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# Fractional order chaotic systems synchronization

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بسم الله الرحمن الرحيم

# الإهداء

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رافع السماء المستحق فداؤه،مشرف العلم والعلماء الذي شمل العالمين إنعامه

وعمّ جميع المخلوقات إكرامه وبعد:

فأهدي ثمرة سنيني وجهدها

رفقائي أولًا وثانيًا وثالثًا وعاشرًا وأخيراً... ماما الغالية أخواتي الحبيبات

إلى قصيدة القلوب المشهود فضلها، مفتاح الدروب البهي ظلها، بطلة الاحلام المستحيل مثلها، وعروس الأيام المكتوب عدلها، هدية الحياة المرجو نيلها، هي ربيع البيت وأغنية أركانه،ضحكة ليله وبهجة نهاره

**أمي الغالية أدامك الله تاج فوق رأسي**

الحبيبة التي طالما كانت أمًا لي.أختي الراحلة ومن تركت الشوق في قلبي وكنت حسرة لي،علمتني الوقوف بعد كل سقوط وأخذت منك الحكمة ،أنت التي جمعت لي كل لحظات الفرح والحزن ،"سماح"بفضلك أنا هنا الأن على عتبة التخرج،إلى من أفضلها على نفسي، ولما لا؟؟ فقد ضحيت من أجلي ولم تذخري جهدا في سبيل إسعادي .

**رحمة الله عليك أختي الحبيبة سماح**

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# ABSTRACT

In recent years, the synchronization of chaotic systems has received significant interest and research attention due to its potential applications in various fields, such as secure communication, biological and physical systems, and structural engineering. A wide variety of approaches have been proposed for achieving synchronization of chaotic systems, including complete synchronization, generalized synchronization, and projective synchronization. In this work, we focus on the problem of synchronization of fractional-order chaotic linear systems using suitable controllers and stability theory of linear integer order systems. The study emphasizes the importance of synchronization in chaotic systems, particularly in the context of fractional-order systems, and the simulation results validate the effectiveness of the proposed approach in achieving synchronization of fractional-order chaotic systems.

***Keywords:***

Associated with the topic include dynamical system, chaos, strange attractor, integer-order system, fractional-order system, and synchronization.

# RESUME

Ces dernières années, la synchronisation des systèmes chaotiques a suscité un intérêt considérable et une attention de recherche en raison de ses applications potentielles dans divers domaines, tels que la communication sécurisée, les systèmes biologiques et physiques, et l'ingénierie structurale. Une grande variété d'approches ont été proposées pour atteindre la synchronisation des systèmes chaotiques, notamment la synchronisation complète, la synchronisation généralisée et la synchronisation projective. Dans ce travail, nous nous concentrons sur le problème de la synchronisation des systèmes linéaires chaotiques d'ordre fractionnaire en utilisant des contrôleurs appropriés et la théorie de la stabilité des systèmes linéaires d'ordre entier. L'étude met l'accent sur l'importance de la synchronisation dans les systèmes chaotiques, en particulier dans le contexte des systèmes d'ordre fractionnaire, et les résultats de simulation valident l'efficacité de l'approche proposée pour atteindre la synchronisation des systèmes chaotiques d'ordre fractionnaire.

## Mots-clés:

Système dynamique, Calcul fractionnaire, Chaos, Système d'ordre fractionnaire, Contrôleurs, Synchronisation, Contrôle actif.

# ملخص

في السنوات الأخيرة، استقطب تزامن الأنظمة الفوضوية اهتمامًا كبيرًا وانتباهًا بحثيًا بسبب تطبيقاته المحتملة في مختلف المجالات، مثل الاتصال الآمن والأنظمة الحيوية والفيزيائية والهندسة الإنشائية. وقد تم اقتراح مجموعة واسعة من النهج لتحقيق تزامن الأنظمة الفوضوية، بما في ذلك التزامن الكامل والتزامن العام والتزامن الإسقاطي. في هذا العمل، نركز على مشكلة تزامن الأنظمة الفوضوية ذات الرتبة الكسرية باستخدام متحكمات مناسبة ونظرية الاستقرار للأنظمة ذات الرتبة الصحيحة الخطية.وتسلط هذه الدراسة الضوء على أهمية التزامن في الأنظمة الفوضوية، وبشكل خاص في سياق الأنظمة ذات الرتبة الكسرية، وتؤكد نتائج المحاكاة فعالية النهج المقترح في تحقيق تزامن الأنظمة الفوضوية ذات الرتبة الكسرية

## الكلمات المفتاحية:

النظام الديناميكي، الحساب التفاضلي الكسري، الفوضى، النظام ذو الرتبة الكسرية، التزامن، التحكم النشط.

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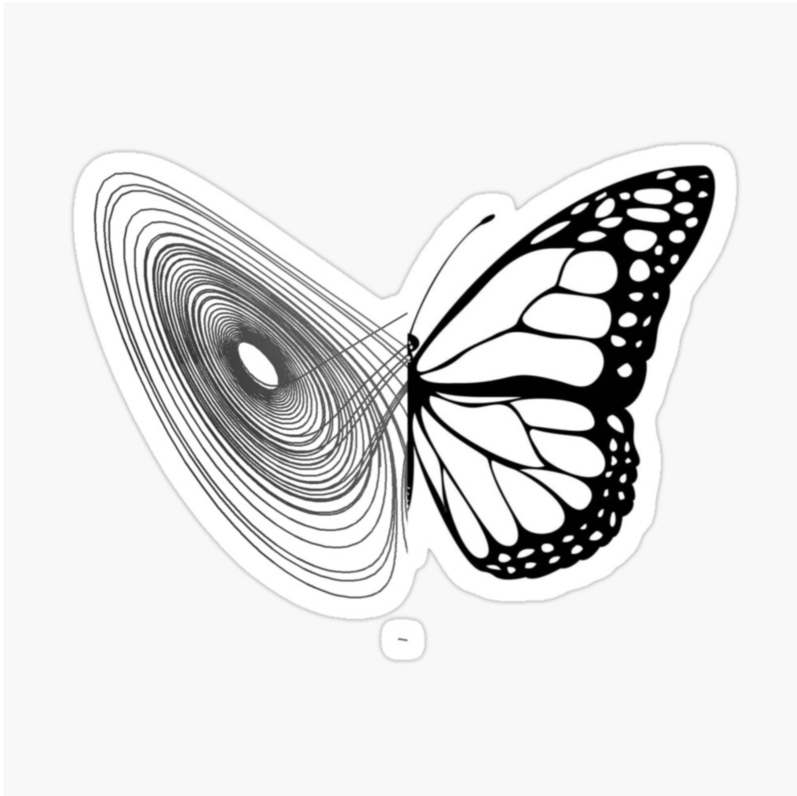

**The butterfly effect** is a concept in chaos theory that suggests that small changes in initial conditions can lead to significant differences in the outcome of a complex system. The term "butterfly effect" was coined by mathematician and meteorologist Edward Lorenz in the early 1960s, based on a hypothetical scenario involving a butterfly's wings in Brazil potentially causing a tornado in Texas.

The idea is that a small change in the initial conditions of a system can cause a chain reaction of events that ultimately lead to a vastly different outcome. For example, a small shift in the wind direction or humidity level could cause a storm to form in a different location, or a slight change in the trajectory of a rocket could cause it to miss its target by a large margin.

The butterfly effect has important implications for a wide range of fields, from weather forecasting to economics to biology. It suggests that even seemingly insignificant events can have far-reaching consequences, and that predicting the behavior of complex systems can be extremely challenging.

Despite its name, the butterfly effect does not suggest that butterflies are actually capable of causing tornadoes. Rather, it is a metaphor for the idea that small events can have large consequences, and that the behavior of complex systems can be highly sensitive to initial conditions.

Today, the butterfly effect remains an important concept in the study of chaos and complexity, and continues to inspire new research and applications in a wide range of fields.

# General Introduction

Since the introduction of the term "chaos" by T. Li and J. Yorke in 1975, chaotic phenomena and behavior have been observed in many natural and modeled systems in various fields such as physics, chemistry, biology, ecology and more. The concept of chaos provides a more comprehensive understanding of the fundamental characteristics. The methods for analyzing chaotic systems have matured over time, allowing for the broad use of nonlinear models in science and technology. One advantage of nonlinear models is their ability to describe complex behavior with a small number of variables and parameters. Engineering applications that benefit from these models include lasers and plasma technologies, mechanical and chemical engineering, system engineering and telecommunications [40].

Fractional order systems have become an active area of research in recent years, with the chaotic dynamics of these systems gaining significant attention. It has been demonstrated that fractional order systems, which generalize many well-known systems, can exhibit chaotic behavior, such as the fractional Duffing system [41], fractional Chua system [42]-[43], fractional Rössler system [44], fractional Chen system [45]-[46]-[47], fractional Lorenz system [48], fractional Arneodo's system [49]-[50], fractional Lü system [51], fractional Newton-Leipnik system [52] and fractional Chen-Lee system [53].

The recent years have seen a significant interest in the chaotic dynamics of fractional-order systems, particularly in the control and synchronization of these systems. Studies have shown that chaotic fractional-order systems can be synchronized [54]-[55]. However, in many literature sources, synchronization among fractional-order systems is mainly explored through numerical simulations based on the stability criteria of linear fractional-order systems, as demonstrated in [57]-[57], or through Laplace transform theory, as presented in [58]-[59]. The chaos synchronization technique is based on the notion that two chaotic systems might evolve on different attractors, but when they are synchronized, they start on various attractors and eventually follow the same course. The synchronization between two systems can be achieved when the trajectories of two systems are matched [60]–[61]

Various control techniques have been developed for controlling and synchronizing fractional-order chaotic systems, including active control, impulsive control, adaptive control, passive control, and sliding mode control.

The problem of synchronizing a fractional-order chaotic system is considered in this work. By using appropriate controllers and the stability theory of linear integer-order systems, the synchronization of the fractional-order chaotic system is achieved. Finally, the corresponding simulation results are presented in Matlab to demonstrate the effectiveness of the proposed method.

This thesis consists of three chapters.

Chapter 1: provides definitions and preliminaries on dynamical systems, chaos, chaotic systems, synchronization and types of synchronization. Additionally, basic definitions and properties of fractional derivatives are presented, along with numerical methods for solving fractional differential equations.

Chapter 2: discusses examples of fractional-order chaotic systems.

Finally, Chapter 3 presents a study on the synchronization between two 3D and 4D fractional-order chaotic systems.

# Chapter 1

# Preliminaries

## 1.1 Introduction

In this chapter, we introduce some prelimanaries about dynamical systems and Chaos theory,Choatic systems, synchronization and types of synchronization. Also,Basic definitions and properties of fractional derivative are given with numerical method for solving fractional differential equations.

## 1.2 Dynamic Systems

A dynamical system consists of an abstract phase space or state space, whose coordinates describe the state at any instant, and a dynamical rule that specifies the immediate future of all state variables, given only the present values of those same state variables [1].

Mathematically, a dynamical system can be described by an initial value problem, which implies that there is a concept of time and that a state at one time evolves into a state or a set of states at a later time. Therefore, states can be organized according to time, and time can be represented by a single quantity.

