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Doctoral Thesis
Option: Stationary Problems

Theme

Existence and multiplicity of solutions for certain nonlinear elliptic problems

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Contents

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Dedication

I dedicate this thesis,

To my parents, thank you for instilling in me a love for learning and for always believing in my abilities.

To my brother and sisters, thank you for supporting me along the way.

To my friends and colleagues, thank you for the hours of discussions, debates, and shared moments of laughter and camaraderie.

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ملخص

تتعلق الاشكالية التي تم تناولها في هذه الرسالة بدراسة بعض المسائل الاهليلجية غير الخطية. يعتمد نهجنا على طرق التغاير. أولا، لقد درسنا نظاما حرجا من نوع شرودينقر ـ كيرشوف يتضمن مؤثر p ـ لابلاس الكسري مع شروط حدود ديريشليت، حيث أثبتنا وجود حلين ضعيفين باستخدام منوعة نهاري و طريقة الألياف. ثانيا، اعتبرنا فئة من مسائل $p(x)$ ـ لابلاسيان. بواسطة تطبيق نظرية مونتان باس و نظرية فونتان، على التوالي، تم الحصول على وجود و تعدد الحلول.

الكلمات المفتاحية : $p(x)$ ـ لابلاسيان، p ـ لابلاسيان الكسري، شرط باليه ـ سميل، طرق التغاير، نظرية مونتان باس، نظرية فونتان، منوعة نهاري، طريقة الألياف.

Résumé

La problématique abordée dans cette thèse concerne l'étude de certains problèmes elliptiques non linéaires. Notre approche est basée sur des méthodes variationnelles. Tout d'abord, nous avons étudié un système critique de type Shrödinger-Kirchhoff faisant intervenir l'opérateur p-Lplacien fractionnaire avec conditions aux limites de Dirichlet, où nous avons prouvé l'existence de deux solutions faibles en utilisant la variété de Nehari et la méthode de fibering. Deuxièmement, nous avons considéré une classe de problèmes $p(x)$-laplacien. En appliquant le théorème du col de la montagne et le théorème de la fontaine, respectivement, l'existence et la multiplicité des solutions ont été obtenues.
Mots clés: $p\left(x\right)$-Laplacien, p-Laplacien fractionnaire, condition de Palais-Smale, méthodes variationelles, théorème du col de la montagne, théorème de la fontaine, variété de Nehari, méthode de fibering.

Abstract

The problem addressed in this thesis concerns the study of some nonlinear elliptic problems. Our approach is based on variational methods. First, we studied a critical Schrödinger-Kirchhoff type system involving the fractional p-Laplacian operator with Dirichlet boundary conditions, where we proved the existence of two weak solutions by using the Nehari manifold and the fibering method. Second, we considered a class of $p(x)$-Laplacian problems. By applying the mountain pass theorem and fountain theorem, respectively, the existence and multiplicity of solutions were obtained.
Keywords: $p\left(x\right)$-Laplacian, fractional p-Laplacian, Palais-Smale condition, variational methods, mountain pass theorem, fountain theorem, Nehari Manifold, fibering method.

Notations

$\forall$: for all.

$\exists$: there exists.

$N \geq 1$: Dimension of the space domain.

$\mathbb{R}^N$: the N-dimensional Euclidean space.

Ω : An open bounded set of $\mathbb{R}^N$.

$\frac{\partial}{\partial x_i}$: the partial derivative with respect to the i-th component of x.

X : a Banach space.

X': the dual space of X .

$\partial\Omega$: boundary of Ω.

a.e.: abbreviation for almost everywhere.

$< .,. >$: the duality pairing between X and X'.

$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right)$: the gradient of u.

$\Delta u = \sum\limits_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$: the Laplacian of u.

div $u = \sum\limits_{i=1}^{N} \frac{\partial u}{\partial x_i}$: the divergence of u.

$\to$: strong convergence.

$\rightharpoonup$: weak convergence.

$\hookrightarrow$: continuous embedding.

$||.||$: the norm.

$\oplus$: direct sum.

$C(\Omega)$: the space of continuous real-valued functions on Ω.

$C_0^\infty(\Omega)$ or $D(\Omega)$: infinitely differentiable functions with compact support on Ω.

$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), p(x) > 1, \forall\, x \in \overline{\Omega}\}$.

$p^- = \inf_{\overline{\Omega}} p(x)$ and $p^+ = \sup_{\overline{\Omega}} p(x)$, for $p \in C_+(\overline{\Omega})$.

$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R},$ is measurable : $\int_\Omega |u(x)|^{p(x)} dx < \infty\}$.

$W^{1,p(x)}(\Omega) = \{f \in L^{p(x)}(\Omega) : |\nabla f| \in L^{p(x)}(\Omega)\}$.

For $s \in (0,1), W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\}$.

$B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$.

Introduction

Recently, research in nonlinear analysis has mainly focused on nonlinear elliptic boundary value problems. The description of various phenomena in science and engineering requires the use of problems of this kind.

There are different methods that have been used to prove the existence results for this kind of problems, one of these is the variational method. This method is applied to problems with a variational structure.

In this thesis, our main purpose is to study the existence and multiplicity of solutions for some nonlinear elliptic boundary value problems using variational methods. The variational method finds solutions of equations by considering solutions as critical points of an appropriately chosen function (called energy functional of the problem).

Lately, problems with nonlocal operators (fractional elliptic operators) have attracted considerable attention from many authors. This type of operators plays an essential role in our real life and in the description of many different phenomena, such as in physics, finance, optimization, and population dynamics.

In this context, this thesis deals with the multiplicity of solutions for critical Schrödinger-Kirchhoff type systems involving the fractional p-Laplacian operator. More precisely,, we consider the following system:

$$\begin{cases} K_1(\,\|w\|_{R_1}^p\,)\left((-\Delta)_p^s w + R_1(x)\,|w|^{p-2}\,w\right) = b_1(x)\,|w|^{p_s^*-2}\,w + \lambda f(x,w,z) \text{ in } \Omega, \\ K_2(\,\|z\|_{R_2}^p\,)\left((-\Delta)_p^s z + R_2(x)\,|z|^{p-2}\,z\right) = b_2(x)\,|z|^{p_s^*-2}\,z + \lambda g(x,w,z) \text{ in } \Omega, \\ w, z > 0 \text{ in } \Omega, \\ w = z = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $N > ps, 0 < s < 1 < q < p, p_s^* = \frac{Np}{N-sp}$, $\lambda > 0$, the functions $b_1, b_2, R_1, R_2, K_1, K_2, f$ and g are assumed to satisfy some suitable assumptions.

The operator $(-\Delta)_p^s$ represents the fractional p-Laplacian operator, defined by

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, x \in \mathbb{R}^N,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. A typical feature of problem (0.1) is the nonlocality, in the sense that the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^N$. Therefore, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$. Moreover, the presence of the Kirchhoff functions K_1 and K_2 implies that the first two equations in (0.1) are no longer pointwise equalities, therefore it is often called nonlocal problem.

The system described in equation (0.1) is associated with the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{0.2}$$

presented by Kirchhoff [38] in 1883. The parameters L, h, E, ρ and P_0 are constants. The model (0.2) extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations.

In the last decades, much interest has grown in elliptic equations of Kirchhoff-type involving critical exponents see [32,35,45] for the setting of bounded domains and [36,40,41] for the context set in $\mathbb{R}^n$. Fiscella and Valdinoci in [33] studied the existence of nonnegative solutions for the following problem of Kirchhoff-type

$$\begin{cases} M\left(\iint\limits_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2} u \text{ in } \Omega, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.3}$$

where λ is a positive real number, $(-\Delta)^s$ represents the fractional Laplace operator, the functions M and f are continuous. Since then, fractional problems of Kirchhoff-type have been extensively explored by numerous researchers. In particular, in reference [42], the authors employed Ekeland's variational principle in conjunction with the mountain pass theorem to establish the existence of solutions and analyze their asymptotic behavior for the

Schrödinger-Kirchhoff type system described below:

$$\begin{cases} F([(u,v)]_{s,p}^p + \|u,v\|_{p,G}^p)\left(\mathcal{L}_p^s u + G(x)\,|u|^{p-2}\,u\right) = \mu H_u(x,u,v) + \frac{t}{p_s^*}|v|^w|u|^{t-2}u \text{ in } \mathbb{R}^n, \\ F([(u,v)]_{s,p}^p + \|u,v\|_{p,G}^p)\left(\mathcal{L}_p^s v + G(x)\,|v|^{p-2}\,v\right) = \mu H_v(x,u,v) + \frac{w}{p_s^*}|u|^t|v|^{w-2}v \text{ in } \mathbb{R}^n, \end{cases} \tag{0.4}$$

where $t+w = p_s^*$, $G : \mathbb{R}^n \to \mathbb{R}^+$, $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous, H_u and H_v are two Caratheodory functions and μ is a positive parameter.

The thesis also concerns the existence and multiplicity of solutions for a class of $p(x)$-Laplacian problems. As a particular case, we study the following class of Steklov boundary value problems involving the $p(x)$-Laplacian operator

$$\begin{cases} -div\left(h\,(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}\,u = S\,(x,u) \text{ in } \Omega, \\ \\ h\,(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} + l(x)|u|^{w(x)-2}u = Q(x,u) \text{ on } \partial\Omega, \end{cases} \tag{0.5}$$

and

$$\begin{cases} -div\left(h\,(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}\,u = S\,(x,u) + \mu\,|u|^{\delta(x)-2}\,u \text{ in } \Omega, \\ \\ h\,(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} + l(x)|u|^{w(x)-2}u = Q(x,u) \text{ on } \partial\Omega, \end{cases} \tag{0.6}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$, is a bounded domain with a Lipschitz boundary $\partial\Omega$. The symbol ∂v is the outer normal derivative on $\partial\Omega$. The functions $p(x), \delta(x), w, S, Q, h$ and l are assumed to satisfy some suitable assumptions, μ is a positive parameter and the operator

$$(-\Delta)_{p(x)}v(x) = -div(|\nabla v|^{p(x)-2}\nabla v),$$

is the $p(x)$-Laplacian. This operator, with $p > 1$, is an extension of the p-Laplace operator $(-\Delta)_p v = -div(|\nabla v|^{p-2}\nabla v)$. However, unlike the p-Laplace operator, the $p(x)$-Laplace operator exhibits a more complex non-linearity and lacks homogeneity.

In the last two decades, nonlinear partial differential problems with nonstandard growth conditions have garnered more and more attention because of their great interest in other fields, such as electrorheological fluids [47], thermorheological fluids [9] and image restoration [18]. For more details, we refer to [24, 29].

The rapid development of variable exponent Lebesgue and Sobolev spaces aided in appearing some physical models. These spaces were introduced in the 1930s by Orcliz [44].

They were initially investigated for theoretical interest. Later, they appeared in the study of functionals of the calculus of variations with nonstandard growth.

In recent years, problems of type (0.5) and (0.6) have been investigated by many papers (see [3–5, 10, 11, 16, 17, 29, 52]). For example, Z. Yücedag [52] studied problem (0.5) with $l(x) = -1$, and $Q(x,u) = 0$ and she showed in this case that problem (0.5) has at least one nontrivial weak solution. In a recent paper [17], Chammem et al. considered problems (0.5) and (0.6) in the case:

$$S(x,u) = v_1(x)h_1(u) \text{ and } Q(x,u) = v_2(x)h_2(u),$$

and they obtained results on existence and multiplicity of solutions.

This thesis consists of four chapters that are briefly presented below.

In the **first chapter**, we give all the necessary tools that will be needed in the course of this work.

In the **second chapter**, we present critical point theory and variational methods including mountain pass theorem, fountain theorem, Nehari manifold, and fibering method.

In the **third chapter**, we study the existence of multiple solutions for problem (0.1) by using the Nehari manifold method and the fibering maps analysis.

In the **fourth chapter**, we prove the existence of a nontrivial weak solution of problem (0.5) by using the mountain pass theorem. Moreover, we show that problem (0.6) has infinitely many pairs of weak solutions by means of fountain theorem.

CHAPTER 1

Preliminaries

Contents

In this chapter, we briefly recall some definitions and preliminary notions of the necessary function spaces and describe their basic properties. Moreover, we provide some notions about operators and differentiability, and we present some results about integration theory such as the fundamental convergence criteria.

Note that the results introduced in this chapter are not given in full generality, they will be presented as our study requires.

1.1 Function spaces

We start by recalling the space of continuous functions.

1.1.1 Space of continuous functions

For an open domain Ω of $\mathbb{R}^N$, the function $u : \Omega \to \mathbb{R}$ is defined as continuous if

$$\forall x_0 \in \Omega, \forall \varepsilon > 0, \exists \sigma > 0,$$

such that

$$\|x - x_0\| < \sigma \Longrightarrow |w(x) - w(x_0)| < \varepsilon,$$

here, $||.||$ is the well-known Euclidean norm.
We introduce the following notations:

$$C(\Omega) := \{w : \Omega \to \mathbb{R} \text{ is continuous}\},$$

$$C\left(\overline{\Omega}\right) := \left\{w : \Omega \to \mathbb{R} \text{ is continuous and extends continuously to } \overline{\Omega}\right\}.$$

We define the norm over $C(\Omega)$, by

$$\|w\|_C = \sup_{x\in\Omega} |w(x)|.$$

1.1.2 Standard Lebesgue and Sobolev spaces

We present fundamental facts concerning classical Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$. These properties are well-established and can be found in various references such as [1,12,13,21,34].

Everywhere in this part, we consider Ω as an open subset of $\mathbb{R}^N$, endowed with the measure of Lebesgue dx.

Definition 1.1 *Let $1 \leq p < \infty$. We set*

$$L^p(\Omega) = \left\{ h : \Omega \longrightarrow \mathbb{R} \text{ is measurable, and } \int_\Omega |h|^p \, dx < +\infty \right\}, \text{ [12]}$$

we define the norm of h in $L^p(\Omega)$ by

$$\|h\|_{L^p} = \|h\|_p = \left(\int_\Omega |h|^p \, dx \right)^{1/p}. \text{ [12]}$$

Definition 1.2 *We define the space of essentially bounded functions*

$$L^\infty(\Omega) = \{ g : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \exists \text{ a constant } S, |g(x)| \leq S \text{ a.e on } \Omega \}, \text{ [12]}$$

the norm in this space is defined as

$$\|g\|_{L^\infty} = \|g\|_\infty = \inf \{ S; |g| \leq S \text{ a.e on } \Omega \}.$$

Remark 1.1 *Let $g \in L^\infty(\Omega)$, then*

$$|g| \leq \|g\|_{L^\infty} \text{ a.e. on } \Omega. \text{ [12]}$$

Theorem 1.1 *Let $1 \leq p \leq \infty$. Suppose that $u \in L^p$ and $v \in L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then, one has*

$$\int_\Omega |uv| \leq \|u\|_{L^p} \|v\|_{L^{p'}}. \text{ [12]}$$

Definition 1.3 *[12] For $1 \leq p \leq \infty$, The Sobolev space $W^{1,p}(\Omega)$ is defined as the set of functions $w \in L^p(\Omega)$ such that the weak derivative ∇w exists in the distributional sense and also belongs to $L^p(\Omega)$.*
The norm over $W^{1,p}(\Omega)$ is introduced as follows

$$\|w\|_{W^{1,p}} = \|w\|_{L^p} + \|\nabla w\|_{L^p}, \text{ [12]}$$

where $1 \leq p < \infty$.

Proposition 1.1 *[12] The functional spaces* $W^{1,q}(\Omega)$ *and* $L^q(\Omega)$ *with* $1 \leq q \leq \infty$ *are Banach spaces. Moreover, they are reflexive for* $1 < q < \infty$, *and typically separable for* $1 \leq q < \infty$.

Further properties of these spaces can be derived in sections 1.1.3 and 1.1.4, by considering them as a special case of the variable exponent spaces properties.

1.1.3 Variable exponent Lebesgue spaces

We introduce fundamental information about variable exponent Lebesgue spaces. For a more comprehensive understanding of this topic, we recommend referring to the following references [6–8, 19, 20, 23, 30, 39, 46, 48].

Let Ω be a bounded open subset of $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), p(x) > 1, \forall\, x \in \overline{\Omega}\}.$$

For any variable exponent $p \in C_+(\overline{\Omega})$, we define

$$p^- = \inf_{\overline{\Omega}} p(x), p^+ = \sup_{\overline{\Omega}} p(x).$$

We also denote $C_+(\partial\Omega)$ and q^-, q^+ for every $q(x) \in C(\partial\Omega)$. We introduce the variable exponent Lebesgue spaces as $L^{p(x)}(\Omega)$ and $L^{q(x)}(\partial\Omega)$ with the following definitions:
$L^{p(x)}(\Omega)$: comprises measurable real-valued functions $f : \Omega \to \mathbb{R}$, for which

$$\vartheta(f) = \int_\Omega |f(x)|^{p(x)} dx,$$

is finite.
The space $L^{q(x)}(\partial\Omega)$: comprises measurable real-valued functions $f : \partial\Omega \to \mathbb{R}$ with

$$\vartheta_\partial(f) = \int_{\partial\Omega} |f(x)|^{q(x)} d\sigma,$$

is finite.

