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Theme

# Study of double phase elliptic problem

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بسم الله الرحمن الرحيم

# شكر وعرفان

سبحانك اللهم لا علم لنا إلا ما علمتنا، نشكر الله ونحمده فضل نعمه علينا، نعمة العقل التي أنار بها دربنا وفكرنا ونعمة الذاكرة التي حفظنا بها سرنا وجهرنا. والصلاة والسلام على قدوة المربين نبينا محمد وعلى آله وصحبه أجمعين.

إن من تمام شكر الله، شكر أهل الفضل والبر، وعملا بقول نبيه محمد صلى الله عليه وسلم: " من لم يشكر القليل لم يشكر الكثير ومن لم يشكر الناس لم يشكر الله " 'رواه أحمد والترميذي'

واخص بالشكر الجزيل لأساتذة الخير الذين علمو بلا شك أن العلم من أجمل العبادات وافضلها، كما أتقدم بالشكر الجزيل إلى كل من ساعدني وساهم في تكويني طيلة مشواري الدراسي من أساتذة التعليم الإبتدائي وصولا إلى أساتذة التعليم العالي والبحث العلمي في قسم الرياضيات والإعلام الآلي بجامعة الشيخ الشهيد العربي التبسي، وأخص بالذكر الأستاذة المشرفة المحترمة " زديري صنية " على كل ماقدمته لي من معلومات وتوجيهات قيمة ساهمت في إثراء بحثي العلمي، فهي برهان للذين بذلوا شاق الجهد.

كما أشكر أعضاء لجنة المناقشة التي شرفتني بقبولها مناقشة مذكرتي، كل من الأستاذ "عكروت كمال " رئيسا و الأستاذة " مزهود رشيدة " ممتحنا الذين لا شك أنهم سيفيضون عليا بتوجيهاتهم القيمة وملاحظاتهم السديدة.

وفي الأخير أشكر كل من قدم لي يد العون و المساعدة من قريب أو بعيد ولو بكلمة طيبة أو بتوجيهة أو حتى بدعوة في ظهر الغيب لهم جزيل الشكر والعرفان .

ولكم مني فائق التقدير والإحترام

# إهداء

الحمد لله على منه وإمتنانه والشكر له على
نعمه وإنعامه حمدا كثيرا طيبا. الذي انعم عليا
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مذكرتي.
إلى كل هؤلاء أهدي ثمرة جهدي

# Abstract

In this memoir, we study quasilinear elliptic equations and systems with double phase operator. We prove the existence of a weak solution by applying the theory of pseudomonotone operators. Furthermore, Imposing some additional linear condition the gradient variable the uniqueness of the solution is obtained.

**Keywords** : Elliptic system, Doube phase problems, pseudomonotone operators, Existence results, Uniqueness.

# Résumé

Dans ce mémoire, nous étudions les équations elliptiques quasilinéaires et les systèmes avec des opérateurs elliptiques de double phase. Nous prouvons l'existence d'au moins une solution faible en appliquant la théorie d'opérateur pseudomonotone. En imposant des conditions de linéarisation sur la variable de gradient, pour assurer l'unicité de la solution.

**Mots clés** : Système elliptique, Problème de double phase, Opérateur pseudomonotone, Résultat existence, Unicité.

# ملخص

في هذه المذكرة، قمنا بدراسة المعادلات والأنظمة البيضاوية شبه الخطية في وجود مؤثر مزدوج، نثبت وجود حل ضعيف واحد على الأقل من خلال تطبيق نظرية المؤثرات شبه رتيبة. بفرض بعض الشروط الخطية على متغير التدرج في الطرف الايسرالغير خطي يتم الحصول على وحدانية الحلول.

**الكلمات المفتاحية:** نظام بيضوي، مسائل المرحلة المزدوجة، مؤثرات شبه رتيبة، نتائج الوجود، الوحدانية.

# Contents

# Introduction

Partial differential equations are of crucial importance in modelization and description of a wide variety of phenomena such as fluid dynamics, quantum physics, sound, heat, electrostatics, diffusion, gravitation, chemistry, biology, calculator charts and time prediction.

In recent years, authors have interested by elliptic problems called double phase, originally the idea to treat such operators comes from Zhikov [36, 37, 38] who introduced such classes to provide models of strongly anisotropic materials; and also the monograph of Zhikov-Kozlov-Oleinik [39]. In order to describe this phenomenon, he introduced the functional.

$$\omega \mapsto \int \left( |\nabla \omega|^p + \mu(x) |\nabla \omega|^q \right) dx, \tag{1}$$

that generates a double phase operator whose behavior switches between two different elliptic situation, on the set $\{x \in \Omega,\ \mu(x) = 0\}$ the operator will be controlled by gradient of order $p$ and in the case $\{x \in \Omega,\ \mu(x) \neq 0\}$ it is the gradient of order $q$. This reason why it is called double phase operator.

The double phase problems has been studied deeply recently, we refer to the papers of Baroni-Colombo-Mingione [3, 4, 5], Baroni-Kussi-Mingione [6], Colombo-Mingione [11, 12] and the references therein concerning the regularity.

In the works of [13, 27, 28] the integral form (1) arise in the context of functional with non-standard growth, Colasuonno-Squassina [10] studied the corresponding eigenvalue problem of the double phase operator with Dirichlet boundary condition he proved the existence and properties of related variational eigenvalues. By applying variational methods, Liu [24] treated double phase problems and proved existence and multiplicity results.

In our work the problem studied depend a non linearity on the right hand side called convection terms which is functions depends on the gradient of the solution. Our starting point is the work of Averna-Motreanu-Tornatore [1] who considered a $(p, q)$-Laplacian problem with a homogeneous Dirichlet boundary condition.

In this memoir we study the existence and uniquesse of solution of double phase elliptic equation , for the existence we used the theory psudomontone operators (surjectivety result), by conditions on the convection term, in addition a strong

condition on the non-linearity we can prove the uniqueness of solution, see [20], this result is generalized for a system of two equations, the problem treated by the same manner, see [26].
For other existence results on quasilinear equations with dependence on the gradient and the $p$-Laplace or the $(p, q)$-Laplace differential operator we refer to the papers of Bai-Gasiński-Papageorgiou [2], De Figueiredo-Girardi-Matzeu [14],Dupaigne-Ghergu-Radulescu [15] , Faraci-Motreanu-Puglisi [16], and the references therein.
The memoir is divided into three chapters.

In the first chapter we suggest some basic concepts concerning functional farmework, psudomonotone operators, eigenvalue problems and Nemytskij Operator.

In chapter 2, we study the existence and uniqueness results for the following double phase problem with convection term

$$\begin{cases} -\operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u+\mu\left(x\right)\left|\nabla u\right|^{q-2}\nabla u\right)=f\left(x,u,\nabla u\right) \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

Such that $\Omega$ is a bounded domain of $\mathbb{R}^N, N \geq 2$ with a lipschitz boundary $\partial\Omega$. Where $1 < p < q < N$, the function $\mu : \overline{\Omega} \to [0, \infty)$ is Lipschitz continuous. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a carathéodory function that is, $x \mapsto f(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s, \xi) \mapsto f(x, s, \xi)$ is continuous for a. a. $x \in \Omega$.

In the last chapter we study the existence and uniqueness of solution of an elliptic system with double phase operator and convection term, using the same theory in chapter 2.

Chapter 1

# Preliminaries

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the memoire.

## 1.1 Functional spaces

### 1.1.1 Lebesgue spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set of $R^N$

**Definition 1.1.1** [7] *Let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$, *we set*

$$\mathrm{L}^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}, f \ \textit{measurable and} \ |f|^p \in \mathrm{L}^1(\Omega) \right\},$$

*equipped with norm*

$$\|f\|_{\mathrm{L}^p(\Omega)} = \|f\|_p = \left( \int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

*We set*

$$\mathrm{L}^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} / f \ \textit{measurable and} \ \exists c > 0 / |f(x)| \leq c \ \textit{a.e in} \ \Omega \}.$$

*With*

$$\|f\|_{\mathrm{L}^\infty(\Omega)} = \|f\|_\infty = \inf \{ c > 0 / |f(x)| \leq c, \ \textit{a.e in} \ \Omega \}.$$

**Proposition 1.1.1** [7] *Let* $1 < p < \infty$, $L^p$ *is reflexive, separable, and the dual of* $L^q$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.
*If* $p = 1$, $L^1$ *is not reflexive, separable and the dual of* $L^\infty$.
*If* $p = \infty$, $L^\infty$ *is not reflexive, not separable and the dual contains* $L^1$.

## *1.1.2* Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$

**Definition 1.1.2** [7] *The sobolev space* $\mathrm{W}^{1,p}(\Omega)$ *is defined by*

$$\mathrm{W}^{1,p}(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega) ; \nabla u \in (\mathrm{L}^p(\Omega))^N \right\}.$$

*The space* $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\|u\|_{\mathrm{W}^{1,p}(\Omega)} = \left( \|u\|^p_{\mathrm{L}^p(\Omega)} + \|\nabla u\|^p_{\mathrm{L}^p(\Omega)} \right)^{\frac{1}{p}},$$

*if* $p = \infty$, *The space* $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\|u\|_{\mathrm{W}^{1.\infty}(\Omega)} = \max\left(\|u\|_\infty, \|\nabla u\|_\infty\right).$$

**Proposition 1.1.2** [7] $W^{1,p}$ *is Banach space for every* $1 \leq p \leq \infty$. $W^{1,p}$ *is reflexive for* $1 < p \leq \infty$, *and it is Separable for* $1 \leq p < \infty$.