Dynamical systems are deterministic if there is a unique consequent to every state, or stochastic or random if there is a probability distribution of possible consequents (the idealized coin toss has two consequents with equal probability for each initial state)[1].

### 1.2.1 Definition

A dynamical system can be considered to be a model describing the temporal evolution of a system. Its intellectual roots can be traced back to various fields, including mathematics, astronomy, physics, meteorology and biology [1].

### 1.2.2 Classification

Dynamical systems can be classified into two main categories :

- Continuous dynamical system.
- Discrete dynamical system.

**Definition 1.1** *A continuous dynamical system can be mathematically described by a system of differential equations [2]:*

$$\frac{dX}{dt} = f(X, t, p) \text{ where } X \in \mathbb{R}^n \text{and } p \in \mathbb{R}^r$$

where $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ defines the dynamics of the continuous system.

**Definition 1.2** *A discrete dynamical system can be defined as a system of recurrent algebraic equations defined by [3]:*

$$x_{k+1} = f(X_k, p) \text{ where } X_k \in \mathbb{R}^n \text{and } p \in \mathbb{R}^r$$

where $f : \mathbb{R}^n \times \mathbb{Z}^+ \to \mathbb{R}^n$ defines the dynamics of the discrete system.

### 1.2.3 Phase space

The phase space is a geometric representation of the state variables of a system and it can be visualized in any number of dimensions. The state variables of a system determine its future behavior and the phase space is a way to visualize how the system evolves over time. The phase space can be used to analyze the long-term behavior of a system, such as whether the system exhibits periodic or chaotic behavior and can help predict the behavior of the system under different conditions [4].

### 1.2.4 Phase portrait

Dynamical regimes, such as a resting state or periodic oscillation, correspond to geometric objects, such as a point or a closed curve, in the phase space. Evolution of a dynamical system corresponds to a trajectory (or an orbit) in the phase space. Different initial states result in different trajectories. The set of of all trajectories forms the phase portrait of a dynamical system, though in practice, only representative trajectories are considered. Since it is usually impossible to derive an explicit formula for the solution of a nonlinear equation, the analysis of phase portraits provides an extremely useful way for visualizing and understanding qualitative features of solutions [5].

## 1.3 Attractor

An attractor is a mathematical concept used to describe the behavior of a dynamical system. It is a set of states towards which the system tends to evolve over time. When the system's trajectories get close enough to the attractor, they will remain close even if they are slightly disturbed. A mathematical definition, for an attractor, is given as follows:

**Definition 1.3** *The set A is an attractor if [6]:*

- For any neighborhood $U$ of $A$, there exists a neighborhood $V$ of $A$ such that any solution

$x(x_0; t) = \varphi_t(x_0)$will remain in $U$ if $x_0 \in V$,

- $\cap \varphi_t(V) = A; t \geq 0$,
- There is a dense orbit in $A$.

There are three main types of attractors in the phase space of a dynamical system: point attractors, limit cycle attractors and strange attractors.

**Point attractors**

Point attractors, also known as fixed points, equilibrium points or steady states, are a type of attractor in the phase space of a dynamical system. They are points towards which the system's trajectories converge over time and remain at rest.

**Limit cycle attractors**

Limit cycle attractors are periodic attractors, meaning that the system's trajectories oscillate around a closed curve in the phase space. The limit cycle can be stable or unstable, depending on the dynamics of the system.

**Strange attractors**

Strange attractors are much more complex than other attractors and are referred to as strange when their fractal dimension is a non-integer value. Strange attractors are characterized by:

- Sensitivity to initial conditions (two initially neighboring attractor trajectories always end up moving away from each other, This behavior reflects chaotic behavior), where small differences in initial conditions can lead to vastly different outcomes over time.
- The dimension d of the attractor is fractal (not integer).
- The attractor has zero volume in the phase space.

The term strange attractor was coined by David Ruelle and Floris Takens to describe the attractor resulting from a series of bifurcations of a system describing fluid flow [7]. Certain types of dynamical systems can have attractors that are strange but not chaotic, as has been demonstrated [8].
Each type of attractor has its own unique properties and plays an important role in the behavior of dynamical systems. By studying the attractors of a system, we can gain insights into its long-term behavior and make predictions about its future evolution.

## 1.4 Fractional calculus

In the past few decades, fractional calculus has become a powerful tool for describing the dynamics of complex systems that are commonly found across various fields of science and engineering. Fractional calculus presents a new approach to calculus by introducing fractional operators, which involve derivatives of non-integer orders. These operators extend the concept of integer-order derivatives by accounting for both integer and non-integer orders [9].
In this section, we will outline the most commonly used definitions for general fractional calculus.

**Definition 1.4** *A real function $f(x)$ with $x > 0$ is said to be in the space $C_\mu$, where $\mu \in \mathbb{R}$, if there exists a real number $\lambda > \mu$ such that $f(x) = x\lambda g(x)$, where $g(x) \in C[0, \infty)$. It is said to be in the space $C_\mu^m$ if and only if $f^{(m)} \in C_\mu$ for $m \in \mathbb{N}$.*

**Definition 1.5** *The Riemann–Liouville fractional integral operator of order $\alpha$ of a real function*

$f(x) \in C_\mu$; $\mu \geq -1$, is defined as:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0 \tag{1.1}$$

and $I^\alpha f(x) = f(x)$, where $\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1} dz, \alpha > 0$ is the Gamma function.
Here are some properties of the operator $I^\alpha$, for $\alpha$, $\beta \geq 0$ and $\xi \geq -1$:

- $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$
- $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$
- $I^\alpha x^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\alpha+\xi+1)} x^{\alpha+\xi}$
- The Laplace transform of the Riemann-Liouville fractional integral rule is given by:

$$L(I^\alpha f(x)) = s^{-\alpha} F(s), \alpha > 0.$$

**Definition 1.6** *The Caputo fractional derivative $D^{\alpha}$ of a function $f(x)$, where $\alpha$ is any real number such that $m-1<\alpha\leq m$, $m\in\mathbb{N}$, for $x>0$ and $f\in C_{-1}^{m}$ in terms of $I^{\alpha}$is:*

$$D^{\alpha}f(x)=\frac{1}{\Gamma\left(m-\alpha\right)}\int\limits_{0}^{x}\left(x-t\right)^{m-\alpha-1}f^{(m)}\left(t\right)dt, \tag{1.2}$$

The operators $D^{\alpha}$ have some properties, for $m-1<\alpha\leq m$, $m\in\mathbb{N}$, $\mu\geq-1$ and $f\in C_{\mu}^{m}$

- $D^{\alpha}D^{\beta}f(x)=D^{\alpha+\beta}f(x)$
- $D^{\alpha}D^{-\alpha}f(x)=D^{0}f(x)=f(x)$
- $D^{\alpha}[af(x)+bg(x)]=aD^{\alpha}f(x)+bD^{\alpha}g(x)$
- The Laplace transform of the Caputo fractional derivative rule is expressed as follows:

$$L\left(D^{\alpha}f\left(x\right)\right)=s^{\alpha}F\left(s\right)-\sum_{k=0}^{m-1}s^{\alpha-k-1}f^{(n)}\left(0\right),\left(\alpha>0,m-1<\alpha\leq m\right) \tag{1.3}$$

Particularly, when $\alpha\in]0,1]$, we have:

$$L\left(D_{t}^{\alpha}f\left(t\right)\right)=s^{\alpha}F\left(s\right)-s^{\alpha-1}f\left(0\right). \tag{1.4}$$

## 1.5 Chaos

In the 1970s, the irregular and unpredictable time evolution of many nonlinear and complex systems was given the name chaos. Chaos theory is the study of the behavior of dynamic systems that are highly sensitive to initial conditions. A small change in the initial conditions can have a significant impact on the outcome, even though these systems are deterministic, meaning that their future behavior depends entirely on their initial conditions without any random elements. This behavior is known as deterministic chaos. Although there is no formal or general definition of chaos, it is generally defined as a particular behavior of a dynamic system. The specific characteristics of this behavior will be defined in the following section [10] [11] [12].

### 1.5.1 Properties of chaos

There is a set of properties that summarize the characteristics observed in chaotic systems. These properties are considered mathematical criteria that define chaos. The most popular properties include [13]:

- Sensitivity to initial conditions explains why a small change in the initial conditions of a chaotic system can lead to unpredictable results in the long term. The degree of sensitivity to initial conditions quantifies the chaotic nature of the system.
- Nonlinearity is responsible for the irregular evolution of a chaotic system's behavior.
- determinism: a chaotic system has deterministic fundamental rules and not probabilistic.
- Unpredictability : is due to their sensitivity to initial conditions, which can only be known up to a finite degree of precision.
- the irregularity: is actually a form of hidden order that comprises an infinite number of unstable periodic patterns or movements. This hidden order forms the underlying structure or infrastructure of chaotic systems.
- Strange attractors: characterize the evolution of chaotic systems.

### 1.5.2 Detection of chaos

There are numerical methods that allow the determination of the chaotic behavior of a nonlinear dynamical system. These methods are generally not very numerous, nor spread over a long enough time scale for the system under study. Two of the most commonly used methods are implemented: fractal dimension and Lyapunov exponents.

**Lyapunov exponents**

The Lyapunov exponents are a mathematical tool used to study the dynamics of a system and to determine whether it exhibits chaotic behavior. They are a measure of the rate at which nearby trajectories in phase space diverge or converge, and they can give us information about the stability of the system's attractors.

If all of the Lyapunov exponents are negative, then the system is said to be stable and the trajectories in phase space converge to a fixed point or limit cycle. On the other hand, if any of the Lyapunov exponents are positive, then the system is said to be chaotic and the trajectories in phase space exhibit sensitive dependence on initial conditions, meaning that small changes in the initial conditions can lead to vastly different outcomes [3] [14].

We chose an illustrative example to apply the previous knowledge represented by the new 4-D FOCS, as demonstrated by the fractional-order dynamic Equation :

$$\begin{cases} \frac{d^q x}{d^q t} = -ayz, \\ \frac{d^q y}{d^q t} = bxz, \\ \frac{d^q z}{d^q t} = x - dz, \\ \frac{d^q \varpi}{d^q t} = kx^3 - \varpi. \end{cases} \tag{1.5}$$

We examine the Lyapunov exponents of the proposed system which is defined by Equation (1.5). The equation specifies the state variables $x$, $y$, $z$, and $\varpi$, positive constant parameters $a$, $b$, $d$, and $k$, and the value of the fractional-order derivative $q$. We use the parameter values $a = 2.5$, $b = 0.05$, $c = 1.2$, $d = 2$, $k = 0.001$, and $q = 0.9$ to investigate the Lyapunov exponents of the system. The system is initialized with $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = -1$, and $\varpi(0) = 0.3$ as the initial conditions. To determine the Lyapunov exponents of the system, we use numerical methods and present the results in Fig 1.1.

