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Presented by **MERAD Souheib**

Theme

Fixed point theorems for single-valued mappings and multi-valued mappings in generalized metric space

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ABSTRACT

In this thesis, we prove a number of single-valued and multi valued fixed point theorems for different kinds of contractions in complex valued b-metric, like-metric spaces, and b-metric like spaces. Furthermore, we unveil a new definition for a class of metrics spaces,$b_v(\theta)$ and we concentrate on the results of the fixed points. In the end, we use the results obtained to look into whether there is a solution to the integral equation of Fredholm and the local solution of ordinary differential equations in $b_v(\theta)$.

Keywords: , b-metric like-space, Rational type contraction, cyclic-contractions, Complex-valued b-metric space,metric like-space, Set valued analysis, Multivalued mapping, Fixed point, Common fixed point.

RÉSUMÉ

Dans cette thèse, nous prouvons un certain nombre de théorèmes de point fixe à valeur unique et à valeurs multiples pour différents types de contractions dans : la métrique b à valeurs complexes, l'espace de type métrique et l'espace de type b-métrique. De plus, nous introduisons une nouvelle définition pour une classe d'espaces de métriques,$b_v(\theta)$ et nous concentrons sur les résultats des Point fixe. A la fin, nous utilisons les résultats obtenus pour étudier l'existence de la solution de l'équation intégrale de Fredholm et de la solution locale des Équations différentielles ordinaires dans $b_v(\theta)$

Mots-clés: espace similaire métrique, espace similaire b-métrique, contractions cycliques, point fixe commun, contraction de type rationnel, espace b-métrique à valeurs complexes, analyse à valeurs définies, cartographie multi-Valuée, fixe point.

الملخص

في هذه الأطروحة، قمنا بإثبات عدد من نظريات النقطة الثابتة ذات القيمة الواحدة والمتعددة القيم لأنواع مختلفة من التقلصات في الفضاءات المترية، والفضاءات ذات القيمة المترية، والفضاءات ذات القيمة المعقدة. علاوة على ذلك، قدمنا تعريفًا جديدًا لفئة جديدة من الفضاءات المترية (θ) b_v وقمنا بتركيز بعض نتائج النقاط الثابتة هناك. وفي القسم الأخير نستخدم النتائج المكتسبة لدراسة الوجدانية مع حلول مسائل القيمة الأولية و المعادلات التكاملية من نوع فريدهولم مع أمثلة عملية للنظريات.

الكلمات المفتاحية: مساحة مماثلة مترية، مساحة مماثلة ب-مترية، تقلصات دورية، نقطة ثابتة مشتركة، انكماش النسبي، مساحة ب-مترية ذات قيمة معقدة، تحليل محدد القيمة، تحليل متعدد القيم، نقطة الثابتة.

NOTIONS

The following is a list of the most common notations, symbols, and abbreviations:

N	:	The set of natural numbers.
R	:	The set of real numbers.
$\mathbb{C}$	:	The set of complex numbers.
$C([\alpha,\beta],Y)$	:	The set of continuous functions from $[\alpha,\beta]$ to Y
$H(T,S)$	:	The Hausdorff distance between T and S.
$P(Y)$	:	Family of non-empty subsets of Y.
$CL(Y)$	:	The set of al lclosed subsets of Y.
$CB(Y)$	:	set of all closed and bounded subsets of Y.
T	:	$Y-\longrightarrow P(X)$: T is a multivalued map from Y to$P(X)$.
$Dom(T)$	:	The domain of T.
$Graph(T)$	:	The graph of T.
$Im(T)$	:	The image of T
$(Y,$	$\precsim$	$,d)$: Complex valued metric space.
$b_v(s)$	:	polygonal b-metric space
$b_v(\theta)$	:	Extended polygonal b-metricspace

Contents

Introduction

Fixed point theory has a rich history dating back to the late 19th century. with pivotal contributions from mathematicians like Brouwer, Banach, and Kakutani. The theory initially emerged as a tool to understand the existence and properties of solutions to equations and systems of equations. Over the decades, it has evolved and diversified to include various extensions and applications in many fields of science, such as engineering, physics, computer science, economics and telecommunication optimization problems, making it a cornerstone of mathematical analysis.[1],[2],[3],[4],[5],[6],[7],[8].

In the realm of single-valued mappings [9], fixed points hold a significant place, particularly in metric spaces. Metric spaces provide the foundational framework for understanding distance and convergence, and they play a fundamental role in the study of single-valued mappings. Beyond standard metric spaces, developments have led to specialized spaces such as b-metric spaces.[10], [11],[12],[13],rectangular b-metric space [14],[15],[16], extended b-metric space [17],extended rectangular b-metric [18],[19], b-metric like space[20],[21],[22],[23] ,[24],[25],[26]), and complex metric spaces [27],[28],[29],[30][31], [32],[33],[34],[35], [36],[37],[38],[39], , each offering unique perspectives and applications for fixed point theory. Multivalued mappings introduce a fascinating dimension to fixed point the- ory. These mappings can have multiple points, known as fixed sets, which do not necessarily reduced to a single value. The study of fixed points in multival- ued mappings are instrumental in solving problems involving non-uniqueness and discontinuity, making it a valuable tool in mathematical analysis and opti- mization problems. [40],[41],[42],[43], [44],[45],[46],[47],[48]. The journey of mul- tivalued fixed point theory began with the pioneering work of Nadler [49], who The existence of multi-valued fixed points is established using the following for- mula Hausdorff metric.

Fixed point theorems are a rich area of mathematical research, encompassing various types of contractions and their applications in diverse settings. One of the most well-known is cyclic contraction mapping. It involves a set of maps that satisfy the contraction condition within a specific cycle of mappings. Such conditions are instrumental in modeling scenarios where entities interact sequentially, leading to cyclical dependencies. In particular, cyclic contraction mappings are essential in addressing problems involving dynamic systems and feedback loops.

Moreover, cyclical contractions extend beyond traditional metric spaces to more complex spaces, such as metric-like spaces or partial metric spaces. These versatile conditions enable the exploration of fixed points in settings where standard metric structures may not be applicable, offering novel approaches to solving equations and studying dynamic behaviors [50],[51],[52, 53, 54, 55, 56, 57, 58, 59, 60, 61],

Moving to quasicontractions, [62, 63], which generalize the classical contraction mapping concept. These maps exhibit a controlled, but not necessarily strict contraction, leading to the Banach-Myhill-Nash Theorem. Contractive Conditions, on the other hand, define what is appropriate in maps that ensure the existence of fixed points, this is crucial to functional analysis. Hybrid contractions, such as Geraghty-type contractions, blend different con- traction concepts and are highly applicable to solving real-world problems in economics and optimization. Dass-Gupta-Jaggi Types of contractive mappings further generalizations, accommodating more extensive spaces, and non-standard metric structures. [64, 65, 66, 67]

In the realm of contractive type mappings and related fixed points, the landscape becomes increasingly diverse, with a multitude of theorems tailored to distinct contexts and spaces. These theorems are invaluable. tools for addressing a wide range of mathematical problems and finding fixed points in various applications, from functional analysis to dynamic systems, thus demonstrating the remarkable depth and versatility of the fixed-point theory.

This thesis is divided into five chapters, as follows:

Chapter 1: We briefly discussed certain concepts related to metric spaces. and multivalued analysis, with some examples. then define some types of contractions, whether in single-valued or multi-valued maps.

Chapter 2: We present new fixed point findings on α_L^{ψ}-rational-contraction mappings in metric-like spaces, in addition to practical examples of theorems.

Chapter 3: We offer or provide in complex-valued b-metric spaces certain fixed point theorems of rational type contraction besides examples and concrete illustrations of theorems.

Chapter 4: This chapter focuses on the common fixed point theorem for multi-valued generalized contractive mappings.

Chapter 5: This chapter devotes some fixed point results in the new $b_v(\theta)$-metric spaces, as well as examples and applications for some Fredholm type integral equations and initial value problems.

1

Preliminaries

The notations, definitions, and initial properties used in this thesis are presented in this chapter.

In this section, we start with some definitions and ideas that will help us in the talks that follow.

1.1 Metric spaces

We give some metric spaces definitions, characteristics, and examples.

Definition 1. *Assume Υ be a non-empty set. A function $\vartheta : \Upsilon \times \Upsilon \longrightarrow [0, +\infty)$ such that for every $\dot{x}, \dot{y}, z \in \Upsilon$, we have the following statement:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0 \Leftrightarrow \dot{x} = \dot{y}$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$

(c) $\vartheta(\dot{x}, \dot{y}) \leq \vartheta(\dot{x}, z) + \vartheta(z, \dot{y})$.

the couple (Υ, ϑ) is considered a metric space.

Definition 2. *[81] Assume Υ be non-empty set, $s \geq 1$ and the mapping $\vartheta : \Upsilon \times \Upsilon \longrightarrow [0; +\infty)$, satisfies:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x} = \dot{y}, for\ every\ \dot{x}, \dot{y} \in \Upsilon$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$ *for every* $\dot{x}, \dot{y} \in \Upsilon$

(c) $\vartheta(\dot{x}, \dot{y}) \leqslant s[\vartheta(\dot{x}, z) + \vartheta(z, \dot{y})]$ *for every* $\dot{x}, \dot{y}, z \in \Upsilon$,

then ϑ *is considered a b-metric on* Υ *and the couple* (Υ, ϑ) *is called a b-metric space with coefficient* s.

Definition 3. *[17] Assume* Υ *be a non-empty set and* $\theta : \Upsilon \times \Upsilon \to [1; \infty)$. *A function* $\vartheta_\theta : \Upsilon \times \Upsilon \to [0; \infty)$ *is considered an extended b-metric space if for every* $\dot{x}, \dot{y}, z \in \Upsilon$,*it satisfies:*

(θ_1) $\vartheta_\theta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x} = \dot{y}$, *for every* $\dot{x}, \dot{y} \in \Upsilon$

(θ_2) $\vartheta_\theta(\dot{x}, \dot{y}) = \vartheta_\theta(\dot{y}, \dot{x})$ *for every* $\dot{x}, \dot{y} \in \Upsilon$

(θ_3) $\vartheta_\theta(\dot{x}, \dot{y}) \leqslant \theta(\dot{x}, \dot{y})[\vartheta(\dot{x}, z) + \vartheta(z, \dot{y})]$.

Then the couple $(\Upsilon, \vartheta_\theta)$ *is called extended b-metric space.*

Remark 4. *remark That a b-metric is a peculiar instance of the extended b-metric when* $\theta(\dot{x}, \dot{y}) = s$, *for* $s \geq 1$.

Definition 5. *[82] Assume* Υ *be non-empty set and the mapping* $\vartheta : \Upsilon \times \Upsilon \to [0; +\infty)$ *such that for every* $\dot{x}, \dot{y} \in \Upsilon$ *and all distinct point* $z, u \in \Upsilon \setminus \{\dot{x}, \dot{y}\}$ *satisfies:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x}, \dot{y}$, *for every* $\dot{x} = \dot{y} \in \Upsilon$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$ *for every* $\dot{x}, \dot{y} \in \Upsilon$

(c)$\vartheta(\dot{x}, \dot{y}) \leqslant \vartheta(\dot{x}, z) + \vartheta(z, u) + \vartheta(u, \dot{y})$

Thus, the couple (Υ, ϑ) *is referred to as a rectangular metric space, and* ϑ *is referred to as a rectangular metric on* Υ.

Definition 6. *Assume* Υ *be non-empty set.and the mapping* $\vartheta : \Upsilon \times \Upsilon \to [0; +\infty)$ *such that for every* $\dot{x}, \dot{y} \in \Upsilon$ *and every discrete point* $z, u \in \Upsilon \setminus \{\dot{x}, \dot{y}\}$ *satisfies:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x}, \dot{y}$, *for every* $\dot{x} = \dot{y} \in \Upsilon$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$ *for every* $\dot{x}, \dot{y} \in \Upsilon$

(c) There is a real number $s \geqslant 1$ *such that*

$$\vartheta(\dot{x}, \dot{y}) \leqslant s[\vartheta(\dot{x}, z) + \vartheta(z, u) + \vartheta(u, \dot{y})]$$

Thus, the couple (Υ, ϑ) *is referred to as a rectangular b-metric space with coefficient* s, *and* ϑ *is referred to as a rectangular b-metric on* Υ.

Definition 7. *[18] Assume* Υ *be a non-empty set and* $\theta : \Upsilon \times \Upsilon \to [1; \infty)$.*A function* $\vartheta_\theta : \Upsilon \times \Upsilon \to [0; \infty)$*such that for every* $\dot{x}, \dot{y} \in \Upsilon$ *and all distinct point* $z, u \in$

$\Upsilon \backslash \{\dot{x}, \dot{y}\}$*satisfies:*
(θ_1) $\vartheta_\theta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x} = \dot{y}$, *for all* $\dot{x}, \dot{y} \in \Upsilon$
(θ_2) $\vartheta_\theta(\dot{x}, \dot{y}) = \vartheta_\theta(\dot{y}, \dot{x})$*for every* $\dot{x}, \dot{y} \in \Upsilon$
(θ_3) $\vartheta_\theta(\dot{x}, \dot{y}) \leqslant \theta(\dot{x}, \dot{y})[\vartheta(\dot{x}, z) + \vartheta(z, u) + \vartheta(u, \dot{y}).]$

Then ϑ *is considered a extended rectangular b-metric on X and the couple* (Υ, ϑ) *is considered a extended rectangular b-metric space.*

Definition 8. *[88] Assume* Υ *be a non-empty set. A function* $\vartheta : \Upsilon \times \Upsilon \longrightarrow [0, +\infty)$ *such that for every* $\dot{x}, \dot{y} \in \Upsilon$, *we have the following assertions:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0 \Longrightarrow \dot{x} = \dot{y}$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$

(c) $\vartheta(\dot{x}, \dot{y}) \leq \vartheta(\dot{x}, z) + \vartheta(z, \dot{y})$

The couple (Υ, ϑ) *is considered a metric like-space. A metric-like* ϑ *on* Υ *meets every metrics requirements, with the exception of* $\vartheta(\dot{x}, \dot{x})$ *may be positive for some* $\dot{x} \in \Upsilon$

Definition 9. *[87]Assume* Υ *be a non-empty set,* $s \geq 1$ *a fixed real number, A function* $\vartheta : \Upsilon \times \Upsilon \longrightarrow [0, +\infty)$ *a mapping. Then,* (Υ, ϑ) *claims to be b-metric like space if,* for *every* $\dot{x}, \dot{y} \in \Upsilon$. *The ensuing statements are true:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0 \Longrightarrow \dot{x} = \dot{y}$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$

(c) $\vartheta(\dot{x}, \dot{y}) \leq s[\vartheta(\dot{x}, z) + \vartheta(z, \dot{y})]$

Thus, a couple (Υ, ϑ) *is a b-metric-like space, and* ϑ *is a b-metric-like on* Υ.

Assume $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the following holds:

(i) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$;

(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;

(iii) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;

(iv) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$.

We will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii), and (iv) is satisfied; also we will write $z_1 \prec z_2$ if only (iv) is satisfied.

Notice that $0 \precsim z_1 \lnsim z_2 \implies |z_1| < |z_2|$ and $z_1 \precsim z_2, z_2 \prec z_3 \implies z_1 \prec z_3$.

The following definition is recently introduced by Azam et al. [70].

Definition 10. *Assume Υ be a non-empty set, A function $\vartheta : \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ is called complex valued metric space if for every $\dot{x}, \dot{y}, z \in \Upsilon$, the following statements hold true:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0 \iff \dot{x} = \dot{y}$,

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$,

(c) $\vartheta(\dot{x}, \dot{y}) \precsim \vartheta(\dot{x}, z) + \vartheta(z, \dot{y})$.

We identify the couple (Υ, ϑ) as complex valued metric space.

Example 11. [33] *Assume $\Upsilon = \mathbb{C}$. Define the mapping $\vartheta : \Upsilon \times \Upsilon \to \mathbb{C}$ by*

$$\vartheta(z_1, z_2) = \exp(ik) |z_1 - z_2|^2,$$

where $k \in \left[0, \frac{\pi}{2}\right]$. Then (Υ, ϑ) is a complex valued metric space.

Definition 12. [72] *Assume Υ be a non-empty set, $s \geq 1$ a fixed real number, A function $\vartheta : \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ is called complex valued b-metric space if for every $\dot{x}, \dot{y}, z \in \Upsilon$, the following statements hold true:*

(a) $\vartheta(\dot{x}, \dot{y}) = 0 \iff \dot{x} = \dot{y}$

(b) $\vartheta(\dot{x}, \dot{y}) = \vartheta(\dot{y}, \dot{x})$

(c) $\vartheta(\dot{x}, \dot{y}) \precsim s\,[\vartheta(\dot{x}, z) + \vartheta(z, \dot{y})]$

The complex valued b-metric space is the couple (Υ, ϑ).