## *1.1.3* $W_0^{1,p}(\Omega)$ Space

**Definition 1.1.3** [7] *For* $1 \leq p < +\infty$ *we define the space* $W_0^{1,p}(\Omega)$ *as being the closure of* $D(\Omega)$ *in* $W^{1,p}(\Omega)$, *and we write*

$$\mathrm{W}_0^{1,p}(\Omega) = \overline{D(\Omega)}^{\mathrm{W}^{1,p}}.$$

## *1.1.4* Musielak-Orlicz space

Let $\mathcal{H} : \Omega \times [0, +\infty) \to [0, +\infty)$ be the function

$$(x, t) \mapsto t^p + \mu(x) t^q,$$

where $1 < p < q < N$, and

$$\frac{q}{p} < 1 + \frac{1}{N}, \ \mu : \overline{\Omega} \to [0, \infty) \text{ is Lipschitz continuous.} \tag{1.1}$$

We set

$$\rho_{\mathcal{H}}(\Omega) := \int\limits_\Omega \mathcal{H}(x, |u|)\, dx = \int\limits_\Omega \left(|u|^p + \mu(x)|u|^q\right) dx.$$

**Definition 1.1.4** [24] *The Musielak-Orlicz space* $L^{\mathcal{H}}(\Omega)$ *is defined by*

$$\mathrm{L}^{\mathcal{H}}(\Omega)=\left\{u \mid u:\Omega\to\mathbb{R}, \text{ is measurable and } \rho_{\mathcal{H}}(u):=\int_{\Omega}\mathcal{H}(x,|u|)\,dx<+\infty\right\}.$$

*Equipped with the norm*

$$\|u\|_{\mathcal{H}}=\inf\left\{\tau>0:\rho_{\mathcal{H}}\left(\frac{u}{\tau}\right)\le 1\right\}.$$

**Proposition 1.1.3** [24] *The space* $L^{\mathcal{H}}(\Omega)$ *a separable, uniformly convex and so a reflexive Banach space. Furthermore we define*

$$\mathrm{L}^{q}_{\mu}(\Omega)=\left\{u \mid u:\Omega\to\mathbb{R} \text{ is measurable and } \int_{\Omega}\mu(x)|u|^{q}\,dx<+\infty\right\},$$

*and endow it with the semi norm*

$$\|u\|_{q,\mu}=\left(\int_{\Omega}\mu(x)|u|^{q}\,dx\right)^{\frac{1}{q}}.$$

*In the same way we define* $L^{q}_{\mu}(\Omega,\mathbb{R}^{N})$.
*From Colasuonno-Squassina* [10]*,we have the continuous embeddings*

$$\mathrm{L}^{q}(\Omega)\hookrightarrow\mathrm{L}^{\mathcal{H}}(\Omega)\hookrightarrow\mathrm{L}^{p}(\Omega)\cap\mathrm{L}^{q}_{\mu}(\Omega).$$

*For* $u\neq 0$ *we that* $\rho_{\mathcal{H}}\left(\frac{u}{\|u\|_{\mathcal{H}}}\right)=1$ *and so, it follows that*

$$\min\left\{\|u\|^{p}_{\mathcal{H}},\|u\|^{q}_{\mathcal{H}}\right\}\le\|u\|^{p}_{p}+\|u\|^{q}_{q}\le\max\left\{\|u\|^{p}_{\mathcal{H}},\|u\|^{q}_{\mathcal{H}}\right\}. \tag{1.2}$$

**Definition 1.1.5** [24] *The Musielak-Orlicz sobolev space* $W^{1,H}(\Omega)$ *defined by*

$$\mathrm{W}^{1,\mathcal{H}}(\Omega)=\left\{u\in\mathrm{L}^{\mathcal{H}}(\Omega):|\nabla u|\in\mathrm{L}^{\mathcal{H}}(\Omega)\right\},$$

*equipped with the norm*

$$\|u\|_{1,\mathcal{H}}=\|\nabla u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}}.$$

*where* $\|\nabla u\|_{\mathcal{H}}=\||\nabla u|\|_{\mathcal{H}}$.
*By* $W^{1,\mathcal{H}}_{0}(\Omega)$ *we denote the completion of* $C^{\infty}_{0}(\Omega)$ *in* $W^{1,\mathcal{H}}$ *and thanks to* (1.1) *we have an equivalent norm on* $W^{1,\mathcal{H}}_{0}(\Omega)$ *given by*

$$\|u\|_{1,\mathcal{H},0}=\|\nabla u\|_{\mathcal{H}},$$

**Proposition 1.1.4** [24] *Both space* $W^{1,\mathcal{H}}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega)$ *are uniformly convex, and so, reflexive Banach space.*
*In addition it is known that the embedding*

$$\mathrm{W}_0^{1,\mathcal{H}}(\Omega) \hookrightarrow \mathrm{L}^r(\Omega), \tag{1.3}$$

*is compact where* $r < p^*$*, with* $p^*$ *being the critical exponent to* $p$ *given by*

$$p^* := \frac{Np}{N-p}, \tag{1.4}$$

*recall that* $1 < p < N$*. From* (1.2) *we directly obtain that*

$$\min\left\{\|u\|_{1,\mathcal{H},0}^p, \|u\|_{1,\mathcal{H},0}^q\right\} \leq \|u\|_p^p + \|u\|_{q,\mu}^q \leq \max\left\{\|u\|_{1,\mathcal{H},0}^p, \|u\|_{1,\mathcal{H},0}^q\right\}, \tag{1.5}$$

*for all* $u \in W_0^{1,\mathcal{H}}(\Omega)$.

**Proposition 1.1.5** [21] *Let* $1 < p < q < N$, $\frac{Nq}{N+q-1} < p$, $\mu(x) \in L^\infty(\Omega)$, $\mu(x) \geq 0$ *for a. a.* $x \in \Omega$ *be satisfied and let*

$$p^* := \frac{Np}{N-p} \text{ and } p_* = \frac{(N-1)p}{N-p},$$

*be the critical exponents to* $p$*. Then the following embeddings hold*
$(i)$ $\mathrm{L}^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $\mathrm{W}^{1,\mathcal{H}} \hookrightarrow W^{1,r}(\Omega)$ *are continuous for all* $r \in [1,p]$;
$(ii)$ $\mathrm{W}^{1,\mathcal{H}} \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1,p^*]$;
$(iii)$ $\mathrm{W}^{1,\mathcal{H}} \hookrightarrow L^r(\Omega)$ *is compact for all* $r \in [1,p^*)$;
$(iv)$ $\mathrm{W}^{1,\mathcal{H}} \hookrightarrow L^r(\partial\Omega)$ *is continuous for all* $r \in [1,p_*]$;
$(v)$ $\mathrm{W}^{1,\mathcal{H}} \hookrightarrow L^r(\partial\Omega)$ *is compact for all* $r \in [1,p_*)$
$(vi)$ $\mathrm{L}^{\mathcal{H}}(\Omega) \hookrightarrow L_\mu^q(\Omega)$ *is continuous;*
$(vii)$ $\mathrm{L}^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

## 1.2 Monotone operators

**Definition 1.2.1** [9] *Let* $X$ *be real Banach space, and let* $A : X \to X^*$ *be an operator.*
$(i)$ $A$ *is called monotone if and only if*

$$\langle Au - Av, u - v\rangle \geq 0 \text{ for all } u, v \in X.$$

$(ii)$ $A$ *is called strictly monotone if and only if*

$$\langle Au - Av, u - v\rangle > 0 \text{ for } u, v \in X \text{ with } u \neq v.$$

*(iii) A is called strongly monotone if and only if there is the constant* $c > 0$ *such that*

$$\langle Au - Av, u - v\rangle \geq c\left\| u - v\right\|^2 \textit{ for all } u, v \in X.$$

*(iv) A is called uniformly monotone if and only if*

$$\langle Au - Av, u - v\rangle \geq \alpha\left(\left\| u - v\right\|\right)\left\| u - v\right\| \textit{ for all } u, v \in X.$$

*Where the continuous function* $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is strictly monotone increasing with* $\alpha(0) = 0$ *and* $\alpha(t) \to +\infty$ *as* $t \to +\infty$

**Definition 1.2.2** [9] *Let X be a real Banach space, and let* $A : X \to X^*$ *be an operator A is called hemicontinuous if for all* $u, v \in X$, *the maps* $t \to \langle A(u + tv), v\rangle$ *is continuous from* $\mathbb{R}$ *in* $\mathbb{R}$.

