial state* $y_0 \in H$ *and desired state* $y_d \in H$*, there exists a control function* $u \in L^2(0, T, U)$ *such that,*

$$\|y(T) - y_d\|_H \leq \epsilon \quad \text{for all} \quad \epsilon > 0. \tag{1.8}$$

Remark 2

- *The definition* (3) *is equivalent to*

$$\overline{Im(F_t)} = H. \tag{1.9}$$

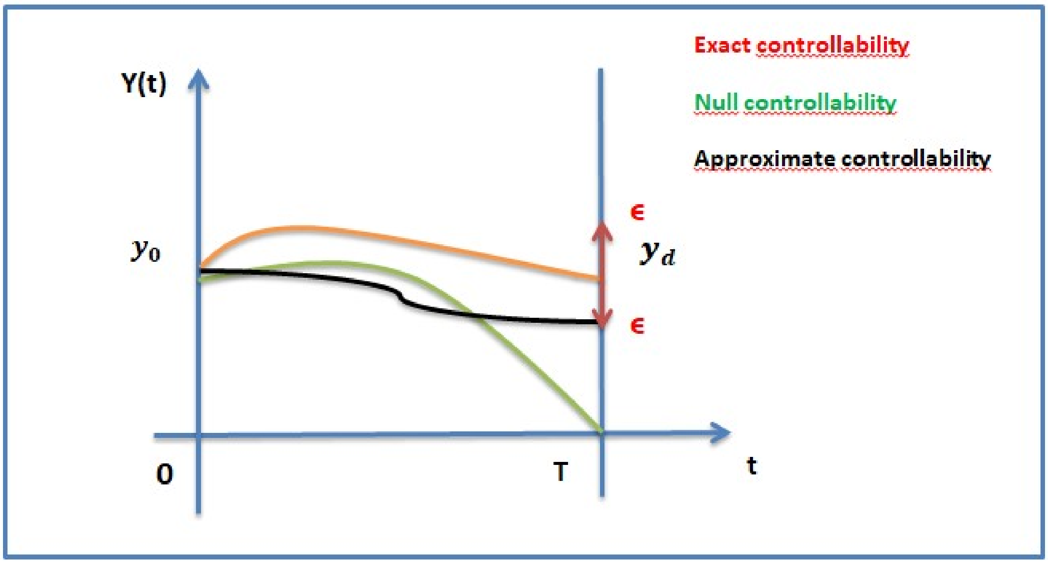

Figure 1.2: Different notions of controllability

1.1.3 Regional Controllability

In applications, few dynamic systems are controllable on the whole domain, hence the need to study this concept only on a part of the domain. For this, the concept of regional controllability is defined.

Let y_d is a given desired state, the problem of regional controllability is whether one can find a control $u \in L^2(0, T, U)$ to bring the system state (1.1) from y_0 to y_d on the region ω of the domaine Ω.

Definition 4 *The system* (1.1) *is exactly regionally controllable on* ω *iff, for all* $y_d \in L^2(\omega)$*, there exists a control function* $u \in L^2(0, T, U)$ *such that*

$$\chi_\omega y(T) = y_d. \tag{1.10}$$

Definition 5 *The system* (1.1) *is approximately regionally controllable on* ω *if and only if, for all* $y_d \in L^2(\omega)$ *and* $\epsilon \geq 0$*, there exists a control function* $u \in L^2(0, T, U)$ *such that*

$$\|y(T)|_\omega - y_d\|_{L^2(\omega)} \leq \epsilon. \tag{1.11}$$

Remark 3

– *The system* (1.1) *will also be said* $\omega-$*exactly (resp.* $\omega-$*weakly) controllable, where* χ_ω *indicates the restriction of* y *to* ω.

Here is some charachterizations of regional controllability:

Proposition 2

– *The system* (1.1) *is* $\omega-$*exactly regionally controllable iff*

$$Im_{\chi_\omega} F_t = H.$$

– *The system* (1.1) *is* $\omega-$*weakly regionally controllable if and only if*

$$\overline{Im_{\chi_\omega} S_t} = H(\omega) \implies \ker S_t^* \chi_\omega = \{0\}.$$

Remark 4

– *A system that is exactly (resp. weakly) controllable is exactly (resp. weakly) regionally controllable.*

– *A system that is exactly (resp. weakly) regionally controllable on* ω_1 *is exactly (resp. weakly) regionally controllable on* ω_2 *for all* $\omega_2 \subset \omega_1$

Let's give two examples of the situation (classical). These are two types of influential control that are often considered in the literature: in one case, control acts as a localized source term in the equation, while in the second, control works on part of the boundary conditions. Examples below relate to heat and wave equations with Dirichlet boundary conditions.

Example 1 ***Distributed control***

Let $\Omega \subset C^2$*, and* ω *is a non-empty open subset of* Ω*, we consider the heat/wave equation with Dirichlet Boundary conditions*

heat equation

$$\begin{cases} y_t - \Delta y = u\chi_\omega & in \quad \Omega \times (0,T), \\ y(x,0) = y_0 & in \quad \Omega, \\ y = 0 & on \quad \partial\Omega, \end{cases}$$

wave equation

$$\begin{cases} y_{tt} - \Delta y = u\chi_\omega & in \quad \Omega \times (0,T), \\ y(x,0) = y_0 & in \quad \Omega, \\ y_t(x,0) = y_1 & in \quad \Omega, \\ y = 0 & on \quad \partial\Omega. \end{cases}$$

- *In the first equation,* y *is the state given by the function* $y(.,t)$*, for instance in* $L^2(\Omega)$*.*

Example 1 ***Boundary control***

Let $\Omega \subset C^2$, and $\Gamma_0 \subset \partial\Omega$ we consider the heat/wave equation with non-homogeneous Dirichlet boundary control

heat equation

$$\begin{cases} y_t - \Delta y &= 0 & in & \Omega \times (0,T), \\ y(x,0) &= y_0 & in & \Omega, \\ y &= u\chi_\omega & on & \partial\Omega, \end{cases}$$

wave equation

$$\begin{cases} y_{tt} - \Delta y &= 0 & in & \Omega \times (0,T), \\ y(x,0) &= y_0 & in & \Omega, \\ y_t(x,0) &= y_1 & in & \Omega, \\ y &= u\chi_\omega & on & \partial\Omega. \end{cases}$$

- *The equations are the same in the previous, here the control in a part Γ_0 of the boundary $\partial\Omega$, for instance $u \in C_0^\infty(\Gamma_0 \times (0,T))$ for the first equation, and $u \in L^2(\Gamma_0 \times (0,T))$ for the second equation.*

1.1.4 Observability

Determining the state of a distributed parameter system from measurements is of great importance when one seeks to apply closed-loop control to such a system. The obtained measurements are expressed by the output function.

$$z(t) = CS(t)y_0 + CS_t u.$$

This output is the sum of a free response with y_0 to be determined and a controlled response with zero initial state. Since the system is linear, we can study the observation of y_0 by assuming $u = 0$. The objective is therefore to determine y_0, the solution of the equation.

$$z(t) = CS(t)y_0 = Ky_0 \quad t \in (0,T).$$

K is a bounded linear operator, the adjoint operator is given by

$$K^* z = \int_0^T S^*(t)C^* z(t)dt.$$

Definition 6 *The system* (1.1) *augmented by the output* (1.3) *is said to be exactly observable on* $(0,T)$ *if* $H^* \subset ImK^*$.

Definition 7 *The system* (1.1) *augmented by the output* (1.3) *is said to be weakly* *observable on* $(0,T)$ *if* $\ker K = \{0\}$.

Definition 8 *The system* (1.1) *augmented by the output* (1.3) *is said to be* $\omega-$*weakly observable on* $(0,T)$ *if* $\ker K\chi_\omega^* = \{0\}$.

1.1.5 Control optimal

In this sebsection, we will determine the optimal control to achieve a given target. In the case where the system (1.1) is controllable, there will generally be an infinity of controls which answer the question.

- Among these controls is there one, which is of minimum standard?
- Can we explicitly determine this control according to the various parameters of the problem?

Optimization is used to find the control that gives controllability with a minimum cost given by a function

$$J(u) = \int_0^T \|u\|^2 \, dt,$$

defined on the control space U.

Let $y_d \in H^1(\Omega)$ be a desired state. We pose the problem of transferring, at a lower cost, the system (1.1) from y_0 to y_d at time T. Thus the question becomes:

Is there a minimum energy control $u \in L^2(0,T,U)$ such that $y(T) = y_d$?

The optimal control problem can be formulated as follows

$$\begin{cases} \min_{u\in U_{ad}} J(u) & = & \min_{u\in U_{ad}} \int_0^T \|u\|^2 \, dt, \\ U_{ad} & = & \{u \in U / y(T) = y_d\}. \end{cases} \tag{1.12}$$

The objectives of this theory are:

1) Study the existence of $u \in U_{ad}$ which realizes the minimum in (1.12), we then say that u is the optimal control.

2) Description, giving the necessary and sufficient conditions for u to be optimal control.

3) Obtain the properties of the optimal control(s) from (2).

We impose

$$G = \{g \in H^1(\Omega), \quad \text{such that} \quad g = 0 \ \text{ in } \quad \omega\},$$

$$\overline{G} = \{g \in H^1(\Omega), \quad \text{such that} \quad g = 0 \ \text{ in } \quad \Omega \backslash \omega\}.$$

We consider the system

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) &= Ay(x,t) + Bu(t) & \text{in} \quad Q, \\ y(x,0) &= y_0 & \text{in} \quad \Omega, \\ y &= 0 & \text{on} \quad \Sigma. \end{cases} \tag{1.13}$$

The construction method is based on the following three steps:

- **Step 01**

For $\Phi_0 \in \overline{G}$, we consider the system:

$$\begin{cases} \frac{\partial \Phi}{\partial t}(x,t) &= -A^*\Phi(x,t) & \text{in} \quad Q, \\ \Phi(x,0) &= \Phi_0 & \text{in} \quad \Omega, \\ \Phi &= 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.14}$$

which admits a unique solution $\Phi \in L^2(0, T, H^1(\Omega)) \cap C^0(0, T, L^2(\Omega))$.

- **Step 02**

Consider the system

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) &= A\Psi(x,t) + BB^*\Phi & \text{in} \quad Q, \\ \Psi(x,0) &= \Psi_0 & \text{in} \quad \Omega, \\ \Psi &= 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.15}$$

For $\Phi_0 \in \overline{G}$, the equation (1.14) gives Φ, then the equation (1.15) gives $\Psi(x, T)$.

Then, we define the operator M by

$$M\Phi_0 = P\left(\Psi(T)\right), \quad \text{where} \quad P = \chi_\omega^* \chi_\omega$$

M is an affine operator which decomposes as: $M\Phi_0 = P\left(\Psi_1(T) + \Psi_2(T)\right)$.

where $\Psi_1(T)$ and $\Psi_2(T)$ are solutions of the following systems

$$\begin{cases} \frac{\partial \Psi_1}{\partial t}(x,t) = A\Psi_1(x,t) + BB^*\Phi & \text{in } Q, \\ \Psi_1(x,0) = \Psi_0 & \text{in } \Omega, \\ \Psi_1 = 0 & \text{on } \Sigma, \end{cases} \tag{1.16}$$

and

$$\begin{cases} \frac{\partial \Psi_2}{\partial t}(x,t) = A\Psi_2(x,t) + BB^*\Phi & \text{in } Q, \\ \Psi_2(x,0) = 0 & \text{in } \Omega, \\ \Psi_2 = 0 & \text{on } \Sigma, \end{cases} \tag{1.17}$$

respectively.

- **Step 03**

We define the linear, bounded and symmetric operator $\Lambda : \overline{G} \to \overline{G}^*$ by:

$$\forall \Phi_0 \in \overline{G}, \Lambda \Phi_0 = P\Psi_2(T),$$

with these notations, the problem of regional controllability leads to the resolution of the equation

$$\Lambda \Phi_0 = P\left(y_d - \Psi_1(T)\right). \tag{1.18}$$

Multiplying equation (1.18) by Φ_0, we get

$$\langle \Lambda \Phi_0, \Phi_0 \rangle = \int_0^T \|B^*\Phi(t)\|^2 \, dt. \tag{1.19}$$

To ensure the existence of the solution of equation (1.18), we introduce the application

$$\Phi_0 \in \overline{G} \to \int_0^T \|B^*\Phi(t)\|^2 \, dt, \tag{1.20}$$

which defines a semi-norm on G. We then have the result:

Proposition 3 *If the system* (1.13) *is* ω*-weakly controllable. Then, the equation* (1.18) *admits a unique solution* $\Phi_0 \in \overline{G}$*, and the control that transfers* (1.13) *in* G *at time* T *is given by:*

$$u^*(t) = B^*\Phi(x,t). \tag{1.21}$$

Proof. see [9]. ■

1.1.6 Regional controllability and penalization

We assume that U_{ad} is the non-empty set, therefore the system (1.13) is exactly regionally controllable on U_{ad}. We want to solve the following optimization problem

$$(OC)\begin{cases} \min_{u\in U_{ad}} J(u) = \min_{u\in U_{ad}} \int_0^T \|u(t)\|^2 \, dt, \\ \text{with } U_{ad} = \{u \in U, y(T) - y_d \in G\}. \end{cases} \tag{1.22}$$

For all $\epsilon > 0$, consider the Penalty problem

$$(OC)\begin{cases} \min_{u\in C} J_\epsilon(u, y), \\ \text{with } J_\epsilon(u, y) = \left(\int_0^T \|u(t)\|^2 \, dt + \frac{1}{2\epsilon} \int_0^T \|y'(t) - Ay(t) - Bu(t)\|^2 \, dt\right), \end{cases} \tag{1.23}$$

where C is the set of pairs (u, y) satisfying

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) - Ay(x, t) - Bu(t) \in L^2(0, T; H), \\ y(x, 0) = y_0, u \in U, \\ y(T) - y_d \in G. \end{cases} \tag{1.24}$$

So, we have the following result

Proposition 4 *For all $\epsilon > 0$, the problem (1.24) admits a unique solution that we denote by (u_ϵ, y_ϵ). The sequence $(u_\epsilon, y_\epsilon)_\epsilon$ converges weakly to (u^*, y^*) when ϵ tends to zero. Moreover u^* is the solution of problem (1.22) given by*

$$u^*(t) = B^*p(t),$$

where $p(t)$ and $y(t)$ are solutions of the following optimality system

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) = Ay(x, t) - Bu(t) \quad in \quad (0, T), \\ y(x, 0) = y_0, u \in U, \\ \frac{\partial p}{\partial t}(x, t) + A^*p(x, t) \quad in \quad (0, T), \\ p(t) \in G. \end{cases} \tag{1.25}$$

Proof. see [9] ■

1.1.7 The Hilbert Uniqueness method

The Hilbert Uniqueness Method is a mathematical technique introduced by Jacques-Louis Lions in 1988 [32]. It has been of great interest to scientists working on controllability in the fields of partial differential equations and general dynamic systems. The method is based on uniqueness results and Hilbert spaces constructed in infinitely many ways. It has been applied to both parabolic and hyperbolic systems.

The HUM provides a powerful, constructive means for solving a wide variety of exact controllability problems for partial differential equations.