Example 13. [72]*Assume $\Upsilon = [0, 1]$. Define the mapping $\vartheta : \Upsilon \times \Upsilon \to \mathbb{C}$ by*

$$\vartheta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y}|^2 + i\,|\dot{x} - \dot{y}|^2,$$

for every $\dot{x}, \dot{y} \in \Upsilon$.Then, with $s = 2$, (Υ, ϑ) is a complex valued b-metric space.

Definition 14. *[87]Assume* $\{\dot{x}_n\}$ *be a sequence with the coefficient* s *in a b-metric-like space* (Υ, ϑ) *. Then:*

(i) *if* $\lim_{n\to\infty} \vartheta(\dot{x}_n, \dot{x}) = \vartheta(\dot{x}, \dot{x})$ *,then the sequence* $\{\dot{x}_n\}$ *is said to be convergent to* $\dot{x}$ *.*

(ii) *if* $\lim_{n,m\to\infty} \vartheta(\dot{x}_n, \dot{x}_m)$ *exists and is finite, then the sequence* $\{\dot{x}_n\}$ *is said to be* ϑ*-Cauchy in* (Υ, ϑ) *.*

One says that a b-metric-like space (Υ, ϑ) *is* ϑ*-complete if for every d-Cauchy sequence* $\{\dot{x}_n\}$ *in* Υ *there exists an* $\dot{x} \in \Upsilon$*, such that* $\lim_{n,m\to\infty} \vartheta(\dot{x}_n, \dot{x}_m) = \vartheta(\dot{x}, \dot{x}) = \lim_{n\to\infty} \vartheta(\dot{x}_n, \dot{x})$ *.*

Lemma 15. *Assume that* $\lim_{n\to\infty} \vartheta(\dot{x}_n, \dot{x}_{n+1}) = 0$*, for a sequence* $\{\dot{x}_n\}$ *on a complete b-metric space* (Υ, ϑ) *with* $s \geq 1$ *.*

If $\lim_{n,m\to\infty} \vartheta(\dot{x}_n, \dot{x}_m) \neq 0$*, there exist* $\varepsilon > 0$ *and two sequences* $\{m_k\}_{k=1}^{+\infty}, \{n_k\}_{k=1}^{+\infty}$ *of positive integers with* $n_k > m_k > k$ *such that*

$$\begin{aligned} \vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right) &\geq \varepsilon, \vartheta\left(\dot{x}_{m_k}, \dot{x}_{n_k-1}\right) < \varepsilon, \frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup \vartheta\left(\dot{x}_{n_k-1}, \dot{x}_{m_k-1}\right) \leq s\varepsilon, \\ \frac{\varepsilon}{s} \lim_{k\to\infty} \sup \vartheta\left(\dot{x}_{n_k-1}, \dot{x}_{m_k}\right) &\leq \varepsilon, \frac{\varepsilon}{s^2} \lim_{k\to\infty} \sup \vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k-1}\right) \leq \varepsilon s^2 \end{aligned}$$

Definition 16. *[92]Assume* (Υ, ϑ) *be a b-metric-like space with the coefficient* s*, denoted as* (Υ, ϑ)*. A sequence* $\{\dot{x}_n\}$ *is referred to as* $0-d-$*Cauchy sequence if* $\lim_{n,m\to\infty} \vartheta(\dot{x}_n, \dot{x}_m) = 0$.
A space (Υ, ϑ) *is considered to be* $0-\vartheta-$*complete when it satisfies the condition that every* $0-\vartheta-$*Cauchy sequence is complete.*
The sequence within the set Υ *eventually converges to a specific point* $\dot{x}, where \dot{x}$ *belongs to* Υ *satisfying the condition* $\vartheta(\dot{x}, \dot{x}) = 0$.

In the sequel, we give the following result to prove that certain Picard sequences are Cauchy see:[89, 90], and the definitions of known notions in existing literature as well as some known results.

Lemma 17. *Assume* $\{\dot{x}_n\}$ *denote a sequence in the b-metric-like space* (Υ, ϑ)*, coefficient* s *must be greater than or equal to 1. such that*

$$\vartheta(\dot{x}_n, \dot{x}_{n+1}) \leq q\vartheta(\dot{x}_{n-1}, \dot{x}_n)$$

where q within the interval $\left[0; \frac{1}{s}\right)$*, and* $n \in N$*, so we can conclude that* $\{\dot{x}_n\}$ *is a* ϑ*-Cauchy sequence in* (Υ, ϑ) *such that* $\lim_{n,m\to\infty} \vartheta(\dot{x}_n, \dot{x}_m) = 0$

Remark 18. *In the context of b-metric-like, it is important to acknowledge that the aforementioned lemma remains applicable. Each value of q within the range of [0,1) is associated with a designated space, as referenced in [86].*

Definition 19. *Let* (Υ, ϑ) *be b-metric space with complex values, as assumed in*[72]

(i) *An element* $\dot{x}$ *in the set* Υ *is referred to as an interior point. Whenever there is a value of r greater than zero, the set A is a subset of* Υ,

such that $B(\dot{x}, r) = \{\Upsilon \in \Upsilon : \vartheta(\dot{x}, \dot{y}) < r\} \subseteq A$.

(ii) *A point* $\dot{x} \in \Upsilon$ *is considered a limit point of a set A whenever for every* $0 < r \in \mathbb{C}$, $B(\dot{x}, r) \cap (A - \{\dot{x}\}) \neq \phi$.

(iii) *A subset* $A \subseteq \Upsilon$ *is considered an open set whenever each element of A is an interior point of a set A.*

(iv) *A subset* $A \subseteq \Upsilon$ *is called closed set whenever each limit point of A belongs to A.*

(v) *A sub-basis for Hausdorff topology* τ *on* Υ *is a family*

$$F = \{B(\dot{x}, r) : x \in \Upsilon \; and \; 0 < r\}.$$

Definition 20. [72]*Assume* (Υ, ϑ) *be a b-metric space with complex values, and Assume* $\{\dot{x}_n\}$ *represent a sequence in* Υ *and* $\dot{x} \in \Upsilon$.

(i) For every positive c in the set C, there exists a corresponding $N \in \mathbb{N}$ such that for every $n > N$, $\vartheta(\dot{x}_n, \dot{x}) < c$, then $\{\dot{x}_n\}$ is referred to as being The sequence converges to $\dot{x}$. We denote this by $\lim_{n\to\infty} \dot{x}_n = \dot{x}$ or $\{\dot{x}_n\} \to \dot{x}$ as $n \to \infty$.

(ii) If for every $c \in C$, with $0 < c$ there is $N \in \mathbb{N}$ such that for every $n > N$, $\vartheta(\dot{x}_n, \dot{x}_{n+m}) < c$ where $m \in \mathbb{N}$, then$\{\dot{x}_n\}$ is said to be Cauchy sequence.

(iii) If each Cauchy sequence within the set Υ converges within Υ,then The space (Υ, ϑ)) is often referred to as a complete complex-valued b-metric space.

Lemma 21. [72]*Consider a complex-valued b-metric space* (Υ, ϑ) *and Let* $\{\dot{x}_n\}$ *be a given sequence in* Υ*. The convergence of* $\{\dot{x}_n\}$ *to* $\dot{x}$ *is characterized by the following condition* $|\vartheta(\dot{x}_n, \dot{x})| \to 0$ *as* $n \to \infty$.

Lemma 22. [72] *Assume* (Υ, ϑ) *be a complex valued b-metric space, if we consider a sequence* $\{\dot{x}_n\}$ *in* Υ*. then it can be said that* $\{\dot{x}_n\}$ *is the Cauchy sequence is defined if and only if* $|\vartheta(\dot{x}_n, \dot{x}_{n+m})| \to 0$ *as* $n \to \infty$ *, where* $m \in \mathbb{N}$.

Definition 23. *[75]Assume* $\{\dot{x}_n\}$ *defines a sequence in a b-metric space* (Υ, ϑ) *with the coefficient* s*. Hence, we have:*

(i) *The convergence of the sequence* $\{\dot{x}_n\}$ *to* $\dot{x}$ *is defined as follows:* $\lim_{n\to\infty} \vartheta\,(\dot{x}_n, \dot{x}) \to 0$;

(ii) *The sequence* $\{\dot{x}_n\}$ *is considered to be Cauchy in* (Υ, ϑ) *within the context of* $\lim_{n,m\to\infty} \vartheta\,(\dot{x}_n, \dot{x}_m) \to 0$

(iii) *One says that a b-metric space* (Υ, ϑ) *is complete if for every Cauchy sequence* $\{\dot{x}_n\}$ *in* Υ *is convergent.*

Definition 24. *[80] Assume* (Υ, ϑ) *be a complete b-metric space. In the sequel, we use the following notations:*

$CB(Y) = \{$*is defined as a non-empty closed and bounded subset of* $\Upsilon\}$,
$\vartheta(A, B) = \inf\{\vartheta(a, b) : a \in A, b \in B\}$,
$\delta(A, B) = \sup\{\vartheta(a, B) : a \in A\}$,
$\delta(B, A) = \sup\{\vartheta(b, A) : b \in B\}$,
$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}$.

The metric ϑ*, gives rise to the Hausdorff metric* H*, as can be observed.*

Moving ahead, we define $F(T)$ *as the collection of all fixed points of a multi-valued mapping* T*, that is:*
$F(T) = \left\{p \in \Upsilon : p \in T_p\right\}$

Definition 25. *A fixed point of the multi-valued function is defined as a point* $\dot{x}_0 \in \Upsilon$*, where* $T : \Upsilon \to CB(Y)$ *if* $\dot{x}_0 \in T\dot{x}_0$.

Lemma 26. *Assume*(Υ, ϑ) *be a b-metric space that is complete. For any* A, B *and* $C \in CB(Y)$ *and any* $\dot{x}, \dot{y} \in \Upsilon$*, one has the following:*

1 $\vartheta(\dot{x}, B) \le \vartheta(\dot{x}, b)$ *,for any* $b \in B$.

2 $\delta(A, B) \le H(A, B)$.

3 $\vartheta(\dot{x}, B) \le H(A, B)$*, for any* $\dot{x} \in A$.

4 $H(A, A) = 0$.

5 $H(A, B) = H(B, A)$.

6 $H(A, C) \leq s[H(A, B) + H(B, C)]$.

7 $\vartheta(\dot{x}, A) \leq s[\vartheta(\dot{x}, \dot{y}) + \vartheta(\dot{x}, A)]$.

8 $\vartheta(A, B) \leq \delta(A, B)$.

Lemma 27. *[75]Assume(Υ, ϑ) be a complete b-metric space and Assume $\{\dot{x}_n\}$ be a sequence in Υ such that*

$$\vartheta\left(\dot{x}_{n+1}, \dot{x}_{n+2}\right) \leq \lambda \vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right), \textit{for every } n = 0, 1, 2, ...$$

where $0 \leq \lambda < 1$. Then $\{\dot{x}_n\}$ is a Cauchy sequence in Υ.

1.2 Various types of contractions

Some types of contractions in single-valued and multi-valued maps are defined in this section.

Definition 28. ([68]) *A mapping $T : \Upsilon \to \Upsilon$ where (Υ, ϑ) is a metric space is said to be weakly C-contractive or a weak C-contraction if for every $\dot{x}, \dot{y} \in \Upsilon$,*

$$(\vartheta\left(T\dot{x}, T\dot{y}\right)) \leq \frac{1}{2}[\vartheta(\dot{x}, T\dot{y} + \vartheta(\dot{y}, T\dot{x}) - \psi\left[\vartheta(\dot{x}, T\dot{y}) + \vartheta(\dot{y}, T\dot{x})\right].$$

where $\psi \in [0, \infty)^2 \to [0, \infty)$,is a continuous mapping such that $\psi\left(\dot{x}, \dot{y}\right) = 0$ if and only if $\dot{x}, \dot{y} = 0$.

Denote Ω as the class of all function $\Psi : [0; \infty) \to [0; \infty)$, satisfying the following condition:

(1) Ψnon-decreasing and continuous;

(2) $\lim_{n\to\infty} \Psi^n(t) = 0$ for every $t > 0$.

Definition 29. *,[89] Assume (Υ, ϑ) be a b-metric-like space, $p \in N$, $B_1, B_2, ..., B_p$ be ϑ-closed subsets of Υ, $\Upsilon = B_1 \cup ... \cup B_p$ and $\alpha : \Upsilon \times \Upsilon \to [0; \infty)$ be a mapping. We say that $T : \Upsilon = \Upsilon \to \Upsilon$ is cyclic α_L^{Ψ}-rational contractive mapping if :*

(1) $T(B_i) \subseteq B_{i+1}, i = 1, 2, ..., p$, where $B_{p+1} = B_1$;

(2) for any $\dot{x} \in B_i$ and $\Upsilon \in B_{i+1}$, $i = 1, 2, ..., p$, where $B_{p+1} = B_1$ and $\alpha(\dot{x}, T\dot{x})\,\alpha(\dot{y}, T\dot{y}) \geq 1$, holds

$$(\vartheta(T\dot{x}, T\dot{y})) \leq \Psi(M(\dot{x}, \dot{y})) - LM(\dot{x}, \dot{y}) \tag{1}$$

Where $\Psi \in \Omega$, $L \in (0; 1)$ and

$$M_d(\dot{x}, \dot{y}) = \max\left\{ \begin{array}{c} \vartheta(\dot{x}, \dot{y}), \frac{\vartheta(\dot{x},\dot{y})}{2s} \frac{\vartheta(\dot{x},T\dot{x})\vartheta(\dot{y},T\dot{y})}{1+\vartheta(\dot{x},\dot{y})}, \frac{\vartheta(\dot{y},T\dot{y})[1+\vartheta(\dot{x},T\dot{x})]}{1+\vartheta(\dot{x},\dot{y})}, \\ \frac{\vartheta(\dot{x},T\dot{y})+\vartheta(\dot{y},T\dot{x})}{4s} \end{array} \right\}.$$

Definition 30. ([91]) *Assume $\Pi = \{\phi : [0, +\infty) \to [0, +\infty) : \phi$ is increasing and continuous$\}$ and $\Omega = \{\psi : [[0, +\infty) \to [0, +\infty) : \psi$ is increasing and lower semi-continuous$\}$.*
A triple (ϕ, ψ, f) is said to be monotone if

$$\dot{x} \leq \Upsilon \text{ implies } f(\phi(\dot{x}), \psi(\dot{x})) \leq f(\phi(\Upsilon), \psi(\Upsilon)),$$

for any $\dot{x}, \dot{y} \in [0, +\infty)$.

Definition 31. ([91])*Assume (Υ, ϑ) be a metric-like space, $p \in \mathbb{N}$, $B_1, B_2, ..., B_p$ be ϑ-closed subsets of Υ, $\Upsilon = B_1 \cup ... \cup B_p$ and $\alpha : \Upsilon \times \Upsilon \to [0, +\infty)$ be a mapping. We say that $T : \Upsilon \to \Upsilon$ is cyclic(α, f, ϕ, ψ) contractive mapping if:*
$T(B_j) \subseteq B_{j+1}$, for every $j = 1, 2, ..., p$, where $B_p = B_{p+1}$

for any $\dot{x} \in B_i$ and $\Upsilon \in B_i$, $(i = 1, 2, ..., p)$, where $B_{i+1} = B_1$ and $\alpha(\dot{x}, T\dot{x})\,\alpha(\dot{y}, T\dot{y}) \geq 1$
we have

$$\phi(\vartheta(T\dot{x}, T\dot{y})) \leq f(\phi(M_d(\dot{x}, \dot{y})), \psi(M_d(\dot{x}, \dot{y}))), \tag{1.1}$$

where $\phi \in \Pi, \psi \in \Omega$ and $f \in C$ such that the triple (ϕ, ψ, f) is monotone and

$$M_d(\dot{x}, \dot{y}) = \frac{a\vartheta(\dot{x}, \dot{y}) + b\vartheta(\dot{x}, T\dot{x}) + c\vartheta(\dot{y}, T\dot{y}) + e\frac{\vartheta(\dot{x}, T\dot{y}) + \vartheta(\dot{y}, T\dot{x})}{2}}{m},$$

with $a, b, c, e \geq 0$ and $a + b + c + 2e = m < 1$.

Definition 32. *Assume(Υ, ϑ) be a complete b-metric space with coefficient $s \geq 1$, and $T, S : \Upsilon \to CB(\Upsilon)$ be two mapping on Υ can be classified as generalized rational contractive mappings if there exist the control functions $\phi, \varphi, \psi : \Upsilon \times \Upsilon \to [0; 1)$ such that:*

$$\begin{aligned} H(T\dot{x}, S\dot{y}) \leq\ & \phi(\dot{x}, \dot{y})\vartheta(\dot{x}, \dot{y}) + \varphi(\dot{x}, \dot{y})[\vartheta(\dot{x}, S\dot{y}) + \vartheta(\dot{y}, T\dot{x})] \\ & + \psi(\dot{x}, \dot{y})\frac{\vartheta(\dot{x}, T\dot{x})\vartheta(\dot{y}, T\dot{y})}{1 + \vartheta(\dot{x}, S\dot{y}) + \vartheta(\dot{y}, T\dot{x}) + \vartheta(\dot{x}, \dot{y})}, \end{aligned} \tag{1.2}$$

for every $\dot{x}, \dot{y} \in \Upsilon$.