We define the norms over these spaces, by

$$||f||_{L^{p(x)}(\Omega)} = |f|_{L^{p(x)}(\Omega)} = \inf\{\gamma > 0 : \int_{\Omega} |\frac{f(x)}{\gamma}|^{p(x)} dx \leq 1\},$$

and

$$||f||_{L^{p(x)}(\partial\Omega)} = |f|_{L^{p(x)}(\partial\Omega)} = \inf\{\lambda > 0 : \int_{\partial\Omega} |\frac{f(x)}{\lambda}|^{p(x)} d\sigma \leq 1\},$$

where, $d\sigma$ represents the surface measure on $\partial\Omega$.

Proposition 1.2 *([27,30]).*

(1) $L^{p(x)}(\Omega)$, *equipped with its norm, is a separable and uniformly convex Banach space.*

(2) *Let* $z \in C_+(\overline{\Omega})$*. The Hölder inequality holds, namely, if* $f \in L^{z(x)}(\Omega)$ *and* $g \in L^{w(x)}(\Omega)$*, with* $\frac{1}{z(x)} + \frac{1}{w(x)} = 1$*. Then, we have*

$$|\int_{\Omega} fg dx| \leq (\frac{1}{z^-} + \frac{1}{w^-})|f|_{z(x)}|g|_{w(x)}.$$

(3) *If* $z, w \in C_+(\overline{\Omega})$ *with* $z(x) \leq w(x)$*, so the continuous embedding from the space* $L^{w(x)}(\Omega)$ *to* $L^{z(x)}(\Omega)$ *holds.*

Theorem 1.2 *[30] For any* $f \in L^{p(x)}(\Omega)$*, one has*

(1) $\vartheta(f) < 1 (resp > 1, = 1)$ *if and only if* $|f|_{L^{p(x)}(\Omega)} < 1$ $(resp > 1, = 1)$,

(2) $|f|^{p^+}_{L^{p(x)}(\Omega)} \leq \vartheta(f) \leq |f|^{p^-}_{L^{p(x)}(\Omega)}$, *if* $|f|_{L^{p(x)}(\Omega)} < 1$,

(3) $|f|^{p^-}_{L^{p(x)}(\Omega)} \leq \vartheta(f) \leq |f|^{p^+}_{L^{p(x)}(\Omega)}$, *if* $|f|_{L^{p(x)}(\Omega)} > 1$.

Proposition 1.3 *(see [48]) For any* $f \in L^{p(x)}(\partial\Omega)$*, one has*

(1) *If* $|f|_{L^{p(x)}(\partial\Omega)} < 1$*, then* $|f|^{p^+}_{L^{p(x)}(\partial\Omega)} \leq \vartheta_\partial(f) \leq |f|^{p^-}_{L^{p(x)}(\partial\Omega)}$,

(2) *If* $|f|_{L^{p(x)}(\partial\Omega)} > 1$*, then* $|f|^{p^-}_{L^{p(x)}(\partial\Omega)} \leq \vartheta_\partial(f) \leq |f|^{p^+}_{L^{p(x)}(\partial\Omega)}$.

Lemma 1.1 *([26]) Let* $f \in L^{w(y)}(\mathbb{R}^N)$ *with* $f \neq 0$*. Let* q *be in* $L^\infty(\Omega)$ *with* $1 \leq q(y).w(y) \leq \infty$*, for a.e* $y \in \Omega$*. Then,*

(1) $|f|^{w^-}_{q(y)w(y)} \leq ||f|^{q(y)}|_{w(y)} \leq |f|^{w^+}_{q(y)w(y)}$, *if* $|f|_{q(y)w(y)} \geq 1$,

(2) $|f|^{w^+}_{q(y)w(y)} \leq ||f|^{q(y)}|_{w(y)} \leq |f|^{w^-}_{q(y)w(y)}$, *if* $|f|_{q(y)w(y)} \leq 1$.

1.1.4 Variable exponent Sobolev spaces

We present fundamental information about Sobolev spaces with variable exponent. For an exposition of these concepts, we refer to [23,26,30].

Let Ω denote a bounded open set of $\mathbb{R}^N$. The space is characterized by the following definition:

$$W^{1,p(x)}(\Omega) = \{g \in L^{p(x)}(\Omega) \text{ such that } |\nabla g| \in L^{p(x)}(\Omega)\},$$

where the function $p(x)$: is continuous, defined on Ω and takes values in the interval $[1, +\infty[$. We define the norm over $W^{1,p(x)}(\Omega)$, by

$$\|f\|_{W^{1,p(x)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega \left(|\tfrac{f(x)}{\lambda}|^{p(x)} + |\tfrac{\nabla f(x)}{\lambda}|^{p(x)}\right) dx \leq 1\}.$$

The subspace $W_0^{1,p(x)}(\Omega)$ is defined as the closure of the space $C_0^\infty(\Omega)$ in the functional space $W^{1,p(x)}(\Omega)$.

Proposition 1.4 *(see [26, 30, 51])*
(1) *Whenever $p(x) \in C_+(\overline{\Omega})$, the functional spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are Banach spaces, moreover they are separable and reflexive.*
(2) *Suppose $w \in C_+(\overline{\Omega})$ where $w(x) < p^*(x)$, for every $x \in \overline{\Omega}$. In this case, the continuous and compact embedding from the functional space $W^{1,p(x)}(\Omega)$ to $L^{w(x)}(\Omega)$ holds, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

(3) *Suppose $w \in C_+(\partial\Omega)$ where $w(x) < p_*(x)$ for every $x \in \partial\Omega$. In this case, the trace embedding from the space $W^{1,p(x)}(\Omega)$ to $L^{w(x)}(\partial\Omega)$ is continuous and compact,where*

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

1.1.5 Fractional Sobolev spaces

Definition 1.4 *[21] The fractional Sobolev space $W^{s,p}(\Omega)$ on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, where $s \in]0,1[$ and $1 \leq p < \infty$, is defined as follows:*

$$W^{s,p}(\Omega) = \left\{ w \in L^p(\Omega) \mid \int_\Omega \int_\Omega \frac{|w(x) - w(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\}.$$

Proposition 1.5 *[21] The space $W^{s,p}(\Omega), 0 \leq s < 1,$ is a Banach space, endowed with the following norm*

$$\|v\|_{W^{s,p}(\Omega)} = (\|v\|_{L^p(\Omega)}^p + [\|v\|_{s,p}']^p)^{\frac{1}{p}},$$

where $[\|v\|_{s,p}']^p = \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy.$

Proposition 1.6 *[21] The functional space $W^{s,p}(\Omega)$ has a local nature, which means that the product φv belongs to $W^{s,p}(\Omega)$, for each v in the functional space $W^{s,p}(\Omega)$ and each $\varphi \in D(\Omega)$.*

Proposition 1.7 *[21] The space $D(\mathbb{R}^N)$ is dense in $W^{s,p}(\Omega)$.*

Theorem 1.3 *[22] Let $\Omega \subseteq \mathbb{R}^N$ be an extension domain for $W^{s,p}$, $s \in (0,1)$ and $p \in [1, +\infty)$ with $N > ps$. So, for any r such that $p \leq r \leq p_s^*$, the continuous embedding from $W^{s,p}(\Omega)$ into $L^r(\Omega)$ holds. In other words, there exists $C_r > 0$ such that for each $v \in W^{s,p}(\Omega)$, one has*

$$\|v\|_{L^r(\Omega)} \leq C_r \|v\|_{W^{s,p}(\Omega)},$$

where r belongs to $[p, p_s^*]$. Whenever Ω is bounded, so the embedding from $W^{s,p}(\Omega)$ to $L^q(\Omega)$ is continuous if q belongs to $[1, p_s^*]$.

Theorem 1.4 *[21] Let $s \in [0, 1[$ and $p > 1$. Suppose Ω is a bounded Lipschitz open domain of $\mathbb{R}^N$, $N \geq 1$. Then, we have:*

- *If $sp < N$, so for any $q < Np/(N - sp)$, the compact embedding from the space $W^{s,p}(\Omega)$ into L^q holds.*
- *If $sp = N$, so for any $q < \infty$, the compact embedding from $W^{s,p}(\Omega)$ into L^q holds.*
- *If $sp > N$, so for $\lambda < s - N/p$, the compact embedding from the functional space $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ holds.*

1.2 Some classes of operators

Definition 1.5 *Consider $(W, \|.\|)$ as a real reflexive separable Banach space, and let W' denote its topological dual space. We say that $T : W \to W'$,*

- *is a* ***continuous operator*** *if $\|Tx_n - Tx\|_{W'} \to 0$ when $\|x_n - x\|_W \to 0$.*

- *is a* ***compact operator*** *if for any bounded set A in W, the image set $T(A)$ is relatively compact in W'. In other words, the closure of $T(A)$ is compact in W'.*

- *is a* ***coercive operator*** *if*
$$\lim_{\|x\|\to+\infty} \frac{\langle T(x), x\rangle}{\|x\|} = +\infty.$$

- *is a* ***monotone operator*** *if*
$$\langle Tz - Tv, z - v\rangle \geq 0, \forall z, v \in W \text{ with } z \neq v.$$

- *is a* ***strictly monotone operator*** *if*
$$\langle Tz - Tv, z - v\rangle > 0, \forall z, v \in W \text{ with } z \neq v.$$

- *is a* ***bounded operator*** *if it maps any bounded set to a bounded set.*

- *is a* ***semi-continuous operator***
$$\text{if } w_n \to w \text{ when } n \to \infty \text{ implies } Tw_n \rightharpoonup Tw \text{ when } n \to \infty.$$

- *is a* ***strongly continuous operator***
$$\text{if } v_n \rightharpoonup v \text{ implies } Tv_n \to Tv \text{ when } n \to \infty.$$

Definition 1.6 *[43] Let W be a reflexive space, D be a nonempty subset of W and $T : D \to W'$ be a mapping. We define T to be a mapping of $(S)_+$ type if, for every sequence $\{u_n\}_{n\geq 1} \subset D$ such that u_n converges weakly to u in W and $\limsup_{n\to\infty}\langle T(u_n), (u_n - u)\rangle \leq 0$, so $u_n \to u$ strongly in W.*

1.3 Derivatives

There are several notions of derivatives for functions defined on Banach spaces. We start with the directional derivative.
Let W be a Banach space.

Definition 1.7 *[37] **(Directional derivative)** Let U be a subset of W and $G : U \to \mathbb{R}$. If $v \in U$ and $h \in W$ we have $v + th \in U$. The function G has (at v) a derivative in the direction of the vector h, denoted as $G'_z(u)$, if the following limit exists*

$$\lim_{t \to \overset{+}{0}} \frac{G(v+th) - G(v)}{t}, \text{ for every } t > 0 \text{ small enough.}$$

Definition 1.8 *[37] **(Gateaux derivative)** Let U be a subset of W and $G : U \to \mathbb{R}$. The function G is Gateaux differentiable at $v \in U$, if there exists an element $l \in W'$ such that in each direction $z \in W$ where $G(v + tz)$ exists for $t > 0$ small enough, the directional derivative $G'_z(v)$ exists and we have*

$$\langle G'(v), z\rangle = \langle l, z\rangle = \lim_{t \to \overset{+}{0}} \frac{G(v+tz) - G(v)}{t}.$$

Definition 1.9 *[37] **(Frechet derivative)** Let U be a subset of W and $G : U \to \mathbb{R}$. The function G is said to be Frechet differentiable at $u \in U$, if there exists an element $l \in W'$, such that:*

$$\forall v \in U \; G(v) - G(u) = \langle l, v - u\rangle + o(v - u).$$

1.4 Some convergence criteria

Theorem 1.5 *[12] Let the sequence $(g_n)_{n\in\mathbb{N}}$ of $L^p(\Omega)$ and consider a function g that belongs to $L^p(\Omega)$ such that*

$$\|g_n - g\|_p \underset{n \longrightarrow \infty}{\longrightarrow} 0.$$

So, there exist a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ and a function h in $L^p(\Omega)$ satisfying the following conditions:

- *$g_{n_k}(x) \longrightarrow g(x)$ a.e on Ω,*
- *$|g_{n_k}(x)| \le h(x) \; \forall k$, a.e. on Ω.*

Theorem 1.6 *[12] **(Lebesgue's dominated convergence theorem)** Consider the sequence of functions (u_n) in $L^1(\Omega)$ such that*

- *$u_n(x)$ converges almost everywhere to u on Ω.*

- *We suppose that there exists $v \in L^1(\Omega)$ such that for all n, we have*

$$|u_n(x)| \leq v(x), \quad a.e. \text{ on } \Omega.$$

Then $u \in L^1(\Omega)$ and

$$\|u_n - u\|_{L^1} \longrightarrow 0.$$

Lemma 1.2 *[49] **(Brezis–Lieb Lemma)**. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $1 \leq p < \infty$ and $(u_n) \subset L^p(\Omega)$ a bounded sequence. Assume that $u_n \to u$ a.e. on Ω, then*

$$\lim_{n\to\infty} \left(\|u_n\|_p^p - \|u_n - u\|_p^p\right) = \|u\|_p^p .$$

Now, we give the definition of a Carathéodory function.

Definition 1.10 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain,$N \geq 1$. We say that $f = f(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function if for all $\xi \in \mathbb{R}^N$*

$$f(.,\xi) : \Omega \to \mathbb{R},$$

is measurable and

$$f(x,.) : \mathbb{R}^N \to \mathbb{R},$$

is continuous for almost every $x \in \Omega$.

CHAPTER 2

Critical point theory and variational methods

Contents

This chapter addresses critical point theory and variational methods that will be used to obtain the main results in this thesis.

2.1 Critical point theory

2.1.1 Critical points and Lagrange multiplier

Definition 2.1 (Homogeneous function) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. Then f is said to be homogeneous of degree k if for any $s > 0$ and any vector $x \in \mathbb{R}^n$, the function satisfies the equation:*

$$f(sx) = s^k f(x).$$

Definition 2.2 (Coercivity) *The function f is coercive if*

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

Definition 2.3 *[37] **(Critical point)** Let W be a Banach space, U an open subset of W and $J : U \to \mathbb{R}$ a function of class C^1. Let $w \in U$, if $J'(w) = 0$, then w is called a critical point of J. On the other hand, w is a regular point of J, if it is not a critical point.*

Let $c \in \mathbb{R}$, if there exists $u \in U$ with $J(u) = c$ and $J'(u)$=0, therefore c is called a critical value of J. On the other hand, we consider c as a regular value of J, if it does not correspond to a critical value.

Definition 2.4 *[37] **(Lagrange multiplier)** Let W be a Banach space, and consider functions $F : W \to \mathbb{R}$ and $J : W \to \mathbb{R}$ that belong to class C^1, and a set:*

$$S = \{u \in W : F(u) = 0\},$$

where $F'(u) \neq 0$ for any $u \in S$. The critical value of J on S is defined as the real number c such that there exists $u \in S$ and $\lambda \in \mathbb{R}$ satisfying:

$$J(u) = c \text{ and } J'(u) = \lambda F'(u).$$

The real λ is referred to as the Lagrange multiplier for c, while u represents a critical point of J on the set S.

Proposition 2.1 *[37] Under the assumptions and notations of definition 2.4, we assume that $u_0 \in S$ and $J(u_0) = \inf_{v \in S} J(v)$. So, there exists $\lambda \in \mathbb{R}$ such that:*

$$J'(u_0) = \lambda F'(u_0).$$

2.1.2 Palais-Smale condition

Definition 2.5 *[37] Let J be a functional defined on a Banach space W and belonging to the class $C^1(W, \mathbb{R})$ and let $a \in \mathbb{R}$. The Palais-Smale condition is satisfied by J at the level a, means that every $(u_n)_n \in W$, satisfying the following condition*

$$J(u_n) \to a \text{ with } J'(u_n) \text{ converges to } 0 \text{ in the dual space } W',$$

possesses a convergent subsequence.

Remark 2.1 *The sequence $(u_n)_n$ is referred to as the Palais-Smale sequence.*

2.1.3 Mountain pass theorem

We introduce the following theorem:

Theorem 2.1 *[49] Let Y be a Banach space, $\Phi : Y \to \mathbb{R}$ a function belongs to the class C^1, $e \in Y$ and $r > 0$ with $\|e\| > r$. Suppose that*

(1) $b = \inf_{||u||=r} \Phi(u) > \Phi(0) \geq \Phi(e)$,

and

(2) The function Φ satisfies the $(PS)_c$ condition.

