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# Doctoral Thesis
## Option: Stationary Problems

## Theme

# Control and Synchronization of Chaotic Systems in Dimension 3 or More

**Presented by Ms. HAMRI Douaa**

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# Abstract

In this work we studied the control and the synchronization of the chaotic systems of fractional order.
We presented a new fractional system which based on the definition of the Caputo derivative and which displays a chaotic behavior from a specific value of commensurable minimum order, in which the theoretical and numerical solution representation of this system is given using the Adams-Bashforth-Moulton algorithm which uses to solve fractional order systems numerically. Also the full hybrid projective synchronization of state (FSHPS) is studied between the new $3D$ chaotic fractional order system and the hyper-chaotic fractional order Lorenz system. The results show that FSHPS is successfully performed between the two systems, indicating that this method can be used to synchronize similar chaotic fractional-order systems in other applications.

**Keywords :** Dynamical system, chaos, fractional order chaotic system, full-state hybrid projective synchronization, Adams-Bashforth-Moulton algorithm.

# ملخص

في هذا العمل، درسنا التحكم والمزامنة للأنظمة الفوضوية ذات الرتب الكسرية.

قدمنا نظامًا كسريًا جديدًا يعتمد على تعريف مشتق Caputo والذي يعرض سلوكًا فوضويًا عند قيمة محددة لأقل ترتيب متناسب، حيث يتم تقديم تمثيل الحل النظري والعددي لهذا النظام باستخدام خوارزمية Adams-Bashforth-Moulton التي تستخدم لحل أنظمة الترتيب الكسري عدديًا. كما تمت دراسة المزامنة الإسقاطية الهجينة الكاملة للحالة (FSHPS) بين نظام الترتيب الكسري الفوضوي ثلاثي الأبعاد الجديد و نظام لورنز ذو الترتيب الكسري شديد الفوضى. يمكن إستخدام هذه الطريقة لمزامنة أنظمة فوضوية كسرية مماثلة في تطبيقات أخرى، تم التحقق من النتائج باستعمال المحاكاة العددية في الماتلاب.

**الكلمات المفتاحية :** النظام الديناميكي، الفوضى، النظام الكسري الفوضوي، التزامن الإسقاطي الهجين الكامل للحالة، خوارزمية Adams-Bashforth- Moulton.

# Résumé

Dans ce travail on a étudié le contrôle et la synchronisation des systèmes chaotiques d'ordre fractionnaires.
On a présenté un nouveau système fractionnaire qui basé sur la définition de la dérivée de Caputo et qui affiche un comportement chaotique à partir d'une valeur spécifique d'ordre minimal commensurable, dans lequel la représentation de solution théorique et numérique de ce système est donnée en utilisant l'algorithme Adams Bashforth Moulton qui utilise pour résoudre numériquement les systèmes d'ordre fractionnaire. Aussi la synchronisation projective hybride complète de l'état (FSHPS) est étudiée entre le nouveau système chaotique à ordre fractionnaire 3D et le système de Lorenz hyper-chaotique à ordre fractionnaire. Les résultats montrent que la FSHPS est réalisée avec succès entre les deux systèmes, ce qui indique que cette méthode peut être utilisée pour synchroniser des systèmes chaotiques à ordre fractionnaire similaires dans d'autres applications.

**Mots clés :** Système dynamique, chaos, système chaotique d'ordre fractionnaire, synchronisation projective hybride complète de l'état, Adams-Bashforth-Moulton algorithme.

# ***Dedication***

*To the loving memory of my beloved late grandmother,*

*Chouchou.*

*To myself, who has embraced the challenges, overcome obstacles, and persevered with unwavering determination.*

*To my ever-supportive father.*

*To my incredible mother.*

*To my amazing sisters.*

# Thanks

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# General Introduction

The work presented in this thesis concerns the study of the control and synchronization of fractional nonlinear continuous chaotic systems in dimension $3$ or more.
In classical mechanics, a system that exhibits changes or transformations over time is typically referred to as a dynamical system, characterized by its ability to change its state or behavior over time, as captured by the two principles:
- Causal: means that the future of the system relies solely on past or present events.
- Deterministic: that is to say that an "initial condition" given at the "present" instant will correspond to each subsequent instant one and only one possible "future" state. The French mathematician Laplace formulated the concept of determinism which was the basis of Newtonian mechanics. Analytical, geometrical, topological, and numerical methods for examining differential equations and iterated mappings are a broad range of dynamical systems theory. Depending on the type of state variables involved, dynamical systems can be categorized as either discrete or continuous, in which a continuous dynamical system means that the variables describing the system can change at any moment in time, and in a discrete dynamical system, the variables change in discrete or distinct steps over time.
Chaos is a characteristic exhibited by nonlinear systems, which is characterized by unstable dynamical behavior that is bounded, easily influenced by the system's starting conditions, and includes unpredictable and recurring movements that are indefinite in nature. Among the systems whose future is difficult to predict because of their sensitive dependence on very simple changes in their initial state: is the butterfly effect used by Lorenz, this is often referred to as the butterfly effect, where even small changes in initial conditions, such as a butterfly flapping its wings, can lead to significant impacts on atmospheric behavior and cause chaotic dynamics to emerge with the passage of time. Economics, for example, with evidence of the occurrence of many unexpected economic crises, the science of wars by not predicting the results of wars..etc. In fields such as electrical circuits, mechanical systems, and control systems, engineers have applied different nonlinear analysis techniques to investigate and design nonlinear systems. In particular, tools for stability analysis of nonlinear systems, with an emphasis on Lyapunov's method, for the stability of feedback systems, and nonlinear feedback control tools, including linearization, gain scheduling, exact feedback linearization, Lyapunov redesign, backstepping, sliding mode control, and adaptive control [1].
A system that is nonlinear and has a single Lyapunov exponent is classified as chaotic, whereas a system with multiple positive Lyapunov exponents is known as hyperchaotic. As a result, the hyperchaotic attractor exhibits motion in several directions, unlike the chaotic attractor, which

moves in just one direction.

The synchronization of nonlinear oscillators is a phenomenon that has attracted the attention of researchers since the observation and description of this phenomenon by Huygens in 1673, in an example of two coupled mechanical systems. It can be defined as the process of adjusting the behavior of multiple dynamical systems so that they converge to a common or synchronized state. This can involve either complete synchronization, where all of the systems reach exactly the same state, or partial synchronization, where the systems exhibit similar behavior but not necessarily identical states. Synchronization refers to the coordination of activities or events to ensure that they occur in a desired order or at the same time. In many domains, such as engineering, biology, and social systems, synchronization is essential for the proper functioning and behavior of complex systems.

Since its introduction, chaotic synchronization has been widely investigated by numerous researchers [2], leading to the proposal of various applications for suppressing chaos, monitoring and controlling dynamical systems, and communication purposes [3]. To achieve chaos synchronization, the objective is to synchronize the slave system variables with the corresponding chaotic master system variables as time progresses. Many research works have been carried out and different methods proposed for the synchronization of chaotic systems, these include complete synchronization [4], lag synchronization [5], anti-synchronization [6], hybrid synchronization [7], projective synchronization [8], hybrid projective synchronization [9], modified projective synchronization [10], combination synchronization [11], combination-combination (C-C) synchronization [12], and compound synchronization [13], etc. Pecora and Carroll showed the possibility of synchronizing chaotic systems using a common pilot signal [14]. The synchronization of chaotic systems has been widely applied to cryptography and the secure transmission of information [15], for this, and in order to be able to decrypt the message, it is first necessary to synchronize the chaotic systems (the master or the transmitter and the slave or the receiver). Some techniques that have been applied include nonlinear feedback control [16], time-delay feedback control [17], delay feedback control [18], adaptive control [19], optimal control [20], back-stepping design [21], stochastic sampled-data control [22], impulsive control [23], and active control [24]. Li and Deng have summarized the theory and techniques of synchronization in [25].

Due to the importance of fractional order systems (FOSs) in both theoretical study and practical applications, such systems attract increasing attention, especially with respect to system identification [26], stability analysis [27], controller synthesis [28], and numerical computing [29], etc. The advantages of fractional calculus have been shown that the fractional order models of real systems are regularly more adequate than usually used integer-order models, among these advantages of utilizing real fractional order systems is that they provide more degrees of freedom in their models, and also incorporate a "memory" characteristic within these models [30].

The fractional-order chaotic system [31] is one of the commonly employed applications of fractional calculus.

The control and synchronization of chaotic systems with fractional-order derivatives have been the subject of numerous commendable studies [32]. Deng and Li [33] were the first to investigate the potential for achieving synchronization in systems that use fractional-order derivatives (specifically in the case of the fractional Lü system).

The objective of the first chapter is to provide an introduction to the fundamental concepts related to continuous nonlinear dynamical systems and the theory of chaos, including their properties such as attractors, stability notions, and bifurcations. Furthermore, it covers the various mathematical techniques employed to analyze chaotic behavior, such as strange attractors, sensitivity to initial conditions, and Lyapunov exponents.

The second chapter intends to offer an overview of synchronization and its various types and different methods utilized for controlling systems.

In chapter three, we delve into the topic of fractional-order chaotic systems. Initially, we provide a succinct overview of fractional order systems, including historical context, definitions, and fundamental concepts. Subsequently, we present a comprehensive overview of established fractional-order chaotic systems.

The fourth chapter introduces a new three-dimension chaotic system of fractional order, in which we demonstrate that the system displays chaotic behavior from a specific value of minimal commensurate order. The system is represented theoretically and numerically using the Adams-Bashforth-Moulton algorithm. Additionally, this chapter examines FSHP synchronization between the new system of fractional order and the Lorenz system exhibiting hyper-chaotic behavior with fractional-order dynamics using this type of synchronization and Lyapunov theory to ensure the stability of fractional-order systems. At last, numerical simulations are presented as proof of the efficiency of the suggested controller, using the improved Adams–Bashforth–Moulton algorithm.

# Chapter 1

# General notions on dynamical systems and chaos

## 1.1 Introduction

A dynamical system is a mathematical framework used to describe the behavior of a system over time. It consists of a set of variables that change over time according to a set of rules or equations. The deterministic evolution of the dynamical system can then be modeled in two distinct ways:
- Discrete evolution, on the other hand, is a type of dynamical system where the state of the system can only change at specific points in time, and the behavior of the system is described using difference equations. Discrete dynamical systems are often used in computer science, economics, and other fields to model systems that change over time in a more "step-by-step" fashion.
- Continuous evolution is a type of dynamical system where the variables change continuously over time, and the behavior of the system is described using differential equations. Continuous dynamical systems are often used in physics, engineering, and other fields to model natural and artificial systems that change over time.

The theoretical study of these continuous models is fundamental because it is particularly useful for modelling complex phenomena that are difficult to study experimentally, such as weather patterns, population dynamics, and the behavior of complex biological systems. In short, the importance of continuous dynamical systems lies in their ability to provide a quantitative framework for studying and understanding the behavior of complex systems over time.

This chapter aims to introduce the basic notions concerning continuous nonlinear dynamical systems and theory of chaos and their properties such as attractors, notions of stability, and bifurcations, and the various mathematical tools which are used for us to characterize chaotic behavior, such as strange attractors, sensitivity to initial conditions, and Lyapunov exponents.

## 1.2 Definitions of a dynamical system

**Definition 1.1** *A dynamical system is a mathematical system that changes and develop over time in such a way that the system's present state is determined by its past states, with a direct dependency on previous time points, it is based on the foundations of decision making, feedback mechanism analysis, and simulation.*

### 1.2.1 Phase space and Poincaré section

**Definition 1.2 (State Space)** *[34], [35] The state space, also called **phase space**, refers to the collection of all possible states of a dynamical system. It can also be defined as an abstract space in which each variable represents a dimension necessary for the description of the system at a given time. It can be a vector space, a differential variety or a measurable space, etc. Phase space can be constructed by the iteration states in the space $(x_n, x_{x+1})$, and the graph shown in the phase space can be represented by the graph of the $1D$ function. The intersection of trajectories in phase space is incompatible with the deterministic character of the system.*
*the phase space is a purely abstract space that represents all possible states of the system in a mathematical way, which has as many dimensions as parameters in the dynamical system studied. Thus one could very well imagine finding oneself manipulating a $196$-dimensional phase space if the dynamical system analyzed involved $196$ initial conditions (all geometrical difficulties aside...). Nevertheless, the number of coordinates can be reduced by using a technique developed by Henri Poincare: an observation plane $\Sigma$ with dimension $d-1$ transforms the continuous trajectory into a succession of discontinuous crossing points through $\Sigma$. This plane is a section of Poincare. In addition to the reduction in the dimension of the phase space $d$ in $d-1$, this method makes it possible to reduce the number of data to be manipulated by keeping only the points of intersection of the trajectories $\Gamma$ with the section $\Sigma$ (Figure $1.1$). The rest of the points of the trajectory being ignored, the dynamics are thus easier to study.*

Figure 1.2 provides a better comprehension of the phase space's graphical representation. In Figure $1.2(a)$, the system eventually reaches an equilibrium state after several oscillations, represented by loops that converge towards a point in the phase space. Figure $1.2(b)$ shows a system that repeats periodically, which corresponds to a cyclic orbit in the phase space. In Figure $1.2(c)$, the system has a more complex motion that repeats after three distinct oscillations, referred to as a cycle of period $3$. This results in more intricate loops in the phase space. Finally, Figure $1.2(d)$ exhibits chaotic behavior, and its phase space takes on the butterfly-wing shape of the Lorenz strange attractor.

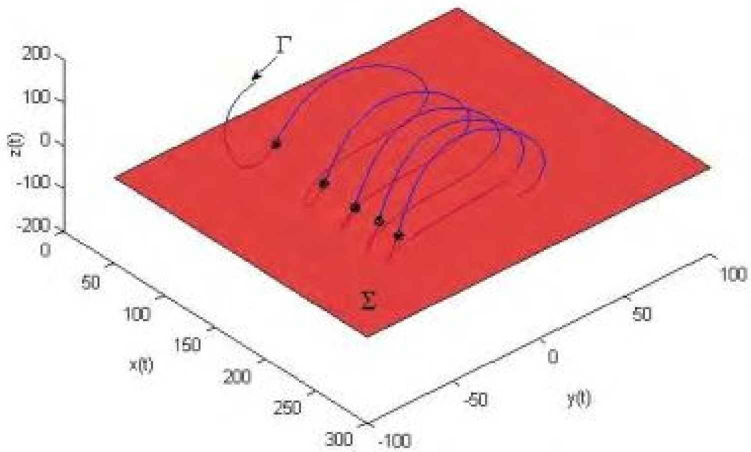



Figure 1.1: Poincare section: the phase trajectory Γ intersects the plane Σ.

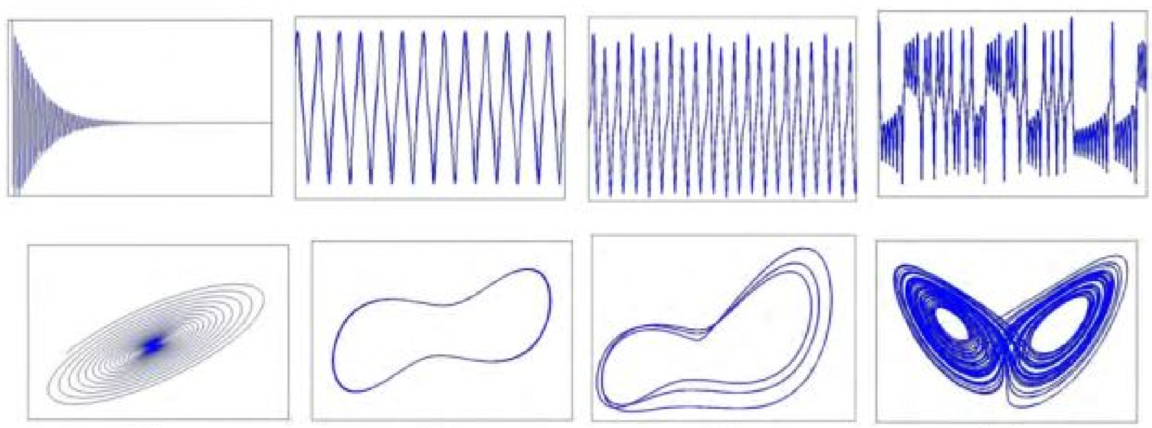



Figure 1.2: Some examples of phase space.

**Definition 1.3 (Orbit)** *An orbit of a dynamical system is the trajectory of the states travelled from the initial state $x_0$ in the state space where the system's behavior is captured. The dynamics of the system can be observed by its orbits in phase space.*

**Definition 1.4 (Phase Portrait)** *The phase portrait is the collection of all trajectories or curves that represent the possible solutions.*

## 1.3 Classification of dynamical systems

### 1.3.1 Continuous and discrete dynamical systems

Depending on the nature of the variables that define the state in dynamical systems, it can be divided into discrete or continuous systems.

**Definition 1.5 (continuous dynamical system [1])** *A differential equation can be used to describe the behavior of most nonlinear continuous-time dynamical systems*

$$\dot{x} = f(x, t, u), \tag{1.1}$$

*where the state variables denoted by $x \in U \subseteq \mathbb{R}^n$ embody the dynamical system's memory of its past, $t \in \mathbb{R}$, and $u \in V \subseteq \mathbb{R}^p$ denote specified input variables.*
*Unforced states refer to cases of continuous-time nonlinear dynamical systems where there is no explicit input variable $u$ influencing the system's behavior*

$$\frac{dx}{dt} = f(x, t). \tag{1.2}$$

*An unforced state equation doesn't always imply that the input to the system is zero. It's possible that the input has been defined as a given function of the time equation $u = \gamma(t)$ , a feedback function of the state $u = \gamma(t)$, or both, $u = \gamma(x, t)$. Substituting $u = \gamma$ in (1.1) eliminates $u$ and results in an unforced state equation.*

**Definition 1.6 (discrete dynamical system)** *A discrete-time nonlinear dynamical system has its states only at regularly distributed instants. It can be described by a difference equation or a map*

$$y\,(i+1) = f(y\,(i)\,, \mu),\ y\,(i) \in U \subset \mathbb{R}^n, \mu \in V \subset \mathbb{R}^p, i = 0, 1, 2, ... \tag{1.3}$$

### 1.3.2 Autonomous and non-autonomous systems

**Definition 1.7 (Autonomous system)** *Systems that are capable of adapting to unforeseen events during their operation are referred to as "autonomous". The technology of autonomous systems has the potential to bring about transformational changes, with potential cost and risk reduction benefits, and an autonomous system, is responsible for more complex tasks, the details of which have been left to its own initiative out of necessity or a desire to simplify its use.*

**Definition 1.8 ([36])** *If the behavior of the dynamical system (1.1) remains unchanged over time, it is referred to as an autonomous system, that is*

$$\frac{dx}{dt} = f(x). \tag{1.4}$$

*So an autonomous system is not affected by any external input and its behavior solely relies on its initial conditions.*
*Autonomous systems exhibit behavior that remains unchanged when the time origin is shifted, as the state equation's right-hand side is not affected by changing the time variable from $t$ to $\tau = t - a$. A non-autonomous system, on the other hand, is referred to as time-varying or non-time-invariant.*

**Definition 1.9 (Non-autonomous System)** *Non-autonomous systems are also of great interest, such that an appropriate controller can change its behavior, and since systems subjected to external inputs, including of course periodic inputs, are very common.*

**From Non-autonomous to Autonomous: A Conversion Process**

Every non-autonomous dynamical system can be transformed into an autonomous system by increasing the phase space dimension by one [37].

**Example 1.1** *The following system*

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -g \sin y_1 + F \cos wt, \end{aligned} \tag{1.5}$$

*has a phase space of dimension 2 and is considered non-autonomous. By making the substitution $t = y_3$, the following system can be obtained*

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -g \sin y_1 + F \cos wy_3, \\ \dot{y}_3 &= 1, \end{aligned} \tag{1.6}$$

*the dimension of the phase space of this dynamical system is 3 and it is autonomous.*

### 1.3.3 Linear and non-linear systems

Mathematics distinguishes between linear and nonlinear systems, and chaos can only emerge in the latter. Consequently, the study of chaos or dynamical systems is synonymous with nonlinear dynamics.

### 1.3.4 Deterministic and random systems

**Definition 1.10 (Deterministic system)** *A deterministic system can be distinguished from a random system in that the former has no randomness involved in its development of future states. Given* *perfect knowledge of the initial conditions and the law of evolution of the system, the future behavior of all times can be determined. The concept of determinism was formulated by Laplace, a French mathematician and was the basis of Newtonian mechanics. In engineering, most systems are considered deterministic, with noise being considered a known random process that adds to the deterministic system.*

**Definition 1.11 (Random System)** *A random system is run to some degree by chance. Given complete information about the dynamics and the initial state, it is impossible to accurately predict the future evolution of the system (although it may be possible to determine the statistics of the future evolution, that is, the probability that the system is in particular states at certain times).*

### 1.3.5 Conservative and dissipative systems

Two kinds of dynamical systems are separate, namely, conservative and dissipative.

**Definition 1.12 (Conservative system)** *Frictionless systems, called conservative or Hamiltonian, have their own interest and usefulness. It is from their study the absence of attractor, the trajectories evolve on surfaces of constant energy tori of KAM [38]. The volume of a conservative system in phase space remains constant as the system evolves over time. Therefore, equation* $(1.4)$ *can be considered a conservative system only if the divergence of the function f is zero* $\nabla.f = 0$.

**Example 1.2 ([37])** *The system described by*

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_1^3 - y_1, \end{aligned} \tag{1.7}$$

*is considered conservative because the divergence of the function* $f$ *is equal to zero, i.e.* $\nabla.f = 0$ .

**Definition 1.13 (Dissipative system)** *The analysis and synthesis of control laws for linear and nonlinear dynamical systems are greatly facilitated by the use of dissipative systems. A crucial feature of these systems is that the total energy stored within them gradually decreases over time. The presence of an "internal friction" in dissipative systems has as its corollary the existence of an "attractor", that is to say of an asymptotic limit (for* $t \to +\infty$*) of the solutions, hence, the system* $(1.4)$ *is dissipative if* $\nabla.f < 0$ .

**Example 1.3 ([37])** *For Lorenz system*

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1), \\ \dot{y}_2 &= \rho y_1 - y_2 - y_1 y_3, \\ \dot{y}_3 &= y_1 y_2 - \beta y_3, \end{aligned} \tag{1.8}$$

*it is a dissipative system, as indicated by the fact that the parameters* $\alpha, \rho$ *and* $\beta$ *are all greater than zero, and the divergence of the vector field is negative, specifically* $\nabla.f = -\alpha - 1 - \beta < 0$.

## 1.4 Dynamical behavior types

### 1.4.1 Steady state (equilibrium point)

Steady state of $(1.2)$ is a state $x = x^*$ in the phase space where the system remains unchanged for all future time, given that it starts at $x^*$, it is the simplest deterministic behavior.
The examination of a linear system's behavior near the equilibrium point $x = 0$ is significant since, in several cases, the nonlinear system's local behavior near an equilibrium point can be inferred by linearizing the system around that point and analyzing the behavior of the resulting linear system. The effectiveness of this approach depends on how the different qualitative phase portraits of a linear system persist under perturbations [36].