A model of wave equation with boundary control action

Let $\Omega \subset \mathcal{R}^n$, with $\Gamma = \partial\Omega$ and $\Gamma_0 \subset \Gamma$, we consider the following wave equation where the control acts on a parts of the boundary

$$\begin{cases} \partial_t^2 y - \Delta y &= 0 & \text{in} & \Omega \times (0,T), \\ y\,(x,0) &= y_0(x) & \text{in} & \Omega, \\ \partial_t y\,(x,0) &= y_1(x) & \text{in} & \Omega, \\ y\,(x,t) &= v\chi_{\Gamma_0} & \text{on} & \Gamma \times (0,T). \end{cases} \tag{1.26}$$

The problem of exact controllability of our system is to find u such that

$$y(x,T) = \partial_t y(x,T) = 0 \quad \text{in} \quad \Omega.$$

In the following steps, the HUM method is described

- **Step** 01**: Forward equation**

We consider Φ solution of the following equation with $(\Phi_0, \Phi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$

$$\begin{cases} \partial_t^2 \Phi - \Delta \Phi &= 0 & \text{in} & \Omega \times (0,T), \\ \Phi\,(x,0) &= \Phi_0 & \text{in} & \Omega, \\ \partial_t \Phi\,(x,0) &= \Phi_1 & \text{in} & \Omega, \\ \Phi\,(x,t) &= 0 & \text{on} & \Gamma \times (0,T), \end{cases} \tag{1.27}$$

- **Step** 02**: backward equation**

Let's define z to be solution to the following adjoint equation

$$\begin{cases} \partial_t^2 z - \Delta z & = & 0 & \text{in} & \Omega \times (0,T), \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t z(x,T) & = & 0 & \text{in} & \Omega, \\ z(x,t) & = & \frac{\partial \Phi}{\partial \nu} & \text{on} & \Gamma \times (0,T). \end{cases} \tag{1.28}$$

where $\frac{\partial \Phi}{\partial \nu} = \nabla \Phi . \nu$ is the normal derivative.

- **Step** 03**: The operator** Λ

We define the operator Λ which associates with (Φ_0, Φ_1) as follow:

$$\Lambda(\Phi_0, \Phi_1) = (\partial_t z(0), -z(0)) . \tag{1.29}$$

By multiplying the equation of (1.28) by $\Phi(x,t)$ solution of (1.27) and by integrating on $\Omega \times (0,T)$, we get

$$\langle \Lambda(\Phi_0, \Phi_1), (\Phi_0, \Phi_1) \rangle = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\Gamma. \tag{1.30}$$

Introducing the basic idea: the uniqueness property

$$\frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \implies (\Phi_0, \Phi_1) = 0 \quad \text{in} \quad \Omega, \tag{1.31}$$

when the uniqueness property holds, we can introduce the Hilbert space

$$F \quad \text{is the completion of} \quad \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \quad \text{with the norm}$$

$$\|(\Phi_0, \Phi_1)\|_F = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\Gamma.$$

By construction, Λ is an isomorphism from F to F' (where F' is the dual space of F).

So, if $(y_1, -y_0) \in F'$, then the equation $\Lambda(\Phi_0, \Phi_1) = (y_1, -y_0)$ has a unique solution in F. We deduce that the control is given by $u = \frac{\partial \Phi}{\partial \nu}$.

1.1.8 Unique continuation property

The unique continuation property (UCP) is a fundamental concept that plays an important role in the theory of partial differential equations (PDEs) and uniqueness of solutions for PDEs. The UCP states that if a solution to a PDE vanishes on a subset of the domain with non-empty interior, then the solution must be identically zero on the entire domain.

In other words, the UCP establishes the uniqueness of solutions to PDEs by showing that if two solutions coincide on a subset of the domain with non-empty interior, then the solutions must be identical on the entire domain.

The UCP has a long history dating back to the classical results of Holmgren theorem and Carleman inequality at the beginning of the twentieth century. It has been used to study a wide range of PDEs, including elliptic, hyperbolic, and parabolic equations. The UCP is a powerful tool in the study uniqueness of PDEs, as it allows one to establish the uniqueness of solutions without having to solve the PDE explicitly.

Holmgren's uniqueness theorem

Holmgren's uniqueness theorem is a result in the theory of partial differential equations. It provides a uniqueness result in the class of analytic solutions to a large class of Cauchy problems for partial differential equations. The theorem gives uniqueness under less restrictive assumptions on the data and the solution. In simpler terms, the theorem states that if a solution to a partial differential equation is real-analytic in a neighborhood, then it is real-analytic everywhere [16].

Consider the following hyperbolic equation:

$$\begin{cases} \partial_t^2 y - \Delta y & = & 0 & \text{in} & \Omega \times (0,T), \\ y(x,0) & = & y_0(x) & \text{in} & \Omega, \\ \partial_t y(x,0) & = & y_1(x) & \text{in} & \Omega, \\ y(x,t) & = & 0 & \text{on} & \Gamma \times (0,T), \end{cases} \tag{1.32}$$

we have the following result

Theorem 5 *If $y = y(x,t)$ satisfy (1.32) for $(y_0, y_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ and we have*

$$\frac{\partial y}{\partial \nu} = 0 \quad on \quad \Gamma_0 \times (0,T).$$

Then

$$y = 0 \quad in \quad \Omega \times (0,T).$$

Proof. [31] ■

Carleman inequalities

Carleman estimates are used to study inverse problems. They provide a way to estimate the stability of solutions to inverse problems and can be used to determine the uniqueness and stability of the solution to an inverse problem.

The Carleman inequalities are a set of inequalities that play an important role in the study of inverse problems and partial differential equations (PDEs). They are named after the Swedish mathematician Torsten Carleman, who introduced them in the 1920s.

The main idea behind the Carleman inequalities is to use a weighting function to provide control over the solution of a PDE. By constructing a suitable weighting function, one can re-write the PDE in a new form in which the solution can be estimated more easily.

For a clearer view of its many applications to inverse problems see [5].

1.2 An overview about the sentinel method

1.2.1 Notion of the sentinel

The notion of sentinels introduced by J. L. LIONS (1992) is based on three considerations:

- The state of a system governed by a boundary value problem with incomplete data;
- A system for observing the evolution of the system;

- A functional, called sentinel, is intended to detect missing data.

The following general introduction is devoted to a formal presentation of these three points. Their development is the subject of this thesis.

– Boundary value problem with incomplete data

To introduce the definition of sentinel, let's consider the following situation. We suppose that y is the solution of the following equation, where $\mathcal{A}$ is elliptic of the 2nd order given by

$$\frac{\partial y}{\partial t} + \mathcal{N} = \xi + \lambda\hat{\xi} \quad \text{in} \quad \Omega \times (0,T), \tag{1.33}$$

where ξ is given in a suitable space, denotes by Y and $\lambda\hat{\xi}$ is not knownand with λ small.

We also assume (for the moment) that the boundary conditions are known, for example

$$y = 0 \quad \text{on} \quad \Gamma \times (0,T). \tag{1.34}$$

We suppose that the coefficients of $\mathcal{A}$ and that the open set Ω are known, but that the initial data are incomplete. If we designate by $y(x,0)$ the initial condition is expressed in the form

$$y(x,0) = y_0 + \tau\hat{y_0} \quad \text{in} \quad \Omega, \tag{1.35}$$

where y_0 is given, and $\hat{y_0}$ remains in the unit ball of a suitable Hilbert or Banach space and with τ small.

We aim to give a method allowing to obtain information on $\lambda\hat{\xi}$ which is not affected by the variations of the initial data. We also establish a distinction between the term $\lambda\hat{\xi}$ which is called pollution term "important term" and the term $\tau\hat{y_0}$ which is called incomplete data "unimportant term" and which we do not seek to identify.

Naturally, to hope to be able to obtain some information, it is necessary to get additional information on the state y; "observe y".

– State observation

The problem of estimating missing data consists of observing the state y of a system on a partially accessible domain and having experimental measurements. It is assumed that all observations are made in a time interval $(0, T)$ and in a domain (arbitrarily small) called the observatory $\mathcal{O}$. We define $\mathcal{O}$ and $y(x, t, \lambda, \tau) = y_{obs}$ on $\mathcal{O}$. Several types of observations can be distinguished depending on the types of observatories.

- distributed

$$\mathcal{O} \subset \Omega. \tag{1.36}$$

- Boundary

$$\mathcal{O} \subset \Gamma = \partial\Omega. \tag{1.37}$$

- We can also consider observatories dependent on time

$$\mathcal{O} = \mathcal{O}(t), \quad t \in (0, T). \tag{1.38}$$

- ponctual observations refer to the situation where measurements of the system's state are taken at specific points in the observatory domain $\mathcal{O}$. In this case, the available data consists of a set of discrete values of yobs at specific spatial locations within $\mathcal{O}$, and at specific times. The goal of estimating missing data in this scenario is to infer the unobserved values of y outside of the measured points, using the available data and any available knowledge about the underlying physics or dynamics of the system.

To fix ideas, consider the case of distributed observation (1.36), we assume the observed state on $\mathcal{O} \times (0, T)$, and we therefore have

$$y(x, t; \lambda, \tau) = m_0(x, t) \quad \text{on} \quad \mathcal{O} \times (0, T), \tag{1.39}$$

where m_0 is given.

– The sentinel functional

Now, our inverse problem is formulated as follows

Can we obtain information from the data of m_0 about the important term $\lambda\hat{\xi}$ that is independent of the unimportant term $\tau\hat{y}_0$?

A standard idea is to take an average value. Let h_0 be a function given on $\mathcal{O}\times(0,T)$ such that

$$h_0 \geq 0, \quad \int_0^T \int_{\mathcal{O}} h_0(x,t)dxdt = 1. \tag{1.40}$$

We then consider

$$\mathcal{N}(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} h_0(x,t)y(\lambda,\tau)dxdt. \tag{1.41}$$

We seek to determine the important term $\lambda\hat{\xi}$ independently of the unimportant term $\tau\hat{y}_0$, but there is in general no reason for $\mathcal{A}$ to be independent at the first order of $\mathcal{N}(\lambda,\tau)$ should be independent of τ. In other words, there is no reason to get

$$\frac{\partial \mathcal{N}}{\partial \tau}(\lambda,\tau) = 0. \tag{1.42}$$

The idea of Jacque Louis Lions was to add another term in (1.41) and we set

$$S(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} (h_0+u)y(\lambda,\tau)dxdt, \tag{1.43}$$

where $u = u(x,t)$ is a function to be determined such that:

$$\frac{\partial S}{\partial \tau}(\lambda,\tau)\bigg|_{\lambda=0,\tau=0} = 0, \tag{1.44}$$

and

$$\|u\|_{L^2(\mathcal{O}\times(0,T))} = \text{minimum}. \tag{1.45}$$

The functional defined by $(1.43)-(1.45)$ is called the sentinel.

Remark 5

– *Condition* (1.44)*, expresses the insensitivity of the functional S with respect to τ (to the first order), and condition* (1.45) *expresses that we move away from the mean "as little as possible".*

– *The choice $u = -h_0$ gives rise to* (1.44)*. Therefore, under very general assumptions, the problem* $(1.44)-(1.45)$ *admits a unique solution. But it will be necessary to ensure* *that under suitable conditions $u \neq -h_0$, the functional $S(\lambda, \tau)$ not being likely to bring us much information.*

1.2.2 Description of the sentinel method

The sentinel method, proposed by Jacques-Louis Lions, is a powerful tool for solving certain types of inverse problems. The method is based on the existence of a sentinel functional, which is closely related to the solution of an optimal control problem.

The idea is to first using the adjoint problems to obtain the equivalence between the existence of a sentinel functional and an optimal control problem. Then, solve the optimal control problem, which allows us to define the control function by using various method (The Hilbert Uniqueness method in the hyperbolic case, the penalization method in the parabolic case, ...). Once we have the control function, it becomes easier to obtain some information about the important term in the equation which we are going to identify. This important term is often a source term or a boundary/initial condition or a coefficient in the main operator that is not explicitly known.

Exemple: The sentinel method to a parabolic equation with incomplete data

Let $\Omega \subset R^n$, $n \geq 1$ be a bounded open subset, with $\partial\Omega = \Gamma$ of class C^∞. For a fixed time $T > 0$, let's consider $Q = \Omega \times]0, T[$ the space-time cylinder, and the lateral surface $\Sigma = \Gamma \times]0, T[$.

Let $y = y(x, t)$ be the solution of the following equation with incopmlete data:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} + Ay + f(y) & = & \xi + \lambda\widehat{\xi} & \text{in} & Q, \\ y(x, 0) & = & y_0(x) + \tau\widehat{y}_0(x) & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Sigma. \end{cases} \tag{1.46}$$

where:

- The source term is not known, but we know its structure of the form $\xi + \lambda\widehat{\xi}$, with

$$\xi \ \textbf{ is known in } \ L^2(Q) \ \text{ and } \ \lambda\widehat{\xi} \ \textbf{ is unknown in } \ L^2(Q).$$

- The initial condition is not known, but we know its structure of the form $y_0(x) + \tau\widehat{y}_0(x)$, with

$$y_0 \text{ **is known in** } L^2(\Omega) \text{ and } \tau\widehat{y}_0 \text{ **is unknown in** } L^2(\Omega).$$

- The reals λ and τ are arbitrarily small.

The inverse problem: Can we get information about the $\lambda\widehat{\xi}$ (the important term) without any traying to calculate $\tau\widehat{y}_0$ (The unimportant term) from the knowledge of y in $O \times (0, T)$?

Remark 6

−We can commute the situation by assuming

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} + Ay + f(y) & = & \xi + \tau\widehat{\xi} & in \quad Q, \\ y(x,0) & = & y_0(x) + \lambda\widehat{y}_0(x) & in \quad \Omega, \\ y & = & 0 & on \quad \Sigma \end{cases}$$

we are looking for information on $\lambda\widehat{y}_0$ (important term) independent of $\tau\widehat{\xi}$(unimportant term).

Define the sentinel functional

Let $h_0 \in O \times (0, T)$ we define S as follow

$$S(\lambda, \tau) = \int_0^T \int_O (h_0 + u) y(x, t; \lambda, \tau) dx dt \tag{1.47}$$

where $u \in O \times (0, T)$ is the control function.

Definition 9 *We say that S is a sentinel of Lions defined by h_0, if there exists a control u such that the pair (u, S) verifies:*

− The condition of insensitivity with respect to missing terms, i.e.

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \forall \hat{y}_0 \in L^2(\Omega). \tag{1.48}$$

– *u with minimal norm, i.e.*

$$\|u\|^2_{L^2(O\times(0,T))} = min. \tag{1.49}$$

Equivalence to an optimal control problem

The condition of "insensitivity" of the sentinel with respect to the missing terms is equivalent to

$$\int_0^T \int_O (h_0 + u) y_\tau(x,t;\lambda,\tau) dxdt = 0, \tag{1.50}$$

where $y_\tau = \frac{\partial y}{\partial \tau}$ is solution of the equation

$$\begin{cases} \frac{\partial y_\tau(x,t)}{\partial t} + Ay_\tau + f'(y) & = & 0 & \text{in} & Q, \\ y_\tau(x,0) & = & \widehat{y}_0(x) & \text{in} & \Omega, \\ y_\tau & = & 0 & \text{on} & \Sigma. \end{cases}$$

Let $q = q(x,t)$ be the adjoint state which is the solution of the following backward problem

$$\begin{cases} \frac{\partial q(x,t)}{\partial t} + Aq + f'(y)q & = & (h_0 + u)\chi_O & \text{in} & Q, \\ q(x,T) & = & 0 & \text{in} & \Omega, \\ q & = & 0 & \text{on} & \Sigma. \end{cases} \tag{1.51}$$

Proposition 6 *The existence of the sentinel* (1.47) *is equivalent to solving the optimal control problem* (1.51) *with* (1.49) *which satisfies*

$$q(x,0) = 0. \tag{1.52}$$

Proof. It's easy to see, by multiplying the first equation of the system (1.51) by y_τ and then integrating by parts with some calculations ■

Study the optimal control problem

The problem of finding a sentinel S such that (1.51) holds is equivalent to the following optimal control problem:

{find u with minimal norm in $L^2(O \times (0,T))$ such that we have (1.51) and (1.52)}.