So, c is a critical value of Φ, where c is given as follows

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$
$$\text{and } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

2.1.4 Fountain theorem

Remark 2.2 *Since X be a reflexive and separable Banach space, there exist $\{e_i\}_{i=1}^{\infty} \subset X$ and $\{e_j^*\}_{j=1}^{\infty} \subset X^*$ such that*

$$X = \overline{span\{e_i, i = 1, 2, \dots\}},\ X^* = \overline{span\{e_j^*, j = 1, 2, \dots\}} \tag{2.1}$$

where

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{2.2}$$

For $t = 1, 2 \dots$, denote

$$X_t = span\{e_t\},\ Y_t = \bigoplus_{i=1}^{t} X_i,\ Z_t = \overline{\bigoplus_{i \geq t} X_i}. \tag{2.3}$$

Proposition 2.2 *(Fountain theorem, see [28, 49]). Assume that*

(H_1) X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspaces X_t, Y_t and Z_t are given by (2.3). If for every, $t = 1, 2, ...,$, there exists $\rho_t > \gamma_t > 0$, such that

(H_2) $\inf_{u \in Z_t, \|u\| = \gamma_t} \varphi(u) \to \infty$ as $t \to \infty$,

(H_3) $\max_{u \in Y_t, \|u\| = \rho_t} \varphi(u) \leq 0$,

(H_4) φ satisfies (PS) condition for every $c > 0$.

Then, φ has a sequence of critical values tending to $+\infty$.

2.2 The Nehari Manifold

Nehari's variational method has proven to be highly valuable in critical point theory. He introduced this method by investigating a boundary value problem associated with a specific nonlinear second-order ordinary differential equation defined on an interval $[a, b]$. Nehari demonstrated the existence of a nontrivial solution to this problem, which can be acquired through the constrained minimization for the corresponding energy functional.

The Nehari manifold method can be described as follows:

Considering a Banach space E and suppose $J \in C^1(E, \mathbb{R})$ be a continuously differentiable functional. If $z \neq 0$ and $J^{'}(z) = 0$. So z belongs to the set

$$\mathcal{N} = \left\{ z \in E \backslash \{0\} : \left\langle J^{'}(z), z \right\rangle = 0 \right\}.$$

This set $\mathcal{N}$ serves as a natural constraint for finding nontrivial solutions. It is referred to as the Nehari manifold. Put

$$c := \inf_{z \in \mathcal{N}} J(z).$$

Under suitable conditions on J, one expects that there exists a $z_0 \in \mathcal{N}$ where c is attained, and z_0 is a critical point of the function J.

2.3 Fibering method

The fibering method, also known as the decomposition method, was introduced by Pohozaev in the late 1990s as a technique to study certain variational problems.
Let $A : X \to Y$ be a nonlinear operator between two Banach spaces. We consider the equation

$$A(v) = h. \tag{2.4}$$

This method is based on the representation of the equation's solutions as

$$v = su,$$

where $s \in \mathbb{R}$, $s \neq 0$ in some open $J \subseteq \mathbb{R}$. To fully understand the fibering method, we provide a comprehensive description starting with the definition of the fiber map. We define the fiber map $\phi(s) : \mathbb{R}^{+} \to \mathbb{R}$,

$$\phi(s) = J(sv).$$

Next, we compute the first and second derivatives of $\phi(s)$, denoted as $\phi'(s)$ and $\phi''(s)$ respectively. These derivatives play an important role in partitioning the set $\mathcal{N}$ into three distinct parts $\mathcal{N}^0$, $\mathcal{N}^+$ and $\mathcal{N}^-$, and corresponding respectively, to points of inflection, local minima, and local maxima of ϕ defined as follows:

$$\mathcal{N}^0 = \{u \in \mathcal{N} : \phi''(1) = 0\},$$

$$\mathcal{N}^{+} = \{u \in \mathcal{N} : \phi''(1) > 0\},$$

$$\mathcal{N}^{-} = \{u \in \mathcal{N} : \phi''(1) < 1\}.$$

The choice of using $\phi''(1)$ in these definitions is deliberate. It ensures that if v is a local minimum for J, then ϕ has a local minimum at $s = 1$. By partitioning $\mathcal{S}$ into these three parts, we can analyze the behavior of the solutions and explore the existence of local minima, local maxima, and points of inflection.

CHAPTER 3

Existence of solutions for a Schrödinger-Kirchhoff system involving the fractional p- Laplacian and critical nonlinearities

Contents

In this chapter, by using the Nehari manifold method and the fibering maps analysis, we prove the existence of two nontrivial weak solutions for a specific type of Schrödinger-Kirchhoff elliptic system. This system involves critical exponents and the nonlocal fractional p-Laplacian. In the first part, we introduce our problem and discuss its variational formulation. Moreover, we state our main result. Then, in the next part, we present the Nehari manifold structure with the analysis of fibering maps, which is related to our problem. In the last part, we establish the proof of our main result. The results of this chapter are based on our paper [31].

In the sequel, i will represent either the number 1 or 2.

3.1 Introduction

We take into consideration the following system

$$\begin{cases} K_1(\,\|w\|_{R_1}^p\,)\left((-\Delta)_p^s w + R_1(x)\,|w|^{p-2}\,w\right) = b_1(x)\,|w|^{p_s^*-2}\,w + \lambda f(x,w,z) \text{ in } \Omega, \\ K_2(\,\|z\|_{R_2}^p\,)\left((-\Delta)_p^s z + R_2(x)\,|z|^{p-2}\,z\right) = b_2(x)\,|z|^{p_s^*-2}\,z + \lambda g(x,w,z) \text{ in } \Omega, \\ w, z > 0 \text{ in } \Omega, \\ w = z = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

here, $\|.\|_{R_1}$ and $\|.\|_{R_2}$ will be introduced later in accordance with (3.8), Ω is a Lipschitz bounded domain in $\mathbb{R}^N$, $N > ps, 0 < s < 1 < q < p, p_s^* = \frac{Np}{N-sp}$, $\lambda > 0$, the weight functions b_1 and b_2 are bounded and positive on the domain Ω, and $(-\Delta)_p^s$ represents the fractional p-Laplacian operator, which is defined as

$$(-\Delta)_p^s\, u = 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+ps}}dy, x \in \mathbb{R}^N,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, for further details about this operator, we advise the reader to see [22]. We suppose that the function $R_i : \Omega \to (0,\infty)$ is continous and there is $R_i > 0$ with $\inf_\Omega R_i \geq R_i$. Also, we suppose that $K_i : (0,\infty) \to (0,\infty)$ is continuous and satisfy certain conditions:

$(H_1)\lim_{t\to+\infty} t^{1-\frac{p_s^*}{p}} K_i(t) = 0.$

(H_2) There exists $k_i > 0$ such that for all $t > 0$,

$$K_i(t) \geq k_i.$$

(H_3) There exists $\theta_i \in [1, \frac{p_s^*}{p}[$ such that for all $t > 0$,

$$K_i(t)t \leq \theta_i \widehat{K}_i(t),$$

where $\widehat{K}_i(t) = \int_0^t K_i(s)ds$. Let

$$k = \min(k_1, k_2). \tag{3.2}$$

$$\theta = \max(\theta_1, \theta_2). \tag{3.3}$$

Moreover, the functions f and g belong to the class $C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}, [0, \infty[)$ and have homogeneity of degree $(q-1)$. This means that for every t greater than zero and (x, w, z) belonging to the set $\Omega \times \mathbb{R} \times \mathbb{R}$, the following equations hold:

$$\begin{cases} f(x, tw, tz) = t^{q-1} f(x, w, z), \\ g(x, tw, tz) = t^{q-1} g(x, w, z). \end{cases} \tag{3.4}$$

More specifically, we present the function $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies the equations:

$$H_w(x, w, z) = f(x, w, z) \text{ and } H_z(x, w, z) = g(x, w, z),$$

where H_w (respectively, H_z) represents the partial derivative of H with respect to w (respectively, z). It is worth noting that the function H is in the class $C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and satisfies:

$$H(x, tw, tz) = t^q H(x, w, z)(t > 0), (x, w, z) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \tag{3.5}$$

$$qH(x, w, z) = uf(x, w, z) + vg(x, w, z), \tag{3.6}$$

$$|H(x, w, z)| \leq \gamma(|w|^q + |z|^q), \text{ for some constant } \gamma > 0. \tag{3.7}$$

Before presenting our result, we would like to introduce some notations. For $s \in (0,1)$, we introduce the functional space

$$W^{s,p}(Q) = \left\{ z \in L^p(\Omega) \text{ and } \frac{z(x)-z(y)}{|x-y|^{\frac{N}{p}+s}} \in L^p(Q, dxdy) \right\},$$

equipped with the norm

$$\|z\|_{W^{s,p}(Q)} = \left(\|z\|^p_{L^p(\Omega)} + \int_Q \frac{|z(x)-z(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

where $\Omega^c = \mathbb{R}^N \setminus \Omega$, $Q = \mathbb{R}^{2N} \diagdown (\Omega^c \times \Omega^c)$. The fact that $(W^{s,p}(Q), \|w\|_{W^{s,p}(Q)})$ is a uniformly convex Banach space is a widely recognized result.

Next, $L^p(\Omega, R_i)$ represents the Lebesgue space of functions $w : \Omega \to \mathbb{R}$, such that $R_i(x)|w|^p \in L^1(\Omega)$. It is equipped with the following norm

$$\|w\|_{p,R_i} = \left(\int_\Omega R_i(x)|w|^p dx \right)^{\frac{1}{p}}.$$

Let us denote by $W^{s,p}_{R_i}(Q)$ the completion of $C_0^\infty(Q)$ with respect to the norm

$$\|w\|_{R_i} = \left(\|w\|^p_{p,R_i} + \int_Q \frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}. \tag{3.8}$$

According to Theorem 1.3, we have the knowledge that the embedding $W^{s,p}_{R_i}(Q) \hookrightarrow L^r(\Omega)$ is continuous for every $r \in [p, p_s^*]$. This implies the existence of a positive constant $C_r > 0$ such that

$$\|w\|_r \leq C_r \|w\|_{R_i} \text{ for all } w \in W^{s,p}_{R_i}(Q).$$

From (Lemma 2.1 of [50]), the embedding of $W^{s,p}_{R_i}(Q)$ in $L^r(\Omega)$ is compact for $1 \leq r < p_s^*$. Let $W = W^{s,p}_{R_1}(Q) \times W^{s,p}_{R_2}(Q)$, endowed with

$$\|(w,z)\| = (\|w\|^p_{R_1} + \|z\|^p_{R_2})^{\frac{1}{p}}.$$

Then, $(W, \|.\|)$ is a reflexive Banach space. For simplicity, we denote

$$A(w,z) = \|(w,z)\|^p.$$

Definition 3.1 *We say that the pair of functions* $(w,z) \in W$ *is a weak solution of* (3.1), *if*

$$K_1\left(\|w\|^p\right)\left(\int\limits_Q \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy + \int\limits_\Omega R_1(x)\,|w|^{p-2}\,wvdx\right)$$

$$+K_2\left(\|z\|^p\right)\left(\int\limits_Q \frac{|z(x)-z(y)|^{p-2}(z(x)-z(y))(u(x)-u(y))}{|x-y|^{N+ps}}dxdy + \int\limits_\Omega R_2(x)\,|z|^{p-2}\,zudx\right)$$

$$= \int\limits_\Omega (b_1(x)\,|w|^{p_s^*-2}\,wv + b_2(x)\,|z|^{p_s^*-2}\,zu)dx + \lambda\int\limits_\Omega (H_w(x,w,z)v + H_z(x,w,z)u)dx,$$

for any $(v,u) \in W$.

We consider the Euler-Lagrange functional $J_\lambda : W \to \mathbb{R}$, defined by

$$J_\lambda(w,z) = \frac{1}{p}(\widehat{K}_1(A_1(w)) + \widehat{K}_2(A_2(z))) - \frac{1}{p_s^*}B(w,z) - \lambda C(w,z), \tag{3.9}$$

where

$$\begin{aligned} A_i\left(u\right) &= \|u\|_{V_i}^p\,, u \in W \\ B\left(w,z\right) &= \int\limits_\Omega (b_1(x)\,|w|^{p_s^*} + b_2(x)\,|z|^{p_s^*})dx, \end{aligned}$$

and

$$C\left(w,z\right) = \int\limits_\Omega H(x,w,z)dx.$$

It is easily seen that, $J_\lambda \in C^1(W,\mathbb{R})$, and furthermore, $J_\lambda' : W \to W'$ is given by

$$\langle J_\lambda'(w,z),(w,z)\rangle = A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z)) - B(w,z) - \lambda q C(w,z), \tag{3.10}$$

where W' is the dual space of W.

Hence, critical points of J are weak solutions of problem (3.1).

Let S_{p,R_i} represent the best Sobolev constant for the embedding from the functional space $W^{s,p}_{R_i}(Q)$ to $L^{p_s^*}(\Omega)$

$$S_{p,R_i} = \inf_{u \in W^{s,p}_{R_i}(Q)\setminus\{0\}} \frac{\|w\|^p_{R_i}}{\|w\|^p_{p_s^*}}. \tag{3.11}$$

Put

$$S = \min(S_{p,R_1}, S_{p,R_2}). \tag{3.12}$$

Now, we announce our main result.

Theorem 3.1 *Suppose that* $s \in (0,1), ps < N$, $1 < q < p < p_s^*$ *and equations* (3.4), (3.5), (3.6), (3.7) *hold. If* K_i *satisfies* $(H_1) - (H_3)$, *then there exists a positive value* λ^* *such that for all* λ *in the interval* $(0, \lambda^*)$, *the problem* (3.1) *possesses at least two nontrivial weak solutions.*

3.2 The Nehari manifold method and fibering maps analysis

In this section, we present some fundamental results regarding the Nehari manifold and the analysis of fibering maps. These results will be utilized in the subsequent section. Since J_λ is not bounded from below on W, we aim to establish the boundedness from below of the energy functional J_λ on a specific set of W, referred to as the Nehari manifold. We define this manifold as follows:

$$\mathcal{N}_\lambda = \{(w,z) \in W\setminus\{(0,0)\}, \langle J'_\lambda(w,z),(w,z)\rangle_W = 0\},$$

it is evident that an element $(w,z) \in \mathcal{N}_\lambda$, is equivalent to

$$A_1(w)M_1(A_1(w)) + A_2(z)M_2(A_2(z)) - B(w,z) - \lambda q C(w,z) = 0, \tag{3.13}$$

where A_1, A_2, B and C are introduced in Section 3.1.

Hence, from (3.10), we note that elements in $\mathcal{N}_\lambda$ are equivalent to nontrivial critical points that represent solutions of problem (3.1). To gain a better understanding of $\mathcal{N}_\lambda$, it is beneficial to analyze it in terms of the stationary points of the fibering maps $\varphi_{w,z} : (0,\infty) \to \mathbb{R}$. These fibering maps are given as follows:

$$\varphi_{w,z}(t) = \frac{1}{p}(\widehat{K}_1(t^p A_1(w)) + \widehat{K}_2(t^p A_2(z))) - \frac{t^{p_s^*}}{p_s^*}B(w,z) - \lambda t^q C(w,z) = J_\lambda(tw,tz).$$

A simple computation shows that

$$\begin{aligned}\varphi'_{w,z}(t) &= t^{p-1}(A_1(w)K_1(t^pA_1(w)) + A_2(z)K_2(t^pA_2(z)))\\ &\quad -t^{p_s^*-1}B(w,z) - \lambda q t^{q-1}C(w,z),\end{aligned}$$

$$\begin{aligned}\varphi''_{w,z}(t) &= (p-1)t^{p-2}\left(A_1(w)K_1(t^pA_1(w)) + A_2(z)K_2(t^pA_2(z))\right)\\ &\quad +pt^{2p-2}\left((A_1(w))^2K_1'(t^pA_1(w)) + (A_2(z))^2K_2'(t^pA_2(z)\right)\\ &\quad -(p_s^*-1)t^{p_s^*-2}B(w,z) - \lambda q(q-1)t^{q-2}C(w,z).\end{aligned}$$

We recommend referring to the following sources [14,15,25], for more details and properties about these maps.
It can be easily observed that for any $t>0$,

$$\varphi'_{w,z}(t) = \langle J'_\lambda(tw,tz),(w,z)\rangle_W = \frac{1}{t^2}\langle J'_\lambda(tw,tz),(tw,tz)\rangle_W.$$

So, the pair (tw,tz) belongs to the Nehari manifold $\mathcal{N}_\lambda$, is equivalent to $\varphi'_{w,z}(t)=0$. In the particular situation, if $t=1$ we get $(w,z)\in\mathcal{N}_\lambda$, is equivalent to $\varphi'_{w,z}(1)=0$. Therefore, from (3.13), we get

$$\begin{aligned}\varphi''_{w,z}(1) &= (p-1)\left(A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z))\right)\\ &\quad +p\left((A_1(w))^2K_1'(A_1(w)) + (A_2(z))^2K_2'(A_2(z))\right)\\ &\quad -(p_s^*-1)B(w,z) - \lambda q(q-1)C(w,z)\\ &= p\left((A_1(w))^2K_1'(A_1(w)) + (A_2(z))^2K_2'(A_2(z))\right) - (p_s^*-p)B(w,z) - \lambda q(q-p)C(w,z) \quad (3.14)\\ &= p\left((A_1(w))^2K_1'(A_1(w)) + (A_2(z))^2K_2'(A_2(z))\right)\\ &\quad -(p_s^*-p)\left(A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z))\right) + \lambda q(p_s^*-q)C(w,z) \quad (3.15)\\ &= p\left((A_1(w))^2K_1'(A_1(w)) + (A_2(z))^2K_2'(A_2(z))\right)\\ &\quad +(p-q)\left(A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z))\right) - (p_s^*-q)B(w,z). \quad (3.16)\end{aligned}$$