Definition 33. *Assume Υ be a non-empty set. A mapping $T : \Upsilon \to \Upsilon$ is said to be an α-admissible mapping if $\alpha(\dot{x}, \dot{y}) \geq 1$ implies*
$\alpha(T\dot{x}, T\dot{y}) \geq 1$, for every $\dot{x}, \dot{y} \in \Upsilon$ and $\alpha : \Upsilon \times \Upsilon \to [0; \infty)$.
Further T called α-continuous on Υ if $\lim_{n\to\infty} \dot{x}_n = \dot{x}$ implies $\lim_{n\to\infty} T\dot{x}_n = T\dot{x}$ for any sequence $\{\dot{x}_n\}$ for Υ which $\alpha(\dot{x}_n, \dot{x}_{n+1}) \geq 1$; $n \in N$.

2

New fixed point results on α_L^{ψ}-rational contraction mappings in -metric-like spaces

This chapter will discuss our findings from([89]), which include new findings on α_L^{ψ}-rational contraction and cyclic α_L^{ψ}-rational contractive mappings defined in complete b-metric-like spaces. To demonstrate the applicability of our theoretical findings, an example is provided.

2.1 Fixed point on α_L^{ψ}-rational contraction mappings in -metric-like spaces

Theorem 34. *Let* (Υ, ϑ) *be a* $\vartheta - complete\ b{-}metric$ *like space and* $\alpha : \Upsilon \times \Upsilon \to [0, +\infty)$ *be a mapping. Assume that* $T : \Upsilon \to \Upsilon$ *is an* α_L^{ψ}*-contractive mapping satisfying the following assertions:*
(i) T *is an* α*-admissible mapping,*
(ii) $\alpha(\dot{x}_0, T\dot{x}_0) \geq 1$ *for an element* $\dot{x}_0$ *in* Υ*,*
(iii) T *is* α*-continuous, or;*
(iv) if $\{\dot{x}_n\}$ *is a sequence in* Υ *such that* $\alpha(\dot{x}_n, \dot{x}_{n+1}) \geq 1$
for all $n \in \mathbb{N}$ *and* $\dot{x}_n \to \dot{x}$ *as* $n \to +\infty$*, then* $\alpha(\dot{x}_n, T\dot{x}_n) \geq 1$*.*
Then T *admits a fixed point in* Υ*.*

Moreover, if
(v) $\alpha(\dot{x}, \dot{x}) \geq 1$, *whenever* $\dot{x} \in Fix(T)$, *then* T *admits a unique fixed point.*

proof Let start with define the sequence $\dot{x}_n = T_n \dot{x}_0$, where $\dot{x}_0$ is the given point for which $\alpha(\dot{x}_0, T\dot{x}_0) \geq 1$. Since T is an α-admissible mapping, we get that

$$\alpha(\dot{x}_1, T\dot{x}_1) = \alpha(T\dot{x}_0, TT\dot{x}_1) \geq 1.$$

Continuing this process, we get $\alpha(\dot{x}_n, T\dot{x}_n) \geq 1$ for all $n \in \mathbb{N}$, and so,

$$\alpha(\dot{x}_n, T\dot{x}_n)\,\alpha(\dot{x}_{n-1}, T\dot{x}_{n-1}) \geq 1, \ for\ all\ n \in \mathbb{N}.$$

If $\dot{x}_n = \dot{x}_{n-1}$ for some $n \in \mathbb{N}$, $\dot{x}_{n-1}$ is a fixed point of T.
Therefore, assume that $\dot{x}_{n-1} \neq \dot{x}_n$ for all $n \in \mathbb{N}$. Hence, we have that

$$\vartheta(\dot{x}_{n-1}, \dot{x}_n) > 0, for\ all\ n \in \mathbb{N}.$$

In order to prove that the sequence $\{\dot{x}_n\}$ is a ϑ-Cauchy sequence, we have

$$\Psi(\vartheta(\dot{x}_n, \dot{x}_{n+1})) \leq \Psi(M(\dot{x}_{n-1}, \dot{x}_n)) - LM(\dot{x}_{n-1}, \dot{x}_n) \tag{2.1}$$

where

$$\begin{aligned} M(\dot{x}_{n-1}, \dot{x}_n) &= \max\left\{\begin{array}{c} \vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_n), \frac{\vartheta(\dot{x}_{n-1}, \dot{x}_n)\vartheta(\dot{x}_n, \dot{x}_{n+1})}{1+\vartheta(\dot{x}_{n-1}, \dot{x}_n)}, \frac{\vartheta(\dot{x}_n, \dot{x}_{n+1})[1+\vartheta(\dot{x}_{n-1}, \dot{x}_n)]}{1+\vartheta(\dot{x}_{n-1}, \dot{x}_n)}, \\ \frac{\vartheta(\dot{x}_{n-1}, \dot{x}_{n+1})+\vartheta(\dot{x}_n, \dot{x}_{n+1})}{4s} \end{array}\right\} \\ &\leq \max\left\{\vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_{n+1}), \frac{3}{4}\vartheta(\dot{x}_{n-1}, \dot{x}_n) + \frac{1}{4}\vartheta(\dot{x}_n, \dot{x}_{n+1})\right\} \\ &\leq \max\{\vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_{n+1})\}, \end{aligned}$$

Hence, we get

$$\Psi(\vartheta(\dot{x}_n, \dot{x}_{n+1})) \leq \Psi(\max\{\vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_{n+1})\}) - L\max\{\vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_{n+1})\}.$$

If

$$\max\{\vartheta(\dot{x}_{n-1}, \dot{x}_n), \vartheta(\dot{x}_n, \dot{x}_{n+1})\} = \vartheta(\dot{x}_n, \dot{x}_{n+1})\ for\ some\ n \in \mathbb{N},$$

we have

$$\begin{aligned} \Psi\left(\vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right)\right) &\leq \Psi\left(\vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right)\right) - L\vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right) \\ 0 &\leq -L\vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right). \end{aligned} \tag{2.2}$$

Which contradiction.
Hence, we get

$$\vartheta\left(\dot{x}_{n-1}, \dot{x}_n\right) \geq \vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right).$$

So, there exists

$$\lim_{n\to+\infty} \vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right) = \vartheta_k \geq 0.$$

Assumeting $n \to +\infty$ in (2.2), we obtain that

$$\begin{aligned} \lim_{n\to+\infty} \Psi\left(\vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right)\right) &\leq \lim_{n\to+\infty} \left[\Psi\left(\vartheta\left(\dot{x}_{n-1}, \dot{x}_n\right)\right) - L\vartheta\left(\dot{x}_{n-1}, \dot{x}_n\right)\right] \\ \Psi\left(\vartheta_k\right) &\leq \Psi\left(\vartheta_k\right) - L\vartheta_k. \end{aligned}$$

Thus

$$\lim_{n\to+\infty} \vartheta\left(\dot{x}_n, \dot{x}_{n+1}\right) = 0.$$

Now, if $\lim_{n,m\to+\infty} \vartheta\left(\dot{x}_n, \dot{x}_m\right) \neq 0$, we have sequences $\{m_k\}$ and $\{n_k\}$ such that

$$\lim_{k\to+\infty} \vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right) = \varepsilon > 0.$$

Assume $u = \dot{x}_{n_k}$ and $v = \dot{x}_{m_k}$ in (1), we get

$$\Psi\left(\vartheta\left(\dot{x}_{n_k+1}, \dot{x}_{m_k+1}\right)\right) \leq \Psi\left(M\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right)\right) - LM\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right), \tag{2.3}$$

where

$$\begin{aligned} M\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right) &= \max\left\{ \begin{array}{c} \vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right), \vartheta\left(\dot{x}_{m_k}, \dot{x}_{n_k+1}\right), \frac{\vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right)\vartheta\left(\dot{x}_{m_k}, \dot{x}_{m_k+1}\right)}{1+\vartheta\left(\dot{x}_{n_k}, \dot{x}_{n_k+1}\right)}, \\ \frac{\vartheta\left(\dot{x}_{m_k}, \dot{x}_{m_k+1}\right)\left[1+\vartheta\left(\dot{x}_{n_k}, \dot{x}_{n_k+1}\right)\right]}{1+\vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k}\right)}, \\ \frac{\vartheta\left(\dot{x}_{n_k}, \dot{x}_{m_k+1}\right)+\vartheta\left(\dot{x}_{m_k}, \dot{x}_{n_{k+1}}\right)}{4s} \end{array} \right\} \\ &\to \max\left\{\varepsilon, \frac{\varepsilon}{2s}, 0, 0, \frac{\varepsilon}{4s}\right\}. \end{aligned}$$

So, as $n \longrightarrow +\infty$ in (3.9) , we have

$$\Psi(\varepsilon) \leq \Psi(\varepsilon) - L\varepsilon,$$

which is paradoxical.

Thus, the sequence$\{\dot{x}_n\}$ is a cauchy and

$$\lim_{n,m\longrightarrow+\infty} \vartheta(\dot{x}_n, \dot{x}_m) = 0.$$

This proves that a single point exists. $\dot{x}^* \in \Upsilon$ such that

$$\vartheta(\dot{x}^*, \dot{x}^*) = \lim_{n\longrightarrow+\infty} \vartheta(\dot{x}_n, \dot{x}^*) = \lim_{n,m\longrightarrow+\infty} \vartheta(\dot{x}_n, \dot{x}_m) = 0.$$

We will now demonstrate that $\dot{x}^*$ is fixed point of T i.e., $T\dot{x}^* = \dot{x}^*$, so it's evident if T is α-continuous.
Additionally, consider that for any sequence $\dot{x}_n$ in Υ and for all $n \in \mathbb{N}$,if $\alpha(\dot{x}_n, \dot{x}_{n+1}) \geq 1$ and $\lim_{n\longrightarrow+\infty} \dot{x}_n = \dot{x}^*$, then

$$\alpha(\dot{x}^*, T\dot{x}^*) \geq 1.$$

Assume $\vartheta(\dot{x}^*, T\dot{x}^*) > 0$. Knowing

$$\alpha(\dot{x}_n, T\dot{x}_n)\,\alpha(\dot{x}^*, T\dot{x}^*) \geq 1.$$

As per the specified contractual terms, we have

$$\Psi(\vartheta(T\dot{x}, T\dot{x}^*)) \leq \Psi(M(\dot{x}, \dot{x}^*)) - LM(\dot{x}, \dot{x}^*).$$

where

$$\begin{aligned} M(\dot{x}, \dot{x}^*) &= \max\left\{\begin{array}{c} \vartheta(\dot{x}_n, \dot{x}^*), \vartheta(\dot{x}^*, \dot{x}_{n+1}), \frac{\vartheta(\dot{x}_n,\dot{x}^*)\vartheta(\dot{x}^*,T\dot{x}^*)}{1+\vartheta(\dot{x}_n,\dot{x}_{n+1})}, \\ \frac{\vartheta(\dot{x}^*,T\dot{x}^*)[1+\vartheta(\dot{x}_n,\dot{x}_{n+1})]}{1+\vartheta(\dot{x}_n,\dot{x}_{n+1})}, \\ \frac{\vartheta(\dot{x}_n,T\dot{x}^*)+\vartheta(\dot{x}^*,\dot{x}_{n+1})}{4s} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} \vartheta(\dot{x}_n, \dot{x}^*), \vartheta(\dot{x}^*, \dot{x}_{n+1}), \frac{\vartheta(\dot{x}_n,\dot{x}^*)\vartheta(\dot{x}^*,T\dot{x}^*)}{1+\vartheta(\dot{x}_n,\dot{x}_{n+1})}, \\ \vartheta(\dot{x}^*, T\dot{x}^*), \\ \frac{s[\vartheta(\dot{x}_n,\dot{x}^*)+\vartheta(\dot{x}^*,T\dot{x}^*)]+\vartheta(\dot{x}^*,\dot{x}_{n+1})}{4s} \end{array}\right\} \\ &\longrightarrow \max\left\{0, 0, 0, \vartheta(\dot{x}^*, T\dot{x}^*), \frac{\vartheta(\dot{x}^*, T\dot{x}^*)}{4}\right\} = \vartheta(\dot{x}^*, T\dot{x}^*), \text{ as } n \longrightarrow +\infty. \end{aligned}$$

Allowing the limit to $n \longrightarrow +\infty$, We obtain

$$\Psi\left(\vartheta\left(\dot{x}^{*}, T\dot{x}^{*}\right)\right) \leq \Psi\left(\vartheta\left(\dot{x}^{*}, T\dot{x}^{*}\right)\right) - L\vartheta\left(\dot{x}^{*}, T\dot{x}^{*}\right),$$

which again is contradictory.
This implies that $\vartheta\left(\dot{x}^{*}, T\dot{x}^{*}\right) = 0$, namely, we demonstrate that $T\dot{x}^{*} = \dot{x}^{*}$.
Lastly, to demonstrate the fixed point of the map T is uniqueness, Let's claim that $\zeta, \eta\ (\zeta \neq \eta)$ are two fixed point of T.
Then,we get $\vartheta(\zeta, \eta) > 0$, $\alpha(\zeta, \zeta) \geq 1$, $\alpha(\eta, \eta) \geq 1$.
Moreover, since

$$\alpha(\zeta, \zeta)\,\alpha(\eta, \eta) \geq 1,$$

We acquire

$$\Psi(\vartheta(\zeta, \eta)) \leq \Psi(M(\zeta, \eta)) - LM(\zeta, \eta),$$

in which

$$\begin{aligned} M(\zeta, \eta) &= \max\left\{\begin{array}{c} \vartheta(\zeta, \eta), \frac{\vartheta(\eta,\zeta)}{2s}, \frac{\vartheta(\zeta,\eta)\vartheta(\eta,\eta)}{1+\vartheta(\zeta,\zeta)}, \\ \frac{\vartheta(\eta,\eta)[1+\vartheta(\zeta,\zeta)]}{1+\vartheta(\zeta,\eta)}, \\ \frac{\vartheta(\zeta,\eta)+\vartheta(\eta,\zeta)}{4s} \end{array}\right\} \\ &= \max\left\{\vartheta(\zeta, \eta), \frac{\vartheta(\eta, \zeta)}{2s}, 0, 0, \frac{\vartheta(\eta, \zeta)}{2s}\right\} = \vartheta(\zeta, \eta). \end{aligned}$$

Thus

$$\Psi(\vartheta(\zeta, \eta)) \leq \Psi(\vartheta(\zeta, \eta)) - L\vartheta(\zeta, \eta),$$

which is paradoxical.The proof is now complete.

Remark 35. *It is noteworthy to observe that the case $s = 1$ means that (Υ, d) is actually a complete metric-like space and we get the results of [55].*

2.2 Example

Example 36. *Given a constant $s = 4$, Assume $\Upsilon = \mathbb{R}$ be a b-metric-like space Define the function $\vartheta : \mathbb{R}^2 \longrightarrow: [0, +\infty)$ by .$\vartheta(\dot{x}, \dot{y}) = (|\dot{x}| + |\dot{y}|)^3$. (Υ, ϑ) is obviously a complete b-metric-like space. Assume that*

$$B_1 = (-\infty, 0],\ B_2 = [0, +\infty)$$

and

$$\Upsilon = B_1 \cup B_2.$$

Define $T : \Upsilon \to \Upsilon$ *and* $\sigma : \Upsilon \times \Upsilon \to [0, +\infty)$ *by*

$$T\dot{x} = \begin{cases} \dot{x}^2 & if \quad \dot{x} \in (-\infty, 1) \\ -\frac{x}{6} & if \quad \dot{x} \in [-1, 0] \\ -\frac{\dot{x}^2}{7} & if \quad \dot{x} \in [0, 1] \\ -x & if \quad \dot{x} \in (1, +\infty) \end{cases} \quad and\ \sigma(\dot{x}, \dot{y}) = \begin{cases} |\dot{x}| + |\dot{y}| + 1, & if\ \dot{x}, \dot{y} \in [-1; 1] \\ 0, & otherwise. \end{cases}$$

Also, define $\Psi : [0, +\infty) \to [0, +\infty)$ *by* $\Psi(t) = \frac{1}{2}t$ *and* $L = \frac{1}{6}$.*Clearly,* $T(B_1) \subset B_2$ *and* $T(B_2) \subset B_1$.