A classical method to solving this problem is the penalization method see [9].

Information given by the sentinel

If we assume that the state $y(\lambda,\tau)$ depends differently on λ and τ, we can write formally

$$S(\lambda,\tau) \simeq S(0,0) + \frac{\partial S}{\partial \lambda}(0,0),$$

(since by definition of the sentinel $\frac{\partial S}{\partial \tau}(0,0) = 0$). Using (1.39), we can therefore write

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) \simeq \int_0^T \int_{\mathcal{O}} (h_0 + u)(m_0 - y_0)\,dxdt.$$

where y_0 is the solution of (1.46) with $\lambda = \tau = 0$.

Chapter 2

Sentinel method for some hyperbolic identification problems

In this second chapter, our objective is to study the hyperbolic system with missing data in the sense that we don't know the initial data and some coefficients in the main operator. We will apply the sentinel method to obtain information on the important terms that are not affected by the variations of the unimportant term around the initial data. We have divided this chapter into two sections, in the first section, we applied the sentinel method to get some information about the potential coefficient independently of the initial conditions in the wave equation [17]. In the second, we applied the sentinel method for an acoustic wave equation with incomplete data to identify the bulk modulus coefficient from boundary observation [18].

2.1 Identifying the potential coefficient in a wave equation with incomplete data

The aim of this section is to apply the sentinel method for a wave equation with incomplete data, where we do not know the potential coefficient and the initial conditions. Actually, we want to obtain some information on the potential coefficient independently of the initial conditions from an observation of the data in the boundary. gives us through the adjoint system that the existence of the sentinel is equivalent to an optimal control problem. We solve this optimal control problem by using the Hilbert uniqueness method (HUM) [17].

2.1.1 Position of the problem

The direct problem

Let $\Omega \subset \mathbb{R}^n$, be a bounded open domain, its boundary $\partial\Omega = \Gamma$ be of class C^2. For fixed time $T > 0$, we take $Q = \Omega \times [0, T]$, and $\Sigma = \Gamma \times [0, T]$, we consider the following initial-boundary value problem for the wave equation

$$\begin{cases} \partial_t^2 y - \Delta y + p(x)\, y & = & 0 & \text{in} & Q, \\ y(x,0) & = & f(x) & \text{in} & \Omega, \\ \partial_t y(x,0) & = & g(x) & \text{in} & \Omega, \\ y(x,t) & = & 0 & \text{on} & \Sigma, \end{cases} \tag{2.1}$$

where $p \in L^\infty(\Omega)$ is the potential coefficient only dependent on x, and $(f, g) \in H_0^1(\Omega) \times L^2(\Omega)$. Under these assumptions, there exists a unique solution y to (2.1) such that $y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. This is the direct problem for the wave equation [5], [52].

Physically, the equation (2.1) describes the interaction of the medium with the disturbance if $y(x,t)$ is a measure of the magnitude of the disturbance at the point x and time t, $p(x)$ is a spatial coefficient representing some physical property of the medium. Here the function f and g indicates that the medium is initially disturbed [49].

The inverse problem

We assume that the obtained model contains a potential coefficient $p(x)$ that isn't completely known, we can write it in the form

$$p(x) = p_0(x) + \lambda \widehat{p_0}(x),$$

where $p_0 \in L^\infty(\Omega)$ is given (known), and the term $\lambda \widehat{p_0}$ so-called important term is unknown.

As well as the initial conditions, have the form

$$f(x) = y_0(x) + \tau_0 \widehat{y}_0(x),$$

and

$$g(x) = y_1(x) + \tau_1 \widehat{y}_1(x),$$

where the function $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ both are known, and where $\tau_0 \widehat{y}_0, \tau_1 \widehat{y}_1$ so-called unimportant term, both are unknown and we assume that $\|\widehat{y}_0\|_{H_0^1(\Omega)} \leq 1$, $\|\widehat{y}_1\|_{L^2(\Omega)} \leq 1$.

The parameters λ, τ_0 and τ_1 are real numbers sufficiently small.

We formulate our inverse problem as follows

Can we obtain information on the unknown important term $\lambda \widehat{p_0}(x)$ without any attempt at computing the two unimportant terms $\tau_0 \widehat{y}_0(x)$ and $\tau_1 \widehat{y}_1(x)$ of the initial data in the following equation

$$\begin{cases} \partial_t^2 y - \Delta y + (p_0 + \lambda \widehat{p_0})\, y &= 0 & \text{in} \quad Q, \\ y(x,0) &= y_0(x) + \tau_0 \widehat{y}_0(x) & \text{in} \quad \Omega, \\ \partial_t y(x,0) &= y_1(x) + \tau_1 \widehat{y}_1(x) & \text{in} \quad \Omega, \\ y(x,t) &= 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.2}$$

from the knowledge of solution measured on a non-empty part of the boundary $O \subset \Gamma$

$$y_{obs} = \frac{\partial y}{\partial \nu}(x,t) \quad \text{for all} \quad (x,t) \in O \times [0,T], \tag{2.3}$$

where $\frac{\partial y}{\partial \nu}(x,t) = \sum_{i=1}^{n} \nu_i(x) \frac{\partial y}{\partial x_i}(x,t)$ is the normal derivative, and $\nu(x)$ is the unit outward normal vector to Γ at x.

The driving force behind this issue is the aim to obtain physical characteristics, such as the density of an inhomogeneous medium, without taking into account the medium's interaction with the initial perturbation. This can be achieved by observing the medium with perturbations created at its boundary.

2.1.2 Definition of the sentinel

Let h_0 is a given function on $O \times (0,T)$, we define the functional S as follow:

$$S(\lambda, \tau_0, \tau_1) = \int_0^T \int_O h_0 \frac{\partial y}{\partial \nu}(x,t,\lambda,\tau_0,\tau_1)\, d\Gamma dt$$

$$+ \int_0^T \int_\Gamma u \frac{\partial y}{\partial \nu}(x,t,\lambda,\tau_0,\tau_1)\, d\Gamma dt,$$

where $u \in L^2(\Gamma \times [0,T])$ a control function is to be found.

The functional $S(\lambda, \tau_0, \tau_1)$ can be written in a compact form as follows

$$S(u;\lambda,\tau_0,\tau_1) = \int_0^T \int_\Gamma (h_0 \chi_O + u) \frac{\partial y}{\partial \nu} d\Gamma dt, \tag{2.4}$$

where χ_O denote now and in the sequel characteristic function for the open subset O.

The notion of sentinel introduced by J. L. Lions in [33] is as follows

Definition 10 $S(u;\lambda,\tau_0,\tau_1)$ *is said a sentinel defined by* h_0 *if these conditions are satisfied:*

1) S *is stationary to the first order with respect to unimportant term* $\tau_0 \widehat{y}_0$, $\tau_1 \widehat{y}_1$

$$\frac{\partial S}{\partial \tau_0}(0,0,0) = 0, \qquad \textit{and} \qquad \frac{\partial S}{\partial \tau_1}(0,0,0) = 0. \tag{2.5}$$

2) *The control* u *is of minimal norm in* $L^2(\Gamma \times (0,T))$ *among "the admissible controls," i.e.*

$$\|u\|^2_{L^2(\Gamma\times(0,T))} = \min_{\widetilde{u} \in U_{ad}} \|\widetilde{u}\|^2_{L^2(\Gamma\times(0,T))}, \tag{2.6}$$

where

$$U_{ad} = \left\{ \widetilde{u} \in L^2(\Gamma \times [0,T]),\ \textit{such that}\ (\widetilde{u}, S(\widetilde{u}))\ \textit{satisfies}\ (2.5) \right\}.$$

2.1.3 Equivalence to an optimal control problem

In this subsection, we can show that having such control function u satisfying (2.5) and (2.6), is equivalent to an optimal control problem.

Denote by

$$y_{\tau_0} = \frac{\partial y}{\partial \tau_0}(\lambda,\tau_0,\tau_1)\bigg|_{\lambda=0,\tau_0=0,\tau_1=0},$$

and

$$y_{\tau_1} = \frac{\partial y}{\partial \tau_1}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0},$$

the solution of the systems

$$\begin{cases} \partial_t^2 y_{\tau_0} - \Delta y_{\tau_0} + p_0(x)\, y_{\tau_0} & = & 0 & \text{in} & Q, \\ y_{\tau_0}(x,0) & = & \widehat{y}_0(x) & \text{in} & \Omega, \\ \partial_t y_{\tau_0}(x,0) & = & 0 & \text{in} & \Omega, \\ y_{\tau_0}(x,t) & = & 0 & \text{on} & \Sigma, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \partial_t^2 y_{\tau_1} - \Delta y_{\tau_1} + p_0(x)\, y_{\tau_1} & = & 0 & \text{in} & Q, \\ y_{\tau_1}(x,0) & = & 0 & \text{in} & \Omega, \\ \partial_t y_{\tau_1}(x,0) & = & \widehat{y}_1(x) & \text{in} & \Omega, \\ y_{\tau_1}(x,t) & = & 0 & \text{on} & \Sigma, \end{cases} \tag{2.8}$$

respectively.

It is immediately deduced that the insensibility condition (2.5) can be rewritten in a different form as follows

$$\int_0^T \int_\Gamma (h_0 \chi_O + u) \frac{\partial y_{\tau_0}}{\partial \nu} d\Gamma dt = 0, \tag{2.9}$$

and

$$\int_0^T \int_\Gamma (h_0 \chi_O + u) \frac{\partial y_{\tau_1}}{\partial \nu} d\Gamma dt = 0. \tag{2.10}$$

Now, let's introduce the adjoint state q given by

$$\begin{cases} \partial_t^2 q - \Delta q + p_0(x)\, q & = & 0 & \text{in} & Q, \\ q(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t q(x,T) & = & 0 & \text{in} & \Omega, \\ q(x,t) & = & h_0 \chi_O + u & \text{on} & \Sigma. \end{cases} \tag{2.11}$$

Then, we have the following proposition

Proposition 7 *Let q be the solution of* (2.11)*. Then, having a sentinel defined in* (2.4) *of the equation* (2.2)*, is equivalent to finding a control u that makes the adjoint state q*

verifies the following null controllability property

$$\partial_t q\,(x,0) = q\,(x,0) = 0 \quad in \quad \Omega. \tag{2.12}$$

Proof. By multiplying both sides of the first equation in (2.11) by y_{τ_0} solution of (2.9) and by integrating by parts and applying Green's formula, we get the following

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \left(\partial_t^2 q - \Delta q + p_0 q\right) y_{\tau_0}\, dx dt \\
&= \int_\Omega \left[y_{\tau_0} \partial_t q\right]_0^T dx - \int_\Omega \left[q \partial_t y_{\tau_0}\right]_0^T dx \\
&\quad + \int_0^T \int_\Omega \left(\partial_t^2 y_{\tau_0} - \Delta y_{\tau_0} + p_0 y_{\tau_0}\right) q\, dx dt \\
&\quad + \int_0^T \int_\Gamma q \frac{\partial y_{\tau_0}}{\partial \nu} d\Gamma dt + \int_0^T \int_\Gamma y_{\tau_0} \frac{\partial q}{\partial \nu} d\Gamma dt.
\end{aligned}$$

Considering that y_{τ_0} is solution of (2.7), therefore

$$\left\langle \partial_t q\,(x,0)\,, \widehat{y}_0\,(x)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_0^T \int_\Gamma \left(h_0 \chi_O + u\right) \frac{\partial y_{\tau_0}}{\partial \nu} d\Gamma dt,$$

where $\langle .,.\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It follows from (2.9), that

$$\left\langle \partial_t q\,(x,0)\,, \widehat{y}_0\,(x)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0 \qquad \text{for all } \|\widehat{y}_0\|_{H_0^1(\Omega)} \leq 1.$$

Hence, we find that

$$\partial_t q\,(x,0) = 0, \qquad \text{in } H^{-1}(\Omega).$$

Consequently

$$\partial_t q\,(x,0) = 0, \qquad a.e \text{ in } \Omega.$$

In the same way as before, by multiplying both sides of the first equation in (2.11) by

y_{τ_1} solution of (2.8) and by integrating by parts and applying Green's formula, we find

$$\int_{\Omega} q\,(x,0)\,\widehat{y}_1\,(x)\,dx = \int_0^T \int_{\Gamma} (h_0\chi_O + u)\,\frac{\partial y_{\tau_1}}{\partial \nu} d\Gamma dt.$$

And from (2.10), we obtain

$$\int_{\Omega} q\,(x,0)\,\widehat{y}_1\,(x)\,dx = 0 \qquad \text{for all } \ \|\widehat{y}_1\|_{L^2(\Omega)} \leq 1.$$

So

$$q\,(x,0) = 0, \qquad \text{in } L^2(\Omega).$$

Therefore

$$q\,(x,0) = 0, \qquad a.e \text{ in } \Omega.$$

■

Remark 7 *The problem* (2.11) – (2.12) *with* (2.6) *is an optimal control problem, this problem has been studied by many authors, see for instance [14] and [58].*

2.1.4 Study of the optimal control problem

From the above, we conclude that having the sentinel functional is equivalent to solving an optimal control problem, which is equivalent to the existence of a unique pair (u, q) such that we have (15), (16) with (12).

Our main result is stated as follows

Theorem 8 *Given* $h_0 \in L^2\,(O \times [0, T])$ *and* $p_0\,(x) \in L^\infty\,(\Omega)$*. Then, there exists a control function* u *of minimal norm in* $L^2\,(\omega \times [0, T])$ *such that the solution* q *to problem* (2.11) *satisfies* (2.12).

Before proving this theorem, we introduce q_0 and z as solutions of

$$\left\{\begin{array}{rcccc} \partial_t^2 q_0 - \Delta q_0 + p_0(x)\, q_0 & = & 0 & \text{in} & Q, \\ q_0(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t q_0(x,T) & = & 0 & \text{in} & \Omega, \\ q_0(x,t) & = & h_0 \chi_O & \text{on} & \Sigma, \end{array}\right. \tag{2.13}$$

and

$$\left\{\begin{array}{rcccc} \partial_t^2 z - \Delta z + p_0(x)\, z & = & 0 & \text{in} & Q, \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t z(x,T) & = & 0 & \text{in} & \Omega, \\ z(x,t) & = & u & \text{on} & \Sigma, \end{array}\right. \tag{2.14}$$

respectively.

It's clear that

$$q(u) = q_0 + z(u).$$

We obtain that (2.12) is equivalent to

$$\left\{\begin{array}{c} z(u)(x,0) = -q_0(x,0), \\ \partial_t z(u)(x,0) = -\partial_t q_0(x,0) \quad \text{in } \Omega. \end{array}\right. \tag{2.15}$$

Thus the optimal control problem has a new form

Find a control function u of minimal norm in $L^2(\Gamma \times (0,T))$ such that the solution z of the system (2.14) satisfies (2.51).

To prove Theorem 1, we consider Φ solution of

$$\left\{\begin{array}{rcccc} \partial_t^2 \Phi - \Delta \Phi + p_0(x)\, \Phi & = & 0 & \text{in} & Q, \\ \Phi(x,0) & = & \Phi^0(x) & \text{in} & \Omega, \\ \partial_t \Phi(x,0) & = & \Phi^1(x) & \text{in} & \Omega, \\ \Phi(x,t) & = & 0 & \text{on} & \Sigma, \end{array}\right. \tag{2.16}$$

with $p_0 \in L^\infty(\Omega)$.