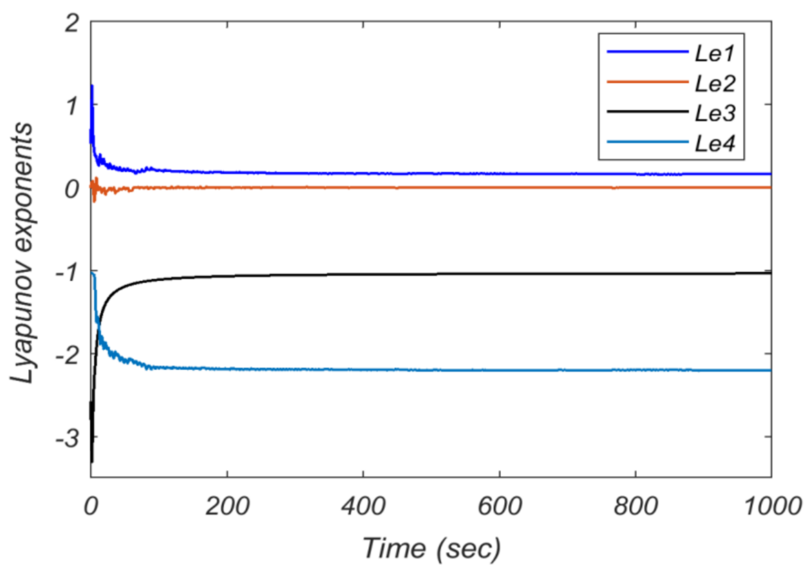


Figure 1.1: The Lyapunov exponents of the new 4-D FOCS.

**Fractal dimension**

The fractal dimension is the fundamental parameter used to describe self-similar patterns and processes, as it indicates the level of complexity of natural objects. This method involves measuring the dimension of the reconstructed attractor of the system studied. By calculating the dimension of the attractor of the system, we can determine whether or not it is constructed in a fractal manner. If the result of the calculation is a positive non-integer value, it indicates the presence of a strange attractor in the system.

Several dimensions have been proposed, including the Kolmogorov dimension, the correlation dimension, and the Lyapunov dimension. While there are slight differences between these dimensions, they all characterize the attractor as a strange object with its fractal dimension [3] [14].

## 1.6 Stability of fractional-order systems

A fractional-order system is a system described by fractional differential equations that contain one or more fractional derivatives. The stability of fractional systems differs from that of integer-order systems, and there are specific stability criteria for fractional systems. The most well-known stability criterion for fractional systems is attributed to Matignon [15].
Consider the following Fractional-order system:

$$\begin{cases} {}_0^cD_t^{\alpha_i} x_i(t) = f_i(x_1(t), x_1(t), ..., x_n(t), t), \\ \qquad x_i(0) = c_i, \quad i = 1, 2, ..., n, \end{cases} \tag{1.6}$$

where $D_t^\alpha$ denotes the Caputo fractional derivative and $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. The vector representation of (1.6) is:

$$D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x}), \tag{1.7}$$

where $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]$ for $0 < \alpha_i \leq 1$, $(i = 1, 2, ..., n)$ and $x \in \mathbb{R}^n$

- The system (1.6) is called a commensurate order system when $\alpha_1 = \alpha_2 = ... = \alpha_n$, otherwise it is an incommensurate order system.
- The system (1.6) is called non-autonomous system if and only if $f$ depends explicitly on $t$ otherwise it is called autonomous system.

### 1.6.1 Equilibrium point

The equilibrium points of system (1.6) are calculated via solving the following equation [15]:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

### 1.6.2 Stability of autonomous linear systems

In this subsection we review some important results of the stability theorems for fractional-order systems [16].

Consider the following linear system of fractional differential equation

$$\begin{cases} D^{\alpha}\mathbf{x} = \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{1.8}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\alpha = [\alpha_1, \alpha_2, ..., \alpha_i, ..., \alpha_n]^T$ $(0 < \alpha i \leq 1$ for $i = 1, 2, ..., n)$ indicates the fractional orders

**Case 1**: When $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$, then the autonomous fractional-order system (1.8) is asymptotically stable if

$$| \arg\left(spec(A)\right) | > \alpha\frac{\pi}{2}.$$

In this case the components of the state decay towards $0$ like $t^{-\alpha}$. In case of $\alpha = 1$, the above stability agrees with the well-known results for ordinary linear differential systems [17].

**Case 2**: Suppose that $\alpha_i$ are rational numbers between $0$ and $1$ but not identically equal to each other. Let $M$ be the least common multiple of the dominators $u_i$ of $\alpha_i$'s, where $\alpha_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i$, $v_i \in N$, for $i = 1, 2, ..., n$. Then system (1.8) is asymptotically stable if all the roots $\lambda$'s of the equation [18].

$$\det(diag(\lambda^{M_{\alpha_1}}, \lambda^{M_{\alpha_2}}, ..., \lambda^{M_{\alpha_n}}) - A) = 0$$

satisfy $| \arg(\lambda) | > \frac{\pi}{2M}$.

## 1.7 Fractional Numerical Method

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). We only derive the predictor–corrector scheme for drive–response systems. This scheme is the generalization of Adams–Bashforth–Moulton one. We interpret the approximate solution of nonlinear fractional-order differential equations using this algorithm in the following way [19].

Consider for $\alpha \in (m-1; m]$; m the differential equation

$$D^{\alpha}y\left(t\right) = f\left(t, y\left(t\right)\right), \quad 0 \leq t \leq T, \tag{1.9}$$

with initial conditions:

$$y^{(k)}\left(0\right) = y_0^{(k)}, \quad k = 0, 1, ..., m-1.$$

is equivalent to the Volterra integral equation

$$y\left(t\right) = \sum_{k=0}^{m-1} y_0^{(k)}\frac{t^k}{k!} + \frac{1}{\Gamma\left(\alpha\right)}\int\limits_0^t \left(t-\tau\right)^{\alpha-1} f\left(\tau, y\left(\tau\right)\right)d\tau. \tag{1.10}$$

Consider the uniform grid $\{t_n = nh/n = 0, 1, ..., N\}$ for some integer $N$ and $h = T/N$.
Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j), j = 1, 2, ..., n$ and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$\begin{aligned} y_h(t_{n+1}) &= \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^p(t_{n+1})\right) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_n(t_j)\right), \end{aligned} \tag{1.11}$$

Where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & if \;\; j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & if \;\; 1 \le j \le n, \\ 1, & if \;\; j = n+1, \end{cases} \tag{1.12}$$

The preliminary approximation $y_h^p(t_{n+1})$ is called predictor and is given by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, y_n(t_j)\right) \tag{1.13}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right) \tag{1.14}$$

The error estimate is

$$\max_{j=0,1,...N} |y(t_j) - y_h(t_j)| = O(h^p), \tag{1.15}$$

where $p = \min(2, 1+\alpha)$.

## 1.8 Synchronization

Synchronization of chaotic systems is a phenomenon that occurs when two or more chaotic systems are coupled or when a chaotic system drives another chaotic system. Because of the butterfly effect which causes exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, the synchronization of chaotic systems is a challenging research problem in the chaos literature [20].

### 1.8.1 Synchronisation method

The idea underlying the phenomenon of synchronization of chaos is that two chaotic systems may evolve on different attractors, but when coupled, they initially start on different attractors

and then somehow eventually follow a common trajectory. Such synchronization between two systems is achieved when the trajectories of the systems are equal, which is the case when one of the two systems changes its trajectory to follow that of the other system or when both systems follow a new common trajectory.
The technique of chaos synchronization is based on the idea that two chaotic systems can evolve on different attractors, but when they are synchronized, they start from different initial conditions and eventually follow the same trajectory. Synchronization between two systems can be achieved when the trajectories of the two systems are matched [21] .

### 1.8.2 Types of Synchronisation

The groundbreaking discovery by Pecora and Carroll has led to the definition of various synchronization types. The main objective of this section is to introduce different synchronization types and the most effective methods for achieving synchronization. To define these types, we can consider chaotic master-slave systems represented by the following equations:

$$D_t^{\alpha} X(t) = F(X(t)) \tag{1.16}$$

$$D_t^{\beta} Y(t) = G(Y(t)) + U \tag{1.17}$$

Here, $X(t) \in \mathbb{R}^n$and $Y(t) \in \mathbb{R}^m$ are the state vectors of the master and slave systems, respectively. $F : \mathbb{R}^n \to \mathbb{R}^n$and $G : \mathbb{R}^m \to \mathbb{R}^m$ are functions, and $(u_i)_{1 \leq i \leq m}$ is a vector controller. Additionally, $D_t^{\alpha}$and $D_t^{\beta}$ are the Caputo fractional derivatives of orders $\alpha$ and $\beta$. Furthermore, several types of synchronization are commonly studied in dynamical systems, and here are some of the most common types:

#### Complete synchronization

Complete synchronization occurs when a controller U is designed to drive the slave system (1.17) to synchronize with the master system (1.16), such that the Euclidean norm of the difference between their state vectors approaches zero as time approaches infinity. This can be represented by [19]:

$$\lim_{t \to \infty} \parallel Y(t) - X(t) \parallel = 0 \tag{1.18}$$

Here, $X(t)$ and $Y(t)$ are the state vectors of the master system and the slave system, respectively.

#### Anti-synchronization

Anti-synchronization occurs when a controller U is designed to drive the slave system (1.17) to anti-synchronize with the master system (1.16), such that the Euclidean norm of the sum of their state vectors approaches zero as time approaches infinity. This can be represented by:[21]

$$\lim_{t \to \infty} \| Y(t) + X(t) \| = 0 \tag{1.19}$$

#### Projective synchronization

Projective synchronization occurs when the state variables of a chaotic slave system synchronize with a constant multiple of the state variables of a master chaotic system. This can be expressed as:[22]

$$\exists \alpha_i \neq 0, \lim_{t \to \infty} \| y_i(t) - \alpha_i x_i(t) \| = 0, \forall (x(0); y(0)), i = 1, 2, ..., n \tag{1.20}$$

Here, $x_i(t)$ and $y_i(t)$ are the state variables of the master system and the slave system, respectively, and $\alpha_i$ is a constant for each $i$.

#### Full-state hybrid projective synchronization

The master systems (1.16) and the slave system (1.17) are said to be full state hybrid function projective synchronized (FSHFPS), if there exist a controller $U$ and differentiable functions $\alpha_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$, such that the synchronization errors:[23]

$$e_i(t) = y_i(t) - \sum_{j=1}^{n} \alpha_{ij}(t) x_j(t), \quad i = 1, 2, ..., m \tag{1.21}$$

satisfy $\lim_{t \to \infty} e_i(t) = 0$.

#### Generalized synchronization

Generalized synchronization occurs when the master and slave systems synchronize with each other with respect to a certain function $(\phi)$. This can be represented by:[24]

$$\lim_{t \to \infty} \| Y(t) - \phi X(t) \| = 0$$

Here, $X(t)$ and $Y(t)$ are the state vectors of the master system (1.16) and the slave system (1.17), respectively.

#### Q-S synchronization

Q-S synchronization involves the master system (1.16) and slave system (1.17) synchronizing with each other in a certain dimension, with the aid of two functions $Q$ and $S$. $Q - S$ synchronization can be achieved if there exists a controller $U$ and two functions $Q : \mathbb{R}^n \to \mathbb{R}^d$ and

$S : \mathbb{R}^m \rightarrow \mathbb{R}^d$, such that the following condition is satisfied [25]:

$$\lim_{t\to\infty} \parallel Q(x) - S(y) \parallel = 0$$

Here, $X(t)$ and $Y(t)$ are the state vectors of the master system and the slave system, respectively, and $Q$ and $S$ map the state vectors to a lower-dimensional space of dimension $d$.