### 1.4.2 Periodic behavior (periodic orbit or limit cycle)

Periodic behavior, also known as **oscillation**, occurs when a behavior repeats itself after a certain period of time denoted by $T > 0$ and continues to do so indefinitely. The undamped pendulum is a classic example of a system exhibiting periodic behavior in the form of simple harmonic motion, which will oscillate indefinitely in the absence of disturbances. Another example of periodic dynamics in the cardiac electrophysiology is the alternation of cardiac action potential duration (APD), that is, a pathological condition of the heart.

**Definition 1.14** *A periodic solution of the system* $\dot{y} = f(y)$ *is one that repeats itself after a fixed period of time* $T > 0$*, meaning that for all* $t$ *in the real numbers,*

$$\forall t \in \mathbb{R},\ y(t+T) = y(t). \tag{1.9}$$

*The set* $L = \{y(t) : t \in [0, T]\}$ *is a closed curve in state space and is called a periodic orbit or a limit cycle.*

### 1.4.3 Quasi-periodic behavior

Quasi-periodicity is characterized by a change in periodic behavior, where the system exhibits a combination of periodic behaviors that cannot be expressed as a ratio of whole numbers, resulting in incalculable periods. This behavior is represented by a state space torus.

**Example 1.4** *An example is in the solar system (sun-earth-moon-Jupiter) with several different periods. The moon orbits the earth in a month* ($30$ *days), the earth takes one year to complete an orbit around the sun* ($365$ *days), and Jupiter orbits the sun in about* $12$ *years* ($4380$ *days).*

### 1.4.4 Random behavior

Random behavior is non-deterministic, even if we knew everything about a system at a given time in perfect detail, we would still not be able to predict the state at a future time, where this behavior characterizes by a lack of pattern or predictability. Initial conditions or any other factors do not influence events that are truly random, they are determined by chance alone.

### 1.4.5 Chaotic behavior

Chaos is irregular behavior over time, it is neither a fixed point nor a cycle. However, not all irregular behavior over time is chaotic. This theory is said to be deterministic in the sense that when the law of evolution is known, the future of the system is perfectly determined once its initial state is known.

## 1.5 Chaos

### 1.5.1 Chaotic system

In a deterministic system that is not chaotic, similar initial conditions lead to similar trajectories, however, chaotic systems are also deterministic and subject to a law of evolution, but their evolution is incredibly complex and completely irregular, and this leads to unpredictable long-term behavior, which is sensitive to initial conditions that evolve in a bounded region, and which possess an infinity number of dense non-periodic trajectories (see Figure 1.3).

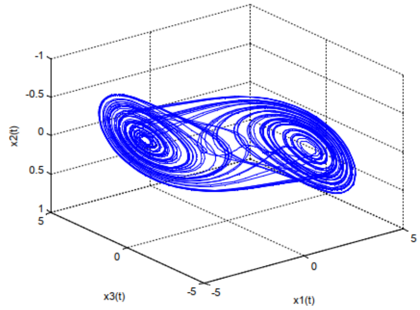



Figure 1.3: Behavior of Chua's chaotic attractor in $3D$ view

## Chaotic system characteristics

Chaos theory, as described by Keller [39], is defined as "the qualitative study of the unstable periodic characteristics of motion in a deterministic dynamical system." Keller acknowledges the absence of a general definition for a chaotic dynamical system, but identifies three essential properties: radical sensitivity to initial conditions, the potential for highly disordered behavior, and, despite this disorder, the system remains deterministic, adhering to laws that fully describe its motion [40].

We briefly present the characteristics that help to understand the salient points of a chaotic system.

1. **Initiale conditions sensibility:**

   The evolution of a chaotic dynamical system is unpredictable with the understanding that it exhibits an extremely high level of sensitivity to the initial conditions. Thus, two initially neighboring phase trajectories always diverge from each other, regardless of their initial proximity. Although we are dealing with deterministic systems, it is impossible to predict their behavior in the long term. The sensitivity to the initial conditions can be quantified thanks to the Lyapunov exponents and therefore lends itself well to the study of concrete examples.

   **Definition 1.15** *Suppose $(Y, d')$ is a metric space, and $A : Y \to Y$ a continuous map. If there exists a constant such that*

   $$\forall y \in Y, \forall \varepsilon > 0, \exists x \in Y, \exists n \in \mathbb{N} : d'(y, x) < \varepsilon \text{ et } d'(A^m(y), A^n(x)) > b, \tag{1.10}$$

   *then we can say that the topological dynamical system $(Y, A)$ exhibits sensitivity to initial conditions.*

   *This definition expresses that for any initial condition, there are points as close as we want to it whose associated orbits will move away from the initial trajectory.*

2. **Chaotic or strange attractor** :

   Strange attractors are unique from other phase-space attractors in that one does not know exactly where on the attractor the system will be. Two points on the attractor that are near each other at one time will be arbitrarily far apart at later times. The only restriction is that the state of system remain on the attractor. Strange attractors are also unique in that they never close on themselves — the motion of the system never repeats (non-periodic). The motion we are describing on these strange attractors is what we mean by chaotic behavior.

3. **Lyapunov exponents** [40]:

   We can define the Lyapunov exponent as an aid to characterize solutions, especially those which are unstable. The evolution of a chaotic flow is difficult to apprehend because the divergence of the trajectories on the attractor is rapid, this is why we try to estimate or even measure the speed of divergence or convergence, this speed is called the Lyapunov exponent .

   The Lyapunov exponent is used to measure the degree of stability of a system and to quantify the sensitivity to the initial conditions of a chaotic system. The phase space dimension is equivalent to the quantity of Lyapunov exponents and they are usually indexed from largest to smallest $\lambda_1, \lambda_2, \lambda_3, ...$

   The onset of chaos requires Lyapunov exponents to fulfil three conditions ([41] and [42]):

   •The existence of a **positive exponent** is required to account for the divergence of the trajectories.

   •To account for the folding of the trajectories, the system must possess at least **one negative exponent.**

   •A chaotic system is dissipative when the summation of its exponents is negative, i.e. it loses energy.

   The value of the largest Lyapunov exponent quantifies the degree of chaos of the system, but the fact that the three conditions stated above are met is not enough to conclude that a system is chaotic. It remains essential to compare the results of the Lyapunov exponent calculation with those provided by other nonlinear analysis tools. A method of approximating Lyapunov exponents is Wolf's algorithm [43], this algorithm makes it possible to calculate the Lyapunov exponents from the effective calculation of the divergence of two trajectories after $t$ time step with respect to the disturbance introduced in parallel, and this within an attractor.

| Attractor Type | Sign of Lyapunov's exponents |
|---|---|
| Fixed point | $-, -, -$ |
| Periodic limit cycle | $0, -, -$ |
| Quasi-periodic limit cycle | $0, 0, -$ |
| Strange attractor | $+, 0, -$ |

Table 1.1: Characterization of attractors

**Definition 1.16** *A system is classified as chaotic if it is nonlinear and has exactly one Lyapunov exponent, whereas a system with multiple positive Lyapunov exponents is known as hyperchaotic. Therefore, unlike the chaotic attractor that extends in only one direction, the hyperchaotic attractor expands in multiple directions.*

4. **Fractal dimension:**

   The fractal dimension is employed to describe the "strange attractors" that are characteristic of chaotic systems.

   Unlike regular geometric shapes, which have integer dimensions, chaotic attractors have non-integer or fractal dimensions, reflecting their intricate and non-repeating pattern that is self-similar structure at different scales, and the fractal dimension is employed to quantify the degree of complexity in the attractor.

   The fractal dimension of a chaotic system is established by analyzing the behavior of the system over time and plotting its phase space trajectory. The phase space trajectory is a plot of the system's state variables (such as position and velocity) over time, and the shape of the trajectory can reveal the underlying dynamics of the system.

   The fractal dimension of the system's attractor is then calculated by examining how the trajectory fills up the phase space as time progresses. If the system's behavior is chaotic, the trajectory will cover a complex, non-repeating pattern in the state space, and the fractal dimension will reflect the degree of intricacy in this pattern.

   In summary, the fractal dimension is employed in chaos theory to quantify the complexity of the patterns that arise in the behavior of chaotic systems, particularly the strange attractors that characterize their long-term behavior.

### 1.5.2 Attractors

**Definition 1.17** *Attractors are points of stable equilibrium, they attract all adjacent trajectories.*

#### Attractors types

1. **Equilibrium point attractor**

   The "equilibrium point" attractor $x^*$ is a dynamical system's ordinary differential equation $\frac{dx}{dt} = Ax$ solution, which is a point in the trajectory phase space.

   The system can exhibit six distinct qualitative phase portraits that correspond to different types of equilibria:

**Case 1.1** *An equilibrium point* $x^*$ *is described as a* ***hyperbolic*** *point if the matrix* $A$ *does not have a zero or imaginary eigenvalue.*

**Case 1.2** *An equilibrium point* $x^*$ *is described as a* ***saddle*** *point if at least one eigenvalue has negative real part and at least one has positive real part.*

**Case 1.3** *An equilibrium point* $x^*$ *is described as a (stable or unstable)* ***node*** *point if all eigenvalues have the same signs (both eigenvalues are negative or positive respectively).*

**Case 1.4** *An equilibrium point* $x^*$ *is described as a (stable or unstable)* ***focus*** *or center point if eigenvalues are complex conjugate pairs (when the real part of the eigenvalues is* $< 0$*,* $> 0$ *or* $= 0$ *respectively).*

2. **Periodic orbit (limit cycle) attractor**

   The "limit cycle" attractor is a closed trajectory in phase space towards which the trajectories tend. It is, therefore, a periodic solution associated with a periodic behavior of the system.

   Limit cycles are a form of periodic motion that can be observed in nonlinear systems within their phase space. According to [44], these cycles manifest as isolated periodic state trajectories. Limit cycles can either be stable, attracting neighbouring states towards them, or unstable, repelling neighbouring states instead.

2. **Quasi-periodic attractor**

The "quasi-periodic" attractor represents the motions resulting from two or more independent oscillations, and it is represented by a "torus" similar to an object in a state space with a dimension greater than or equal to three.

4. **Strange attractor**

The "strange" attractors are much more complex than the others, an attractor associated with dissipative chaotic systems called strange because of the strangeness of the unpredictable and infinitely complex behavior at all scales of its structure. We speak of strange attractors when the fractal dimension is not whole.

**Example 1.5 (Strange attractor)**

In [45], Ruelle and Takens introduced a type of attractor, the characteristics of this strange attractor are:

a. In phase space, the attractor has zero volume.

b. The dimension of the attractor, indicated by $D$, is a non-integer and is instead a fractal value, with $2 < D < m$, where $m$ indicates the dimension of the phase space.

c. The impact of initial conditions on system behavior: two trajectories of the initially neighboring attractor always end up deviating from each other.

### 1.5.3 Stability concepts

Stability represents an essential aspect when examining linear and nonlinear dynamical systems. It is a concept that has given rise to different terminologies which will be briefly recalled in order to specify in what sense the term stability is used in this thesis.

#### Studying the Stability of Equilibrium Points

The notion of stability corresponds to the idea of behavior that lasts over time and makes it possible to formalize the following question: at a point close to a point of equilibrium $x$ what happens to the solution trajectory? This question is important because in practice the initial conditions present uncertainties, it would be desirable that two close initial conditions, lead to close trajectories for any time and even for infinitely long times. A natural way to approach this question would be to solve the differential equation and examine the behavior of solutions. But in general, one cannot solve differential equations.
A first approach for the study of these systems consists in seeking the points of equilibrium, that is to say the stationary solutions not presenting any temporal evolution.
Once the equilibrium points have been determined, examining its stability is crucial, that is to say how a slight disturbance at these points influences the evolution of the system.
When examining dynamical systems, several types of stability problems arise. We consider the following dynamical system

$$\dot{y} = h(y, t) \tag{1.11}$$

where $h$ is a nonlinear function.

#### Local stability (Linear stability) analysis

**Definition 1.18** *A system* (1.11) *possesses a* ***stable*** *equilibrium point* $y^*$ *if any solutions beginning in the vicinity of that point remain close by, indicating that the system is resistant to perturbations, i.e.*

$$\forall \varepsilon > 0, \exists \rho > 0 : \|y(t_1) - y_e\| < \rho \Longrightarrow \|y(t, y(t_1)) - y_e\| < \varepsilon, \forall t \geq t_0. \tag{1.12}$$

**Definition 1.19** *In a system* $(1.11)$, *an equilibrium point* $y^*$ *is considered* ***asymptotically stable*** *when all solutions beginning at nearby points not only remain close by but also converge towards the equilibrium point as time goes to infinity. i.e.*

$$\exists\rho > 0 : \|y(t_1) - y_e\| < \rho \Longrightarrow \lim_{t\to\infty} \|y(t, y(t_1)) - y_e\| = 0. \tag{1.13}$$

**Definition 1.20** *A system* $(1.11)$ *with an equilibrium point* $x^*$ *is considered to be* ***exponential stable*** *if*

$$\begin{aligned} &\forall\varepsilon > 0, \exists\rho > 0 : \|y(t_1) - y_e\| < \rho \\ &\Longrightarrow \|y(t, y(t_1)) - y_e\| < b\,\|y(t_1) - y_e\| \exp(-ct), \forall t \geq t_0. \end{aligned} \tag{1.14}$$

The previous definitions of stability of an equilibrium point are local, they only concern the orbits close to a point of equilibrium ($\|y - b\| \leq \eta$).

**Global stability (nonlinear stability) analysis** We present here two methods of Lyapunov to study the stability.

1. **Direct method**

   The direct method is difficult to implement but, on the other hand, it is much more general in scope. This concept is rooted in the definition of a specific function, known as the Lyapunov function and denoted $V(x)$, which is reducing along the trajectories of the system.

   **Theorem 1.1 (Lyapunov's function and global stability)** *Assuming that* $y^*$ *is a steady state of the system* $(1.11)$, *the condition for the existence of a Lyapunov function* $V : \mathbb{R}^n \to [0, +\infty[$ *of class* $C^1$ *is satisfied if:*

   - $V(y^*) = 0$ *and* $V(y) > 0$ *for* $y \neq y^*$.

   - $V$ *decreases along all trajectories* $(\frac{dV}{dt} \leq 0)$.

   *Then* $y^*$ *exhibits stability according to the Lyapunov definition.*

   *Furthermore, if* $y \neq y^*$ *, then* $y^*$ *demonstrates asymptotic stability based on the Lyapunov definition.*

   *If we again assume that* $V$ *tends to infinity when* $y \in \mathbb{R}^n$ *tends to infinity (in norm), then all the trajectories, even those that start far from* $y^*$ *, tend to* $y^*$ *(we say that* $y^*$ *is globally asymptotically stable), but if* $(\frac{dV}{dt} > 0)$. *for* $y \neq y^*$ *then* $y^*$ *is unstable.*

**Remark 1.1** *There is no general method for determining a Lyapunov function.*

2. **Indirect method (Linearization)** [36]

   The indirect method of Lyapunov, to study the stability around an equilibrium point $y^*$, consists in studying the linear system

$$\dot{y} = Cy \tag{1.15}$$

   with, the matrix

$$C = Df(0) = \frac{\partial f}{\partial y}(y)\Bigg|_{y=0} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}_{y=0} \tag{1.16}$$

   is the Jacobian matrix of $f$ at $0$, has all distinct eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, then the solution of $(1.15)$ is

$$y = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i, \tag{1.17}$$

   where $v_i$ the eigenvector associated with $\lambda_i$.

   When the real parts of the eigenvalues of matrix $C$, also known as the characteristic exponents of the equilibrium point $0$, are negative, the origin is deemed to be asymptotically stable.

   When the matrix $C$ has eigenvalues with positive real values, the origin is considered unstable.

   In cases where matrix $C$ has eigenvalues with zero real values, and the remaining eigenvalues have negative real values, linearization cannot establish the stability characteristics of the origin.

### 1.5.4 Bifurcation

The fundamental aspect of the examination of dynamical systems is the notion of bifurcation, at specific values of the system's control parameters that are considered critical, the solution of the differential equation changes qualitatively: it is said to have a bifurcation [46]. Bifurcation is a non-linear effect, intimately tied to the phenomenon of multiple solution to non-linear equation. In this light it is a local theory, dealing with the local fluctuation of solutions to a given problem. A point of. intersection of two or more solution branches will be called a bifurcation point.

#### Bifurcation types

In general, all the destabilizations of a system come down to three types of generic bifurcations (however there are rare special cases) that we will present on three simple one-dimensional cases:

⟶ Node-pass bifurcation (or saddle node).

⟶ Fork bifurcation (or pitchfork).

⟶ Hopf bifurcation.

### Bifurcation diagram

Bifurcation diagram summarizes all the information on the bifurcation and thus makes it possible to understand how the system evolves.

### Feedback

Feedback are modifications made to an element in a system result in alterations to its state. These changes can then propagate to other interconnected elements, and the impacts may propagate back to the original element. This phenomenon is known as feedback. Feedback can be categorized into two types:

**Definition 1.21** 1) *Positive or self-reinforcing feedback, which amplifies the current changes in the system.*
2) *Negative or self-correcting feedback, which seeks equilibrium by counteracting the changes occurring in the system.*

Complex systems are "complex" due to the numerous feedback loops and interactions among the system's various components.

## 1.6 Examples of continuous chaotic systems

### Lorenz system

Edward N. Lorenz, a meteorologist and mathematician, developed the Lorenz system while researching thermal fluctuations within an air cell [47]. It was created with the following dynamical equations [48]

$$\begin{aligned} \dot{y}_1 &= -\alpha(y_2 - y_1), \\ \dot{y}_2 &= \gamma y_1 - y_2 - y_1 y_3, \\ \dot{y}_3 &= y_1 y_2 - \delta y_3, \end{aligned} \tag{1.18}$$

where $y_1, y_2,$ and $y_3$ are the state variables, and $\alpha, \gamma,$ $\delta$ are positive parameters. The presence of limit cycles and sensitivity to initial conditions are characteristic of a Lorenz system, which demonstrates a pattern of periodic doubling as a parameter is changed in one direction, ultimately resulting in chaotic behavior. The trajectories of the Lorenz chaotic attractor are represented by a butterfly or figure-eight shape, as depicted in Figure 1.4 ([48]).

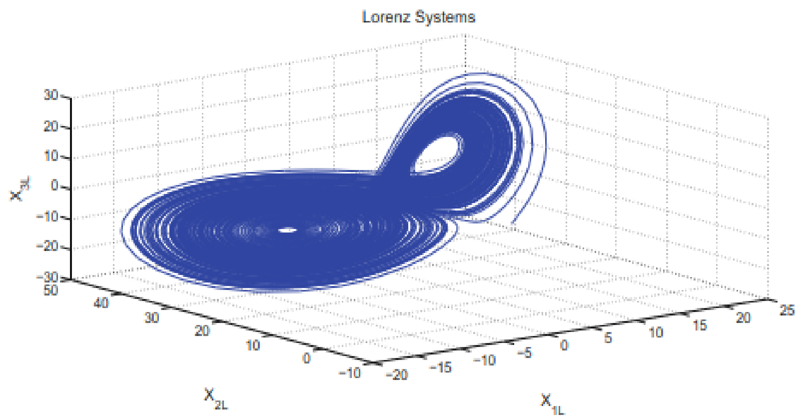



Figure 1.4: Chaotic attractor for the Lorenz system.

## Rössler system

Otto Rössler created the Rössler attractor in 1976 [49], and later, its equations proved useful in modelling equilibria in chemical reactions

$$\begin{aligned} \dot{y}_1 &= -(y_2 + y_3), \\ \dot{y}_2 &= y_1 + \alpha y_2, \\ \dot{y}_3 &= \beta + y_3(y_1 - \gamma), \end{aligned} \tag{1.19}$$

where $y_1, y_2$, and $y_3$ are the state variables, $\alpha$, $\beta$, and $\gamma$ are the parameters.
The Rössler chaotic attractor depicted in Figure 1.5 ([48]) displays a single manifold for the parameter values $\alpha = \beta = 0.2$ and $\gamma = 0.5$.

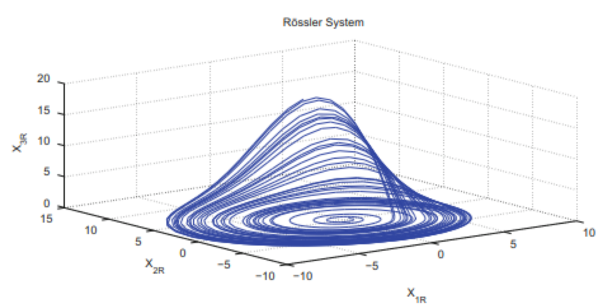



Figure 1.5: Chaotic attractor for the Rössler system.

## Chua system

In the realm of experimental chaos, Chua's system stands out as the initial system that was observed in a laboratory setting [50]. Its chaotic behavior has been validated both through computer simulations and rigorous mathematical proofs. A collection of dynamical equations define this electric circuit system [48]

$$\begin{aligned}\dot{y}_1 &= (y_2 - ag(y_1)),\\ \dot{y}_2 &= y_1 - y_2 + y_3,\\ \dot{y}_3 &= -by_2,\end{aligned} \tag{1.20}$$

here, $g$ represents a function that is piecewise-linear in nature and $a$ and $b$ are bifurcation parameters.

$$g(y) = \begin{cases} n_1(y+1) - n_0, & y < -1, \\ n_0 y \quad , & -1 \leq y \leq 1, \\ n_1(y-1) + n_0, & y > 1. \end{cases} \tag{1.21}$$

Figure 1.6 depicts the behavior of the Chua system.

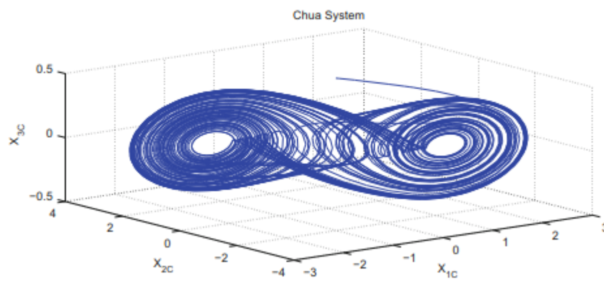



Figure 1.6: Chaotic attractor for the Chua system.

## Van der Pol system

One can describe the autonomous system of the Van der Pol oscillator as follows: [48]

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 - \epsilon(x_1^2 - 1)x_2.\end{aligned} \tag{1.22}$$

The Van der Pol system's behavior is depicted in Figure 1.7.

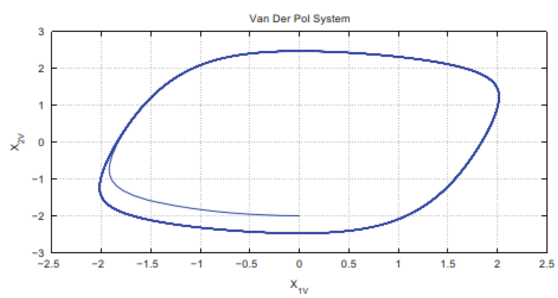



Figure 1.7: Chaotic attractor for the Van der Pol system.

