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Theme

Study of a fourth order elliptic problem

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شكر وتقدير

الحمد لله على منه وامتنانه والشكر له على نعمه وإنعامه حمدا كثيرا طيبا الذي أنعم علي بنعمة العلم وسهل لي طريقا أبغي فيه علما و وفقني في إنهاء عملي المتواضع هذا، والصلاة والسلام على الحبيب المصطفى الذي بلغ الرسالة وأدى الأمانة ونصح الأمة وعملا بقوله صلى الله عليه وسلم :

(مَنْ لَّمْ يَشْكُرِ القَلِيلْ لَّمْ يَشْكُرِ الكَثِيرْ وَ مَنْ لَّمْ يَشْكُرِ النَّاسْ لَّمْ يَشْكُرِ الله)

رواه أحمد الترمذي.

أتقدم بأسمى عبارات الشكر والتقدير إلى كل من علمني ومن أزال غيمة جهل مررت بها برياح العلم الطيبة. إلى كل من علمني علما به أنتفع وأدبا به أرتفع. بدءا من معلمي الابتدائي وصولا إلى أساتذة التعليم العالي والبحث العلمي بقسم الرياضيات و الإعلام الآلي بجامعة الشهيد الشيخ العربي التبسي - تبسة وأخص بالذكر الأستاذ المؤطر - **عكروت كمال** - مع كامل تقديري واحترامي لما بذله من مجهودات قيمة وما حضني به من احترام ومعاملة حسنة للوصول إلى إنجاز هذا العمل البسيط.

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* فـردي نــهال *

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* فــردي نـــهال *

ملخص

العمل المقدم في هذه المذكرة مكرس لدراسة وجود وتعدد الحلول غير البديهية لمشكلة إهليجية من الدرجة الرابعة (بي هارمونيك). يتم الحصول على النتائج باستخدام منوعة نيهارى و مبدأ التنوع لإيكلاند .

الكلمات المفتاحية: مؤثر بي هارمونيك ، منوعة نيهارى ، مبدأ التنوع لإيكلاند

Résumé

Le travail présenté dans ce mémoire est consacré à l´étude de l´existence et la multiplicité de solutions non triviales d´un problème elliptique d'ordre quatre (p-biharmonique).
Les résultats sont obtenus en utilisant la variété de Nehari et le principe variationnelle d´Ekeland.
Mots-clés: Opérateur p-biharmonique, la variété de Nehari, le principe variationnel d'Ekeland.

Abstract

The work presented in this memory is devoted to the study of the existence and multiplicity of nontrivial solutions for a fourth-order elliptic problem (p-biharmonic).
The results are obtained by using the Nehari manifold and Ekeland's variational principle.
Keywords: p-biharmonic operator, Nehari manifold, Ekeland variational principle.

Contents

Introduction

The topic of partial differential equations is one of the most important subjects since the concept of calculus appeared in the correspondence of Leibniz and Lopital in 1695. Because this topic has several important applications in the real world. Physics, financial, mechanics, chemistry. And other several phenomena in several fields that can be studied as a equation such as nuclear reactor dynamics, thermoelasticity, mechanical vibrations, biological tissues, entropy, and diffusion.

In the last ten years, several authors have used a Nehari manifold to solve problems involving sign-changing weight functions. We suggest you read [5, 14] for semilinear elliptic equations, [4, 15] for elliptic problems with nonlinear boundary conditions, [17] for problems in R^N, [7] for Kirchhoff type problems, and [4, 15, 16] for elliptic systems. Meanwhile, positive solutions of semilinear biharmonic equations with Navier boundary on bounded domain in R^N are extensively studied, for example [1, 18],and so on.There are a lot of papers about solving nontrivial problems involving biharmonic or p-biharmonic equations [6, 12, 13, 19] and references therein,There are fewer solutions to p-biharmonic equations with Dirichlet boundary conditions on bounded domains.In [8], the Kirchhoff function can be taken to be the same everywhere, and it has been proven that there are infinitely many solutions to an equation that is governed by the p(x) -polyharmonic operator.Under Dirichlet conditions, we can solve problems by using variational methods.

Our interest in the work is related to The p-biharmonic operator Δ_p^2 which recently attracted many researchers Looking for positive solutions, The main purpose of this study is concerned with multiple solutions of the p-biharmonic equation involving concave-convex nonlinearities and sign-changing weight function and the combined effect of concave and convex nonlinearities on the number of nontrivial solutions of the form

$$\begin{cases} \Delta_p^2 u = |u|^{q-2} u + \lambda f(x) |u|^{r-2} u & in\ \Omega \\ u \quad = \nabla u = 0 & on\ \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$,$1 < r < p < q < p_2^*(p_2^* = \frac{Np}{N-2p}$ if $p < \frac{N}{2}$, $p_2^* = \infty$ if $p \geq \frac{N}{2})$, $\lambda > 0$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous function which changes sign in $\overline{\Omega}$.By means of the Nehari manifold, we prove that there are at least two nontrivial solutions for the problem.

In the first chapter, we start by giving some basic notions, that concern the functional framework necessary to obtain the results of the existence of solutions for the considered problem.

In the second chapter, we present the critical point theory, Nehari manifold, and Ekeland's variational principle.

In the third chapter, we study the fourth-order elliptic problem by using the Nehari manifold and the Ekeland's variational principle.

CHAPTER 1

Preliminaries

Contents

In the first chapter, we begin by giving some basic notions, which concern the functional framework necessary to obtain the results of the existence of solutions for the problem under consideration.

1.1 $L^p(\Omega)$ space

Let Ω be an open set of $\mathbb{R}^N$, equipped with the Lebesgue measure dx, and let p be a positive real number. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values on $\mathbb{R}$, it is provided with the norm

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|dx.$$

Definition 1.1 *We define $L^p(\Omega)$ the space of the class of all measurable functions f, defined on Ω, for which*

$$\int_\Omega |f(x)|^p dx < +\infty,$$

equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left[\int_\Omega |f(x)|^p dx\right]^{1/p}.$$

Definition 1.2 *We also define the space $L^\infty(\Omega)$ by*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}\ , f\ measurable\ ,\ \exists c > 0\ , so\ that\ |f(x)| < c\ \ a.e\ on\ \Omega\},$$

it will be equipped with the essential-sup norm

$$\|f\|_{L^\infty(\Omega)} = ess\sup_{x\in\Omega} |f(x)| = \inf\{c\ ,\ |f(x)| < c\ \ a.e\ on\ \Omega\}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f1_k \in L^p(\Omega)$ for any compact $k \subset \Omega$.

1.1.1 Hölder inequality and L^p completeness

If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ where the real numbers p and p' satisfy $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,we have Hölder inequality:

$$\int_{\Omega} |f(x) g(x)| dx \leq \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^{p'} dx \right)^{\frac{1}{p'}}.$$

Theorem 1.1 *[3] The space $L^p(\Omega)$ is Banach spaces if $1 \leq p \leq \infty$ (complete normed space), separable space if $1 \leq p < \infty$, and $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$.*

1.2 $W^{m,p}(\Omega)$ space

When $\alpha \in \mathbb{N}^n$,we denote by $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n$ the length of α and we denote

$$\partial^{\alpha} u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u,$$

in all that follows $\partial^{\alpha} u$ (or$D^{\alpha} u$) denotes the weak derivative of a function $u \in L^1_{loc}(\Omega)$.

Definition 1.3 *[3]*

We define the space $W^{m,p}(\Omega), m \geq 2$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \textit{such that } \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m,\ \partial^{\alpha} u \in L^p(\Omega), |\alpha| \leq m\},$$

equipped by the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^{\alpha} u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

Remark 1.1 *For $p = 2$, it is customary to replace the notation $W^{m,2}(\Omega)$ by $H^m(\Omega)$.*

Proposition 1.1 *[11]*

The space $W^{m,p}(\Omega)$ provided with the norm defined by

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, 1 \leq p < \infty \\ \\ \max\limits_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)}, p = +\infty, \end{cases}$$

is a Banach space, and for $p \in]1, \infty[$*, this space is convex, so it is a reflexive space. The space* $H^m(\Omega)$*. endowed with the scalar product*

$$(u, v) = \sum_{|\alpha|\leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)},$$

is a Hilbert space.