### Active control Method

The active control method for synchronizing chaotic systems was proposed by Bai and Lonngren [26].This effective technique has demonstrated its power not only for synchronizing identical systems but also for synchronizing non-identical systems. Furthermore, this method offers remarkable simplicity for implementing the algorithm [27] [28].

The active control method is a technique for synchronizing chaotic systems. It involves selecting a control matrix $(C)$ that can drive the error between the trajectories of the master and slave systems to converge to zero. For global synchronization to occur, the control matrix $C$ needs to be selected such that the real part of the eigenvalues of $A - C$ is negative, where $A$ is the Jacobian matrix of the master system. In other words, the condition for global synchronization is $\mathrm{Re}(eig(A - C)) < 0$.

To synchronize two chaotic systems, a master system (1.16) and a slave system (1.17), the error between their trajectories must converge to zero as time tends towards infinity. This error can be defined as:

$$e(t) = Y(t) - X(t), \tag{1.22}$$

Thus, the error system can be expressed as:

$$D_t^q e(t) \quad = D_t^q Y(t) - D_t^q X(t) = G(Y(t)) - F(x(t)) + U. \tag{1.23}$$

If we can represent $G(Y(t)) - F(X(t))$ as follows:

$$G(y(t)) - F(X(t)) = Ae(t) + N(X(t), Y(t)), \tag{1.24}$$

Then, the error system can be formulated as:

$$D_t^q e(t) = Ae(t) + N(X(t), Y(t)) + U, \tag{1.25}$$

Here, $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $N$ is a nonlinear function. The controller $U$ can be defined as:

$$U = \mathbf{V} - N(X(t), Y(t)), \tag{1.26}$$

where $\mathbf{V}$ is **the active controller**, defined by:

$$\mathbf{V} = -Ce(t), \tag{1.27}$$

Here, $C$ is an unknown control matrix. Consequently, we arrive at the final expression for the error:

$$D_t^q e(t) = (A - C)e(t). \tag{1.28}$$

Therefore, the synchronization problem between the master system (1.16) and the slave system (1.17) can be reformulated as a problem of zero-stability of the system (1.28).

#### General method

We decompose the drive system (1.16) and response system (1.17) as [16]:

$$D_t^\alpha X(t) = Ax + F(X(t)) \tag{1.29}$$

and

$$D_t^\beta Y(t) = By + G(Y(t)) + U \tag{1.30}$$

where $A$, $B \in \mathbb{R}^{n\times n}$ the parameter matrices for the linear components in the drive system and response system, respectively,
$F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ the nonlinear components in the drive system and response system, respectively
In the definition of PS, the synchronization error is defined as

$$e = y - \chi x, \tag{1.31}$$

where $\chi$ is called the scaling factor. If there exists a constant $\chi(\chi \neq 0)$ such that

$$\lim_{t\to\infty} \parallel e_i(t) \parallel = \lim_{t\to\infty} \parallel Y(t) - \chi X(t) \parallel \to 0 \tag{1.32}$$

If the PS is attained for arbitrary initial conditions $x(0)$ and $y(0)$ between the drive and response systems, then it is said to be achieved. Our objective is to find a suitable and effective controller function $U$ to ensure that the drive system (1.16) and response system (1.17) with non-identical orders approach the PS.

To explain the general method in detail, we separate the controller function $U(x, y)$ into two sub-controllers $U_1(x, y)$ and $U_2(x, y)$, i.e., $U(x, y) = U_1(x, y) + U_2(x, y)$. Our basic concept is to transform the response system into an equivalent fractional-order system with fractional orders being equal to the orders of the corresponding states in the drive system. Hence we propose the following form for the sub-controller $U_1(x, y)$

$$U_1(x, y) = (D^{-(\alpha-\beta)} - I)\,[g(y)]\,, \tag{1.33}$$

where $I$ is the identity operator. By submitting sub-controller (1.33) into system (1.17), we can rewrite the response system as follows

$$D^{\beta}y = D^{-(\alpha-\beta)}g(y) + U_2(x, y). \tag{1.34}$$

By applying the fractional derivative of order $\alpha - \beta$ to both the left and right sides of Eq (1.34), we obtain

$$\begin{aligned} D^{(\alpha-\beta)}[D^{\beta}y] &= D^{\alpha}y \\ &= D^{(\alpha-\beta)}[D^{-(\alpha-\beta)}g(y) + U_2(x, y)] \\ &= g(y) + D^{(\alpha-\beta)}[U_2(x, y)]. \end{aligned} \tag{1.35}$$

Note that $\alpha_i - \beta_i$ satisfies $\alpha_i - \beta_i \in [0, 1)$. According to the property (1) of fractional calculus in section (1.4), the above statement holds. By introducing the sub-controller $U_1(x, y)$, we then reduce the problem to the synchronization of fractionalorder systems with identical orders.
There are numerous techniques available in literature to achieve the synchronization of fractional-order systems. In this case, the active control technique is employed, and a nonlinear observer with the following form is proposed for the sub-controller $U_2(x, y)$

$$U_2(x, y) = D^{-(\alpha-\beta)}[\chi(A - B)x - G(y) + \chi F(x) - \mathbf{L}e], \tag{1.36}$$

where $\mathbf{L} = [\mathbf{L}_1, \mathbf{L}_2, ..., \mathbf{L}_n]^T \in \mathbb{R}^{n\times n}$ is a matrix for the control parameters and $\mathbf{L}_i = [\mathbf{L}_{i1}, \mathbf{L}_{i2}, ..., \mathbf{L}_{in}]$ is the control vector in each state controller.
To demonstrate the effectiveness of the active control approach-based controller mentioned above, we provide the following theoretical analysis. Since system (1.35) is equivalent to the response system (1.17), substituting Eq (1.36) into system (1.35) yields the response system in the following form

$$D^{\alpha}y = By + G(y) + D^{(\alpha-\beta)}D^{-(\alpha-\beta)}[\chi(A - B)x - G(y) + \chi F(x) - \mathbf{L}e]. \tag{1.37}$$

By recalling the fractional calculus properties discussed in section (1.4), we derive the error dynamic system, which can be expressed as follows

$$\begin{aligned} D^{\alpha}e &= D^{\alpha}(y-\chi x) \\ &= D^{\alpha}y - D^{\alpha}(\chi x) \\ . &= By + G(y) - \chi[Ax + F(x)] + [\chi(A-B)x - G(y) + \chi F(x) - \mathbf{L}e] \\ &= (B-\mathbf{L})e \end{aligned} \tag{1.38}$$

For the linear error system $D^{\alpha}e$, it is always possible to select suitable control matrices $\mathbf{L}$ in order to ensure that all the eigenvalues $\lambda_i (i = 1, 2, ..., n)$ of $(B - \mathbf{L})$(Case 1 in Subsection 1.6.2) or all the roots of $\det(diag(\lambda^{M_{\alpha_1}}, \lambda^{M_{\alpha_2}}, ..., \lambda^{M_{\alpha_n}}) - (B - \mathbf{L})) = 0$.
(Case 2 in Subsection 1.6.2) satisfies the stability theorems of the fractional-order system $D^{\alpha}e$, which ensures the asymptotic stability of the error system (1.38). As a result, the synchronization of chaotic systems (1.16) and (1.17) with different orders is guaranteed. Also, various choices of the matrix $\mathbf{L}$ are possible, which proves the efficiency of the proposed method.

**The PS between two different 3-D commensurate fractional-order chaotic systems**

In this subsection, we choose the well-known fractional-order Lorenz system and fractional-order Chua's circuit as the drive system and response system, respectively. The chaos dynamics in the classical fractional-order Lorenz system have been studied in [29] and it is found that the commensurate fractional-order Lorenz system of order $0.99$ still behaves chaotically with the default parameters as $\sigma = 10$, $\rho = 28$ and $b = 8/3$. In [30], Hartley et al. investigated the nonlinear dynamics of a particular fractional-order Chua's circuit where the system has a cubic nonlinearity which yields similar behaviour to that of the classical Chua's circuit with piecewise-linear nonlinearity. Their results verified that the commensurate fractional-order Chua's circuit with order $0.9$ can yield chaos.

In this case we consider the PS between the classical fractional-order Lorenz system and fractional-order Chua's circuit by using the former to drive the latter. The drive and response systems are given as follows:

$$\begin{cases} D^{0.99}x_1 = 10(y_1 - x_1), \\ D^{0.99}y_1 = 28x_1 - x_1z_1 - y_1 \\ D^{0.99}z_1 = x_1y_1 - 8z_1/3 \end{cases} \tag{1.39}$$

and

$$\begin{cases} D^{0.9}x_2 = a_2\left[y_2 + x_2 - 2x_2^3\right] + u_1 \\ D^{0.9}y_2 = x_2 - y_2 + z_2 + u_2 \\ D^{0.9}z_2 = -b_2y_2 + u_3 \end{cases} \tag{1.40}$$

According to the proposed method, we can derive the controller function as follows:

$$\begin{cases} u_1(t) = (D^{-0.09} - I)[a_2 y_2 + a_2 x_2/7] - 2a_2 x_2^3 7 + D^{-0.09}[\chi((10-a_2)y_1 - (10+a_2/7)x_1) - \mathbf{L}_1 e] \\ u_2(t) = (D^{-0.09} - I)[x_2 - y_2 + z_2] + D^{-0.09}[\chi(27x_1 - z_1) - \chi x_1 z_1 - \mathbf{L}_2 e] \\ u_3(t) = (D^{-0.09} - I)[-b_2 y_2] + D^{-0.09}[\chi(b_2 y_1 - 8z_1/3) + \chi x_1 y_1 - \mathbf{L}_3 e], \end{cases} \tag{1.41}$$

where $\mathbf{e} = [e1, e2, e3]^T = [x_2 - \chi x_1, y2 - \chi y_1, z_2 - \chi z_1]^T$, $\mathbf{L}_i = [\mathbf{L}_{i1}, \mathbf{L}_{i2}, \mathbf{L}_{i3}](i = 1, 2, 3)$.
The parameters for Chua's circuit are set to be $a_2 = 12.75$, $b_2 = 100/7$ as same in [30]. If the gain matrix is chosen as $\mathbf{L} = [\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3]^T = diag(10, 10, 10)$.
then the error system is calculated as follows:

$$\mathbf{D}^{0.99}\mathbf{e} = (\mathbf{B} - \mathbf{L})\mathbf{e} = \begin{pmatrix} -8.1786 & 12.75 & 0 \\ 1 & 11 & 1 \\ 0 & -14.2857 & -10 \end{pmatrix} \times \begin{pmatrix} e_1 \\ e_2 \\ e3 \end{pmatrix} \tag{1.42}$$

The eigenvalues of the matrix $(\mathbf{B} - \mathbf{L})$ are $\lambda_{1,2} = -11.2388 \pm 2.5205i$ and $\lambda_3 = -6.7011$, which all satisfy $|\arg(\lambda i)| > \alpha i\pi/2 = 0.99\pi/2$. According to the stability theorems of the fractional-order system (Case 1in Subsection 1.6.2), the error system is asymptotically stable and the synchronization is guaranteed.