## Duffing system

G. Duffing introduced the Duffing oscillator in 1918 to describe the dynamics of a classical particle in a double-well potential, a differential equation of a single variable is utilized for this purpose, known as the Duffing equation [51]. According to [48], if the equation is forced, its most general form is able to be represented as a system of first-order ordinary differential equations:

$$\begin{aligned} \dot{y}_1 &= y_2. \\ \dot{y}_2 &= y_1 - y_1^3 - \delta y_2 + \gamma \cos(\omega t). \end{aligned} \tag{1.23}$$

The trajectories of a chaotic Duffing system are illustrated in Figure 1.8.

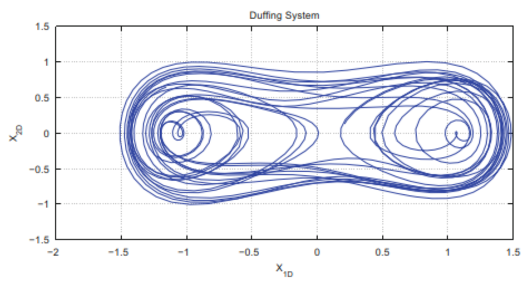



Figure 1.8: Chaotic attractor for the Duffing system.

## Rikitake system

The Rikitake chaotic system is a type of autonomous chaotic system with a quadratic formulation. that exists in three dimensions. Despite its simplicity, this system has the ability to generate two-scroll chaotic attractors with a high degree of complexity [48]

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + x_2 x_3, \\ \dot{x}_2 &= -\mu x_2 + (x_3 - b) x_1, \\ \dot{x}_3 &= 1 - x_1 x_2, \end{aligned} \tag{1.24}$$

where we will assume that the parameters $b$ and $\mu$ in the equation are nonnegative.
We can observe the Rikitake system's chaotic attractor in Figure 1.9.

# 1.7 Conclusion

The focus of this chapter is to provide a concise reminder of continuous nonlinear dynamical systems and their general properties.

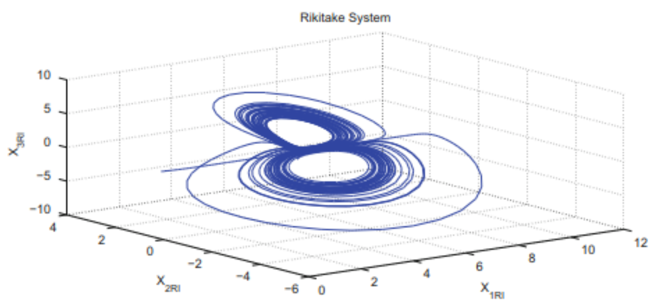



Figure 1.9: Chaotic attractor for the Rikitake system.

# Chapter 2

# Synchronization and some methods of control

The goal of this chapter is to present an introduction to synchronization and various control methods.

## 2.1 Introduction

Controlling and synchronizing chaotic systems present a difficult challenge. Synchronization is the action of phasing to create simultaneity between several operations, depending on time. Historically, it was Chrystiaan Huyghens who discovered this phenomenon in 1673 thanks to two clocks suspended from an unfixed beam: as he explained in his memoirs “Horologium Oscillatorium”, after a certain time the movement of the beam causes the entry into the phase of the oscillations of the two pendulums.

This phenomenon has been an area of ongoing research since the early days of physics, as referenced in [52].

Synchronization can be studied using a variety of mathematical tools, including differential equations, graph theory, and control theory. It finds its practical use in various disciplines, including physics, biology, computer science, and telecommunications, and is an important area of research in both pure and applied mathematics.

Control methods are used to influence or regulate the behavior of systems or processes, often with the goal of achieving synchronization.

Efforts in research have been dedicated to solving the problems of controlling and synchronizing chaos in numerous dynamical systems [53], [54].

According to recent studies cited in [55]-[59], synchronization is also possible in chaotic fractional order systems.
Numerical simulations are commonly used to study synchronization among fractional order systems in many literatures, by utilizing stability conditions derived from linear fractional order systems, as seen in previous works such as [60] and [61]. Additionally, some studies are based on the Laplace transform theory, as shown in [59] and [62].

## 2.2 Synchronization theory

### 2.2.1 General definition of synchronization

**Definition 2.1 ([63])** *"Synchronous" comes from the Greek words* $\chi\rho'ovo\varsigma$ *"chronos" (time) and* $\sigma'\upsilon\upsilon$ *"syn" (same/common), which, translated directly, means "sharing the common time" or "occurring in the same time". This term, along with "synchronization" and "synchronized", refers to various phenomena in natural sciences, engineering, and social life. These phenomena may seem different, but they often follow universal laws.*

**Definition 2.2** *Synchronization involves the coordination of two or more events or processes so that they occur simultaneously or in a specific order.*

**Example 2.1** *A real-life example of synchronization is the coordination of traffic signals at a busy intersection. Traffic lights are synchronized so that they change in a specific sequence, allowing vehicles to flow smoothly through the intersection without colliding. This synchronization helps to prevent accidents and reduce traffic congestion.*

**Example 2.2** *The synchronization of heartbeats among members of a choir or audience, where the rhythm of the music helps to synchronize the heartbeats of individuals, studies have shown that when people listen to music together, their heartbeats can synchronize with the beat of the music. This phenomenon is known as "entrainment," and it occurs because the human body is sensitive to rhythmic stimuli. For example, imagine a group of people attending a concert and listening to the same piece of music. As the music plays, their heartbeats may begin to synchronize with the beat of the music, creating a shared physiological response. This synchronization can create a sense of unity and connectedness among the members of the group, as they experience the music together in a synchronized way. Similarly, in a choir, members often sing together in a synchronized manner, following the same tempo and rhythm. As they sing, their breathing patterns and heart rates can also become synchronized, further enhancing the feeling of unity and cohesion among the group.*

### 2.2.2 Mathematical definition of synchronization

**Definition 2.3** *Synchronization is used to understand the sensitivity based on the initial conditions. It has been proven that synchronizing two or more chaotic systems proves the ability to follow closely the same movement of these dual systems together.*

The concept of synchronization is manifested when two dynamical systems follow an identical way as a function of time. One of the most popular synchronization configurations is the master-slave configuration for which a dynamical system, called the slave system, follows the rhythm and the trajectory imposed by another dynamical system, called the master system. Hence the following definition :

**Definition 2.4 ([64])** *A system is considered a slave system:*

$$\dot{y}_1(t) = h_1(y_1(t)), \quad y_1(t) \in \mathbb{R}^n \tag{2.1}$$

*synchronizes with a master system :*

$$\dot{y}_2(t) = h_2(y_2(t)), \quad y_2(t) \in \mathbb{R}^n, \tag{2.2}$$

*if for any pair of initial conditions* $(y_1(0), y_2(0))$,

$$\lim_{t \to +\infty} |y_2(t) - y_1(t)| = 0. \tag{2.3}$$

**Theorem 2.1** *Synchronization between the master system and the slave system occurs only when the conditional Lyapunov exponents of the slave system have negative values.*

### 2.2.3 Chaos synchronization

Along with the great advances made in chaos theory, the prospects of the use of chaos in various applications have motivated researchers to investigate the question of the possibility of synchronizing chaos.

Chaos synchronization is the phenomenon in which two or more chaotic systems evolve in such a way that their trajectories become identical or asymptotically approach each other. This synchronization can be achieved through a suitable coupling between the systems, or through the use of control techniques [65].

**Example 2.3** 1. ***In secure communication systems** [66]**:*** *One application of chaos synchronization is in secure communication systems, where chaotic signals are utilized for transmitting information in a secure way. A well-known example of this is the Lorenz-based encryption* *scheme, in which two Lorenz chaotic systems are coupled and synchronized to generate a secure key for encryption.*

2. ***In controlling chaotic systems*** *[67]: Another application of chaos synchronization is in controlling chaotic systems, where a master chaotic system is utilized to control the behavior of a slave chaotic system. This has been employed, as an illustration, to stabilize the motion of a chaotic pendulum.*

3. ***In studying complex networks*** *[68]: Chaos synchronization has also been used to study complex networks, where a network of coupled chaotic systems is used to model a complex system such as the brain or the climate system. By synchronizing the chaotic systems in the network, researchers can examine the emergent dynamics of the network and how it responds to external stimuli.*

### 2.2.4 Different types of synchronization

The types of coupling can be listed as indicated:

1. *Unidirectional coupling*, to achieve it, two separate subsystems are formed by partitioning the original system with one subsystem designated as the driver and the other as the follower. The dynamics of the follower are then forced to mirror those of the driver, without influencing the behavior of the latter. As such, this configuration is often referred to as "master-slave." An illustration of this arrangement can be observed in the context of secure communications.

2. In *bidirectional coupling*, the behavior of two subsystems mutually influences each other's trajectories as they are connected in a certain way. Unlike unidirectional coupling, this approach involves reciprocal coupling between the subsystems. A prime instance of bidirectional coupling is observed in lasers that feature feedback mechanisms [69].

In this section, in the sense of an extensive bibliographic study, we have collected different types and patterns of synchronization.

#### Complete (Full) synchronization

Complete synchronization (**CS**) refers to the phenomenon in which two or more oscillators evolve in such a way that their trajectories become identical, both in phase and amplitude. This type of synchronization is important in various fields of application, including secure communication systems [70].

**Definition 2.5** *We consider a master chaotic system represented by*

$$\dot{Y}_1 = F_1(Y_1(t)), \tag{2.4}$$

*in which $Y_1(t)$ is the state vector of the $m$-dimensional master system, and the slave system represented by the following formula*

$$\dot{Y}_2 = F_2(Y_2(t)) + U, \tag{2.5}$$

*in which $Y_2(t)$ is the state vector of the $k$-dimensional slave system, $F_1 : \mathbb{R}^m \to \mathbb{R}^m, F_2 : \mathbb{R}^k \to \mathbb{R}^k$ and $U = (u_i)_{i=1}^k \in \mathbb{R}^k$ determine the control vector.*
*The complete synchronization error is defined by*

$$e(t) = Y_2(t) - Y_1(t) \textit{ such that } \lim_{t\to+\infty} \|e(t)\| = 0. \tag{2.6}$$

*where $\|.\|$ the Euclidean norm.*

- *If $F_1 = F_2$, The relationship leads to a state of complete synchronization, where the two systems evolve in a completely identical way over time.*
- *If $F_1 \neq F_2$, it is a non-identical complete synchronization.*

  *So synchronization, also known as* ***CS****, occurs when there is a complete coincidence between the state variables of the two systems that are synchronized.*

**Example 2.4** *Two identical chaotic systems can become completely synchronized if they are coupled in such a way that their trajectories become identical [70].*

### Anti-synchronization

Anti-synchronization is a phenomenon that happens when two coupled dynamical systems evolve in such a way that their states become opposite or out-of-phase. More precisely, it is the process of adjusting the behavior of two or more dynamical systems so that they evolve in a coordinated manner, but with their states being opposite or negatively correlated.
Anti-synchronization can be characterized by the property that the coupled systems exhibit an oscillatory behavior in which their states alternate between being opposite or out-of-phase with a fixed amplitude. This is in contrast to synchronization, where the difference between the states oscillates around zero with zero phase shift [71].
Then, the anti-synchronization error is presented by :

$$e(t) = Y_2(t) + Y_1(t). \tag{2.7}$$

**Example 2.5** *An example of anti-synchronization can be found in two coupled Chua oscillators. In this case, the two oscillators exhibit opposite dynamics, meaning that their states evolve in an anti-phase manner [72].*

### Shifted (delayed) synchrobization

Two non-identical chaotic dynamical systems are said to exhibit shifted synchronization if and only if time is shifted. Specifically, it means that the synchronized states of the systems are shifted or delayed with respect to each other by a certain amount of time [73].

**Definition 2.6** *If the state variables $Y_2(t)$ of the chaotic slave system converge to the state variables $Y_1(t)$ of the master system in lagged time, we say that there is delayed synchronization as indicated by the following relation :*

$$\lim_{t\to+\infty} \|Y_2(t) - Y_1(t-\varepsilon)\| = 0, \quad (\text{or } \lim_{t\to+\infty} \|Y_2(t) - Y_1(t+\varepsilon)\| = 0), \ \forall Y_1(0), \tag{2.8}$$

*where $\varepsilon$ is an extremely small positive value*

**Example 2.6** *As for an example of the application of this method, one well-known case is the synchronization of two chaotic systems with a time delay, as described in the paper [74]. In this paper, the authors show how the dynamics of two coupled chaotic systems can be synchronized using a feedback control scheme that involves a time delay. Specifically, they demonstrate that the chaotic trajectories of the two systems can be made to coincide with each other, even when there is a notable time delay between them. This technique has found applications in a range of fields, including secure communication and chaos-based cryptography.*

### Generalized synchronization

Generalized synchronization (**GS**) is a type of synchronization in which two or more chaotic systems exhibit similar but not necessarily identical dynamics, meaning that their states evolve in a synchronized manner but may differ by a scaling factor, a phase shift, or some other transformation [75].

**Definition 2.7** *To define the GS, a set of two master-slave systems is represented by*

$$\begin{cases} \dot{X}_1 = F_1(X_1(t)), \\ \dot{X}_2 = F_2(X_2(t)) + U. \end{cases} \tag{2.9}$$

*where $X_1(t) \in \mathbb{R}^m$, $X_2(t) \in \mathbb{R}^k$ represent the states of the master system and the slave system, respectively, $F_1 : \mathbb{R}^m \to \mathbb{R}^m$, $F_2 : \mathbb{R}^k \to \mathbb{R}^k$ and $U = (u_i)_{i=1}^k \in \mathbb{R}^k$ is a controller to be determined.*

*If there is a function $\Psi : \mathbb{R}^m \to \mathbb{R}^k$, such that all the trajectories of the master system and of the slave system, with the initial conditions $x_1(0)$ and $x_2(0)$ verify*

$$\lim_{t \to +\infty} \|X_2(t) - \Psi(X_1(t))\| = 0, \forall x_1(0), x_2(0), \tag{2.10}$$

*then, the above master-slave systems achieve generalized synchronization with respect to the function $\Psi$.*

**Example 2.7** *An example of generalized synchronization can be found in a coupled Henon map and a Lorenz model driven by a Rossler model. The results consistently demonstrate that the proposed measure successfully detects the direction of coupling, thereby highlighting its superior performance [76].*

### Projective synchronization

Projective synchronization (**PS**) is a type of synchronization in which two or more chaotic systems exhibit similar but transformed dynamics, meaning that their states evolve in a synchronized manner but with a nonlinear transformation applied to one or more of the state variables [77].

**Definition 2.8** *We say that we have **PS** if the state variables $y_i(t)$ of the chaotic slave system $Y(t) = (y_i(t))_{1 \le i \le n}$ synchronize with a constant multiple of l state $x_i(t)$ of the master chaotic system $X(t) = (x_i(t))_{1 \le i \le n}$ so that :*

$$\exists \alpha_i \neq 0, \quad \lim_{t \to +\infty} \|y_i(t) - \alpha_i x_i(t)\| = 0, \ \forall (x(0), y(0)), \ i = 1, 2, \dots, n. \tag{2.11}$$

**Remark 2.1**

- *The case where all $\alpha_i$'s $= 1$ represents a complete synchronization case.*
- *A complete anti-synchronization case is represented when all $\alpha_i$'s $= -1$.*

**Example 2.8** *An example of projective synchronization can be found in two coupled Lorenz oscillators, where one oscillator is subject to a nonlinear transformation. In this case, the two oscillators exhibit similar but transformed dynamics, meaning that their states evolve in a synchronized manner but with a nonlinear transformation applied to one or more of the state variables [78].*

### Generalized projective synchronization

Generalized projective synchronization (**GPS**) is a synchronization type that allows a master system to synchronize with a response system in a more flexible way, by adding a scaling factor $\alpha$ to the synchronization error, in other words, that the master and the response system become proportional.

**Definition 2.9 ([79])** *One may examine the chaotic system provided below:*

$$\begin{cases} \dot{Y}_1 = F_1(Y_1(t)), \\ \dot{Y}_2 = F_2(Y_2(t)) + U. \end{cases} \tag{2.12}$$

*in which $Y_1(t), Y_2(t) \in \mathbb{R}^m$ denote the state vectors of the master (or driver) and slave (or response) systems, respectively. Additionally, $F_1, F_2 : \mathbb{R}^m \to \mathbb{R}^m$ are vectors fields, and $U = (u_i)_{i=1}^m \in \mathbb{R}^m$ is a vector controller whose values are yet to be determined.*
*If a constant $c \neq 0$ exists with the aim that*

$$\lim_{t\to+\infty} \|Y_1(t) - cY_2(t)\| = 0, \tag{2.13}$$

*then we say that there is a **GPS** and we call $c$ a **scaling factor**.*

**Example 2.9** *In reference [80], a straightforward yet effective control approach for achieving generalized projective synchronization is demonstrated on a unified chaotic system. Results of numerical simulations indicate the high efficacy of this method, and it can be applied to other chaotic systems as well.*

### Full state projective synchronization

Full state projective synchronization (**FSPS**) refers to a category of synchronization where two or more chaotic systems synchronize not only their trajectories but also their scaling factors and bias parameters. In this type of synchronization, the difference between the state variables of the synchronized systems is proportional to a matrix multiplication of the state variables of one of the systems. Full state projective synchronization has been thoroughly investigated studied in the literature due to its potential applications in secure communication and data encryption [81].

**Definition 2.10** *For the chaotic system (2.12), we say that there is an **FSPS**, if there is a non zero constant $g$, with the aim that*

$$\lim_{t\to+\infty} \|Y(t) - gX(t)\| = 0, \tag{2.14}$$

*that is,*

$$\lim_{t\to+\infty} \|y_j(t) - gx_j(t)\| = 0, \ j = \overline{1.m}, \tag{2.15}$$

*thus, the **FSPS** of system (2.12) is achieved.*

**Example 2.10** *The paper [81] addresses the problem of achieving full-state projective synchronization in chaotic continuous-time systems. The authors propose a nonlinear observer control approach to achieve this synchronization, aiming to overcome the challenges posed by chaotic dynamics. The* *article likely discusses the theoretical framework and methodology of the proposed observer control technique and provides numerical simulations or experimental results demonstrating its effectiveness in achieving full-state projective synchronization in chaotic systems.*

**Inverse full state hybrid projective synchronization**

Inverse full state hybrid projective synchronization (IFSHPS) is another type of synchronization between two chaotic systems, in which the drive system is synchronized to the inverse response system up to a linear transformation. Like FSHPS, IFSHPS also involves both continuous and discrete feedback controls [82].

**Definition 2.11** *The master and the slave systems in* (2.12) *are said to be inverse full state hybrid projective synchronized, if there exists a controller* $U = (u_j)_{1\leq j\leq k}$ *and given real numbers* $(\gamma_{ji})_{1\leq j\leq m, 1\leq i\leq k}$, *such that the synchronization errors*

$$e_j(t) = x_j(t) - \sum_{i=1}^{k} \gamma_{ji} y_i(t), \quad j = \overline{1, m}, \tag{2.16}$$

*achieve that the limit of* $e_j(t)$ *approaches* $0$ *as* $t$ *approaches* $+\infty$.
*If* $(\gamma_{ji}(t))_{1\leq j\leq m, 1\leq i\leq k}$, *are given differentiable functions, then the system* (2.12) *are supposedly IFSHP synchronized.*

**Example 2.11** *The article [82] addresses the IFSHP synchronization of the proposed fractional-order system, potentially exploring different synchronization techniques and their effectiveness in achieving synchronization. The numerical or analytical results presented in the article may demonstrate the synchronization behavior of the fractional-order system and provide insights into its potential applications in real-world scenarios.*

**Q-S synchronization**

Q-S synchronization is considered a generalization of all types of previous synchronizations [83].

**Definition 2.12** *We say that a master system* $Y_1(t) \in \mathbb{R}^m$ *and a slave system* $Y_2(t) \in \mathbb{R}^k$ *are in Q-S synchronization in dimension* $d$, *if there is a controller* $U = (u_j)_{1\leq j\leq k}$ *and two functions* $Q : \mathbb{R}^m \to \mathbb{R}^d$, $S : \mathbb{R}^d \to \mathbb{R}^k$ *such as synchronization error*

$$e(t) = Q(Y_1(t)) - S(Y_2(t)), \tag{2.17}$$

*verifies* $\lim_{t\to+\infty} \|e(t)\| = 0$.

**Example 2.12** *In the paper [84], the authors investigate the synchronization behavior of the fractional-order unified system, in which this article delves into the QS synchronization technique employed, explaining how it can be applied to achieve synchronization in the fractional-order unified system. Numerical simulations or analytical results are provided to illustrate the effectiveness of the proposed synchronization approach, shedding light on the dynamics and potential applications of the synchronized fractional-order unified system.*

### Phase synchronization

Phase synchronization signifies the phenomenon where the phases of two or more oscillators become locked to each other, while the amplitudes of the oscillators may still differ. This type of synchronization is important in many natural and engineered systems, such as the synchronization of neurons in the brain [76].

**Example 2.13** *Two pendulum clocks that are hung on the same wall can become phase-synchronized if they are close enough to each other. In this case, the pendulums will eventually swing in unison, even if they started with slightly different amplitudes [85].*

### Anti-phase synchronization

Anti-phase synchronization signifies the phenomenon where the phases of two or more systems (oscillators) become locked to each other, but the systems (oscillators) are in anti-phase, meaning that they oscillate in opposite directions. This type of synchronization is important in many applications, such as the synchronization of the heartbeats of two people [86].

**Example 2.14** *In the study by Strogatz [86], the researchers investigate the phenomenon of anti-phase synchronization using the flashing patterns of fireflies. Fireflies are known for their ability to synchronize their flashing signals, creating mesmerizing light displays in certain regions. The researchers focus on understanding the specific case of anti-phase synchronization, where the fireflies flash in alternating patterns. Through field observations and mathematical modelling, Strogatz et al. demonstrates how fireflies achieve anti-phase synchronization. They reveal that the behavior arises from the inherent properties of the firefly's nervous system and their interaction with each other. The study provides insights into the mechanisms underlying anti-phase synchronization in biological systems and explores its relevance in fields such as neuroscience and collective behavior.*

### Modiefed projective synchronization

Modiefed projective synchronization (**MPS**) involves the synchronization of the drive and response systems to the extent of a scaling factor and a time delay. The scaling factor and time delay are modified from the usual definitions of projective synchronization. This type of synchronization has been studied in several systems, including the Chua's circuit and the Rössler system [87], and Li [88] introduced modified projective synchronization in prior work.

**Definition 2.13** *MPS is distinguished by a scaling matrix that two systems synchronize in parallel. For the chaotic systems* (2.12)*, If there is a nonzero constant matrix:* $\beta = diag(\beta_1, \beta_2, \ldots, \beta_m)$, *so that*

$$\lim_{t \to +\infty} \|Y_1(t) - \beta Y_2(t)\| = 0, \tag{2.18}$$

*then we call this synchronization **MPS**, we refer to* $\beta$ *as a "**scaling matrix**".*

**Remark 2.2** *Clearly, **CS** and **projective synchronizations** can be viewed as particular instances of **MPS** where* $\beta_1 = \beta_2 = \ldots = \beta_n = 1$ *and* $\beta_1 = \beta_2 = \ldots = \beta_n$*, respectively.*

**Example 2.15** *In the study by G.H. Li [87], the author explores modified projective synchronization in chaotic systems. The concept of projective synchronization involves driving a response system to synchronize with a drive system up to a certain transformation. Li introduces modifications to this synchronization scheme, aiming to achieve enhanced control over the synchronization process. By applying suitable control techniques and adjusting system parameters, Li demonstrates the successful achievement of modified projective synchronization in chaotic systems. The study contributes to the understanding and practical application of synchronization phenomena in chaotic systems, opening up possibilities for controlling and manipulating chaotic dynamics in various fields, including secure communication and data encryption.*

### Function projective synchronization

Function projective synchronization (**FPS**), as described in [89], involves a scaling function matrix.