Corollary 1.1 ***(Poincare inequality)***

Let Ω *be an open and bounded set of* $\mathbb{R}^N$*, Then there exists a constant* $C\,(C\,(\Omega, p))$ *such that*

$$\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}, \forall u \in W^{1,p}(\Omega), 1 \leq p < +\infty.$$

1.3 Notions on operators

Let $(X, \|\cdot\|)$ be a real Banach space and let X' be topological dual.

Definition 1.4 *Let* $A : X \to X'$*, we say that :*

- ***Continuous*** *if* $\|Ax_n - Ax\|_{X'} \to 0$ *when* $\|x_n - x\|_X \to 0$.

- ***Compact*** *if* $A(\overline{B}_X)$ *is relatively compact in* X'*, where* B_X *denotes the ball unit in* X.

- ***Coercive*** *if*

$$\lim_{\|x\|\to+\infty} \frac{\langle A(x), x\rangle}{\|x\|} = +\infty.$$

- ***Monotone*** *if*

$$\langle Au - Av, u - v\rangle \geq 0, \forall u, v \in X \; with \; u \neq v.$$

- ***Strictly monotone*** *if*

$$\langle Au - Av, u - v\rangle > 0, \forall u, v \in X \; with \; u \neq v.$$

- ***Bounded*** *if the image by A of any bounded subset of X is a bounded subset of X'.*

- ***Semi-continuous***

$$if \; u_n \rightharpoonup u \; when \; n \to \infty \; implies \; Au_n \rightharpoonup Au \; when \; n \to \infty.$$

- ***Strongly continuous***

$$if \; u_n \to u \; when \; n \to \infty \; implies \; Au_n \to Au \; when \; n \to \infty.$$

1.4 Week derivative

Definition 1.5 *[11](Directional derivative)*

Let w be a part of a Banach space X and $F : w \to \mathbb{R}$ a real valued function. If $u \in w$ and $z \in X$ we have $u + tz \in w$, we say that F admits (at the point u) a derivative in the direction z if the limit

$$\lim_{t\to 0^+} \frac{F(u + tz) - F(u)}{t}, \; for \; all \; t > 0 \; small \; enough$$

exists. We will denote this limit $F'_z(u)$. The Gateaux differential generalizes the idea of a directional derivative.

Definition 1.6 *[11](Gateaux derivative)*

Let w be a part of a Banach space X and $F : w \to \mathbb{R}$. If $u \in w$, we say that F is Gateaux differentiable in u, if there exists $l \in X'$ or $F(u+tz)$ for $t > 0$ small enough. The Gateaux differential is defined

$$\langle l, z \rangle = \lim_{t \to 0^+} \frac{F(u+tz) - F(u)}{t}.$$

Where $F'(u) := l$.

Definition 1.7 *[11] **(Frechet derivative)***

Let X be a Banach space, w an open space in X, and F a function. If $u \in w$, we say that F is differentiable (or derivable) in u (in the sense of Frechet) if there exists $l \in X'$, such that:

$$\forall v \in w \; F(v) - F(u) = \langle l, v-u \rangle + \sigma(v-u).$$

If F is differentiable, l is unique and we denote by $F'(u) := l$. The set of differentiable functions $w \to \mathbb{R}$ will be denoted by $C^1(w, \mathbb{R})$.

1.5 Convergence criteria

Theorem 1.2 *[3]**(Lebesgue's dominated convergence)***

Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy

- $f_n(x) \to f \; a.e, on \; \Omega,$

- *There is a function $g \in L^1(\Omega)$ such that for all n,*

$$|f_n(x)| \leq g(x), \; a.e. on \; \Omega.$$

Then

$$f \in L^1(\Omega) \; and \; \|f_n - f\|_{L^1} \to 0.$$

Theorem 1.3 *[11](Vitali's convergence theorem)*

Let $f_1, f_2, \cdots$ be L^p-integrable function on some measure space, for $1 \leq p < \infty$. The sequence $\{f_n\}$ converges in L^p to a measurable function f if and only if

- *The sequence $\left\{ f_n \right\}$ converges to f in measure.*
- *The functions $\left\{ |f_n|^p \right\}$ are uniformly,integrable.*
- *For every $\epsilon > 0$, there exists a set E of finite measure, such that $\int_{E^c} |f_n|^p < \epsilon$ for all n.*

Theorem 1.4 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that*

$$\|f_n - f\|_p \underset{n\to\infty}{\to} 0.$$

Then, there exist a subsequence $(f_{n_k})_k \in \mathbb{N}$ and a function $h \in L^p(\Omega)$ such that

- $f_{n_k}(x) \to f(x)$ *a.e on* Ω,
- $|f_{n_k}(x)| \leq h(x) \forall k,$ *a.e. on* Ω.

Lemma 1.1 *[3] (Fatou's Lemma)*

Let (f_n) a sequence of functions in $L^1(\Omega)$ that satisfy

- *For all n, $f_n \geq 0$,*
- $\sup_n \int f_n < \infty$,

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f(x)dx \leq \lim_{n\to\infty} \inf \int_\Omega f_n(x)dx.$$

Lemma 1.2 *[2] (Brezis-Lieb).*

Let Ω be an open bounded set in $\mathbb{R}^n$ and $1 < p < +\infty$, $(f_n)_n$ is sequence of measurable functions such that $f_n \to f$ a.e. in $L^p(\Omega)$, then

$$f \in L^p(\Omega) \text{ and } \|f\|_p^p = \|f_n\|_p^p - \|f_n - f\|_p^p + \sigma(1).$$

Definition 1.8 *[3]*

Let $f : D \to \mathbb{R}$ *and let* $x_0 \in D$*. We say that* f *is lower semi-continuous function (l.s.c) at* x_0 *if for every* $\epsilon > 0$*, there exist* $\delta > 0$ *such that*

$$f(x_0) - \epsilon < f(x)\ forall\ x \in B(x_0; \delta) \cap D.$$

Or equivalently

$$\lim_{x \to x_0} \inf f(x) \geq f(x_0).$$

CHAPTER 2

Critical point theory

Contents

2.1 Critical point

Definition 2.1 ***(Homogeneous function)***

Let f be a function of n variables defined on a set S for which $(tx_1, \cdots, tx_n) \in S$ whenever $t > 0$ and $(tx_1, \cdots, tx_n) \in S$ Then f is homogeneous of degree k if

$$f(tx_1, \cdots, tx_n) = t^k f(x_1, \cdots, x_n) \ forall \ (x_1, \cdots, x_n) \in S \ and \ all \ t > 0.$$

Definition 2.2 ***(Coercivity)***

f is a coercive function if

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

Definition 2.3 ***(Critical point)*** *[11]*

A point $(u, v) \in E$ is critical for J_λ if $J'_\lambda(u, v) = 0$, otherwise (u, v) is regular. If $J_\lambda(u, v) = c$ for some critical point$(u, v) \in E$ of J_λ, the value c is critical, otherwise c is regular.

Let E be a Banach space, $\Phi \in C^1(E, \mathbb{R})$ is a set of constraints:

$$\mathcal{N} = \{v \in E : \Phi(v) = 0\}.$$

Definition 2.4 ***(Lagrange multiplier)*** *[11]*

we suppose that for all $u \in \mathcal{N}$, we have $\Phi'(u) \neq 0$. If $J \in C^1(E, \mathbb{R})$ we say that $c \in \mathbb{R}$ is critical value of J on $\mathcal{N}$, if there exists $u \in \mathcal{N}$, and $\lambda \in \mathbb{R}$ such that

$$J(u) = c \ and \ J'(u) = \lambda \Phi'(u).$$

The point u is a critical point of J on $\mathcal{N}$ and the real λ is called the Lagrange multiplier for the critical value c (or the critical point u).
When X is a functional space and the equation $J'(u) = \lambda\Phi'(u)$ corresponds to a partial derivative equation, we say that $J'(u) = \lambda\Phi'(u)$ is the Euler-Lagrange equation (or the Euler's equation) satisfied by the critical point u on the constraint $\mathcal{N}$.