**Simulation and results**

In the numerical simulations, the initial conditions for the drive and response systems are $(x_1(0), y_1(0), z_1(0))^T = (3, 4, 5)^T$ and $(x_2(0), y_2(0), z_2(0))^T = (-6, -10, 13)^T$, respectively.
Here we choose the scaling factor as $\chi = -1$. Note that with such a scaling factor the PS is degraded into the anti-synchronization. Synchronization errors are shown in Fig.1.2.

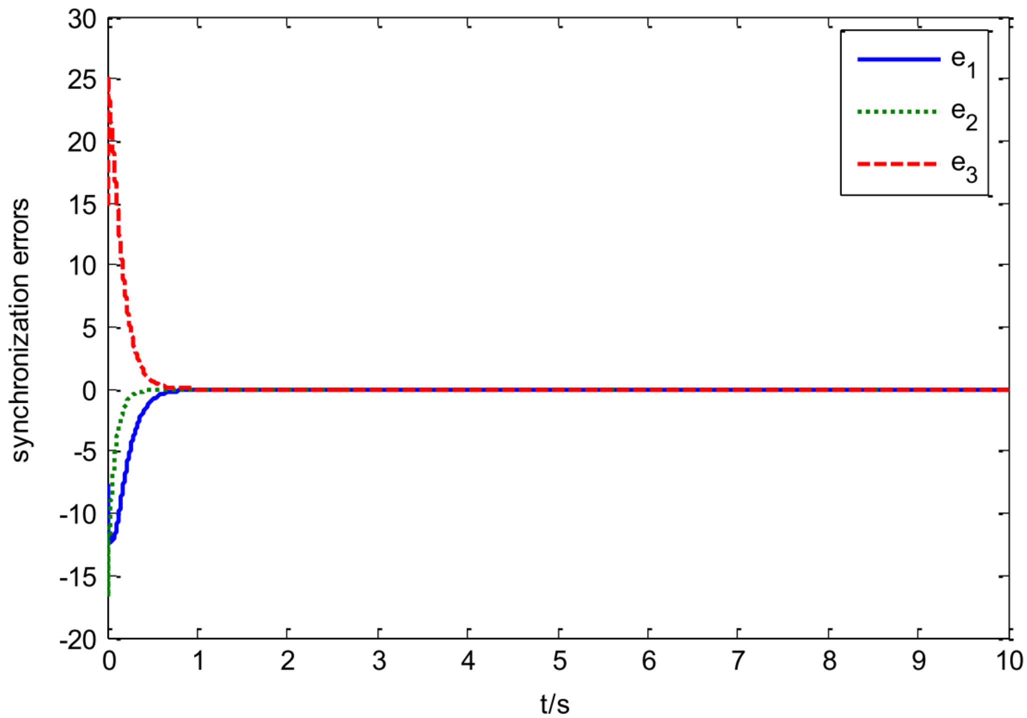


Figure 1.2: Time-History of the synchronization errors $e_i(t)$

## 1.9 Conclusion

This chapter covers some preliminaries on dynamical systems and chaos theory, chaotic systems, synchronization, and types of synchronization. Additionally, it provides basic definitions and properties of fractional derivatives, along with a numerical method for solving fractional differential equations.

# Chapter 2

# Examples of chaotic system of fractional orders

## 2.1 Introduction

This chapter presents and analyzes multiple examples of $3D$ fractional-order chaotic systems that display complex and unpredictable behavior. These systems are used as simulation applications to study and understand complex systems. The examples demonstrate the potential of fractional-order systems in modeling complex phenomena and the chaotic behavior can be used to design novel control strategies.

**Example 2.1** *The fractional order happiness system is given by the following nonlinear equations[31]:*

$$\begin{cases} D^q x = y, \\ D^q y = z, \\ D^q z = -x - b(1 - x^2)y - az + f(t). \end{cases}$$

This system exhibits chaotic behavior when $q = 0.97$, system parameters $(a, b) = (5, 0.5)$, and initial condition $(x_0, y_0, z_0) = (1, 1, 1)$,without external circumstance $(f(t) = 0)$.
The chaotic attractors of this system are shown in Fig.2.1-2.2.

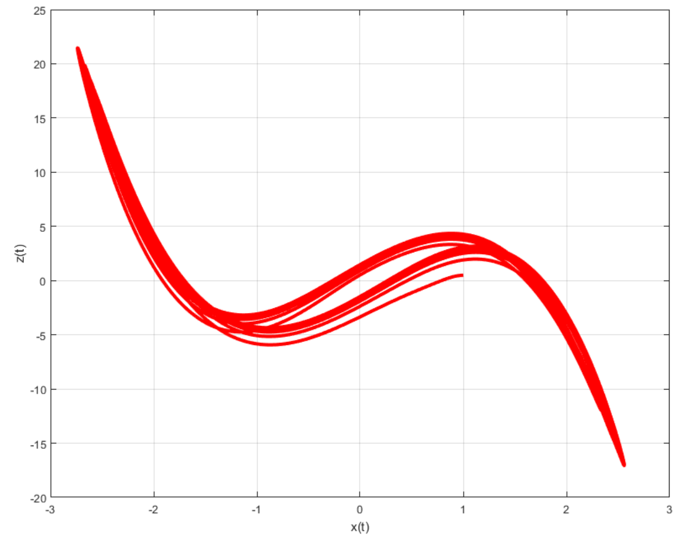


Figure 2.1: Chaotic attractor of system (2.2) in x-z plan.

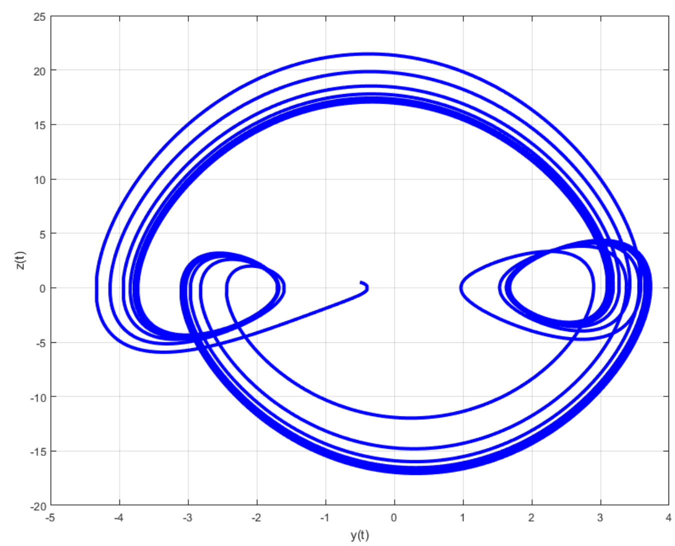


Figure 2.2: Chaotic attractor of system (2.2) in y-z plan.

**Example 2.2** *The Fractional Order Financial Chaotic System is given by the following nonlinear equations [32] :*

$$\begin{cases} D^q x = z + (y-a)x, \\ D^q y = 1 - b(y+x^2) - cx^6, \\ D^q z = -x - z. \end{cases} \tag{2.1}$$

This system exhibits chaotic behavior when $q = 0.98$, system parameters $(a, b, c) = (7.6, 0.3, 4.82)$, and initial condition $(x_0, y_0, z_0) = (0.4, 0.2, 0.5)$.
The chaotic attractors of the financial system are shown in Fig.2.3-2.4 and 2.5.

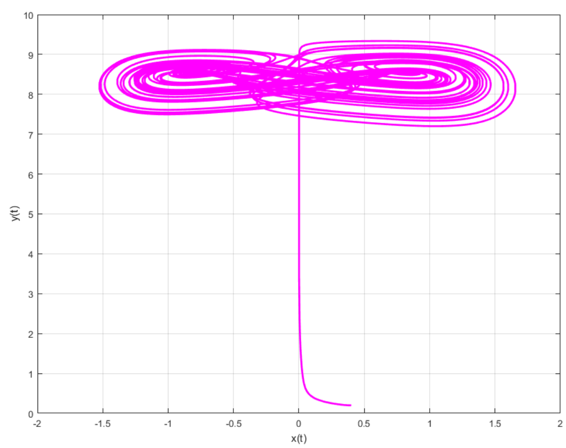


Figure 2.3: Chaotic attractor of system (2.1) in x-y plan.

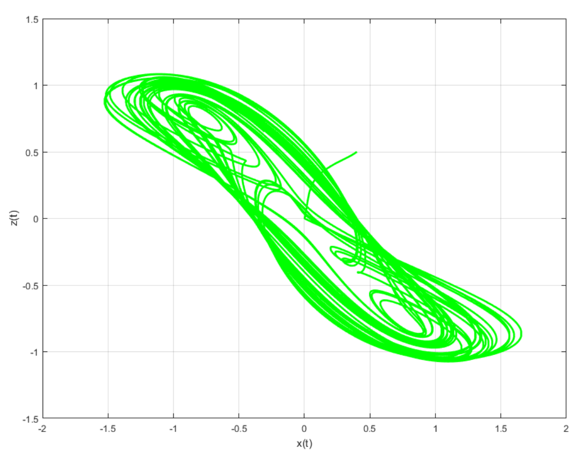


Figure 2.4: Chaotic attractor of system (2.1) in x-z plan.

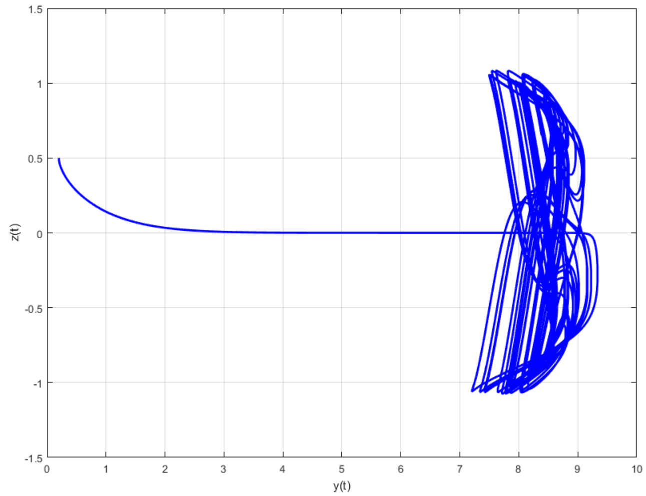


Figure 2.5: Chaotic attractor of system (2.1) in y-z plan.

**Example 2.3** *we consider the chaotic system recalled in Lu et al. (2004) and described by the following nonlinear equations [33]:*

$$\begin{cases} D^q x = ax - by - yz, \\ D^q y = cx, \\ D^q z = -dz + y^2. \end{cases}$$

This system exhibits chaotic behavior when $q = 0.94$, system parameters $(a, b, c, d) = (-2, -6.4, 1, 1)$, and initial condition $(x_0, y_0, z_0) = (0.2, 0.1, 0.2)$.
The chaotic attractors of this system are shown in Fig2.6-2.7 and 2.8.