To achieve a multiplicity of solutions, we can split the Nehari manifold $\mathcal{N}_\lambda$ into three distinct parts.

$$\mathcal{N}_\lambda^0 = \left\{(w,z) \in \mathcal{N}_\lambda \text{ such that } \varphi''_{w,z}(1) = 0\right\} = \left\{(w,z) \in W : \varphi'_{w,z}(1) = 0 \text{ and } \varphi''_{w,z}(1) = 0\right\},$$

$$\mathcal{N}_\lambda^+ = \left\{(w,z) \in \mathcal{N}_\lambda \text{ such that } \varphi''_{w,z}(1) > 0\right\} = \left\{(w,z) \in W : \varphi'_{w,z}(1) = 0 \text{ and } \varphi''_{w,z}(1) > 0\right\},$$

$$\mathcal{N}_\lambda^- = \left\{(w,z) \in \mathcal{N}_\lambda \text{ such that } \varphi''_{w,z}(1) < 0\right\} = \left\{(w,z) \in W : \varphi'_{w,z}(1) = 0 \text{ and } \varphi''_{w,z}(1) < 0\right\}.$$

Lemma 3.1 *Assume that (w_0, z_0) is a local minimizer for J_λ on $\mathcal{N}_\lambda$, and $(w_0, z_0) \notin \mathcal{N}_\lambda^0$. So, (w_0, z_0) is a critical point of J_λ.*

Proof Let (w_0, z_0) represents a local minimizer for J_λ on $\mathcal{N}_\lambda$, then it follows that (w_0, z_0) satisfies the optimization problem

$$\begin{cases} \min\limits_{(w,z)\in\mathcal{N}_\lambda} J_\lambda(w,z) = J_\lambda(w_0,z_0), \\ \\ \beta(w_0,z_0) = 0, \end{cases}$$

where

$$\beta(w,z) = A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z)) - B(w,z) - \lambda q C(w,z).$$

By applying the Lagrange multipliers theorem, we can conclude that there is a real number δ, such that

$$J'_\lambda(w_0,z_0) = \delta\beta'(w_0,z_0). \tag{3.17}$$

As $(w_0, z_0) \in \mathcal{N}_\lambda$, we obtain

$$\delta\langle\beta'(w_0,z_0),(w_0,z_0)\rangle_W = \langle J'_\lambda(w_0,z_0),(w_0,z_0)\rangle_W = 0. \tag{3.18}$$

Moreover, from (3.13) and the constraint $\beta(w_0, z_0) = 0$, we obtain

$$\begin{aligned}\langle \beta'(w_0, z_0), (w_0, z_0)\rangle_W &= p\left((A_1(w_0))^2 K_1'(A_1(w_0)) + (A_2(z_0))^2 K_2'(A_2(z_0))\right) \\ &\quad -(p_s^* - p)B(w_0, z_0) - \lambda q(q-p)C(w_0, z_0) \\ &= \varphi''_{w_0,z_0}(1).\end{aligned}$$

Since $(w_0, z_0) \notin \mathcal{N}_\lambda^0$, so $\varphi''_{w_0,z_0}(1) \neq 0$. Thus, by (3.18) we find that $\delta = 0$. Hence, by substitution of δ in (3.17), we get $J_\lambda'(w_0, z_0) = 0$. Therefore, Lemma 3.1 is proved.

To gain a better understanding of the Nehari manifold and fibering maps, we introduce the function $\psi_{w,z} : (0, \infty) \to \mathbb{R}$, defined as

$$\psi_{w,z}(t) = t^{p-q}\left(A_1(w)K_1(t^p A_1(w)) + A_2(z)K_2(t^p A_2(z))\right) - t^{p_s^*-q}B(w,z) - \lambda q C(w,z). \tag{3.19}$$

We observe that

$$t^{q-1}\psi_{w,z}(t) = \varphi'_{w,z}(t).$$

Thus, it is evident that, $(tw, tz) \in \mathcal{N}_\lambda$ is equivalent to

$$\psi_{w,z}(t) = 0. \tag{3.20}$$

By a straightforward computation, we have

$$\begin{aligned}\psi'_{w,z}(t) &= (p-q)t^{p-q-1}\left(A_1(w)K_1(t^p A_1(w)) + A_2(z)K_2(t^p A_2(z))\right) \\ &\quad + p t^{2p-q-1}\left(A_1^2(w)K_1'(t^p A_1(w)) + A_2^2(z)K_2'(t^p A_2(z))\right) - (p_s^* - q)t^{p_s^*-q-1}B(w,z),\end{aligned}$$

therefore, we see that, if $(tw, tz) \in \mathcal{N}_\lambda$, then

$$t^{q-1}\psi'_{w,z}(t) = \varphi''_{w,z}(t). \tag{3.21}$$

So, $(tw, tz) \in \mathcal{N}_\lambda^+$, $\left(\text{respectively, } (tw, tz) \in \mathcal{N}_\lambda^-\right)$ is equivalent to $\psi'_{w,z}(t) > 0$, (respectively, $\psi'_{w,z}(t) < 0$).

Put

$$\lambda_* = \frac{(kS)^{\frac{p_s^*-q}{p_s^*-p}}}{\gamma q|\Omega|^{\frac{p_s^*-q}{p_s^*}}}\left(\frac{p_s^*-p}{p_s^*-q}\right)\left(\frac{p-q}{(p_s^*-q)b}\right)^{\frac{p-q}{p_s^*-p}}. \tag{3.22}$$

Lemma 3.2 *Let $(w,z)\in \mathcal{N}_\lambda$ and suppose that (H_1) and (H_2) hold. So, there exists $\lambda_*>0$ and two unique values $t_1>0$ and $t_2>0$, such that $(t_1w,t_1z)\in\mathcal{N}_\lambda^+$, $(t_2w,t_2z)\in\mathcal{N}_\lambda^-$ for any λ within the interval $(0,\lambda_*)$.*

Proof By (H_1) and (3.19), we note that $\psi_{w,z}(t)\to -\lambda q C(w,z)$ when $t\to 0^+$, $\psi_{w,z}(t)\to-\infty$ when $t\to\infty$. On the other hand, by combining (3.7), (3.11) with the Hölder inequality, we get

$$\begin{aligned} B(w,z) &\le \|b_1\|_\infty \|w\|_{p_s^*}^{p_s^*} + \|b_2\|_\infty \|z\|_{p_s^*}^{p_s^*} \le b(\|w\|_{p_s^*}^{p_s^*}+\|z\|_{p_s^*}^{p_s^*}) \\ &\le b(S_{p,R_1}^{-\frac{p_s^*}{p}}(A_1(w))^{\frac{p_s^*}{p}} + S_{p,R_2}^{-\frac{p_s^*}{p}}(A_2(z))^{\frac{p_s^*}{p}}) \le S^{-\frac{p_s^*}{p}} b(A(w,z))^{\frac{p_s^*}{p}}, \end{aligned} \tag{3.23}$$

$$C(w,z)\le \gamma(\|w\|_q^q+\|z\|_q^q)\le \gamma\,|\Omega|^{\frac{p_s^*-q}{p_s^*}}(\|w\|_{p_s^*}^q+\|z\|_{p_s^*}^q)\le \gamma S^{-\frac{q}{p}}\,|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(w,z))^{\frac{q}{p}}, \tag{3.24}$$

where $b=\max(\|b_1\|_\infty,\|b_2\|_\infty)$ and S is defined in (3.12). Therefore, by using (H_2), and by combining (3.23) with (3.24), we get

$$\begin{aligned} \psi_{w,z}(t) &\ge t^{p-q}(k_1A_1(w)+k_2A_2(z)) - t^{p_s^*-q}S^{-\frac{p_s^*}{p}} b(A(w,z))^{\frac{p_s^*}{p}} \\ &\quad -\lambda q\gamma S^{-\frac{q}{p}}\,|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(w,z))^{\frac{q}{p}} \\ &\ge k t^{p-q}A(w,z) - t^{p_s^*-q}S^{-\frac{p_s^*}{p}} b(A(w,z))^{\frac{p_s^*}{p}} \\ &\quad -\lambda q\gamma S^{-\frac{q}{p}}\,|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(w,z))^{\frac{q}{p}} \\ &\ge (A(w,z))^{\frac{q}{p}} F_{w,z}(t), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} F_{w,z}(t) &= k t^{p-q}(A(w,z))^{\frac{p-q}{p}} - t^{p_s^*-q}S^{-\frac{p_s^*}{p}} b(A(w,z))^{\frac{p_s^*-q}{p}} \\ &\quad -\lambda q\gamma S^{-\frac{q}{p}}\,|\Omega|^{\frac{p_s^*-q}{p_s^*}}, \end{aligned}$$

with k is given in (3.2). Since $1<q<p<p_s^*$, we note that $\lim_{t\to 0^+}F_{w,z}(t)<0$ and $\lim_{t\to\infty}F_{w,z}(t)=-\infty$. So, By a straightforward computation we can demonstrate that $F_{w,z}(t)$ has a unique critical point at

$$t_{max}(w,z)=\left(\frac{k}{S^{-\frac{p_s^*}{p}}b}\left(\frac{p-q}{p_s^*-q}\right)\right)^{\frac{1}{p_s^*-p}}(A(w,z))^{\frac{-1}{p}}, \tag{3.26}$$

moreover

$$F_{w,z}(t_{max}) = q\gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^*}} (\lambda_* - \lambda), \tag{3.27}$$

with λ_* is determined by (3.22).

If we consider λ to be less than λ_*, then, we have

$$F_{w,z}(t_{max}) > 0,$$

therefore, from (3.25), we get

$$\psi_{w,z}(t_{max}) \geq (A(w,z))^{\frac{q}{p}} F_{w,z}(t_{max}) > 0. \tag{3.28}$$

As a result of analyzing the variation of $\psi_{w,z}(t)$, we can deduce the existence of two unique values $t_1 < t_{max}(w,z)$ and $t_2 > t_{max}(w,z)$, such that $\psi'_{w,z}(t_1) > 0$, $\psi'_{w,z}(t_2) < 0$, and

$$\psi_{w,z}(t_1) = 0 = \psi_{w,z}(t_2).$$

Finally, it can be deduced from (3.20) and (3.21) that $(t_1 w, t_1 z) \in \mathcal{N}_\lambda^+$ and $(t_2 w, t_2 z) \in \mathcal{N}_\lambda^-$.

The fact that sets $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty can be observed from Lemma 3.2. Now, in the subsequent lemma, we give a property associated with $\mathcal{N}_\lambda^0$.

Lemma 3.3 *Assume that (H_2) is satisfied, then for every λ values in the interval $(0, \lambda_*)$, we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof Let us proceed by employing a proof by contradiction. Suppose that there is λ greater than zero with $0 < \lambda < \lambda_*$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. That is there exists $(w_0, z_0) \in \mathcal{N}_\lambda^0$. From (H_2), (3.15) and (3.24), we get

$$\begin{aligned}
0 = \varphi''_{w,z}(1) &= p\left((A_1(w))^2 K_1'(A_1(w)) + (A_2(z))^2 K_2'(A_2(z))\right) \\
&\quad -(p_s^* - p)\left(A_1(w) K_1(A_1(w)) + A_2(z) K_2(A_2(z))\right) + \lambda q(p_s^* - q) C(w,z) \\
&\leq p\left((A_1(w))^2 K_1'(A_1(w)) + (A_2(z))^2 K_2'(A_2(z))\right) \\
&\quad -(p_s^* - p)\left(k_1 A_1(w) + k_2 A_2(z)\right) + \lambda q(p_s^* - q) C(w,z) \\
&\leq p\left((A_1(w))^2 K_1'(A_1(w)) + (A_2(z))^2 K_2'(A_2(z))\right) \\
&\quad -(p_s^* - p) k A(w,z) + \lambda q(p_s^* - q)\gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^*}} (A(w,z))^{\frac{q}{p}}.
\end{aligned} \tag{3.29}$$

On the other hand, from (H_2), (3.16) and (3.23), one has

$$
\begin{aligned}
0=\varphi''_{w,z}(1) &= p\left((A_1(w))^2K_1'(A_1(w))+(A_2(z))^2K_2'(A_2(z))\right)\\
&\quad +(p-q)\left(A_1(w)K_1(A_1(w))+A_2(z)K_2(A_2(z))\right)-(p_s^*-q)B(w,z)\\
&\geq p\left((A_1(w))^2K_1'(A_1(w))+(A_2(z))^2K_2'(A_2(z))\right)\\
&\quad +(p-q)\left(k_1A_1(w)+k_2A_2(z)\right)-(p_s^*-q)B(w,z)\\
&\geq p\left((A_1(w))^2M_1'(A_1(w))+(A_2(z))^2M_2'(A_2(z))\right)\\
&\quad +(p-q)kA(w,z)-(p_s^*-q)S^{-\frac{p_s^*}{p}}b(A(w,z))^{\frac{p_s^*}{p}}.
\end{aligned}
\tag{3.30}
$$

Then, combining (3.29) and (3.30), we get

$$
\lambda \geq \frac{k(A(w,z))^{\frac{p-q}{p}}-S^{-\frac{p_s^*}{p}}b(A(w,z))^{\frac{p_s^*-q}{p}}}{q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}}. \tag{3.31}
$$

Now, we define the function H on $(0,\infty)$, by

$$
H(t)=\frac{kt^{\frac{p-q}{p}}-S^{-\frac{p_s^*}{p}}bt^{\frac{p_s^*-q}{p}}}{q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}},
$$

since $p_s^*>p>q>1$, then, we observe that $\lim\limits_{t\to 0^+}H(t)=0$ and $\lim\limits_{t\to\infty}H(t)=-\infty$, thus, a straightforward calculation demonstrates that the function H achieves its maximum at

$$
\tilde{t}=\left(\left(\frac{p-q}{p_s^*-q}\right)\frac{kS^{\frac{p_s^*}{p}}}{b}\right)^{\frac{p}{p_s^*-p}},
$$

and

$$
\max_{t>0}H(t)=H(\tilde{t})=\lambda_*. \tag{3.32}
$$

Hence, by equations (3.31) and (3.32), we have

$$
\lambda \geq \max_{t>0}H(t)=\lambda_*,
$$

and this contradicts the assumption that $0<\lambda<\lambda_*$. Then, $\mathcal{N}_\lambda^0=\emptyset$, for $\lambda\in(0,\lambda_*)$.

Lemma 3.4 *Let $(H_2)-(H_3)$ hold. So J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof If $(w, z) \in \mathcal{N}_\lambda$, (3.13) implies that

$$B(w, z) = A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z)) - \lambda q C(w, z),$$

this implies

$$\begin{aligned} J_\lambda(w, z) = & \frac{1}{p}(\widehat{K}_1(A_1(w)) + \widehat{K}_2(A_2(z))) - \frac{1}{p_s^*}(A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z))) \\ & -\lambda \left(1 - \frac{q}{p_s^*}\right) C(w, z). \end{aligned}$$

Moreover, By virtue of assumptions $(H_2), (H_3)$ and from (3.24), we get

$$\begin{aligned} J_\lambda(w, z) \geq & \frac{1}{\theta_1 p} A_1(w)K_1(A_1(w)) + \frac{1}{\theta_2 p} A_2(z)K_2(A_2(z)) - \frac{1}{p_s^*} A_1(w)K_1(A_1(w)) \\ & - \frac{1}{p_s^*} A_2(z)K_2(A_2(z)) - \lambda \left(1 - \frac{q}{p_s^*}\right) C(w, z) \end{aligned}$$

$$\begin{aligned} & \geq \left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right) (A_1(w)K_1(A_1(w)) + A_2(z)K_2(A_2(z))) - \lambda \left(1 - \frac{q}{p_s^*}\right) C(w, z) \\ & \geq \left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right) (k_1 A_1(w) + k_2 A_2(z)) - \lambda \left(1 - \frac{q}{p_s^*}\right) C(w, z) \\ & \geq k \left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right) A(w, z) - \lambda \left(1 - \frac{q}{p_s^*}\right) \gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^* - q}{p_s^*}} (A(w, z))^{\frac{q}{p}}, \end{aligned}$$

where θ is given in (3.3). By considering $q < p$ and $p_s^* > \theta p$, it can be concluded that, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.