For all $(\Phi^0, \Phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique solution $\Phi = \Phi(\Phi^0, \Phi^1) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. Moreover, there exists a constant $C > 0$ independent of Φ^0 and Φ^1, such that

$$\left\| \frac{\partial \Phi}{\partial \nu} \right\|_{L^2(\Gamma \times [0,T])} \leq C \left(\left\| \Phi^0 \right\|_{H_0^1(\Omega)} + \left\| \Phi^1 \right\|_{L^2(\Omega)} \right) \tag{2.17}$$

holds [52].

Let us introduce for Φ solution of (2.16) this inequality called the observability inequality, which plays an important role to prove the controllability.

Proposition 9 *Let $p_0 \in L^\infty(\Omega)$ and*

$$T > 2\rho \qquad \text{with } \rho = \max_{x \in \overline{\Omega}} |x| . \tag{2.18}$$

Then there exists a constant $C = C(\Omega, T, a_0) > 0$ such that

$$\left\| \Phi^0 \right\|_{H_0^1(\Omega)} + \left\| \Phi^1 \right\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \Phi}{\partial \nu} \right\|_{L^2(\Gamma \times [0,T])}, \tag{2.19}$$

for all $\Phi^0 \in H_0^1(\Omega)$ and $\Phi^1 \in L^2(\Omega)$.

Proof. See [29], [30] and [52]. ∎

Remark 8 *The condition (2.18) is necessary for estimating two functions Φ^0 and Φ^1, and too much for determining either of Φ^0 and Φ^1.*

We introduce z solution of

$$\begin{cases} \partial_t^2 z - \Delta z + p_0(x) z &= 0 \quad \text{in} \quad Q, \\ z(x,T) &= 0 \quad \text{in} \quad \Omega, \\ \partial_t z(x,T) &= 0 \quad \text{in} \quad \Omega, \\ z(x,t) &= \frac{\partial \Phi}{\partial \nu} \quad \text{on} \quad \Sigma. \end{cases} \tag{2.20}$$

Now, we go back to prove the previous Theorem, we divide the proof into three steps.

Proof.

Step one: We define a linear operator Λ by

$$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow H^{-1}(\Omega) \times L^2(\Omega),$$

$$\Lambda\left\{\Phi^0,\Phi^1\right\}=\left\{-\partial_t z\left(0\right),z\left(0\right)\right\}, \tag{2.21}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Step two: By multiplying both side of the first equation of (2.20) by Φ solution of (2.16), and by integrating by parts and applying Green's formula, we obtain

$$\left\langle\Lambda\left\{\Phi^0,\Phi^1\right\},\left\{\Phi^0,\Phi^1\right\}\right\rangle=\int_0^T\int_\Gamma\left|\frac{\partial\Phi}{\partial\nu}\right|^2 d\Gamma dt,$$

where $\langle .,.\rangle$ is the duality product between $H_0^1(\Omega)\times L^2(\Omega)$ and $H^{-1}(\Omega)\times L^2(\Omega)$. Furthermore, Λ is clearly a positive and self-adjoint operator.

This leads to the introduction of the following semi norm

$$\left|\left\{\Phi^0,\Phi^1\right\}\right|_{H_0^1(\Omega)\times L^2(\Omega)}=\left(\int_0^T\int_\Gamma\left|\frac{\partial\Phi}{\partial\nu}\right|^2 d\Gamma dt\right)^{\frac{1}{2}}. \tag{2.22}$$

We want to show that the previous semi-norm (3.20) is a norm on the space set of initial data $\{\Phi^0,\Phi^1\}$.

We must only prove:

$$\int_0^T\int_\Gamma\left|\frac{\partial\Phi}{\partial\nu}\right|^2 d\Gamma dt=0 \implies \Phi=0 \text{ in } Q.$$

From the above proposition using the observability inequality (2.19), we can show that the previous semi-norm (3.20) is a norm, denoted by

$$\left\|\left\{\Phi^0,\Phi^1\right\}\right\|_{H_0^1(\Omega)\times L^2(\Omega)}=\left(\int_0^T\int_\Gamma\left|\frac{\partial\Phi}{\partial\nu}\right|^2 d\Gamma dt\right)^{\frac{1}{2}}. \tag{2.23}$$

Furthermore, it is clear from (2.17) and (2.19), that the norm (3.21) is equivalent to the usual norm of $H_0^1(\Omega)\times L^2(\Omega)$.

Step three: We must show that the operator Λ is an isomorphism from $H_0^1(\Omega)\times L^2(\Omega)$ to $H^{-1}(\Omega)\times L^2(\Omega)$. The norm (2.23) define by this scalar product $\left\langle\Lambda\left\{\widetilde{\Phi^0},\widetilde{\Phi^1}\right\},\{\Phi^0,\Phi^1\}\right\rangle$ and define a Hilbert space on the set of initial data, which is equivalent to (the Hilbert space) $H_0^1(\Omega)\times L^2(\Omega)$.

So by Riesz Representation Theorem we conclude that Λ is an isomorphism from

$H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$. So (2.21) has a unique solution given by

$$\left\{\Phi^0, \Phi^1\right\} = \Lambda^{-1}\left\{\partial_t q_0(0), -q_0(0)\right\}. \tag{2.24}$$

Subsequently, the control is given by

$$u = \frac{\partial \Phi}{\partial \nu} \text{ on } \Sigma, \tag{2.25}$$

where Φ is the solution of (2.16). ■

Finally, we have the existence of the sentinel (2.4) given in the following form

$$S(\lambda, \tau_0, \tau_1) = \int_0^T \int_\Gamma \left(h_0 \chi_O + \frac{\partial \Phi}{\partial \nu}\right) \frac{\partial y}{\partial \nu} d\Gamma dt. \tag{2.26}$$

2.1.5 Information given about the important term

Here, we are interested in estimating the important term, for this, we consider m_0 the measured state of the system on the observatory O during the interval $(0, T)$, then the sentinel observer associated with the state m_0 is given by

$$S_{obs}(\lambda, \tau_0, \tau_1) = \int_0^T \int_\Gamma \left(h_0 \chi_O + \frac{\partial \Phi}{\partial \nu}\right) m_0(x,t)\, d\Gamma dt. \tag{2.27}$$

Theorem 10 *The information obtained about the important term is given as follows*

$$-\int_0^T \int_\Omega \lambda \widehat{p_0}(x)\, y_0 q dx dt = S_{obs}(\lambda, \tau_0, \tau_1) - S(0,0,0), \tag{2.28}$$

where

$$S(0,0,0) = S(\lambda, \tau_0, \tau_1)|_{\lambda=0, \tau_0=0, \tau_1=0}, \text{ and } y_0 = y(\lambda, \tau_0, \tau_1)|_{\lambda=0, \tau_0=0, \tau_1=0}.$$

Proof. According to Taylor's formula, we have

$$S(\lambda, \tau_0, \tau_1) \simeq S(0,0,0) + \tau_0 \frac{\partial S}{\partial \tau_0}(0,0,0) + \tau_1 \frac{\partial S}{\partial \tau_1}(0,0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0,0), \quad \text{for } \lambda,\ \tau_0 \text{ and } \tau_1 \text{ small.} \tag{2.29}$$

Because $\left.\frac{\partial S}{\partial \tau_0}(\lambda, \tau_0, \tau_1)\right|_{\lambda=0, \tau_0=0, \tau_1=0} = \left.\frac{\partial S}{\partial \tau_1}(\lambda, \tau_0, \tau_1)\right|_{\lambda=0, \tau_0=0, \tau_1=0} = 0$ and $S(\lambda, \tau_0, \tau_1)$

is observed, then we have:

$$\lambda \frac{\partial S}{\partial \lambda}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} \simeq S_{obs}(\lambda, \tau_0, \tau_1) - S(0,0,0), \tag{2.30}$$

where

$$\frac{\partial S}{\partial \lambda}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} = \int_0^T \int_\Gamma (h_0 \chi_O + \frac{\partial \Phi}{\partial \nu}) \frac{\partial y_\lambda}{\partial \nu} d\Gamma dt, \tag{2.31}$$

with y_λ defined by $y_\lambda = \frac{\partial y}{\partial \lambda}(0,0,0)$, which is the unique solution of

$$\begin{cases} \partial_t^2 y_\lambda - \Delta y_\lambda + p_0 y_\lambda & = & -\widehat{p_0}(x)\, y_0 & \text{in} & Q, \\ y_\lambda(x,0) & = & 0 & \text{in} & \Omega, \\ \partial_t y_\lambda(x,0) & = & 0 & \text{in} & \Omega, \\ y_\lambda(x,t) & = & 0 & \text{on} & \Sigma. \end{cases} \tag{2.32}$$

Multiplying both sides of the first equation in (2.32) by $q = q(x,t)$ solution of (2.11)

$$\int_0^T \int_\Omega \left(\partial_t^2 y_\lambda - \Delta y_\lambda + p_0 y_\lambda\right) q dx dt = -\int_0^T \int_\Omega \widehat{p_0}(x)\, y_0 q dx dt.$$

By integrating by parts and applying Green's formula, we get

$$\int_0^T \int_\Gamma (h_0 \chi_O + \frac{\partial \Phi}{\partial \nu}) \frac{\partial y_\lambda}{\partial \nu} d\Gamma dt = -\int_0^T \int_\Omega \widehat{p_0}(x)\, y_0 q dx dt, \tag{2.33}$$

combining (2.33) and (2.30), we obtain the information about the important term given in the previous theorem.

■

2.2 Identifying the bulk modulus coefficient in the acoustic equation with incomplete data

In this section, we examine how to get some information about the bulk modulus coefficient in an acoustic equation when there is incomplete data, specifically when only partial information about the bulk modulus coefficient and initial conditions is available. The objective is to obtain information about the bulk modulus coefficient, disregarding the missing initial conditions, by using the solution measured on a part of the boundary. To accomplish this, we employ the sentinel method, which involves finding a sentinel functional that is equivalent to an optimal control problem. We use the HUM method to solve this problem, allowing us to determine the control of the minimal norm. Our study offers a physical perspective on determining the bulk modulus coefficient in an acoustic equation using boundary observations, emphasizing the significance of controlling the system's behavior to acquire precise information about material properties.

2.2.1 Position of the problem

The direct problem

Let $\Omega \subset \mathbb{R}^n$, an open bounded domain, its boundary Γ be of class C^2. For fixed time $T > 0$, we take $Q = \Omega \times [0, T]$, and $\Sigma = \Gamma \times [0, T]$. We consider the following acoustic wave equation, which describes the behavior of sound waves in a medium

$$\begin{cases} \partial_t^2 y - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right) & = & 0 & \text{in} & \Omega \times (0, T), \\ y(0) & = & B(x) & \text{in} & \Omega, \\ \partial_t y(0) & = & C(x) & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Gamma \times (0, T), \end{cases} \tag{2.34}$$

where $\partial_t^2 y = \frac{\partial^2 y}{\partial t^2}$, and $a_{ij}(x) = a_{ij}(x)$ for all $i, j = 1, ..., n$ are C^∞ function on $\mathbb{R}^n$, which satisfies the following uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i,j=1}^n \xi_i^2, x \in \Omega, \text{for some } \alpha > 0. \tag{2.35}$$

Given $B \in H_0^1(\Omega)$ and $C \in L^2(\Omega)$, the problem (2.34) a admits unique solution $y \in C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$ [53].

We assume that for all $i, j = 1, ..., n$ we have $a_{ij}(x) = a(x)\delta_{ij}$, where δ_{ij} is the Kroniker index, so our equation becomes written in the following divergence form

$$\begin{cases} \partial_t^2 y - div\,(a\,(x)\,\nabla y) & = & 0 & \text{in} & \Omega \times (0,T)\,, \\ y\,(0) & = & B(x) & \text{in} & \Omega, \\ \partial_t y\,(0) & = & C(x) & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Gamma \times (0,T)\,, \end{cases} \tag{2.36}$$

where $a \in C^1(\overline{\Omega})$, we assume that $a(x) > 0$ for all $x \in \overline{\Omega}$.

For equation (2.36), if the coefficient a, and the functions B and C are known, then we can prove that the problem has a unique solution, this is a direct problem.

The inverse problem

We're dealing with the identification of the coefficient a, where we assume that it's partially known, and has the form

$$a(x) = a_0\,(x) + \lambda \widehat{a}_0\,(x)\,,$$

where a_0 is given known, and the term $\lambda \widehat{a}_0$ is unknown and it's the important term.

Additionally, we assume that the initial conditions are partially known and that their forms are

$$B\,(x) = y_0\,(x) + \tau_0 \widehat{y}_0\,(x)\,,$$

and

$$C\,(x) = y_1\,(x) + \tau_1 \widehat{y}_1\,(x)\,,$$

where the functions $(y_0, y_1) \in H_0^1\,(\Omega) \times L^2\,(\Omega)$ are known, and where $\tau_0 \widehat{y}_0, \tau_1 \widehat{y}_1$ both are unknown, It's the unimportant terms, with $\|\widehat{y}_0\,(x)\|_{H_0^1(\Omega)} \leq 1$ and $\|\widehat{y}_1\,(x)\|_{L^2(\Omega)} \leq 1$.

The parameters λ, τ_0 and τ_1 are real numbers sufficiently small.

We are interested in the following inverse problem

Can we get information on the important term $\lambda \widehat{a}_0$ regardless of calculating the unimportant term $\tau_0 \widehat{y}_0$ and $\tau_1 \widehat{y}_1$ in the following equation

$$\begin{cases} \partial_t^2 y - div\left(\left(a_0\left(x\right)+\lambda\widehat{a}_0\left(x\right)\right)\nabla y\right) & = & 0 & \text{in} & \Omega\times\left(0,T\right), \\ y\left(0\right) & = & y_0+\tau_0\widehat{y}_0 & \text{in} & \Omega, \\ \partial_t y\left(0\right) & = & y_1+\tau_1\widehat{y}_1 & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Gamma\times\left(0,T\right), \end{cases} \tag{2.37}$$

from the knowledge of the conormal derivative

$$\left.\frac{\partial y}{\partial\nu_a}\right|_{O\times(0,T)}, \tag{2.38}$$

where O is a non-empty open subset of Γ (the observatory), and $\frac{\partial y}{\partial\nu_a} = a(x)\nabla y.\nu(x)$ is the conormal derivative, and $\nu\left(x\right)$ is the unit outward normal vector to Γ at x.

When we measure the conormal derivative on the boundary of a region, we are effectively measuring the rate of change of the acoustic pressure across that boundary. This rate of change is directly related to the normal acoustic velocity, which in turn depends on the bulk modulus coefficient. Therefore, by measuring the conormal derivative on the boundary of a medium, We can achieve our goal of obtaining some information [4], [10].

Physically, this inverse problem consists of the determination of the bulk modulus in the acoustic equation (2.37) which is considered in a nonhomogeneous medium. The bulk modulus is a fundamental physical parameter that describes the compressibility of a medium, or the resistance of a material to compression. In the context of the acoustic wave equation, the bulk modulus coefficient plays a crucial role in the behavior of sound waves as they propagate through that medium [54].

Determining the bulk modulus coefficient is important for a wide range of applications, including geophysics, medical imaging, and non-destructive testing. For example, in geophysics, the bulk modulus coefficient is used to map the subsurface of the earth, which is important for locating natural resources such as oil and gas. In medical imaging, the bulk modulus coefficient is used to determine the acoustic properties of tissue, which is important for imaging and diagnosing various diseases. In non-destructive testing, the bulk modulus coefficient is used to detect and identify flaws or defects in materials without damaging them, see [4], [10]. Also, we refer to [36] and [57].