**Definition 1.2.3** [9] *Let X be real Banach space, and let* $A : X \to X^*$ *be an operator. A is called coercive if and only if*

$$\lim_{\|u\|\to\infty} \frac{\langle Au, v\rangle}{\|u\|} = +\infty,$$

## 1.3 Pseudomonotone Operators

**Definition 1.3.1** [9] *The operator* $A : X \to X^*$ *is pseudomonotone if and only if* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0 \text{ implies } Au_n \rightharpoonup Au \text{ and } \langle Au_n, u_n\rangle \to \langle Au, u\rangle\,.$$

**Lemma 1.3.1** [9] *Let* $A, B : X \to X^*$ *be operators on the real reflexive Banach space X. Then the following implications hold*
*(i) If A is monotone and hemicontinuous, then A is pseudomonotone.*
*(ii) If A is strongly continuous, then A is pseudomonotone.*
*(iii) If A and B are pseudomonotone,* $A + B$ *is pseudomonotone.*

**Theorem 1.3.1** [20] *Let X be a real, reflexive Banach space, and let* $A\colon X \to X^*$ *be a pseudomonotone, bounded, and coercive operator,and* $b \in X^*$. *Then a solution of the equation* $Au = b$ *exists.*

For the proof of this theorem see [8], it was proved by using the Galerkin method.it is summarized in the following steps:

**Step1** Solution of Galerkin equations, take a sequence $(e_k)_k$ of linearly independent vectors in $V$, such that setting

$$V_n := span\left\{e_1, ..., e_2\right\},$$

yields $V = \overline{U_n V_n}$. We are looking for a solution $u_n \in V_n$, which is of the form

$$u_n = \sum_{k=1}^{n} c_k^n e_k,$$

and which solves the Galerkin equations

$$\langle A(u_n) - f, e_k\rangle = 0 \text{ for } k \in \{1, ..., n\}.$$

**Step2** A priori estimates, we show that $(u_n)$ is bounded.
**Step3** Weak convergence.
We show that there is a subsequenc $(u_n)$ with

$$u_n \rightharpoonup u \text{ as } n \to \infty.$$

**Step4** We show that $u$ is a solution of the original equation $Au = b$, $u \in X$. see [4]

**Theorem 1.3.2** [7]*(Lebesgue's dominated convergence) Let* $(f_n)$ *be a sequence of functions in* $\mathrm{L}^1(\Omega)$ *that satisfy*

$f_n(x) \to f$ a. e, on $\Omega$, there is a function $g \in L^1(\Omega)$ such that for all $n$,

$$|f_n(x)| \leq g(x), \text{ a. e. on } \Omega.$$

Then

$$f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1} \to 0.$$

**Lemma 1.3.2** [7] *(Fatou's Lemma)*
*Let* $(f_n)$ *a sequence of functions in* $\mathrm{L}^1(\Omega)$ *that satisfy, for all* $n$, $f_n \geq 0$,

$\sup_n \int f_n < \infty$, for almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f(x)\,dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,dx.$$

## 1.4 Nemytskij Operator

**Definition 1.4.1** [9] *(Carathéodory Function) Let* $\Omega \subset \mathbb{R}^N$, $N \geq 1$, *be a nonempty measurable set, and let* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, *and* $u : \Omega \to \mathbb{R}^m$ *The function* $f$ *is called a Carathéodory function if the following two conditions are satisfied*
*(i)* $x \mapsto f(x, s)$ *is measurable in* $\Omega$ *for all* $s \in \mathbb{R}^m$.
*(ii)* $s \mapsto f(x, s)$ *is continuous on* $\mathbb{R}^m$ *for a.e.*$x \in \Omega$.

**Definition 1.4.2** [9] *(Nemytskij Operator) Let* $\Omega \subset \mathbb{R}^N$, $N \geq 1$, *be a nonempty measurable set, and let* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, *and* $u : \Omega \to \mathbb{R}^m$ *be a given function. Then the superposition or Nemytskij operator* $F$ *assigns* $u \mapsto f \circ u$; *i. e.,* $F$ *is given by*

$$Fu(x) = (f \circ u)(x) = f(x, u(x)) \ \textit{for } x \in \Omega.$$

**Lemma 1.4.1** [9] *Let* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, *be a Carathéodory function that satisfies a growth condition of the form*

$$|f(x, s)| \leq k(x) + c\sum_{i=1}^{m} |s_i|^{\frac{p_i}{q}}, \ \forall s = (s_1, ..., s_m) \in \mathbb{R}^m, \ a.\ e.x \in \Omega,$$

*for some positive constant* $c$ *and some* $k \in \mathrm{L}^q(\Omega)$, *and* $1 \leq q$, $p_i < \infty$ *for all* $i = 1, ...., m$. *Then the Nemytskij operator* $F$ *defined by*

$$Fu(x) = f(x, u_1(x), ...., u_m(x)),$$

*is continuous and bounded from* $\mathrm{L}^{p_1}(\Omega) \times ... \times \mathrm{L}^{p_m}(\Omega)$ *into* $\mathrm{L}^q(\Omega)$. *Here* $u$ *denotes the vector function* $u = (u_1, .....u_m)$. *Furthermore,*

$$\|Fu\|_{\mathrm{L}^q(\Omega)} \leq c\left(\|k\|_{\mathrm{L}^q(\Omega)} + \sum_{i=1}^{m} \|u_i\|_{\mathrm{L}^{p_i}(\Omega)}^{\frac{p_i}{q}}\right).$$

## 1.5 Eigenvalue problems

For $1 < p < \infty$, the $p$-Laplacian of a function $f$ on an open bounded domain $\Omega$ is defined by

$$\Delta_p f = \operatorname{div}\left(|\nabla f|^{p-2} \nabla f\right).$$

**Lemma 1.5.1** *Let* $V$ *be a closed subspace of* $\mathrm{W}^{1,p}(\Omega)$ *and* $\mathrm{W}_0^{1,p}(\Omega) \subseteq V \subseteq \mathrm{W}^{1,p}(\Omega)$. *Then it holds*
*(i)* $-\Delta_p : V \to V^*$*is continuous bounded and has the* $(S_+)$*-property. i e, if every*

*sequence* $\{u_n\}_n$ *in* $V$ *such that* $u_n \rightharpoonup u$ *and* $\lim_{n\to\infty}\sup\langle -\Delta_p u_n, u_n - u\rangle \leq 0$ *has a converngent subsequence* $\{u_{n_k}\}_k$ *such that* $u_{n_k} \to u$.
*(ii)* $-\Delta_p : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{-1,q}(\Omega)$ *is*
*a) strictly monotone if* $1 < p < \infty$.
*b) strongly monotone if* $p = 2$.
*c) uniformly monotone if* $2 < p < \infty$.

**Definition 1.5.1** *we say that* $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, *is an eigenfunction of the operator* $-\Delta_p u$ *if:*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi dx = \lambda\int_\Omega |u|^{p-2}u.\varphi dx, \tag{1.6}$$

*for all* $\varphi \in C_0^\infty(\Omega)$. *The corresponding real number* $\lambda$ *is called eigenvalue.*
*Let* $\lambda_{1,p}$ *defined by*

$$\lambda_{1,p} = \inf_{u\in \mathrm{W}_0^{1,p}(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx}, \tag{1.7}$$

*equivalent to*

$$\lambda_{1,p} = \inf\left\{\int_\Omega |\nabla u|^p\, dx : \int_\Omega |u|^p\, dx = 1, u \in \mathrm{W}_0^{1,p}(\Omega), u \neq 0\right\}, \tag{1.8}$$

$\lambda_{1,p}$ *is the first eigenvalue of* $p$*-laplacian operator with homogeneous Dirichlet conditions at the edge.*

## *1.6* Some Inequalities

### Holder's Inequality

Let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, then one has

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p(\Omega)} \times \|v\|_{L^q(\Omega)}.$$

### Monotonicity Inequality

Let $1 < p < \infty$. Consider the vector-valued function $a : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by

$$a(\xi) = |\xi|^{p-2}\xi \text{ for } \xi \neq 0,\ a(0) = 0.$$

If $1 < p < 2$, then we have

$$(a(\xi) - a(\xi')).(\xi - \xi') > 0 \text{ for all } \xi, \xi' \in \mathbb{R}^N,\ \xi \neq \xi'.$$

If $2 \leq p < \infty$, then a constant $c > 0$ exists such that

$$\left(a\left(\xi\right) - a\left(\xi'\right)\right).\left(\xi - \xi'\right) \geq c\left|\xi - \xi'\right|^{p} \text{ for all } \xi \in \mathbb{R}^{N}.$$

**Young's Inequality**
Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^{p}}{p} + \frac{b^{q}}{q} \ \left(a, b \geq 0\right).$$

Chapter 2

# Existence and uniqueness results for double phase problems with convection term

## 2.1 Introduction

In this chapter, we study the existence and uniqueness results for double phase problems with convection term

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)=f\ (x,u,\nabla u), \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

whereas $\Omega$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $1<p<q<N$, the function $\mu\colon \overline{\Omega}\to[0,\infty)$ is supposed to be Lipschitz continuous and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function.

## 2.2 Definition and notations

We give the following two definitions before we give our main result.