Theorem 2.1 *[11]*
Let$(E, \|.\|)$ be a Banach space, Ω an open in E and $J : \Omega \to \mathbb{R}$ a differentiable function on Ω and $\Phi \in C^1(\Omega, \mathbb{R}^n)$ of components $\Phi_1, \cdots, \Phi_n$. Given a point in $\mathbb{R}^n$, we set $K = \Phi^{-1}(a)$ which we assume not empty, if at a point $u_0 \in K$

$$J(u_0) = \inf_{x \in K} J(u),$$

and if moreover the differential $\Phi'(u_0) \in L(E, \mathbb{R}^n)$ is surjective then there exist real numbers $\lambda_1, \cdots, \lambda_n$ for which

$$J'(u_0) = \sum_{i=1}^{n} \lambda_i \Phi_i'(u_0).$$

2.2 Nehari Manifold

Nehari has introduced a variational method very useful in critical point theory and eventually came to bear his name. He considered a boundary value problem for a certain nonlinear second-order ordinary differential equation in an interval $[a, b]$ and proved that it has a non-trivial solution which may be obtained by constrained minimization. To describe Nehari's method in an abstract setting, let E be a Banach space and $J \in C^1(E, \mathbb{R})$ a functional. The Frechet derivative of J at u, $J'(u)$, is an element of the dual space E'. Suppose $u \neq 0$ is a critical point of J, i.e.,$J'(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} = \{u \in E \setminus \{0\} : \langle J'(u), u \rangle = 0\}.$$

So $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of $J(u)$ by minimizing the energy functional J on the constraint $\mathcal{N}$ is called the Nehari manifold. Set

$$c := \inf_{n \in \mathcal{N}} J(u).$$

$u \in \mathcal{N}$ Under appropriate conditions on J one hopes that c is attained at some $u_0 \in \mathcal{N}$ and that u_0 is a critical point.

2.3 Ekeland's variational principle

In general, it is not clear that a bounded and lower semi-continuous functional E actually attains its infimum. The analytic function $f(x) = \arctan x$, for example, neither attains its infimum nor its supremum on the real line.

A variant due to Ekeland of Dirichlet's principle, however, permits one to construct minimizing sequences for such functionals E whose elements u_n each minimize a functional E_m, for a sequence of functionals $\{E_m\}$ converging locally uniformly to E.

Theorem 2.2 *[10]*

Let E be a reflexive Banach space with norm $\|\cdot\|$, and $J : E \to \mathbb{R}$ is coercive and weakly lower semi-continuous on E , that is, suppose the following conditions are fulfilled:

- $J(u, v) \to \infty$ *as* $\|(u, v)\| \to \infty, (u, v) \in E$.
- *For any $(u, v) \in E$, any sequence (u_n, v_n) in E such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E there holds $J(u, v) \leq \liminf_{n \to \infty} J(u_n, v_n)$. Then J is bounded from below on E and attains its infimum in E such that*

$$J(u_0, v_0) = \inf_E J.$$

Theorem 2.3 *[9]*

Let M be a complete metric space with metric d, and let $J : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, *bounded from below, and* $\neq \infty$. *Then for any* $\epsilon,\ \delta > 0$, *any* $u \in M$ *with*

$$J(u) \leq \inf_{M} J(u) + \epsilon,$$

there is an element $v \in M$ *strictly minimizing the functional*

$$J_v(w) \leq J(w) + \frac{\epsilon}{\delta} d(v, w).$$

Moreover, we have

$$J(v) \leq J(u), d(u, v) \leq \delta.$$

CHAPTER 3

Fourth order elliptic problem

Contents

In this chapter, we are looking at multiple solutions to a p-biharmonic equation:

$$\begin{cases} \Delta_p^2 u = |u|^{q-2} u + \lambda f(x) |u|^{r-2} u \quad in\ \Omega \\ u \quad = \nabla u = 0 \quad on\ \partial\Omega \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$,$1 < r < p < q < p_2^*(p_2^* = \frac{N_p}{N-2p}$ if $p < \frac{N}{2}$, $p_2^* = \infty$ if $p \geq \frac{N}{2}$), $\lambda > 0$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous function which changes sign in $\overline{\Omega}$.

We know that the corresponding energy functional of problem (3.1) is

$$J_\lambda(u) = \frac{1}{p} \int_\Omega |\Delta u|^p \, dx - \frac{1}{q} \int_\Omega |u|^q \, dx - \frac{\lambda}{r} \int_\Omega f(x) |u|^r \, dx,$$

where $u \in W_0^{2,p}(\Omega)$ with the norm $\|u\| = (\int_\Omega |\Delta u|^p \, dx)^{\frac{1}{p}}$,and J_λ is a $\mathbf{C}^1$ functional and the critical points of J_λ are the weak solutions of problem (3.1).

The following is the main result of this paper.

Theorem 3.1 *There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem (3.1) has at least two nontrivial solutions.*

3.1 Preliminaries

Throughout this section, we denote by S the best Sobolev constant for the embedding of $W_0^{2,p}(\Omega)$ in $L^q(\Omega)$. We consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda(\Omega) = \inf \{J_\lambda(u) \mid u \in \mathcal{M}_\lambda(\Omega)\},$$

where

$$\mathcal{M}_\lambda(\Omega) = \left\{u \in W_0^{2,p}(\Omega) \backslash \{0\} \mid \langle J'_\lambda(u), u \rangle = 0\right\}.$$

Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u \rangle = \|u\|^p - \int_\Omega |u|^q \, dx - \lambda \int_\Omega f(x) |u|^r \, dx.$$

Then for $u \in \mathcal{M}_\lambda(\Omega)$,

$$\langle \psi'_\lambda(u), u \rangle = p\,\|u\|^p - q \int_\Omega |u|^q\,dx - \lambda r \int_\Omega f(x)\,|u|^r\,dx.$$

We may split $\mathcal{M}_\lambda(\Omega)$ into three parts:

$$\mathcal{M}^+_\lambda(\Omega) = \left\{ u \in \mathcal{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u \rangle > 0 \right\},$$

$$\mathcal{M}^0_\lambda(\Omega) = \left\{ u \in \mathcal{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u \rangle = 0 \right\},$$

$$\mathcal{M}^-_\lambda(\Omega) = \left\{ u \in \mathcal{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u \rangle < 0 \right\}.$$

Now, we give the following lemmas.

Lemma 3.1 *there exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1), \mathcal{M}^0_\lambda(\Omega) = \emptyset$.*

Proof We consider the following two cases.

Case (I). $u \in \mathcal{M}_\lambda(\Omega)$ and $\int_\Omega f(x)\,|u|^r\,dx = 0$. We have

$$\|u\|^p - \int_\Omega |u|^q\,dx = 0.$$

Thus,

$$\langle \psi'_\lambda(u), u \rangle = p\,\|u\|^p - q \int_\Omega |u|^q\,dx = (p - q)\,\|u\|^p < 0,$$

and so $u \notin \mathcal{M}^0_\lambda(\Omega)$.

Case (II).$u \in \mathcal{M}_\lambda(\Omega)$ and $\int_\Omega f(x)\,|u|^r\,dx \neq 0$.

suppose that $\mathcal{M}^0_\lambda(\Omega) \neq \emptyset$ for all $\lambda > 0$. If $u \in \mathcal{M}^0_\lambda(\Omega)$, then we have

$$\begin{aligned} 0 &= \langle \psi'_\lambda(u), u \rangle = p\,\|u\|^p - q \int_\Omega |u|^q\,dx - \lambda r \int_\Omega f(x)\,|u|^r\,dx \\ &= (p - r)\,\|u\|^p - (q - r) \int_\Omega |u|^q\,dx. \end{aligned}$$

Thus,

$$\|u\|^p = \frac{q-r}{p-r}\int_\Omega |u|^q\,dx, \tag{3.2}$$

and

$$\lambda\int_\Omega f(x)\,|u|^r\,dx = \|u\|^p - \int_\Omega |u|^q\,dx = \frac{q-p}{p-r}\int_\Omega |u|^q\,dx. \tag{3.3}$$