By Lemma (3.3), we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$. Furthermore, Lemma (3.4), enables us to define,

$$\alpha_\lambda^- = \inf_{(w,z) \in \mathcal{N}_\lambda^-} J_\lambda(w, z) \text{ and } \alpha_\lambda^+ = \inf_{(w,z) \in \mathcal{N}_\lambda^+} J_\lambda(w, z).$$

3.3 Proof of the main result

Proposition 3.1 *If $(H_2) - (H_3)$ hold, then there exist a positive value t_0 and an element $(w_0, z_0) \in W \backslash \{0\}$, with $(w_0, z_0) > 0$ in $\mathbb{R}^N$, such that*

$$\frac{1}{p}(\widehat{K}_1(A_1(w_0)t_0^p) + \widehat{K_2}(A_2(z_0)t_0^p)) - \frac{t_0^{p_s^*}}{p_s^*} B(w_0, z_0) = \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{s p_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}}. \tag{3.33}$$

Proof For every $(w,z) \in W \setminus \{0\}$, we define $\zeta_{w,z} : (0,\infty) \to \mathbb{R}$, as

$$\begin{aligned}\zeta_{w,z}(t) &= \frac{1}{p}(\widehat{K}_1(A_1(tw)) + \widehat{K_2}(A_2(tz))) - \frac{1}{p_s^*}B(t(w,z)) \\ &= \frac{1}{p}(\widehat{K}_1(t^p A_1(w)) + \widehat{K}_2(t^p A_2(z))) - \frac{t^{p_s^*}}{p_s^*}B(w,z).\end{aligned}$$

From (H_3) it can be shown that $\lim_{t\to 0^+}\zeta_{w,z}(t) \geq 0$ and $\lim_{t\to\infty}\zeta_{w,z}(t) = -\infty$. It is obvious that ζ belongs to the class C^1, moreover, from (H_2) and (H_3), we obtain

$$\begin{aligned}\zeta_{w,z}(t) &\geq \frac{t^p}{\theta_1 p}A_1(w)K_1(t^p A_1(w)) + \frac{t^p}{\theta_2 p}A_2(z)K_2(t^p A_2(z)) - \frac{t^{p_s^*}}{p_s^*}B(w,z) \\ &\geq \frac{t^p}{\theta p}(A_1(w)K_1(t^p A_1(w)) + A_2(z)K_2(t^p A_2(z))) - \frac{t^{p_s^*}}{p_s^*}B(w,z) \\ &\geq \frac{t^p}{\theta p}(k_1 A_1(w) + k_2 A_2(z)) - \frac{t^{p_s^*}}{p_s^*}B(w,z) \\ &\geq \frac{k}{\theta p}t^p A(w,z) - \frac{t^{p_s^*}}{p_s^*}B(w,z) = \omega_{w,z}(t).\end{aligned}$$

Since $\lim_{t\to 0}\omega_{w,z}(t) = 0$ and $\lim_{t\to\infty}\omega_{w,z}(t) = -\infty$. Then, $\omega_{w,z}$ attains its global maximum at

$$t_* = \left(\frac{kA(w,z)}{\theta B(w,z)}\right)^{\frac{1}{p_s^*-p}}.$$

Moreover, from (3.23) and the fact that $p_s^* > \theta p$, we have

$$\begin{aligned}\sup_{t>0}\omega_{w,z}(t) &= \omega_{w,z}(t_*) \\ &= \left(\frac{p_s^*-p}{pp_s^*}\right)\left(\frac{k}{\theta}\right)^{\frac{p_s^*}{p_s^*-p}}(A(w,z))^{\frac{p_s^*}{p_s^*-p}}(B(w,z))^{-\frac{p}{p_s^*-p}} \\ &= \left(\frac{p_s^*-p}{pp_s^*}\right)\left(\frac{k}{\theta}\right)^{\frac{p_s^*}{p_s^*-p}}\left((A(w,z))^{-\frac{p_s^*}{p}}B(w,z)\right)^{-\frac{p}{p_s^*-p}} \\ &= \frac{s}{N}\left(\frac{k}{\theta}\right)^{\frac{N}{sp}}(A(w,z))^{\frac{N}{sp}}(B(w,z))^{-\frac{N}{sp_s^*}} \\ &\geq \frac{s}{N}b^{\frac{-N}{sp_s^*}}(\frac{kS}{\theta})^{\frac{N}{sp}} \\ &\geq \left(\frac{s}{N} - \frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}(\frac{kS}{\theta})^{\frac{N}{sp}} \\ &= \left(\frac{p_s^*-\theta p}{\theta pp_s^*}\right)b^{\frac{-N}{sp_s^*}}(\frac{kS}{\theta})^{\frac{N}{sp}} > 0.\end{aligned} \tag{3.34}$$

Then, from (3.34) and the variations of the functions $\zeta_{w,z}$ and $\omega_{w,z}$, we have

$$\sup_{t>0}\zeta_{w,z} \geq \sup_{t>0}\omega_{w,z} \geq \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right) b^{\frac{-N}{sp_s^*}}(\frac{kS}{\theta})^{\frac{N}{sp}}.$$

Therefore, there exists $t_0 > 0$, that satisfy

$$\zeta_{w,z}(t_0) = \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right) b^{\frac{-N}{sp_s^*}}(\frac{kS}{\theta})^{\frac{N}{sp}}.$$

This concludes the proof of Proposition 3.1.

Put

$$L = (p-q)\left(\frac{k}{q}\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)\right)^{-\frac{q}{p-q}}\left(\frac{p_s^*-q}{\theta p^2}\right)^{\frac{p}{p-q}}\left(\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\right)^{\frac{p}{p-q}}. \tag{3.35}$$

Proposition 3.2 *Assume that conditions $(H_2)-(H_3)$ hold. If $p_s^* > p > q > 1$. Then, every Palais Smale sequence $\{(w_k, z_k)\} \subset W$ for J_λ at level c, with*

$$c < \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right) b^{\frac{-N}{sp_s^*}}\left(\frac{kS}{\theta}\right)^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}}L, \tag{3.36}$$

has a convergent subsequence.

Proof Let $\{(w_n, z_n)\}$ be a Palais Smale sequence for J_λ at level c, meaning that the functional $J_\lambda(w_n, z_n) \to c$ and $J'_\lambda(w_n, z_n) \to 0$, as $n \to \infty$. From Lemma (3.4), we observe that $\{(w_n, z_n)\}$ is bounded in W. So, there exist a subsequence indicated by $\{(w_n, z_n)\}$, satisfying the following properties:

$$\begin{aligned}
&(w_n, z_n) \rightharpoonup (w_*, z_*) \quad \text{weakly in } W, \quad \|w_n\|_{R_1} \to \mu, \|z_n\|_{R_2} \to \eta,\\
&(w_n, z_n) \rightharpoonup (w_*, z_*) \quad \text{weakly in } L^{p_s^*}(\Omega) \times L^{p_s^*}(\Omega),\\
&(w_n, z_n) \to (w_*, z_*) \text{ strongly in } L^q(\Omega) \times L^q(\Omega),\ 1 \leq q < p_s^*,\\
&(w_n, z_n) \to (w_*, z_*) \text{ a.e. in } \Omega,
\end{aligned} \tag{3.37}$$

as $n \to \infty$, where $\mu > 0$ and $\eta > 0$. Since $1 \leq q < p_s^*$. So, by Theorem 1.5, there exists $l_1(x), l_2(x) \in L^q(\Omega)$ such that

$$|w_n(x)| \leq l_1(x),\ |z_n(x)| \leq l_2(x) \text{ a.e. in } \Omega.$$

Thus, using the dominated convergence theorem, we deduce that

$$C(w_n, z_n) \longrightarrow C(w_*, z_*), \text{ as } n \to \infty. \tag{3.38}$$

Furthermore, based on the Brezis-Lieb Lemma (Lemma 1.2), we have the following relations:

$$A_1(w_n) = A_1(w_n - w_*) + A_1(w_*) + o(1),$$

$$A_2(z_n) = A_2(z_n - z_*) + A_2(z_*) + o(1),$$

and

$$B(w_n, z_n) = B\left(w_n - w_*, z_n - z_*\right) + B(w_*, z_*) + o(1),$$

as $n \to \infty$. Consequently, as the value of n tends to infinity, we get

$$\begin{aligned}
o(1) &= \langle J'_\lambda(w_n, z_n), (w_n - w_*, z_n - z_*)\rangle_W \\
&= K_1(A_1(w_n)) \left(\int_Q \frac{|w_n(x) - w_n(y)|^{p-1}\left((w_n - w_*)(x) - (w_n - w_*)(y)\right)}{|x-y|^{N+ps}} dxdy \right. \\
&\quad \left. + \int_\Omega R_1(x)\, |w_n|^{p-1}\, (w_n - w_*)dx \right) - \int_\Omega b_1(x)|w_n|^{p_s^*-1}(w_n - w_*)dx \\
&\quad + K_2(A_2(z_n)) \left(\int_Q \frac{|z_n(x) - z_n(y)|^{p-1}\left((z_n - z_*)(x) - (z_n - z_*)(y)\right)}{|x-y|^{N+ps}} dxdy \right. \\
&\quad \left. + \int_\Omega R_2(x)\, |z_n|^{p-1}\, (z_n - z_*)dx \right) - \int_\Omega b_2(x)|z_n|^{p_s^*-1}(z_n - z_*)dx \\
&\quad - \lambda \int_\Omega (H_w(x, w_n, z_n)(w_n - w_*) + H_z(x, w_n, z_n)(z_n - z_*))dx \\
&= K_1(\mu^p)(\mu^p - A_1(w_*)) + K_2(\eta^p)(\eta^p - A_2(z_*)) - \int_\Omega (b_1(x)|w_n|^{p_s^*} + b_2(x)|z_n|^{p_s^*})dx \\
&\quad + \int_\Omega (b_1(x)|w_*|^{p_s^*} + b_2(x)|z_*|^{p_s^*})dx \\
&\quad - \lambda \int_\Omega (H_w(x, w_n, z_n)(w_n - w_*) + H_z(x, w_n, z_n)(z_n - z_*))dx + o(1) \\
&= K_1\left(\mu^p\right) A_1\left(w_n - w_*\right) + K_2\left(\eta^p\right) A_2\left(z_n - z_*\right) - B(w_n - w_*, z_n - z_*)
\end{aligned}$$

$$-\lambda \int_{\Omega} (H_w(x,w_n,z_n)(w_n-w_*)+H_z(x,w_n,z_n)(z_n-z_*))dx+o(1),$$

therefore,

$$K_1(\mu^p)\lim_{n\to\infty} A_1(w_n-w_*)+K_2(\eta^p)\lim_{n\to\infty} A_2(z_n-z_*)=\lim_{n\to\infty} B(w_n-w_*,z_n-z_*)$$
$$+\lim_{n\to\infty}\lambda\int_{\Omega}(H_w(x,w_n,z_n)(w_n-w_*)+H_z(x,w_n,z_n)(z_n-z_*))dx.$$

By (3.7), (3.37) and Holder's inequality, we get

$$\begin{aligned}
&\int_{\Omega}(H_w(x,w_n,z_n)(w_n-w_*)+H_z(x,w_n,z_n)(z_n-z_*))dx\\
&\leq \gamma q\int_{\Omega}|w_n|^{q-1}(w_n-w_*)dx+\gamma q\int_{\Omega}|z_n|^{q-1}(z_n-z_*)dx\\
&\leq \gamma q\|w_n\|_q^{q-1}\|w_n-w_*\|_q+\gamma q\|z_n\|_q^{q-1}\|z_n-z_*\|_q\\
&\leq C_q\gamma q\|w_n\|_{R_1}^{q-1}\|w_n-w_*\|_q+C_q\gamma q\|z_n\|_{R_2}^{q-1}\|z_n-z_*\|_q,
\end{aligned}$$

such that C_q is a positive constant. So, we obtain

$$\lim_{n\to\infty}\int_{\Omega}(H_w(x,w_n,z_n)(w_n-w_*)+H_z(x,w_n,z_n)(z_n-z_*))dx=0. \tag{3.39}$$

Thus, from (3.39), we deduce that

$$K_1(\mu^p)\lim_{n\to\infty} A_1(w_n-w_*)+K_2(\eta^p)\lim_{n\to\infty} A_2(z_n-z_*)=\lim_{n\to\infty} B(w_n-w_*,z_n-z_*).$$

Let us denote

$$a:=\lim_{n\to\infty} B(w_n-w_*,z_n-z_*)=K_1(\mu^p)\lim_{n\to\infty} A_1(w_n-w_*)+K_2(\eta^p)\lim_{n\to\infty} A_2(z_n-z_*).$$

Therefore, we aim to prove that (w_n,z_n) converges strongly to (w_*,z_*) which means that $a=0$. We assume by contradiction that $a>0$. Thus, from (H_2), we get

$$\begin{aligned}
A_1(w_n-w_*)K_1(\mu^p)+A_2(z_n-z_*)K_2(\eta^p)&\geq k_1A_1(w_n-w_*)+k_2A_2(z_n-z_*)\\
&\geq kA(w_n-w_*,z_n-z_*).
\end{aligned}\tag{3.40}$$

Using (3.23), we get

$$A(w_n - w_*, z_n - z_*) \geq S b^{-\frac{p}{p_s^*}} \left(B\left(w_n - w_*, z_n - z_*\right)\right)^{\frac{p}{p_s^*}}. \tag{3.41}$$

So, using (3.40) and (3.41), we obtain

$$A_1(w_n - w_*)K_1(A_1(w_*)) + A_2(z_n - z_*)K_2(A_2(z_*)) \geq kSb^{-\frac{p}{p_s^*}} \left(B\left(w_n - w_*, z_n - z_*\right)\right)^{\frac{p}{p_s^*}}.$$

Which implies that

$$a \geq b^{\frac{-N}{sp_s^*}} (kS)^{\frac{N}{sp}}. \tag{3.42}$$

So, from (H_3), (3.38) and (3.42), we get

$$\begin{aligned}
c = \lim_{n\longrightarrow\infty} J_\lambda(w_n, z_n) &= \lim_{n\longrightarrow\infty} \left(J_\lambda(w_n, z_n) - \frac{1}{p_s^*} \langle J'_\lambda(w_n, z_n), (w_n, z_n)\rangle_W \right) \\
&= \lim_{n\longrightarrow\infty} \left[\frac{1}{p} \left(\widehat{K}_1\left(A_1(w_n)\right) + \widehat{K}_2\left(A_2(z_n)\right) \right) - \frac{1}{p_s^*} A_1(w_n)K_1(A_1(w_n)) \right. \\
&\quad \left. - \frac{1}{p_s^*} A_2(z_n)K_2\left(A_2(z_n)\right) - \lambda \left(\frac{p_s^* - q}{p_s^*} \right) C(w_n, z_n) \right] \\
&\geq \lim_{n\longrightarrow\infty} \left[\frac{1}{\theta_1 p} A_1(w_n)K_1(A_1(w_n)) + \frac{1}{\theta_2 p} A_2(z_n)K_2\left(A_2(z_n)\right) \right. \\
&\quad - \frac{1}{p_s^*} \left(A_1(w_n)K_1(A_1(w_n)) + A_2(z_n)K_2\left(A_2(z_n)\right)\right) \\
&\quad \left. - \lambda \left(\frac{p_s^* - q}{p_s^*} \right) C(w_n, z_n) \right] \\
&\geq \lim_{n\longrightarrow\infty} \left[\left(\frac{1}{\theta p} - \frac{1}{p_s^*} \right) \left(A_1(w_n)K_1(A_1(w_n)) + A_2(z_n)K_2\left(A_2(z_n)\right)\right) \right. \\
&\quad \left. - \lambda \left(\frac{p_s^* - q}{p_s^*} \right) C(w_n, z_n) \right] \\
&= \lim_{n\longrightarrow\infty} \left[\left(\frac{p_s^* - \theta p}{\theta p p_s^*} \right) A_1(w_n)K_1(\mu^p) + \left(\frac{p_s^* - \theta p}{\theta p p_s^*} \right) A_2(z_n)K_2(\eta^p) \right. \\
&\quad \left. - \lambda \left(\frac{p_s^* - q}{p_s^*} \right) C(w_n, z_n) \right] \\
&= \lim_{n\longrightarrow\infty} \left[\left(\frac{p_s^* - \theta p}{\theta p p_s^*} \right) \left(A_1(w_n - w_*)K_1(\mu^p) + A_2(z_n - z_*)K_2(\eta^p)\right) \right. \\
&\quad \left. + \left(\frac{p_s^* - \theta p}{\theta p p_s^*} \right) \left(A_1(w_*)K_1(\mu^p) + A_2(z_*)K_2(\eta^p)\right) - \lambda \left(\frac{p_s^* - q}{p_s^*} \right) C(w_n, z_n) \right] \\
&= \left(\frac{s}{N} - \frac{\theta - 1}{\theta p} \right) a + \left(\frac{s}{N} - \frac{\theta - 1}{\theta p} \right) \left(A_1(w_*)K_1(\mu^p) + A_2(z_*)K_2(\eta^p)\right)
\end{aligned}$$

$$
\begin{aligned}
&-\lambda\left(\frac{p_s^*-q}{p_s^*}\right)C(w_*,z_*)\\
&\geq\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)a+\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)(k_1A_1(w_*)+k_2A_2(z_*))\\
&-\lambda\left(\frac{p_s^*-q}{p_s^*}\right)C(w_*,z_*)\\
&\geq\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}(kS)^{\frac{N}{sp}}+\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)kA(w_*,z_*)\\
&-\lambda\left(\frac{p_s^*-q}{p_s^*}\right)C(w_*,z_*).
\end{aligned}
$$

On the other hand from (3.24), and considering the fact that $\theta p < p_s^*$, we get

$$
\begin{aligned}
c &\geq \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}(kS)^{\frac{N}{sp}}+\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)kA(w_*,z_*)\\
&\quad-\lambda\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\left(\frac{p_s^*-q}{\theta p}\right)(A(w_*,z_*))^{\frac{q}{p}}\\
&=\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}(kS)^{\frac{N}{sp}}+h\left(A(w_*,z_*)\right),
\end{aligned}
\tag{3.43}
$$

where h is defined as

$$
h(\xi)=\left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)k\xi-\lambda\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\left(\frac{p_s^*-q}{\theta p}\right)\xi^{\frac{q}{p}}.
$$

A straightforward calculation demonstrates that h attains its minimum at

$$
\xi_0=\left(\lambda q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\left(\frac{p^*-q}{kp}\right)\frac{1}{\frac{s}{N}\theta p-(\theta-1)}\right)^{\frac{p}{p-q}},
$$

and

$$
\inf_{\xi>0} h(\xi)=h(\xi_0)=-\lambda^{\frac{p}{p-q}}L,
\tag{3.44}
$$

where L is defined in (3.35).