**Definition 2.14** *For the system* (2.12)*, the error term is described by*

$$e(t) = Y_1(t) - K(t)Y_1(t), \tag{2.19}$$

*where* $K(t) = (k_j(t))_{j=1}^{m}$ *is a continuously differentiable function that is bounded and non-zero for all* $t$*. The synchronization is termed **FPS** and* $K(t)$ *is referred to as a "**scaling function matrix**" when* $\lim_{t \to +\infty} \|e(t)\|$ *approaches zero.*

**Remark 2.3** ***FPS*** *is a broader concept than PS, as it permits the synchronization of two master-slave systems to a scaling function instead of just a constant, as is the case with PS. Employing this function could enhance the security of communication, because it is obvious that the scale unpredictability of the function in the FPS method can further improve the communication security.*

**Example 2.16** *The article [90] presents a method for achieving FPS between two different chaotic systems with indeterminate parameters. The authors derive a control law that uses a linear combination of the drive and response systems's state variables to achieve FPS. The control law is designed to be robust to parameter uncertainties, through simulations of the Lorenz and Chen systems, the authors illustrate the efficacy of their approach.*

### Modified function projective synchronization

MFPS is a novel form of synchronization in which the synchronized dynamical states are attuned to a designated scaling function matrix, as stated in [91].
Equation (2.12) presents the definitions of the drive system and the response system. Subsequently, we introduce the error vector $e(t)$, defined as

$$e(t) = Y_1(t) - \Gamma(t)Y_2(t), \tag{2.20}$$

where $e(t) = (e_1, e_2, \dots e_m)^T$, $\Gamma(t)$ is a $m$-order diagonal matrix, $\Gamma(t) = diag(\gamma_1(t), \gamma_2(t), \dots, \gamma_m(t))$ and $\gamma_j(t)$ is a continuously differentiable function that is bounded and non-zero $\forall t$.

**Definition 2.15 (MFPS)** *The system* (2.12) *is considered to be exhibiting* ***MFPS*** *when a scaling function matrix* $\Gamma(t)$ *exists so that* $\lim_{t\to+\infty} \|e(t)\|$ *approaches zero.*

**Remark 2.4** ***MFPS*** *is a broader concept than* ***MPS*** *and* ***FPS,*** *where* $\gamma_1(t) = \gamma_2(t) = \dots = \gamma_n(t) = \gamma(t)$, $\gamma_1(t) = \gamma_1, \gamma_2(t) = \gamma_2, \dots, \gamma_n(t) = \gamma_n$, *respectively.*

**Remark 2.5** *The lack of predictability of the scaling function matrix in* ***MFP*** *can increase the security of communications, making it a potential technique for achieving greater security.*

**Example 2.17** *The article [91] proposes a modified method for achieving FPS between chaotic systems. The authors modify the standard FPS approach by introducing a nonlinear feedback control term that incorporates a quadratic function of the error for the drive and response systems. The authors show through simulations that their modified FPS method achieves better synchronization performance than the standard FPS approach between the Lorenz and Rössler systems.*

### Generalized function projective synchronization

Generalized function projective synchronization (**GFPS**) involves the synchronisation of the drive and response systems until a scaling function matrix.
To define the **GFP** synchronization, the error term is presented by:

$$e(t) = Y_2(t) - B(t)Y_1(t), \tag{2.21}$$

where $e(t) = (e_1, e_2, \dots e_m)^T$, $B(t) = diag(b_1(t), b_2(t), \dots, b_m(t))$ is reversible and differentiable and $b_j(t) : \mathbb{R}^+ \to \mathbb{R}$ are continuously differentiable functions that is bounded and non-zero $\forall t$.

**Definition 2.16** *For the chaotic system* (2.12) *consisting of a master and slave system,* ***GFPS*** *occurs when a vector function $U(Y_1, Y_2, t)$ and a scaling function matrix $B(t)$ of order $m$ exist so that $\lim_{t\to+\infty} \|e(t)\|$ approaches zero, where the norm $\|.\|$ is induced by the matrix norm.*

**Remark 2.6**
- *If $b_1(t) = b_2(t) = \dots = b_m(t)$,* ***FPS*** *is achieved.*
- *If $b_j(t) = b_j,\ (j = \overline{1, m})$ where $b_i \in \mathbb{R}$ and non-zero, then* ***GFPS*** *is simplified to* ***GPS****.*
- *If $B = \alpha I_m$ where $\alpha \in \mathbb{R}$ and $I_m$ is the identity matrix, the* ***GFPS*** *simplifies to* ***PS****.*

**Example 2.18** *The paper [92] presents a new approach to achieve adaptive* ***GFPS*** *of uncertain chaotic systems. The proposed method uses a sliding mode control technique and an adaptive control law to synchronize two chaotic systems, one with known parameters and the other with uncertain parameters. The generalized function projective synchronization ensures that a linear combination of the synchronization errors for the two systems converges to zero. Numerical simulations of the Lorenz and Chen chaotic systems are carried out to demonstrate the efficacy of the proposed method.*

### Hybrid projective synchronization

Hybrid PS was introduced by Manfeng Hu et al [93]. This method allows for the drive and response systems to synchronize with **different scale factors**.
To define **HPS**, the error term is offered as:

$$e(t) = K(t)Y_2(t) - Y_1(t), \tag{2.22}$$

where $e(t) = (e_1, e_2, \dots e_m)^T$, $K(t) = diag(k_1, k_2, \dots, k_m)$ is a scaling matrix, and $k_j$ is a scale non-zero factor $\forall t$.

**Definition 2.17** *If the system* (2.12) *has a reversible scaling matrix $K$ of order $m$ such that $\lim_{t\to+\infty} \|e(t)\|$ is zero, then we say that there is* ***HPS****.*

**Remark 2.7** *It is easy to see that **CS, AS, PS** and **MPS** are special cases of **HPS**:*

- *If $H = I$, where $I$ is the unit matrix, then the synchronization is called **CS**.*
- *If $H = -I$, then the synchronization is named **AS**.*
- *If $H = \alpha I$, and $\alpha \neq \pm 1$ a constant real different from zero, then the synchronization is called **PS**.*
- *If $H = diag(h_1, h_2, \ldots, h_n)$ and $h_1, h_2, \ldots, h_n$ are different nonzero constants, then the synchronization is named **MPS.***

**Example 2.19** *In the article [93], the authors investigate the hybrid projective synchronization of two different chaotic complex nonlinear systems. They propose a control method based on projective synchronization and adaptive control techniques, which allows for the synchronization of two chaotic systems with different structures and parameters. Numerical simulations are conducted by the authors to demonstrate the efficacy of their proposed method, with the Lorenz and Chen chaotic systems used as examples. The results show that the proposed method is able to obtain HPS for the two chaotic systems with high accuracy and robustness. The authors propose that their method could be used in cryptography and secure communication systems.*

### Hybrid function projective synchronization

Zhang and Li [94] developed a novel type of synchronization denoted by **HFPS**, where both master and slave systems synchronize until a scaling function matrix.
To define **HFPS** synchronization, the error term is offered as :

$$e(t) = Y_2(t) - K(t)Y_1(t), \tag{2.23}$$

where $e(t) = (e_1, e_2, \ldots e_m)^T$, $K(t)$ is a scaling function matrix, $K(t) = diag(k_1, k_2, \ldots, k_m)$, and $k_j(t)$ is a continuously differentiable function that bounded and non-zero $\forall t$.

**Definition 2.18** *The system (2.12) is considered to have a **HFPS** if a reversible scaling matrix $K$ can be found so that $\lim\limits_{t \to +\infty} \|e(t)\| = 0$.*

**Example 2.20** *The paper [94] presents a method for achieving HFPS of chaotic systems with uncertain time-varying parameters. The method uses Fourier series expansion to approximate the unknown functions, which can effectively reduce the effects of parameter uncertainties. The proposed control scheme is easy to implement and has good robustness and synchronization accuracy. The efficacy of the proposed method is demonstrated through numerical simulations.. Overall, the article provides a useful contribution to the field of synchronization of chaotic systems.*

**Full state hybrid projective synchronization**

Full state hybrid projective synchronization (FSHPS) is a type of synchronization between two chaotic systems, where the states of the drive system and response system achieve synchronized up to a linear transformation. The synchronization is said to be hybrid because it involves both continuous and discrete feedback controls [95].

**Definition 2.19 ([96])** *We have two chaotic systems, the master and slave, which are described by*

$$D_t^{\alpha_j} x_j(t) = f_j(X(t)), j = \overline{1, m}, \tag{2.24}$$

*and the response system can be expressed as*

$$D_t^{\alpha_j} y_j(t) = \sum_{i=1}^{n} b_{ji} y_i(t) + g_j(Y(t)) + u_j, j = \overline{1, n}, \tag{2.25}$$

*with* $X(t) = (x_1, x_2, ..., x_m)^T$ *and* $Y(t) = (y_1, y_2, ..., y_n)^T$ *are the state vectors of the systems* (2.24) *and* (2.25) *respectively*, $f_j : R^m \longrightarrow R^m$, $j = \overline{1, m}$ *and* $g_i : R^n \longrightarrow R^n$, $i = \overline{1, n}$ *are sets of nonlinear functions,* $0 < \alpha_j < 1$, $D_t^{\alpha_j}$ *is the Caputo fractional derivative of order* $\alpha_j$.
*The synchronization of the two systems* (2.24) *and* (2.25) *is called FSHP if we can find a controller* $U = (u_j)_{1 \le j \le m}$ *and real numbers* $(c_{jk})_{1 \le j \le m, 1 \le k \le n}$, *with the aim that the synchronization errors*

$$e_j(t) = y_j(t) - \sum_{k=1}^{n} c_{jk} x_k(t), \quad j = \overline{1, m}, \tag{2.26}$$

*match the specification that* $\lim_{t \to +\infty} e_j(t) = 0$.
*If* $(c_{jk})_{1 \le j \le m, 1 \le k \le n}$ *are given differentiable functions, then the master system* (2.24) *and the slave system* (2.25) *are referred to as FSHP synchronized.*

**Remark 2.8**

- ***FSHP** synchronization is a generalization of projective synchronization [97].*
- *IThe definition of FSHP synchronization covers full synchronization, anti-synchronization, and projective synchronization by simply assigning the matrix* $C = (c_{jk})_{1 \le j \le m, 1 \le k \le n}$ *the values* $I, -I$ *and* $\beta I$ ($\beta$ *is a constant), respectively. This relationship can be easily observed.*

**Example 2.21** *In the study by M. Hu, Z. Xu, and R. Zhang [95], the authors investigate full state hybrid projective synchronization in continuous-time chaotic (hyperchaotic) systems. The concept of hybrid synchronization combines both complete synchronization and projective synchronization, allowing for more flexible and versatile control over the synchronization process. Hu, Xu, and Zhang propose a novel synchronization scheme that achieves full state hybrid projective synchronization, where all the state variables of the drive and response systems exhibit synchronized behaviors.*

*Through numerical simulations and analysis, they demonstrate the effectiveness of their proposed scheme in achieving robust synchronization in chaotic and hyperchaotic systems. This research expands the understanding of synchronization phenomena in complex dynamical systems and offers potential applications in secure communication, signal processing, and information transmission.*

## 2.3 Control methods for synchronization

The specific mathematical formulation of the control law is dependent on the nature of the systems being synchronized and the intended performance criteria. This section is devoted to the presentation of various methods to achieve different kinds of synchronization.

### Active controller method

Active control technique is a type of control system create methodology that utilizes feedback control to obtain the required performance of the system. It involves designing a controller that actively manipulates the inputs of the system aiming to achieve the required output where it manipulates the input signal $u(t)$ to achieve the desired output $x_2(t)$. The control law is usually modeled in the form $u(t) = F_1(x_1(t), x_2(t))$, where $x_1(t)$ is the state vector, $x_2(t)$ is the output vector, and $F_1$ is a function that maps these signals to the input signal $u(t)$ [98].
Consider two chaotic systems to be synchronized, master and slave, defined by :

$$\dot{X}_1 = F_1(X_1(t)), \tag{2.27}$$
$$\dot{X}_2 = F_2(X_2(t)) + U. \tag{2.28}$$

where $X_1(t) \in \mathbb{R}^m, X_2(t) \in \mathbb{R}^m$ are the states of the master and slave systems, respectively, $F_1, F_2 : \mathbb{R}^m \to \mathbb{R}^m$, and $U = (u_j)_{1\leq j\leq m}$ is a controller to be determined. For the two systems to be synchronized, the error between the trajectories of the two systems must converge towards zero when time tends towards infinity. This error is obtained as follows :

$$\dot{e}(t) = \dot{X}_2(t) - \dot{X}_1(t) = F_2(X_2(t)) - F_1(X_1(t)) + U. \tag{2.29}$$

If we can write the quantity $F_2(X_2(t)) - F_1(X_1(t))$ as follows :

$$F_2(X_2(t)) - F_1(X_1(t)) = Be(t) + M(X_1(t), X_2(t)), \tag{2.30}$$

the error can be expressed as follows :

$$\dot{e}(t) = Be(t) + M(X_1(t), X_2(t)) + U, \tag{2.31}$$

where $B \in \mathbb{R}^{m\times m}$ is a constant matrix and $M$ a nonlinear function.

The $U$ controller is offered as follows :

$$U = V - M(X_1(t), X_2(t)), \tag{2.32}$$

where $V$ is the active controller defined by :

$$V = -\Lambda e(t), \tag{2.33}$$

where $\Lambda$ is an unknown control matrix. So, we get the following final formula of the error :

$$\dot{e}(t) = (B - \Lambda)e(t). \tag{2.34}$$

So the problem of synchronization for the master system (2.27) and the slave system (2.28) is changed into a zero-stability problem of the system (2.34). Now the following Theorem is an immediate result of the theory of the stability of continuous linear dynamical systems.

**Theorem 2.2** *If the eigenvalues of* $(B - \Lambda)$ *lie inside the unit disk, the control law* (2.32) *achieves global synchronization between the master system* (2.27) *and the slave system* (2.28)*. Conversely, global synchronization is not achieved unless the control matrix* $\Lambda$ *is chosen to satisfy this condition.*

## Backstepping method

The Backstepping Method is a type of control system design methodology that is used for non-linear systems. It involves designing a control law for a system by recursively transforming the original system into a simpler one until a linear system is obtained [99].

This is done by introducing new state variables and using the Lyapunov stability theory to guarantee the stability of the system. The control law is usually designed and calculated such that the positive Lyapunov function $V$ stabilize the system $(V < 0)$.

Consider that the master system and the slave system are defined in the following way

$$\begin{cases} \dot{x}_1 = g_1(x_1, x_2), \\ \dot{x}_2 = g_2(x_1, x_2, x_3), \\ \ldots.. \\ \dot{x}_m = g_m(x_1, x_2, x_3, \ldots, x_m). \end{cases} \tag{2.35}$$

and

$$\begin{cases} \dot{y}_1 = g_1(y_1, y_2), \\ \dot{y}_2 = g_2(y_1, y_2, y_3), \\ \ldots.. \\ \dot{y}_m = g_m(y_1, y_2, y_3, \ldots, y_m). \end{cases} \tag{2.36}$$

where $g_1$ is a linear function, $g_i$ $(i = \overline{2,m})$ are nonlinear functions and $u$ is a controller which must be chosen appropriately to obtain synchronization between the systems (num syst $\dot{x}$) and (num syst $\dot{y}$).
The synchronization error is given as follows

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ \ldots\ldots \\ e_m = y_m - x_m. \end{cases} \tag{2.37}$$

Thus, the dynamics of the error system is written :

$$\begin{cases} \dot{e}_1 = h_1(e_1, e_2), \\ \dot{e}_2 = h_2(e_1, e_2, e_3), \\ \ldots\ldots \\ \dot{e}_m = h_m(e_1, e_2, \ldots, e_m) + u. \end{cases} \tag{2.38}$$

where $h_1$ is a linear function, $h_j$, $j = \overline{2,m}$ are nonlinear functions.
The goal is to calculate a control law $u$ which ensures the convergence of the system $e_j$, $j = \overline{2,m}$ towards the origin using the backstepping algorithm. For this, the error system (numerror) must be decomposed into subsystems :

$$e_1, (e_1, e_2), (e_1, e_2, e_3), \ldots, (e_1, e_2, \ldots, e_m),$$

and for each subsystem, we define a positive Lyapunov function $V : V_k(e_k, u_k, \alpha_k)$, where $k$ is the order of the subsystem, $u_k$ and $\alpha_k$ represent, respectively, the control law and the virtual controller of the subsystem of order $k$, $u_k$ and $\alpha_k$ are calculated each time such that : $V_k < 0$.

## Slip mode method

The slip mode method is a control strategy used to regulate the behavior of nonlinear systems. This approach involves the introduction of a "slip" variable that controls the system's dynamics by adjusting the phase difference between two oscillators. The slip variable is designed to maintain synchronization by constraining the phase difference between the oscillators to remain within a specified range. The method is particularly effective for systems with uncertain parameters or disturbances, as it can compensate for these factors and ensure stability. The slip mode method is a type of sliding mode control, which was previously extensively conducted in the control literature. For instance, Shtessel et al. [100] provide a detailed treatment of sliding mode control and its applications to uncertain systems.

**Example 2.22** *In the study by Ablay [101], the author investigates the sliding mode control of uncertain unified chaotic systems. Unified chaotic systems are a class of complex systems that exhibit chaotic behavior, and dealing with their uncertainties is essential for control purposes. Ablay proposes a sliding mode control approach to effectively stabilize and control these uncertain unified chaotic systems. The sliding mode control technique ensures robustness against uncertainties and disturbances, allowing for precise tracking of desired trajectories. Through mathematical analysis and simulations, Ablay demonstrates the effectiveness of the proposed control strategy in achieving stable and controlled behavior in uncertain unified chaotic systems. This research contributes to the understanding of chaos control methods and provides practical insights for the design and implementation of robust control systems in various applications, including communication systems and secure data transmission.*

## Phase reduction

This technique involves approximating the behavior of a high-dimensional oscillator by its dynamics in a low-dimensional phase space. By reducing the system to a phase oscillator, one can analyze the stability and bifurcations of the synchronization manifold and predict the emergence of different types of synchronization, in the paper [102], Strogatz provides an overview of different approaches to analyzing synchronization in populations of coupled oscillators, including the phase reduction technique.

**Example 2.23** *This method has the potential to be utilized for study synchronization in a broad variety of systems, including biological oscillators, electronic circuits, and chemical reactions. Here is an example of a paper that uses phase reduction to analyze synchronization in various network topologies [103], in this paper, the authors focus on phase reduction techniques, which provide a powerful tool for analyzing the behavior of coupled oscillators in both simple and complex networks.*

## Feedback control

Feedback control is a method of control in which a system or process is monitored, and the output is fed back into the system to adjust the input and regulate the output. This method is commonly used in engineering systems, such as control systems for aircraft, vehicles, and robots [104].

**Example 2.24** *Here's an example to illustrate feedback control: suppose we want to regulate the temperature of a room to a specific setpoint. We could use a thermostat as a feedback control mechanism. The thermostat measures the room's temperature and compares it to the desired setpoint. If the temperature is below the setpoint, the thermostat turns on the heating system to increase the temperature. If the temperature is above the setpoint, the thermostat turns off the heating system.*

*The thermostat continuously adjusts the heating system based on the difference between desired and actual temperatures.*

## 2.4 Conclusion

We first presented general and mathematical definitions of synchronization and an overview of chaos synchronization. We then defined several synchronization types of chaotic (hyper-chaotic) dynamical systems with examples. Finally, we gave control methods to achieve synchronization of dynamical systems. The study of the integer-order and fractional-order chaotic systems synchronization is a subject that has been treated recently, it gives rise to new synchronization schemes ensuring security in telecommunications.

# Chapter 3

# Some chaotic dynamical systems with fractional orders

## 3.1 Inroduction

The intricate and unpredictable dynamical behaviors exhibited by chaotic and hyperchaotic systems are among their most complex properties, this makes fractional order calculus an attractive area of study for implementing circuits that mimic these systems. Fractional order calculus can give a more accurate description of various nonlinear phenomena that may be overlooked by integer order calculus.

Fractional calculus, which involves the use of fractional integrals and derivatives, has been employed in controlling dynamical systems, particularly when the controlled system or controller is modeled using fractional differential equations. Despite the fact that the total order of such systems with fractional derivatives is typically less than three, they can still exhibit chaotic behavior, including the presence of strange attractors, in their mathematical models.

Chapter three is on fractional-order chaotic systems. It begins with a brief overview of fractional order systems, covering historical background, definitions, and fundamental concepts. The chapter also includes a comprehensive review of popular fractional-order chaotic systems.

### 3.1.1 Historical overview

As early as 1695, during the time when classical calculus (i.e. calculus of derivatives and integrals of integer order) was just established by Leibniz and Newton, there was a correspondence between Leibniz and L'Hôpital where they discussed the interpretation of the derivative of order one half. This sparked the interest of many renowned mathematicians such as Euler (1738), Laplace (1820), Fourier (1822) and Lagrange (1849) to work on similar questions, which ultimately led to the development of fractional calculus as a field of study.

### 3.1.2 Application of fractional systems

For example, applications of feedback control include the modelling of biological tissues [105] and neuron systems [106] in medicine, as well as the use of fractional calculus in continuous-time finance in financial markets [107]. Additionally, the fractional Burgers equation has been introduced as a relevant model for anomalous diffusions, such as diffusion in complex phenomena, relaxations in viscoelastic mediums, and propagation of acoustic waves in gas-filled tubes (see [108] and [109]). These concepts have found applications in various fields, including viscoelastic systems, electrochemistry, and biological and economic systems *(see* [110]-[112]).

## 3.2 Mathematical background of fractional calculus

Fractional calculus involves generalizing differentiation and integration to non-integer order operators represented as ${}_aD_t^q$, where $q$ represents the fractional order and $a$ and $t$ are the limits of the operation. This concept has been extensively studied by mathematicians since the time of Leibniz and Newton in 1695, leading to the development of a field known as fractional calculus, as referenced in [113]

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & \Re(q) > 0 \\ 1, & \Re(q) = 0 \\ \int_a^t (d\tau)^{-q}, & \Re(q) < 0 \end{cases} . \tag{3.1} $$

It is generally assumed that $q \in \mathbb{R}$, however, it is also possible for $q$ to be a complex number, as noted in [114].

To delve deeper into fractional calculus, we suggest referring to [115] and [116].

### 3.2.1 Specific functions related to fractional calculation

In this section, we will discuss the Gamma, Beta and Mittag-Leffler functions, which constitute one of the essential tools in fractional calculus theory.