Moreover,

$$\begin{aligned}\frac{q-p}{p-r}\,\|u\|^p &= \|u\|^p - \int_\Omega |u|^q\,dx = \lambda\int_\Omega f(x)\,|u|^r\,dx\\ &\leq \lambda\,\|f\|_{L^{q^*}}\,\|u\|^r_{L^q} \leq \lambda\,\|f\|_{L^{q^*}}\,S^r\,\|u\|^r,\end{aligned}$$

where $q^* = \frac{q}{q-r}$.This implies

$$\|u\| \leq (\lambda(\frac{q-r}{q-p})\,\|f\|_{L^{q^*}}\,S^r)^{\frac{1}{p-r}}. \tag{3.4}$$

Let $I_\lambda : \mathcal{M}_\lambda(\Omega) \to \mathbb{R}$

be given by

$$I_\lambda(u) = K(q,r)\left(\frac{\|u\|^q}{\int_\Omega |u|^q\,dx}\right)^{\frac{p}{q-p}} - \lambda\int_\Omega f(x)\,|u|^r\,dx,$$

where $K(q,r) = (\frac{q-p}{q-r})(\frac{p-r}{q-r})^{\frac{p}{q-p}}$.Then $I_\lambda(u) = 0$ for all $u \in \mathcal{M}^0_\lambda(\Omega)$. Indeed, from (3.2) and (3.3) it follows that for $u \in \mathcal{M}^0_\lambda(\Omega)$, we have

$$\begin{aligned}I_\lambda(u) &= K(q,r)\left(\frac{\|u\|^q}{\int_\Omega |u|^q\,dx}\right)^{\frac{p}{q-p}} - \lambda\int_\Omega f(x)\,|u|^r\,dx\\ &= \left(K(q,r)(\frac{q-r}{p-r})^{\frac{q}{q-p}} - \frac{q-p}{p-r}\right)\int_\Omega |u|^q\,dx\\ &= 0.\end{aligned} \tag{3.5}$$

However, by(3.4),the Hölder and sobolev inequality, for $u \in \mathcal{M}_\lambda^0(\Omega)$,

$$\begin{aligned}
I_\lambda(u) &\geq K(q,r)\left(\frac{\|u\|^q}{\int_\Omega |u|^q\,dx}\right)^{\frac{p}{q-p}} - \lambda \|f\|_{L^{q^*}} \|u\|_{L^q}^r \\
&\geq \|u\|_{L^q}^r \left(K(q,r)(\frac{\|u\|^q}{S^{\frac{r(q-p)+pq}{p}}\|u\|^{\frac{r(q-p)+pq}{p}}})^{\frac{p}{q-p}} - \lambda \|f\|_{L^{q^*}}\right) \\
&= \|u\|_{L^q}^r \left(K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\|u\|^{-r} - \lambda \|f\|_{L^{q^*}}\right) \\
&\geq \|u\|_{L^q}^r \left\{K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\lambda^{\frac{-r}{p-r}}[(\frac{q-r}{q-p})\|f\|_{L^{q^*}} S^r]^{\frac{-r}{p-r}} - \lambda \|f\|_{L^{q^*}}\right\}.
\end{aligned}$$

This implies that for λ sufficiently small we have $I_\lambda(u) > 0$ for all $u \in \mathcal{M}_\lambda^0(\Omega)$, this contradicts (3.5). Thus, we can conclude that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $\mathcal{M}_\lambda^0(\Omega) = \emptyset$.

Lemma 3.2 *If $u \in \mathcal{M}_\lambda^+(\Omega)$,then $\int_\Omega f(x)\,|u|^r\,dx > 0$.*

Proof For $u \in \mathcal{M}_\lambda^+(\Omega)$, we have

$$\|u\|^p - \int_\Omega |u|^q\,dx - \lambda \int_\Omega f(x)\,|u|^r\,dx = 0,$$

and

$$\|u\|^p > \frac{q-r}{p-r}\int_\Omega |u|^q\,dx.$$

Thus,

$$\lambda \int_\Omega f(x)\,|u|^r\,dx = \|u\|^p - \int_\Omega |u|^q\,dx > \frac{q-p}{p-r}\int_\Omega |u|^q\,dx > 0.$$

This completes the proof.

By Lemma 3.1, for $\lambda \in (0, \lambda_1)$, we write $\mathcal{M}_\lambda(\Omega) = \mathcal{M}_\lambda^+(\Omega) \bigcup \mathcal{M}_\lambda^-(\Omega)$ and define

$$\alpha_\lambda^+(\Omega) = \inf_{u \in \mathcal{M}_\lambda^+(\Omega)} J_\lambda(u), \qquad \alpha_\lambda^-(\Omega) = \inf_{u \in \mathcal{M}_\lambda^-(\Omega)} J_\lambda(u) \quad .$$

The following lemma shows that the minimizers on $\mathcal{M}_\lambda(\Omega)$ are the critical points for J_λ. We write $(W_0^{2,p}(\Omega))^*$ is the dual space of $W_0^{2,p}(\Omega)$.

Lemma 3.3 *For $\lambda \in (0, \lambda_1)$,if u_0 is a local minimizer for J_λ on $\mathcal{M}_\lambda(\Omega)$, then $J_\lambda'(u) = 0$ in $(W_0^{2,p}(\Omega))^*$.*

Proof If u_0 is a local minimizer for J_λ on $\mathcal{M}_\lambda(\Omega)$, then u_0 is a solution of the optimization problem

$$minimize\ \ J_\lambda(u)\ \ subject\ to\ \ \psi_\lambda(u) = 0.$$

Hence, by the theory of lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that

$$J'_\lambda(u_0) = \theta\, \psi'_\lambda(u_0) \ \text{ in } \ (W_0^{2,p}(\Omega))^*.$$

Thus,

$$\langle J'_\lambda(u_0), u_0\rangle = \theta\ \langle \psi'_\lambda(u_0), u_0\rangle\,. \tag{3.6}$$

Since $u_0 \in \mathcal{M}_\lambda(\Omega)$, so $\langle J'_\lambda(u_0), u_0\rangle = 0$. Moreover, since $\mathcal{M}^0_\lambda(\Omega) = \emptyset$, so $\langle \psi'_\lambda(u_0), u_0\rangle \neq 0$ and by (3.6) $\theta = 0$. This completes the proof.

For $u \in W_0^{2,p}(\Omega)$,we write

$$t_{mex} = \left(\frac{(p-r)\,\|u\|^p}{(q-r)\int_\Omega |u|^q\,dx}\right)^{\frac{1}{q-p}}.$$

Then we have the following lemma.

Lemma 3.4 *Let* $q^* = \frac{q}{q-r}$ *and* $\lambda_2 = (\frac{p-r}{q-r})^{\frac{p-r}{q-p}}(\frac{q-p}{q-r})S^{\frac{p(r-q)}{q-p}}\ \|f\|^{-1}_{L^{q^*}}$. *Then for each* $u \in W_0^{2,p}(\Omega)\backslash\{0\}$ *and* $\lambda \in (0, \lambda_2)$, *we have*

(i) There is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in \mathcal{M}^-_\lambda(\Omega)$ and $J_\lambda(t^-u) = \max_{t\geq t_{\max}} J_\lambda(tu)$;

(ii) $t^-(u)$ is a continuous function for nonzero u;

(iii) $\mathcal{M}^-_\lambda(\Omega) = \left\{u \in W_0^{2,p}(\Omega)\backslash\{0\} \mid \frac{1}{\|u\|}t^-(\frac{u}{\|u\|}) = 1\right\}$;

(iv) If $\int_\Omega f(x)\,|u|^r\,dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in M^+_\lambda(\Omega)$ and $J_\lambda(t^+u) = \min_{0\leq t\leq t^-} J_\lambda(tu)$.