**Definition 2.2.1** *Let $X$ be a reflexive Banach space, $X^*$its dual space and denote by $\langle .,.\rangle$ its duality pairing. Let $A\colon X\to X^*$, then*
*(a) $A$ satisfies $(S_+)$-property if $u_n\rightharpoonup u$ in $X$ and $\limsup\limits_{n\to\infty}\langle Au_n,\ u_n-u\rangle\leq 0$ imply $u_n\to u$ in $X$;*
*(b) $A$ is called pseudomonotone operator if $u_n\rightharpoonup u$ in $X$ and $\limsup\limits_{n\to\infty}\langle A(u_n),u_n-u\rangle\leq 0$ imply $Au_n\rightharpoonup Au$ and $\langle Au_n,\ u_n\rangle\to\langle Au,\ u\rangle$.*
*Our existence result is based on the following surjectivity result for pseudomonotone operators, see, e.g. Carl-Le-Motreanu [9].*

**Definition 2.2.2** *We say that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of problem $(1,1)$ if it satisfies*

$$\int\limits_{\Omega} \left( |\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) . \nabla \varphi dx = \int\limits_{\Omega} f(x, u, \nabla u) \varphi dx, \tag{2.2}$$

*for all test functions $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$. by the embedding (1.3) and the fact that $p < q$ along with (1.5)we easily see that a weak solution of (2.2) is well-defined.*

Let $A : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the operator defined by

$$\langle A(u), \varphi \rangle_{\mathcal{H}} := \int\limits_{\Omega} \left( |\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) . \nabla \varphi dx, \tag{2.3}$$

Where $\langle ., . \rangle_{\mathcal{H}}$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$.The properties of the operator $A : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ are summarized in the following proposition, see Liu-Dai [18]

**Proposition 2.2.1** *The operator $A$ defined by (2.3) is bounded, continuous, monotone (hence maximal monotone) and of type $(S_+)$.*

**Proof.**

1) $A$ is bounded. For convenience in writing we set $\lambda_1 := \|u\|$, $\lambda_2 := \|v\|$. By Hölder's inequality and Young's inequality, we have that

$$\begin{aligned}
&\left| \left\langle \tfrac{A(u)}{\lambda_1}, \tfrac{v}{\lambda_2} \right\rangle \right| = \left| \int\limits_{\Omega} \left| \tfrac{\nabla u}{\lambda_1} \right|^{p-2} \tfrac{\nabla u}{\lambda_1} \tfrac{\nabla v}{\lambda_2} dx + \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla u}{\lambda_1} \right|^{q-2} \tfrac{\nabla u}{\lambda_1} \tfrac{\nabla v}{\lambda_2} dx \right|, \\
&\leq \left( \int\limits_{\Omega} \left| \tfrac{\nabla u^p}{\lambda_1} \right| dx \right)^{\frac{p-1}{p}} \left( \int\limits_{\Omega} \left| \tfrac{\nabla v^p}{\lambda_2} \right| dx \right)^{\frac{1}{p}} + \left( \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla u}{\lambda_1} \right|^q dx \right)^{\frac{q-1}{q}} \left( \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla v}{\lambda_2} \right|^q dx \right)^{\frac{1}{q}}, \\
&\leq \tfrac{p-1}{p} \int\limits_{\Omega} \left| \tfrac{\nabla u}{\lambda_1} \right|^p dx + \tfrac{q-1}{q} \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla u}{\lambda_1} \right|^q dx + \tfrac{1}{p} \int\limits_{\Omega} \left| \tfrac{\nabla v}{\lambda_2} \right| + \tfrac{1}{q} \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla v}{\lambda_2} \right|^q dx, \\
&\leq \tfrac{q-1}{q} \left( \int\limits_{\Omega} \left| \tfrac{\nabla u}{\lambda_1} \right|^p dx + \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla u}{\lambda_1} \right|^q dx \right) + \tfrac{1}{p} \left( \int\limits_{\Omega} \left| \tfrac{\nabla v}{\lambda_2} \right|^p + \int\limits_{\Omega} \mu(x) \left| \tfrac{\nabla v}{\lambda_2} \right|^q dx, \right), \\
&\leq \tfrac{q-1}{q} + \tfrac{1}{q} \leq 2.
\end{aligned}$$

Hence, we have that

$$\|A(u)\|_{X^*} = \sup_{\|v\| \leq 1} |\langle A(u), v \rangle| \leq 2 \|u\|_X,$$

which implies that $A$ is bounded.

2) $A$ is continuous

Suppose that $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. For all $v \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|v\| = 1$ by the Hölder inequality,

$$|\langle A(u_j) - A(u), \upsilon\rangle| \leq \left\| |\nabla u_j|^{p-2} \nabla u_j - |\nabla u|^{p-2} \nabla u \right\|_{p'} \|\nabla \upsilon\|_p$$
$$+ \left\| \mu(x) \left| |\nabla u_j|^{q-2} \nabla u_j - |\nabla u|^{q-2} \nabla u \right| \right\|_{q'} \|\nabla \upsilon\|_{q,\mu},$$

Since $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$, $\nabla u_j \to \nabla u$ in $L^p(\Omega) \cap L^q_\mu(\Omega)$, and $\|\nabla \upsilon\|_p, \|\nabla \upsilon\|_{q,\mu}$ are uniformly bounded, according to Theorem (Lebesgue's dominated convergence)

$$\lim_{u_j \to \infty} |\langle A(u_j) - A(u), \upsilon\rangle| \leq 0 \Rightarrow A(u_j) \underset{j \to \infty}{\to} A(u).$$

3) $A$ is monotone
$\forall u, \upsilon \in W_0^{1,\mathcal{H}}(\Omega)$

$$\begin{aligned} &\langle Au - A\upsilon, u - \upsilon\rangle = \int_\Omega (|\nabla u|^p + |\nabla \upsilon|^p) + \mu(x)(|\nabla u|^q + |\nabla \upsilon|^q) dx \\ &-\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \upsilon + |\nabla \upsilon|^{p-2} \nabla \upsilon \nabla u) dx \\ &-\int_\Omega \mu(x)(|\nabla u|^{q-2} \nabla u \nabla \upsilon + |\nabla \upsilon|^{q-2} \nabla \upsilon \nabla u)) dx, \end{aligned} \tag{*}$$

by using inequatity of Young we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \upsilon dx \leq \int_\Omega |\nabla u|^{p-1} |\nabla \upsilon| dx \leq \int_\Omega \left( \frac{|\nabla u|^p}{s} + \frac{|\nabla \upsilon|^p}{p} \right) dx, s = \frac{p}{p-1}.$$

It follows

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \upsilon dx + \int_\Omega |\nabla \upsilon|^{p-2} \nabla \upsilon \nabla u dx \leq \int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla \upsilon|^p dx,$$

$$\int_\Omega \mu(x)(|\nabla u|^{q-2} \nabla u \nabla \upsilon + |\nabla \upsilon|^{q-2} \nabla \upsilon \nabla u)) dx \leq \int_\Omega \mu(x) |\nabla u|^q dx + \int_\Omega \mu(x) |\nabla \upsilon|^q dx,$$

by substitution in $(*)$ finds

$$\langle Au - A\upsilon, u - \upsilon\rangle \geq 0.$$

4) $A$ verifiy $(S_+)$-propriety, assume that $\{u_n\} \subset X$, $u_n \rightharpoonup u$ and

$$\limsup_{n \to +\infty} \langle A(u_n) - A(u), u_n - u\rangle \leq 0$$

■

**A special case**: A special case of the operator $A$ defined by (2.3) occurs when $\mu \equiv 0$. This leads to the operator
$A_p : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ defined by

$$\langle A_p(u), \varphi\rangle_p := \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi dx,$$

where $\langle .,. \rangle_p$ is the duality pairing between $W_0^{1,p}(\Omega)$ and its dual space $W_0^{1,p}(\Omega)^*$. This operator is the well-known p-Laplace differential operator.
Another special case happens when $\mu \equiv 1$, that is, $A_{q,\,p}$ :$W_0^{1,\,q}(\Omega)$ →$W^{1,\,q}(\Omega)^*$ defined by

$$\langle A_{q,p}(u), \varphi \rangle_{q\ p} := \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi dx + \int_\Omega |\nabla u|^{q-2} \nabla u . \nabla \varphi dx, \tag{2.4}$$

where $\langle .,. \rangle_{q\ p}$ stands for the duality pairing between $W^{1,q}(\Omega)$ and its dual space $W_0^{1,q}(\Omega)^*$ , is the so-called $(q,\ p)$-Laplace differential operator.

## $\boxed{2.3}$ Existence result

We suppose the following hypotheses:
(H) $f\ : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that
(i) There exists $\alpha \in L^{\frac{q_1}{q_1-1}}(\Omega)$ and $a_1, a_2 \geq 0$ such that

$$|f(x,s,\xi)| \leq a_1 |\xi|^{p\frac{q_1-1}{q_1}} + a_2 |s|^{q_1-1} + \alpha(x), \tag{2.4}$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$ with the critical exponent $p^*$given in (1.4).
(ii) There exists $\omega \in L^1(\Omega)$ and $b_{1,} b_2 \geq 0$ such that

$$f(x,s,\xi)\, s \leq b_1 |\xi|^p + b_2 |s|^p + \omega(x), \tag{2.5}$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$. Moreover,

$$b_1 + b_2 \lambda_{1,\,p}^{-1} < 1, \tag{2.6}$$

where $\lambda_{1,\,p}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the $p$ -Laplacien.