### The Gamma function

**Definition 3.1** *The Euler's Gamma function* $\Gamma(w)$ *is defined by*

$$\Gamma(w) = \int_0^\infty t^{w-1}e^{-t}dt, \quad (\mathrm{Re}(w) > 0), \tag{3.2}$$

*where* $t^{w-1} = e^{(w-1)\log(t)}$, $\Gamma(1) = 1$ *and* $\Gamma(0_+) = +\infty$. $\Gamma(w)$ *is a monotonic, strictly decreasing function for* $0 < w \leq 1$, *and this integral is convergent for* $w \in \mathbb{C}, (\mathrm{Re}(w) > 0)$.

**Proposition 3.1** *The Gamma function exhibits the following properties:*

1. $\forall w \in \mathbb{R}_+^*$ :

$$\Gamma(w+1) = w\Gamma(w) \tag{3.3}$$

2. *Eular's Gamma function generalizes the concept of factorial*

$$\Gamma(m+1) = m!, \quad \forall m \in \mathbb{N}. \tag{3.4}$$

**Proposition 3.2 ([117])** *The limit expression can be used to represent the Gamma function as follows:*

$$\Gamma(w) = \lim_{m \longrightarrow +\infty} \frac{m!m^w}{w(w+1)......(w+m)}. \tag{3.7}$$

### The Beta function

**Definition 3.2** *The definition of the Beta function is given by*

$$B(w,z) = \int_0^1 t^{w-1}(1-t)^{z-1}dt, \quad \forall w, z \in \mathbb{R}^{*+}. \tag{3.8}$$

**Proposition 3.3** *The relationship between Euler's Beta function and Euler's Gamma is given by:*

$$B(w,z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)}. \tag{3.9}$$

### The Mittag-Leffler function

The Mittag-Leffer function naturally embeds the usual exponential.

**Definition 3.3** *G.M. Mittag-Leffler introduced the generalization of the exponential function to a single parameter [118], and denoted by the following function [119]:*

$$E_q(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(qk+1)}, \quad \forall w \in \mathbb{R}, \; q > 0. \tag{3.10}$$

**Definition 3.4** *In fractional calculus theory, the Mittag-Leffer function's 2-parameter plays a very important role. The latter is defined by the following series expansion [119]:*

$$E_{\gamma,\delta}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\gamma k + \delta)}, \quad (\forall w \in \mathbb{R},\ \gamma, \delta > 0). \tag{3.11}$$

**Example 3.1**

$$\begin{aligned}
E_{\gamma,1}(w) &= \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\gamma k + 1)} = E_{\gamma}(w) \\
E_{1,1}(w) &= \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{w^k}{k!} = e^w \\
E_{1,2}(w) &= \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(k + 2)} = \frac{1}{w} \sum_{k=0}^{\infty} \frac{w^{k+1}}{(k+1)!} = \frac{e^w - 1}{w} \\
E_{1,\delta}(w) &= \frac{1}{w^{\delta-1}} \left[ e^w - \sum_{k=0}^{\delta-2} \frac{w^k}{k!} \right] \\
E_{2,1}(w^2) &= \sum_{k=0}^{\infty} \frac{w^{2k}}{\Gamma(2k + 1)} = \sum_{k=0}^{\infty} \frac{w^{2k}}{2k!} = \cosh(w).
\end{aligned}$$

### 3.2.2 Laplace transformation

We will review some fundamental concepts of the Laplace transform. We denote the Laplace transforms with capital letters and the originals with lowercase letters.

**Definition 3.5** *Given a function $h \in L^1(0, \infty)$. we can define a function $H(s)$ of the complex variable $s$ using the Laplace transform:*

$$H(s) = L\{h(t)\}(s) = \int_0^{+\infty} e^{-st} h(t) dt, \tag{3.12}$$

*In this equation, $h(t)$ is the original function of $H(s)$, and $H$ is commonly referred to as the Laplace transform of $h(t)$.*
*A sufficient condition for the existence of the integral (3.12) is that the function $h(t)$ must be of exponential order $c$, meaning there exist $N$, $T > 0$ such that $|h(t)| \leq Ne^{ct}$ for $t > T$ where $T \in \mathbb{R}_+^*$.*

#### Laplace transform Properties

- We can reconstruct the origin $h(t)$ from the Laplace transform $H(s)$ using the inverse Laplace transform

$$h(t) = L^{-1}\{H(s)\}(t) = \int_{a-i\infty}^{a+i\infty} e^{st} H(s) ds, \ a = \mathrm{Re}(s) > a_0, \tag{3.13}$$

where $a_0$ is the absolute convergence index of the integral (3.12).

- Assuming that the Laplace transforms $H(s)$ and $G(s)$ exist for two functions $h(t)$ and $g(t)$, whose domains are limited to $t \geq 0$, the Laplace transform of their convolution, defined as:

$$h(t) * g(t) = \int_0^t h(t-\tau)g(\tau)d\tau = \int_0^t h(t)g(\tau-\tau)d\tau \tag{3.14}$$

is equal to the product of their Laplace transforms:

$$L\{h(t) * g(t); s\} = H(s)G(s).$$

- The Laplace transform of a derivative of an integer order $q$ of the function $h(t)$ :

$$\begin{aligned} L[h^{(q)}(t)](s) &= s^q H(s) - \textstyle\sum_{k=0}^{q-1} s^{q-k-1} h^{(k)}(0) \\ &= s^q H(s) - \textstyle\sum_{k=0}^{q-1} s^k h^{(q-k-1)}(0). \end{aligned} \tag{3.15}$$

**Remark 3.1** *Unfortunately, the Laplace transform technique is not well-suited for handling the Riemann-Liouville fractional derivative because it necessitates knowledge of non-integer order derivatives of the function at $t = 0$, making it challenging to apply. However, this issue is not present in the Caputo definition, also referred to as the smooth fractional derivative in some literature [120].*

### 3.2.3 Fractional derivatives

For the fractional derivation, there are several types of definitions, let us quote the most used in the applications:

#### Fractional derivatives of Grünwald-Letnikov

**Definition 3.6 ([121])** *Let $0 \leq m-1 < q < m$, we can define the left Grünwald -Letnikov derivative of order $q$ as follows:*

$${}_a^G D_+^q h(t) = \lim_{p \to 0^+} \frac{1}{p^q} \sum_{j=0}^{\infty} (-1)^j \binom{q}{j} h(t - jh), \forall t \in \mathbb{R}, \tag{3.16}$$

*while* $(-1)^j \binom{q}{j} = \frac{\Gamma(j-q)}{\Gamma(j+1)\Gamma(-q)}$.

**Definition 3.7** *Let $0 \leq m-1 < q < m$. We can define the right Grünwald -Letnikov derivative of order $q$ as follows:*

$${}_a^G D_-^q h(t) = \lim_{p \to 0^+} \frac{1}{p^q} \sum_{j=0}^{\infty} (-1)^j \binom{q}{j} h(t + jh), \forall t \in \mathbb{R}, \tag{3.17}$$

*while* $(-1)^j \binom{q}{j} = \frac{\Gamma(j-q)}{\Gamma(j+1)\Gamma(-q)}$.

### Fractional derivatives of Riemann-Liouville

**Definition 3.8** *The Riemann-Liouville fractional derivative of a function $h$ of order $q$ (with $m-1 \leq q < m$) on the interval $[a,t]$ is defined as follows, provided the integral exists:*

$$\begin{aligned} {}_a^R D_t^q h(t) &= \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-q-1} h(\tau) d\tau \\ &= \frac{d^m}{dt^m} (I^{m-q} h(t)). \end{aligned} \tag{3.18}$$

**Remark 3.2** •

- *If $h$ is of class $C^m$, then by doing integrations by parts and repeated derivations we obtain*

$$\begin{aligned} {}_a^R D_t^q h(t) &= \sum_{k=0}^{m-1} \frac{h^{(k)}(a)(t-a)^{k-q}}{\Gamma(k-q+1)} + \frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} h^{(n)}(\tau) d\tau \\ &= {}_a^G D_t^q h(t). \end{aligned} \tag{3.19}$$

  *In this case the Grünwald- Letnikov approach and Riemann- Liouville approach are equivalent.*

- *Laplace transform: the Laplace transform of Riemann- Liouville fractional integral satisfies*

$$L\{{}_a^R D_t^q h(t)\} = s^{-q} L\{h(t)\} \tag{3.20}$$

### Fractional derivatives of Caputo

Applied problems in viscoelasticity, solid mechanics and rheology have prompted several authors including Caputo to realize that the definition of fractional derivation in the sense of Riemann -Liouville needs to be revised despite the important role it has played in the development of the fractional calculation. As a solution to the challenges encountered with the RL approach to fractional differential equations when solving physical problems, the Caputo fractional derivative was introduced.
The $q^{th}$ Caputo fractional derivative of $h(t)$ (such that $\frac{d^m h}{dt^m} \in L_1[a,b]$ ) is defined by [122]

$${}_a^C D_t^q h(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} f^{(m)}(\tau) d\tau, \tag{3.21}$$

where $m-1 \leq q < m, \ m \in \mathbb{N}^*$.

**Remark 3.3** • *Laplace transform of Caputo fractional derivative satisfies,*

$$L\{{}_a^C D_t^q h(t)\} = s^q L\{h(t)\} - \sum_{j=0}^{m-1} h^{(j)}(0^+) s^{q-1-j} \tag{3.22}$$

*where $m-1 < q \leq m$. To obtain the Laplace transform of the Caputo fractional derivative, it is necessary to have information about the (bounded) initial values of the function, as well as its integer derivatives of order $j = \overline{1, m-1}$.*

- *Because of its favorable properties regarding Laplace transform and constant derivatives, the Caputo definition has been used exclusively in **Chapter** 4.*

## 3.3 Stability analysis of fractional systems

Control theory has extensively explored the stability analysis of fractional order systems. The literature has derived necessary and sufficient conditions for stability [123]. As systems with memory tend to be more stable than their memory-less counterparts, fractional-order systems are considered to be at least as stable as their integer-order counterparts. In the fractional case, the stability is different from that in the integer one.
Memory systems, in general, are more stable than their low-memory counterparts, making fractional-order differential equations more stable than their corresponding integer-order equations.
This section summarizes the primary stability outcomes, which are crucial for investigating the existence of chaotic attractors and achieving synchronization in fractional order systems.
The stability condition for fractional order systems can be expressed as follows:

**Definition 3.9** *Consider the following $n$ dimensional fractional order system*

$$\begin{cases} D^{q_1} z_1 = h_1(z_1, z_2, \cdots, z_n), \\ D^{q_2} z_2 = h_2(z_1, z_2, \cdots, z_n), \\ \qquad \vdots \\ D^{q_n} z_n = h_n(z_1, z_2, \cdots, z_n), \end{cases} \tag{3.23}$$

*here, $q_j$ represents a rational number between $0$ and $1$, indicating fractional orders, and $D^{q_j}$ refers to the Caputo fractional derivative of orders $q_j$, for $j = \overline{1, n}$. We assume that for $j = \overline{1, m}$, $q_j = k_j/m_j$, where $(k_j, m_j) = 1$, and $k_j, m_j \in \mathbb{N}$. Let $m$ denote the least common multiple of the denominators $m_j$'s of $q_j$'s.*
*The system (3.23) is called as a commensurate order if $q_1 = q_2 = \cdots = q_n$ otherwise an incommensurate order.*

*An equilibrium point of system* (3.23) *is defined as a point* $z^* = (z_1^*, z_2^*, \cdots, z_n^*)$ *where* $h_j(z^*) = 0$ *holds for all* $j = \overline{1, n}$.

### Indirect (linear) method

When the system (3.23) is a linear system, meaning $[h_1(z), h_2(z), \ldots, h_n(z)]^T = [c_{ij}]_{i,j=1}^n z = Cz$, where $z \in \mathbb{R}^n$, the following results apply:

- The commensurate order system described by (3.23) is asymptotically stable if only if [124]

$$|\arg(spec(C))| > q\pi/2. \tag{3.24}$$

- For the incommensurate fractional order dynamical system given by (3.23), asymptotic stability is achieved if all roots $\lambda$ of the equation

$$\det(diag(\lambda^{mq_1}, \lambda^{mq_2}, \ldots, \lambda^{mq_n}) - C) = 0 \tag{3.25}$$

  where $q_i$ is a rational number between 0 and 1,satisfy

$$|\arg(\lambda)| > \rho\pi/2, \tag{3.26}$$

  where $\rho = 1/m$ [125].

### Direct method (Lyapunov)

If function $h_i$ has $2^{nd}$ continuous partial derivatives at an equilibrium point $z^*$, the following results apply:

- For a system (3.23) where $q_1 = q_2 = \cdots = q_n$, the equilibrium point $z^*$ is locally asymptotically stable if all the eigenvalues of the Jacobian matrix

$$J = [\partial_j h_i(z^*)]_{i,j=1}^n = \begin{pmatrix} \partial_1 h_1(z^*) & \partial_2 h_1(z^*) & \cdots & \partial_n h_1(z^*) \\ \partial_1 h_2(z^*) & \partial_2 h_2(z^*) & \cdots & \partial_n h_2(z^*) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 h_n(z^*) & \partial_2 h_n(z^*) & \cdots & \partial_n h_n(z^*) \end{pmatrix}. \tag{3.27}$$

  evaluated at $z^*$ satisfy the following condition [123]

$$|\arg(\mathrm{Ei}\,g(J\,|_{z^*}))| > q\pi/2. \tag{3.28}$$

- For an incommensurate fractional order dynamical system given by (3.23) with $q_i$ being a rational number between 0 and 1, the equilibrium point $z^*$ is asymptotically stable if all roots $\lambda$ of the equation $\det(diag(\lambda^{mq_1}, \lambda^{mq_2}, \ldots, \lambda^{mq_n}) - J\,|_{z^*}) = 0$ satisfy $|\arg(\lambda)| > \rho\pi/2$, where $\rho = 1/m$ [120].

**Lemma 3.1** *[96], [126]If there exists a positive definite function* $V\left(Y\left(t\right)\right)$ *such that* $D_t^{\alpha} V\left(Y\left(t\right)\right) < 0, \forall t > 0$, *the trivial solution of the fractional order system*

$$D_t^{\alpha} Y\left(t\right) = G\left(Y\left(t\right)\right), \tag{3.29}$$

*in which* $D_t^{\alpha}$ *is the Caputo fractional derivative of order* $\alpha$ *,* $0 < \alpha \leq 1$*,* $G : R^m \longrightarrow R^m$*, is asymptotically stable.*

**Lemma 3.2** *[96]*$\forall Y\left(t\right) \in R^m, \forall \alpha \in \left[0, 1\right]$, *and* $\forall\, t > 0$

$$D_t^{\alpha}\left(Y^T\left(t\right) Y\left(t\right)\right) \leq 2Y^T\left(t\right) D_t^{\alpha}\left(Y\left(t\right)\right). \tag{3.30}$$

## 3.4 Adams-Bashforth-Moulton numerical method for calculation of fractional derivatives

A method based on the Adams Bashforth Moulton type predictor corrector scheme has been proposed for the numerical simulation of fractional order systems in [125]. This method is suitable for Caputo derivatives since it only requires initial conditions and has a clear physical meaning for unknown functions. It is based on the fact that the following fractional differential equations can be transformed into integral equations, which can then be solved by numerical methods:

$$D_t^{\alpha} z(t) = h(z(t), t),\ \ z^{(j)}(0) = z_0^{(j)}, j = \overline{0, m-1} \tag{3.31}$$

is equivalent to the Volterra integral equation

$$z(t) = \sum_{j=0}^{[\alpha]-1} z_0^{(j)} \frac{t^j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau, z(\tau)) d\tau. \tag{3.32}$$

By converting the Volterra equation (3.32) to discrete form for time instances $t_n = nh$ ($n = \overline{0, N}$), $h = T_{sim}/N$, and employing the short memory principle (fixed or logarithmic [127]), a satisfactory numerical approximation of the true solution of a fractional differential equation can be obtained while maintaining the order of accuracy.
An improved predictor corrector approach to solve the Fokker-Planck equation has been mentioned in [128]. A variety of numerical algorithms have also been presented in [129].

## 3.5 Fractional order chaotic systems

Studies have demonstrated that fractional order systems, which are extensions of numerous established systems, can also exhibit chaotic behavior. Examples include:

# Fractional Chen system

In 1999, Chen and Ueta proposed the Chen system [130], which shares similarities with the Lorenz system but is not topologically equivalent to it. The Chen system is a chaotic system that exhibits a double scroll attractor. The fractional form of the Chen system is given by [131]

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = \alpha(y-x), \\ \frac{d^{q_2}y}{dt^{q_2}} = (\gamma-\alpha)x - xz + \gamma y, \\ \frac{d^{q_3}z}{dt^{q_3}} = xy - \beta z, \end{cases} \tag{3.33}$$

where $x$, $y$, and $z$ are the state variables, $0 < q_1, q_2, q_3 \leq 1$ are the fractional-order derivatives. The integer order Chen system displays chaotic attractors, for example, when $(\alpha, \beta, \gamma) = (35, 3, 28)$.

For $q_i = q = 0.9, (i = \overline{1,3})$ the fractional order Chen system can display chaotic attractor, and numerical simulation of the Chen system for the parameter values $\alpha = 35$, $\beta = 3$, $\gamma = 28$, the fractional-order $q = 0.9$, and the initial conditions $(-9, -5, 14)$ with $h = 0.005$ and $TSim = 100s$ is depicted in Figure 3.1.

# Fractional Rössler system

The Rössler system is a non-linear system that has the potential to show a chaotic attractor of one scroll [132], its fractional version [48] can be expressed as :

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -(y+z), \\ \frac{d^{q_2}y}{dt^{q_2}} = x + \alpha y, \\ \frac{d^{q_3}z}{dt^{q_3}} = z(x-\gamma) + \beta, \end{cases} \tag{3.34}$$

where $x$, $y$, and $z$ are the state variables, $0 < q_1, q_2, q_3 \leq 1$ are the fractional-order derivatives, and $\alpha, \beta$, and $\gamma$ are the parameters.

Simulations are performed to obtain chaotic behavior of fractional order Rössler system and the results demonstrate that chaos indeed exists with order less than 3. For example : chaotic attractors are found for the parameter values $\alpha = 0.63$, $\beta = 10$, $\gamma = 0.2$, the fractional-order $q_1 = 0.9$, $q_2 = 0.8$, $q_3 = 0.7$, and the initial conditions $(0, 0, 0)$ with $h = 0.005$ and $TSim = 300s$ (see Figure 3.2).

# Fractional Chua system

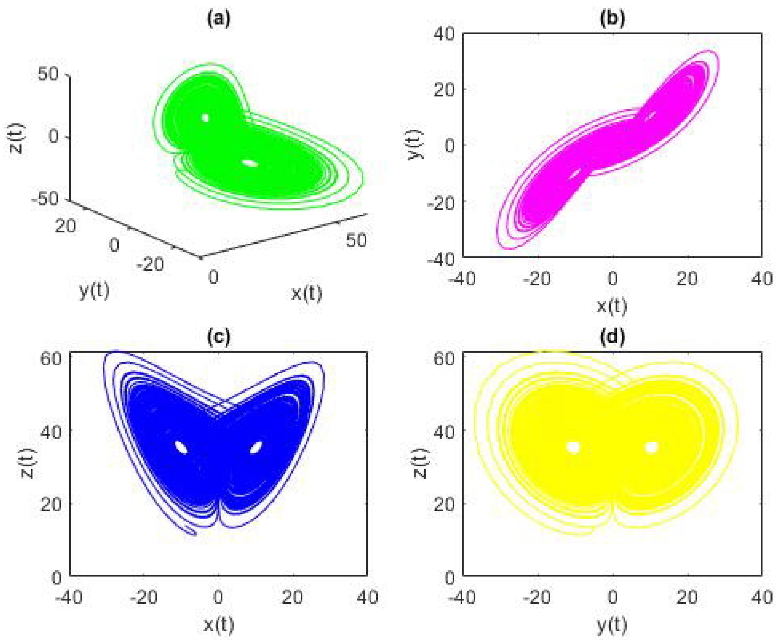



Figure 3.1: Double scroll attractor of Chen system projected in **(a)** $x-y-z$ space, **(b)** $x-y$, **(c)** $x-z$, and **(d)** $y-z$ planes.

Consider the fractional order Chua system

$$\begin{aligned} \frac{d^{\alpha_1}x}{dt^{\alpha_1}} &= \delta(y+\frac{x-2x^3}{7}),\\ \frac{d^{\alpha_2}y}{dt^{\alpha_2}} &= x-y+z,\\ \frac{d^{\alpha_3}z}{dt^{\alpha_3}} &= -\beta y-\gamma z. \end{aligned} \tag{3.35}$$

where $x$, $y$, and $z$ are the state variables, $0<\alpha_1,\alpha_2,\alpha_3\leq 1$ are the fractional-order derivatives. For the parameter values $\delta=10.725$, $\beta=10.593$, and $\gamma=0.268$, the fractional-order $\alpha_1=0.93$, $\alpha_2=0.99$, $\alpha_3=0.92$, and the initial conditions $(x(0),y(0),z(0))=(0.60,1,-0.6)$ with $h=0.005$ and $TSim=60s.$, the system $(3.35)$ presents chaotic behavior (see Figure 3.3).

# Fractional Lorenz system

The fractional order Lorenz system [133]-[135] is given by

$$\begin{aligned} {}_0D_t^{q_1}x &= \sigma(y-x),\\ {}_0D_t^{q_2}y &= x(\rho-z)-y,\\ {}_0D_t^{q_3}z &= xy-\beta z, \end{aligned} \tag{3.36}$$

where $x$, $y$, and $z$ are the state variables, $\sigma$, $\rho$, and $\beta$ are the parameters and $0<q_j\leq 1$ $(j=1,2,3)$ are the fractional order derivatives.

for the parameter values $\sigma=10$, $\rho=28$, and $\beta=8/3$, the fractional-order $q_1=q_2=q_3=$

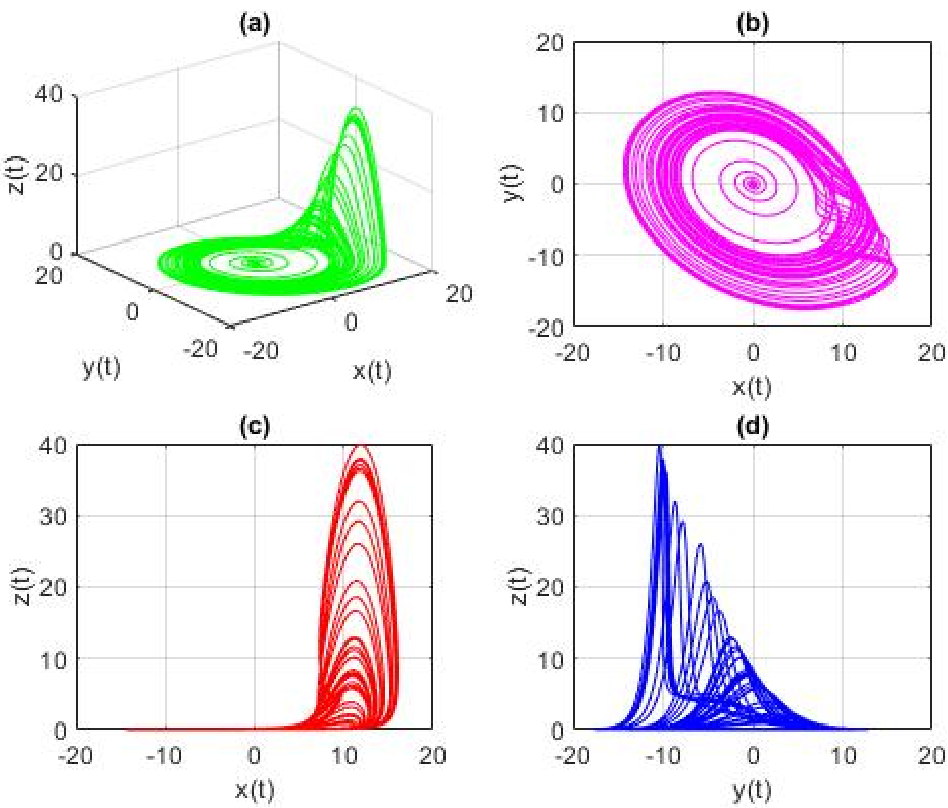



Figure 3.2: Chaotic attractror for the fractional-order Rössler system projected in **(a)** $x-y-z$ space and **(b)** $x-y$, **(c)** $x-z$, and **(d)** $y-z$ planes.