Proof (i) Fix $u \in W_0^{2,p}(\Omega)\backslash\{0\}$, let

$$s(t) = t^{p-r}\left\| u \right\|^p - t^{q-r}\int_\Omega |u|^q \, dx \quad \text{for } t \geq 0.$$

We have $s(0) = 0$, $s(t) \to -\infty$ as $t \to +\infty$ and $s(t)$ achieves its maximum at $t_{\max}$. moreover,

$$\begin{aligned} s(t_{\max}) &= \left(\frac{(p-r)\left\| u \right\|^p}{(q-r)\int_\Omega |u|^q \, dx}\right)^{\frac{p-r}{q-p}} \left\| u \right\|^p - \left(\frac{(p-r)\left\| u \right\|^p}{(q-r)\int_\Omega |u|^q \, dx}\right)^{\frac{q-r}{q-p}} \int_\Omega |u|^q \, dx \\ &= \left\| u \right\|^r \left[\left(\frac{(p-r)\left\| u \right\|^q}{(q-r)\int_\Omega |u|^q \, dx}\right)^{\frac{p-r}{q-p}} - \left(\frac{(p-r)\left\| u \right\|^{\frac{q(p-r)}{q-r}}}{(q-r)(\int_\Omega |u|^q \, dx)^{\frac{p-r}{q-r}}}\right)^{\frac{q-r}{q-p}} \right] \\ &= \left\| u \right\|^r \left[\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} - \left(\frac{p-r}{q-r}\right)^{\frac{q-r}{q-p}} \right] \left(\frac{\left\| u \right\|^q}{\int_\Omega |u|^q \, dx}\right)^{\frac{p-r}{q-p}} \\ &\geq \left\| u \right\|^r \left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \left(\frac{q-p}{q-r}\right) \left(\frac{1}{S^q}\right)^{\frac{p-r}{q-p}}. \end{aligned} \tag{3.7}$$

Case(I). $\int_\Omega f(x)\,|u|^r \, dx \leq 0$.

There is a unique $t^- > t_{\max}$ such that $s(t^-) = \lambda \int_\Omega f(x)\,|u|^r \, dx$ and $s'(t) < 0$.

Now

$$\begin{aligned} &(p-r)\left\| t^- u \right\|^p - (q-r)\int_\Omega \left| t^- u \right|^q dx \\ &= (t^-)^{r+1}\left((p-r)(t^-)^{p-r-1}\left\| u \right\|^p - (q-r)(t^-)^{q-r-1}\int_\Omega |u|^q \, dx\right) \\ &= (t^-)^{r+1} s'(t^-) < 0, \end{aligned}$$

and

$$
\begin{aligned}
&\left\langle J_\lambda'(t^-u), t^-u \right\rangle \\
&= (t^-)^p \|u\|^p - (t^-)^q \int_\Omega |u|^q \, dx - (t^-)^r \lambda \int_\Omega f(x) |u|^r \, dx \\
&= (t^-)^r \left(s(t^-) - \lambda \int_\Omega f(x) |u|^r \, dx \right) = 0.
\end{aligned}
$$

Thus,$t^-u \in \mathcal{M}_\lambda^-(\Omega)$.moreover,since for $t > t_{\max}$,

$$
\frac{d}{dt} J_\lambda(tu) = t^{p-1} \|u\|^p - t^{q-1} \int_\Omega |u|^q \, dx - t^{r-1} \lambda \int_\Omega f(x) |u|^r \, dx = 0 \quad \text{for only } t = t^-,
$$

and

$$
\frac{d^2}{dt^2} J_\lambda(tu) < 0 \quad \text{for } t = t^-.
$$

Therefore, $J_\lambda(t^-u) = \max_{t \geq t_{\max}} J_\lambda(tu)$.

Case (II).$\int_\Omega f(x) |u|^r \, dx > 0$.

By(3.7) and

$$
\begin{aligned}
s(0) = 0 < \lambda \int_\Omega f(x) |u|^r \, dx &\leq \lambda \|f\|_{L^{q^*}} S^r \|u\|^r \\
&\leq \|u\|^r \left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \left(\frac{q-p}{q-r}\right) \left(\frac{1}{S^q}\right)^{\frac{p-r}{q-p}} \\
&\leq s(t_{\max}) \qquad \text{for } \lambda \in (0, \lambda_2),
\end{aligned}
$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$
s(t^+) = \lambda \int_\Omega f(x) |u|^r \, dx = s(t^-),
$$

and

$$
s'(t^+) > 0 > s'(t^-).
$$

We have $t^+u \in \mathcal{M}_\lambda^+(\Omega)$, $t^-u \in \mathcal{M}_\lambda^-(\Omega)$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus

$$J_\lambda(t^- u) = \max_{t \geq t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^+ u) = \min_{0 \leq t \leq t^-} J_\lambda(tu).$$

(ii) By the uniqueness of $t^-(u)$ and the external property of $t^-(u)$, we have that $t^-(u)$ is a continuous function of $u \neq 0$.

(iii) For $u \in \mathcal{M}_\lambda^-(\Omega)$, let $v = \frac{u}{\|u\|}$. By part (i) , there is unique $t^-(v) > 0$ such that $t^-(v)v \in \mathcal{M}_\lambda^-(\Omega)$, that is $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|}u \in \mathcal{M}_\lambda^-(\Omega)$. Since $u \in \mathcal{M}_\lambda^-(\Omega)$, we have $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1$, which implies

$$\mathcal{M}_\lambda^-(\Omega) \subset \left\{ u \in W_0^{2,p}(\Omega) \backslash \{0\} \mid t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1 \right\}.$$

Conversely,let $u \in W_0^{2,p}(\Omega)\backslash\{0\}$ such that $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1$,then

$$t^-(\frac{u}{\|u\|})\frac{u}{\|u\|} \in \mathcal{M}_\lambda^-(\Omega).$$

Thus,

$$\mathcal{M}_\lambda^-(\Omega) = \left\{ u \in W_0^{2,p}(\Omega) \backslash \{0\} \mid t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1 \right\}.$$

(iv) By Case (II) of part (i).

By $f : \overline{\Omega} \to \mathbb{R}$ is continuous function which changes sign in Ω , we have $\Theta = \{x \in \Omega \mid f(x) > 0\}$ is a open set in $\mathbb{R}^N$. Consider the following p-biharmonic equation:

$$\begin{cases} \Delta_p^2 u = |u|^{q-2}\, u & \text{in } \Theta \\ u = \nabla u = 0 & \text{on } \partial\Theta. \end{cases} \tag{3.8}$$

Associated with (3.8), we consider the energy functional

$$K(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \, dx - \frac{1}{q}\int_\Omega |u|^q \, dx,$$

and the minimization problem

$$\beta(\Theta) = \inf \{K(u) \mid u \in N(\Theta)\},$$

where $N(\Theta) = \left\{u \in W_0^{2,p}(\Theta)\backslash\{0\} \mid \langle K'(u), u\rangle = 0\right\}$. Now we prove that problem (3.8) has a nontrivial solution ω_0 such that $K(\omega_0) = \beta(\Theta) > 0$.

Lemma 3.5 *For any $u \in W_0^{2,p}(\Theta)\backslash\{0\}$ there exists a unique $t(u) > 0$ such that $t(u)u \in N(\Theta)$. The maximum of $K(tu)$ for $t \geq 0$ is achieved at $t = t(u)$, The function*

$$W_0^{2,p}(\Theta)\backslash\{0\} \to (0, +\infty) : u \to t(u),$$

is continuous and the map $u \to t(u)u$ defines a homeomorphism of the unit sphere of $W_0^{2,p}(\Theta)$ with $N(\Theta)$.

Proof Let $u \in W_0^{2,p}(\Theta)\backslash\{0\}$ be fixed and define the function $g(t) := K(tu)$ on $[0, \infty)$.Clearly we have

$$\begin{aligned} g'(t) &= 0 \Leftrightarrow tu \in N(\Theta) \\ &\Leftrightarrow \|u\|^p = t^{q-p} \int_\Omega |u|^q \, dx. \end{aligned} \tag{3.9}$$

It is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Therefore $\max_{[0,+\infty)} g(t)$ is achieved at a unique $t = t(u)$ such that $g'(t(u)) = 0$ and $t(u)\,u \in N(\Theta)$. To prove the continuity of $t(u)$, assume that $u_n \to u$ in $W_0^{2,p}(\Theta)\backslash\{0\}$. It is easy to verify that $\{t(u_n)\}$ is bounded. If a subsequence of $\{t(u_n)\}$ converges to t_0, it follows from (3.9) that $t_0 = t(u)$, But then $t(u_n) \to t(u)$. Finally the continuous map from the unit sphere of $W_0^{2,p}(\Theta)$ to $N(\Theta)$, $u \to t(u)\,u$, is inverse to the retraction $u \to \frac{u}{\|u\|}$.