Assume $\dot{x} \in B_1$, $y \in B_2$ *and* $\sigma(\dot{x}, T\dot{x})\,\sigma(\dot{y}, T\dot{y}) \geq 1$.*If* $\dot{x} \notin [-1, 1]$ *or* $y \notin [-1, 1]$, *then* $\sigma(\dot{x}, T\dot{x}) = 0$ *or* $\sigma(\dot{y}, T\dot{y}) = 0$.*That is,* $\sigma(\dot{x}, T\dot{x})\,\sigma(\dot{y}, T\dot{y}) = 0$, *which is a contradiction. Hence* $\dot{x} \in B_1$, $y \in B_2$ *and* $\dot{x}, \dot{y} \in [-1, 1]$.

This implies that $\dot{x} \in [-1, 0]$ *or* $\dot{y} \in [1, 0]$.

Then,

$$\begin{aligned} \Psi(\vartheta(T\dot{x}, T\dot{y})) &= \frac{1}{2}\left(\left|-\frac{\dot{x}}{6}\right| + \left|-\frac{\dot{y}^2}{7}\right|\right)^3 \\ &\leq 2\left(\left|\frac{\dot{x}}{6}\right|^3 + \left|\frac{\dot{y}^2}{7}\right|^3\right) \\ &\leq \frac{1}{3}\left(|\dot{x}|^3 + \left|\dot{y}^2\right|^3\right) \\ &\leq \frac{1}{3}\left(|\dot{x}|^3 + |\dot{y}|^3\right),\ since\ \dot{y} \in [1, 0]. \\ &\leq \frac{1}{3}(|\dot{x}| + |\dot{y}|)^3 \\ &= \frac{1}{3}\vartheta(\dot{x}, \dot{y}) = \Psi(M_d(\dot{x}, \dot{y})) - LM_d(\dot{x}, \dot{y}). \end{aligned}$$

Consequently, T is a cyclic α_L^{ψ}*-rational contractive mapping. It is evident that* $\sigma(0, T0) \geq 1$ *and so the condition* (ii) *of Theorem 34 is satisfied.*

When $\sigma(\dot{x}, y) \geq 1$, *it means that* $\dot{x}, \dot{y} \in [-1, 1]$, *indicating that T is an* α*-admissible mapping and* $\sigma(T\dot{x}, T\dot{y}) \geq 1$.

Consider $\{\dot{x}_n\}$ *as a sequence in* Υ *such that* $\sigma(\dot{x}_n, T\dot{x}_n) \geq 1$ *and* $\dot{x}_n \to \dot{x}$ *as* $n \to +\infty$.

Hence, we have to have $\dot{x}_n \in [-1, 1]$ *and so,* $\dot{x} \in [-1, 1]$, *that is* $a(\dot{x}, T\dot{x}) \geq 1$. *Thus,* *all of Theorem* 34 *requirements are met as result, and* T *has a fixed point* $\dot{x} = 0 \in B_1 \cap B_2$.

3

Some fixed point theorems of rational type contraction in complex valued b-metric spaces

This chapter's goals are to generalize certain findings from the previous literature and establish a common fixed-point theorem for rational-type contractions in the context of complex valued b-metric spaces. Lastly, we provide a compelling illustration to back up our primary findings.

3.1 Fixed point theorems of rational type contraction in complex valued b-metric spaces

Theorem 37. *Assume* (Υ, ϑ) *be a complete complex valued b-metric space with a coefficient* $s \geq 1$, *and* $T : \Upsilon \to \Upsilon$ *be a mappings on* Υ *satisfying the condition*

$$\vartheta(T\dot{x}, T\dot{y}) \precsim a\vartheta(\dot{x}, \dot{y}) + b\frac{\vartheta(\dot{x}, T\dot{x})\,\vartheta(\dot{x}, T\dot{y}) + \vartheta(\dot{y}, T\dot{y})\,\vartheta(\dot{y}, T\dot{x})}{\vartheta(\dot{x}, T\dot{y}) + \vartheta(\dot{y}, T\dot{x})}, \tag{3.1}$$

for all, $\dot{x}$, $\dot{y}$ *in* Υ *and* $a, b \geq 0$, $\vartheta(\dot{x}, T\dot{y}) + \vartheta(\dot{y}, T\dot{x}) \neq 0$ *with* $s(a + b) < 1$. *Then* T *a unique fixed point.*

Proof. Assume $\dot{x}_0 \in \Upsilon$ be an arbitary point in Υ. We define by induction a sequence $\{\dot{x}_n\}$ in Υ such that

$$\dot{x}_{2n+1} = T\dot{x}_{2n}, \text{for all } n \in \mathbb{N}.$$

Now, we show that the sequence $\{\dot{x}_n\}$ is Cauchy:

$$\begin{aligned}
\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right) &= \vartheta(T\dot{x}_{2n}, T\dot{x}_{2n+1}) \\
&\precsim a\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right) \\
&\quad + b\frac{\vartheta\left(\dot{x}_{2n}, T\dot{x}_{2n}\right)\vartheta\left(\dot{x}_{2n}, T\dot{x}_{2n+1}\right) + \vartheta\left(\dot{x}_{2n+1}, T\dot{x}_{2n+1}\right)\vartheta\left(\dot{x}_{2n+1}, T\dot{x}_{2n}\right)}{\vartheta\left(\dot{x}_{2n}, T\dot{x}_{2n+1}\right) + \vartheta\left(\dot{x}_{2n+1}, T\dot{x}_{2n}\right)} \\
&= a\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right) \\
&\quad + b\frac{\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right)\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+2}\right) + \vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right)\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+1}\right)}{\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+2}\right) + \vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+1}\right)} \\
&= (a+b)\,\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right).
\end{aligned}$$

Thus

$$\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right) \precsim (a+b)\,\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right). \tag{3.2}$$

By using lemma (22) thus implies

$$\begin{aligned}
\left|\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right)\right| &\leq \left|(a+b)\,\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right)\right| \\
&\leq (a+b)\left|\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right)\right|.
\end{aligned}$$

Since $a + b < 1$,

$$\left|\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right)\right| \leq (a+b)\left|\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right)\right|. \tag{3.3}$$

Thus for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\left|\vartheta\left(\dot{x}_{2n+1}, \dot{x}_{2n+2}\right)\right| &\leq (a+b)\left|\vartheta\left(\dot{x}_{2n}, \dot{x}_{2n+1}\right)\right| \leq (a+b)^2\left|\vartheta\left(\dot{x}_{2n-1}, \dot{x}_{2n-2}\right)\right| \quad (3.4) \\
&\leq \ldots \leq (a+b)^{2n+1}\left|\vartheta\left(\dot{x}_1, \dot{x}_0\right)\right|.
\end{aligned}$$

Thus for any $m > n, m, n \in \mathbb{N}$,

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s\,|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s\,|\vartheta(\dot{x}_{n+1}, \dot{x}_m)| \\
&\leq s\,|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2\,|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^2\,|\vartheta(\dot{x}_{n+2}, \dot{x}_m)| \\
&\leq s\,|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2\,|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3\,|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + s^3\,|\vartheta(\dot{x}_{n+3}, \dot{x}_m)| \\
&\leq s\,|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2\,|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3\,|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + \ldots + s^{m-n-1}\,|\vartheta(\dot{x}_{m-2}, \dot{x}_{m-1})| + s^{m-n}\,|\vartheta(\dot{x}_{m-1}, \dot{x}_m)|\,.
\end{aligned}
$$

By (3.4), we have

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s\,(a+b)^n\,|\vartheta(\dot{x}_0, \dot{x}_1)| + s^2\,(a+b)^{n+1}\,|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^3\,(a+b)^{n+2}\,|\vartheta(\dot{x}_0, \dot{x}_1)| + \ldots + s^{m-n-1}\,(a+b)^{m-2}\,|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^{m-n}\,(a+b)^{m-1}\,|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{i=1}^{m-n} s^i\,(a+b)^{i+n-1}\,|\vartheta(\dot{x}_0, \dot{x}_1)|\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\,(a+b)^{i+n-1}\,|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{p=n}^{m-1} s^p\,(a+b)^p\,|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\leq \sum_{p=n}^{\infty} [s\,(a+b)]^p\,|\vartheta(\dot{x}_0, \dot{x}_1)| = \frac{[s\,(a+b)]^p}{1-s\,(a+b)}\,|\vartheta(\dot{x}_0, \dot{x}_1)|\,,
\end{aligned}
$$

hence

$$
|\vartheta(\dot{x}_n, \dot{x}_m)| \leq \frac{[s\,(a+b)]^p}{1-s\,(a+b)}\,|\vartheta(\dot{x}_0, \dot{x}_1)| \to 0 \text{ as } m, n \to \infty.
$$

Thus $\{\dot{x}_n\}$ is a Cauchy sequence in Υ. Since Υ is complete, there exists $u \in \Upsilon$ such that $\dot{x}_n \to u$ as $n \to \infty$.

Assume not, then there exists $z \in \Upsilon$ such that

$$
|\vartheta(u, Tu)| = |z| > 0. \tag{3.5}
$$

Consequently, by applying the triangle inequality and (3.1), we find

$$\begin{aligned} z &= \vartheta(u,Tu) \precsim s\vartheta(u,\dot{x}_{2n+2}) + s\left|\vartheta(\dot{x}_{2n+2},Tu)\right| \\ &= s\vartheta(u,\dot{x}_{2n+2}) + s\vartheta(Tu,T\dot{x}_{2n+1}) \\ &\precsim s\vartheta(u,\dot{x}_{2n+2}) + sa\vartheta(u,\dot{x}_{2n+1}) \\ &\quad + sb\frac{\vartheta(u,Tu)\,\vartheta(u,T\dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n+1},T\dot{x}_{2n+1})\,\vartheta(\dot{x}_{2n+1},Tu)}{\vartheta(u,T\dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n+1},Tu)} \\ &= s\vartheta(u,\dot{x}_{2n+2}) + sa\vartheta(u,\dot{x}_{2n+1}) \\ &\quad + sb\frac{\vartheta(u,Tu)\,\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})\,\vartheta(\dot{x}_{2n+1},Tu)}{\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},Tu)}, \end{aligned}$$

it suggests that

$$\begin{aligned} |z| &= |\vartheta(u,Tu)| \\ &\le s\left|\vartheta(u,\dot{x}_{2n+2})\right| + sa\left|\vartheta(u,\dot{x}_{2n+1})\right| \\ &\quad + sb\frac{|\vartheta(u,Tu)|\,|\vartheta(u,\dot{x}_{2n+2})| + |\vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})|\,|\vartheta(\dot{x}_{2n+1},Tu)|}{|\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},Tu)|}. \end{aligned} \tag{3.6}$$

Taking the limit of (3.6) as $n \to \infty$, we get that $|z| = |\vartheta(u,Tu)| \le 0$, a contradiction with (3.5). So$|z| = 0$. Hence

$$Tu = u.$$

To prove the uniqueness of common fixed, assume $v \in \Upsilon$ be a different fixed point of T that is

$$v = Tv.$$

Then

$$\begin{aligned} \vartheta(u,v) &= \vartheta(Tu,Tv) \\ &\precsim a\vartheta(u,v) + b\frac{\vartheta(u,Tu)\,\vartheta(u,Tv) + \vartheta(v,Tv)\,\vartheta(v,Tu)}{\vartheta(u,Tv) + \vartheta(v,Tu)} \\ &= a\vartheta(u,v). \end{aligned}$$

Since $a < 1$, we have $\vartheta(u,v) = 0$

Consequently, we demonstrated that T have a single common fixed point in Υ. □

3.2 Common fixed point theorems of rational type contraction in complex valued b-metric spaces

Theorem 38. *Assume*(Υ, ϑ) *be a complete complex valued b-metric space with a coefficient* $s \geq 1$, *and* $T, S : \Upsilon \longrightarrow \Upsilon$ *be two mappings on* Υ *satisfying the condition*

$$\vartheta(T\dot{x}, S\dot{y}) \precsim a\vartheta(\dot{x}, \dot{y}) + b\frac{\vartheta(\dot{x}, T\dot{x})\,\vartheta(\dot{x}, S\dot{y}) + \vartheta(\dot{y}, S\dot{y})\,\vartheta(\dot{y}, T\dot{x})}{\vartheta(\dot{x}, S\dot{y}) + \vartheta(\dot{y}, T\dot{x})}, \tag{3.7}$$

for all, $\dot{x}$, $\dot{y}$ *in* Υ *and* $a, b \geq 0$, $\vartheta(\dot{x}, S\dot{y}) + \vartheta(\dot{y}, T\dot{x}) \neq 0$ *with* $s(a + b) < 1$. *Then* T *and* S *have a unique common fixed point.*

Proof. Assume $\dot{x}_0 \in \Upsilon$ be an arbitrary point in Υ. We define by induction a sequence $\{\dot{x}_n\}$ in Υ such that

$$\begin{aligned} \dot{x}_{2n+1} &= T\dot{x}_{2n}, \\ \dot{x}_{2n+2} &= S\dot{x}_{2n+1}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By putting $n = 2k$, with $x = \dot{x}_{2k}$ and $\Upsilon = \dot{x}_{2k+1}$ we get

$$\begin{aligned} \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) &= \vartheta(T\dot{x}_{2k}, S\dot{x}_{2k+1}) \\ &\precsim a\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}) + b\frac{\vartheta(\dot{x}_{2k}, T\dot{x}_{2k})\,\vartheta(\dot{x}_{2k}, S\dot{x}_{2k+1}) + \vartheta(\dot{x}_{2k+1}, S\dot{x}_{2k+1})\,\vartheta(\dot{x}_{2k+1}, T\dot{x}_{2k})}{\vartheta(\dot{x}_{2k}, S\dot{x}_{2k+1}) + \vartheta(\dot{x}_{2k+1}, T\dot{x}_{2k})} \\ &= a\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}) + b\frac{\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1})\,\vartheta(\dot{x}_{2k}, \dot{x}_{2k+2}) + \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2})\,\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+1})}{\vartheta(\dot{x}_{2k}, \dot{x}_{2k+2}) + \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+1})} \\ &= (a + b)\,\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}). \end{aligned}$$

Thus

$$\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) \precsim (a + b)\,\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}). \tag{3.8}$$

If $\dot{x}_n = \dot{x}_{n+1}$ for some n, with $n = 2k$ then from (3.8) ,we have $\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) = 0$. So that $\dot{x}_{2k+1} = \dot{x}_{2k+2}$.

For $n = 2k + 1$, utilizing the same justifications as in the case $n = 2k$, we get the same result.

Continuing in this way we can show that $\dot{x}_{2k-1} = \dot{x}_{2k} = \dot{x}_{2k+1} = ...$
Hence $\{\dot{x}_n\}$ is a Cauchy sequence.

Now assume that $\dot{x}_{2k} \neq \dot{x}_{2k+1}$ for all $n \in \mathbb{N}$.

Firstly, we want to show that

$$\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\precsim(a+b)\,\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right),\ \text{for all } n\in\mathbb{N}. \tag{3.9}$$

We need to think about two situations.

Case 1. $n=2k+1$, $k\in\mathbb{N}$.

From (3.8) we have

$$\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\precsim(a+b)\,\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right),\ n=2k+1,k\in\mathbb{N}. \tag{3.10}$$

Case 2. $n=2k$, $k\in\mathbb{N}$.

From (3.8) we have

$$\begin{aligned}\vartheta\left(\dot{x}_{n+1},\dot{x}_{n+2}\right) &\precsim (a+b)\,\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right) \\ &\precsim \vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\precsim(a+b)\,\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right),\ n=2k,k\in\mathbb{N}.\end{aligned} \tag{3.11}$$

So from $(3.10),(3.11)$ We determine that

$$\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\precsim(a+b)\,\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right),\ \text{for all } n\in\mathbb{N}.$$

We thus arrive at that (3.9) holds.

Here, we demonstrate that the sequence $\{\dot{x}_n\}$ is a Cauchy sequence.