**Theorem 2.3.1** [20] *Let $1 < p < q < N$ and let hypotheses (1.1) and (H) be satisfied. Then problem (2.1) admits at least one weak solution $u \in \mathrm{W}_0^{1,\mathcal{H}}(\Omega)$.*

**Proof.** Let $\hat{N}_f : W_0^{1,\mathcal{H}}(\Omega) \subseteq L^{q_1}(\Omega) \to L^{q_1'}(\Omega)$ be the Nemytskij operator associated to $f$ and let $i^* : L^{q_1'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the adjoint operator of the embedding $i$ : $W_0^{1,\mathcal{H}}(\Omega) \to L^{q_1}(\Omega)$. For $u \in W_0^{1,\mathcal{H}}(\Omega)$ we define $N_f\ :\ = i^* \circ \hat{N}_f$ and set

$$\mathcal{A}(u) = A(u) - N_f(u). \tag{2.7}$$

From the growth condition on $f$,see (2.4), we easily that $\mathcal{A} : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ maps bounded sets into bounded sets. Let us now prove that $\mathcal{A}$ is pseudomonotone, see Definition 2.2.1(b).To this end, let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega) \text{ and } \limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u \rangle_{\mathcal{H}} \leq 0. \tag{2.8}$$

From the compact embedding in (1.3)we obtain that

$$u_n \to u \text{ in } L^{q_1}(\Omega), \tag{2.9}$$

since $q_1 < p^*$. Using the strong convergence in $L^{q_1}(\Omega)$, see (2.9), along with Hölder's inequality and the growth condition on $f$ we obtain

$$\lim_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,dx = 0.$$

Therefore, we can pass to the limit in the weak formulation in (2.2) replacing $u$ by $u_n$ and $\varphi$ by $u_n - u$. This gives

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_{\mathcal{H}} = \limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u\rangle_{\mathcal{H}} \le 0. \tag{2.10}$$

From Proposition 2.2.1 we know that $A$ fulfills the $(S_+)$-property and so we conclude, in view of (2.8) and (2.10), that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Hence, because of the continuity of $\mathcal{A}$, we have that $\mathcal{A}(u_n) \to \mathcal{A}(u)$ in $W_0^{1.\mathcal{H}}(\Omega)^*$ which proves that $\mathcal{A}$ is pseudomonoton.
Next we show that the operator $\mathcal{A}$ is coercive, that is,

$$\lim_{\|u\|_{1,\mathcal{H},0}\to\infty} \frac{\langle \mathcal{A}u, u\rangle_{\mathcal{H}}}{\|u\|_{1.\mathcal{H},0}} = +\infty. \tag{2.11}$$

From the representation of the first eigenvalue of the $p$-Laplacian, see (1.8), replacing $r$ by $p$, we have the inequality

$$\|u\|_p^p \le \lambda_{1.p}^{-1} \|\nabla u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega). \tag{2.12}$$

Since $W_0^{1,\mathcal{H}}(\Omega) \subseteq W_0^{1,p}(\Omega)$ and by applying (2.12), (2.5) and (1.2) we derive

$$\begin{aligned}
&\langle \mathcal{A}(u), u) = \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right).\nabla u dx - \int_\Omega f(x,u,\nabla u)\,u dx \\
&\ge \|\nabla u\|_p^p + \|u\|_{q,\mu}^q - b_1\|\nabla u\|_p^p - b_2\|u\|_p^p - \|\omega\|_1 \\
&\ge \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\|\nabla u\|_p^p + \|u\|_{q,\mu}^q - \|\omega\|_1 \\
&\ge \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\left(\|\nabla u\|_p^p + \|u\|_{q,\mu}^q\right) - \|\omega\|_1 \\
&\ge \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\min\left\{\|u\|_{1,\mathcal{H},0,}^p, \|u\|_{1,\mathcal{H},0}^q\right\} - \|\omega\|_1.
\end{aligned}$$

Therefore, since $1 < p < q$ and (2.6), it follows (2.11) and thus, the operator $\mathcal{A}\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ is coercive. Hence, the operator $\mathcal{A}\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ is bounded, pseudomonotone and coercive. Then Theorem 1.3.1 provides $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that $\mathcal{A}(u) = 0$. By the definition of $\mathcal{A}$, see (2.7), the function $u$ turns out to be a weak solution of problem (2.1) which completes the proof. ■

**Example 2.3.1** *The following function satisfies hyootheses* $(H)$*, where for simplicity we drop the* $x$*-dependence*

$$f(s,\xi) = -d_1 |s|^{q_1-2} s + d_2 |\xi|^{p-1} \text{ for all } s \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N,$$

with $1 < q_1 < p^*, d_1 \geq 0$ and

$$0 \leq d_2 < \frac{p}{p-1+\lambda_{1,p}^{-1}}.$$

## 2.4 Uniqueness result

Let us now give sufficient conditions on the perturbation such that problem (1.2) has a unique weak solution. To this end, we need the following stronger conditions on the convection term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.
$(U_1)$ There exists $c_1 \geq 0$ such that

$$\left(f(x,s,\xi) - f(x,t,\xi)\right)(s-t) \leq c_1 |s-t|^2,$$

for a. a. $x \in \Omega$, for all $s,t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.
$(U_2)$ There exists $p \in L^{r'}(\Omega)$ with $1 < r' < p^*$ and $c_2 \geq 0$ such that $\xi \longmapsto f(x;s,\xi) - \rho(x)$ is linear for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and

$$|f\ \ (x,s,\xi) - \rho(x)| \leq c_2 |\xi|,$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$. Moreover,

$$c_1 \lambda_{1,2}^{-1} + c_2 \lambda_{1,2}^{-\frac{1}{2}} < 1, \tag{2.13}$$

where $\lambda_{1,2}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the Laplace differential operator.

**Theorem 2.4.1** [*20*] *Let* (1.1)*,* *(H)* *,* $(U_1)$*, and* $(U_2)$ *be satisfied and let* $2 = p < q < N$*. Then, problem* (2.1) *admits a unique weak solution.*

**Proof.** Let $u,v \in W_0^{1,\mathcal{H}}(\Omega)$ be two weak solutions of (2.1). Taking in both weak formulations the test function $\varphi = u - v$ and subtracting these equations result in

$$\begin{aligned} &\int_\Omega |\nabla(u-v)|^2 dx + \int_\Omega \mu(x) \left(|\nabla u|^{q-2} \nabla u - |\nabla v|^{q-2} \nabla v\right).\nabla(u-v)\,dx \\ &= \int_\Omega \left(f(x,u,\nabla u) - f(x,v,\nabla u)\right)(u-v)\,dx + \int_\Omega \left(f(x,v,\nabla u) - f(x,v,\nabla v)\right)(u-v)\,dx, \end{aligned} \tag{2.14}$$

sine the second term on the left-hand side of (2.14) is nonnegative, we have the simple estimate

$$\begin{aligned} &\int_\Omega \left|\nabla (u-v)\right|^2 dx + \int_\Omega \mu(x)\left(\left|\nabla u\right|^{q-2}\nabla u - \left|\nabla v\right|^{q-2}\nabla u\right).\nabla(u-v)\,dx \\ &\geq \int_\Omega \left|\nabla (u-v)\right|^2 dx. \end{aligned} \tag{2.15}$$

The right-hand side of (2.14) can be estimated via $(U_1)$, $(U_2)$ and Hölder's inequality

$$\begin{aligned} &\int_\Omega \left(f(x,u,\nabla u) - f(x,v,\nabla u)\right)(u-v)\,dx + \int_\Omega \left(f(x,v,\nabla u) - f(x,v,\nabla v)\right)(u-v)\,dx \\ &\leq c_1\left\|u-v\right\|_2^2 + \int_\Omega \left(f\left(x,v,\nabla\left(\tfrac{1}{2}(u-v)^2\right)\right) - \rho(x)\right)dx \\ &\leq c_1\left\|u-v\right\|_2^2 + c_2\int_\Omega \left|u-v\right|\left|\nabla(u-v)\right|dx \\ &\leq \left(c_1\lambda_{1,2}^{-1} + c_2\lambda_{1,2}^{\frac{-1}{2}}\right)\left\|\nabla(u-v)\right\|_2^2. \end{aligned} \tag{2.16}$$

Combining (2.14), (2.15) and (2.16) gives

$$\left\|\nabla(u-v)\right\|_2^2 = \int_\Omega \left|\nabla(u-v)\right|^2 dx \leq \left(c_1\lambda_{1,2}^{-1} + c_2\lambda_{1,2}^{\frac{-1}{2}}\right)\left\|\nabla(u-v)\right\|_2^2. \tag{2.17}$$

Then, by (2.13) and (2.17), we get that $u = v$. ■

**Example 2.4.1** *The following function satisfies hypotheses* $(H)$, $(U_1)$ *and* $(U_2)$, *where for simplicity we drop the* $s$*-dependence,*

$$f(x,\xi) = \sum_{i=1}^{N}\beta_i\xi_i + \rho(x) \ \textit{for a. a. } x\in\Omega \textit{ and for all } \xi\in\mathbb{R}^N,$$

*with* $2 = p \leq q_1 < 2^*$, $\rho \in \mathrm{L}^2(\Omega)$ *and*

$$\left\|\beta\right\|_{\mathbb{R}^N}^2 < \min\left\{1 - \frac{1}{2}\lambda_{1,2}^{-1}, \lambda_{1,2},\right\}$$

where $\beta = (\beta_1, \beta_2, \ldots \beta_N) \in \mathbb{R}^N$.

Chapter

# 3

# Existence and uniqueness of elliptic systems with double phase operators and convection terms

## 3.1 Introduction

In this chapter, we are concerned with the existence and uniqueness of elliptic systems with double phase operators and convection term

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p_1-2}\nabla u+\mu_1(x)|\nabla u|^{q_1-2}\nabla u\right)=f_1(x,u,\upsilon,\nabla u,\nabla\upsilon) \text{ in } \Omega \\ -\operatorname{div}\left(|\nabla \upsilon|^{p_2-2}\nabla \upsilon+\mu_2(x)|\nabla \upsilon|^{q_2-2}\nabla \upsilon\right)=f_2(x,u,\upsilon,\nabla u,\nabla\upsilon) \text{ in } \Omega \\ u=\upsilon=0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $1<p_i<q_i<N$, $\mu_i:\overline{\Omega}\to[0,\infty)$ are Lipschitz continuous and $f_i:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ are Carathéodory function.

## 3.2 Definitions and notations

We give the following definition before we give our main result.