Define

$$c_1 := \inf_{u \in W_0^{2,p}(\Theta)\setminus\{0\}} \max_{t \geq 0} K(tu),$$
$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} K(\gamma(tu)),$$

where $\Gamma := \left\{\gamma \in C([0,1], W_0^{2,p}(\Theta)) : \gamma(0) = 0, K(\gamma(1)) < 0\right\}$.

Lemma 3.6 $\beta(\Theta) = c_1 = c > 0$ *and* c *is a critical value of* K.

Proof The lemma 3.5 implies that $\beta(\Theta) = c_1$. Since $K(tu) < 0$ for $u \in W_0^{2,p}(\Theta)\setminus\{0\}$ and t large, we obtain $c \leq c_1$. The manifold $N(\Theta)$ separates $W_0^{2,p}(\Theta)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover $K(u) \geq 0$ for all u in this component, because $\langle K'(tu), u\rangle \geq 0$ for all $0 \leq t \leq t(u)$. Thus every $\gamma \in \Gamma$ has to cross $N(\Theta)$ and $\beta(\Theta) \leq c$. Since the embedding $W_0^{2,p}(\Theta) \hookrightarrow L^q(\Theta)$ is compact, it is easy to prove that $c > 0$ is a critical value of K and ω_0 a nontrivial solution corresponding to c.

With the help of lemma 3.6, we have the following result.

Lemma 3.7 *(i) There exists* $\widetilde{t} > 0$ *such that*

$$\alpha_\lambda(\Omega) \leq \alpha_\lambda^+(\Omega) < \frac{r-p}{r}\widetilde{t}^p \beta(\Theta) < 0;$$

(ii) J_λ *is coercive and bounded below on* $\mathcal{M}_\lambda(\Omega)$ *for all* $\lambda \in (0, \frac{q-p}{q-r}]$.

Proof (i) Let ω_0 be a nontrivial solution of problem (3.8) such that $K(\omega_0) = \beta(\Theta) > 0$. Then

$$\int_\Omega f(x)\,|\omega_0|^r\,dx = \int_\Theta f(x)\,|\omega_0|^r\,dx > 0.$$

Set $\widetilde{t} = t^+(\omega_0)$ as defined by Lemma 3.4(iv). Hence $\widetilde{t}\omega_0 \in \mathcal{M}_\lambda^+(\Omega)$ and

$$
\begin{aligned}
J_\lambda(\widetilde{t}\omega_0) &= \frac{\widetilde{t}^p}{p}\int_\Omega |\Delta\omega_0|^p\,dx - \frac{\widetilde{t}^q}{q}\int_\Omega |\omega_0|^q\,dx - \frac{\lambda\widetilde{t}^r}{r}\int_\Omega f(x)\,|\omega_0|^r\,dx \\
&= (\frac{1}{p}-\frac{1}{r})\widetilde{t}^p\int_\Omega |\Delta\omega_0|^p\,dx + (\frac{1}{r}-\frac{1}{q})\widetilde{t}^q\int_\Omega |\omega_0|^q\,dx \\
&< \frac{r-p}{r}\widetilde{t}^p\beta(\Theta) < 0.
\end{aligned}
$$

This yields

$$
\alpha_\lambda(\Omega) \le \alpha_\lambda^+(\Omega) < \frac{r-p}{r}\widetilde{t}^p\beta(\Theta) < 0.
$$

(ii) For $u \in \mathcal{M}_\lambda(\Omega)$, we have $\int_\Omega |\Delta u|^p\,dx = \int_\Omega |u|^q\,dx + \int_\Omega f(x)\,|u|^r\,dx$. Then by the Hölder and Young inequality

$$
\begin{aligned}
J_\lambda(u) &= \frac{q-p}{pq}\int_\Omega |\Delta u|^p\,dx - \lambda\frac{q-r}{qr}\int_\Omega f(x)\,|u|^r\,dx \\
&\ge \frac{q-p}{pq}\int_\Omega |\Delta u|^p\,dx - \lambda\frac{q-r}{qr}\,\|f\|_{L^{q^*}}\,S^r\,\|u\|^r \\
&\ge \frac{1}{qp}\left[(q-p)-\lambda(q-r)\right]\|u\|^p - \lambda\frac{(q-r)(p-r)}{qpr}(\|f\|_{L^{q^*}}\,S^r)^{\frac{p}{p-r}}.
\end{aligned}
$$

Thus J_λ is coercive on $\mathcal{M}_\lambda(\Omega)$ and

$$
J_\lambda(u) \ge -\lambda\frac{(q-r)(p-r)}{qpr}(\|f\|_{L^{q^*}}\,S^r)^{\frac{p}{p-r}},
$$

for all $\lambda \in (0, \frac{q-p}{q-r}]$.

3.2 Proof of the main result

For the proof of theorem, we need the following lemmas.

Lemma 3.8 *For $u \in \mathcal{M}_\lambda(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathcal{M}_\lambda(\Omega)$ and*

$$\langle \xi'(0), v \rangle = \frac{p \int_\Omega |\Delta u|^{p-2} \Delta u \Delta v dx - q \int_\Omega |\Delta u|^{q-2} uv dx - r\lambda \int_\Omega f(x) |u|^{r-2} uv dx}{(p-r) \int_\Omega |\Delta u|^p dx - (q-r) \int_\Omega |u|^q dx}, \tag{3.10}$$

for all $v \in W_0^{2,p}(\Omega)$.

Proof For $u \in \mathcal{M}_\lambda(\Omega)$, define a function $F : \mathbb{R} \times W_0^{2,p} \to \mathbb{R}$ by

$$\begin{aligned} F_u(\xi, \omega) &= \langle J'_\lambda(\xi(u-\omega)), \xi(u-\omega) \rangle \\ &= \xi^p \int_\Omega |\Delta(u-\omega)|^p dx - \xi^q \int_\Omega |u-\omega|^q dx - \xi^r \lambda \int_\Omega f(x) |u-\omega|^r dx. \end{aligned}$$

Then $F_u(1,0) = \langle J'_\lambda(u), u \rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_u(1,0) &= p \int_\Omega |\Delta u|^p dx - q \int_\Omega |u|^q dx - r\lambda \int_\Omega f(x) |u|^r dx \\ &= (p-r) \int_\Omega |\Delta u|^p dx - (q-r) \int_\Omega |u|^q dx \neq 0. \end{aligned}$$

According to the implicit function theorem, there exists $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$such that $\xi(0) = 1$ and

$$\langle \xi'(0), v \rangle = \frac{p \int_\Omega |\Delta u|^{p-2} \Delta u \Delta v dx - q \int_\Omega |\Delta u|^{q-2} uv dx - r\lambda \int_\Omega f(x) |u|^{r-2} uv dx}{(p-r) \int_\Omega |\Delta u|^p dx - (q-r) \int_\Omega |u|^q dx},$$

and

$$F_u(\xi(v), v) = 0 \qquad \text{for all } v \in B(0;\epsilon),$$

which is equivalent to

$$\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v) \rangle = 0 \qquad \text{for all } v \in B(0;\epsilon),$$

that is $\xi(v)(u-v) \in \mathcal{M}_\lambda(\Omega)$.

Similarity, we have

Lemma 3.9 *For each $u \in \mathcal{M}_\lambda^-(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0) = 1$,the function $\xi^-(v)(u-v) \in \mathcal{M}_\lambda^-(\Omega)$ and*

$$\big\langle(\xi^-)(0), v\big\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx - q\int_\Omega |u|^{q-2} uv dx - r\lambda\int_\Omega f(x)|u|^{r-2} uv dx}{(p-r)\int_\Omega |\Delta u|^p dx - (q-r)\int_\Omega |u|^q dx}, \tag{3.11}$$

for all $v \in W_0^{2,p}(\Omega)$.

Proof similar to the proof in Lemma 3.8, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathcal{M}_\lambda(\Omega)$ for all $v \in B(0;\epsilon)$. Since

$$\langle\psi_\lambda'(u), u\rangle = (p-r)\|u\|^p - (q-r)\int_\Omega |u|^q dx < 0.$$

Thus by the continuity of the function ψ_λ' and ξ^- , we have

$$\big\langle\psi_\lambda'(\xi^-(v)(u-v)), \xi^-(v)(u-v)\big\rangle = (p-r)\big\|\xi^-(v)(u-v)\big\|^p - (q-r)\int_\Omega \big|\xi^-(v)(u-v)\big|^q dx < 0.$$

If ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathcal{M}_\lambda^-(\Omega)$.