$0.993$,and the initial conditions $(0.1, 0.1, 0.1)$ with $h = 0.005$ and $TSim = 100s$, the chaotic attractor of fractional order Lorenz system is depicted in (Figure 3.4).

# Fractional Lü system

The fractional Lü system is described as follows [136]

$$\begin{aligned} {}_0D_t^{q_1}x &= \alpha(y-x), \\ {}_0D_t^{q_2}y &= -xz+\gamma y, \\ {}_0D_t^{q_3}z &= xy-\beta z, \end{aligned} \tag{3.37}$$

where $x, y$, and $z$ are the state variables, $0 < q_i \leq 1, \;\; i = \overline{1,3}$ are the fractional order derivatives, and $\alpha, \;\; \beta, \gamma$ are the parameters.

The fractional order Lü system exhibit chaotic behavior for the parameter values $\alpha = 36, \beta = 3, \gamma = 20$, the fractional-order $q_1 = 0.985$, $q_2 = 0.99$, $q_3 = 0.98$,and the initial conditions $(0.2, 0.5, 0.3)$ with $h = 0.005$ and $TSim = 200s$. (see Figure 3.5).

# Fractional Liu system

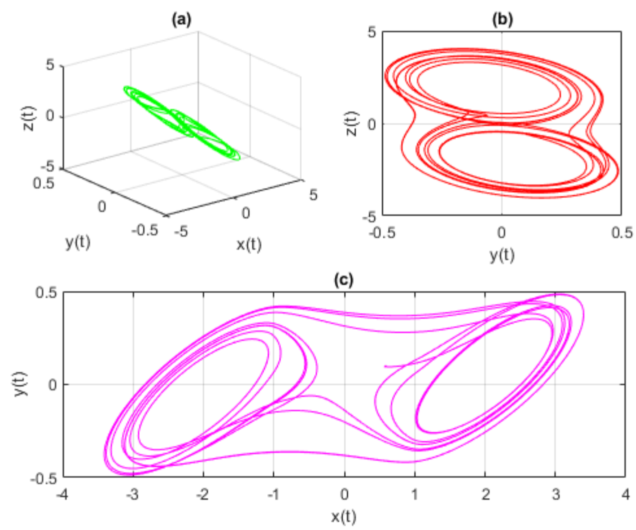



Figure 3.3: Chaotic attractror for the fractional-order Chua system projected in **(a)** $x-y-z$ space and **(b)** $y-z$, and **(c)** $x-y$ planes.

The fractional order Liu system ([137] and [138]) is given by

$$\begin{aligned} D^{q_1}x &= -\alpha x - ey^2, \\ D^{q_2}y &= by - dxz, \\ D^{q_3}z &= -\gamma z + rxy. \end{aligned} \tag{3.38}$$

where $x$, $y$, and $z$ are the state variables, $0 < q_i \leq 1$, $i = \overline{1,3}$ are the fractional order derivatives, and $a$, $b, d, e, r$, and $\gamma$ are the parameters.
For the parameter values $\alpha = 1$, $\beta = 2.5$, $\gamma = 5$, $e = 1$, $d = 4$, $r = 4$, the fractional-order $q_i = 0.95$, $i = \overline{1.3}$ and the initial conditions $(0.2, 0, 0.5)$ with $h = 0.005$ and $TSim = 100s$, a two scroll attractor exists, hence, fractional order Liu system can display chaotic behaviors (see.Figure 3.6 and Table **3.1**).

# Fractional Four-Wing system

The fractional order Four-wing system ([139], [140]) is given by

$$\begin{cases} \frac{d^q x}{dt^q} = \alpha x + \beta y - yz \\ \frac{d^q y}{dt^q} = -\gamma y - z + xz \\ \frac{d^q z}{dt^q} = -x + \delta z + xy \end{cases} \tag{3.39}$$

where $x$, $y$, and $z$ are the state variables, $0 < q_i \leq 1$, $i = \overline{1,3}$ are the fractional order derivatives, and $\alpha$, $\beta$, $\delta$, and $\gamma$ are the parameters.

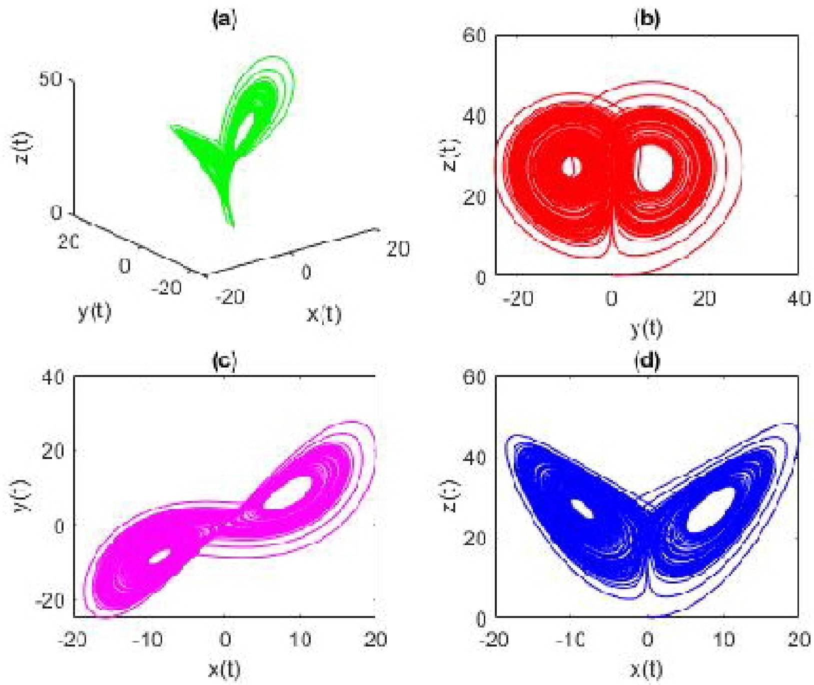



Figure 3.4: Chaotic attractror for the fractional-order Lorenz system projected in **(a)** $x-y-z$ space and **(b)** $y-z$, **(c)** $x-y$, and **(d)** $x-z$ planes.

For the parameter values $\alpha = 6$, $\delta = 5$, $\gamma = 12$, and $\beta = 11$, the fractional-order $q_i = 0.9$, $i = \overline{1.3}$ and the initial conditions $(-20, 3, -4)$ with $h = 0.005$ and $TSim = 300s$, a two scroll attractor exists, hence, fractional order Four-Wing systam can display chaotic behaviors (see Figure 3.7).

# Fractional Newton-Leipnik system

The fractional order Newton–Leipnik systam [141], [134] is given by

$$
\begin{aligned}
{}_0D_t^{q_1}x &= -ax + y + 10yz,\\
{}_0D_t^{q_2}y &= -x - 0.4y + 5xz,\\
{}_0D_t^{q_3}z &= bz - 5xz,
\end{aligned}
\tag{3.40}
$$

where $x$, $y$, and $z$ are the state variables, $0 < q_i \leq 1$ $(j = \overline{1,3})$ are the fractional order derivatives, and $a$ and $b$ are the parameters.

For the parameters value $a = 0.4$ and $b = 0.175$, the fractional-order $q_i = 0.95$, $i = \overline{1.3}$ and the initial conditions $(0.19, 0, -0.18)$ with $h = 0.005$ and $TSim = 200s$, the chaotic attractors of the fractional order Newton–Leipnik system are described in Figure 3.8.

# Fractional Chen Four-Wing system

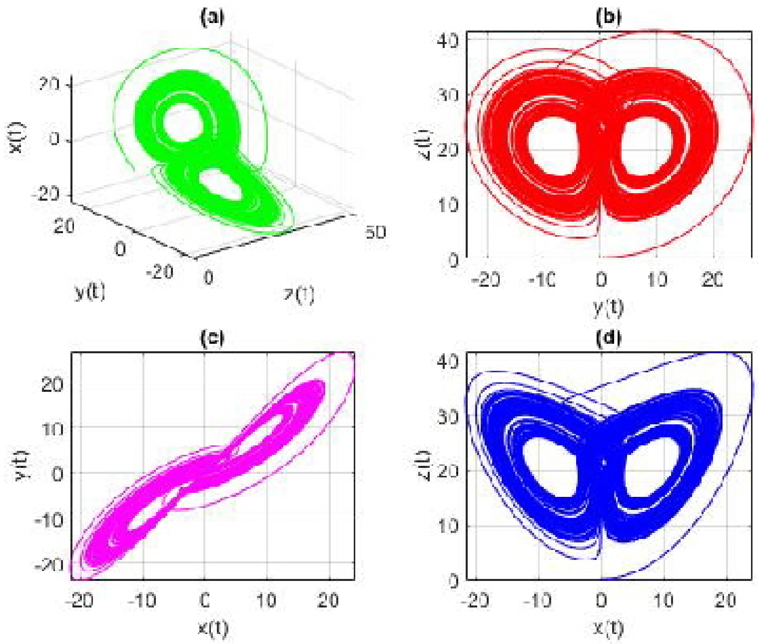



Figure 3.5: Chaotic attractror for the fractional-order Lü system projected in **(a)** $z-y-x$ space and **(b)** $y-z$, **(c)** $x-y$, and **(d)** $x-z$ planes.

The fractional order Chen four-wing systam [142],[138] is provided by

$$\begin{aligned} D^{\alpha_1}x &= ax - yz, \\ D^{\alpha_2}y &= -by + xz + d\,|x|\,, \\ D^{\alpha_3}z &= -cz + xy, \end{aligned} \tag{3.41}$$

when $a=\frac{20}{7}, b=10, d=0.5, c=4$,a two-scroll attractor exists, hence, fractional order Chen Four-Wing systam can display chaotic behaviors (see.Table 3.2).

# Fractional Li system

The fractional order Li system [143], [138] is provided by

$$\begin{aligned} D^{\alpha_1}x &= \beta(y - x), \\ D^{\alpha_2}y &= (\gamma - \beta)x + \gamma y - dxz, \\ D^{\alpha_3}z &= -bz + ey^2, \end{aligned} \tag{3.42}$$

when $\beta=38,\ e=\ 1,\ b=3,\ \ d=1,\ \ \gamma=30,$ a two-scroll attractor exists, hence, fractional order Li system can display chaotic behaviors (see.Table **3.3**).

# Fractional Duffing system

The fractional order Duffing system is provided by

$$\begin{aligned} {}_0D_t^{\alpha_1}x &= y, \\ {}_0D_t^{\alpha_2}y &= x - x^3 - \gamma y + \delta\cos(zt), \end{aligned} \tag{3.43}$$

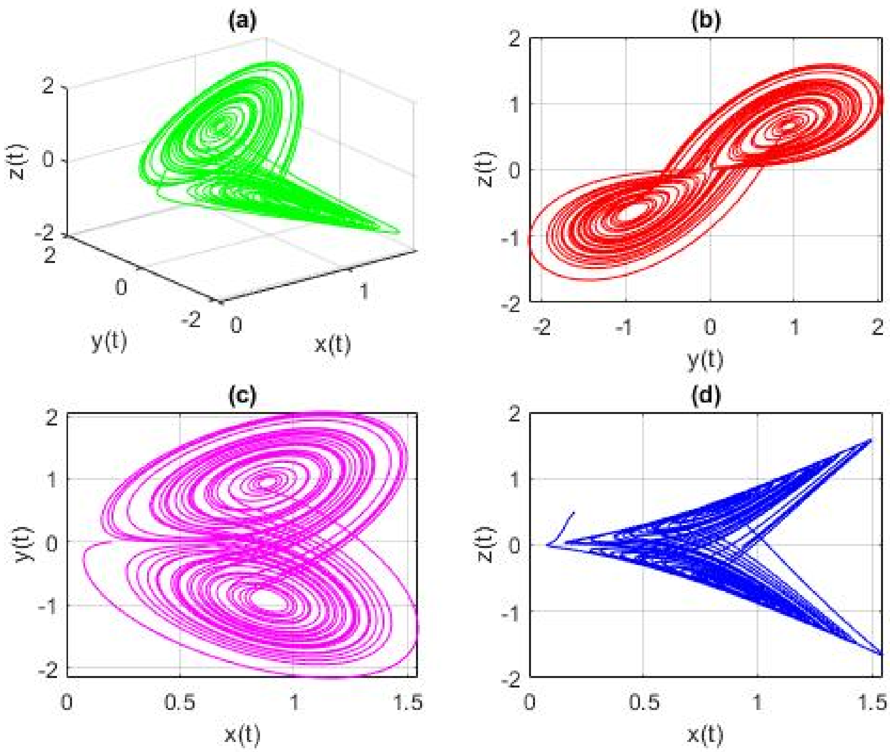



Figure 3.6: Chaotic attractror for the fractional-order Liu system projected in **(a)** $x-y-z$ space and **(b)** $y-z$, **(c)** $x-y$, and **(d)** $x-z$ planes.

where $\alpha_1$, $\alpha_2$ are the fractional order derivatives and $\gamma, \delta, z$ are the system parameters. The limit cycle in the phase plane for the Duffing fractional order oscillator with the parameter values $\gamma = 0.15$, $\delta = 1.3$, and $z = 1$, the fractional order derivatives $\alpha_1 = \alpha_2 = 0.95$, and the initial conditions $(1.0, 1.0)$ with $Tsim = 200s$ and $h = 0.005$ is depicted in Figure 3.9.

# Fractional Lotka Volterra system

The fractional order Lotka Volterra system, ( Petráš [137] and [134] ) is given by

$$\begin{aligned} \frac{d^{q_1}x}{dt^{q_1}} &= ax - bxy + ex^2 - \beta zx^2, \\ \frac{d^{q_2}y}{dt^{q_2}} &= -cy + dxy, \\ \frac{d^{q_3}z}{dt^{q_3}} &= -kz + \beta zx^2, \end{aligned} \tag{3.44}$$

in which $a,\ b,\ c,\ d,\ e,\ k$, and $\beta$ are the model parameters with $a,\ b,\ c,\ d >\ 0$ and $0 < q_j \leq 1$ $(j = \overline{1,3})$ are the fractional order derivatives.

For the parameters $a = b = c = d = \ 1,\ e = \ 2,\ \beta = 2.7$, and $k = \ 3$, the initial condition $[1,\ 1.4\ ,1]$, and at $q_j = \ 0.95$ $(j = \overline{1,3})$ and the initial conditions $(1, 1.4, 1)$ with $h = 0.005$ and $TSim = 200s$, the fractional order Lotka Volterra system exhibit chaotic attractors which described through Figure 3.10.

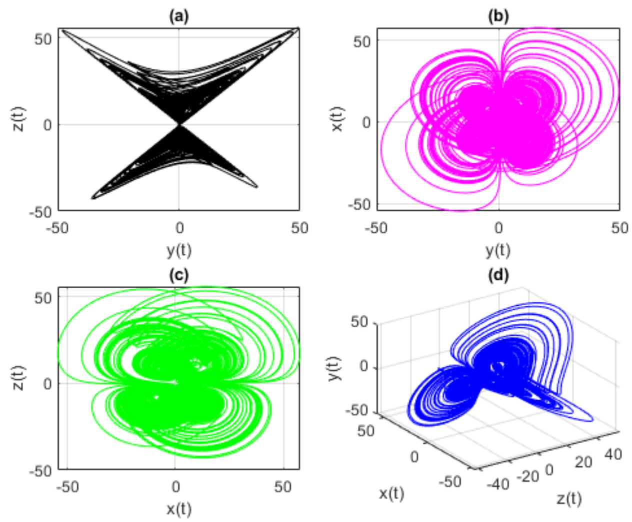



Figure 3.7: Chaotic attractror for the fractional-order Four-wing system projected in **(a)** $y-z$, **(b)** $y-x$, **(c)** $x-z$ planes and **(d)** $z-x-y$ space.

# Fractional Van der Pol system

In 1920, Van der Pol employed the Van der Pol (VPO) model to investigate oscillations in vacuum tube circuits. The nonlinear Van der Pol systam can be expressed in terms of a fractional derivative of order $q$, as shown in [144]

$$
\begin{aligned}
{}_0D_t^{q_1}x(t) &= y, \\
{}_0D_t^{q_2}y(t) &= -x-\varepsilon(x^2-1)y,
\end{aligned}
\tag{3.45}
$$

where $0<q_1,q_2<1$ and $\varepsilon>0$. Figure 3.11 illustrate the limit cycle of the Van der Pol fractional-order oscillator (3.45) in the phase planes $x-y$ and $y-x$ for the fractional-order derivatives $q_1=1.2$ and $q_2=0.8$, the parameter value $\varepsilon=1$, and the initial conditions $(0.2,-0.2)$ with $Tsim=30s$ and $h=0.005$.

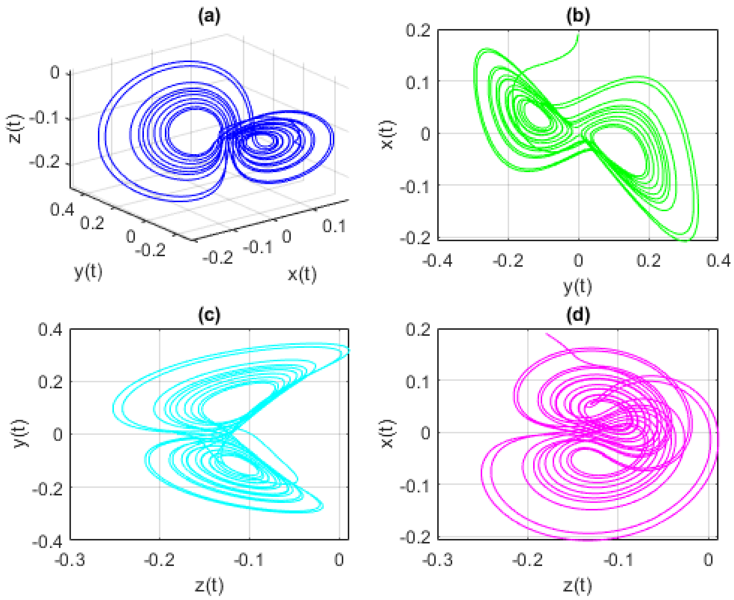



Figure 3.8: Chaotic attractror for thr fractional-order Newton-Leipnik system projected in **(a)** $x - y - z$ space and **(b)** $y - x$, **(c)** $z - y$, and **(d)** $z - x$ planes.

## 3.6 Conclusion

This chapter offers a concise overview, including historical background, fundamental concepts and definitions, and a review of established chaotic systems with fractional order dynamics.

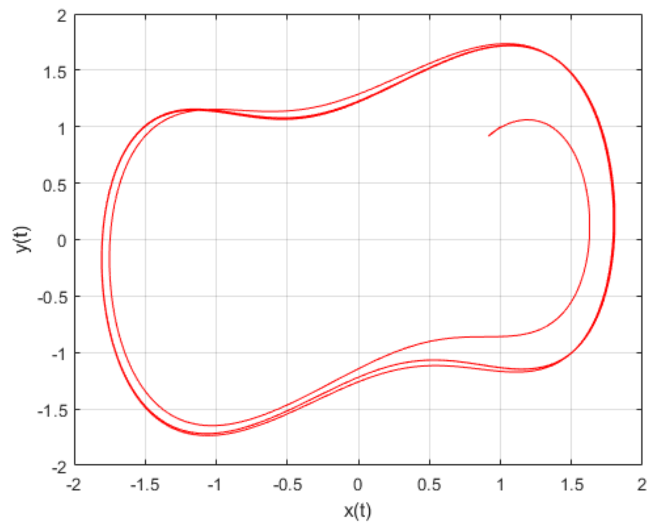



Figure 3.9: Phase trajectory (limit cycle) in the plane $x - y$ for the fractional-order Duffing system.

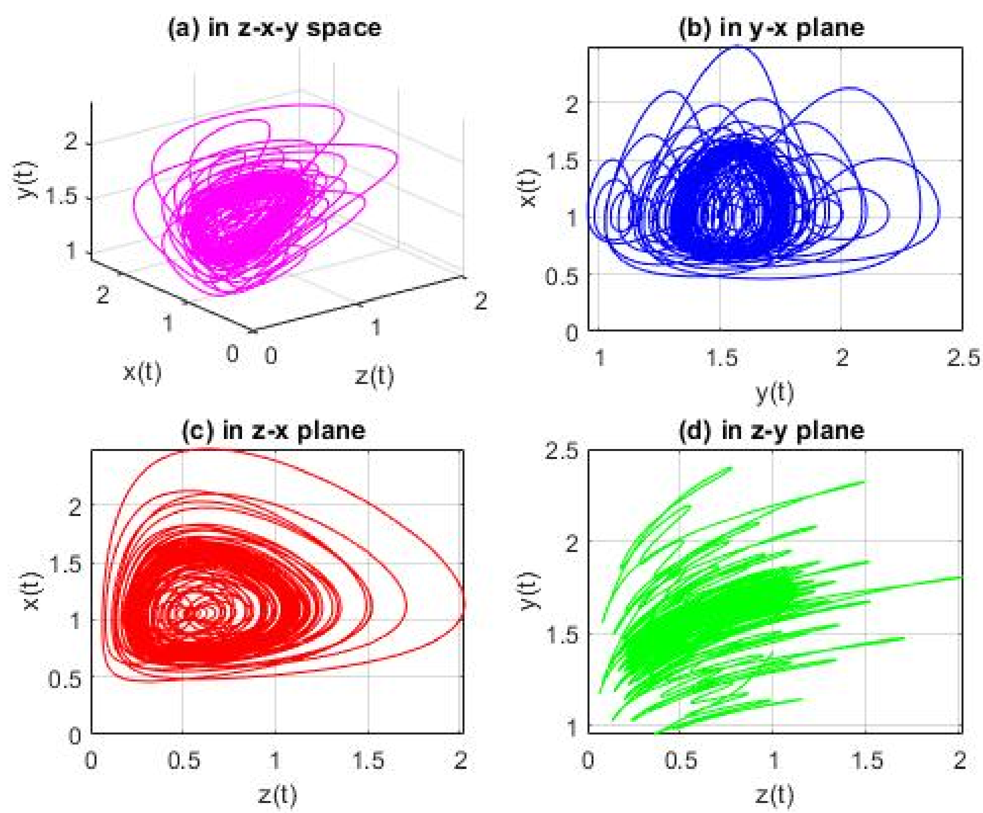



Figure 3.10: Chaotic attractror for the fractional-order Lotka Volterra system projected in **(a)** $z - x - y$ space and **(b)** $y - x$, **(c)** $z - x$, and **(d)** $z - y$ planes.