**Definition 3.2.1** *We say that* $(u,\upsilon)\in \mathrm{W}_0^{1,\mathcal{H}_1}(\Omega)\times\mathrm{W}_0^{1,\mathcal{H}_2}(\Omega)$ *is a weak solution of problem* (3.1) *if*

$$\begin{gathered} \int_\Omega\left(|\nabla u|^{p_1-2}\nabla u+\mu_1(x)|\nabla u|^{q_1-2}\nabla u\right).\nabla\varphi dx=\int_\Omega f_1(x,u,\upsilon,\nabla u,\nabla\upsilon)\varphi dx \\ \int_\Omega\left(|\nabla \upsilon|^{p_2-2}\nabla \upsilon+\mu_2(x)|\nabla \upsilon|^{q_2-2}\nabla \upsilon\right).\nabla\psi dx=\int_\Omega f_2(x,u,\upsilon,\nabla u,\nabla\upsilon)\psi dx, \end{gathered} \tag{3.2}$$

*is satisfied for all test functions* $(\varphi,\psi)\in \mathrm{W}_0^{1,\mathcal{H}_1}(\Omega)\times \mathrm{W}_0^{1,\mathcal{H}_2}(\Omega)$. *Taking the embedding* (1.3) *into account, along with the growth conditions on* $f_1 and f_2$ *, we see that the definition of a weak solution is well defined.*

Our existence result is based on the following surjectivity result for pseudomonotone operators, see, e.g., Carl-Le-Motreanu[5],or Papageorgiou-Winkert [27].

We consider the space $W := W^{1,\mathcal{H}_1}(\Omega)\times W^{1,\mathcal{H}_2}(\Omega)$ endowed with the norm

$$\|(u,v)\|_W := \|u\|_{1,\mathcal{H}_1,0}+\|v\|_{1,\mathcal{H}_2,0}\,,$$

for every $(u,v)\in W_0^{1,\mathcal{H}_1}(\Omega)\times W_0^{1,\mathcal{H}_2}(\Omega)$.

Then we consider the operator

$$A: W^{1,\mathcal{H}_1}(\Omega)\times W^{1,\mathcal{H}_2}(\Omega)\to \left(W^{1,\mathcal{H}_1}(\Omega)\right)^*\times\left(W^{1,\mathcal{H}_2}(\Omega)\right)^*,$$

defined by

$$\begin{aligned}&\langle A(u,v),(\varphi,\psi)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} := \int_\Omega \left(|\nabla u|^{p_1-2}\nabla u+\mu_1(x)|\nabla u|^{q_1-2}\nabla u\right).\nabla\varphi dx\\ &+\int_\Omega\left(|\nabla v|^{p_2-2}\nabla v+\mu_2(x)|\nabla v|^{q_2-2}\nabla v\right).\nabla\psi dx.\end{aligned}\tag{3.3}$$

Where $\langle .,.\rangle_{\mathcal{H}_1\times\mathcal{H}_2}$ is the duality pairing between $W^{1,\mathcal{H}_1}(\Omega)\times W^{1,\mathcal{H}_2}(\Omega)$ and its dual space $\left(W^{1,\mathcal{H}_1}(\Omega)\right)^*\times\left(W^{1,\mathcal{H}_2}(\Omega)\right)^*$. Then next result summarizes the properties of the operator $A$.

**Lemma 3.2.1** *Let* $A:\mathrm{W}_0^{1,\mathcal{H}_1}(\Omega)\times\mathrm{W}_0^{1,\mathcal{H}_2}(\Omega)\to\left(\mathrm{W}_0^{1,\mathcal{H}_1}(\Omega)\right)^*\times\left(\mathrm{W}_0^{1,\mathcal{H}_2}(\Omega)\right)^*$ *be the operator defined by* (3.3). *Then,* $A$ *is bounded, continuous, monotone (hence maximal monotone), and of type* $(S_+)$. *The proof to the one in Liu-Dai* [18]

## 3.3 Existence result

We assume the following hypotheses on the nonlinearities $f_1, f_2$.

$(H)$ $f_1, f_2:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ are Carthéodory functions such that

$(i)$ There exist $\alpha_i\in L^{\frac{r_i}{r_i-1}}(\Omega)$ $(i=1,2)$ such that

$$|f_1(x,s,t,\xi,\zeta)|\le A_1|s|^{a_1}+A_2|t|^{a_2}+A_3|s|^{a_3}|t|^{a_4}+A_4|\xi|^{a_5}+A_5|\zeta|^{a_6}+A_6|\xi|^{a_7}|\zeta|^{a_8}+|\alpha_1(x)|,$$

$$|f_2(x,s,t,\xi,\zeta)|\le B_1|s|^{b_1}+B_2|t|^{b_2}+B_3|s|^{b3}|t|^{b_4}+B_4|\xi|^{b_5}+B_5|\zeta|^{b6}+B_6|\xi|^{b_7}|\zeta|^{b_8}+|\alpha_1(x)|,$$

for a. a. $x\in\Omega$, for all $s,t\in\mathbb{R}$ and for all $\xi,\zeta\in\mathbb{R}^N$, where $Aj$, $B_j$, $j=1,...6$, are nonnegative constants and with $1<r_i<p_i^*$, $i=1,2$. Moreover, the exponents

$a_\ell, b_\ell,\ \ell = 1, ..., 8$, are nonnegative and satisfy the following conditions

$$\begin{array}{l}
(E_1)\ \ a_1 \leq r_1 - 1,\ \ (E_2)\ \ a_2 \leq \frac{r_1-1}{r_1} r_2, \\
(E_3)\ \ \frac{a_3}{r_1} + \frac{a_4}{r_2} \leq \frac{r_1-1}{r_1},\ \ (E_4)\ \ a_5 \leq \frac{r_1-1}{r_1} p_1, \\
(E_5)\ \ a_6 \leq \frac{r_1-1}{r_1} p_2,\ \ (E_6)\ \ \frac{a_7}{p_1} + \frac{a_8}{p_2} \leq \frac{r_1-1}{r_1}, \\
(E_6)\ \ b_1 \leq \frac{r_2-1}{r_2} r_1,\ \ (E_8)\ \ b_2 \leq r_2 - 1, \\
(E_9)\ \frac{b_3}{r_1} + \frac{b_4}{r_2} \leq \frac{r_2-1}{r_2},\ \ (E_{10})\ \ b_5 \leq \frac{r_2-1}{r_2} p_1, \\
(E_{11})\ \ b_2 \leq \frac{r_2-1}{r_2} p_2,\ \ (E_{12})\ \ \frac{b_7}{p_1} + \frac{b_8}{p_2} \leq \frac{r_2-1}{r_2}.
\end{array}$$

$(ii)$ There exist $\omega \in L^1(\Omega)$ and $\Lambda, \Gamma \geq 0$ such that

$$f_1(x, s, t, \xi, \zeta)\, s + f_2(x, s, t, \xi, \zeta)\, t \leq \Lambda \left(|\xi|^{p_1} + |\zeta|^{p_2}\right) + \Gamma \left(|s|^{p_1} + |t|^{p_2}\right) + \omega(x), \tag{3.4}$$

for a. a. $x \in \Omega$, for all $s, t \in \mathbb{R}$ and for all $\xi, \zeta \in \mathbb{R}^N$ and with

$$\Lambda + \Gamma \max\left\{\lambda_{1,p_1}^{-1}, \lambda_{1,p_2}^{-1}\right\} < 1, \tag{3.5}$$

where $\lambda_{1,p_i}$ is the first eigenvalue of the $p_i$-Laplacian, see (1.6).

Let us consider, for example, the third term on the right-hand side of the growth of $f_1$. Applying Hölder's inequality we get

$$A_3 \int\limits_\Omega |u|^{a_3} |v|^{a_4} \varphi dx \leq A_3 \int\limits_\Omega \left(\int\limits_\Omega |u|^{a_3 s_1} dx\right)^{\frac{1}{s_1}} \left(\int\limits_\Omega |v|^{a_4 s_2} dx\right)^{\frac{1}{s_2}} \left(\int\limits_\Omega |\varphi|^{s_3} dx\right)^{\frac{1}{s_3}}, \tag{3.6}$$

where $(u, v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega),\ \varphi \in W_0^{1,\mathcal{H}_1}(\Omega)$ and

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1.$$

Taking $s_3 = r_1$ with $1 < r_1 < p_i^*$ and using $s_1 \leq \frac{r_1}{a_3}$ as well as $s_2 \leq \frac{r_2}{a_4}$ leads to

$$\frac{a_3}{r_1} + \frac{a_4}{r_2} \leq \frac{r_1 - 1}{r_1},$$

which is exactly condition $(E3)$. Note that the conditions in $(H)\,(i)$ are chosen in order to prove our main results by applying the compact embedding (1.3). Of course, for the finiteness of the integrals in the weak formulation (3.2), we can also allow critical growth to have a well defined weak formulation. Now we are ready to formulate and prove our main result in this section.

**Theorem 3.3.1** [*26*] *Let $1 < p_i < q_i < N$, $i = 1, 2$, and let hypotheses* (1.2) *and* $(H)$ *be satisfied. Then, there exists a weak solution $(u, v) \in \mathrm{W}_0^{1,\mathcal{H}_1}(\Omega) \times \mathrm{W}_0^{1,\mathcal{H}_2}(\Omega)$ of problem* (3.1).