By using lemma (22) and (3.8) thus implies

$$\begin{aligned}\left|\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\right| &\leq \left|(a+b)\,\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right)\right| \\ &\leq (a+b)\left|\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right)\right|.\end{aligned}$$

Since $a+b<1$,

$$\left|\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\right|\leq(a+b)\left|\vartheta\left(\dot{x}_{n-1},\dot{x}_{n}\right)\right|. \tag{3.12}$$

Thus for any $m>n, m,n\in\mathbb{N}$,

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s|\vartheta(\dot{x}_{n+1}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^2|\vartheta(\dot{x}_{n+2}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + s^3|\vartheta(\dot{x}_{n+3}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + \ldots + s^{m-n-1}|\vartheta(\dot{x}_{m-2}, \dot{x}_{m-1})| + s^{m-n}|\vartheta(\dot{x}_{m-1}, \dot{x}_m)|.
\end{aligned}
$$

By (3.12), we get

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s(a+b)^n|\vartheta(\dot{x}_0, \dot{x}_1)| + s^2(a+b)^{n+1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^3(a+b)^{n+2}|\vartheta(\dot{x}_0, \dot{x}_1)| + \ldots + s^{m-n-1}(a+b)^{m-2}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^{m-n}(a+b)^{m-1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{i=1}^{m-n} s^i(a+b)^{i+n-1}|\vartheta(\dot{x}_0, \dot{x}_1)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}(a+b)^{i+n-1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{p=n}^{m-1} s^p(a+b)^p|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\leq \sum_{p=n}^{\infty} [s(a+b)]^p|\vartheta(\dot{x}_0, \dot{x}_1)| = \frac{[s(a+b)]^p}{1-s(a+b)}|\vartheta(\dot{x}_0, \dot{x}_1)|.
\end{aligned}
$$

and hence

$$
|\vartheta(\dot{x}_n, \dot{x}_m)| \leq \frac{[s(a+b)]^p}{1-s(a+b)}|\vartheta(\dot{x}_0, \dot{x}_1)| \to 0 \text{ as } m, n \to \infty.
$$

Thus $\{\dot{x}_n\}$is a Cauchy sequence in Υ. Since Υ is complete, there exists some $u \in \Upsilon$ such that $\dot{x}_n \to u$ as $n \to \infty$.
Assume not, then there exists $z \in \Upsilon$ such that

$$
|\vartheta(u, Tu)| = |z| > 0. \tag{3.13}
$$

So by using the triangular inequality and (3.7), we receive

$$\begin{aligned} z &= \vartheta(u,Tu) \precsim s\vartheta(u,\dot{x}_{2n+2}) + s\left|\vartheta(\dot{x}_{2n+2},Tu)\right| \\ &= s\vartheta(u,\dot{x}_{2n+2}) + s\vartheta(Tu,S\dot{x}_{2n+1}) \\ &\precsim s\vartheta(u,\dot{x}_{2n+2}) + sa\vartheta(u,\dot{x}_{2n+1}) \\ &\quad +sb\frac{\vartheta(u,Tu)\,\vartheta(u,S\dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n+1},S\dot{x}_{2n+1})\,\vartheta(\dot{x}_{2n+1},Tu)}{\vartheta(u,S\dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n+1},Tu)} \\ &= s\vartheta(u,\dot{x}_{2n+2}) + sa\vartheta(u,\dot{x}_{2n+1}) \\ &\quad +sb\frac{\vartheta(u,Tu)\,\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})\,\vartheta(\dot{x}_{2n+1},Tu)}{\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},Tu)}, \end{aligned}$$

which implies that

$$\begin{aligned} |z| &= |\vartheta(u,Tu)| \\ &\leq s\left|\vartheta(u,\dot{x}_{2n+2})\right| + sa\left|\vartheta(u,\dot{x}_{2n+1})\right| \\ &\quad +sb\frac{|\vartheta(u,Tu)|\,|\vartheta(u,\dot{x}_{2n+2})| + |\vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})|\,|\vartheta(\dot{x}_{2n+1},Tu)|}{|\vartheta(u,\dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n+1},Tu)|} \end{aligned} \tag{3.14}$$

Exceeding the limit of (3.14) as $n \to \infty$, we get that$|z| = |\vartheta(u,Tu)| \leq 0$, a contradiction with (3.13). So $|z| = 0$. Hence $Tu = u$, Similarly, one can also show that $Su = u$.

To prove the uniqueness of common fixed, Assume $v \in \Upsilon$ be a different common fixed point of S and T that is

$$v = Tv = Sv.$$

Then

$$\begin{aligned} \vartheta(u,v) &= \vartheta(Tu,Sv) \\ &\precsim a\vartheta(u,v) + b\frac{\vartheta(u,Tu)\,\vartheta(u,Sv) + \vartheta(v,Sv)\,\vartheta(v,Tu)}{\vartheta(u,Sv) + \vartheta(v,Tu)} \\ &= a\vartheta(u,v). \end{aligned}$$

Since $a < 1$, we have

$$\vartheta(u,v) = 0.$$

As such, we established that there is only one common fixed point for T and S in Υ.

□

3.3 Common fixed point theorems of cyclic rational type contraction in complex valued b-metric spaces

Theorem 39. *Assume* (Υ, ϑ) *be a complete complex valued b-metric space with a coefficient* $s \geq 1$, *and* $T, S : \Upsilon \longrightarrow \Upsilon$ *be two mappings on* Υ *satisfying the condition*

$$\vartheta(T\dot{x}, S\dot{y}) \precsim a\vartheta(\dot{x}, \dot{y}) + b\frac{\vartheta(\dot{y}, S\dot{y})[1 + \vartheta(\dot{x}, T\dot{x})]}{1 + \vartheta(\dot{x}, \dot{y})} + c\frac{\vartheta(\dot{y}, S\dot{y}) + \vartheta(\dot{y}, T\dot{x})}{1 + \vartheta(\dot{y}, S\dot{y})\,\vartheta(\dot{y}, T\dot{x})}, \tag{3.15}$$

for all, $\dot{x}$, $\dot{y}$ *in* Υ *and* $a, b, c \geq 0$, *and* $s(a + b + c) < 1$. *Then* T *and* S *have a unique common fixed point.*

Proof. Assume $\dot{x}_0 \in \Upsilon$ be an arbitrary point in Υ.We define by induction a sequence $\{\dot{x}_n\}$ in Υ such that

$$\begin{aligned} \dot{x}_{2n+1} &= T\dot{x}_{2n}, \\ \dot{x}_{2n+2} &= S\dot{x}_{2n+1}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By putting $n = 2k$, with $x = \dot{x}_{2k}$ and $\Upsilon = \dot{x}_{2k+1}$ we get

$$\begin{aligned} \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) &= \vartheta(T\dot{x}_{2k}, S\dot{x}_{2k+1}) \\ &\precsim a\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}) + b\frac{\vartheta(\dot{x}_{2k+1}, S\dot{x}_{2k+1})[1 + \vartheta(\dot{x}_{2k}, T\dot{x}_{2k})]}{1 + \vartheta(\dot{x}_{2k}, \dot{x}_{2k+1})} \\ &\quad + c\frac{\vartheta(\dot{x}_{2k+1}, S\dot{x}_{2k+1}) + \vartheta(\dot{x}_{2k+1}, T\dot{x}_{2k})}{1 + \vartheta(\dot{x}_{2k+1}, S\dot{x}_{2k+1})\,\vartheta(\dot{x}_{2k+1}, T\dot{x}_{2k})} \\ &= a\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}) + b\frac{\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2})[1 + \vartheta(\dot{x}_{2k}, \dot{x}_{2k+1})]}{1 + \vartheta(\dot{x}_{2k}, \dot{x}_{2k+1})} \\ &\quad + c\frac{\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) + \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+1})}{1 + \vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2})\,\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+1})} \\ &= a\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}) + b\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) + c\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}). \end{aligned}$$

Thus

$$\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) \precsim \frac{a}{1 - (b + c)}\vartheta(\dot{x}_{2k}, \dot{x}_{2k+1}). \tag{3.16}$$

If $\dot{x}_n = \dot{x}_{n+1}$ for some n, with $n = 2k$ then from (3.16) , we have $\vartheta(\dot{x}_{2k+1}, \dot{x}_{2k+2}) = 0$. So that $\dot{x}_{2k+1} = \dot{x}_{2k+2}$.

For $n = 2k+1$, by using the same arguments as in the case $n = 2k$, we get the same resulat.

Continuing in this way we can show that $\dot{x}_{2k-1} = \dot{x}_{2k} = \dot{x}_{2k+1} = ...$

Hence $\{\dot{x}_n\}$ is a Cauchy sequence.

Now assume that $\dot{x}_{2k} \neq \dot{x}_{2k+1}$ for all $n \in \mathbb{N}$.

Firstly we want to show that

$$\vartheta(\dot{x}_n, \dot{x}_{n+1}) \precsim \frac{a}{1-(b+c)} \vartheta(\dot{x}_{n-1}, \dot{x}_n), \text{ for all } n \in \mathbb{N}. \tag{3.17}$$

There are two cases which we have to consider.

Case 1. $n = 2k+1$, $k \in \mathbb{N}$.

From (3.16) we have

$$\vartheta(\dot{x}_n, \dot{x}_{n+1}) \precsim \frac{a}{1-(b+c)} \vartheta(\dot{x}_{n-1}, \dot{x}_n), \ n = 2k+1, k \in \mathbb{N}. \tag{3.18}$$

Case 2. $n = 2k$, $k \in \mathbb{N}$.

From (3.16) we have

$$\begin{aligned} \vartheta(\dot{x}_{n+1}, \dot{x}_{n+2}) &\precsim \frac{a}{1-(b+c)} \vartheta(\dot{x}_n, \dot{x}_{n+1}) \precsim \vartheta(\dot{x}_n, \dot{x}_{n+1}) \\ &\precsim \frac{a}{1-(b+c)} \vartheta(\dot{x}_{n-1}, \dot{x}_n), \ n = 2k, k \in \mathbb{N} \end{aligned} \tag{3.19}$$

So from $(3.18), (3.19)$ we conclude that

$$\vartheta(\dot{x}_n, \dot{x}_{n+1}) \precsim \frac{a}{1-(b+c)} \vartheta(\dot{x}_{n-1}, \dot{x}_n), \text{ for all } n \in \mathbb{N}$$

Thus we obtain that (3.17) holds.

Now, we show that the sequence $\{\dot{x}_n\}$ is a Cauchy sequence.

By using lemma (22) and (3.17) thus implies

$$\begin{aligned} |\vartheta(\dot{x}_n, \dot{x}_{n+1})| &\leq \left| \left(\frac{a}{1-(b+c)} \right) \vartheta(\dot{x}_{n-1}, \dot{x}_n) \right| \\ &\leq \left(\frac{a}{1-(b+c)} \right) |\vartheta(\dot{x}_{n-1}, \dot{x}_n)|. \end{aligned}$$

Since $a + b < 1$,

$$|\vartheta(\dot{x}_n, \dot{x}_{n+1})| \leq h \, |\vartheta(\dot{x}_{n-1}, \dot{x}_n)|. \tag{3.20}$$

Where $h = \frac{a}{1-(b+c)} < \frac{1}{s} \leq 1$, because $s(a+b+c) < 1$.
Thus for any $m > n.m, n \in \mathbb{N}$,

$$\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s|\vartheta(\dot{x}_{n+1}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^2|\vartheta(\dot{x}_{n+2}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + s^3|\vartheta(\dot{x}_{n+3}, \dot{x}_m)| \\
&\leq s|\vartheta(\dot{x}_n, \dot{x}_{n+1})| + s^2|\vartheta(\dot{x}_{n+1}, \dot{x}_{n+2})| + s^3|\vartheta(\dot{x}_{n+2}, \dot{x}_{n+3})| \\
&\quad + \ldots + s^{m-n-1}|\vartheta(\dot{x}_{m-2}, \dot{x}_{m-1})| + s^{m-n}|\vartheta(\dot{x}_{m-1}, \dot{x}_m)|.
\end{aligned}$$

By (3.20), we get

$$\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq s(h)^n|\vartheta(\dot{x}_0, \dot{x}_1)| + s^2(h)^{n+1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^3(h)^{n+2}|\vartheta(\dot{x}_0, \dot{x}_1)| + \ldots + s^{m-n-1}(h)^{m-2}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\quad + s^{m-n}(h)^{m-1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{i=1}^{m-n} s^i(h)^{i+n-1}|\vartheta(\dot{x}_0, \dot{x}_1)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\vartheta(\dot{x}_n, \dot{x}_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}(h)^{i+n-1}|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&= \sum_{p=n}^{m-1} s^p(h)^p|\vartheta(\dot{x}_0, \dot{x}_1)| \\
&\leq \sum_{p=n}^{\infty}[s(h)]^p|\vartheta(\dot{x}_0, \dot{x}_1)| = \frac{[s(h)]^p}{1-s(h)}|\vartheta(\dot{x}_0, \dot{x}_1)|,
\end{aligned}$$

and hence

$$|\vartheta(\dot{x}_n, \dot{x}_m)| \leq \frac{[s(h)]^p}{1-s(h)}|\vartheta(\dot{x}_0, \dot{x}_1)| \to 0 \text{ as } m, n \to \infty.$$

Thus $\{\dot{x}_n\}$is a Cauchy sequence in Υ. Since Υ is complete, there exists some $u \in \Upsilon$ such that $\dot{x}_n \to u$ as $n \to \infty$.
Assume not, then there exists $z \in \Upsilon$ such that

$$|\vartheta(u, Tu)| = |z| > 0. \tag{3.21}$$

So by using the triangular inequality and (3.15), we receive

$$\begin{aligned}
z &= \vartheta(u, Tu) \precsim s\vartheta(u, \dot{x}_{2n+2}) + s|\vartheta(\dot{x}_{2n+2}, Tu)| \\
&= s\vartheta(u, \dot{x}_{2n+2}) + s\vartheta(Tu, S\dot{x}_{2n+1}) \\
&\precsim s\vartheta(u, \dot{x}_{2n+2}) + sa\vartheta(u, \dot{x}_{2n+1}) \\
&\quad + sb\frac{\vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1})[1 + \vartheta(u, Tu)]}{1 + \vartheta(u, \dot{x}_{2n+1})} + sc\frac{\vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n+1}, Tu)}{1 + \vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1})\vartheta(\dot{x}_{2n+1}, Tu)},
\end{aligned}$$

which implies that

$$\begin{aligned}
|z| &= |\vartheta(u, Tu)| \\
&\leq s|\vartheta(u, \dot{x}_{2n+2})| + sa|\vartheta(u, \dot{x}_{2n+1})| \\
&\quad + sb\frac{|\vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1})||1 + \vartheta(u, Tu)|}{|1 + \vartheta(u, \dot{x}_{2n+1})|} \\
&\quad + sc\frac{|\vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1})| + |\vartheta(\dot{x}_{2n+1}, Tu)|}{|1 + \vartheta(\dot{x}_{2n+1}, S\dot{x}_{2n+1})\vartheta(\dot{x}_{2n+1}, Tu)|}
\end{aligned} \tag{3.22}$$

Taking the limit of (3.22) as $n \to \infty$, we get that $|z| = |\vartheta(u, Tu)| \leq sc|\vartheta(u, Tu)|$, a contradiction since $sc < 1$. So$|z| = 0$. Hence $Tu = u$.

Similarly, we get

$$|\vartheta(u, Su)| \leq s(b + c)|\vartheta(u, Su)|.$$

Since $s(b + c) < 1$, $|\vartheta(u, Su)| = 0$ thus $Su = u$.

To prove the uniqueness of common fixed, Assume $v \in \Upsilon$ be another common fixed point of S and T that is

$$v = Tv = Sv.$$

Then

$$\begin{aligned}
\vartheta(u, v) &= \vartheta(Tu, Sv) \\
&\precsim a\vartheta(u, v) + b\frac{\vartheta(v, Sv)[1 + \vartheta(u, Tu)]}{1 + \vartheta(u, v)} + c\frac{\vartheta(v, Sv) + \vartheta(v, Tu)}{1 + \vartheta(v, Sv)\vartheta(v, Tu)}, \\
&= (a + c)\vartheta(u, v).
\end{aligned}$$

Since $0 < a + c < 1$, we have $\vartheta(u, v) = 0$.

Thus, we proved that T and S have a unique common fixed point in Υ. This completes the proof.

□

3.4 Example

The following example illustrates the result of 37.

Example 40. *Assume $\Upsilon = [0, 1]$. Define the mapping $\vartheta : \Upsilon \times \Upsilon \rightarrow \mathbb{C}$ by*

$$\vartheta(\dot{x}, \dot{y}) = 3\left\{|\dot{x} - \dot{y}|^3 + i\,|\dot{x} - \dot{y}|^3\right\},$$

for all $\dot{x}, \dot{y} \in \Upsilon$. Then (Υ, ϑ) is a complex valued b-metric space with $s = 4$

To verify that (Υ, ϑ) is a complete complex valued b-metric space with $s = 4$, it is enough to verify the triangular inequality condition:

$$\begin{aligned}
\frac{1}{3}\vartheta(\dot{x}, \dot{y}) &= |\dot{x} - \dot{y}|^3 + i\,|\dot{x} - \dot{y}|^3 \\
&= |\dot{x} - \dot{y} + z - z|^3 + i\,|\dot{x} - \dot{y} + z - z|^3 \\
&\preccurlyeq 2^2\left(|\dot{x} - z|^3 + |z - \dot{y}|^3\right) + i2^2\left(|\dot{x} - z|^3 + |z - \dot{y}|^3\right) \\
&\preccurlyeq 4\left[\left(|\dot{x} - z|^3 + i\,|\dot{x} - z|^3\right) + \left(|z - \dot{y}|^3 + i\,|z - \dot{y}|^3\right)\right] \\
&= 4\left[\vartheta(\dot{x}, z) + \vartheta(z, \dot{y})\right].
\end{aligned}$$

Therefore $s = 4$.