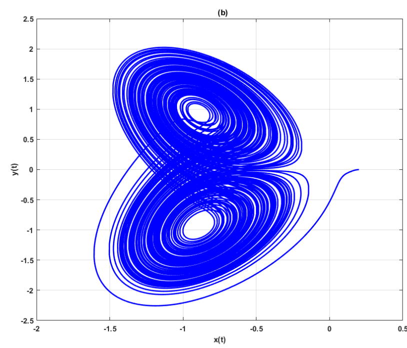
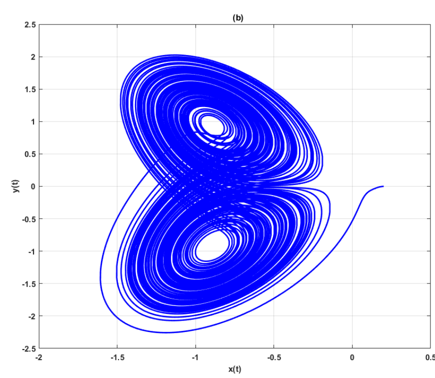
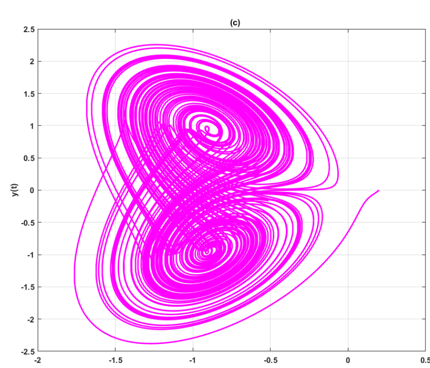
| | Liu System |
|---|---|
| | $a = \quad e = \ 1, b = \quad 2.5, d = r = \quad 4, \ \gamma = 5$ |
| $q = 0.8$ |  |
| $q = 0.95$ |  |
| $q = 1$ |  |

Table 3.1: Simulation results for Liu system.

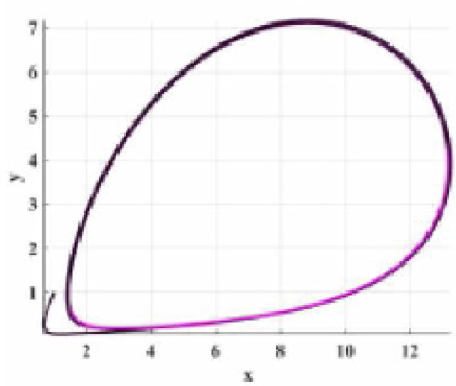
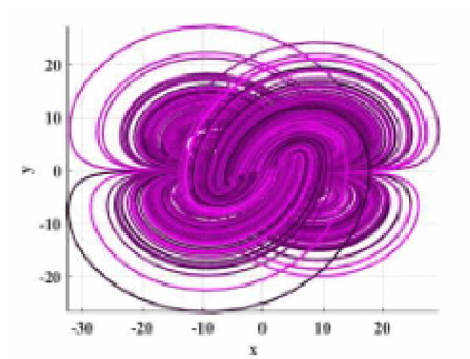
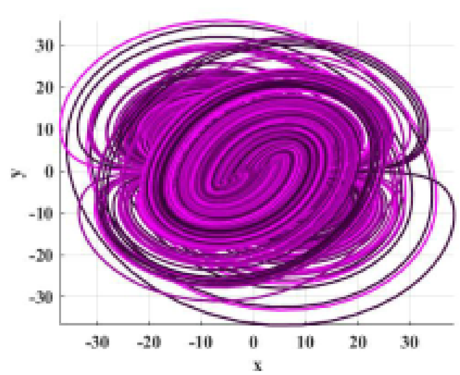
| | Chen Four-Wing System<br>$a = \frac{20}{7}, b = 10, d = 0.5, c = 4$ |
|---|---|
| $\alpha = 0.8$ | $(a)$<br> |
| $\alpha = 0.9$ |  |
| $\alpha = 1$ |  |

Table 3.2: Simulation results for Chen Four-Wing system.

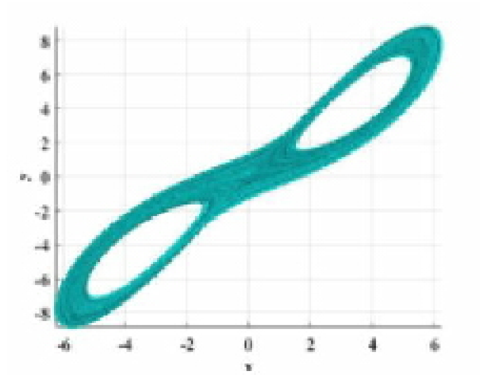
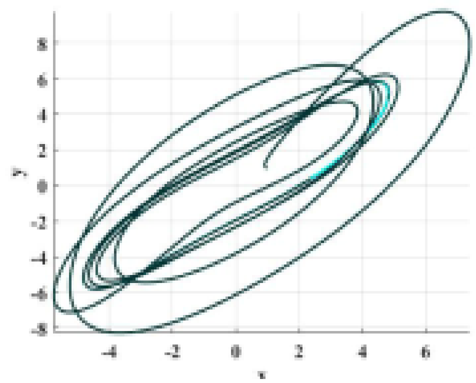
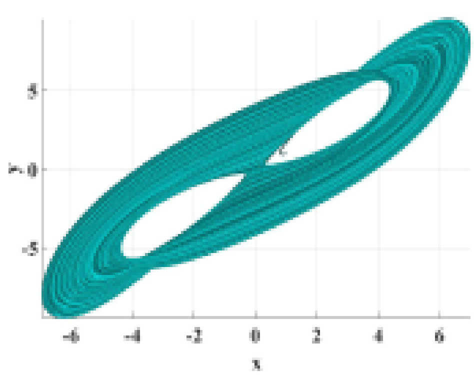
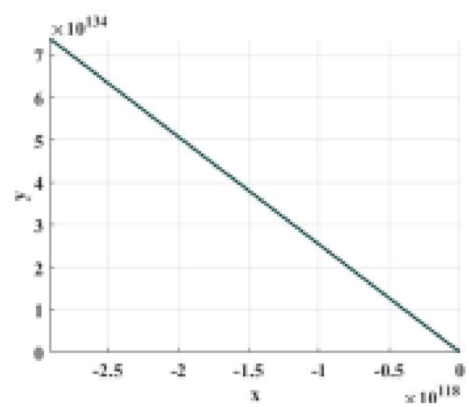
| Li System<br>$h = 2^{-8}, \beta = 8,\ e = 1,\ b = 1.5,\ d = 1,\ \gamma = 8$ | | | |
|---|---|---|---|
| $\alpha = 0.8$ |  | $\alpha = 0.95$ |  |
| $\alpha = 0.9$ |  | $\alpha = 1$ |  |

Table 3.3: Simulation results for Li system.

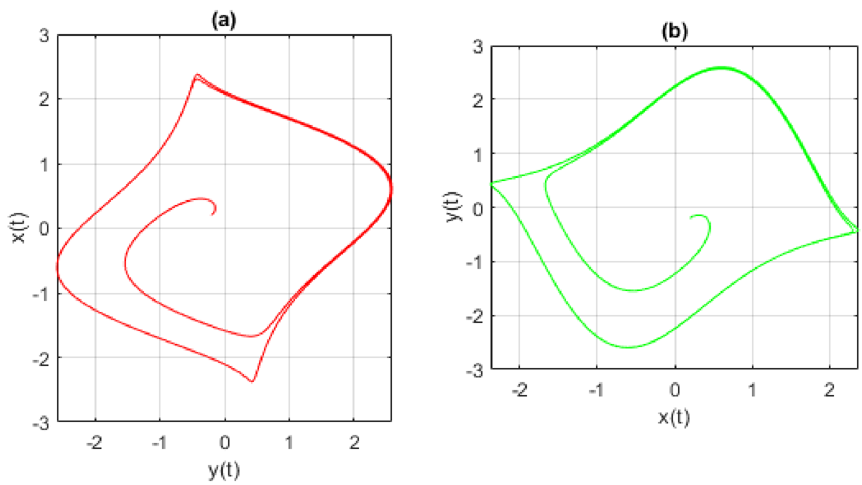



Figure 3.11: Limit cycle for the FrVPO system in $(\mathbf{a})$ $y - x$ and $(\mathbf{b})$ $x - y$ phase planes.

# Chapter 4

# Analysis and synchronization study of a novel three-dimensional chaotic system with fractional-order dynamics.

## 4.1 Introduction

This work introduces a novel $3D$ chaotic system in fractional order form [96]. The system displays chaotic behavior when the order of commensurability is kept to a minimum, and both theoretical and numerical solution representations are provided utilizing the Adams Bashforth Moulton algorithm. The study explores FSHP synchronization of the new three-dimensional chaotic system with fractional order and the Lorenz system exhibiting hyper chaotic behavior with fractional-order dynamics, using FSHP synchronization and Lyapunov theory to ensure the stability of fractional-order systems. Ultimately, numerical simulations are presented as proof of the efficiency of the suggested controller, using the improved Adams Bashforth Moulton algorithm.

## 4.2 An overview of the new chaotic system

This investigation study examines the fractional order version of a three-dimensional chaotic system originally presented in [145], which is expressed as equation

$$\begin{cases} D_t^{\alpha_1} x = s(y-x) + \beta yz, \\ D_t^{\alpha_2} y = (\gamma - s)x + \gamma y - xz, \\ D_t^{\alpha_3} z = xy - z, \end{cases} \tag{4.1}$$

where $s, \beta, \gamma \in \mathbb{R}^+$ (where $\beta$ is not equal to $1$).

The first section of this paper demonstrates that system $(4.1)$ exhibits chaotic behavior when the parameter values for $s, \beta$, and $\gamma$ are set to

$$(s, \beta, \gamma) = (15,\ 8/3,\ 10). \tag{4.2}$$

### 4.2.1 dynamics of system behavior

#### Equilibrium points and stability

When the parameters are set to the values specified in $(4.2)$ of the system $(4.1)$, and its equations are set equal to zero and solved as follows:

$$\begin{cases} 15y - 15x + \frac{8}{3}yz = 0, \\ -5x + 10y - xz = 0, \\ xy - z = 0, \end{cases} \tag{4.3}$$

we obtain five equilibrium points [96]

$$\begin{cases} P_0 = (0, 0, 0)\,, \\ P_{1,3} = (\pm 4.049, \mp 3.1659, -12.819)\,, \\ P_{2,4} = (\pm 1.7464, \pm 1.2563, 2.194)\,. \end{cases} \tag{4.4}$$

In order to check the stability of the equilibrium point, we derive the Jacobian matrix of the system $(4.1)$ for each equilibrium point

$$J(P) = \begin{pmatrix} -15 & 15 + \frac{8}{3}z & \frac{8}{3}y \\ -5 - z & 10 & -x \\ y & x & -1 \end{pmatrix}. \tag{4.5}$$

For $P_0$, we obtain

$$J(P_0) = \begin{pmatrix} -15 & 15 & 0 \\ -5 & 10 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.6}$$

Hence, the characteristic equation of the system evaluated at the equilibrium $P_0$ can be computed as:

$$\det(J(P_0) - \lambda I) = 0, \tag{4.7}$$

which simplifies to:

$$-\lambda^3 - 6\lambda^2 + 70\lambda + 75 = 0. \tag{4.8}$$

By solving for the roots of the above equations, we obtain the eigenvalues

$$\lambda_1 = -1,\ \lambda_2 = 6.5139,\ \lambda_3 = -11.514. \tag{4.9}$$

The above suggests that $P_0$ serves as an unstable saddle point.
Using a similar approach, the eigenvalues of the Jacobian matrix at $P_1$ and $P_3$ can be determined

$$\lambda_1 = 3.3706 - 8.3184i,\ \lambda_2 = 3.3706 + 8.3184i,\ \lambda_3 = -12.741, \tag{4.10}$$

and the eigenvalues of the Jacobian matrix at $P_2$ and $P_4$ can be determined

$$\lambda_1 = 1.0824 + 4.5105i, \lambda_2 = 1.0824 - 4.5105i, \lambda_3 = -8.1648, \tag{4.11}$$

consequently, $P_1$ and $P_3$ can be classified as unstable saddle-focus points, while $P_2$ and $P_4$ are also unstable saddle-focus points due to the absence of eigenvalues with a real part equal to zero, and the fact that $\lambda_1, \lambda_2$ are complex.

- If we consider the **commensurate-order** system with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, a critical requirement for the fractional-order nonlinear system (4.1) to exhibit chaotic behavior is that $\alpha > \frac{2}{\pi} \arctan\left(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)}\right)$, where $\lambda$ represents the eigenvalues of the saddle equilibrium point of index two in system (4.1). By using the aforementioned eigenvalues, we can determine the minimum commensurate order that sustains the chaotic nature of system (4.1). The results show that the value of $\alpha$ must exceed 0.75491 for $P_1$ and $P_3$, and 0.85006 for $P_2$ and $P_4$. Thereby implying that the necessary condition for the fractional-order system (4.1) to display chaos is $\alpha > 0.85006$.

- The necessary criterion for system (4.1) to exhibit chaotic oscillations in the **incommensurate** case can be expressed as $\frac{\pi}{2M} - min_j(|arg(\lambda_j(J_P))|) > 0$, where $\lambda_j(J_P)$, $j = \overline{1,3}$, represents the eigenvalues of the Jacobian matrix $J_P$ of the system (4.1) at the equilibrium $P$, $M$ is the LCM of the fractional orders. As an example, if $\alpha_1 = 0.9$, $\alpha_2 = 0.9$, $\alpha_3 = 0.8$, then $M = 10$. The characteristic equation of the system evaluated at the equilibrium $P_j$ can be computed as $det(diag[\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \lambda^{M\alpha_3}] - J_{P_j}) = 0$, which simplifies to $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{P_j}) = 0$ for $j = \overline{1,4}$. We get $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{P_0}) = 0$, $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{P_{1,3}}) = 0$, $det(diag[\lambda^9, \lambda^9, \lambda^8] - J_{P_{2,4}}) = 0$. Solving for the roots of the above equations yields $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 5\lambda^9 - 75\lambda^8 - 75 = 0$, $\lambda^{26} + \lambda^{18} + 5\lambda^{17} - 5.3334\lambda^9 - 3.04 \times 10^{-4}\lambda^8 + 1026.4 = 0$, $\lambda^{26} + \lambda^{18} + 5\lambda^{17} + 3.8412\lambda^9 + 2.094 \times 10^{-3}\lambda^8 + 175.67 = 0$.

From the roots of these equations, we obtain $\lambda = 1.2315$ whose argument is zero which is the minimum argument. Thus, the necesary stability condition holds since $\frac{\pi}{2M} - 0 > 0$.

### 4.2.2 Using Adams–Bashforth–Moulton algorithm on the novel system

Through the utilization of the Adams-Bashforth-Moulton algorithm as described in [146], the new fractional-order chaotic system $(4.1)$ can be expressed as stated in: [96]

$$\begin{cases} x_{m+1} = x_0 + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \begin{pmatrix} a\left(y^{\beta}_{m+1} - x^{\beta}_{m+1}\right) + by^{\beta}_{m+1}z^{\beta}_{m+1} \\ +\sum_{j=1}^{m} a_{1,j,m+1}\left(a\left(y_j - x_j\right) + by_j z_j\right) \end{pmatrix}, \\ \\ y_{m+1} = y_0 + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \begin{pmatrix} (c-a)\,x^{\beta}_{m+1} + cy^{\beta}_{m+1} - x^{\beta}_{m+1}z^{\beta}_{m+1} \\ +\sum_{j=1}^{m} a_{2,j,m+1}\left((c-a)\,x_j + cy_j - x_j z_j\right) \end{pmatrix}, \\ \\ z_{m+1} = z_0 + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \left(x^{\beta}_{m+1}y^{\beta}_{m+1} - z^{\beta}_{m+1} + \sum_{j=1}^{m} a_{3,j,m+1}\left(x_j y_j - z_j\right)\right), \end{cases} \tag{4.12}$$

in which

$$\begin{cases} x^{\beta}_{m+1} = x_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=1}^{m} b_{1,j,m+1}\left(a\left(y_j - x_j\right) + by_j z_j\right), \\ y^{\beta}_{m+1} = y_0 + \frac{1}{\Gamma(\alpha_2)} \sum_{j=1}^{m} b_{2,j,m+1}\left((c-a)\,x_j + cy_j - x_j z_j\right), \\ z^{\beta}_{m+1} = z_0 + \frac{1}{\Gamma(\alpha_3)} \sum_{j=1}^{m} b_{3,j,m+1}\left(x_j y_j - z_j\right), \end{cases} \tag{4.13}$$

with

$$\begin{cases} b_{1,j,m+1} = \frac{h^{\alpha_1}}{\alpha_1}\left((m-j+1)^{\alpha_1} - (m-j)^{\alpha_1}\right), \\ b_{2,j,m+1} = \frac{h^{\alpha_2}}{\alpha_2}\left((m-j+1)^{\alpha_2} - (m-j)^{\alpha_2}\right), \\ b_{3,j,m+1} = \frac{h^{\alpha_3}}{\alpha_3}\left((m-j+1)^{\alpha_3} - (m-j)^{\alpha_3}\right), \end{cases} \tag{4.14}$$

and

$$a_{i,j,m+1} = \begin{cases} (m)^{\alpha_i+1} - (m-\alpha_i)\,(m+1)^{\alpha_i}, j = 0, \\ (m-j+2)^{\alpha_i+1} - (m-j)^{\alpha_i+1} - 2\,(m-j+1)^{\alpha_i+1}, 1 \le j \le n, \quad i = \overline{1,3}, \\ 1, j = m+1. \end{cases} \tag{4.15}$$

The Adams-Bashforth-Moulton algorithm can be employed to numerically solve a fractional-order system.

As shown in Figure 4.1, the simulation result for the fractional-order system $(4.1)$ projected onto the $x - z$ plane, was computed using $\alpha_1 = \alpha_2 = \alpha_3 = 0.88$, simulation time $Tsim = 100s$, time step $h = 0.005$, $s = 15$, $\beta = 8/3$, $\gamma = 10$, and initial conditions $(x(0), y(0), z(0)) = (1, -1, 2)$ to produce a double-scroll attractor.

Figure 4.2 shows the simulation result for the fractional-order system $(4.1)$ projected onto the $z - y$, with the parameter values $s = 15$, $\beta = 8/3$, $\gamma = 10$, fractional orders $q_1 = q_2 = q_3 = 0.95$, and initial conditions $(x(0), y(0), z(0)) = (-2, 5, -10)$.

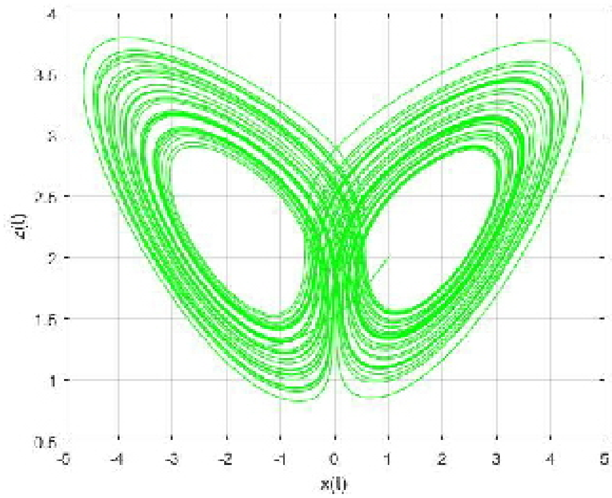



Figure 4.1: A double-scroll attractor.for the fractional-order system $(4.1)$ projected onto the $x-z$ plane.

Figure 4.3 also shows the simulation result for the fractional-order system $(4.1)$ projected onto the $y-z$, with the parameter values $a=15, b=8/3, c=10$, fractional orders $q_1=q_2=q_3=0.95$, and initial conditions $(x(0), y(0), z(0))=(-10,5,-10)$ .
Simulation results for the fractional-order system $(4.1)$ projected onto the $z-x-y$ plane and the $x-y-z$ plane for $q_1=q_2=0.9$, $q_3=0.8$, $(x(0), y(0), z(0))=(1,-1,2)$ are depicted in Figure 4.4 and Figure 4.5 respectively.

### 4.2.3 Lyapunov exponents

Lyapunov exponents are used to measure the exponential rates of divergence and convergence of nearby trajectoiries, which is an important characterstic to judge whether the system is chaotic or not.
The existence of at least one positive Lyapunov exponent implies that the system is chaotic.
For the parameter values $s=15$, $\beta=8/3$, $\gamma=10$ and the initial conditions $(1,-1,2)$ for the fractional order $(4.1)$ system , the Lyapunov exponents (LEs ) were computed using MATLAB in the two cases as follows:

**Case 4.1** *For the commensurate case with* $q_1=0.98, q_2=0.98, q_3=0.98$*, (LEs ) are obtained as:*

$$L_1=0.7240,\ L_2=-0.00,\ L_3=-7.3507. \tag{4.16}$$

**Case 4.2** *For the incommensurate case with* $q_1=0.9, q_2=0.9, q_3=0.8$*, (LEs) are:*

$$L_1=1.6124,\ L_2=-0.0015,\ L_3=-12.4935. \tag{4.17}$$

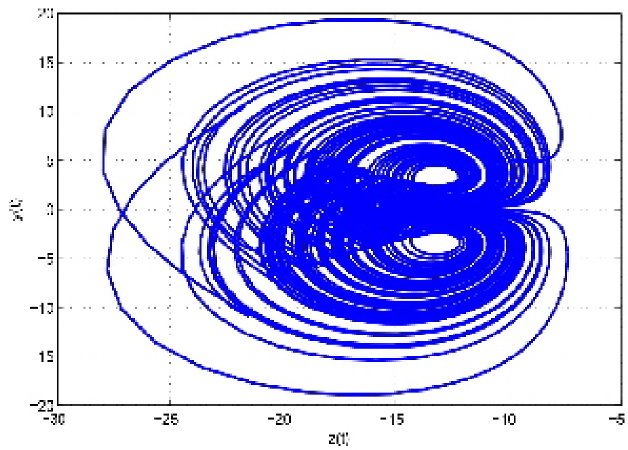



Figure 4.2: Simulation result for the fractional-order system $(4.1)$ projected onto the $z - y$ plane.

The Lyapunov spectrium for the two cases is shown in Figure.4.6 and Figure.4.7 respectively.