**Proof.** Let

$$\hat{N}_{f_i} : W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to L^{r_1'}(\Omega) \times L^{r_1'}(\Omega),$$

be the Nemytskij operator associated to $f_i$. Moreover, let

$$j_i^* : L^{r_1'}(\Omega) \times L^{r_2'}(\Omega) \to \left(W_0^{1.\mathcal{H}_1}(\Omega)\right)^* \left(W_0^{1.\mathcal{H}_2}(\Omega)\right)^*,$$

be the adjoint operator for the embedding

$$j_i : W_0^{1.\mathcal{H}_1}(\Omega) \times W_0^{1.\mathcal{H}_2}(\Omega) \to L^{r_1}(\Omega) \times L^{r_2}(\Omega).$$

We then define

$$N_{f_i} := j_i^* \circ \hat{N}_{f_i} : W_0^{1.\mathcal{H}_1}(\Omega) \times W_0^{1.\mathcal{H}_2}(\Omega) \to \left(W_0^{1.\mathcal{H}_1}(\Omega)\right)^* \left(W_0^{1.\mathcal{H}_2}(\Omega)\right)^*,$$

which is well defined by hypotheses $(H)\,(i)$. We set

$$\mathcal{A}(u,v) := A(u,v) - N_{f_1}(u,v) - N_{f_2}(u,v). \tag{3.7}$$

Our aim is to apply Theorem 1.3.1, so, we need to show that $\mathcal{A}$ is bounded, pseudomonotone and coercive.

**1**) $\mathcal{A}$ is bounded

The boundedness of $\mathcal{A}$ follows directly from the boundedness of $A$ and the growth conditions on $f_1$and $f_2$ stated in (H) (i).

**2**) $\mathcal{A}$ is pseudomonotone.

To this end, let $\{(u_n, v_n)\}_{n\in N} \subset W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u,v) \text{ in } W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega), \tag{3.8}$$

and

$$\limsup_{n\to\infty} \langle \mathcal{A}(u_n, v_n), (u_n - u, v_n - v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} < 0. \tag{3.9}$$

Taking the compact embedding (1.3) into account yields

$$u_n \to u \text{ in } L^{r_1}(\Omega) \text{ and } v_n \to v \ in\ L^{r_2}(\Omega), \tag{3.10}$$

since $r_1 < p_1^*$ and $r_2 < p_2^*$, respectively. We want to show that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega f_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u)\,dx = 0, \\ &\lim_{n\to\infty} \int_\Omega f_2(x, u_n, v_n, \nabla u_n, \nabla v_n)(v_n - v)\,dx = 0. \end{aligned} \tag{3.11}$$

Let us consider the first expression in (3.11). By the growth condition (H) (i) it follows

$$\begin{aligned} &\int_\Omega f_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u)\,dx \\ &\le \int_\Omega (A_1 |u_n|^{a_1} + A_2 |v_n|^{a_2} + A_3 |u_n|^{a_3} |v_n|^{a_4} + A_4 |\nabla u_n|^{a_5} \\ &+ A_5 |\nabla v_n|^{a_6} + A_6 |\nabla u_n|^{a_7} |\nabla v_n|^{a_8} + |\alpha_1(x)|\,|u_n - u|\,dx. \end{aligned} \tag{3.12}$$

Applying Hölder's inequality, (3.10) and condition $(E_1)$ and $(E_2)$, respectively, we obtain

$$\begin{aligned} &A_1 \int_\Omega |u_n|^{a_1} |u_n - u| \, dx \leq A_1 \left( \int_\Omega |u_n|^{a_1 r_1'} \, dx \right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\leq C_1 \left( 1 + \|u_n\|_{r_1}^{r_1 - 1} \right) \|u_n - u\|_{r_1} \to 0, \end{aligned}$$

and

$$\begin{aligned} &A_2 \int_\Omega |\upsilon_n|^{a_2} |u_n - u| \, dx \leq A_2 \left( \int_\Omega \upsilon_n^{a_2 r_1'} dx \right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\leq C_2 \left( 1 + \|\upsilon_n\|_{r_2}^{\frac{r_2}{r_1'}} \right) \|u_n - u\|_{r_1} \to 0, \end{aligned}$$

for some $C_1, C_2 > 0$. Moreover, Hölder's inequality with exponents $x_1, y_1, z_1 > 1$ such that

$$x_1 a_3 \leq r_1, \ y_1 a_4 \leq r_2, \ z_1 = r_1, \ \frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{z_1} = 1,$$

gives, by hypothesis $(E_3)$,

$$A_3 \int_\Omega |u_n|^{a_3} |\upsilon_n|^{a_4} |u_n - u| \, dx \leq A_3 \|u_n\|_{a_3 x_1}^{a_3} \|\upsilon_n\|_{a_4 y_1}^{a_4} \|u_n - u\|_{r_1} \to 0.$$

Next we apply Hölder's inequality with exponents $r_1, r_1'$ and use $(E_4)$ and $(E_5)$ to get

$$\begin{aligned} &A_4 \int_\Omega |\nabla u_n|^{a_5} |u_n - u| \, dx \leq A_4 \left( \int_\Omega |\nabla u_n|^{a_5 r_1'} \, dx \right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\leq C_3 \left( 1 + \|\nabla u_n\|_{p_1}^{\frac{p_1}{r_1'}} \right) \|u_n - u\|_{r_1} \to 0, \end{aligned}$$

and

$$\begin{aligned} &A_5 \int_\Omega |\nabla \upsilon_n|^{a_6} |u_n - u| \, dx \leq A_5 \left( \int_\Omega |\nabla \upsilon_n|^{a_6 r_1'} \, dx \right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\leq C_4 \left( 1 + \|\nabla \upsilon_n\|_{p_2}^{\frac{p_2}{r_1'}} \right) \|u_n - u\|_{r_1} \to 0, \end{aligned}$$

for some $C_3, C_4 > 0$. Furthermore, condition $(E_6)$ allows us to apply Hölder's inequality with exponents $x_2, y_2, z_2 > 1$ such that

$$x_2 a_7 \leq p_1, \ y_2 a_8 \leq p_2, \ z_2 = r_1, \ \frac{1}{x_2} + \frac{1}{y_2} + \frac{1}{z_2} = 1,$$

in order to have

$$A_6 \int_\Omega |\nabla u_n|^{a_7} |\nabla \upsilon_n|^{a_8} (u_n - u) \, dx \leq A_6 \|u_n\|_{a_7 x_2}^{a_7} \|\nabla \upsilon_n\|_{a_8 y_2}^{a_8} \|u_n - u\|_{r_1} \to 0.$$

Since both $\|\nabla u_n\|_{a_7 x_2}$ and $\|\nabla v_n\|_{a_8 y_2}$ are bounded. Finally, for the last term in (3.12) we have

$$\int\limits_{\Omega} |\alpha_1(x)| (u_n - u)\, dx \leq \|\alpha_1\|_{r_1'} \|u_n - u\|_{r_1} \to 0.$$

Combining all the calculations above give

$$\lim_{n\to\infty} \int\limits_{\Omega} f_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u)\, dx = 0.$$

Applying similar arguments proves that

$$\lim_{n\to\infty} \int\limits_{\Omega} f_2(x, u_n, v_n, \nabla u_n, \nabla v_n)(v_n - v)\, dx = 0.$$

Hence, (3.11) is fulfilled. We now take the weak formulation (3.2), replace $u$ by $u_n$, $v$ by $v_n$, $\varphi$ by $u_n - u$ and $\psi$ by $v_n - v$ and use (3.9) as well as (3.11) in order to have

$$\limsup_{n\to\infty} \langle A(u_n, v_n), (u_n - u, v_n - v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} = \limsup_{n\to\infty} \langle \mathcal{A}(u_n, v_n), (u_n - u, v_n - v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} \leq 0. \tag{3.13}$$

Since $A$ satisfies the $(S_+)$-property, see Lemma 3.2.1, we derive from (3.8) and (3.13) that

$$(u_n, v_n) \to (u, v) \text{ in } W_0^{1.\mathcal{H}_1}(\Omega) \times W_0^{1.\mathcal{H}_2}(\Omega).$$

Since $\mathcal{A}$ is continuous we have $\mathcal{A}(u_n, v_n) \to \mathcal{A}(u, v)$ in $\left(W_0^{1.\mathcal{H}_1}(\Omega)\right)^* \times \left(W_0^{1.\mathcal{H}_2}(\Omega)\right)^*$, which proves that $\mathcal{A}$ is pseudomonotone.
**3**) $\mathcal{A}$ is coercive.
First of all taking into account the representation(1.8) and replacing $r$ by $p_1$ and $p_2$, respectively, we have

$$\|u\|_{p_1}^{p_1} \leq \lambda_{1,p_1}^{-1} \|\nabla u\|_{p_1}^{p_1} \text{ and } \|v\|_{p_2}^{p_2} \leq \lambda_{1,p_2}^{-1} \|\nabla v\|_{p_2}^{p_2}, \tag{3.14}$$

for all $(u, v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$. Note that $W_0^{1,\mathcal{H}_1}(\Omega) \subseteq W_0^{1,p_1}(\Omega)$ and $W_0^{1,\mathcal{H}_2}(\Omega) \subseteq W_0^{1,p_2}(\Omega)$