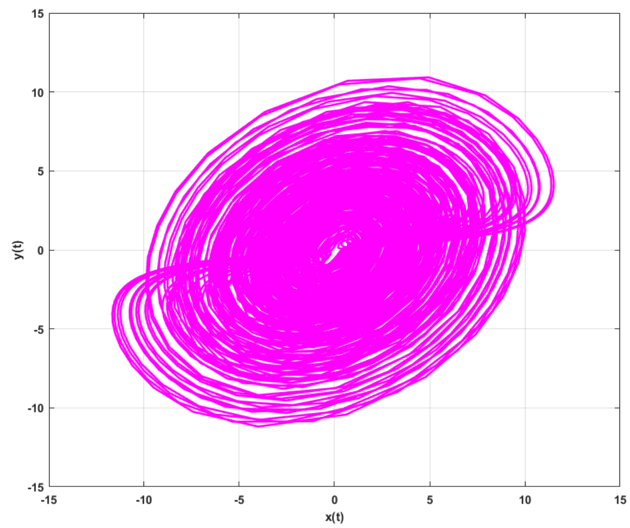


Figure 2.6: Chaotic attractor of system (2.4) in x-y plan.

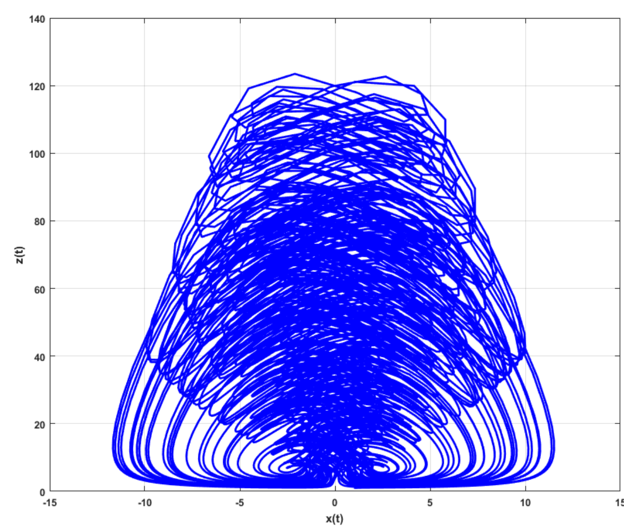


Figure 2.7: Chaotic attractor of system (2.4) in x-z plan.

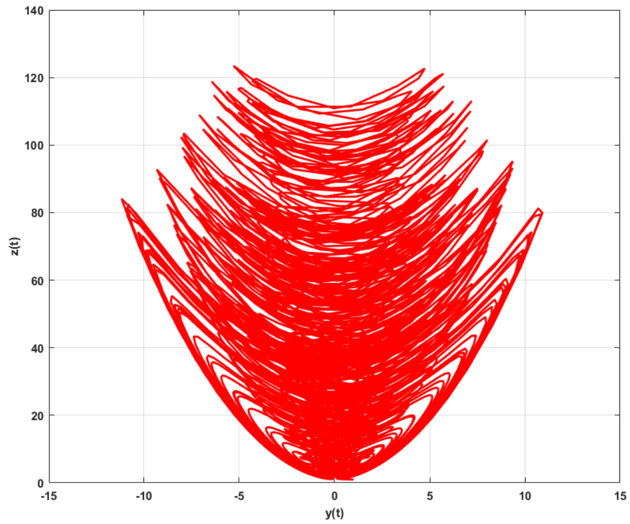


Figure 2.8: Chaotic attractor of system (2.4) in y-z plan.

**Example 2.4** *we consider the fractional differential system described as follows [34]:*

$$\begin{cases} D^q x = ax - dyz, \\ D^q y = -by + xz, \\ D^q z = -cz + xyz + k. \end{cases}$$

This system exhibits chaotic behavior when $q = 0.95$, system parameters $(a, b, c, d, k) = (4, 9, 4, 1, 4)$, and initial condition $(x_0, y_0, z_0) = (1, 1, 1)$.
The chaotic attractors of this system are shown in Fig.2.9-2.10 and 2.11.

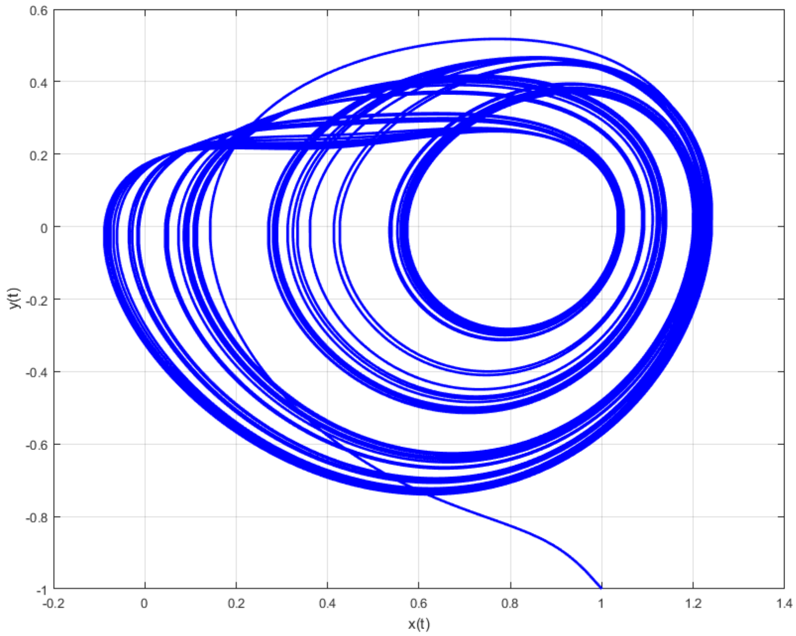


Figure 2.9: Chaotic attractor of system (2.5) in x-y plan.

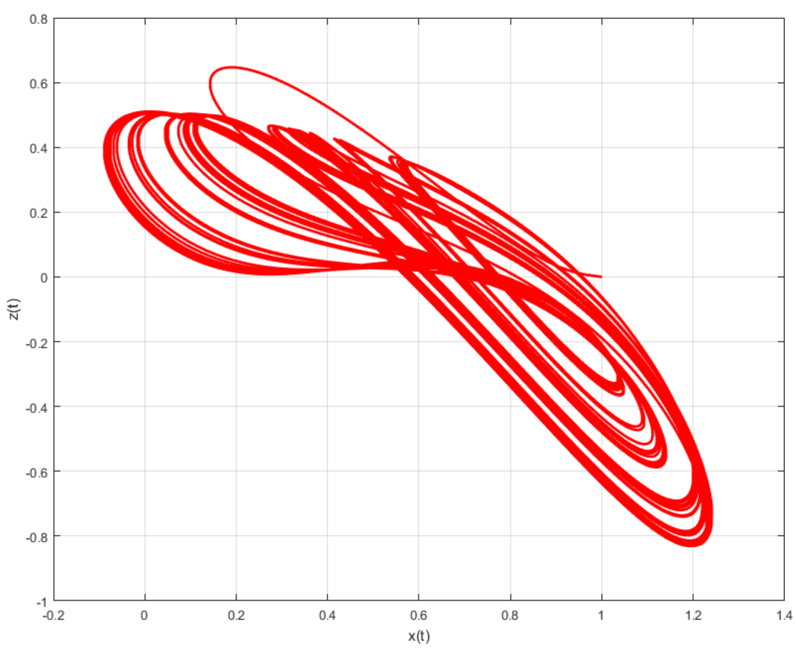


Figure 2.10: Chaotic attractor of system (2.5) in x-z plan.

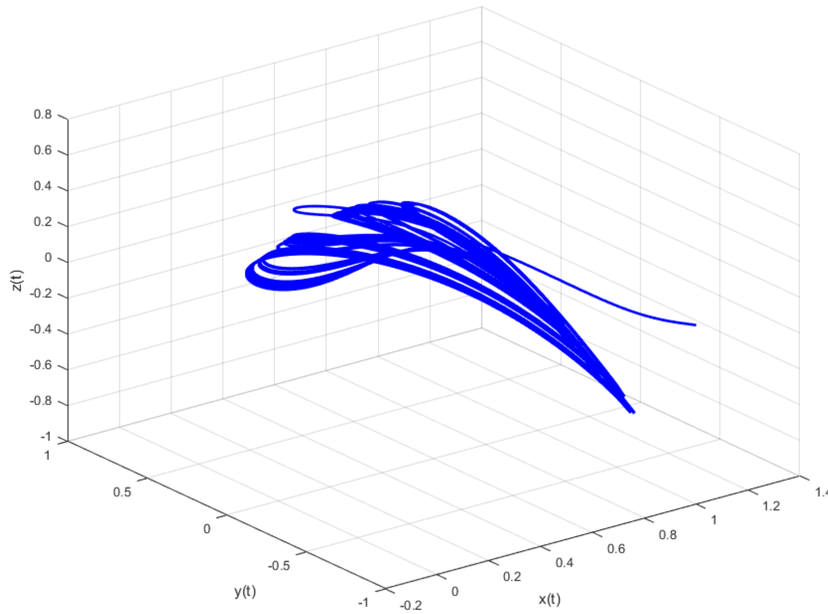


Figure 2.11: Chaotic attractor of system (2.5) in x-y-z space.

**Example 2.5** *The fractional order Rössler system is given by the following nonlinear equations [35]:*

$$\begin{cases} D^q x = -y - z, \\ D^q y = x + ay, \\ D^q z = bx - cz + xz. \end{cases} \tag{2.2}$$

*This system exhibits chaotic behavior when* $q = 0.95$*, system parameters* $(a, b, c) = (0.5, 0.2, 10)$*, and initial condition* $(x_0, y_0, z_0) = (0.5, 1.5, 0.1)$*.The chaotic attractors of this system are shown in Fig.2.12-2.13 and 2.14.*

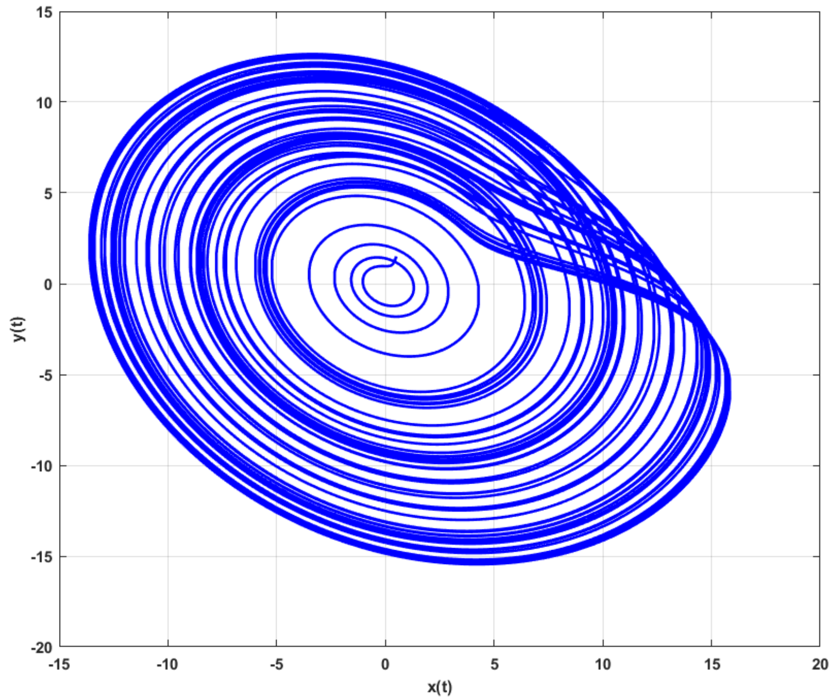


Figure 2.12: Chaotic attractor of system (2.3) in x-y plan.