Therefore, by (3.43), (3.44), and using $1\leq\theta$, we get

$$
\begin{aligned}
c &\geq \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}(kS)^{\frac{N}{sp}}-\lambda^{\frac{p}{p-q}}L\\
&\geq \left(\frac{s}{N}-\frac{\theta-1}{\theta p}\right)b^{\frac{-N}{sp_s^*}}\left(\frac{kS}{\theta}\right)^{\frac{N}{sp}}-\lambda^{\frac{p}{p-q}}L.
\end{aligned}
$$

This contradicts (3.36). Hence, $a=0$. So, we deduce that $(w_n,z_n)\to(w_*,z_*)$ strongly in W.

This ends the proof.

Proposition 3.3 *Assume that* $(H_2) - (H_3)$ *hold, then, there exist a positive value* t_0*,* $\lambda^* > 0$*, and* $(w_0, z_0) \in W$*, such that*

$$J_\lambda(t_0 w_0, t_0 z_0) \leq \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}} L, \tag{3.45}$$

holds for λ *is within the interval* $(0, \lambda^*)$*. In particular*

$$\alpha_\lambda^- < \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}} L. \tag{3.46}$$

Proof We put

$$\lambda_{**} = \left(\frac{1}{L}\left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}}\right)^{\frac{p-q}{p}}.$$

Then, for every λ in the interval $(0, \lambda_{**})$, we have

$$\left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}} L > 0. \tag{3.47}$$

By (3.33), there exists t_0 and $(w_0, z_0) \in W \setminus \{0\}$, such that

$$\begin{aligned} J_\lambda(t_0 w_0, t_0 z_0) &= \frac{1}{p}\left(\widehat{K}_1(t_0^p A_1(w_0)) + \widehat{K}_2(t_0^p A_2(z_0))\right) - \frac{t_0^{p^*}}{p^*} B(w_0, z_0) - \lambda t_0^q C(w_0, z_0) \\ &= \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda t_0^q C(w_0, z_0). \end{aligned} \tag{3.48}$$

Put

$$\lambda_{***} = \left(\frac{t_0^q C(w_0, z_0)}{L}\right)^{\frac{p-q}{q}},$$

therefore, for every $\lambda \in (0, \lambda_{***})$, we obtain

$$-\lambda t_0^q C(w_0, z_0) < -\lambda^{\frac{p}{p-q}} L. \tag{3.49}$$

Thus, from (3.48) and (3.49), we get

$$J_\lambda(t_0 w_0, t_0 z_0) < \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}} L.$$

Hence, (3.45) hold true.

Finally, by choosing

$$\lambda^* = \min(\lambda_*, \lambda_{**}, \lambda_{***}).$$

Then, we can ensure that for all values of λ in the interval $(0, \lambda^*)$, the fibering maps analysis $\varphi_{w,z}(t) = J_\lambda(tw, tz)$, leads to

$$\alpha_\lambda^- < \left(\frac{s}{N} - \frac{\theta - 1}{\theta p}\right) b^{\frac{-N}{sp_s^*}} (\frac{kS}{\theta})^{\frac{N}{sp}} - \lambda^{\frac{p}{p-q}} L.$$

Proof of Theorem (3.1) By Lemma 3.4, J_λ is bounded from below on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Therefore, there are two sequences $\{(w_k^+, z_k^+)\} \in \mathcal{N}_\lambda^+$ and $\{(w_k^-, z_k^-)\} \in \mathcal{N}_\lambda^-$, such that as k approaches infinity:

$$J_\lambda(w_k^+, z_k^+) \longrightarrow \inf_{(w,z)\in\mathcal{N}_\lambda^+} J_\lambda(w, z) = \alpha_\lambda^+,$$

and

$$J_\lambda(w_k^-, z_k^-) \longrightarrow \inf_{(w,z)\in\mathcal{N}_\lambda^-} J_\lambda(w, z) = \alpha_\lambda^-.$$

Using the fibering maps analysis $\varphi_{w,z}(t)$ we can deduce that $\alpha_\lambda^+ < 0$ and $\alpha_\lambda^- > 0$. Furthermore, according to Propositions (3.2) and (3.3), we have

$$J_\lambda(w_k^+, z_k^+) \longrightarrow J_\lambda(w_*^+, z_*^+) = \inf_{(w,z)\in\mathcal{N}_\lambda^+} J_\lambda(w, z) = \alpha_\lambda^+, J_\lambda'(w_k^+, z_k^+) \longrightarrow 0,$$

and

$$J_\lambda(w_k^-, z_k^-) \longrightarrow J_\lambda(w_*^-, z_*^-) = \inf_{(w,z)\in\mathcal{N}_\lambda^-} J_\lambda(w, z) = \alpha_\lambda^-, J_\lambda'(w_k^-, z_k^-) \longrightarrow 0,$$

as k tends to infinity. Therefore, (w_*^+, z_*^+) (respectively, (w_*^-, z_*^-)) is a minimizer of J_λ on $\mathcal{N}_\lambda^+$ (respectively, on $\mathcal{N}_\lambda^-$). So, by Lemma 3.1, problem (3.1) admits two solutions $(w_*^+, z_*^+) \in \mathcal{N}_\lambda^+$ and $(w_*^-, z_*^-) \in \mathcal{N}_\lambda^-$ in W. Moreover, since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, then (w_*^+, z_*^+) and (w_*^-, z_*^-) are distinct. Finally, the facts that $\alpha_\lambda^+ < 0$ and $\alpha_\lambda^- > 0$ imply that (w_*^+, z_*^+) and (w_*^-, z_*^-) are two nontrivial solutions for the problem (3.1). This finishes the proof.

CHAPTER 4

Multiplicity results for some Steklov problems involving $p(x)$-Laplacian operator

Contents

In this chapter, we prove the existence and multiplicity results for a class of $p(x)$-Laplacian problems using variational methods. The results of this chapter are based on our paper [2].

4.1 Introduction

We consider the boundary value problems

$$\begin{cases} L_{p(x)}u + |u|^{p(x)-2}\, u = S\,(x,u) \ \text{ in } \Omega, \\ \\ h\,(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial \nu} + l(x)|u|^{w(x)-2}u = Q(x,u) \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

and

$$\begin{cases} L_{p(x)}u + |u|^{p(x)-2}\, u = S\,(x,u) + \mu\,|u|^{\delta(x)-2}\, u \text{ in } \Omega, \\ \\ h\,(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial \nu} + l(x)|u|^{w(x)-2}u = Q(x,u) \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

where

$$L_{p(x)}u = -div(h(x)|\nabla u|^{p(x)-2}\nabla u)),$$

Ω is a bounded domain in $\mathbb{R}^N$, $(N \geq 2)$, the symbol $\partial\Omega$ represents a Lipschitz boundary of Ω, ν is the outer normal to $\partial\Omega$,

$$(-\Delta)_{p(x)}u(x) = -div(|\nabla u|^{p(x)-2}\nabla u),$$

is the $p(x)$-Laplace operator, $p(x), \delta(x) : \overline{\Omega} \to (1, +\infty))$ are bounded continuous functions, $w(x) \in C(\partial\Omega, (1, +\infty))$, the parameter μ is positive, $S : \Omega \times \mathbb{R} \to \mathbb{R}$ and $Q : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $z(x) \neq \gamma(x) \neq w(y)$, for every $x \in \overline{\Omega}$ and every $y \in \partial\Omega$, h and l are continuous functions with

$$h_1 \leq h(x) \leq h_2, \ \text{ and } l_1 \leq l(x) \leq l_2, \tag{4.3}$$

where the constants h_1, h_2, l_1 and l_2 are positive.
We define

$$\|u\| = \inf\{\kappa > 0 : \int_\Omega (h(x)|\tfrac{u(x)}{\kappa}|^{p(x)} + l(x)|\tfrac{\nabla u(x)}{\kappa}|^{p(x)})dx \leq 1\}, \text{ for } u \in W^{1,p(x)}(\Omega),$$

from (4.3), we can verify that $\|u\|$ is an equivalent norm on the variable exponent space $W^{1,p(x)}(\Omega)$.

In the sequel, positive constants $ci, i = 1, 2, ...$, are employed to represent values that may vary from one line to another.

4.2 Existence result

Our aim is to demonstrate the existence of a nontrivial weak solution for problem (4.1) using the mountain pass theorem. We will now present the hypotheses regarding problem (4.1) as follows:

(A_0) $\exists C_1 > 0$ and $\alpha \in C_+(\overline{\Omega})$, such that

$$|S\left(x, u\right)| \leq C_1 \left(1 + |u|^{\alpha(x)-1}\right), \text{ for all } (x, u) \in \Omega \times \mathbb{R},$$

where

$$1 < \alpha\left(x\right) < p^*\left(x\right). \tag{4.4}$$

(A_1) $\exists C_2 > 0$ and $\beta \in C_+(\partial\Omega)$, such that

$$|Q\left(x, u\right)| \leq C_2 \left(1 + |u|^{\beta(x)-1}\right), \text{ for all } (x, u) \in \partial\Omega \times \mathbb{R},$$

where

$$1 < \beta\left(x\right) < p_*\left(x\right), q\left(x\right) < p_*\left(x\right). \tag{4.5}$$

(A_2) As $u \to 0$, we have

$S\left(x, u\right) = o\left(|u|^{p^+-1}\right)$, for any $x \in \Omega$.

(A_3) As $u \to 0$, we have

$Q\left(x, u\right) = o\left(|u|^{p^+-1}\right)$, for any $x \in \partial\Omega$.

(A_4) There exist a constant $K_1 > 0$ and $\theta_1 > p^+$ such that for any $x \in \Omega$,

$$0 < \theta_1 \widehat{S}(x, u) \leq S(x, u)u, \ |u| \geq K_1,$$

where $\widehat{S}(x,t)=\int_0^t S(x,s)ds$.

(A_5) There exist a constant $K_2>0, \theta_2>p^+$ such that

$$0<\theta_2\widehat{Q}(x,u)\leq Q(x,u)u,\ |u|\geq K_2,\ \text{for all } x\in\partial\Omega,$$

where $\widehat{Q}(x,t)=\int_0^t Q(x,s)ds$.

We denote

$$\Sigma(u)=\int_\Omega (h\,(x)\,|\nabla u|^{p(x)}+|u|^{p(x)})dx,\ \text{for every } u\in W^{1,p(x)}\,(\Omega)\,.$$

Proposition 4.1 *(see [17,30,52]) There exist $\xi_1>0,\xi_2>0$, such that*

$(i)\Sigma(u)\geq 1 \Longrightarrow \xi_1||u||^{p^-}\leq\Sigma(u)\leq\xi_2||u||^{p^+}$,

$(ii)\ \Sigma(u)\leq 1 \Longrightarrow \xi_1||u||^{p^+}\leq\Sigma(u)\leq\xi_2||u||^{p^-}$.

Definition 4.1 *A function $u\in X=W^{1,p(x)}(\Omega)$, is called a weak solution to problem (4.1), if*

$$\int_\Omega h\,(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla v+\int_\Omega|u|^{p(x)-2}uvdx-\int_\Omega S(x,u)vdx+\int_{\partial\Omega}l(x)|u|^{w(x)-2}uvd\sigma$$
$$-\int_{\partial\Omega}Q(x,u)vd\sigma=0,$$

for any $v\in X$.

Let us define the energy functional $\Psi: X\to\mathbb{R}$,

$$\Psi(u)=I(u)+G(u)-\phi(u),$$

where

$$I(u)=\int_\Omega\frac{h\,(x)\,|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx,$$
$$G(u)=\int_{\partial\Omega}\frac{l(x)|u|^{w(x)}}{w(x)}d\sigma,$$

and

$$\phi(u)=\int_\Omega\widehat{S}(x,u)dx+\int_{\partial\Omega}\widehat{Q}(x,u)d\sigma.$$

Proposition 4.2 *(see [52])*
$I \in C^1(X,\mathbb{R})$, Then

$$\langle I'(u), v\rangle = \int_\Omega \left(h(x)|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv\right)dx,$$

for all $u, v \in X$. Moreover, I' is a mapping of type (S_+).

Proposition 4.3 *([30,52])*
$G \in C^1(X,\mathbb{R})$, and

$$\langle G'(u), v\rangle = \int_{\partial\Omega} l(x)|u|^{w(x)-2}uvd\sigma, \text{ for all } u, v \in X.$$

Moreover, the function G and its derivative are sequentially weakly-strongly continuous. On other words, if $u_n \rightharpoonup u$ in X it follows that $G(u_n) \to G(u)$ and $G'(u_n) \to G'(u)$.

Remark 4.1 *From Lemma 1.1 and proposition 1.4 , and under assumptions $(A_0), (A_1)$, it is easy to show that $\phi \in C^1(X,\mathbb{R})$ and*

$$\langle \phi'(u), v\rangle = \int_\Omega S(x,u(x))v(x)dx + \int_{\partial\Omega} Q(x,u(x))v(x)d\sigma, \text{ for all } u, v \in X.$$

Hence, from Proposition 4.2, Proposition 4.3 and remark 4.1, it is easily seen that $\Psi \in C^1(X,\mathbb{R})$. Furthermore, for each $u, v \in X$, we have the following expression:

$$\langle \Psi'(u), v\rangle = \langle I'(u), v\rangle + \langle G'(u), v\rangle + \langle \phi'(u), v\rangle.$$

Thus, the critical points of Ψ are equivalent to the weak solutions of (4.1).
The result we have obtained is as follows

Theorem 4.1 *Suppose that $(A_0) - (A_5)$ are verified. If $\min(\theta_1, \theta_2) > w^+$ and $\min(\alpha^-, \beta^-) > p^+$, then problem (4.1) has a nontrivial weak solution.*

The proof of our result is structured into multiple lemmas.

Lemma 4.1 *Let $\min(\theta_1, \theta_2) > w^+$, and assume that $(A_0), (A_1), (A_4)$ (A_5) are verified, then, Ψ satisfies the (PS) condition.*

Proof Suppose that $\{u_n\} \subset X$ is a sequence such that

$$\Psi\left(u_n\right) \to c \text{ as } n \to \infty, \tag{4.6}$$

where $c > 0$ is constant.

$$\Psi^{'}\left(u_n\right) \to 0, \text{in } X^*, \text{as } n \to \infty. \tag{4.7}$$

From (4.6) there exist $N_1 > 0$, such that

$$\left|\Psi\left(u_n\right)\right| \leq N_1. \tag{4.8}$$

By (4.7) there exists $N_2 > 0$ such that

$$\Big|\langle\Psi^{'}\left(u_n\right), u_n\rangle\Big| \leq N_2. \tag{4.9}$$

Now, we need to show that $\{u_n\}$ is bounded. By contradiction, we assume that $\|u_n\| \geq 1$. Using (4.8) and (4.9), we get for $\theta := \min\left(\theta_1, \theta_2\right)$

$$\begin{aligned} N_1 &\geq \Psi\left(u_n\right) \\ &\geq \frac{1}{p^+}\Sigma\left(u_n\right) + \frac{1}{w^+}\int_{\partial\Omega} l\left(x\right)\left|u_n\right|^{w(x)} d\sigma - \phi\left(u_n\right) \\ &\geq \frac{1}{p^+}\Sigma\left(u_n\right) + \frac{1}{\theta}\int_{\partial\Omega} l\left(x\right)\left|u_n\right|^{w(x)} d\sigma - \phi\left(u_n\right), \end{aligned} \tag{4.10}$$

and

$$N_2 \geq -\langle\Psi^{'}\left(u_n\right), u_n\rangle = -\Sigma\left(u_n\right) - \int_{\partial\Omega} l\left(x\right)\left|u_n\right|^{w(x)} d\sigma + \langle\phi^{'}\left(u_n\right), u_n\rangle. \tag{4.11}$$

Using (4.10), (4.11) and Proposition 4.1, we get

$$\begin{aligned} \theta N_1 + N_2 &\geq \left(\frac{\theta}{p^+} - 1\right)\Sigma\left(u_n\right) - \theta\phi\left(u_n\right) + \langle\phi^{'}\left(u_n\right), u_n\rangle \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\left\|u_n\right\|^{p^-} + \int_\Omega \Big(S\left(x, u_n\right)u_n - \theta_1\widehat{S}\left(x, u_n\right)\Big)\, dx \\ &\quad + \int_{\partial\Omega}\Big(Q\left(x, u_n\right)u_n - \theta_2\widehat{Q}\left(x, u_n\right)\Big)\, d\sigma. \end{aligned}$$

Hence, from hypotheses $(A_4)-(A_5)$, we have

$$\theta N_1+N_2\geq\left(\frac{\theta}{p^+}-1\right)\xi_1\left\|u_n\right\|^{p^-}.$$

Using $\theta=\min\left(\theta_1,\theta_2\right)>p^+$, we obtain a contradiction as $n\to\infty$.