2.2.2 Definition of sentinel

For $h_0 \in L^2(O \times [0,T])$, we have

$$S(\lambda, \tau_0, \tau_1) = \int_0^T \int_\Gamma (h_0 \chi_O + u\chi_\omega) \frac{\partial y}{\partial \nu_a} d\Gamma dt, \tag{2.39}$$

where u is a control function to be determined as follows

1) For all $\tau_0 \widehat{y}_0$ and $\tau_1 \widehat{y}_1$

$$\frac{\partial S}{\partial \tau_0}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} = \frac{\partial S}{\partial \tau_1}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} = 0. \tag{2.40}$$

2) The control u is of minimal norm in $L^2(\omega \times (0;T))$ among "the admissible controls", i.e.

$$\|u\|^2_{L^2(\omega\times(0,T))} = \min_{\widetilde{u}\in U_{ad}} \|\widetilde{u}\|^2_{L^2(\omega\times(0;T))}, \tag{2.41}$$

where

$$U_{ad} = \left\{ \widetilde{u} \in L^2(\omega \times (0,T)),\ \text{such that}\ \left(\widetilde{u}, S(\widetilde{u})\right)\ \text{satisfies (2.40)} \right\}.$$

2.2.3 Equivalence to an optimal control problem

We denote by $y_{\tau_0} = \frac{\partial y}{\partial \tau_0}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0}$ the solution of

$$\begin{cases} \partial_t^2 y_{\tau_0} - div\left(a_0(x)\nabla y_{\tau_0}\right) = 0 & \text{in } \Omega \times (0,T), \\ y_{\tau_0}(x,0) = \widehat{y}_0(x) & \text{in } \Omega, \\ \partial_t y_{\tau_0}(x,0) = 0 & \text{in } \Omega, \\ y_{\tau_0} = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{2.42}$$

and by $y_{\tau_1} = \frac{\partial y}{\partial \tau_1}(\lambda, \tau_0, \tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0}$ the solution of

$$\begin{cases} \partial_t^2 y_{\tau_1} - div\left(a_0(x)\nabla y_{\tau_1}\right) = 0 & \text{in } \Omega \times (0,T), \\ y_{\tau_1}(x,0) = 0 & \text{in } \Omega, \\ \partial_t y_{\tau_1}(x,0) = \widehat{y}_1(x) & \text{in } \Omega, \\ y_{\tau_1} = 0 & \text{on } \Gamma \times (0,T). \end{cases} \tag{2.43}$$

Remark 9 *The condition 1 (the insensitivity condition) could be written in the following form*

$$\int_{\Gamma\times(0,T)} (h_0\chi_O + u\chi_w)\frac{\partial y_{\tau_0}}{\partial \nu_a} d\Gamma dt = 0, \tag{2.44}$$

and

$$\int_{\Gamma\times(0,T)} (h_0\chi_O + u\chi_w)\frac{\partial y_{\tau_1}}{\partial \nu_a} d\Gamma dt = 0. \tag{2.45}$$

In this subsection, we will show that having the sentinel for (2.37) is related to solving an optimal control problem.

Here is our first main result

Proposition 11 *The existence of the sentinel defined in* (2.39) *for the problem* (2.37) *is related to solving the following null- controllability problem*

$$\begin{cases} \partial_t^2 q - div\,(a_0\,(x)\,\nabla q) & = & 0 & in & \Omega\times(0,T)\,, \\ q\,(x,T) & = & 0 & in & \Omega, \\ \partial_t q\,(x,T) & = & 0 & in & \Omega, \\ q & = & h_0\chi_O + u\chi_w & on & \Gamma\times(0,T)\,, \end{cases} \tag{2.46}$$

which satisfies the following property (null controllability property)

$$\begin{cases} q(x,0) & = & 0 \\ \partial_t q(x,0) & = & 0 \end{cases}, \tag{2.47}$$

in Ω.

Proof. By multiplying both sides of the first equation in (2.46) by y_{τ_0} solution of (2.42) and by integrating by parts and by an application of Green's formula, we find

$$\begin{aligned} 0 &= \int_0^T\int_\Omega \left(\partial_t^2 q - div\,(a_0\,(x)\,\nabla q)\right) y_{\tau_0} dx dt \\ &= \int_\Omega \left[y_{\tau_0}\partial_t q\right]_0^T dx - \int_\Omega \left[q\partial_t y_{\tau_0}\right]_0^T dx \\ &\quad + \int_0^T\int_\Omega \left(\partial_t^2 y_{\tau_0} - div\,(a_0\,(x)\,\nabla y_{\tau_0})\right) q dx dt \\ &\quad + \int_0^T\int_\Gamma q\frac{\partial y_{\tau_0}}{\partial \nu_a} d\Gamma dt + \int_0^T\int_\Gamma y_{\tau_0}\frac{\partial q}{\partial \nu_a} d\Gamma dt. \end{aligned}$$

Considering that y_{τ_0} is solution of (2.42), therefore

$$\langle \partial_t q\,(x,0)\,,\widehat{y}_0\,(x)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_0^T \int_\Gamma \left(h_0\chi_O + u\chi_\omega\right)\frac{\partial y_{\tau_0}}{\partial \nu}d\Gamma dt,$$

where $\langle .,.\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
It follows from (2.44), that

$$\langle \partial_t q\,(x,0)\,,\widehat{y}_0\,(x)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0 \qquad \text{for all } \|\widehat{y}_0\|_{H_0^1(\Omega)} \leq 1.$$

Hence, we find

$$\partial_t q\,(0) = 0, \qquad a.e. \text{ in } \Omega.$$

In the same way as before, multiplying both sides of the first equation in (2.46) by y_{τ_1} solution of (2.43) and by integrating by parts and by an application of Green's formula, we find

$$\int_\Omega q\,(0)\,\widehat{y}_1\,(x)\,dx = \int_0^T \int_\Gamma \left(h_0\chi_O + u\chi_\omega\right)\frac{\partial y_{\tau_1}}{\partial \nu_a}d\Gamma dt.$$

And from (2.45), we obtain

$$\int_\Omega q\,(0)\,\widehat{y}_1\,(x)\,dx = 0, \qquad \text{for all } \;\|\widehat{y}_1\,(x)\|_{L^2(\Omega)} \leq 1.$$

So

$$q\,(x,0) = 0, \qquad \text{in } L^2(\Omega).$$

Therefore

$$q\,(x,0) = 0, \qquad a.e \text{ in } \Omega.$$

■

2.2.4 Study of the optimal control problem

In the previous subsection, we demonstrated that the existence of a sentinel functional was related to the solution of the optimal control problem. For this reason, we are interested in solving this optimal control problem and will state our main result as follows

Theorem 12 *Given $h_0 \in L^2(O \times [0,T])$. Then, there exists a control function u of minimal norm in $L^2(\omega \times [0,T])$ such that the solution q of the problem* (2.46) *satisfies* (3.8) *and ω verifies the following geometrical condition*

$$\omega = \{x \in \Gamma, H(x).v(x) > 0\},$$

where $\nu(x)$ is the unit outward normal vector to Γ at x, and $H(x)$ is defined in [53].

The problem (2.46) - (3.8) is an optimal control problem, we use the method of HUM introduced by J. L. Lions [31] to establish this control with the minimal norm in $L^2(\omega \times [0,T])$, It's a constructive method, based on uniqueness results and on the construction of an isomorphism operator.

Note that we can write $q(u) = q_0 + z(u)$, where q_0 and z are solution of the systems

$$\left\{ \begin{array}{rclcc} \partial_t^2 q_0 - div\left(a_0(x)\nabla q_0\right) & = & 0 & \text{in} & \Omega \times (0,T), \\ q_0(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t q_0(x,T) & = & 0 & \text{in} & \Omega, \\ q_0 & = & h_0\chi_O & \text{on} & \Gamma \times (0,T), \end{array} \right. \tag{2.48}$$

and

$$\left\{ \begin{array}{rclcc} \partial_t^2 z - div\left(a_0(x)\nabla z\right) & = & 0 & \text{in} & \Omega \times (0,T), \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ \partial_t z(x,T) & = & 0 & \text{in} & \Omega, \\ z & = & u\chi_w & \text{on} & \Gamma \times (0,T), \end{array} \right. \tag{2.49}$$

respectively.

Obviously, (3.8) could be written

$$\begin{cases} q(x,0) &= z(u)(x,0) + q_0(x,0) \\ \partial q(x,0) &= \partial_t z(u)(x,0) + \partial_t q_0(x,0) \end{cases} \text{ in } \Omega. \tag{2.50}$$

Take into consideration that $z = z(u)$ solution of (2.49) is the state which satisfies the following

$$\begin{cases} z(u)(x,0) = -q_0(x,0) \\ \partial_t z(u)(x,0) = -\partial_t q_0(x,0) \end{cases} \text{ in } \Omega. \tag{2.51}$$

Introduce Φ solution of the following system

$$\begin{cases} \partial_t^2 \Phi - div\,(a_0(x)\nabla\Phi) &= 0 & \text{in} & \Omega \times (0,T), \\ \Phi(x,0) &= \Phi^0(x) & \text{in} & \Omega, \\ \partial_t \Phi(x,0) &= \Phi^1(x) & \text{in} & \Omega, \\ \Phi &= 0 & \text{on} & \Gamma \times (0,T), \end{cases} \tag{2.52}$$

where $(\Phi_0, \Phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$. The system (2.52) admits unique solution $\Phi \in C([0,T]; (H_0^1(\Omega) \times L^2(\Omega)))$. More precisely the following inequality, for all $T > 0$ and a constant $C > 0$

$$\int_{\Gamma_0 \times (0,T)} \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma dt \leq CT \left(||\Phi_0||^2_{H_0^1(\Omega)} + ||\Phi_1||^2_{L^2(\Omega)} \right), \tag{2.53}$$

holds [53].

Moreover, there exist a constants $c > 0$ such that

$$\int_{\Gamma_0 \times (0,T)} \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma dt \geq c \left(||\Phi_0||^2_{H_0^1(\Omega)} + ||\Phi_1||^2_{L^2(\Omega)} \right), \tag{2.54}$$

[53].

In addition, let Ψ be the solution to the following problem

$$\begin{cases} \partial_t^2 \Psi - div\,(a_0(x)\nabla\Psi) &= 0 & \text{in} & \Omega \times (0,T), \\ \Psi(x,T) &= 0 & \text{in} & \Omega, \\ \partial_t \Psi(x,T) &= 0 & \text{in} & \Omega, \\ \Psi &= \frac{\partial \Phi}{\partial \nu_a}\chi_w & \text{on} & \Gamma \times (0,T). \end{cases} \tag{2.55}$$

Now, we go back to prove the previous theorem

Proof. We define a linear operator Λ by

$$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow H^{-1}(\Omega) \times L^2(\Omega),$$

$$\Lambda\left\{\Phi^0, \Phi^1\right\} = \{-\partial_t z(x,0), z(x,0)\}, \tag{2.56}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Multiplying the both side of the first equation of (2.55) by Φ solution of (2.52), and by integrating by parts and by an application of Green's formula, we obtain

$$\left\langle \Lambda\left\{\Phi^0, \Phi^1\right\}, \left\{\Phi^0, \Phi^1\right\}\right\rangle = \int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma dt,$$

where $\langle .,. \rangle$ denotes the duality product between $H_0^1(\Omega) \times L^2(\Omega)$ and $H^{-1}(\Omega) \times L^2(\Omega)$. Moreover, it's clear that Λ is a positive and self-adjoint operator.

This leads to the introduction of the following semi norm

$$\left|\left\{\Phi^0, \Phi^1\right\}\right|_{H_0^1(\Omega)\times L^2(\Omega)} = \left(\int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma dt\right)^{\frac{1}{2}}. \tag{2.57}$$

To prove that Λ is an isomorphism, we have to show that the previous semi-norm (3.20) is a norm on the set of initial data $\{\Phi^0, \Phi^1\}$, we have to prove that

$$\int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma dt = 0 \implies \Phi = 0 \text{ in } Q.$$

Take $\omega = \Gamma_0$, from the inequality (2.54) (observability inequality), it is easy to show that the previous semi-norm is a norm, denoted by

$$\left\| \left\{ \Phi^0, \Phi^1 \right\} \right\|_{H_0^1(\Omega)\times L^2(\Omega)} = \left(\int\limits_0^T \int\limits_\omega \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma dt \right)^{\frac{1}{2}}. \tag{2.58}$$

Furthermore, it is clear from (2.53) and (2.54) that the norm is equivalent to the usual norm of $H_0^1(\Omega) \times L^2(\Omega)$.

We must show that the operator Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$ .The norm (3.21) define by this scalar product $\langle \Lambda \left\{ \widetilde{\Phi^0}, \widetilde{\Phi^1} \right\}, \{\Phi^0, \Phi^1\} \rangle$ and define a Hilbert space on the set of initial data, which is equivalent to (the Hilbert space) $H_0^1(\Omega) \times L^2(\Omega)$.

From Riesz Representation Theorem we conclude that is Λ an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$ [46].
So (2.56) has a unique solution given by

$$\left\{ \Phi^0, \Phi^1 \right\} = \Lambda^{-1} \left\{ \partial_t q_0(x, 0), -q_0(x, 0) \right\}. \tag{2.59}$$

Thus, the control is given by the restriction

$$u = \frac{\partial \Phi}{\partial \nu_a} \chi_\omega, \tag{2.60}$$

where Φ is the solution of (2.52). ∎

So, the sentinel (2.39) is given by

$$S(\lambda, \tau_0, \tau_1) = \int_0^T \int_\Gamma (h_0 \chi_O + \frac{\partial \Phi}{\partial \nu_a} \chi_\omega) \frac{\partial y}{\partial \nu_a} d\Gamma dt. \tag{2.61}$$

2.2.5 Information given about the imporant term

In this section, we will present the main result of this paper that allows giving information on the important term.

Let be $y_\lambda = \frac{\partial y}{\partial \lambda}$ a unique solution of the following system

$$\left\{ \begin{array}{rcl} \partial_t^2 y_\lambda - div\left(a_0\left(x\right)\nabla y_\lambda\right) - div\left(\widehat{a}_0\left(x\right)\nabla y_0\right) & = 0 & \text{in} \quad \Omega \times (0,T), \\ y_\lambda\left(x,0\right) & = 0 & \text{in} \quad \Omega, \\ \partial_t y_\lambda\left(x,0\right) & = 0 & \text{in} \quad \Omega, \\ y_\lambda & = 0 & \text{on} \quad \Gamma \times (0,T), \end{array} \right. \tag{2.62}$$

where y_0 is the solution of (2.37) when $\lambda = \tau_0 = \tau_1 = 0$.

On the other hand, we consider the sentinel associated to the measure m_0 given by

$$S_{obs}\left(\lambda,\tau\right) = \int_0^T \int_\Gamma \left(h_0\chi_O + u\chi_\omega\right) m_0\left(x,t,\lambda,\tau\right) dxdt, \tag{2.63}$$

where m_0 is the measured state of the system on the observatory O through the time interval $[0,T]$.