Now, define $T : \Upsilon \rightarrow \Upsilon$ as $T\dot{x} = \frac{\dot{x}}{4}, T\dot{y} = \frac{\dot{y}}{4}$, for all $\dot{x}, \dot{y} \in \Upsilon$. Then

$$\begin{aligned}
\vartheta(T\dot{x}, T\dot{y}) &= \vartheta\left(\frac{\dot{x}}{4}, \frac{\dot{y}}{4}\right) \\
\frac{1}{3}\vartheta(T\dot{x}, T\dot{y}) &= \left\{\left|\frac{x}{4} - \frac{y}{4}\right|^3 + i\left|\frac{\dot{x}}{4} - \frac{\dot{y}}{4}\right|^3\right\} \\
&= \frac{1}{4}\left\{|\dot{x} - \dot{y}|^3 + i\,|\dot{x} - \dot{y}|^3\right\} \\
\vartheta(T\dot{x}, T\dot{y}) &= \frac{3}{4}\vartheta(\dot{x}, \dot{y}),
\end{aligned}$$

Under the condition (3.1) , we have

$$\vartheta(T\dot{x},T\dot{y}) \lesssim \frac{1}{3}\vartheta\left(\frac{\dot{x}}{4},\frac{\dot{y}}{4}\right)+\frac{1}{4}\frac{\vartheta\left(\dot{x},\frac{\dot{x}}{4}\right)\vartheta\left(\dot{x},\frac{\dot{y}}{4}\right)+\vartheta\left(\dot{y},\frac{y}{4}\right)\vartheta\left(\dot{y},\frac{x}{4}\right)}{\vartheta\left(\dot{x},\frac{y}{4}\right)+\vartheta\left(\dot{y},\frac{x}{4}\right)}.$$

Then

$$s\left(a+b\right)=4\left(\frac{1}{4}.\frac{1}{3}\right)=\frac{1}{3}<1.$$

It is easily and clearly verified that the map T satisfies contractive condition (3.1) of Theorem 37 with the coefficients $s=4, a=\frac{1}{3}$ and $b=\frac{1}{4}$.

4

Common fixed point theorem for multi-valued generalized contractive mappings

The common fixed point theorem for multi-valued generalized contractive mappings, including control functions of two variables, is the main goal of this chapter.

4.1 Common fixed point theorem for multi-valued generalized contractive mappings

Theorem 41. *Assume*(Υ, ϑ) *be a complete b-metric space with coefficient* $s \geq 1$*, and* $T, S : \Upsilon \to CB(\Upsilon)$ *be generalized rational contractive mappings on* Υ *satisfying the following conditions:*

(a) $\phi(ST\dot{x}, \dot{y}) \leq \phi(\dot{x}, \dot{y})$ *and* $\phi(\dot{x}, ST\dot{y}) \leq \phi(\dot{x}, \dot{y})$
$\varphi(ST\dot{x}, \dot{y}) \leq \varphi(\dot{x}, \dot{y})$ *and* $\varphi(\dot{x}, ST\dot{y}) \leq \varphi(\dot{x}, \dot{y})$
$\psi(ST\dot{x}, \dot{y}) \leq \psi(\dot{x}, \dot{y})$ *and* $\psi(\dot{x}, ST\dot{y}) \leq \psi(\dot{x}, \dot{y})$

(b) $\phi(\dot{x}, \dot{y}) + 2s\varphi(\dot{x}, \dot{y}) + s\psi(\dot{x}, \dot{y}) \prec 1.$
then T *and* S *have a unique common fixed point.*

Proof. Assume $\dot{x}_0 \in \Upsilon$ be an arbitrary point in Υ.We define by induction a sequence $\{\dot{x}_n\}$ in Υ such that

$$\begin{aligned} \dot{x}_{2n+1} &= T\dot{x}_{2n}, \\ \dot{x}_{2n+2} &= S\dot{x}_{2n+1,} \text{ for all } n \in N. \end{aligned}$$

Now by (1.2), we have

$$\begin{aligned} \vartheta(T\dot{x}_{2n}, S\dot{x}_{2n+1}) &\leq H(T\dot{x}_{2n}, S\dot{x}_{2n+1}) \leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) \\ &\quad +\varphi(\dot{x}_{2n}, \dot{x}_{2n+1})[D(\dot{x}_{2n}, S\dot{x}_{2n+1}) + D(\dot{x}_{2n+1}, T\dot{x}_{2n})] \\ &\quad +\psi(\dot{x}_{2n}, \dot{y}) \frac{D(\dot{x}_{2n}, T\dot{x}_{2n}) \vartheta(\dot{x}_{2n+1}, T\dot{x}_{2n+1})}{1 + \vartheta(\dot{x}_{2n}, S\dot{x}_{2n+1}) + D(\dot{x}_{2n+1}, T\dot{x}_{2n}) + \vartheta(\dot{x}_{2n}, \dot{x}_{2n+1})} \\ &\leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) \\ &\quad +\varphi(\dot{x}_{2n}, \dot{x}_{2n+1})[D(\dot{x}_{2n}, \dot{x}_{2n+2}) + D(\dot{x}_{2n+1}, \dot{x}_{2n+1})] \\ &\quad +\psi(\dot{x}_{2n}, \dot{x}_{2n+1}) \frac{D(\dot{x}_{2n}, \dot{x}_{2n+1}) \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2})}{1 + \vartheta(\dot{x}_{2n}, \dot{x}_{2n+2}) + D(\dot{x}_{2n+1}, \dot{x}_{2n+1}) + \vartheta(\dot{x}_{2n}, \dot{x}_{2n+1})} \\ &\leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) + \varphi(\dot{x}_{2n}, \dot{x}_{2n+1})[D(\dot{x}_{2n}, \dot{x}_{2n+2})] \\ &\quad +\psi(\dot{x}_{2n}, \dot{x}_{2n+1}) \frac{D(\dot{x}_{2n}, \dot{x}_{2n+1}) \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2})}{1 + \vartheta(\dot{x}_{2n}, \dot{x}_{2n+2}) + \vartheta(\dot{x}_{2n}, \dot{x}_{2n+1})} \\ &\leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) \\ &\quad +s\varphi(\dot{x}_{2n}, \dot{x}_{2n+1})[D(\dot{x}_{2n}, \dot{x}_{2n+1}) + D(\dot{x}_{2n+1}, \dot{x}_{2n+2})] \\ &\quad +s\psi(\dot{x}_{2n}, \dot{x}_{2n+1}) \frac{D(\dot{x}_{2n}, \dot{x}_{2n+1}) \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2})}{s + \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2})} \\ &\leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) + s\varphi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) \\ &\quad +s\varphi(\dot{x}_{2n}, \dot{x}_{2n+1}) \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2}) + s\psi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}). \end{aligned} \tag{4.1}$$

Proposition 1 yields the following conclusion:

$$\vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2}) \leq \phi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) + s\varphi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

$$+s\varphi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n+1}, \dot{x}_{2n+2}) + s\psi(\dot{x}_{2n}, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

$$\leq \phi(\dot{x}_0, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1}) + s\varphi(\dot{x}_0, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

$$+s\varphi(\dot{x}_0, \dot{x}_{2n+1}) D(\dot{x}_{2n+1}, \dot{x}_{2n+2}) + s\psi(\dot{x}_0, \dot{x}_{2n+1}) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

$$\leq \phi(\dot{x}_0, \dot{x}_1) D(\dot{x}_{2n}, \dot{x}_{2n+1}) + s\varphi(\dot{x}_0, \dot{x}_1) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

$$+s\varphi(\dot{x}_0, \dot{x}_1) \vartheta(\dot{x}_{2n+1}, \dot{x}_{2n+2}) + s\psi(\dot{x}_0, \dot{x}_1) D(\dot{x}_{2n}, \dot{x}_{2n+1})$$

,

which implies that

$$\vartheta\left(\dot{x}_{2n+1},\dot{x}_{2n+2}\right)\leq\lambda D\left(\dot{x}_{2n},\dot{x}_{2n+1}\right). \tag{4.2}$$

Where $\lambda=\frac{\phi(\dot{x}_0,\dot{x}_1)+s\varphi(\dot{x}_0,\dot{x}_1)+s\psi(\dot{x}_0,\dot{x}_1)}{1-s\varphi(\dot{x}_0,\dot{x}_1)}\in[0;1)$.

A similar computation verifies that

$$\vartheta\left(\dot{x}_{n},\dot{x}_{n+1}\right)\leq\lambda D\left(\dot{x}_{n},\dot{x}_{n-1}\right). \tag{4.3}$$

Hence, by Lemma , we obtain that $\{\dot{x}_n\}$,is a Cauchy sequence in (Υ,ϑ) .By completeness of (Υ,ϑ) there exists $u\in\Upsilon$ such that $\lim_{n\to\infty}\dot{x}_n=u$.

Now, we show that u is a fixed point of T. From (1.2), we have

$$\begin{aligned}
\vartheta\left(u,Tu\right) &\leq s\left[\vartheta\left(u,S\dot{x}_{2n+1}\right)+H\left(S\dot{x}_{2n+1},Tu\right)\right]\leq \\
& s\begin{pmatrix} \vartheta\left(u,S\dot{x}_{2n+1}\right)+\phi\left(u,\dot{x}_{2n+1}\right)\vartheta\left(u,\dot{x}_{2n+1}\right) \\ \\ +\varphi\left(u,\dot{x}_{2n+1}\right)\left[\vartheta\left(u,S\dot{x}_{2n+1}\right)+\vartheta\left(\dot{x}_{2n+1},Tu\right)\right] \\ +\psi\left(u,\dot{x}_{2n+1}\right)\frac{\vartheta(u,Tu)\vartheta(\dot{x}_{2n+1},S\dot{x}_{2n+1})}{1+\vartheta(u,S\dot{x}_{2n+1})+\vartheta(\dot{x}_{2n+1},Tu)+\vartheta(u,\dot{x}_{2n+1})}\end{pmatrix} \\
&\leq s\begin{pmatrix} \vartheta\left(u,\dot{x}_{2n+2}\right)+\phi\left(u,\dot{x}_{2n+1}\right)\vartheta\left(u,\dot{x}_{2n+1}\right) \\ \\ +\varphi\left(u,\dot{x}_{2n+1}\right)\left[\vartheta\left(u,\dot{x}_{2n+2}\right)+\vartheta\left(\dot{x}_{2n+1},Tu\right)\right] \\ +\psi\left(u,\dot{x}_{2n+1}\right)\frac{\vartheta(u,Tu)\vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})}{1+\vartheta(u,\dot{x}_{2n+2})+\vartheta(\dot{x}_{2n+1},Tu)+\vartheta(u,\dot{x}_{2n+1})}\end{pmatrix} \\
&\leq s\begin{pmatrix} \vartheta\left(u,\dot{x}_{2n+2}\right)+\phi\left(u,\dot{x}_{1}\right)\vartheta\left(u,\dot{x}_{2n+1}\right) \\ \\ +\varphi\left(u,\dot{x}_{1}\right)\left[\vartheta\left(u,\dot{x}_{2n+2}\right)+\vartheta\left(\dot{x}_{2n+1},Tu\right)\right] \\ +\psi\left(u,\dot{x}_{1}\right)\frac{\vartheta(u,Tu)\vartheta(\dot{x}_{2n+1},\dot{x}_{2n+2})}{1+\vartheta(u,\dot{x}_{2n+2})+\vartheta(\dot{x}_{2n+1},Tu)+\vartheta(u,\dot{x}_{2n+1})}\end{pmatrix}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we get

$$\begin{aligned}
\vartheta(u,Tu) &\leq s\begin{pmatrix} \vartheta(u,u)+\phi(u,\dot{x}_1)\,\vartheta(u,u)+\varphi(u,\dot{x}_1)\left[\vartheta(u,u)+\vartheta(u,Tu)\right] \\ +\psi(u,\dot{x}_1)\,\frac{\vartheta(u,Tu)\vartheta(u,u)}{1+\vartheta(u,u)+\vartheta(u,Tu)+\vartheta(u,u)} \end{pmatrix} \\
&\leq s\varphi(u,\dot{x}_1)\,\vartheta(u,Tu) \\
&\leq \left[\phi(u,\dot{x}_1)+2s\varphi(u,\dot{x}_1)+s\psi(u,\dot{x}_1)\right]\vartheta(u,Tu) \\
&< \vartheta(u,Tu).
\end{aligned}$$

which is a contradiction. Thus, u is a fixed point of T. Similarly, we can also show that $\dot{u}$ is a fixed point of S, by using

$$\vartheta(u,Su) \leq s\left[\vartheta(u,\dot{x}_{2n+1})+H(\dot{x}_{2n+1},Su)\right]. \tag{4.4}$$

Now, we prove that u is a unique.We assume that there exists another common fixed $\dot{x}^*$ of T and S, i.e.,

$$\dot{x}^* \in F(T)\cap F(S).$$

where $\dot{x}^* \neq \dot{x}$.Now, from (1.2), we have

$$\begin{aligned}
\vartheta(\dot{x},\dot{x}^*) &= \vartheta(T\dot{x},S\dot{x}^*) \leq H(T\dot{x},S\dot{x}^*) \\
&\leq \phi(\dot{x},\dot{x}^*)\,\vartheta(\dot{x},\dot{x}^*)+\varphi(\dot{x},\dot{x}^*)\left[\vartheta(\dot{x},S\dot{x}^*)+\vartheta(\dot{x}^*,T\dot{x})\right] \qquad (4.5)\\
&\quad +\psi(\dot{x},\dot{x}^*)\,\frac{\vartheta(\dot{x},T\dot{x})\,\vartheta(\dot{x}^*,T\dot{x}^*)}{1+\vartheta(\dot{x},S\dot{x}^*)+\vartheta(\dot{x}^*,T\dot{x})+\vartheta(\dot{x},\dot{x}^*)} \\
&= \left[\phi(\dot{x},\dot{x}^*)+2\phi(\dot{x},\dot{x}^*)\right]\vartheta(\dot{x},\dot{x}^*) \\
&\leq \left[\phi(\dot{x},\dot{x}^*)+2s\phi(\dot{x},\dot{x}^*)\right]\vartheta(\dot{x},\dot{x}^*).
\end{aligned}$$

Since $\phi(\dot{x},\dot{x}^*)+2s\phi(\dot{x},\dot{x}^*) < 1$, we have

$$\vartheta(\dot{x},\dot{x}^*) = 0. \tag{4.6}$$

Thus, $\dot{x} = \dot{x}^*$. □

Corollary 42. *[73]Assume(Υ,ϑ) be a complete b-metric space with coefficient $s \geq 1$, and $T : \Upsilon \rightarrow CB(\Upsilon)$.*

If there exists a control function $\phi : \Upsilon\times\Upsilon \rightarrow [0;I]$ such that:

- $\phi(T\dot{x},\dot{y}) \leq \phi(\dot{x},\dot{y})$ and $\phi(\dot{x},S\dot{y}) \leq \phi(\dot{x},\dot{y})$;

- $\phi(\dot{x}, \dot{y}) < 1$;
- $\vartheta(T\dot{x}, T\dot{y}) \leq \phi(\dot{x}, \dot{y})\,\vartheta(\dot{x}, \dot{y})$;

then T has a unique fixed point.

4.2 Application

In this segment, we showcase a typical instance of utilizing fixed point techniques to examine the existence of solutions in integral equations. To sum up, we present an overview of the background and notation utilized in this context.

Assume $\Upsilon = C([0; I]; R)$ be the set of real continuous functions defined on $[0; I]$, where $I > 0$, and Assume $d : \Upsilon \times \Upsilon \to [0, \infty)$ be given by

$$\vartheta(\dot{x}, \dot{y}) = \dot{x}_{0<t<I} \left|\dot{x}(t) - \dot{y}(t)\right|^m,$$

for all $\dot{x}, \dot{y} \in \Upsilon$. Then$(\Upsilon, \vartheta)$ is a complete b-metric space.
Consider the integral equation

$$\dot{x}(t) = p(t) + \lambda \int_0^I f(t, s) K(s, \dot{x}(s)) \text{ for } t, s \in [a, b]. \tag{4.7}$$

Where λ is a constant, $K : [0; I] \times R \to R$ and $p(t) : [0; I] \to R$ are two continuous functions and $f : [0; I] \times [0; I] \to [0, \infty)$ is a function such that $f(\dot{x}, .) \in L^1([0; I])$ for all $t \in [0; I]$.
Consider the operator $T : \Upsilon \to \Upsilon$ defined by

$$T(\dot{x})(t) = p(t) + \lambda \int_0^I f(t, s) K(s, \dot{x}(s)).$$

Then we prove the following existence result.