Consequently, it can be deduced that $\{u_n\}$ is bounded X. So, there exist a subsequence denoted by $\{u_n\}$, and a component $u\in X$ such that, $\{u_n\}$ converges to u weakly in X.

By considering $p^+<\alpha\left(x\right)<p^*\left(x\right)$ and $w\left(x\right)<p_*\left(x\right)$, according to Proposition 1.4, we deduce that

$$\begin{cases}u_n\to u, \text{ strongly in } L^{\alpha(x)}\left(\Omega\right),\\ u_n\to u, \text{ strongly in } L^{p^+}\left(\Omega\right),\\ u_n\to u, \text{ strongly in } L^{w(x)}\left(\Omega\right).\end{cases}$$

To prove that $u_n\to u$ strongly in X, we have

$$\begin{aligned}\left\langle\Psi'\left(u_n\right),u_n-u\right\rangle=\left\langle I'\left(u_n\right),u_n-u\right\rangle&+\int_{\partial\Omega}l\left(x\right)\left|u_n\right|^{w(x)-2}u_n\left(u_n-u\right)d\sigma\\&-\int_{\Omega}S\left(x,u_n\right)\left(u_n-u\right)dx-\int_{\partial\Omega}Q\left(x,u_n\right)\left(u_n-u\right)d\sigma.\end{aligned}$$

From the Hölder's inequality and by Proposition 1.4 and Lemma 1.1, we get

$$\begin{aligned}\int_{\partial\Omega}l\left(x\right)\left|u_n\right|^{w(x)-1}\left|u_n-u\right|d\sigma&\leq l_2\left|u_n-u\right|_{L^{w(x)}}\left|\left|u_n\right|^{w(x)-1}\right|_{L^{\frac{w(x)}{w(x)-1}}}\\&\leq l_2\left|u_n-u\right|_{L^{w(x)}}\max\left(\left|u_n\right|_{L^{w(x)}}^{w^+-1},\left|u_n\right|_{L^{w(x)}}^{w^--1}\right)\\&\leq c_1\left|u_n-u\right|_{L^{w(x)}}\max\left(\left\|u_n\right\|^{w^+-1},\left\|u_n\right\|^{w^--1}\right).\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int_{\partial\Omega}l\left(x\right)\left|u_n\right|^{w(x)-2}u_n\left(u_n-u\right)d\sigma=0. \tag{4.12}$$

Using (A_0), Proposition 1.4 and Lemma 1.1, and the Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_{\Omega}S\left(x,u_n\right)\left(u_n-u\right)dx\right|&\leq\int_{\Omega}C_1\left|u_n-u\right|dx+\int_{\Omega}C_1\left|u_n\right|^{\alpha(x)-1}\left|u_n-u\right|dx\\&\leq C_1\left|\Omega\right|^{\frac{p^+-1}{p^+}}\left|u_n-u\right|_{L^{p^+}(\Omega)}+C_1\left|u_n-u\right|_{L^{\alpha(x)}}\left|\left|u_n\right|^{\alpha(x)-1}\right|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}}\\&\leq C_1\left|\Omega\right|^{\frac{p^+-1}{p^+}}\left|u_n-u\right|_{L^{p^+}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&+C_1 \left|u_n-u\right|_{L^{\alpha(x)}} \max\left(\left|u_n\right|_{L^{\alpha(x)}}^{\alpha^+-1}, \left|u_n\right|_{L^{\alpha(x)}}^{\alpha^--1}\right) \\
\leq\ & C_1 \left|\Omega\right|^{\frac{p^+-1}{p^+}} \left|u_n-u\right|_{L^{p^+}(\Omega)} \\
&+C_1 \left|u_n-u\right|_{L^{\alpha(x)}} \max\left(\left|\left|u_n\right|\right|^{\alpha^+-1}, \left|\left|u_n\right|\right|^{\alpha^--1}\right).
\end{aligned}
$$

It follows that

$$\lim_{n\to\infty}\int_{\Omega} S\left(x,u_n\right)\left(u_n-u\right)dx=0. \tag{4.13}$$

Similarly,

$$\lim_{n\to\infty}\int_{\partial\Omega} Q\left(x,u_n\right)\left(u_n-u\right)d\sigma=0. \tag{4.14}$$

Since $\langle \Psi'\left(u_n\right), u_n-u\rangle \to 0$ and by using (4.12) – (4.14), we get

$$\langle I'\left(u_n\right), u_n-u\rangle \to 0,$$

Now, letting $n\to\infty$, we obtain

$$\langle I'\left(u\right), u_n-u\rangle \to 0.$$

Thus

$$\lim_{n\to\infty}\langle I'\left(u_n\right)-I'\left(u\right), u_n-u\rangle = 0.$$

So, it follows from Proposition 4.2 that $u_n\to u$ strongly in X.

Lemma 4.2 *Let* $\min\left(\alpha^-,\beta^-\right)>p^+$, *and assume that* $\left(A_0\right)-\left(A_3\right)$ *hold, then there exist* $\rho>0, r>0$ *such that, for* $u\in X$, *we have*

$$\Psi\left(u\right)\geq\rho \textit{ for all } u\in X \textit{ with } \|u\|=r.$$

Proof Let $u\in X$, be such that $\|u\|<1$. Assumptions $\left(A_0\right)$ and $\left(A_2\right)$ imply

$$\left|\widehat{S}\left(x,u\right)\right|\leq\varepsilon_1\left|u\right|^{p^+}+C\left(\varepsilon_1\right)\left|u\right|^{\alpha(x)}, \text{ for every } \left(x,u\right)\in\Omega\times\mathbb{R}. \tag{4.15}$$

Similarly, from hypothesis $\left(A_1\right)$ and $\left(A_3\right)$, we get

$$\left|\widehat{Q}\left(x,u\right)\right|\leq\varepsilon_2\left|u\right|^{p^+}+C\left(\varepsilon_2\right)\left|u\right|^{\beta(x)}, \text{ for any } \left(x,u\right)\in\partial\Omega\times\mathbb{R}. \tag{4.16}$$

Then, using (4.15),(4.16), Theorem 1.2 and proposition 1.3, we obtain

$$\begin{aligned}
\Psi(u) &\geq \frac{1}{p^+}\Sigma(u) - \int_\Omega \left(\varepsilon_1 |u|^{p^+} + C(\varepsilon_1)|u|^{\alpha(x)}\right) dx - \int_{\partial\Omega} \left(\varepsilon_2 |u|^{p^+} + C(\varepsilon_2)|u|^{\beta(x)}\right) d\sigma \\
&\geq \frac{1}{p^+}\Sigma(u) - \int_\Omega \varepsilon_1 |u|^{p^+} dx - C(\varepsilon_1) \max\left(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\right) - \int_{\partial\Omega} \varepsilon_2 |u|^{p^+} d\sigma \\
&\quad - C(\varepsilon_2) \max\left(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\right).
\end{aligned}$$

Since $1 < p^+ < \alpha^-$ and $1 < p^+ < \beta^-$, then by Proposition 1.4, we get

$$\Psi(u) \geq \frac{1}{p^+}\Sigma(u) - (\varepsilon_1 c_1 + \varepsilon_2 c_2)\|u\|^{p^+} - c_3 C(\varepsilon_1)\|u\|^{\alpha^-} - c_4 C(\varepsilon_2)\|u\|^{\beta^-}$$

So, by Proposition 4.1, we get

$$\begin{aligned}
\Psi(u) &\geq \frac{\xi_1}{p^+}\|u\|^{p^+} - (\varepsilon_1 c_1 + \varepsilon_2 c_2)\|u\|^{p^+} - c_3 C(\varepsilon_1)\|u\|^{\alpha^-} - c_4 C(\varepsilon_2)\|u\|^{\beta^-} \\
&\geq \|u\|^{p^+}\left(\frac{\xi_1}{p^+} - \varepsilon_1 c_1 - \varepsilon_2 c_2 - c_3 C(\varepsilon_1)\|u\|^{\alpha^- - p^+} - c_4 C(\varepsilon_2)\|u\|^{\beta^- - p^+}\right).
\end{aligned}$$

Choose ε_1 and ε_2 small enough that $0 < \varepsilon_1 c_1 + \varepsilon_2 c_2 < \frac{\xi_1}{2p^+}$, we get

$$\begin{aligned}
\Psi(u) &\geq \|u\|^{p^+}\left(\frac{\xi_1}{2p^+} - c_3 C(\varepsilon_1)\|u\|^{\alpha^- - p^+} - c_4 C(\varepsilon_2)\|u\|^{\beta^- - p^+}\right) \\
&\geq \|u\|^{p^+}\left(\frac{\xi_1}{2p^+} - \eta\|u\|^{\lambda}\right).
\end{aligned}$$

$$\lambda = \min\left(\alpha^- - p^+, \beta^- - p^+\right),$$

and

$$\eta = c_3 C(\varepsilon_1) + c_4 C(\varepsilon_2).$$

Choose $\|u\| = r$ small enough and since $\alpha^-, \beta^- > p^+$, we get

$$\frac{\xi_1}{2p^+} - \eta r^\lambda > 0.$$

Hence

$$\Psi(u) \geq r^{p^+}\left(\frac{\xi_1}{2p^+} - \eta r^\lambda\right) = \rho > 0.$$

Lemma 4.3 *Let* $\min(\theta_1,\theta_2) > w^+$, *and assume that* $(A_4), (A_5)$ *are verified, then, there exists* $e_1 \in X$ *such that*

$$\|e_1\| > r, \text{ and } \Psi(e_1) < 0.$$

Proof By (A_4) and (A_5), $\exists m_1 > 0$ and $m_2 > 0$ such that

$$\widehat{S}(x,t) \geq m_1 |t|^{\theta_1}, (x,t) \in \Omega \times \mathbb{R}, \tag{4.17}$$

and

$$\widehat{Q}(x,t) \geq m_2 |t|^{\theta_2}, (x,t) \in \partial\Omega \times \mathbb{R}. \tag{4.18}$$

Let $t > 1$ be large enough and $e \in X$, such that $\int_\Omega |e|^{\theta_1} dx > 0$,. Then, we have

$$\begin{aligned}\Psi(te) = &\int_\Omega \frac{h(x)|\nabla(te)|^{p(x)} + |te|^{p(x)}}{p(x)} dx + \int_{\partial\Omega} l(x) \frac{|te|^{w(x)}}{w(x)} d\sigma \\ &- \int_\Omega \widehat{S}(x,te)\,dx - \int_{\partial\Omega} \widehat{Q}(x,te)\,d\sigma.\end{aligned}$$

So, using (4.17) and (4.18), we get

$$\begin{aligned}\Psi(te) \leq &\frac{t^{p^+}}{p^-} \int_\Omega h(x)|\nabla(e)|^{p(x)} + |e|^{p(x)} dx + \frac{t^{w^+}}{w^-} l_2 \int_{\partial\Omega} |e|^{w(x)} d\sigma \\ &- m_1 t^{\theta_1} \int_\Omega |e|^{\theta_1} dx - m_2 t^{\theta_2} \int_{\partial\Omega} |e|^{\theta_2} d\sigma.\end{aligned}$$

Using the fact that $\min(\theta_1,\theta_2) > \max(q^+, p^+)$, we obtain

$$\Psi(te) \to -\infty, \text{ as } t \to \infty.$$

Then, we deduce that there exists $t_1 > 0$ and $e_1 = t_1 e$, with $\|e_1\| > r$ and $\Psi(e_1) < 0$.

Proof of Theorem4.1 Based on Lemma 4.2, Lemma 4.3, and Lemma 4.1, we can infer that Ψ fulfills all the conditions outlined in Theorem 2.1. Hence, in accordance with the mountain pass theorem, the problem (4.1) possesses a nontrivial weak solution.

4.3 Multiplicity result

We will assume through this section that $\delta \in C_+\left(\overline{\Omega}\right)$ with

$$\delta^- < p^-.$$

For problem (4.2), we assume that $(A_0) - (A_1)$ and $(A_4) - (A_5)$ are verified, and that

(A_6) $S(x, -u) = -S(x, u)$, for any $(x, u) \in (\Omega \times \mathbb{R})$.

(A_7) $Q(x, -u) = -Q(x, u)$, for any $(x, u) \in (\partial\Omega \times \mathbb{R})$.

We consider the functional $\Phi_\mu : X \to \mathbb{R}$ associated to (4.2), defined by:

$$\Phi_\mu(u) = \Psi(u) - \mu \int_\Omega \frac{|u|^{\delta(x)}}{\delta(x)},$$

where Ψ is introduced in section 4.2.

Remark 4.2 $\Phi_\mu \in C^1(X, \mathbb{R})$. *Moreover, the critical points of the functional Φ_μ represent the weak solutions of problem* (4.2).

Proposition 4.4 *(see [51]). Define*

$$\beta_t = \sup\{|u|_{L^{\beta(x)}(\Omega)} : ||u|| = 1, u \in Z_t\},$$

where $\beta(x)$ belongs to the space $C_+(\overline{\Omega})$ with $\beta(x) < p^*(x)$, for any $x \in \overline{\Omega}$,

$$w_t = \sup\{|u|_{L^{w(x)}(\partial\Omega)} : ||u|| = 1, u \in Z_t\},$$

where $w(x)$ belongs to the space $C_+(\partial\Omega)$ such that $w(x) < p_*(x)$, for every $x \in \partial\Omega$.

Then, $\lim_{t\to\infty} \beta_t = 0, \lim_{t\to\infty} w_t = 0.$

Lemma 4.4 *Let $\min(\theta_1, \theta_2) > w^+$ and suppose that the assumptions $(A_0), (A_1), (A_4)$ and (A_5) are verified. So, for all $\mu > 0$, Φ_μ satisfies the (PS) condition.*

Proof Let $\{u_n\} \subset X$, be a sequence and

$$\Phi_\mu(u_n) \to c, \tag{4.19}$$

where $c > 0$ is constant.

$$\Phi_{\mu}^{'}(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty. \tag{4.20}$$

By (4.19) and (4.20), there exist $M_1 > 0$ and $M_2 > 0$, such that

$$|\Phi_{\mu}(u_n)| \leq M_1, \tag{4.21}$$

and

$$\left|\langle \Phi_{\mu}^{'}(u_n), u_n \rangle\right| \leq M_2. \tag{4.22}$$

Now, we need to prove that $\{u_n\}$ is bounded. By contradiction, we assume that $\|u_n\| \geq 1$. By (4.21), we have

$$\begin{aligned} M_1 \geq \Phi_{\mu}(u_n) &= I(u_n) + G(u_n) - \phi(u_n) - \mu \int_{\Omega} \frac{|u_n|^{\delta(x)}}{\delta(x)} dx \\ &\geq \frac{1}{p^+} \Sigma(u_n) + \frac{1}{w^+} \int_{\partial\Omega} l(x) |u_n|^{w(x)} d\sigma - \phi(u_n) - \mu \int_{\Omega} \frac{|u_n|^{\delta(x)}}{\delta^-} dx \\ &\geq \frac{1}{p^+} \Sigma(u_n) + \frac{1}{\theta} \int_{\partial\Omega} l(x) |u_n|^{w(x)} d\sigma - \phi(u_n) - \mu \int_{\Omega} \frac{|u_n|^{\delta(x)}}{\delta^-} dx, \end{aligned} \tag{4.23}$$

where $\theta = \min(\theta_1, \theta_2)$.

On the other hand, from (4.22), we get

$$M_2 \geq -\langle \Phi_{\mu}^{'}(u_n), u_n \rangle = -\Sigma(u_n) - \int_{\partial\Omega} l(x) |u_n|^{w(x)} d\sigma + \langle \phi^{'}(u_n), u_n \rangle + \mu \int_{\Omega} |u_n|^{\delta(x)} dx. \tag{4.24}$$

Using (4.23), (4.24), assumptions $(A_4) - (A_5)$ and Proposition 4.1, we get

$$\begin{aligned} \theta M_1 + M_2 &\geq \left(\frac{\theta}{p^+} - 1\right) \Sigma(u_n) - \theta \left(\phi(u_n) + \mu \int_{\Omega} \frac{|u|^{\delta(x)}}{\delta^-} dx\right) + \langle \phi^{'}(u_n), u_n \rangle \\ &\quad + \mu \int_{\Omega} |u_n|^{\delta(x)} dx \\ &\geq \left(\frac{\theta}{p^+} - 1\right) \xi_1 \|u_n\|^{p^-} + \int_{\Omega} \left(S(x, u_n) u_n - \theta_1 \widehat{S}(x, u_n)\right) dx \\ &\quad + \int_{\partial\Omega} \left(Q(x, u_n) u_n - \theta_2 \widehat{Q}(x, u_n)\right) d\sigma + \mu \int_{\Omega} \left(1 - \frac{\theta}{\delta^-}\right) |u_n|^{\delta(x)} dx \\ &\geq \left(\frac{\theta}{p^+} - 1\right) \xi_1 \|u_n\|^{p^-} + \mu \int_{\Omega} \left(1 - \frac{\theta}{\delta^-}\right) |u_n|^{\delta(x)} dx. \end{aligned}$$

Thus,

$$\theta M_1 + M_2 \geq \left(\frac{\theta}{p^+} - 1\right) \xi_1 \|u_n\|^{p^-} + \mu \left(1 - \frac{\theta}{\delta^-}\right) \int_\Omega |u_n|^{\delta(x)} dx. \tag{4.25}$$

By Proposition 1.4, $\exists c > 0$, such that

$$\int_\Omega |u_n|^{\delta(x)} dx \leq |u_n|^l_{L^{\delta(x)}(\Omega)} \leq c \|u_n\|^l. \tag{4.26}$$

where $l = \delta^-$ or δ^+.