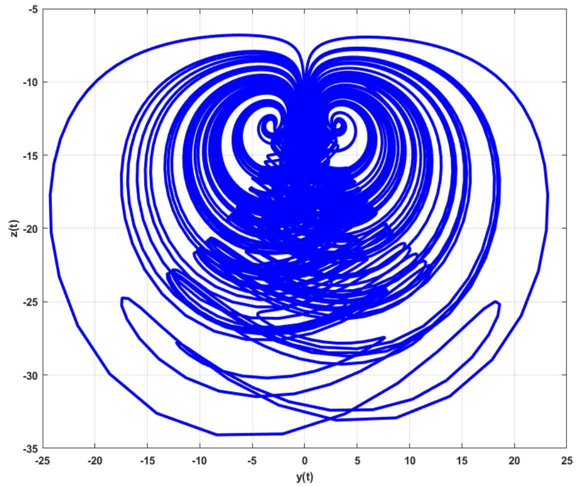



Figure 4.3: Simulation result for the fractional-order system $(4.1)$ projected onto the $y - z$ plane.

## 4.3 Full-State Hybrid Projective Synchronization of fractional order systems

This section explores the synchronization of the new fractional-order chaotic system $(4.1)$ and the fractional-order hyper-chaotic Lorenz system through full-state hybrid projective.
The novel fractional-order chaotic system is used as the driving system, Thus, we cons ider the novel fractional-order chaotic system as the driving system given by [96]

$$\begin{cases} D_t^{\alpha_1} x_1 = a(x_2 - x_1) + b x_2 x_3, \\ D_t^{\alpha_2} x_2 = (c - a) x_1 + c x_2 - x_1 x_3, \\ D_t^{\alpha_3} x_3 = x_1 x_2 - x_3, \end{cases} \tag{4.18}$$

where $a = 15$, $b = 8/3$, $c = 10$, and the fractional-order hyper-chaotic Lorenz system as the response system [96]

$$\begin{cases} D_t^{\alpha_1} y_1 = a\,(y_2 - y_1) + y_4 + u_1 \\ D_t^{\alpha_2} y_2 = c y_1 - y_2 - y_1 y_3 + u_2 \\ D_t^{\alpha_3} y_3 = -b y_3 + y_1 y_2 + u_3 \\ D_t^{\alpha_4} y_4 = -y_2 y_3 + d y_4 + u_4, \end{cases} \tag{4.19}$$

where the parameter values are $a = 10$, $b = 28$, $c = 8/3$, $d = -1$, and the fractional orders of the system are $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.98, 0.98)$.

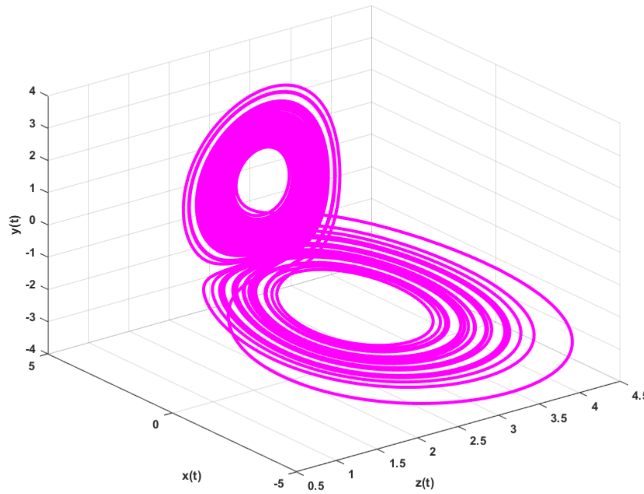



Figure 4.4: Simulation result for the fractional-order system $(4.1)$ projected onto the $z-x-y$ space.

In accordance with Definition 2.19, the state errors for $(4.18)$ and $(4.19)$ are given by

$$e_i = y_i - \sum_{j=1}^{3} \beta_{ij} x_j, i = \overline{1,4}. \tag{4.20}$$

Taking the Caputo fractional derivative of both sides with respect to time $t$, we obtain:

$$D_t^{\alpha_i} e_i = D_t^{\alpha_i} y_i - D_t^{\alpha_i} \left( \sum_{j=1}^{3} \beta_{ij} x_j \right), i = \overline{1,4}. \tag{4.21}$$

Thus, the error dynamical system can be represented as:

$$\begin{cases} D_t^{\alpha_1} e_1 = D_t^{\alpha_1} y_1 - D_t^{\alpha_1} \left( \sum_{j=1}^{3} \beta_{1j} x_j \right) \\ D_t^{\alpha_2} e_2 = D_t^{\alpha_2} y_2 - D_t^{\alpha_2} \left( \sum_{j=1}^{3} \beta_{2j} x_j \right) \\ D_t^{\alpha_3} e_3 = D_t^{\alpha_3} y_3 - D_t^{\alpha_3} \left( \sum_{j=1}^{3} \beta_{3j} x_j \right) \\ D_t^{\alpha_4} e_4 = D_t^{\alpha_4} y_4 - D_t^{\alpha_4} \left( \sum_{j=1}^{3} \beta_{4j} x_j \right), \end{cases} \tag{4.22}$$

In equation $(4.22)$, the error dynamical system is described using the fractional-order system $(4.11)$ and the driving system $(4.12)$. The system can be written as:

$$\begin{cases} D_t^{\alpha_1} e_1 = a\left(y_2 - y_1\right) + y_4 + u_1 - D_t^{\alpha_1}\left(\beta_{11} x_1 + \beta_{12} x_2 + \beta_{13} x_3\right) \\ D_t^{\alpha_2} e_2 = c y_1 - y_2 - y_1 y_3 + u_2 - D_t^{\alpha_2}\left(\beta_{21} x_1 + \beta_{22} x_2 + \beta_{23} x_3\right) \\ D_t^{\alpha_3} e_3 = -b y_3 + y_1 y_2 + u_3 - D_t^{\alpha_3}\left(\beta_{31} x_1 + \beta_{32} x_2 + \beta_{33} x_3\right) \\ D_t^{\alpha_4} e_4 = -y_2 y_3 + d y_4 + u_4 - D_t^{\alpha_4}\left(\beta_{41} x_1 + \beta_{42} x_2 + \beta_{43} x_3\right). \end{cases} \tag{4.23}$$

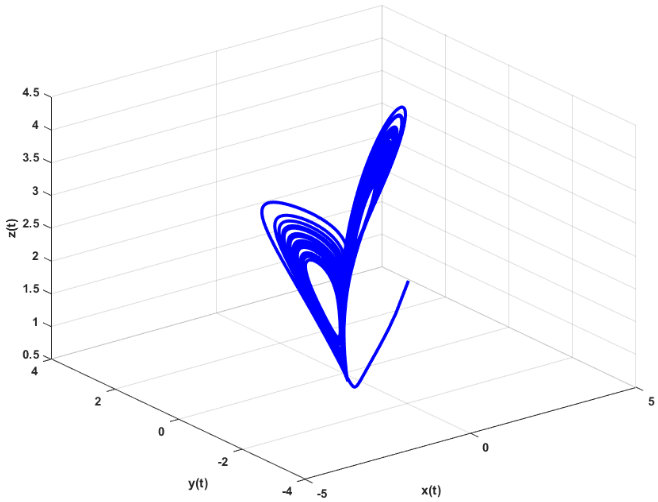



Figure 4.5: Simulation result for the fractional-order system $(4.1)$ projected onto the $x-y-z$ space.

This system can be rewritten as

$$\begin{cases} D_t^{\alpha_i} e_i = \sum_{j=1}^{4} a_{ij} e_j(t) + R_i + u_i, \ i = \overline{1,4}, \end{cases} \tag{4.24}$$

where

$$\begin{cases} R_1 = -D_t^{\alpha_1}(\beta_{11}x_1 + \beta_{12}x_2 + \beta_{13}x_3) + \sum_{j=1}^{4} a_{1j}(y_j(t) - e_j(t)) \\ R_2 = -y_1y_3 - D_t^{\alpha_2}(\beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3) + \sum_{j=1}^{4} a_{2j}(y_j(t) - e_j(t)) \\ R_3 = y_1y_2 - D_t^{\alpha_3}(\beta_{31}x_1 + \beta_{32}x_2 + \beta_{33}x_3) + \sum_{j=1}^{4} a_{3j}(y_j(t) - e_j(t)) \\ R_4 = -y_2y_3 - D_t^{\alpha_4}(\beta_{41}x_1 + \beta_{42}x_2 + \beta_{43}x_3) + \sum_{j=1}^{4} a_{4j}(y_j(t) - e_j(t)). \end{cases} \tag{4.25}$$

To rewrite the error system $(4.24)$ in a compact form, we can write it as:

$$\begin{bmatrix} D_t^{\alpha_1} e_1 \\ D_t^{\alpha_2} e_2 \\ D_t^{\alpha_3} e_3 \\ D_t^{\alpha_4} e_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_{4u_1} \end{bmatrix}, \tag{4.26}$$

which can be written in a more compact form as:

$$D_t^{\alpha} e = Ae + R + U, \tag{4.27}$$

where $R = (R_i)_{i=\overline{1,4}}$, $A = (a_{ij})_{4\times 4}$, $U = (u_i)_{i=\overline{1,4}}$, $e = (e_1, e_2, e_3, e_4)^T$.

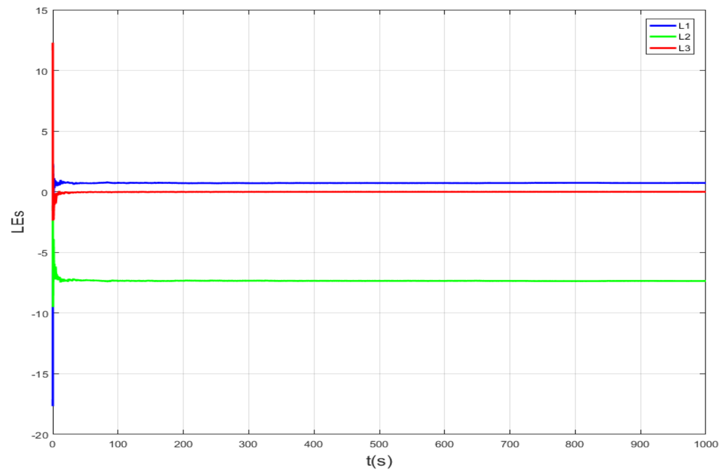



Figure 4.6: Lyapunouv spectrium for the commensurate case with $q_1 = q_2 = q_3 = 0.98$.

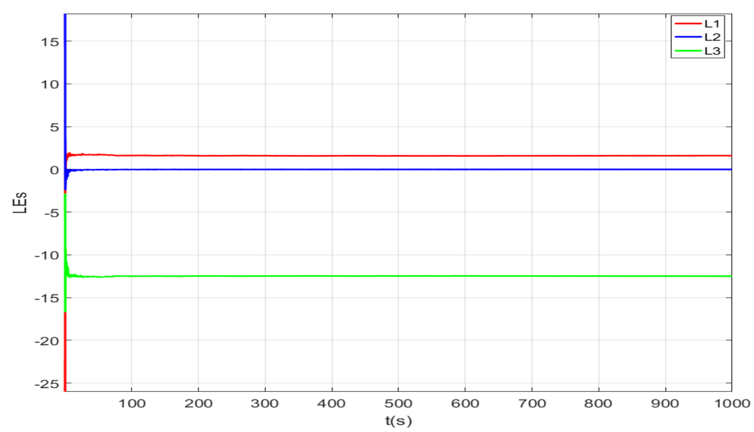



Figure 4.7: Lyapunouv spectrium for the incommensurate case with $q_1 = q_2 = 0.9$, and $q_3 = 0.8$,

**Theorem 4.1 ([96])** *The full-state hybrid projective synchronization between the master system* $(4.18)$ *and the slave system* $(4.19)$ *occurs under the following control law:*

$$U = -(De + R), \tag{4.28}$$

where $D$ is a $4 \times 4$ feedback gain matrix selected so that $A - D$ is a negative definite matrix.
**Proof.** By substituting equation $(4.28)$ into $(4.27)$, we obtain:

$$D_t^{\alpha} e = (A - D)\, e, \tag{4.29}$$

where: $A = (a_{ij})\,, D = (d_{ij})$ are two $4 \times 4$ matrices and $e = (e_1, e_2, e_3, e_4)^T$ is the error vector of the system. If we select the feedback gain matrix $D$ such that $A - D$ is a negative definite matrix, then all the eigenvalues $\lambda_i, i = \overline{1,4}$, of $A - D$ stay in the left-half plane, i.e., $Re(\lambda_i) < 0$, and selecting a candidate Lyapunov function as

$$V = \sum_{i=1}^{4} \frac{1}{2} e_i^2, \tag{4.30}$$

then the time Caputo fractional derivative of order $0.98$ of the Lyapunov function $V$ along the trajectory of the system $(4.29)$ is as follows

$$D_t^{0.98} V = \sum_{j=1}^{4} D_t^{0.98} \left( \frac{1}{2} e_i^2 \right). \tag{4.31}$$

applying Lemma 3.2, we get $(4.31)$ is negative

$$D_t^{0.98} V \leq \sum_{j=1}^{4} e_i D_t^{0.98} e_i \tag{4.32}$$

$$= \lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 e_3^2 + \lambda_4 e_4^2 < 0. \tag{4.33}$$

which implies, according to Lemma 3.1, the asymptotic stability of the trivial solution of the fractional-order system $(4.29)$. Therefore, the full-state hybrid projective synchronization between the two systems is achieved. ■

## 4.4 Numrical simulation

The approach mentioned earlier outlines the technique for achieving FSHPS, which is as follows [96]

$$A = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 8/3 & -1 & 0 & 0 \\ 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.34}$$

and

$$\begin{cases} R_1 = 10e_1 - 10e_2 - e_4 + 25x_1 - 35x_2 + 5x_3 - 10y_1 + 10y_2 + y_4 - 5x_1x_2 + 2x_1x_3 - \frac{8}{3}x_2x_3 \\ R_2 = e_2 - \frac{8}{3}e_1 + 35x_1 - 40x_2 - 3x_3 + \frac{8}{3}y_1 - y_2 + 3x_1x_2 + x_1x_3 - \frac{16}{3}x_2x_3 - y_1y_3 \\ R_3 = 28e_3 + 75x_1 - 90x_2 + 2x_3 - 28y_3 - 2x_1x_2 + 3x_1x_3 - \frac{32}{3}x_2x_3 + y_1y_2 \\ R_4 = e_4 + 100x_1 - 110x_2 + 7x_3 - y_4 - 7x_1x_2 + 2x_1x_3 - 16x_2x_3 - y_2y_3, \end{cases} \tag{4.35}$$

and finally, the equation

$$(u_1, u_2, u_3, u_4)^T = -\left( R + D\,(e_1, e_2, e_3, e_4)^T \right), \tag{4.36}$$

can be expressed as:

$$\begin{cases} u_1 = 35x_2 - 25x_1 - 15e_1 - 5x_3 + 10y_1 - 10y_2 - y_4 + 5x_1x_2 - 2x_1x_3 + \frac{8}{3}x_2x_3 \\ u_2 = 40x_2 - 35x_1 - 5e_2 + 3x_3 - \frac{8}{3}y_1 + y_2 - 3x_1x_2 - x_1x_3 + \frac{16}{3}x_2x_3 + y_1y_3 \\ u_3 = 90x_2 - 75x_1 - 30e_3 - 2x_3 + 28y_3 + 2x_1x_2 - 3x_1x_3 + \frac{32}{3}x_2x_3 - y_1y_2 \\ u_4 = 110x_2 - 100x_1 - 10e_4 - 7x_3 + y_4 + 7x_1x_2 - 2x_1x_3 + 16x_2x_3 + y_2y_3 \end{cases} \tag{4.37}$$

for the chosen $(\beta_{ij})_{4\times 3} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & -3 \\ 4 & 3 & 2 \\ 6 & 2 & 7 \end{pmatrix}, D = \begin{pmatrix} 5 & 10 & 0 & 1 \\ 8/3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$

Then the error system is given by

$$\begin{pmatrix} D_t^{\alpha_1}e_1 \\ D_t^{\alpha_2}e_2 \\ D_t^{\alpha_3}e_3 \\ D_t^{\alpha_4}e_4 \end{pmatrix} = \begin{pmatrix} -15 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{4.38}$$

The matrix $(A - D)$ has eigenvalues of $\lambda_1 = -15$, $\lambda_2 = -5$, $\lambda_3 = -30$, $\lambda_4 = -10$, all of which are negative. This indicates that the error system is asymptotically stable and the synchronization between systems (4.18) and (4.19) has been achieved.

To demonstrate the effectiveness of the proposed controller, we utilized the improved classical Adams-Bashforth-Moulton method and solved system (4.38) with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.98, 0.98)$ and initial conditions of $(e_1(0), e_2(0), e_3(0), e_4(0)) = (-2, 1, -5, 1)$. The time history of the synchronization errors $e_1(t); e_2(t); e_3(t); e_4(t)$ is illustrated in: Figure 4.8, Figure 4.9,.Figure 4.10, and Figure 4.11,

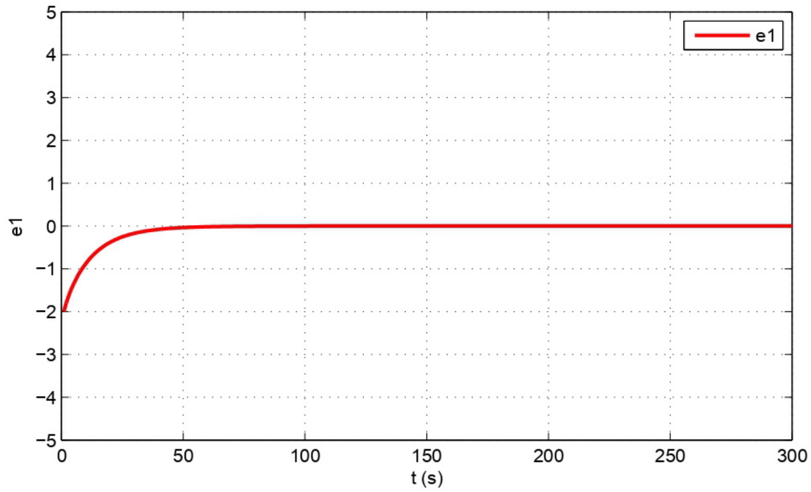



Figure 4.8: Time evolution of the error signal $e_1(t)$ under the action of the controller $(4.29)$ is shown for FSHP.

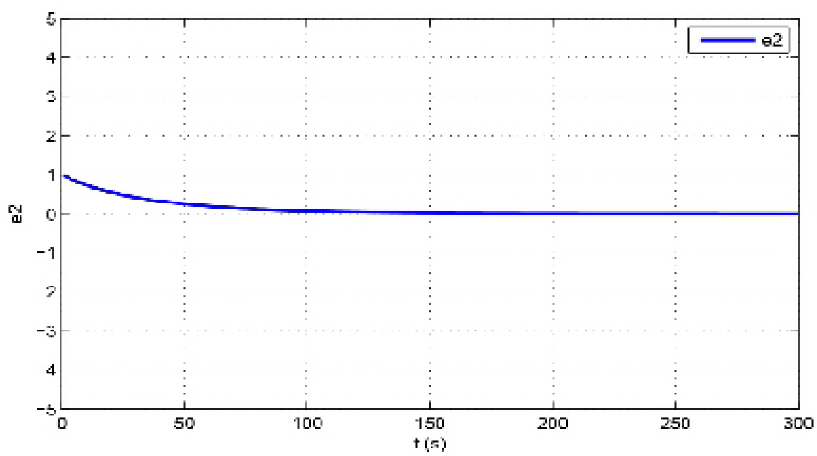



Figure 4.9: Time evolution of the error signal $e_2(t)$ under the action of the controller $(4.29)$ is shown for FSHP.

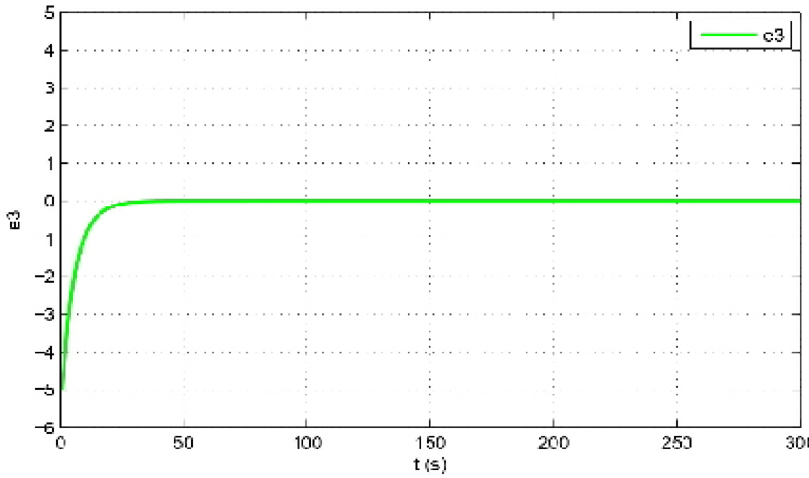



Figure 4.10: Time evolution of the error signal $e_3(t)$ under the action of the controller $(4.29)$ is shown for FSHP.

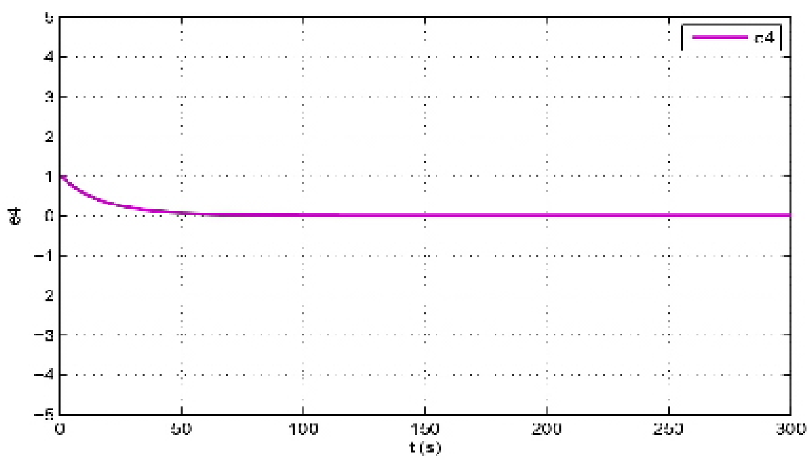



Figure 4.11: Time evolution of the error signal $e_4(t)$ under the action of the controller $(4.29)$ is shown for FSHP.

## 4.5 Conclusion

We establish that the system displays chaotic behavior beyond a certain minimum commensurate order value, and present a hypothetical and computational solution using the Adams–Bashforth–Moulton algorithm. Moreover, we investigate the feasibility of full-S hybrid PS between the new three-dimensional fractional-order system and the Lorenz hyper-chaotic system of fractional-order. This is done by utilizing the definition of FSHPS and Lyapunov theory for linear fractional-order system stability. The proposed controller's efficacy is validated through numerical simulations conducted in Matlab using the improved Adams–Bashforth–Moulton algorithm.

# General Conclusion and Perspectives

We studied in this thesis the control and synchronization of integer and fractional nonlinear continuous chaotic systems in dimension $3$ or more.

With our findings presented, we have successfully attained our objectives: a novel fractional system based on the Caputo derivative definition is introduced and has been shown to display chaotic behavior when the order of commensurability is kept to a minimum. The Adams–Bashforth–Moulton algorithm is employed to provide both theoretical and numerical solutions for this system, and a thorough examination of full-state hybrid projective synchronization is conducted.

As prospects, we are interested in conducting additional research on the practical applications of this fractional system.

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