Theorem 13 *The information given by the sentinel about the important term $\lambda\widehat{a}_0\left(x\right)$ is as follows*

$$\int_0^T \int_\Omega div\left(\lambda\widehat{a}_0\left(x\right)\nabla y_0\right) q dxdt \simeq \int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega)\left(m_0 - \frac{\partial y_0}{\partial \nu_a}\right) d\Gamma dt. \tag{2.64}$$

Proof. According to Taylor's formula, we have

$$S\left(\lambda,\tau_0,\tau_1\right) \simeq S\left(0,0,0\right) + \tau_0 \frac{\partial S}{\partial \tau_0}\left(0,0,0\right) + \tau_1 \frac{\partial S}{\partial \tau_1}\left(0,0,0\right) + \lambda \frac{\partial S}{\partial \lambda}\left(0,0,0\right), \tag{2.65}$$

for λ, τ_0 and τ_1 small.

Due to (2.40), and by considering that $S\left(\lambda,\tau_0,\tau_1\right)$ is observed. Then, we have

$$\lambda \left.\frac{\partial S}{\partial \lambda}(\lambda,\tau_0,\tau_1)\right|_{\lambda=0,\tau_0=0,\tau_1=0} \simeq S_{obs}\left(\lambda,\tau_0,\tau_1\right) - S\left(0,0,0\right), \tag{2.66}$$

where

$$\frac{\partial S}{\partial \lambda} = \int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega) \frac{\partial y_\lambda}{\partial \nu_a} d\Gamma dt,$$

with y_λ is the unique solution of (2.62).

Multiplying both sides of the first equation in (2.62) by $q = q\left(x,t\right)$ solution of

(2.46), then we have

$$\int_0^T \int_\Omega \left\{ \partial_t^2 y_\lambda - div\left(a_0\left(x\right)\nabla y_\lambda\right) - div\left(\widehat{a}_0\left(x\right)\nabla y_0\right)\right\} q = 0.$$

Integrating by parts and by an application of Green's formula, we obtain

$$\int_0^T \int_\Omega div\left(\widehat{a}_0\left(x\right)\nabla y_0\right) q dx dt = \int_0^T \int_\Gamma (h_0 \chi_O + u\chi_\omega)\frac{\partial y_\lambda}{\partial \nu_a} d\Gamma dt, \tag{2.67}$$

combining (2.67) and (2.66) , we obtain the information about the important term given in (2.64).

■

Chapter 3

Sentinel method for identification problems of schrödinger and diffusion equations

In this third chapter, we use the sentinel method to get some information about the the potential coefficient in the Schrödinger equation with incomplete initial condition and the diffusion coefficient in diffusion equation with icomplete data.

3.1 Identifying the potential coefficient in the Schrödinger equation with incomplete initial condition

This section delves into an inverse problem of the Schrödinger equation, a fundamental equation in quantum mechanics. Specifically, we focus on incomplete data, where there are missing terms in the potential term and the initial condition. The potential term is a critical part of the equation, representing the potential energy of the system under investigation. Our objective is to obtain valuable information about this potential term without the need to determine the unknown initial condition. To achieve this, we employ the sentinel method. Our research shows that the existence of this functional is connected to solving an optimal control problem, which we accomplish using the Hilbert Uniqueness method. By using this approach, we are able to gain insights into the potential coefficient, which can provide significant benefits in a wide range of applications.

3.1.1 Position of the problem

Let $\Omega \subset \mathbb{R}^n$, an open bounded domain, its boundary Γ be of class C^2. We denote by $Q = \Omega \times (0,T)$, and by $\Sigma = \Gamma \times [0,T]$, for a fixed time $T > 0$.

Consider the following Schrödinger equation with incomplete data

$$\begin{cases} i\frac{\partial y}{\partial t} + \Delta y + (p_0(x) + \lambda \widehat{p}_0(x))y &= 0 & \text{in} & Q, \\ y\,(x,0) &= y_0(x) + \tau \widehat{y}_0(x) & \text{in} & \Omega, \\ y\,(x,t) &= 0 & \text{on} & \Sigma, \end{cases} \tag{3.1}$$

where $i = \sqrt{-1}$, and we assume that

- The potential coefficient has a structure $p_0(x) + \lambda \widehat{p}_0(x)$, where in this structure $p_0 \in L^\infty(\Omega)$ is known, whereas $\lambda \widehat{p}_0(x)$ is unknown, called the important term.
- The initial condition also has the same structure $y_0(x) + \tau \widehat{y}_0(x)$, where in this structure $y_0 \in H_0^1(\Omega)$ is known, whereas $\tau \widehat{y}_0(x)$ is unkhnown, called the unimportant term, and we have $\|\widehat{y}_0(x)\|_{H_0^1(\Omega)} \leq 1$.
- The parameters λ and τ are real numbers sufficiently small.

Giving some observation $\frac{\partial y}{\partial \nu} = y_{obs}$ on a non-empty part Γ_0 of the boundary Γ, where $\frac{\partial y}{\partial \nu}$ is the normal derivative, and $\nu(x)$ is the unit outward normal vector to Γ at x.
Problem: Without caring to evaluate the unimportant term $\tau \widehat{y}_0(x)$, can we obtain some information about the important term $\lambda \widehat{p}_0(x)$ from the knowledge of the observation y_{obs}?

This means that we are interested in determining the potential energy of the particle, from the knowledge of the flux on a part of the boundary, regardless of its initial state. [6].

3.1.2 Definition of the sentinel

The basic idea of the sentinel method is to build a functional S that achieves certain conditions, this functional links between the solutions of the state equation with the observation and a control function, defined as follow

Let h_0 be a well-known function in $L^2(\Gamma_0 \times (0,T))$, we define the sentinel functional S, which is depending to parameters λ and τ, as follows:

$$S(\lambda, \tau) = \int_0^T \int_{\Gamma_0} (h_0 + v) \frac{\partial \bar{y}}{\partial \nu} d\Gamma_0 dt, \tag{3.2}$$

where v is a control function defined on $L^2(\Gamma_0 \times (0,T))$ to be determined as follows

- S is insensitive at the first order with respect to the missed term $\tau \widehat{y}_0$, i.e.

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \qquad \forall v \in L^2(\Gamma_0 \times (0,T)), \tag{3.3}$$

- v has a minimal norm in $L^2(\Gamma_0 \times (0,T))$, i.e.

$$\|v\|_{L^2(\Gamma_0 \times (0,T))} = \min. \tag{3.4}$$

We use here and in the sequel the notations $(\bar{.})$ to indicate the conjugate of the complexe number.

3.1.3 Equivalence to an optimal control problem

We denote by $y_\tau = \frac{\partial y}{\partial \tau}$, the solution of the system

$$\begin{cases} i\frac{\partial y_\tau}{\partial t} + \Delta y_\tau + p_0(x) y_\tau &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ y_\tau(x,0) &= \widehat{y}_0(x) \quad \text{in} \quad \Omega, \\ y_\tau(x,t) &= 0 \quad \text{on} \quad \Gamma \times (0,T). \end{cases} \tag{3.5}$$

In this section, we shall transform sentinel existence problem (3.2) which satisfies (3.3)-(3.4) to an optimal control problem for the adjoint state.

We start by the insensibility condition (3.3), as from the definition of the sentinel

(3.2) and the condition (3.3), we have

$$\int_0^T \int_{\Gamma_0} (h_0 + v) \frac{\partial \bar{y}_\tau}{\partial \nu} d\Gamma_0 dt = 0, \tag{3.6}$$

where $\bar{y}_\tau$ denotes the complex conjugate of y_τ solution of (3.5).

We define the following system (adjoint system) by

$$\begin{cases} i\frac{\partial q}{\partial t} + \Delta q + p_0(x) q = 0 & \text{in} \quad \Omega \times (0,T), \\ q(x,T) = 0 & \text{in} \quad \Omega, \\ q(x,t) = \begin{cases} h_0 + v & \text{on} \quad \Gamma_0 \times (0,T) \\ 0 & \text{on} \quad \Gamma \diagdown \Gamma_0 \times (0,T). \end{cases} \end{cases} \tag{3.7}$$

Our first main result is the following

Proposition 14 *The existence of the sentinel defined in* (3.2) *for the system* (3.1) *is equivalent to solving the eqution* (3.38) *which satisfies the following property (nul controllability property)*

$$q(x,0) = 0 \ \text{in} \ \ \Omega. \tag{3.8}$$

Proof.

Multiplying the both sides of the first equation in (3.38) by $\bar{y}_\tau$, then

$$\int_0^T \int_\Omega \left(i\frac{\partial q}{\partial t} + \Delta q + p_0(x) q \right) \bar{y}_\tau dx dt = 0. \tag{3.9}$$

After using the integration by part with Green's formula over $\Omega \times (0,T)$, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \left(i\frac{\partial q}{\partial t} + \Delta q + p_0(x) q \right) \bar{y}_\tau dx dt \\ &= \int_\Omega i\bar{y}_\tau(x,T) q(x,T) dx - \int_\Omega i\bar{y}_\tau(x,0) q(x,0) dx \\ &+ \int_0^T \int_\Omega \overline{\left(i\frac{\partial y_\tau}{\partial t} + \Delta y_\tau + p_0 y_\tau \right)} q dx dt \\ &+ \int_0^T \int_\Gamma \bar{y}_\tau \frac{\partial q}{\partial \nu} d\Gamma dt - \int_0^T \int_\Gamma q \frac{\partial \bar{y}_\tau}{\partial \nu} d\Gamma dt. \end{aligned}$$

Considering that y_τ is solution of (3.5), therefore

$$i\left\langle q\left(x,0\right),\bar{\bar{y}}_0\left(x\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}=\int_0^T\int_{\Gamma_0}\left(h_0+u\right)\frac{\partial\bar{y}_\tau}{\partial\nu}d\Gamma_0dt,$$

where $\langle .,.\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

From (3.6), we get

$$i\left\langle q\left(x,0\right),\bar{\bar{y}}_0\left(x\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}=0 \qquad \text{for all } \left\|\widehat{\bar{y}}_0\right\|_{H_0^1(\Omega)}\leq 1.$$

Hence, we find that

$$q\left(x,0\right)=0, \qquad \text{in } H^{-1}(\Omega).$$

Consequently

$$q\left(x,0\right)=0, \qquad a.e \text{ in } \Omega.$$

■

3.1.4 Study of the optimal control problem

Through the previous section, we found that the existence of the sentinel (3.2) is related to solving an optimal control problem (3.38)-(3.8). In this section, we will study the optimal control problem (3.38)-(3.8) by applying a method introduced by J. L. Lions [31] called the Hilbert Uniqueness method (HUM), which is systematic method, based on a reverse inequality (uniqueness result) and Hilbert space to be created.

Let's state our second main result as follow

Theorem 15 *Under some geometrical conditions upon Γ_0, find a control function v of minimal norm in $L^2\left(\Gamma_0\times(0,T)\right)$ such that the solution q of the system* (3.38) *satisfies* (3.8).

We make the decomposition $q=q_0+z$, where q_0 and z are solution of the following systems

$$\begin{cases} i\frac{\partial q_0}{\partial t} + \Delta q_0 + p_0(x)q_0 & = & 0 & \text{in} & \Omega \times (0,T), \\ q_0(x,T) & = & 0 & \text{in} & \Omega, \\ q_0(x,t) & = & \begin{cases} h_0 & \text{on} & \Gamma_0 \times (0,T), \\ 0 & \text{on} & \Gamma \diagdown \Gamma_0 \times (0,T), \end{cases} \end{cases} \tag{3.10}$$

and

$$\begin{cases} i\frac{\partial z}{\partial t} + \Delta z + p_0(x)z & = & 0 & \text{in} & \Omega \times (0,T), \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ z(x,t) & = & \begin{cases} v & \text{on} & \Gamma_0 \times (0,T), \\ 0 & \text{on} & \Gamma \diagdown \Gamma_0 \times (0,T), \end{cases} \end{cases} \tag{3.11}$$

respectively.

It's clear from (3.8) that

$$z(v)(x,0) = -q_0(x,0) \quad \text{in} \quad \Omega. \tag{3.12}$$

The optimal control problem we will address, has a new form as follows

Find a control function v of minimal norm in $L^2(\Gamma_0 \times (0,T))$ such that the solution z of the system (3.11) satisfies (3.12).

There are several ways to deal with this problem.

As is customary, we introduce Φ solution of the following system

$$\begin{cases} i\frac{\partial \Phi}{\partial t} + \Delta \Phi + p_0(x)\Phi & = & 0 & \text{in} & \Omega \times (0,T), \\ \Phi(x,0) & = & \Phi^0(x) & \text{in} & \Omega, \\ \Phi(x,t) & = & 0 & \text{on} & \Gamma \times (0,T). \end{cases} \tag{3.13}$$

Proposition 15 *For all $T > 0$ and under some geometrical conditions upon Γ_0, there exist two constants $C_1 > 0$, and $C_2 > 0$, such that*

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\sigma dt \leq C_1 \left\| \Phi^0 \right\|_{H_0^1(\Omega)}, \tag{3.14}$$

and

$$\left\| \Phi^0 \right\|_{H_0^1(\Omega)} \leq C_2 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\sigma dt, \tag{3.15}$$

hold for every Φ *solution of the problem* (3.13).

Also, we need this lemma

Lemma 1 *Let* $q \in C^2(\bar{Q}, \mathbb{R}^n)$, *for all solution of* (3.13), *with* $\Phi^0 \in H_0^1(\Omega)$ *the following identity holds*

$$\frac{1}{2}\int_0^T \int_\Gamma (q.\nu)\left|\frac{\partial \Phi}{\partial \nu}\right|^2 d\sigma dt = \frac{1}{2} Im \int_\Omega \left(\Phi q.\nabla\bar{\Phi}\right) dxdt\Big|_0^T + Im \int_0^T \int_\Omega \left(q_t.\nabla\Phi\bar{\Phi}\right) dxdt$$

$$+\frac{1}{2} Re \int_0^T \int_\Omega \left(\Phi\nabla(div_x q).\nabla\bar{\Phi}\right) dxdt + Re \int_0^T \int_\Omega \sum_{j,k}\left(\frac{\partial q_k}{\partial x_j}\frac{\partial\bar{\Phi}}{\partial x_k}\frac{\partial \Phi}{\partial x_j}\right) dxdt$$

$$Re \int_0^T \int_\Omega p_0 \Phi q.\nabla\bar{\Phi} dxdt + \frac{1}{2} Re \int_0^T \int_\Omega p_0 \left|\Phi\right|^2 (div_x q) dxdt, \tag{3.16}$$

where $div_x q = \sum_{j=1}^n \frac{\partial q_j}{\partial x_j}$, *and we denote by* Re *and* Im *the real and the imaginary part of the complexe number.*

Proof. It's easy to prove the lemma by multiplying (3.13) by $q.\nabla\bar{\Phi} + \frac{1}{2}\bar{\Phi}(div_x q)$, taking the real part and integrating it by parts and by an application of Green's formula, we get the identity (3.16). ■

We go back to prove the proposition (4).

Proof. We demonstrate each inequality in a step

Step 1 Proof of the inequality (3.1.4).