Applying these facts along with (3.14), (3.4), and (1.4) leads to

$$\begin{aligned}
&\langle \mathcal{A}(u,\upsilon),(u,\upsilon)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} = \int_\Omega \left(|\nabla u|^{p_1-2}\nabla u + \mu_1(x)|\nabla u|^{q_1-2}\nabla u\right).\nabla u dx\\
&+\int_\Omega \left(|\nabla \upsilon|^{p_1-2}\nabla \upsilon + \mu_2(x)|\nabla \upsilon|^{q_2-2}\nabla \upsilon\right).\nabla \upsilon dx\\
&-\int_\Omega f_1(x,u,\upsilon,\nabla u,\nabla \upsilon)u dx - \int_\Omega f_2(x,u,\upsilon,\nabla u,\nabla \upsilon)\upsilon dx\\
&\geq \|\nabla u\|_{p_1}^{p_1} + \|\nabla u\|_{q_1,\mu_1}^{q_1} + \|\nabla \upsilon\|_{p_2}^{p_2} + \|\nabla \upsilon\|_{q_2,\mu_2}^{q_2}\\
&-\Lambda\left(\|\nabla u\|_{p_1}^{p1} + \|\nabla \upsilon\|_{p_2}^{p_2}\right) - \Gamma\left(\|u\|_{p_1}^{p_1} + \|\upsilon\|_{p_2}^{p_2}\right) - \|\omega\|_1\\
&\geq \left(1-\Lambda-\Gamma\lambda_{1,p_1}^{-1}\right)\|\nabla u\|_{p_1}^{p_1} + \|\nabla u\|_{q_1,\mu_1}^{q_1}\\
&+\left(1-\Lambda-\Gamma\lambda_{1,p_2}^{-1}\right)\|\nabla \upsilon\|_{p_2}^{p_2} + \|\nabla \upsilon\|_{q_2,\mu_2}^{q_2} - \|\omega\|_1\\
&\geq \left(1-\Lambda-\Gamma\max\left\{\lambda_{1,p_1}^{-1},\lambda_{1,p_2}^{-1}\right\}\right)\left(\min\left\{\|u\|_{1,\mathcal{H}_1,0}^{p_1},\|u\|_{1,\mathcal{H}_1,0}^{q_1}\right\}\right)\\
&+\min\left\{\|\upsilon\|_{1,\mathcal{H}_2,0}^{p_2},\|\upsilon\|_{1,\mathcal{H}_2,0}^{q_2}\right\} - \|\omega\|_{L^1(\Omega)}.
\end{aligned}$$

Since $1 < p_i < q_i$ and condition (3.5) holds, it follows that $\mathcal{A}$ is coercive.
From the Claims 1-3 we see that $\mathcal{A}$ is bounded, pseudomonotone and coercive. Therefore, by Theorem 1.3.1, there exists $(u,\upsilon) \in W_0^{1,H_1}(\Omega)\times W_0^{1,H_2}(\Omega)$ such that $\mathcal{A}(u,\upsilon) = 0$. Taking into account the definition of $\mathcal{A}$, see equation (3.7), it follows that $(u,\upsilon)$ is a weak solution of problem (3.1). That finishes the proof. ■

## 3.4 Uniqueness result

Now we consider the uniqueness of solutions of (3.1). To this end, let $f : \Omega\times\mathbb{R}^2\times\left(\mathbb{R}^N\right)^2 \to \mathbb{R}^2$ be the vector field defined by:

$$f(x,s,\xi) = (f_1(x,s,\xi), f_2(x,s,\xi)),$$

for a.a. $x\in\Omega$, for all $s\in\mathbb{R}^2$ and for all $\xi\in\left(\mathbb{R}^N\right)^2$. We suppose the following conditions on $f$ :
$(U_1)$ There exists $c_1 \geq 0$ such that

$$(f(x,s,\xi) - f(x,t,\xi)).(s-t) \leq c_1|s-t|^2,$$

for a.a. $x\in\Omega$, for all $s,t\in\mathbb{R}^2$ and for all $\xi\in\left(\mathbb{R}^N\right)^2$.
$(U_2)$ There exist $\rho = \left(\rho_{1,}\rho_2\right)$ with $\rho_i \in L^{s_i}(\Omega)$, $1 < s_i < p_i^*$ and $c_2 \geq 0$ such that $f(x,s,.) - \rho(x)$ is linear on $\left(\mathbb{R}^N\right)^2$ for a.a. $x\in\Omega$, and for all $s\in\mathbb{R}^2$ and

$$|f(x,s,\xi) - \rho(x)| \leq c_2|\xi|,$$

for a.a. $x\in\Omega$, for all $s\in\mathbb{R}^2$ and for all $\xi\in\left(\mathbb{R}^N\right)^2$.

**Theorem 3.4.1** [26]*Let* (1.2) , $(H)$ , $(U_1)$, *and* $(U_2)$ *be satisfied. If* $2 = p_i \leq q_i \leq N$ *for* $i = 1, 2$ *and*

$$c_1\lambda_{1,2}^{-1} + c_2\left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}} < 1, \tag{3.15}$$

*then there exists a unique weak solution of problem* (3.1) .

**Proof.** Let $u = (u_1, u_2)\,, v = (v_1, v_2) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ be two weak solutions of (3.1). Considering the weak formulation for $u$ and $v$, choosing $\varphi = u_1 - v_1$ as well as $\psi = u_2 - v_2$ and subtracting the related equations gives

$$\begin{aligned}
&\int_\Omega |\nabla(u_1 - v_1)|^2 dx + \int_\Omega |\nabla(u_1 - v_1)|^2 dx\\
&+\int_\Omega \mu_1(x)\left(|\nabla u_1|^{q_1-2}\nabla u_1 - |\nabla v_1|^{q_1-2}\nabla v_1\right).\nabla(u_1 - v_1)\,dx\\
&+\int_\Omega \mu_2(x)\left(|\nabla u_2|^{q_2-2}\nabla u_2 - |\nabla v_2|^{q_2-2}\nabla v_2\right).\nabla(u_2 - v_2)\,dx\\
&= \int_\Omega (f(x, u, \nabla u) - f(x, v, \nabla v)).(u - v)\,dx\\
&+\int_\Omega (f(x, v, \nabla u) - \rho(x) - f(x, v, \nabla v) + \rho(x)).(u - v)\,dx.
\end{aligned} \tag{3.16}$$

By the monotonicity of $\xi \mapsto |\xi|^{q_i-2}\xi$ we see that the third and the fourth integral on the left hand side of (3.16) are nonnegative, that is,

$$\begin{aligned}
&\int_\Omega |\nabla(u_1 - v_1)|^2 dx + \int_\Omega |\nabla(u_2 - v_2)|^2 dx\\
&+\int \mu_1(x)\left(|\nabla u_1|^{q_1-2}\nabla u_1 - |\nabla v_1|^{q_1-2}\nabla v_1\right).\nabla(u_1 - v_1)\,dx\\
&+\int_\Omega \mu_2(x)\left(|\nabla u_2|^{q_2-2}\nabla u_2 - |\nabla v_2|^{q_2-2}\nabla v_2\right).\nabla(u_2 - v_2)\,dx\\
&\int_\Omega |\nabla(u_1 - v_1)|^2 dx + \int_\Omega |\nabla(u_2 - v_2)|^2 dx\\
&= \|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2 .
\end{aligned} \tag{3.17}$$

On the other side, by applying $(U_1)$ to the first integral on the right hand side of (3.16) and $(U_2)$ to the second we obtain along with Hölder's inequality

$$\begin{aligned}
&\int_\Omega (f(x, u, \nabla u) - f(x, v, \nabla u)).(u - v)\,dx\\
&+\int_\Omega (f(x, v, \nabla u) - \rho(x) - f(x, v, \nabla v) + \rho(x)).(u - v)\,dx\\
&\leq c_1\left(\|u_1 - v_1\|_2^2 + \|u_2 - v_2\|_2^2\right)\\
&+\int_\Omega (f_1(x, v_1, v_2, (u_1 - v_1)\nabla(u_1 - v_1), (u_1 - v_1)\nabla(u_2 - v_2)) - \rho_1(x))dx\\
&+\int_\Omega (f_2(x, v_1, v_2, (u_2 - v_2)\nabla(u_1 - v_1), (u_2 - v_2)\nabla(u_2 - v_2)) - \rho_2(x))\,dx\\
&\leq c_1\lambda_{1,2}^{-1}\left(\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2\right)\\
&+c_2\int_\Omega (|u_1 - v_1| + |u_2 - v_2|)\left(|\nabla(u_1 - v_2)|^2 + |\nabla(u_2 - v_2)|^2\right)^{\frac{1}{2}} dx\\
&\leq \left(c_1\lambda_{1,2}^{-1} + c_2\left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}}\right)\left(\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2\right).
\end{aligned} \tag{3.18}$$

Combining (3.16) , (3.17) and (3.18) gives

$$\begin{aligned} &\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2 \\ &\leq \left(c_1 \lambda_{1,2}^{-1} + c_2 \left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}}\right)\left(\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2\right). \end{aligned} \tag{3.19}$$

Taking (3.15) into account, we see from (3.19)that $u_1 = v_1$ and $u_2 = v_2$ and so the solution of (3.1) is unique ■

# Conclusion

In this memoir, we studied the existence and uniqueness of quasilinear elliptic equation and system with double phase operator, using theory of pseudomonotone operator.

These result can be generalized to more problems with different boundary conditions, it can be treated in other ways, by using fixed point theory or by minimization of energy functional.

studies in this area provide valuable results that will contribute to exploring new horizons for research in this emerging topic, so we looking forward to study the multiplicity of solution of this kind of problems in Nehari Manifold, and extending the study to the double phase problems with variable exponents.

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