Theorem 43. *Assume* $\Upsilon = C([0; I]; R)$ *.Assume that:*

- there exists a continuous function $\phi : \Upsilon \times \Upsilon \to [0; I]$ such that
$$\phi(T\dot{x}, \dot{y}) \leq \phi(\dot{x}, \dot{y}) \text{ and } \phi(\dot{x}, S\dot{y}) \leq \phi(\dot{x}, \dot{y})$$

- there exist $\dot{x}(t,s) : \Upsilon \times \Upsilon \to [0,\infty)$ and $\alpha : \Upsilon \times \Upsilon \to [0,\infty)$ such that if $\alpha(t,s) \geq 1$ for $\dot{x},\dot{y} \in \Upsilon$, then, for every $s \in [0;I]$ and some $\lambda > 0$ one has

$$\begin{aligned}
|K(s,x(s)) - K(s,\dot{y}(s))| &\leq \phi(\dot{x},\dot{y})^{\frac{1}{m}} x(t,s)(|\dot{x}(s) - \dot{y}(s)|), \\
\left\| \int_0^I f(t,s)\dot{x}(t,s) \right\|_\infty &\leq \frac{1}{\lambda},
\end{aligned}$$

for all $t,s \in [0,I]$. Then the integral equation has a unique solution.

Proof. Define $T : \Upsilon \times \Upsilon \to \Upsilon$ by $T\dot{x}(t) = \int_0^I K(s,\dot{x}(s))\,ds$ for $t,s \in [0,I]$. So Υ is fixed point of T if and only if its a unique solution of the integral equation. So for all $\dot{x},\dot{y} \in \Upsilon$ we have

$$\begin{aligned}
\vartheta(T\dot{x},T\dot{y}) &= |T\dot{x} - T\dot{y}|^m \leq \left(\left|\lambda \int_0^I f(t,s)K(s,\dot{x}(s))\,ds - \lambda \int_0^I f(t,s)K(s,\dot{y}(s))\,ds\right|\right)^m \\
&\leq \left(\left|\lambda \int_0^I f(t,s)[K(s,\dot{x}(s)) - K(s,\dot{y}(s))]\,ds\right|\right)^m \\
&\leq \left(\lambda \int_0^I f(t,s)[|K(s,\dot{x}(s))|\,ds - |K(s,\dot{y}(s))|]\,ds\right)^m \\
&\leq \left(\lambda \int_0^I f(t,s)\phi(\dot{x},\dot{y})^{\frac{1}{m}} \dot{x}(t,s)(|\dot{x}(s) - \dot{y}(s)|)\,ds\right)^m \\
&\leq \left(\int_0^I \phi(\dot{x},\dot{y})^{\frac{1}{m}} (|\dot{x}(s) - \dot{y}(s)|)\,ds\right)^m \\
&= \phi(\dot{x},\dot{y})\,\vartheta(\dot{x},\dot{y}).
\end{aligned}$$

Hence, all the assumptions of Corollary 1 are satisfied, and T has a unique fixed point in Υ Which is a solution of the integral equation . □

5

Some fixed point results in the new $b_v(\theta)$-metric spaces with applications

As a generalization of metric space, rectangular metric space, and b-metric

space, rectangular b-metric space, polygonal metric space, and $b_v(s)$-metric space,we provide the new idea of extended polygonal b-metric space, also known as $b_v(s)$-metric space. Furthermore,

We prove several fixed point findings under Banach's contraction condition. for $b_v(\theta)$-metrics spaces.

5.1 New definitions and proprieties

Definition 44. *Assume Υ be a non empty set, $\theta : \Upsilon \times \Upsilon \longrightarrow [1, +\infty)$ is any function and $v \in N$ is a fixed integer. The mapping $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow [1, +\infty)$ is called an extended $b_v(\theta)$-metric function if for all $\dot{x}, \dot{y} \in \Upsilon$ and for all distinct points $\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_{v-1}, \dot{x}_v \in \Upsilon \setminus \{\dot{x}, \dot{y}\}$, it satisfies the following assertions*

(θ_1) $\vartheta_\theta(\dot{x}, \dot{y}) = 0$ *if and only if* $\dot{x} = \dot{y}$

(θ_2) $\vartheta_\theta(\dot{x}, \dot{y}) = \vartheta_\theta(y, \dot{x}$

(θ_3) $\vartheta_\theta(\dot{x}, \dot{y}) \leq \theta(\dot{x}, \dot{y}) [\vartheta_\theta(\dot{x}, \dot{x}_1) + \vartheta_\theta(\dot{x}_1, \dot{x}_2) \cdots + \vartheta_\theta(\dot{x}_{v-1}, \dot{x}_v) + \vartheta_\theta(\dot{x}_v, \dot{y})]$

Then, the pair $(\Upsilon, \vartheta_\theta)$ is called extended polygonal b-metric space of order v or simply extended $b_v(\theta)$-metric space.

Remark 45. *It is obvious that both the class of the $b_v(s)$-metric spaces and extended b-metric spaces are special cases of the extended $b_v(\theta)$-metric spaces for $\theta \equiv s > 1$ and $v = 1$, respectively. This shows that this type of generalized metric spaces contains all previous spaces exposed in section. It is worth noting that the work in some of generalized spaces is essentially harder.*

Assume us expose some examples of this type of new spaces.

Example 46. *Assume $\Upsilon = F(E, \mathbb{C})$ be the set of complex-valued function defined on the non empty set E and $q \geq 2$ is a fixed positive integer. Define $\theta : \Upsilon \times \Upsilon \longrightarrow [1, \infty)$ and $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ as follows*

$$\theta(\dot{x}, \dot{y}) = (v+1)^{q-1} + |\dot{x}| + |\dot{y}| \text{ and } \vartheta_\theta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|^q .$$

Then, the pair $(\Upsilon, \vartheta_\theta)$ is an extended polygonal b-metric space of order v, where v is any fixed positive integer.

Proof. Observe that the assertions (θ_1)-(θ_2) are obviously fulfilled. Furthermore, we have

$$\begin{aligned}
\vartheta_\theta(\dot{x}, \dot{y}) &= |\dot{x} - \dot{y})|^q = |\dot{x} - \dot{x}_1 + \dot{x}_1 - \dot{x}_2 + \cdots + \dot{x}_{v-1} - \dot{x}_v + \dot{x}_v - \dot{y}|^q \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq (|\dot{x} - \dot{x}_1| + |\dot{x}_1 - \dot{x}_2| + \cdots + |\dot{x}_v - \dot{y}|)^q \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq (v+1)^{q-1} \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right) \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq \left((v+1)^{q-1} - 1\right) \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right) \\
&\quad + |\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q .
\end{aligned}$$

On the other hand, we have

$$|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q \leq (1 + |\dot{x}| + |\Upsilon|) \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right).$$

Consequently, we get the assertion (θ_3) as follows

$$\begin{aligned}
\vartheta_\theta(\dot{x}, \dot{y}) &\leq \left((v+1)^{q-1} - 1\right) \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right) \\
&\quad + (1 + |\dot{x}| + |\Upsilon|) \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right) \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq \left((v+1)^{q-1} + |\dot{x}| + |\dot{y}|\right) \left(|\dot{x} - \dot{x}_1|^q + |\dot{x}_1 - \dot{x}_2|^q + \cdots + |\dot{x}_v - \dot{y}|^q\right) \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq \theta(\dot{x}, \dot{y}) \left(\vartheta(\dot{x}, \dot{x}_1) + \vartheta(\dot{x}_1, \dot{x}_2) + \cdots + \vartheta(\dot{x}_v, \dot{y})\right).
\end{aligned}$$

Hence, the desired result is obtained. □

Example 47. *Assume $\Upsilon = Z$. Define the functions $\theta : \Upsilon \times \Upsilon \longrightarrow [1, \infty)$ and $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ as follows*

$$\theta(\dot{x}, \dot{y}) = \begin{cases} (2v+2)^{p-1} \frac{|\dot{x}^p - \dot{y}^p|}{|\dot{x}-\dot{y})|}, & \text{if } \dot{x} \neq \dot{y}; \\ 1, & \text{if } \dot{x} = \dot{y} \end{cases}$$

and $\vartheta_\theta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|^p$ with p is an odd fixed integer. Then, $(\Upsilon, \vartheta_\theta)$ is an extended $b_v(\theta)$-metric space for $v \geq 1$ is any fixed integer.

Proof. It is obvious that (θ_1) and (θ_2) are satisfied. Assume us checking the assertion (θ_3). For this, we need to prove the following inequality $|a+b|^{p+1} \leq |a+b|^p \, |a^p + b^p|$ for all $a, b \in \mathbb{R}$ such that $a = 0$ or $|a| \geq 1$ and $b = 0$ or $|b| \geq 1$. It is obvious that for $a = 0$ or $b = 0$, the inequality is satisfied as an equality. Assume $|a| \geq 1$ and $|b| \geq 1$. Then, we get the desired result

$$|a+b|^{p+1} = \frac{|a+b|^{p+1}}{|a^p + b^p|} |a^p + b^p| \leq \frac{|a+b|^{p+1}}{|a+b|} |a^p + b^p| = |a+b|^p \, |a^p + b^p| \, .$$

For all $\dot{x}, \dot{y} \in \Upsilon$ and for a fixed integer $v \geq 1$, Assume us setting $a = \dot{x} + \sum_{i=1}^{v} \dot{x}_i$ and $b = -\dot{y} - \sum_{i=1}^{v} \dot{x}_i$. Then, by denoting $\varpi = \sum_{i=1}^{v} \dot{x}_i$, we get

$$\begin{aligned} |\dot{x} - \dot{y})|^{p+1} &= |\dot{x} + \varpi - \dot{y} - \varpi|^{p+1} \leq |\dot{x} + \varpi - \dot{y} - \varpi|^p \left| (\dot{x} + \varpi)^p + (-\dot{y} - \varpi)^p \right| \\ |\dot{x} - \dot{y})|^{p+1} &= |\dot{x} - \dot{x}_1 + \dot{x}_1 - \dot{x}_2 + \cdots + \dot{x}_v - \dot{y}|^p \left| (\dot{x} + \varpi)^p + (-1)^p \, (y + \varpi)^p \right| . \end{aligned}$$

Using Jameson inequality, we obtain

$$\begin{aligned} |\dot{x} - \dot{y})|^{p+1} &\leq (v+1)^{p-1} \left(|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p \cdots + |\dot{x}_v - \dot{y}|^p \right) \times \left| 2^{p-1} \left(\dot{x}^p + \varpi^p - \dot{y}^p - \varpi^p \right) \right| \\ &= (v+1)^{p-1} \, 2^{p-1} \left(|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p \cdots + |\dot{x}_v - \dot{y}|^p \right) |\dot{x}^p - \dot{y}^p| \, . \end{aligned}$$

It follows from the latter inequality

$$\begin{aligned} \vartheta_\theta(\dot{x}, \dot{y}) &\leq |\dot{x} - \dot{y})|^p \leq \left[(v+1)^{p-1} \, 2^{p-1} \frac{|\dot{x}^p - \dot{y}^p|}{|\dot{x} - \dot{y})|} \right] \left(|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p \cdots + |\dot{x}_v - \dot{y}|^p \right) \\ \vartheta_\theta(\dot{x}, \dot{y}) &\leq \left[(2v+2)^{p-1} \frac{|\dot{x}^p - \dot{y}^p|}{|\dot{x} - \dot{y})|} \right] \left(|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p \cdots + |\dot{x}_v - \dot{y}|^p \right) \\ \vartheta_\theta(\dot{x}, \dot{y}) &= \theta(\dot{x}, \dot{y}) \left(d(\dot{x}, \dot{x}_1) + \vartheta(\dot{x}_1 - \dot{x}_2) + \cdots \vartheta(\dot{x}_{v-1}, \dot{x}_v) + \vartheta(\dot{x}_v, \dot{y}) \right) . \end{aligned}$$

where $\theta(\dot{x},\dot{y}) = (2v+2)^{p-1}\frac{|\dot{x}^p-\dot{y}^p|}{|\dot{x}-\dot{y})|} > 1$. Hence, (θ_3) is satisfied. The proof is achieved. □

Remark 48. *If we consider $\vartheta_\theta(\dot{x},\dot{y}) = |\dot{x}-\dot{y})|^p$ with p is an even fixed integer, then by following the same procedure, we can show that $(\Upsilon,\vartheta_\theta)$ is a $b_v(\theta)$-metric space with*

$$\theta(\dot{x},\dot{y}) = \begin{cases} (2v+2)^{p-1}\frac{|\dot{x}^p+\dot{y}^p|}{|\dot{x}-\dot{y})|}, & \text{if } \dot{x} \neq \dot{y};\\ 1, & \text{if } \dot{x} = y. \end{cases}$$

We state now a fixed point theorem in these generalized $b_v(\theta)$-metric spaces which extends [85, Theorem 2.1] with a direct, short and different proof than that given by Mitrovic and Radenovic for theorem 2.1 in [85].

5.2 Fixed point in the new $b_v(\theta)$-metric spaces

In these generalized $b_v(\theta)$-metric spaces, we now prove a fixed point theorem that extends [85, Theorem 2.1]. Our proof is direct, concise, and distinct from the one provided by Mitrovic and Radenovic for theorem 2.1 in [85].

Theorem 49. *Assume $(\Upsilon,\vartheta_\theta)$ be a complete extended $b_v(\theta)$-metric space so that ϑ_θ is a continuous functional. Assume us consider the mapping $T:\Upsilon \longrightarrow \Upsilon$ satisfying the following Banach contraction inequality*

$$\vartheta_\theta(T\dot{x},T\dot{y}) \leq \lambda\, \vartheta_\theta(\dot{x},\dot{y}) \textit{ for all } \dot{x},\dot{y} \in X. \tag{5.1}$$

where $\lambda \in [0,1)$ is a fixed real number,and $M \in \mathbb{Z}_+$. Assume that for each $\dot{x}_0$, we have

$$\theta(\dot{x}_n,\dot{x}_m) < M.$$

where the sequence $\{\dot{x}_n\}_{n=0}^{\infty} = \{T^n\dot{x}_0\}_{n=0}^{\infty}$. Then, T has a unique fixed point.

Proof. Assume $\dot{x}_0 \in \Upsilon$ be arbitrary.$\exists k \in \mathbb{N}$ such that $\lambda^k M < 1$.Define the sequence $\{\dot{x}_n\}$ by $\dot{x}_{n+1} = S\dot{x}_n$ for all $n \in \mathbb{Z}_+$.Where $S = T^k$.We get

$$\vartheta_\theta(S\dot{x},Sy) \leq \lambda^k\, \vartheta_\theta(\dot{x},\dot{y}) \text{ for all } \dot{x},\dot{y} \in X.$$

Assume us prove that $\{\dot{x}_n\}$ is Cauchy sequence in Υ. If $\dot{x}_n = \dot{x}_{n+1}$ for some n,then $\dot{x}_n$ is a fixed point of S. Now assume that $\dot{x}_n \neq \dot{x}_{n+1}$ for all $n \in \mathbb{Z}_+$ and in order to simplify the expository,Assume setting $\vartheta(\dot{x}_n, \dot{x}_{n+1}) = \vartheta_{\theta_n}$ and $\theta(\dot{x}_n, \dot{x}_m) = \theta_{n,m}$. From (7) ,we get

$$\vartheta_\theta(\dot{x}_n, \dot{x}_{n+1}) = \vartheta_\theta(S\dot{x}_{n-1}, S\dot{x}_n) \leq \lambda^k \vartheta_\theta(\dot{x}_{n-1}, \dot{x}_n)$$
$$\vartheta_{\theta_n} \leq \lambda^k \vartheta_{\theta_{n-1}}.$$

Repeating this process, we obtain

$$\vartheta_{\theta_n} \leq \lambda^{kn} \vartheta_{\theta_0}. \tag{8}$$

For the sequence $\{\dot{x}_n\}$,we consider $\vartheta_\theta(\dot{x}_n, \dot{x}_m)$ such that $m = n + pq + 1$ where $q \in N$. Assume denote$\Theta_{n,m} = \vartheta_{j=0}^{q-1} \theta_{n+jp,m}$.