So, it follows from (4.25) and (4.26) that

$$\theta M_1 + M_2 \geq \left(\frac{\theta}{p^+} - 1\right) \xi_1 \|u_n\|^{p^-} - c\mu \left(\frac{\theta}{\delta^-} - 1\right) \|u_n\|^l.$$

Using $\theta > p^+ \geq p^- > l$, we obtain a contradiction as $n \to \infty$. So, we deduce that $\{u_n\}$ is bounded in X. Therefore, there exists $\{u_n\}$ (a subsequence) and $u \in X$ such that $\{u_n\}$ converges weakly to $u \in X$. We omit the rest of the proof because it is similar to the one in Lemma 4.1.

Theorem 4.2 *Assume that* $(A_0) - (A_1)$ *and* $(A_4) - (A_7)$ *hold. If* $\min(\alpha^-, \beta^-) > p^+, \min(\theta_1, \theta_2) > w^+$*, then,* Φ_μ *has a sequence of critical points* $\{\pm u_n\}$ *with*

$$\Phi_\mu(\pm u_n) \to \infty, n \to \infty.$$

Proof Obviously, from (A_6) and (A_7), Φ_μ is an even functional and according to Lemma 4.4, Φ_μ satisfies the (PS) condition. Next, we are going to prove (H_2) and (H_3).

Let $u \in Z_t$ with $\|u\| > 1$, using (A_0) and (A_1), we have

$$\begin{aligned}
\Phi_\mu(u) &\geq \frac{1}{p^+}\Sigma(u) - \int_\Omega C_1(1 + |u|^{\alpha(x)})dx - \int_{\partial\Omega} C_2(1 + |u|^{\beta(x)})d\sigma \\
&\quad - \frac{\mu}{\delta^-} \int_\Omega |u|^{\delta(x)} dx \\
&\geq \frac{\xi_1}{p^+}||u||^{p^-} - C_1 \max\left(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\right) \\
&\quad - C_2 \max\left(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\right) - c_1 \\
&\quad - \frac{\mu}{\delta^-} \max\left(|u|^{\delta^-}_{L^{\delta(x)}(\Omega)}, |u|^{\delta^+}_{L^{\delta(x)}(\Omega)}\right).
\end{aligned}$$

If $|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$ is the maximum of $\{|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\delta^-}_{L^{\delta(x)}(\Omega)}, |u|^{\delta^+}_{L^{\delta(x)}(\Omega)}\}$.

Therefore, by Proposition 4.4, we get

$$\begin{aligned}\Phi_\mu(u) &\geq \frac{\xi_1}{p^+}||u||^{p^-} - c_2(\mu,\delta^-)\,|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} - c_1\\ &\geq \frac{\xi_1}{p^+}||u||^{p^-} - c_2(\mu,\delta^-)\alpha_t^{\alpha^+}||u||^{\alpha^+} - c_1.\end{aligned}$$

Choose $\gamma_t = (\frac{c_2(\mu,\delta^-)\alpha^+\alpha_t^{\alpha^+}}{\xi_1})^{\frac{1}{p^- - \alpha^+}} = ||u||$, then we have

$$\Phi_\mu(u) \geq \xi_1(\frac{1}{p^+} - \frac{1}{\alpha^+})\gamma_t^{p^-} - c_1. \tag{4.27}$$

Since $\alpha_t \to 0$ when $t \to \infty$ and $p^+ < \alpha^- \leq \alpha^+$, it follows that $1/p^+ - 1/\alpha^+ > 0$ and $\gamma_t \to \infty$. Hence, we get $\Phi_\mu(u) \to \infty$ where $u \in Z_t$ and $||u|| = \gamma_t$ as $t \to \infty$. For the remaining cases, the proof is similar. So, (H_2) holds.

Using (4.17) and (4.18), then for any $u \in Y_t$ such that $||u|| = \rho_t > \gamma_t > 1$, we obtain

$$\begin{aligned}\Phi_\mu(u) &\leq \frac{1}{p^-}\Sigma(u) + \frac{l_2}{w^-}\int_{\partial\Omega}|u|^{w(x)}\,d\sigma - \int_\Omega \widehat{S}(x,u)dx - \frac{\mu}{\delta^-}\int_\Omega |u|^{\delta(x)}\,dx\\ &\quad - \int_{\partial\Omega}\widehat{Q}(x,u)d\sigma\\ &\leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{l_2}{w^-}\max\{|u|^{w^-}_{L^{w(x)}(\partial\Omega)}, |u|^{w^+}_{L^{w(x)}(\partial\Omega)}\}\\ &\quad - m_1\int_\Omega |u|^{\theta_1}\,dx - m_2\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma.\end{aligned}$$

If $\max\{|u|^{w^-}_{L^{w(x)}(\partial\Omega)}, |u|^{w^+}_{L^{w(x)}(\partial\Omega)}\} = |u|^{w^+}_{L^{w(x)}(\partial\Omega)}$, then, we have

$$\Phi_\mu(u) \leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{l_2}{w^-}\,|u|^{w^+}_{L^{w(x)}(\partial\Omega)} - m_1\int_\Omega |u|^{\theta_1}\,dx - m_2\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma.$$

Since $\dim Y_t < \infty$, all norms are equivalent in Y_t. Then, we get

$$\Phi_\mu(u) \leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{l_2}{w^-}c_2||u||^{w^+} - c_3||u||^{\theta_1} - c_4||u||^{\theta_2}. \tag{4.28}$$

also, since $\max(w^+, p^+) < \min(\theta_1, \theta_2)$, then, we have $\Phi_\mu(u) \to -\infty$ when $||u|| \to \infty$. For the remaining case, the proof follows a similar approach, and for brevity, we omit it here, therefore (H_3) fulfills. Thus, the proof is complete.

Conclusion

In this thesis, we have studied the existence and multiplicity of solutions for some nonlinear elliptic problems using variational techniques.

In Chapter 3, using the Nehari manifold method and fibering maps analysis, we have proved the existence of two nontrivial solutions for a critical Schrödinger-Kirchhoff type system involving the fractional p-Laplacian in a bounded domain with homogenous Dirichlet boundary conditions.

In Chapter 4, by applying the mountain pass theorem and fountain theorem, we have showed the existence and multiplicity of solutions for a class of $p(x)$-Laplacian problems.

In perspective, the problem considered in the third chapter can be extended to include the case of variable exponents.

Bibliography

[1] ADAMS, R. A., AND FOURNIER, J. J. Sobolev spaces. pure and applied mathematics (amsterdam), 2003.

[2] AKROUT, K., AND FAREH, S. Multiplicity results for some steklov problems involving p(x)-laplacian operator. *Novi Sad Journal of Mathematics Accepted* (Mar. 2023).

[3] ALI, K. B., GHANMI, A., AND KEFI, K. On the steklov problem involving the p (x)-laplacian with indefinite weight. *Opuscula Mathematica 37*, 6 (2017), 779–794.

[4] ALKHUTOV, Y. A., AND SURNACHEV, M. A harnack inequality for a transmission problem with p (x)-laplacian. *Applicable Analysis 98*, 1-2 (2019), 332–344.

[5] ALLAOUI, M., EL AMROUSS, A. R., AND OURRAOUI, A. Existence and multiplicity of solutions for a steklov problem involving the p (x)-laplace operator. *Electron. J. Diff. Equ 132* (2012), 1–12.

[6] ALMEIDA, A., HARJULEHTO, P., HÄSTÖ, P., AND LUKKARI, T. Riesz and wolff potentials and elliptic equations in variable exponent weak lebesgue spaces. *Annali di Matematica Pura ed Applicata (1923-) 194*, 2 (2015), 405–424.

[7] ALMEIDA, A., AND HÄSTÖ, P. Interpolation in variable exponent spaces. *Revista matemática complutense 27*, 2 (2014), 657–676.

[8] ANTONTSEV, S., AND SHMAREV, S. Evolution pdes with nonstandard growth conditions. *Atlantis Studies in Differential Equations 4* (2015), xviii+–409.

[9] ANTONTSEV, S. N., AND RODRIGUES, J. F. On stationary thermo-rheological viscous flows. *ANNALI-UNIVERSITA DI FERRARA SEZIONE 7 52*, 1 (2006), 19.

[10] AYOUJIL, A. On the superlinear steklov problem involving the $p(x)$-laplacian. *Electronic Journal of Qualitative Theory of Differential Equations 2014*, 38 (2014), 1–13.

[11] BEN ALI, K., GHANMI, A., AND KEFI, K. Minimax method involving singular p (x)-kirchhoff equation. *Journal of Mathematical Physics 58*, 11 (2017), 111505.

[12] BREZIS, H. Analyse fonctionnelle. *Théorie et applications* (1983).

[13] BRÉZIS, H. *Functional analysis, Sobolev spaces and partial differential equations*, vol. 2. Springer, 2011.

[14] BROWN, K., AND ZHANG, Y. The nehari manifold for a semilinear elliptic equation with a sign-changing weight function. *Journal of Differential Equations 193*, 2 (2003), 481–499.

[15] BROWN, K. J., AND WU, T.-F. A fibering map approach to a semilinear elliptic boundary value problem. *Electronic Journal of Differential Equations (EJDE)[electronic only] 2007* (2007), Paper–No.

[16] CHAMMEM, R., GHANMI, A., AND SAHBANI, A. Existence of solution for a singular fractional laplacian problem with variable exponents and indefinite weights. *Complex Variables and Elliptic Equations 66*, 8 (2021), 1320–1332.

[17] CHAMMEM, R., GHANMI, A., AND SAHBANI, A. Existence and multiplicity of solutions for some styklov problem involving p (x)-laplacian operator. *Applicable Analysis 101*, 7 (2022), 2401–2417.

[18] CHEN, Y., LEVINE, S., AND RAO, M. Variable exponent, linear growth functionals in image restoration. *SIAM journal on Applied Mathematics 66*, 4 (2006), 1383–1406.

[19] CRUZ-URIBE, D., FIORENZA, A., RUZHANSKY, M., WIRTH, J., AND TIKHONOV, S. *Variable Lebesgue spaces and hyperbolic systems*. Springer, 2014.

[20] CRUZ-URIBE, D. V., AND FIORENZA, A. *Variable Lebesgue spaces: Foundations and harmonic analysis*. Springer Science & Business Media, 2013.

[21] DEMENGEL, F., DEMENGEL, G., AND ERNÉ, R. *Functional spaces for the theory of elliptic partial differential equations*. Springer, 2012.

[22] DI NEZZA, E., PALATUCCI, G., AND VALDINOCI, E. Hitchhiker's guide to the fractional sobolev spaces. *Bulletin des sciences mathématiques 136*, 5 (2012), 521–573.

[23] DIENING, L., HARJULEHTO, P., HÄSTÖ, P., AND RUZICKA, M. *Lebesgue and Sobolev spaces with variable exponents*. Springer, 2011.

[24] DIENING, L., HÄSTÖ, P., AND NEKVINDA, A. Open problems in variable exponent lebesgue and sobolev spaces. *FSDONA04 proceedings* (2004), 38–58.

[25] DRÁBEK, P., AND POHOZAEV, S. I. Positive solutions for the p-laplacian: application of the fibrering method. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics 127*, 4 (1997), 703–726.

[26] EDMUNDS, D., AND RÁKOSNÍK, J. Sobolev embeddings with variable exponent. *Studia Mathematica 3*, 143 (2000), 267–293.

[27] EDMUNDS, D. E., AND RÁKOSNÍK, J. Density of smooth functions in wk, p (x)(?). *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences 437*, 1899 (1992), 229–236.

[28] FAN, X., AND HAN, X. Existence and multiplicity of solutions for p (x)-laplacian equations in rn. *Nonlinear Analysis: Theory, Methods & Applications 59*, 1-2 (2004), 173–188.

[29] FAN, X., ZHANG, Q., AND ZHAO, D. Eigenvalues of p (x)-laplacian dirichlet problem. *Journal of Mathematical Analysis and Applications 302*, 2 (2005), 306–317.

[30] FAN, X., AND ZHAO, D. On the spaces lp (x)(ω) and wm, p (x)(ω). *Journal of mathematical analysis and applications 263*, 2 (2001), 424–446.

[31] FAREH, S., AKROUT, K., GHANMI, A., AND REPOVŠ, D. D. Multiplicity results for fractional schrödinger-kirchhoff systems involving critical nonlinearities. *Advances in Nonlinear Analysis 12*, 1 (2023), 20220318.

[32] FIGUEIREDO, G. M. Existence of a positive solution for a kirchhoff problem type with critical growth via truncation argument. *Journal of Mathematical Analysis and Applications 401*, 2 (2013), 706–713.

[33] FISCELLA, A., AND VALDINOCI, E. A critical kirchhoff type problem involving a nonlocal operator. *Nonlinear Analysis: Theory, Methods & Applications 94* (2014), 156–170.

[34] HAROSKE, D., AND TRIEBEL, H. *Distributions, Sobolev spaces, elliptic equations*, vol. 4. European Mathematical Society, 2007.

[35] HE, X., AND ZOU, W. Ground states for nonlinear kirchhoff equations with critical growth. *Annali di Matematica Pura ed Applicata 193*, 2 (2014), 473–500.

[36] HE, Y., LI, G., AND PENG, S. Concentrating bound states for kirchhoff type problems in r3 involving critical sobolev exponents. *Advanced Nonlinear Studies 14*, 2 (2014), 483–510.

[37] KAVIAN, O. *Introduction à la théorie des points critiques: et applications aux problèmes elliptiques*, vol. 13. Springer, 1993.

[38] KIRCHHOFF, G. Mechanik, teubner, leipzig. *Kirchhoff G* (1883).

[39] KOVÁČIK, O., AND RÁKOSNÍK, J. On spaces lp (x) and wk, p (x). *Czechoslovak mathematical journal 41*, 4 (1991), 592–618.

[40] LIANG, S., AND SHI, S. Soliton solutions to kirchhoff type problems involving the critical growth in rn. *Nonlinear Analysis: Theory, Methods & Applications 81* (2013), 31–41.

[41] LIU, J., LIAO, J.-F., AND TANG, C.-L. Positive solutions for kirchhoff-type equations with critical exponent in rn. *Journal of Mathematical Analysis and Applications 429*, 2 (2015), 1153–1172.

[42] MINGQI, X., RĂDULESCU, V. D., AND ZHANG, B. Combined effects for fractional schrödinger–kirchhoff systems with critical nonlinearities. *ESAIM: Control, Optimisation and Calculus of Variations 24*, 3 (2018), 1249–1273.

[43] MOTREANU, D., MOTREANU, V. V., AND PAPAGEORGIOU, N. S. *Topological and variational methods with applications to nonlinear boundary value problems*, vol. 1. Springer, 2014.

[44] ORLICZ, W. Über konjugierte exponentenfolgen. *Studia Mathematica 3*, 1 (1931), 200–211.

[45] OURRAOUI, A. On a p-kirchhoff problem involving a critical nonlinearity. *Comptes Rendus Mathematique 352*, 4 (2014), 295–298.

[46] RADULESCU, V. D., AND REPOVS, D. D. *Partial differential equations with variable exponents: variational methods and qualitative analysis*, vol. 9. CRC press, 2015.

[47] RUZICKA, M. *Electrorheological fluids: modeling and mathematical theory*. Springer Science & Business Media, 2000.

[48] WEI, Z., AND CHEN, Z. Existence results for the p (x)-laplacian with nonlinear boundary condition. *International Scholarly Research Notices 2012* (2012).

[49] WILLEM, M. *Minimax theorems*, vol. 24. Springer Science & Business Media, 1997.

[50] YANG, L., AND AN, T. Infinitely many solutions for fractional p-kirchhoff equations. *Mediterranean Journal of Mathematics 15* (2018), 1–13.

[51] YAO, J. Solutions for neumann boundary value problems involving p (x)-laplace operators. *Nonlinear Analysis: Theory, Methods & Applications 68*, 5 (2008), 1271–1283.

[52] YÜCEDAĞ, Z. Existence results for steklov problem with nonlinear boundary condition. *Middle east journal of science 5*, 2 (2019), 146–154.