Proposition 3.1 *Let $\lambda_0 = \inf\{\lambda_1, \lambda_2, \frac{q-p}{q-r}\}$, for $\lambda \in (0, \lambda_0)$.*

(i) There exists a minimizing sequence $\{u_n\} \subset \mathcal{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1),$$
$$J_\lambda'(u_n) = o(1), \quad \text{for } (W_0^{2,p}(\Omega))^*;$$

(ii) There exists a minimizing sequence $\{u_n\} \subset \mathcal{M}_\lambda^-(\Omega)$such that

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1),$$
$$J_\lambda'(u_n) = o(1), \quad \text{for } (W_0^{2,p}(\Omega))^*.$$

Proof (i) By Lemma 3.7(ii) and the Ekeland variational principle [10], there exists a minimizing sequence $\{u_n\} \subset \mathcal{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) < \alpha_\lambda(\Omega) + \frac{1}{n}, \tag{3.12}$$

and

$$J_\lambda(u_n) < J_\lambda(\omega) + \frac{1}{n}\|\omega - u_n\| \quad \text{for each } \omega \in M_\lambda(\Omega). \tag{3.13}$$

By taking n enough large, from Lemma 3.7(i), we have

$$\begin{aligned} J_\lambda(u_n) &= (\frac{1}{p} - \frac{1}{q})\|u_n\|^p - (\frac{1}{r} - \frac{1}{q})\lambda \int_\Omega f(x)|u_n|^r\,dx \\ &< \alpha_\lambda(\Omega) + \frac{1}{n} < \frac{r-p}{r}\tilde{t}^p\beta(\Theta) < 0. \end{aligned} \tag{3.14}$$

This implies

$$\|f\|_{L^{q^*}} S^r \|u_n\|^r \geq \int_\Omega f(x)|u_n|^r\,dx > \frac{q(p-r)}{\lambda(q-r)}\tilde{t}^p\beta(\Theta). \tag{3.15}$$

Consequently $u_n \neq 0$ and putting together (3.14),(3.15) and the Hölder inequality, we obtain

$$\|u_n\| \geq \left[\frac{q(p-r)}{\lambda(q-r)}\frac{\tilde{t}^p}{\|f\|_{L^{q^*}} S^r}\beta(\Theta)\right]^{\frac{1}{r}}, \tag{3.16}$$

and

$$\|u_n\| \geq \left[\frac{\lambda p(q-r)}{r(q-p)}\|f\|_{L^{q^*}} S^r\right]^{\frac{1}{p-r}}. \tag{3.17}$$

Now we show that

$$\|J'_\lambda(u_n)\|_{(W_0^{2,p}(\Omega))^*} \to 0 \text{ as } n \to \infty.$$

Applying Lemma 3.8 with u_n to obtain the function ξ_n: $B(0;\epsilon_n) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(\omega)(u_n - \omega) \in \mathcal{M}_\lambda(\Omega)$. Choose $0 < \rho < \epsilon_n$. Let $u \in W_0^{2,p}(\Omega)$ with $u \not\equiv 0$ and let $\omega_\rho = \frac{\rho u}{\|u\|}$. We set $\eta_\rho = \xi_n(\omega_\rho)(u_n - \omega_\rho)$. Since $\eta_\rho \in \mathcal{M}_\lambda(\Omega)$, we deduce from (3.13) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geq -\frac{1}{n}\left\|\eta_\rho - u_n\right\|,$$

and by the mean value theorem, we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n\rangle + o(\|\eta_\rho - u_n\|) \geq -\frac{1}{n}\left\|\eta_\rho - u_n\right\|.$$

Thus,

$$\langle J'_\lambda(u_n), -\omega_\rho\rangle + (\xi_n(\omega_\rho) - 1)\langle J'_\lambda(u_n), (u_n - \omega_\rho)\rangle \geq -\frac{1}{n}\left\|\eta_\rho - u_n\right\| + o(\|\eta_\rho - u_n\|). \tag{3.18}$$

From $\xi_n(\omega_\rho)(u_n - \omega_\rho) \in \mathcal{M}_\lambda(\Omega)$ and (3.18) it follows that

$$-\rho\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle + (\xi_n(\omega_\rho) - 1)\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - \omega_\rho)\rangle \geq -\frac{1}{n}\left\|\eta_\rho - u_n\right\| + o(\|\eta_\rho - u_n\|).$$

Thus,

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{(\xi_n(\omega_\rho) - 1)}{\rho}\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - \omega_\rho)\rangle + \frac{1}{n\rho}\left\|\eta_\rho - u_n\right\| + \frac{o(\|\eta_\rho - u_n\|)}{\rho}. \tag{3.19}$$

Since

$$\left\|\eta_\rho - u_n\right\| \leq \left|\xi_n(\omega_\rho) - 1\right|\left\|u_n\right\| + \rho\left|\xi_n(\omega_\rho)\right|,$$

and

$$\lim_{\rho\to 0}\frac{|\xi_n(\omega_\rho) - 1|}{\rho} \leq \left\|\xi'_n(0)\right\|.$$

If we let $\rho \to 0$ in (3.19) for a fixed n, then by (3.17) we can find a constant $C > 0$, independent of ρ, such that

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|).$$

We are done once we show that $\|\xi'_n(0)\|$ is uniformly bounded in n. By (3.10),(3.17) and Hölder inequality, we have

$$\langle \xi_n'(0), v \rangle \leq \frac{b\,\|v\|}{\left|(p-r)\int_\Omega |\Delta u_n|^p\,dx - (q-r)\int_\Omega |u_n|^q\,dx\right|} \text{for some } b > 0.$$

We only need to show that

$$\left|(p-r)\int_\Omega |\Delta u_n|^p\,dx - (q-r)\int_\Omega |u_n|^q\,dx\right| > c, \tag{3.20}$$

for some $c > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(p-r)\int_\Omega |\Delta u_n|^p\,dx - (q-r)\int_\Omega |u_n|^q\,dx = o(1). \tag{3.21}$$

Combining (3.21) with (3.16), we can find a suitable constant $d > 0$ such that

$$\int_\Omega |u_n|^q\,dx \geq d \qquad \text{for } n \text{ sufficiently large.} \tag{3.22}$$

In addition (3.21), and the fact $\{u_n\} \subset \mathcal{M}_\lambda(\Omega)$ also give

$$\lambda \int_\Omega f(x)\,|u_n|^r\,dx = \|u_n\|^p - \int_\Omega |u_n|^q\,dx > \|u_n\|^p > \frac{q-p}{p-r}\int_\Omega |u_n|^q\,dx > 0,$$

and

$$\|u_n\| \le \left(\lambda(\frac{q-r}{q-p})\,\|f\|_{L^{q^*}}\,S^r\right)^{\frac{1}{p-r}} + o(1). \tag{3.23}$$

This implies

$$\begin{aligned} I_\lambda(u_n) &= K(q,r)\left(\frac{\|u_n\|^q}{\int_\Omega |u_n|^q\,dx}\right)^{\frac{p}{q-p}} - \lambda\int_\Omega f(x)\,|u_n|^r\,dx \\ &= \left(K(q,r)(\frac{q-r}{p-r})^{\frac{q}{q-p}} - \frac{q-p}{p-r}\right)\int_\Omega |u_n|^q\,dx + o(1) \\ &= o(1). \end{aligned} \tag{3.24}$$

However, by (3.22), (3.23) and $\lambda \in (0, \lambda_0)$,

$$\begin{aligned} I_\lambda(u_n) &\ge K(q,r)\left(\frac{\|u_n\|^q}{\int_\Omega |u_n|^q\,dx}\right)^{\frac{p}{q-p}} - \lambda\,\|f\|_{L^{q^*}}\,\|u_n\|_{L^q}^r \\ &\ge \|u_n\|_{L^q}^r\left(K(q,r)\left(\frac{\|u_n\|^q}{S^{\frac{r(q-p)+pq}{p}}\,\|u\|^{\frac{r(q-p)+pq}{p}}}\right)^{\frac{p}{q-p}} - \lambda\,\|f\|_{L^{q^*}}\right) \\ &= \|u_n\|_{L^q}^r\left(K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\,\|u_n\|^{-r} - \lambda\,\|f\|_{L^{q^*}}\right) \\ &\ge \|u\|_{L^q}^r\left\{K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\lambda^{\frac{-r}{p-r}}[(\frac{q-r}{q-p})\,\|f\|_{L^{q^*}}\,S^r]^{\frac{-r}{p-r}} - \lambda\,\|f\|_{L^{q^*}}\right\}, \end{aligned}$$

This contradicts (3.24). We get

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \le \frac{C}{n}.$$

The proof is compleat.