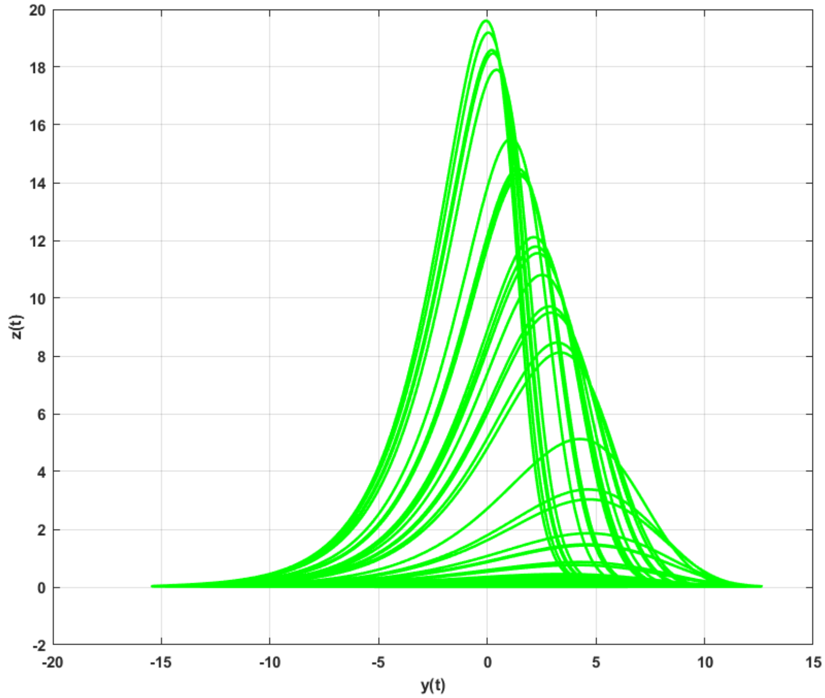


Figure 2.13: Chaotic attractor of system (2.3) in x-z plan.

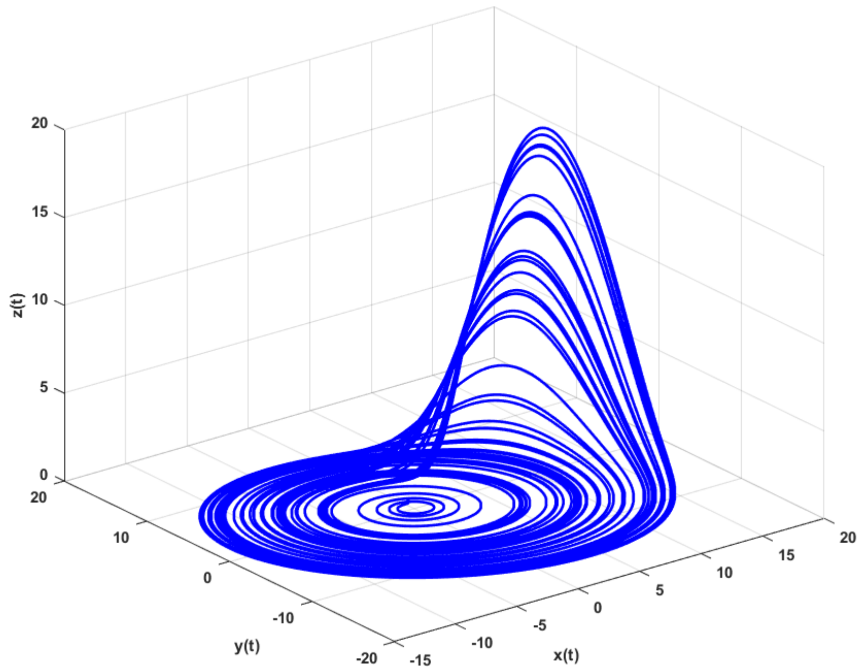


Figure 2.14: Chaotic attractor of system (2.3) in x-y-z space.

## 2.2 Conclusion

In this chapter, we have presented some examples of 3D fractional-order chaotic systems, which illustrate the potential of this approach for modeling complex systems and designing control strategies.

# Chapter 3

# Synchronization of two chaotic systems of fractional orders

## 3.1 Introduction

This chapter focuses on studying the synchronization of a fractional-order chaotic system by designing appropriate sub-controllers based on the stability criteria of the linear system's order. The simulation results presented in the chapter illustrate the success of the proposed approach.

## 3.2 Example in 3D

In this section, we investigate the synchronization between the Newton-Leipnik system [36] [37] and the Lü systems [38].
Assuming that the Newton-Leipnik system drives the Lü system, we define the drive (master) and response (slave) systems as follows:

$$\begin{cases} D^{q_1}x_1 = -a_1x_1 + y_1 + 10y_1z_1, \\ D^{q_2}y_1 = -x_1 - 0.4y_1 + 5x_1z_1, \\ D^{q_3}z_1 = b_1z_1 - 5x_1y_1. \end{cases} \tag{3.1}$$

The parameters are taken as $a_1 = 0.4$, $b_1 = 0.175$, initial condition $= (0.19, 0, -0.18)$, and $0 < q_i \leq 1$ is the order of the derivative. At $q_i = 0.95 (i = 1, 2, 3)$, Eq (3.1) becomes the fractional-order Newton-Leipnik chaotic equation.

and

$$\begin{cases} D^{q_1}x_2 = a_2(y_2 - x_2) + u_1(t), \\ D^{q_2}y_2 = c_2y_2 - x_2z_2 + u_2(t), \\ D^{q_3}z_2 = x_2y_2 - b_2z_2 + u_3(t). \end{cases} \quad (3.2)$$

The parameters are taken as $a_2 = 36$, $b_2 = 3$, $c_2 = 20$, initial condition $= (0.2, 0.5, -1)$, and $0 < q_i \leq 1$ is the order of the derivative. At $q_i = 0.95 (i = 1, 2, 3)$, Eq (3.2) becomes the fractional-order Lü chaotic equation. The unknown terms $u_1$, $u_2$, $u_3$ in (3.2) are active control functions to be determined.

We define the error functions as:

$$e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1. \quad (3.3)$$

Eq. (3.3), together with (3.1) and (3.2), yields the error system.

$$\begin{cases} D^{q_1}e_1 = -a_2e_1 + a_2e_2 + a_2(y_1 - x_1) + a_1x_1 - y_1 - 10y_1z_1 + u_1(t), \\ D^{q_2}e_2 = c_2e_2 + (c_2 + 0.4)y_1 - x_2z_2 + x_1 + 0.4y_1 - 5x_1z_1 + u_2(t), \\ D^{q_3}e_3 = -b_2e_3 - (b_2 + b_1)z_1 + 5x_1y_1 + u_3(t). \end{cases} \quad (3.4)$$

We define the active control functions $u_i(t)$ as:

$$\begin{cases} u_1(t) = V_1(t) - a_2(y_1 - x_1) - a_1x_1 + y_1 + 10y_1z_1, \\ u_2(t) = V_2(t) - (c_2 + 0.4)y_1 + x_2z_2 - x_1 - 0.4y_1 + 5x_1z_1, \\ u_3(t) = V_3(t) + (b_2 + b_1)z_1 - 5x_1y_1. \end{cases} \quad (3.5)$$

The terms $V_i(t)$ are linear functions of the error terms $e_i(t)$. When $u_i(t)$ is chosen according to (3.5), the error system (3.4) becomes:

$$\begin{cases} D^{q_1}e_1 = -a_2e_1 + a_2e_2 + V_1(t) \\ D^{q_2}e_2 = c_2e_2 + V_2(t) \\ D^{q_3}e_3 = -b_2e_3 + V_3(t) \end{cases} \quad (3.6)$$

The control terms $V_i(t)$ are selected to ensure the stability of the system (3.6). There is no unique choice for such functions, but we have chosen:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = -C \begin{pmatrix} e_1 \\ e_2 \\ e3 \end{pmatrix},$$

To ensure stability of the closed loop system, the matrix $C$, which is a $3 \times 3$ constant matrix, should be chosen carefully so that its eigenvalues $\lambda_i$ satisfy the control requirements of the feedback system.

$$|\arg(\lambda_i)| > 0.5\pi\alpha, \ i = 1, 2, 3.$$

Let us choose the matrix $C$ as:

$$C = \begin{pmatrix} -a_2 + 1 & 0 & 0 \\ 0 & c_2 + 1 & 0 \\ 0 & 0 & -b_2 + 1 \end{pmatrix}$$

The eigenvalues of the linear system (3.6) are $-1$,$-1$,$-1$. As a result, the condition that all $q_i \leq 1$ is satisfied, which leads to achieving the required synchronization.

**Simulation and results**

The parameters for the Newton-Leipnik system are $a_1 = 0.4$ and $b_1 = 0.175$, while the parameters for the Lü system are $a_2 = 36$, $b_2 = 3$, and $c_2 = 20$. The experiments are conducted with a fixed fractional order value of $q = 0.95$ for both the drive system (3.1) and the response system (3.2). The initial conditions for the master system (Newton-Leipnik system) and the slave system (Lü system) are set to $x_1(0) = 0.19$, $y_1(0) = 0$, $z_1(0) = -0.18$ and $x_2(0) = 0.2$, $y_2(0) = 0.5$, $z_2(0) = -1$, respectively. Thus, the initial errors are $e_1(0) = 0.01$, $e_2(0) = 0.5$, and $e_3(0) = -0.82$. The errors $e_i(t)$ for the drive and response system are shown in the Fig.3.1.

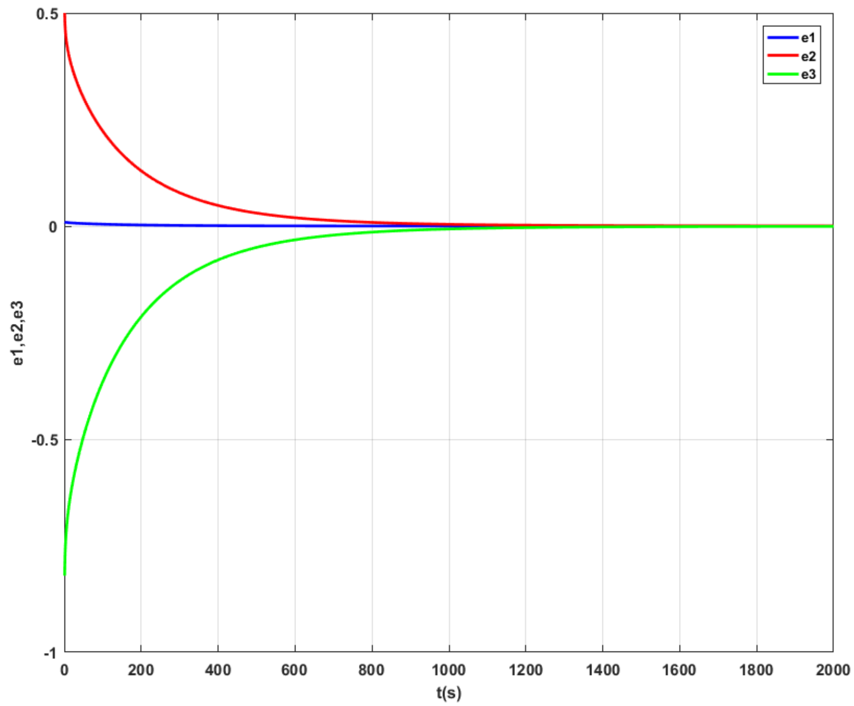


Figure 3.1: Time-history of the errors synchronization between systems (3.1) and (3.2).