In (3.16), we choose $q = q(x) \in C^2(\bar{Q}, \mathbb{R}^n)$ such that $q.\nu = 1$ in Γ, it's easy to obtain

$$\begin{aligned}
\frac{1}{2}\int_0^T \int_\Gamma \left|\frac{\partial \Phi}{\partial \nu}\right|^2 d\sigma dt \;&\leq\; c_1 \left\|q\right\|_{L^\infty(\Omega)} \Big(\left\|\Phi(T)\right\|^2_{L^2(\Omega)} + \left\|\nabla\Phi(T)\right\|^2_{L^2(\Omega)} + \left\|\Phi(0)\right\|^2_{L^2(\Omega)} \\
+ \left\|\nabla\Phi(0)\right\|^2_{L^2(\Omega)}\Big) \;&+\; c_2 \left\|q\right\|_{W^{2,\infty}(\Omega)} \int_0^T \left\|\Phi(t)\right\|_{L^2(\Omega)} \left\|\nabla\Phi(t)\right\|_{L^2(\Omega)} dt \\
&+\; c_3 \left\|q\right\|_{W^{1,\infty}(\Omega)} \int_0^T \left\|\nabla\Phi(t)\right\|^2_{L^2(\Omega)} dt \\
&+\; c_4 \left\|q\right\|_{L^\infty(\Omega)} \left\|p_0\right\|_{L^\infty(\Omega)} \int_0^T \left\|\Phi(t)\right\|_{L^2(\Omega)} \left\|\nabla\Phi(t)\right\|_{L^2(\Omega)} dt \\
&+\; c_5 \left\|q\right\|_{W^{1,\infty}(\Omega)} \left\|p_0\right\|_{L^\infty(\Omega)} \int_0^T \left\|\Phi(t)\right\|^2_{L^2(\Omega)} dt,
\end{aligned}$$

by the conservation of energy, we can easily conclude that, there exists a constant C_1, such that

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\sigma dt \leq C_1 \left\| \Phi^0 \right\|_{H_0^1(\Omega)}.$$

Step 2 Proof of the inequality (3.15).

We refer to ■

We go back to prove our main result of Theorem

Proof. Let's consider z solution of the following problem

$$\begin{cases} i\frac{\partial z}{\partial t} + \Delta z + p_0(x)z & = & 0 & \text{in} & \Omega \times (0,T), \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ z(x,t) & = & \begin{cases} \frac{\partial \Phi}{\partial \nu} & \text{on} & \Gamma_0 \times (0,T), \\ 0 & \text{on} & \Gamma \diagdown \Gamma_0 \times (0,T). \end{cases} \end{cases} \tag{3.17}$$

We start by defining the linear continuous operator Λ by

$$\Lambda : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega),$$

$$\Lambda \Phi_0 = -iz(0), \tag{3.18}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Multiplying the both side of the first equation of (3.17) by $\bar{\Phi}$, taking the real part, integrating it by parts and by an application of Green's formula, we get

$$\langle \Lambda \Phi^0, \Phi^0 \rangle = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\sigma dt, \tag{3.19}$$

where $\langle .,. \rangle$ denotes the duality product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Moreover, it's clear that Λ is a positive and self-adjoint operator.

This leads to the introduction of the following semi norm

$$\left| \Phi^0 \right|_{H_0^1(\Omega)} = \left(\int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\sigma dt \right)^{\frac{1}{2}}. \tag{3.20}$$

We want to show that the previous semi-norm (3.20) is a norm, we must prove only that:

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu} \right|^2 d\Gamma dt = 0 \quad \text{then} \quad \Phi_0 = 0 \quad \text{in } Q.$$

From the inequality (3.15) (observability inequality), it is easy to show that the previous semi-norm is a norm, denoted by

$$\left\| \Phi^0 \right\|_{H_0^1(\Omega)} = \left(\int_0^T \int_{\Gamma_0} \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma dt \right)^{\frac{1}{2}}. \tag{3.21}$$

Furthermore, it is clear from (3.1.4) and (3.15) that the norm is equivalent to the usual norm of $H_0^1(\Omega)$.

It remains to show that the operator Λ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. The norm (3.21) define by the scalar product $\langle \Lambda \widetilde{\Phi^0}, \Phi^0 \rangle$ and define a Hilbert space on the set of initial data, which is equivalent to the norm of $H_0^1(\Omega)$.

From Riesz representation theorem we conclude that is Λ an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ [46].

So, (3.18) has a unique solution given by

$$\Phi^0 = i\Lambda^{-1} q_0(x, 0). \tag{3.22}$$

Thus, the control is given by the restriction on Γ_0

$$u = \frac{\partial \Phi}{\partial \nu} \chi_{\Gamma_0}, \tag{3.23}$$

where Φ is the solution of (3.13). ■

3.1.5 Information given about the important term

Let be $y_\lambda = \frac{\partial y}{\partial \lambda}$ the unique solution of the following system

$$\begin{cases} i\frac{\partial y_\lambda}{\partial t} + \Delta y_\lambda + p_0(x) y_\lambda + \widehat{p}_0(x) y_0 & = 0 \text{ in } \Omega \times (0,T), \\ y_\lambda(x,0) & = 0 \text{ in } \Omega, \\ y_\lambda(x,t) & = 0 \text{ on } \Gamma \times (0,T), \end{cases} \tag{3.24}$$

where y_0 is the solution of (3.1) when $\lambda = \tau = 0$.

In other hand, let's consider the sentinel S_{obs} defined by

$$S_{obs}(\lambda,\tau) = \int_0^T \int_{\Gamma_0} (h_0 + u)\, m_0(x,t,\lambda,\tau)\, d\Gamma_0 dt, \tag{3.25}$$

where m_0 is the measured state of the system on the observatory O through the interval $[0,T]$.

In this section, we will present a result that allows giving information on the important term.

Theorem 16 *The information obtained using the sentinel method about the important term $\lambda \widehat{p}_0(x)$ is*

$$\int_0^T \int_\Omega \lambda \widehat{p}_0(x)\, y_0 \bar{q}\, dx dt \simeq \int_0^T \int_{\Gamma_0} (h_0 + u)\left(m_0 - \frac{\partial \bar{y}_0}{\partial \nu}\right) d\Gamma_0 dt. \tag{3.26}$$

Proof. Acording to Taylor's formula, we have

$$S(\lambda,\tau) \simeq S(0,0) + \tau \frac{\partial S}{\partial \tau}(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0), \tag{3.27}$$

for λ and τ small.

Due to (3.3), and considering that $S(\lambda,\tau)$ is observed. Then, we have

$$\lambda \left.\frac{\partial S}{\partial \lambda}(\lambda,\tau)\right|_{\lambda=0,\tau=0} \simeq S_{obs}(\lambda,\tau) - S(0,0), \tag{3.28}$$

where

$$\frac{\partial S}{\partial \lambda} = \int_0^T \int_{\Gamma_0} (h_0 + u) \frac{\partial \bar{y}_\lambda}{\partial \nu} d\Gamma_0 dt,$$

with y_λ is the unique solution of (3.24).

Multiplying the both sides of the first equation in (3.24) by $\bar{q} = \bar{q}(x,t)$, then we have

$$\int_0^T \int_\Omega \left\{ i\frac{\partial y_\lambda}{\partial t} + \Delta y_\lambda + p_0(x)y_\lambda + \widehat{p}_0(x)y_0 \right\} \bar{q}dxdt = 0.$$

After using the integration it by part and by an application of Green's formula, we obtain

$$\int_0^T \int_\Omega \widehat{p}_0(x)y_0\bar{q}dxdt = \int_0^T \int_{\Gamma_0} (h_0 + u)\frac{\partial \bar{y}_\lambda}{\partial \nu} d\Gamma_0 dt, \tag{3.29}$$

combining (3.29) and (3.63) , we obtain the information about the important term given in the previous theorem.

■

3.2 Identifying the diffusion coefficient in a diffusion equation with incomplete data

In this section, we consider a diffusion equation with an unknown diffusion coefficient, we want to get information about the diffusion coefficient independently of the initial condition. For this aim, we use the sentinel method of Jacques Lious Lions.. We find that the existence of this sentinel is related to an optimal control problem, we solving this optimal control problem by the penalization method. The determination of the diffusion coefficient is crucial for understanding and predicting the behavior of many physical and biological systems.

3.2.1 Position of the problem

The direct problem

Let $\Omega \subset R^n$, $n \geq 1$ be a bounded open subset, with $\partial\Omega = \Gamma$ of class C^∞. For a fixed time $T > 0$, let's consider $Q = \Omega \times]0, T[$ the space-time cylinder, and the lateral surface $\Sigma = \Gamma \times]0, T[$.

Let $y = y(x,t)$ be the solution of the following semi-linear diffusion equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} - div(a_0(x) \nabla y(x,t)) + f(y) & = & 0 & \text{in} & Q, \\ y(x,0) & = & g(x) & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Sigma \end{cases} \tag{3.30}$$

where $a_0(x)$ is the diffusion coefficient.

In the above equation, we assume that

$$a_0 \in L^\infty(\Omega),$$

$$\text{for } \alpha_0, \alpha_1 > 0,\ \alpha_0 \leq a_0(x) \leq \alpha_1 \quad \text{almost everywhere in } \overline{\Omega},$$

$$f : \mathbb{R} \longrightarrow \mathbb{R} \text{ is nonlinear function in } C^2,$$

and

$$g \in L^2(\Omega).$$

Under the previous assumptions, the problem (3.30) has an unique solution y, such that

$$y \in H^1\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega)\right).$$

The diffusion equation (3.30) is a mathematical model that describes the behavior of a wide range of physical and biological systems where a substance or a property is transported from regions of high concentration to regions of low concentration due to random molecular motion. The equation has broad applications in various fields of science and engineering, including physics, chemistry, biology, and environmental science. It can be used to model the diffusion of gases, the spread of pollutants in the atmosphere, the transport of nutrients in biological tissues, and the flow of heat in materials.

Actually, the diffusion coefficient a_0 is a key parameter in the diffusion equation that describes the rate of diffusion of a substance in a given medium. It is a measure of how easily a substance can move through a particular medium, and it depends on various factors such as the size and shape of the molecule, the temperature and pressure of the system, and the interactions between the substance and the medium.

The inverse problem

The diffusion coefficient is often unknown in many physical and biological systems, making it challenging to accurately model and predict the behavior of these systems.

The determination of the diffusion coefficient is crucial for understanding and predicting the behavior of many physical and biological systems. Various methods have been developed to estimate or measure the diffusion coefficient, including experimental techniques, theoretical models, and numerical simulations. However, in many cases, the diffusion coefficient is partially known or unknown, which presents a significant challenge for modeling and prediction.

In this section, we are interested in the diffusion equation with diffusion coefficient that are not completely known $a_0(x) = a(x) + \lambda\widehat{a}(x)$, where

- The function $a(x)$ is a given function in $L^{\infty}(\Omega)$, but the term $\lambda\widehat{a}(x)$ (so-called important term) is unknown.

As well as the initial condition $g(x) = y_0(x) + \tau\widehat{y}_0(x)$, where

- The function $y_0(x)$ is a given function in $L^2(\Omega)$, but the term $\tau\widehat{y}_0(x)$ (so-called unimportant term) is unknown, and we have $\|\widehat{y}_0(x)\|_{L^2(\Omega)} \leq 1$.

- The parameters λ and τ are reals numbers choosen sufficiently small.

Inverse problem

Can we obtain information about the important term $\lambda\widehat{a}$ independently of the unimportant term $\tau\widehat{y}_0$, in the following equation

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} - div((a(x) + \lambda\widehat{a}(x))\nabla y(x,t)) + f(y) & = & 0 & \text{in} & Q, \\ y(x,0) & = & y_0(x) + \tau\widehat{y}_0(x) & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Sigma, \end{cases} \tag{3.31}$$

from the knowledge of some available data y_{obs}?

3.2.2 Definition of the sentinel

Now, we introduce the notion of sentinel, before that, starting from observations. Let $y(\lambda,\tau) = y(x,t,\lambda,\tau)$ be the unique solution of the problem (1). We set $O \subset \Gamma$ (The observatory).

Suppose we have the following boundary observation

$$y_{obs} = \frac{\partial y}{\partial \nu_a}\text{, for all } (x,t) \in O \times (0,T), \tag{3.32}$$

which is a measure taken on a non empty open subset O at the interval time $(0,T)$, and where $\frac{\partial y}{\partial \nu_a} = a(x)\nabla y.\nu$, with ν is the unit exterior normal to Γ.

Now, let h be a given function on $O \times (0,T)$ such that

$$h \geq 0, \quad \int_0^T \int_O h = 1.$$

On the other hand, let ω be an open non empty subset of Γ, for u a control function in $L^2(\omega \times (0,T))$, we consider the functional $S(\lambda,\tau)$ as follows

$$S(\lambda,\tau) = \int_0^T \int_O h \frac{\partial y(x,t,\lambda,\tau)}{\partial \nu_a} d\Gamma dt + \int_0^T \int_\omega u \frac{\partial y(x,t,\lambda,\tau)}{\partial \nu_a} d\Gamma dt.$$

The functional can be written in a compact form as follows

$$S(\lambda,\tau) = \int_0^T \int_\Gamma (h\chi_O + u\chi_\omega) \frac{\partial y}{\partial \nu_a} d\Gamma dt, \tag{3.33}$$

where χ_O and χ_ω denotes the characteristic functions for the open subsets O and ω, respectively.

Definition 11 *S is said a sentinel defined by h if the following conditions are satisfied:*
1) *S is stationary to the first order with respect to missing term $\tau\widehat{y}_0$, i.e.*

$$\left.\frac{\partial S}{\partial \tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0} = 0. \tag{3.34}$$

2) *The control u is of minimal norm in $L^2(\omega \times (0;T))$ among "all the admissible controls", i.e.*

$$\|u\|^2_{L^2(\omega\times(0;T))} = \min_{\widetilde{u}\in U_{ad}} \|\widetilde{u}\|^2_{L^2(\omega\times(0,T))}, \tag{3.35}$$

where U_{ad} is the set of admissible controls defined as follows

$$U_{ad} = \left\{\widetilde{u} \in L^2(\omega \times (0;T)),\ \textit{such that}\ (\widetilde{u}, S(\widetilde{u}))\ \textit{satisfies}\ (3.34)\right\}.$$

3.2.3 Equivalence to an optimal control problem

In the previous section, we gave some basic definitions that related to the sentinel functional. We see that, the existence of the sentinel is linked to the existence of the control function u, for this, we will look for a function u that assuring conditions (3.34) and (3.35).

In this section, we will show that the existence of such control function u satisfying (3.34) and (3.35) is equivalent to an optimal control problem.

Denote by $y_\tau = \frac{\partial y}{\partial \tau}(\lambda, \tau)\big|_{\lambda=0,\tau=0}$ the solution of the system

$$\begin{cases} \frac{\partial y_\tau}{\partial t} - div\left(a\left(x\right)\nabla y_\tau\right) + f'\left(y_0\right)y_\tau & = & 0 & \text{in} & Q, \\ y_\tau\left(x,0\right) & = & \widehat{y}_0 & \text{in} & \Omega, \\ y_\tau & = & 0 & \text{on} & \Sigma, \end{cases} \tag{3.36}$$

where $y_0 = y\left(x,t;0,0\right)$, and $f'(y_0)$ denotes the derivative of f at point y_0.

It's clear that the insensibilty condition (3.34) is equivalent to

$$\int_0^T \int_\Gamma (h\chi_O + u\chi_\omega)\frac{\partial y_\tau}{\partial \nu_a} d\Gamma dt = 0. \tag{3.37}$$

Let's introduce the adjoint state q by

$$\begin{cases} -\frac{\partial q}{\partial t} - div(a\left(x\right)\nabla q) + f'(y_0)q & = & 0 & \text{in} & Q, \\ q(x,T) & = & 0 & \text{in} & \Omega, \\ q & = & h\chi_O + u\chi_\omega & \text{on} & \Sigma. \end{cases} \tag{3.38}$$

Since $h\chi_O + u\chi_\omega \in L^2(\Sigma)$, The problem (3.38) has an unique solution q such that

$$q \in L^2(Q) \cap C(0,T;H^{-1}(\Omega)).$$

Then, we have the following result.