(ii) Similar to the proof of (i), we may prove (ii).

Now, we establish the existence of a local minimum for J_λ on $\mathcal{M}_\lambda^+(\Omega)$.

Theorem 3.2 *Let λ_0 as in Proposition 3.1, then for $\lambda \in (0, \lambda_0)$, the functional J_λ has a minimizer $u_0^+ \in \mathcal{M}_\lambda^+(\Omega)$ and il satisfies*

(i) $J_\lambda(u_0^+) = \alpha_\lambda(\Omega) = \alpha_\lambda^+(\Omega)$ *;*

(ii) u_0^+ *is a nontrivial solution of problem (3.1);*

(iii) $J_\lambda(u_0^+) \to 0$ *as* $\lambda \to 0$ *.*

Proof Let $\{u_n\} \subset \mathcal{M}_\lambda(\Omega)$ is a minimizing sequence for J_λ on $\mathcal{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1),$$
$$J'_\lambda(u_n) = o(1), \quad \text{for } (W_0^{2,p}(\Omega))^*.$$

Then by Lemma 3.7 and the compact imbedding theorem, there exists a subsequence $\{u_n\}$ and $u_0^+ \in W_0^{2,p}(\Omega)$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in} \quad W_0^{2,p}(\Omega),$$
$$u_n \to u_0^+ \quad \text{strongly in} \quad L^q(\Omega),$$

and

$$u_n \to u_0^+ \quad \text{strongly in} \quad L^r(\Omega). \tag{3.25}$$

We firstly show that $\int_\Omega f(x)\left|u_0^+\right|^r dx \neq 0$.If not, by (3.25) we can conclude that

$$\int_\Omega f(x)\left|u_0^+\right|^r dx \neq 0,$$

and

$$\int_\Omega f(x)\left|u_n\right|^r dx \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$\int_\Omega |\Delta u_n|^p\, dx = \int_\Omega |u_n|^q\, dx + o(1).$$

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\int_\Omega |\Delta u_n|^p\, dx - \frac{1}{q}\int_\Omega |u_n|^q\, dx - \frac{\lambda}{r}\int_\Omega f(x)\,|u_n|^r\, dx \\ &= (\frac{1}{p}-\frac{1}{q})\int_\Omega |u_n|^q\, dx + o(1) \\ &= (\frac{1}{p}-\frac{1}{q})\int_\Omega \left|u_0^+\right|^q dx \qquad \text{as} \quad n\to\infty, \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda(\Omega) < 0$ as $n \to \infty$.In particular, $u_0^+ \in \mathcal{M}_\lambda^+(\Omega)$ is a nontrivial solution of problem (3.1) and $J_\lambda(u_0^+) \geq \alpha_\lambda(\Omega)$. We now prove that $u_n \rightharpoonup u_0^+$ strongly in $W_0^{2,p}(\Omega)$. Supposing the contrary, then $\left\|u_0^+\right\| < \liminf_{n\to\infty} \|u_n\|$ and so

$$\begin{aligned} &\left\|u_0^+\right\|^p - \int_\Omega \left|u_0^+\right|^q dx - \lambda\int_\Omega f(x)\left|u_0^+\right|^r dx \\ &< \liminf_{n\to\infty}(\|u_n\|^p - \int_\Omega |u_n|^q\, dx - \lambda\int_\Omega f(x)\,|u_n|^r\, dx) = 0, \end{aligned}$$

this contradicts $u_0^+ \in \mathcal{M}_\lambda(\Omega)$.In fact, if $u_0^+ \in \mathcal{M}_\lambda^-(\Omega)$,by Lemma 3.4, there are unique t_0^+ and t_0^-such that $t_0^+ u_0^+ \in \mathcal{M}_\lambda^+(\Omega)$ and $t_0^- u_0^+ \in \mathcal{M}_\lambda^-(\Omega)$, we have $t_0^+ < t_0^- = 1$.Since

$$\frac{d}{dt}J_\lambda(t_0^+ u_0^+) = 0 \;\text{ and }\; \frac{d^2}{dt^2}J_\lambda(t_0^+ u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+)$. By

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+) \leq J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+),$$

which is a contradiction. By Lemma 3.3, we know that u_0^+ is a nontrivial solution.

Moreover, by Lemma 3.7,

$$0 > J_\lambda(u_0^+) \geq -\lambda \frac{(q-r)(p-r)}{qpr} (\|f\|_{L^{q^*}} S^r)^{\frac{p}{p-r}},$$

it is clear that $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$.

Next, we establish the existence of a local minimum for J_λ on $\mathcal{M}_\lambda^-(\Omega)$.

Theorem 3.3 *Let λ_0 as in Proposition 3.1, then for $\lambda \in (0, \lambda_0)$, the functional J_λ has a minimizer $u_0^- \in \mathcal{M}_\lambda^-(\Omega)$ and it satisfies*

(i) $J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega)$;

(ii) u_0^- *is a nontrivial solution of problem(3.1).*

Proof Let $\{u_n\}$ is a minimizing sequence for J_λ on $\mathcal{M}_\lambda^-(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1),$$
$$J'_\lambda(u_n) = o(1), \quad \text{for } (W_0^{2,p}(\Omega))^*.$$

Then by Proposition 3.1 (ii) and the compact imbedding theorem, there exists a subsequence $\{u_n\}$ and $u_0^- \in \mathcal{M}_\lambda^-(\Omega)$ such that

$$u_n \rightharpoonup u_0^- \quad \text{weakly in} \quad W_0^{2,p}(\Omega),$$
$$u_n \to u_0^- \quad \text{strongly in} \quad L^q(\Omega),$$

and

$$u_n \to u_0^- \quad \text{strongly in} \quad L^r(\Omega). \tag{3.26}$$

We now prove that $u_n \to u_0^-$ strongly in $W_0^{2,p}(\Omega)$. Supposing the contrary, then

$\left\| u_0^- \right\| < \liminf_{n\to\infty} \|u_n\|$ and so

$$\begin{aligned} & \left\| u_0^- \right\|^p - \int_\Omega \left| u_0^- \right|^q dx - \lambda \int_\Omega f(x) \left| u_0^- \right|^r dx \\ < & \liminf_{n\to\infty} (\|u_n\|^p - \int_\Omega |u_n|^q \, dx - \lambda \int_\Omega f(x) \, |u_n|^r \, dx) = 0, \end{aligned}$$

this contradicts $u_0^- \in \mathcal{M}_\lambda^-(\Omega)$. Hence $u_n \to u_0^-$ strongly in $W_0^{2,p}(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega) \qquad \text{as as } n \to \infty.$$

By Lemma 3.3, we know that u_0^- is a nontrivial solution.

Combing with Theorem 3.2 and Theorem 3.3, for problem (3.1) there exist two nontrivial solution u_0^+ and u_0^- such that $u_0^+ \in \mathcal{M}_\lambda^+(\Omega)$, $u_0^- \in \mathcal{M}_\lambda^-(\Omega)$. Since $\mathcal{M}_\lambda^+(\Omega) \cap \mathcal{M}_\lambda^-(\Omega) = \emptyset$, this shows that u_0^+ and u_0^- are different.

Conclusion

In this memory, we have studied the existence and multiplicity of solutions for a fourth-order elliptic problem, using variational techniques, exactly the Nehari manifold method and Ekeland's variational principle under homogenous boundary conditions.

The results obtained in this work can be generalized with other operators and different nonlinearities, in critical and sub-critical cases.

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