## 3.3 Example in 4D

In this section, we investigate the synchronization between the new fractional order hyperchaotic system [40] and the Chen fractional order hyperchaotic system [15].
Assuming that the new fractional order hyperchaotic system is used as the drive system and the Chen fractional order hyperchaotic system serves as the response system:

$$\begin{cases} D^{q_1}x_1 = x_2, \\ D^{q_2}x_2 = -x_1 + x_2x_3 + a, \\ D^{q_3}x_3 = -x_1 - 10x_1x_2 - x_1x_3 + b, \\ D^{q_4}x_4 = -x_1x_2 + x_{4.} \end{cases} \tag{3.7}$$

and

$$\begin{cases} D^{q_1}y_1 = a_2(y_2 - y_1) + y_4 + u_1(t) \\ D^{q_2}y_2 = d_2y_1 - y_1y_3 + c_2y_2 + u_2(t), \\ D^{q_3}y_3 = y_1 - b_2y_3 + u_3(t), \\ D^{q_4}y_4 = y_2y_3 + r_2y_4 + u_4(t). \end{cases} \tag{3.8}$$

For the new FOHS and Chen FOHS, the default system parameters are chosen as $(a, b) = (0, 0)$ and $(a_2, b_2, c_2, d_2, r_2) = (35, 3, 12, 7, 0.5)$, respectively. Experiments are conducted with a fixed fractional order value of $q = 0.98$ for both the drive system (3.7) and response system (3.8). In the following sections, we will use these system parameters and fractional orders in our synchronization analysis.
The unknown terms $u_1$, $u_2$, $u_3$, and $u_4$ in (3.8) are active control functions that need to be determined.
We define the error functions as:

$$e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3, e_4 = y_4 - x_4. \tag{3.9}$$

Eq. (3.9) together with (3.7) and (3.8) yields the error system.

$$\begin{cases} D^{q_1}e_1 = -a_2e_1 + a_2e_2 + a_2(x_2 - x_1) + y - x_2 + u_1(t), \\ D^{q_2}e_2 = d_2e_1 + c_2e_2 + d_2x_1 + c_2x_2 - y_1y_3 + x_1 - x_2x_3 - a + u_2(t), \\ D^{q_3}e_3 = -b_2e_3 - b_2x_3 + x_1 + 10x_1x_2 + x_1x_3 - b + u_3(t), \\ D^{q_4}e_4 = r_2e_4 + y_2y_3 + (r_2 - 1)x_4 + x_1x_2 + u_4(t). \end{cases} \tag{3.10}$$

The active control functions $u_i(t)$ are defined as:

$$\begin{cases} u_1(t) = V_1(t) - a_2(x_2 - x_1) - y + x_2, \\ u_2(t) = V_2(t) - d_2x_1 - c_2x_2 + y_1y_3 - x_1 + x_2x_3 + a, \\ u_3(t) = V_3(t) + b_2x_3 - x_1 - 10x_1x_2 - x_1x_3 + b, \\ u_4(t) = V_4(t) - y_2y_3 - (r_2 - 1)x_4 - x_1x_2 \end{cases} \tag{3.11}$$

The terms $V_i(t)$ represent linear functions of the error terms $e_i(t)$. When we choose $u_i(t)$ as defined in equation (3.11), the error system (3.10) can be expressed as:

$$\begin{cases} D^\alpha e_1 = -a_2e_1 + a_2e_2 + V_1(t) \\ D^\alpha e_2 = d_2e_1 + c_2e_2 + V_2(t) \\ D^\alpha e_3 = -b_2e_3 + V_3(t) \\ D^\alpha e_4 = r_2e_4 + V_4(t) \end{cases} \tag{3.12}$$

The control terms $V_i(t)$ are selected to ensure that the system (3.12) becomes stable. There is no unique selection for these functions, but we choose:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = -C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

Here, $C$ is a constant $4 \times 4$ matrix. To ensure the stability of the closed-loop system, the matrix $C$ must be chosen in such a way that the eigenvalues $\lambda_i$ of $C$ satisfy the control.

$$|\arg(\lambda_i)| > 0.5\pi\alpha, \; i = 1, 2, 3, 4.$$

Let us choose the matrix $C$ as:

$$C = \begin{pmatrix} -a_2 - 1 & a_2 & 0 & 0 \\ d_2 & c_2 + 1 & 0 & 0 \\ 0 & 0 & -b_2 + 1 & 0 \\ 0 & 0 & 0 & r_2 + 1 \end{pmatrix}$$

The eigenvalues of the linear system (3.12) are $-1$,$-1$,$-1$. As a result, the condition that all $q_i \leq 1$ is satisfied, which leads to achieving the required synchronization.

**Simulation and results**

The parameters for the new FOHS system are set to $(a, b) = (0, 0)$, while the parameters for the Chen FOHS system are $(a_2, b_2, c_2, d_2, r_2) = (35, 3, 12, 7, 0.5)$. The initial conditions for the master system (the new FOHS) and the slave system (Lü system) are $x_1(0) = 10$, $x_2(0) = -2$, $x_3(0) = 10$, $x_4(0) = 5$ and $y_1(0) = 0.3$, $y_2(0) = -6$, $y_3(0) = 0$, $y_4(0) =$ 10, respectively. The initial errors are $e_1(0) = 9.7$, $e_2(0) = -4$, $e_3(0) = -10$, and $e_4(0) = 5$. Numerical simulations for the drive system (3.7) and response system (3.8) are performed with $q = 0.98$, and the errors $e_i(t)$ for the drive and response systems are shown in the Fig.3.2.

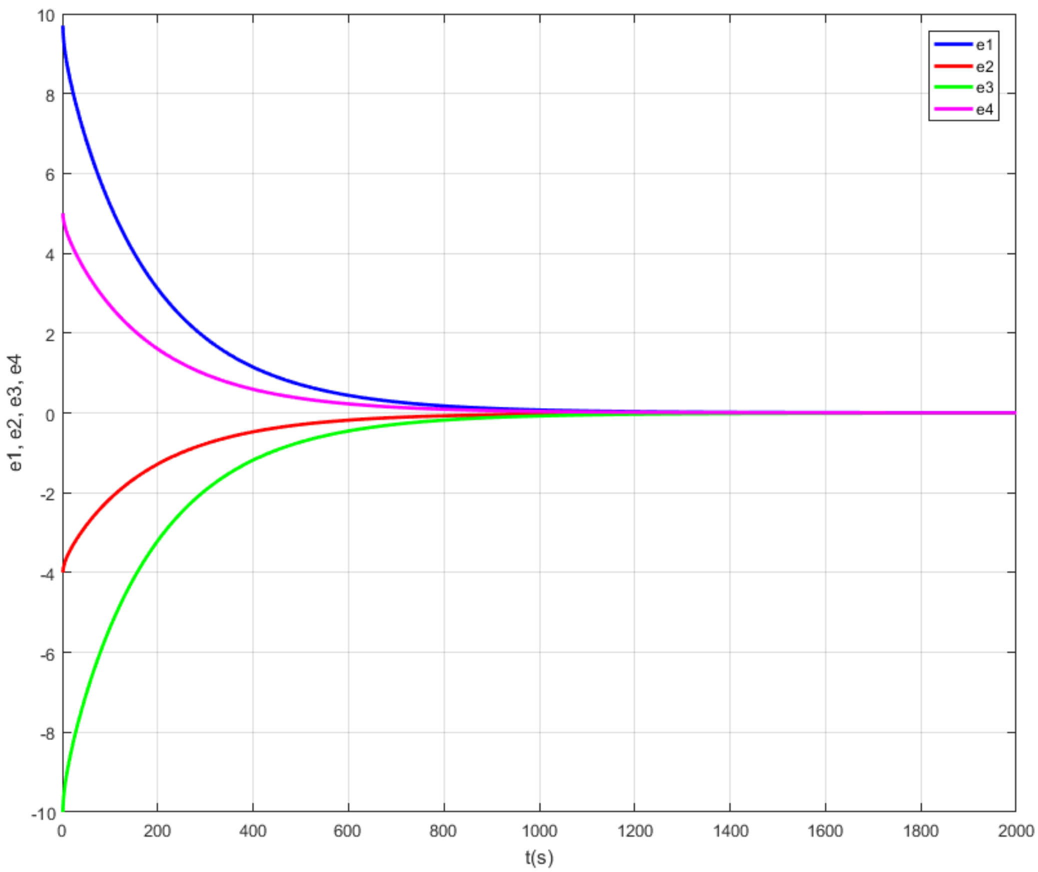


Figure 3.2: Time-history of the errors synchronization between systems (3.7) and (3.8).

## 3.4 Conclusion

In this chapter, we demonstrate the synchronization of two distinct fractional-order chaotic systems using active control and we provide simulation results to showcase the effectiveness of the proposed approach.

# General conclusion

Chaos is a significant nonlinear phenomenon that has been extensively studied in science, mathematics, engineering, and various other fields. Due to its potential applications in different industries, the synchronization of chaotic systems has become a popular research topic. In recent years, various synchronization solutions have been developed for chaotic systems, such as the backstepping design method, sliding mode control, passive control, nonlinear active control, projective synchronization, projective function synchronization, global synchronization, and more. These methods have been applied to both chaotic and hyper-chaotic systems.
In this work, we presented a study of some fractional-order chaotic systems, to achieve the objective of this study we have divided our thesis into three chapters.
the first Chapter, contains some definitions and prelimanaries about: chaos, chaotic systems, synchronization, and types of synchronization. Additionally, basic definitions and properties of fractional derivatives are presented, along with numerical methods for solving fractional differential equations.
The second chapter, introduced some examples of fractional-orders chaotic systems.
The third chapter, present the study of the synchronization between two 3D and 4D fractional-orders chaotic systems. Finally, numerical simulations are given to demonstrate the effectiveness of the proposed method with numerical simulation.

Based on extensive research conducted over the past two decades, it can be concluded that the synchronization of chaotic systems has numerous applications in various fields including biology, chemistry, telecommunications (information security), and physics, among others, and these are just a few examples. However, these examples are only a fraction of the potential fields of application. The synchronization can also be extended to fractional (non-integer) derivative systems, as demonstrated in this thesis. As a perspective, it would be interesting to contribute to and develop this type of synchronization in this research field.

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