Proposition 17 *Let q be the solution of the adjoint state* (3.38)*. Then, the existence of the sentinel defined by* (3.33) *for the equation* (3.31) *is equivalent to find a control u with minimal norm in $L^2(\omega \times (0,T))$ that makes the adjoint state q verifies the following null-controllability property*

$$q(x,0) = 0 \text{ in } \Omega. \tag{3.39}$$

Proof 1 *Let $u \in L^2\left(\omega \times (0;T)\right)$, and $q = q\left(x;t;u\right)$ is the solution of the problem* (3.38)*.*

We multiply the both side of first equation of (3.38) *by y_τ solution of* (3.36)*, and integrating by parts in $\Omega \times (0,T)$*

$$\int_0^T \int_\Omega \left(-\frac{\partial q}{\partial t} - div(a\left(x\right)\nabla q) + f'(y_0)q\right) y_\tau dx dt = 0.$$

We get

$$\begin{aligned}0 &= \int_0^T \int_\Omega \left(\frac{\partial y_\tau}{\partial t} - div\left(a\left(x\right) \nabla y_\tau\right) + f'\left(y_0\right) y_\tau \right) q dx dt \\ &+ \int_\Omega q(x,0) y_\tau(x,0) dx - \int_\Omega q(x,T) y_\tau(x,T) dx \\ &+ \int_0^T \int_\Gamma \frac{\partial y_\tau}{\partial \nu_a} q - \int_0^T \int_\Gamma \frac{\partial q}{\partial \nu_a} y_\tau,\end{aligned}$$

y_τ *is the solution of* (3.36)*, therefore*

$$\int_0^T \int_\Gamma (h\chi_O + u\chi_\omega) \frac{\partial y_\tau}{\partial \nu_a} d\Gamma dt = - \int_\Omega q(x,0) \widehat{y}_0 dx.$$

From (3.37)*, we obtain*

$$\int_\Omega q(x,0) \widehat{y}_0 dx = 0 \ \ for \ all \ \ \|\widehat{y}_0\|_{L^2(\Omega)} \leq 1.$$

i.e. $q(x,0)$ *is orthogonal to unit ball of* $L^2(\Omega)$ *which means that* $q(x,0)$ *is orthogonal to the whol space.*

Consequently

$$q\left(x,0\right) = 0 \quad in \ \Omega.$$

3.2.4 Study of the optimal control problem

From the above, we conclude that the existence of the sentinel insensitive to the unimportant terme is related to solving an optimal control problem which is equivalent to the existence of a unique pair (u,q) such that we have (3.38), (3.39) with (3.35).

So, in this section we are interested in the following optimal control problem

$$(OC) \begin{cases} \min_{(u,q)\in M} \|u\|^2_{L^2(\omega\times(0;T))}, \\ \text{with } M = \left\{u \in L^2\left(\omega \times (0;T)\right), \text{such that } \ (u,q) \ \text{ satisfies } \ (3.38), (3.39)\right\}. \end{cases}$$

We introduce q_0 and z as the solutions of

$$\begin{cases} -\frac{\partial q_0}{\partial t} - div(a(x)\nabla q_0) + f'(y_0)q_0 & = & 0 & \text{in} & Q, \\ q_0(x,T) & = & 0 & \text{in} & \Omega, \\ q_0 & = & h\chi_O & \text{on} & \Sigma, \end{cases} \tag{3.40}$$

and

$$\begin{cases} -\frac{\partial z}{\partial t} - div(a(x)\nabla z) + f'(y_0)z & = & 0 & \text{in} & Q, \\ z(x,T) & = & 0 & \text{in} & \Omega, \\ z & = & u\chi_\omega & \text{on} & \Sigma, \end{cases} \tag{3.41}$$

respectively.

Then, we have

$$q(u) = q_0 + z(u). \tag{3.42}$$

We want to find a control function u such that

$$z(u)(x,0) = -q_0(x,0). \tag{3.43}$$

We think of u as a control function, and we think of $z = z(u)$ as given by (3.41) as the state.

Now, we are dealing with the new following problem.

> **Find** u**; in which** $\|u\|^2_{L^2(\omega\times(0;T))} = \min$, **such that the system with state equation** (3.41) **is moves from** $z(x,T) = 0$ **to** $z(x,0) = -q_0(x,0)$ in Ω.

This is the problem we should solve now by used the penalization method which is a technique for analyzing and solving analytically or numerically constrained optimization problems. It consists in transforming the problem with constraints into a problem or problems of optimization without constraint; the precise meaning of this phrase will appear later.

Penalization

We consider the penalized problem

$$\inf_{(u,z)\in M} J_{\epsilon}(u,z), \tag{3.44}$$

where

$$J_{\epsilon}(u,z) = \frac{1}{2}\int_0^T \int_{\omega} u^2 d\Gamma dt + \frac{1}{2\epsilon}\left\| -\frac{\partial z}{\partial t} - div\left(a(x)\nabla z\right) + f'(y_0)\, z \right\|^2, \tag{3.45}$$

where $\|.\|$ denotes the $L^2(\Omega \times (0,T))$ norm, and $\epsilon > 0$.

In (3.45), z is smooth enough and it satisfies

$$\begin{cases} z(T) = 0, \quad z(0) = -q_0(0) \quad \text{in } \Omega, \\ z = u\chi_{\omega} \text{ on } \Sigma. \end{cases} \tag{3.46}$$

Let u_{ϵ}, z_{ϵ} be the solution of (3.44) where (3.46) holds true.

Then, we have

$$\rho_{\epsilon} = \frac{1}{\epsilon}\left(-\frac{\partial z_{\epsilon}}{\partial t} - div\left(a(x)\nabla z_{\epsilon}\right) + f'(y_0)\, z_{\epsilon} \right), \tag{3.47}$$

where z_{ϵ} be the solution of (3.41) associated to the control u_{ϵ}.

So, we have

$$\int_0^T \int_{\omega} u_{\epsilon}\widehat{u} d\Gamma dt + \int_0^T \int_{\Omega} \rho_{\epsilon}\left(-\frac{\partial \widehat{z}}{\partial t} - div\left(a(x)\nabla \widehat{z}\right) + f'(y_0)\,\widehat{z} \right) dxdt = 0, \tag{3.48}$$

for all $\widehat{z}$ such that

$$\begin{cases} -\frac{\partial \widehat{z}}{\partial t} - div\left(a(x)\nabla \widehat{z}\right) + f'(y_0)\,\widehat{z} & = & 0 & \text{in} & Q, \\ \widehat{z}(x,T) = \widehat{z}(x,0) & = & 0 & \text{in} & \Omega, \\ \widehat{z} & = & \widehat{u}\chi_{\omega} & \text{on} & \Sigma. \end{cases} \tag{3.49}$$

It follows from (3.48) that

$$\begin{cases} \frac{\partial \rho_{\epsilon}}{\partial t} - div\left(a(x)\nabla \rho_{\epsilon}\right) + f'(y_0)\,\rho_{\epsilon} & = & 0 & \text{in} & Q, \\ \rho_{\epsilon} & = & 0 & \text{on} & \Sigma, \end{cases} \tag{3.50}$$

with

$$\frac{\partial \rho_\epsilon}{\partial \nu_a} + u_\epsilon = 0 \quad \text{on } \omega \times (0,T)\,, \tag{3.51}$$

where

$$\frac{\partial \rho_\epsilon}{\partial \nu_a} = a(x)\frac{\partial \rho_\epsilon}{\partial \nu} \text{ with } \frac{\partial \rho_\epsilon}{\partial \nu} \text{ is the derivative of } y \text{ with respect to the normal } \nu.$$

Optimality system

Assuming that, in some topology, $\rho_\epsilon \longrightarrow \rho$, we define ρ solution of

$$\begin{cases} \frac{\partial \rho}{\partial t} - div\left(a(x)\nabla\rho\right) + f'(y_0)\,\rho &= 0 \quad \text{in} \quad Q, \\ \rho(x,0) &= \rho^\circ \quad \text{in} \quad \Omega, \\ \rho &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{3.52}$$

where ρ° is to be found.

We introduce now z solution of (the formel limite of z_ϵ)

$$\begin{cases} -\frac{\partial z}{\partial t} - div(a(x)\nabla z) + f'(y_0)z &= 0 \quad \text{in} \quad Q, \\ z(x,T) &= 0 \quad \text{in} \quad \Omega, \\ z &= -\frac{\partial \rho}{\partial \nu_a} \quad \text{on} \quad \omega \times (0,T)\,, \\ z &= 0 \quad \text{on} \quad \Gamma \diagdown \omega \times (0,T)\,. \end{cases} \tag{3.53}$$

The question is then to find ρ° in such a way that

$$z(x,0) = -q_0(x,0) \quad \text{in} \quad \Omega. \tag{3.54}$$

We define a linear operator Λ by

$$\Lambda : F \longrightarrow F',$$

$$\Lambda(\rho^\circ) = z(x,0)\,, \tag{3.55}$$

where F and F' will be defined later. We multiply the both side of first equation of

(3.53) by ρ solution of (3.52), and integrating by parts in Q, we obtain

$$\int_{\Omega} z(0)\,\rho^{\circ} dx = \int_0^T \int_{\omega} \left| \frac{\partial \rho}{\partial \nu_a} \right|^2 d\Gamma dt.$$

Therefore

$$\langle \Lambda(\rho^{\circ}), \rho^{\circ} \rangle = \int_0^T \int_{\omega} \left| \frac{\partial \rho}{\partial \nu_a} \right|^2 d\Gamma dt,$$

where $\langle ., . \rangle$ denotes the duality product between F and F'.

This leads to the introduction of the following semi norm on F

$$|\rho^{\circ}|_F = \left(\int_0^T \int_{\omega} \left| \frac{\partial \rho}{\partial \nu_a} \right|^2 d\Gamma dt \right)^{\frac{1}{2}}. \tag{3.56}$$

Lemma 2 *The expression* (3.56) *is norm.*

Proof. To show that the semi norm (3.56) is norm, we must prove only that

$$\int_0^T \int_{\omega} \left| \frac{\partial \rho}{\partial \nu_a} \right|^2 d\Gamma dt = 0 \quad \text{then} \quad \rho = 0 \quad \text{in} \quad Q.$$

It's easy to see that by using the uniqueness continuation property for parabolic equation [?]. ■

By dint of the introduction of F, we now have

$$\Lambda \text{ is an isomorphism from } F \text{ to } F', \text{ And } \Lambda^* = \Lambda$$

where the Hilbert space F is the completion of smooth functions for the norm (3.56) and we denote by F' its dual.

Equation (3.55) is equivalent to

$$\Lambda(\rho^{\circ}) = -q_0(x, 0). \tag{3.57}$$

Then

$$\rho^{\circ} = -\Lambda^{-1} q_0(x, 0). \tag{3.58}$$

When we multiply (3.40) by ρ solution of (3.52), we find

$$\langle q_0(x,0), \rho^\circ\rangle = -\int_0^T\int_O h_0 \frac{\partial \rho}{\partial \nu_a}\, d\Gamma dt \tag{3.59}$$

So that

$$q_0(x,0) \in F' \tag{3.60}$$

And (3.55) admits a unique solution.

Hence, the sentinel (3.33) will take the form

$$S(\lambda,\tau) = \int_0^T\int_\Gamma \left(h\chi_O - \frac{\partial\rho}{\partial\nu_a}\chi_\omega\right)\frac{\partial y}{\partial \nu_a} d\Gamma dt. \tag{3.61}$$

3.2.5 Information given about the important term

To show how the sentinel defined above permits to estimate the pollutin term, we consider $m_0 = y_{obs}$ be the measured state of the system on the observatory O during the interval $[0,T]$, then the measured sentinel associate to m_0 is given by

$$S_{obs}(\lambda,\tau) = \int_0^T\int_\Omega \left(h\chi_O - \frac{\partial\rho}{\partial\nu_a}\chi_\omega\right) m_0(x,t,\lambda,\tau)\, dxdt. \tag{3.62}$$

Theorem 18 *The information given by the sentinel about the important term is as follows*

$$\begin{aligned}\int_0^T\int_\Omega div\left(\lambda\widehat{a}(x)\nabla y_0\right) q dxdt &\simeq S_{obs}(\lambda,\tau) - S(0,0)\\ &= \int_0^T\int_\Omega \left(h\chi_O - \frac{\partial\rho}{\partial\nu_a}\chi_\omega\right)\left(m_0 - \frac{\partial y_0}{\partial\nu_a}\right) d\Gamma dt,\end{aligned}$$

where, $S(0,0) = S(\lambda,\tau)|_{\lambda=0,\tau=0}$, *and* $y_0 = y(\lambda,\tau)|_{\lambda=0,\tau=0}$

Proof. We have according to The Taylor's formula

$$S(\lambda,\tau) \simeq S(0,0) + \tau\frac{\partial S}{\partial\tau}(0,0) + \lambda\frac{\partial S}{\partial\lambda}(0,0), \quad \text{for } \lambda,\ \tau \text{ small.}$$

Because $\left.\dfrac{\partial S}{\partial \tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0}=0$. And $S(\lambda,\tau)$ is observed then we have:

$$\lambda \left.\frac{\partial S}{\partial \lambda}(\lambda,\tau)\right|_{\lambda=0,\tau=0} \simeq S_{obs}(\lambda,\tau)-S(0,0). \tag{3.63}$$

And,

$$\left.\frac{\partial S}{\partial \lambda}(\lambda,\tau)\right|_{\lambda=0,\tau=0}=\int_0^T\int_\Omega (h\chi_O-\frac{\partial \rho}{\partial \nu_a}\chi_\omega)\frac{\partial y_\lambda}{\partial \nu_a}d\Gamma dt,$$

where y_λ defined by $y_\lambda=\frac{\partial y}{\partial \lambda}(0,0)$, which is the unique solution of

$$\begin{cases} \dfrac{\partial y_\lambda}{\partial t}-div\,(a(x)\,.\nabla y_\lambda)+f'(y_0)y_\lambda = div\,(\widehat{a}(x)\,\nabla y_0) & \text{in} \quad Q, \\ y_\lambda(0) = 0 & \text{in} \quad \Omega, \\ y_\lambda = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3.64}$$

Multiply (3.64) by $q=q(x,t)$ solution of (3.38) , and integrate by part, we obtain

$$\int_0^T\int_\Omega (h\chi_O-\frac{\partial \rho}{\partial \nu_a}\chi_\omega)\frac{\partial y_\lambda}{\partial \nu_a}d\Gamma dt=\int_0^T\int_\Omega div\,(\widehat{a}(x)\,\nabla y_0)\,q dx dt \tag{3.65}$$

Combining (3.65) and (3.63), we obtain the information about the pollution term given in the previous theorem. ■

Conclusion and perspectives

This thesis is devoted to identification problems, where we have dealt with the identification of the important term which is independent of the variations of the unimportant term. In the first chapter, we presented the sentinel method, which is the most popular strategy and consists in obtaining information on the missing terms from a weighted average of the observation. In the second chapter, we carried out a study of the identification of the bulk modulus and potential coefficient in a wave and acoustic equations with incomplete data respectively, where we used the sentinel method of J. L. Lions to answer the question. This showed that we can estimate the important term independently of other data that we do not want to identify. This method consists in transposing a problem of identification or estimation of incomplete data into an optimal control problem. In the third chapter, we use the same method to identify the diffusion and potential coefficient in a diffusion and schrodinger equations with incomplete data respectively. In solving these problems, it is necessary to have the measured data of the state.

These results open up numerical perspectives of this method. The digital simulation tools available can still be improved to respond to the many current environmental problems. Today, we hope that the development of new techniques will allow a better estimation of unknown parameters and coefficients in other identification problems.

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