Using (θ2)-(θ3) and (8) ,we derive by induction and since $\theta(\dot{x}_n, \dot{x}_m) < M$:

$$\begin{aligned}\vartheta_\theta(\dot{x}_n, \dot{x}_m) &\leq \theta_{n,m}\left(\vartheta_{\theta_n} + \vartheta_{\theta_{n+1}} + \vartheta_{\theta_{n+2}} + \ldots + \vartheta_{\theta_{n+p-1}} + \vartheta_\theta\left(\dot{x}_{n+p}, \dot{x}_m\right)\right)\\ &\leq \theta_{n,m}\left(\lambda^{kn}\vartheta_{\theta_0} + \lambda^{kn+1}\vartheta_{\theta_0} + \lambda^{kn+2}\vartheta_{\theta_0} + \ldots + \lambda^{kn+p-1}\vartheta_{\theta_0}\right)\\ &+ \theta_{n,m} \times \theta_{n+p,m}\left(\vartheta_{\theta_{n+p}} + \vartheta_{\theta_{n+p+1}} + \vartheta_{\theta_{n+p+2}} + \ldots + \vartheta_{\theta_{n+2p-1}} + \vartheta\left(\dot{x}_{n+2p}, \dot{x}_m\right)\right)\\ &.\\ &.\end{aligned}$$

$$\begin{aligned}\vartheta_\theta(\dot{x}_n, \dot{x}_m) &\leq \theta_{n,m}\left(\lambda^{nk}\vartheta_{\theta_0} + \lambda^{(n+1)k}\vartheta_{\theta_0} + \lambda^{(n+2)k}\vartheta_{\theta_0} + \ldots + \lambda^{(n+p-1)k}\vartheta_{\theta_0}\right) +\\ &\theta_{n,m} \times \theta_{n+p,m}\left(\lambda^{(n+p)k}\vartheta_{\theta_0} + \lambda^{(n+p+1)k}\vartheta_{\theta_0} + \lambda^{(n+p+2)k}\vartheta_{\theta_0} + \ldots + \lambda^{(n+2p-1)k}\vartheta_{\theta_0}\right) +\\ &+ \ldots + \vartheta_{\theta_0}\vartheta_{j=0}^{q-1}\theta_{n+jp,m}.\lambda^{nk}\sum_{j=0}^{p}\lambda^{((q-1)p+j)k}\\ \vartheta(\dot{x}_n, \dot{x}_m) &\leq \vartheta_{\theta_0}\left[\begin{array}{c}\theta_{n,m}\left(\lambda^{nk}\left(\frac{1-\lambda^{kp}}{1-\lambda^k}\right)\right) + \theta_{n,m} \times \theta_{n+p,m}\left(\lambda^{(n+p)k}\left(\frac{1-\lambda^{kp}}{1-\lambda^k}\right)\right)\\ +\ldots + \Theta_{n,m}\left(\lambda^{(n+(q-1)p)k}\left(\frac{1-\lambda^{(p+1)k}}{1-\lambda^k}\right)\right)\end{array}\right]\\ &\leq \lambda^{kn}\vartheta_{\theta_0}\left(\frac{1-\lambda^{(p+1)k}}{1-\lambda^k}\right)\left[\sum_{i=0}^{i=q-1}\lambda^{ipk}\vartheta_{j=1}^{j=i+1}\theta_{n+(j-1)p,m}\right]\end{aligned}$$

Taking the limits in both sides of the latter inequality,obtain

$$\lim_{n;m\to\infty}\vartheta_\theta(\dot{x}_n, \dot{x}_m) \leq \lim_{n;m\to\infty}\lambda^{kn}\vartheta_{\theta_0}\left(\frac{1-\lambda^{(p+1)k}}{1-\lambda^k}\right)\left[\sum_{i=0}^{i=q-1}\lambda^{ipk}\vartheta_{j=1}^{j=i+1}\theta_{n+(j-1)p,m}\right].$$

We note that the series $\sum_{i=0}^{i=q-1} \lambda^{ipk} \vartheta_{j=1}^{j=i+1} \theta_{n+(j-1)p,m}$ with positive terms converges according to the ratio test. Indeed,by putting $U_{i,m} = \lambda^{ipk} \vartheta_{j=1}^{j=i+1} \theta_{n+(j-1)p,m}$, we get

$$\frac{U_{i+1,m}}{U_{i,m}} = \frac{\lambda^{(i+1)p} \theta_{n,m} \times \theta_{n+p,m} \times ....\theta_{n+ip,m} \times \theta_{n+(i+1)p,m}}{\lambda^{ip} \theta_{n,m} \times \theta_{n+p,m} \times ....\theta_{n+ip,m}} = \lambda^{pk} \theta_{n+ip,m},$$

since, $\lambda^k \theta(\dot{x}_n, \dot{x}_m) < 1$. Hence $\lim_{n,m\to\infty} \vartheta_\theta(\dot{x}_n, \dot{x}_m) = 0$ and consequently $\{\dot{x}_n\}$ is Cauchy sequence. Since, we deal with a complete space, it follows that there exists $u \in \Upsilon$ such that $\lim_{n\longrightarrow\infty} \dot{x}_n = u$.

We now prove that u is a fixed point of S,so it's a fixed point for T. Indeed, for any $n \in \mathbb{Z}_+$, we have :

$$\begin{aligned}
\vartheta_\theta(Su, u) &\le \theta(Su, u)\left[\vartheta_\theta(u, \dot{x}_n) + \vartheta_\theta(\dot{x}_n, \dot{x}_{n+1}) + ... + \vartheta_\theta\left(\dot{x}_{n+p-2}, \dot{x}_{n+p-1}\right) + \vartheta_\theta\left(\dot{x}_{n+p-1}, Su\right)\right] \\
&\le \theta(Su, u)\left[\vartheta_\theta(u, \dot{x}_n) + \vartheta_{\theta n} + \vartheta_{\theta n+1} + \vartheta_{\theta n+2} + ... + \vartheta_{\theta n+p-2} + \vartheta_\theta\left(S\dot{x}_{n+p-2}, Su\right)\right] \\
&\le \theta(Su, u)\left[\vartheta_\theta(u, \dot{x}_n) + \vartheta_{\theta n} + \vartheta_{\theta n+1} + \vartheta_{\theta n+2} + ... + \lambda^k \theta\left(\dot{x}_{n+p-2}, u\right) \vartheta_\theta\left(\dot{x}_{n+p-2}, u\right)\right]
\end{aligned}$$

By letting $n \longrightarrow \infty$ we get $\vartheta(Su, u) \le 0$. Since $\vartheta(Su, u) \ge 0$, it follows that $\vartheta(Su, u) = 0$. This shows that $Su = u$,i.e. u is a fixed point of S. it remains to prove the uniqueness, by the usual way. For this, Assume v be another fixed point of S. According to (1) we have

$$\vartheta_\theta(u, v) = \vartheta_\theta(Su, Sv) \le \lambda^k \theta(u, v) \vartheta_\theta(u, v) < \vartheta_\theta(u, v),$$

Therefore, we derive $\vartheta_\theta(u, v) = 0$,i.e., $u = v$,and consequently there is a unique fixed point for S,($i.e.$ unique fixed point for T) . □

Remark 50. *This theorem extends many other results obtained under the same contraction condition in some specific generalized metric spaces. For example:*

- *If $\theta(\dot{x}, \dot{y}) = 1$ and v arbitrary positive integer, we obtain the first part of [82, Theorem 2.2] in polygon metric spaces.*
- *If $\theta(\dot{x}, \dot{y}) = s > 1$ and $v = 1$, we obtain for b-metric spaces.*
- *If $\theta(\dot{x}, \dot{y}) = s > 1$ and $v = 2$, we obtain [84, Theorem 2.1] in rectangular b-metric spaces.*
- *If $\theta(\dot{x}, \dot{y}) = s > 1$ and v arbitrary positive integer, we obtain [85, Theorem 2.1] for $b_v(s)$-metric spaces.*

Example 51. *Assume us consider* $\dot{x} = \left[0, \frac{\pi}{2}\right]$, *the functions* $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow R_+$ *and* $\theta : \Upsilon \times \Upsilon \longrightarrow [1, +\infty[$ *defined for all* $\dot{x}, \dot{y} \in \Upsilon$ *as follows*

$$\vartheta_\theta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|^p \text{ for a fixed integer } p \geq 2.$$
$$\theta(\dot{x}, \dot{y}) = (v+1)^{p-1} + |\dot{x} - \dot{y})|\,, \text{ with } v \geq 2 \text{ is every fixed integer.}$$

By Jameson's inequality, we obtain

$$\begin{aligned}
\vartheta_\theta(\dot{x}, \dot{y}) &= |\dot{x} - \dot{y})|^p = |\dot{x} - \dot{x}_1 + \dot{x}_1 - \dot{x}_2 + \cdots + \dot{x}_v - \dot{y}|^p \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq [|\dot{x} - \dot{x}_1| + |\dot{x}_1 - \dot{x}_2| + \cdots + |\dot{x}_v - \dot{y}|]^p \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq (v+1)^{p-1} \left[|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p + \cdots + |\dot{x}_v - \dot{y}|^p\right] \\
\vartheta_\theta(\dot{x}, \dot{y}) &\leq \left((v+1)^{p-1} + |\dot{x} - \dot{y})|\right) \left[|\dot{x} - \dot{x}_1|^p + |\dot{x}_1 - \dot{x}_2|^p + \cdots + |\dot{x}_v - \dot{y}|^p\right] \\
\vartheta_\theta(\dot{x}, \dot{y}) &= \theta(\dot{x}, \dot{y}) \left[\vartheta(\dot{x}, \dot{x}_1) + \vartheta(\dot{x}_1, \dot{x}_2) + \cdots + \vartheta(\dot{x}_v, \dot{y})\right].
\end{aligned}$$

It is worth noting that the pair $(\Upsilon, \vartheta_\theta)$ *is a complete* $b_\theta(v)$*-metric space. Assume us denote*

$$\gamma = \min\left(\frac{2}{\pi}, \sqrt[pv]{(v+1)^{1-p}}\right) < 1,$$

and consider the family of mapping T_δ *defined by* $T_\delta \dot{x} = \sin(\delta \dot{x}$ *with* $\delta \in]0, \gamma[$ *. Observe that*

$$\delta < \sqrt[pv]{(v+1)^{1-p}} \Longleftrightarrow (v+1)^{p-1} < \delta^{-pv}.$$

Now Assume us prove that T_δ *is a contraction with respect to the metric* ϑ_θ *with contraction coefficient* $\lambda = \delta^p < 1$. *Indeed, from Lagrange mean value theorem, we have*

$$\vartheta_\theta(T_\delta \dot{x}, T_\delta y) = |T_\delta \dot{x} - T_\delta \Upsilon|^p = |\sin(\delta \dot{x} - \sin(\delta y)|^p \leq \delta^p\, |\dot{x} - \dot{y})|^p = \delta^p\, \vartheta_\theta(\dot{x}, \dot{y}).$$

On the other hand, Assume us consider the sequence $\{\dot{x}_n\}_{n=0}^{\infty} = \{T^n \dot{x}\}_{n=0}^{\infty}$ *in* Υ *starting from every* $\dot{x} \in \Upsilon$. *We have* $\dot{x}_n \geq 0$ *and* $\dot{x}_n - \dot{x}_{n-1} = \sin \dot{x}_{n-1} - \dot{x}_n \leq 0$, *for ever* $n \in N$, *and consequently the sequence is decreasing. It follows that the sequence* $\{\dot{x}_n\}_{n=0}^{\infty} = \{T^n \dot{x}\}_{n=0}^{\infty}$ *is convergent in the complete usual metric space* (Υ, ϑ), *where the distance* $\vartheta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|$ *for all* $(\dot{x}, \dot{y}) \in \dot{y}^2$. *Therefore, for all* $0 < \delta < \gamma$, *we get :*

$$\lim_{n,m \longrightarrow \infty} \theta(\dot{x}_n, \dot{x}_m) = \lim_{n,m \longrightarrow \infty} \left((v+1)^{p-1} + |\dot{x}_n - \dot{x}_m|\right)$$
$$\lim_{n,m \longrightarrow \infty} \theta(\dot{x}_n, \dot{x}_m) = (v+1)^{p-1} < \delta^{-pv} = \frac{1}{\delta^{pv}} = \frac{1}{(\delta^p)^v} = \frac{1}{\lambda^v}.$$

It follows that for $\delta \in \,]0, \gamma[$*, all conditions of theorem 49 are satisfied. Hence, for every* $\delta \in \,]0, \gamma[$*, there exists a unique fixed point for the mapping* T_δ*. This fixed point is obviously* $\dot{x} = 0$.

Example 52. *Assume* $\Upsilon = [-1, 1]$*. Define* $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow R_+$ *and* $\theta : \Upsilon \times \Upsilon \longrightarrow [1, +\infty[$*, for all* $\dot{x}, \dot{y} \in \Upsilon$ *as* $\vartheta_\theta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|^3$ *and* $\theta(\dot{x}, \dot{y}) = 9 + |\dot{x}| + |\dot{y}|$*. Using Jameson's inequality, we get*

$$\vartheta_\theta(\dot{x}, \dot{y}) = |\dot{x} - \dot{y})|^3 = |\dot{x} - \dot{x}_1 + \dot{x}_1 - \dot{x}_2 + \dot{x}_2 - \dot{y}|^3 \leq 9\left[|\dot{x} - \dot{x}_1| + |\dot{x}_1 - \dot{x}_2| + |\dot{x}_2 - \dot{y}|\right]^3$$
$$\vartheta_\theta(\dot{x}, \dot{y}) \leq (9 + |\dot{x}| + |\dot{y}|)\left[|\dot{x} - \dot{x}_1| + |\dot{x}_1 - \dot{x}_2| + |\dot{x}_2 - \dot{y}|\right]^3$$
$$\vartheta_\theta(\dot{x}, \dot{y}) = \theta(\dot{x}, \dot{y})\left[\vartheta(\dot{x}, \dot{x}_1) + \vartheta(\dot{x}_1, \dot{x}_2) + \vartheta(\dot{x}_2, \dot{y})\right].$$

Then, $(\dot{x}, \vartheta_\theta)$ *is complete* $b_v(\theta)$*-metric space with* $v = 2$*. Consider the mapping* $T : \Upsilon \longrightarrow \Upsilon$ *such that* $T\dot{x} = \frac{\dot{x}^3}{6}$ *for every* $\dot{x} \in \Upsilon$*. Then, we get*

$$\vartheta_\theta(T\dot{x}, T\dot{y}) = |T\dot{x} - T\dot{y}|^3 = \left|\frac{\dot{x}^3}{6} - \frac{\dot{y}^3}{6}\right|^3 = \frac{1}{6^3}\left|\dot{x}^3 - \dot{y}^3\right|^3 = \frac{1}{6^3}\left|\dot{x}^2 + xy + \dot{y}^2\right|^3 |\dot{x} - \dot{y})|^3$$
$$\vartheta_\theta(T\dot{x}, T\dot{y}) \leq \frac{1}{8}|\dot{x} - \dot{y})|^3 = \frac{1}{8}\vartheta_\theta(\dot{x}, \dot{y}).$$

This shows that the contraction condition (5.1) *holds with* $\lambda = \frac{1}{8} < 1$*. We also easily seen that* $T^n\dot{x} = \frac{\dot{x}^{3^n}}{6^{4\times 3^{n-2}}}$ *for all* $n \geq 2$ *and* $\dot{x} \in \dot{x} = [-1, 1]$*. Thus, we obtain*

$$\lim_{n,m\longrightarrow\infty} \theta\left(T^n\dot{x}, T^m\dot{x}\right) = \lim_{n,m\longrightarrow\infty} \left(9 + |T^n\dot{x}| + |T^m\dot{x}|\right)$$
$$\lim_{n,m\longrightarrow\infty} \theta\left(T^n\dot{x}, T^m\dot{x}\right) = \lim_{n,m\longrightarrow\infty} \left(9 + \frac{|\dot{x}|^{3^n}}{6^{4\times 3^{n-2}}} + \frac{|\dot{x}|^{3^m}}{6^{4\times 3^{m-2}}}\right)$$
$$\lim_{n,m\longrightarrow\infty} \theta\left(T^n\dot{x}, T^m\dot{x}\right) = 9 < 64 = \frac{1}{\lambda^2} = \frac{1}{\lambda^v}.$$

Consequently, all conditions of theorem 49 are fulfilled. Therefore, the mapping T has a unique fixed point in $\Upsilon = [-1, 1]$.

5.3 Evaluation of the order of convergence to the

FIXED POINT IN THE GENERALIZED $b_v(s)$-METRIC SPACES

In the following theorem, we give an evaluation of the order of convergence to the fixed point in the generalized $b_v(s)$-metric spaces which is not considered by Mitrovic and Radenovic in [85]. Notice that the first part of theorem 53 is exactly the full statement of [85, Theorem 2.1] and a direct consequence of theorem 49.

Theorem 53. *Assume (Υ, ϑ) be a complete $b_v(s)$-metric space with coefficient $s > 1$ and $T : \Upsilon \longrightarrow \Upsilon$ be a mapping satisfying the following contraction inequality*

$$\vartheta(T\dot{x}, T\dot{y}) \leq \lambda \vartheta(\dot{x}, \dot{y}) \tag{5.2}$$

for all $\dot{x}, \dot{y} \in \Upsilon$, where $\lambda \in [0, 1)$ is a fixed real number. Then, T has a unique fixed point ζ. Moreover, if $s\lambda^v < 1$, the error of approximation of the unique fixed point ζ by the convergent sequence $\{\dot{x}_n\} = \{T^n \dot{x}_0\}$, from the start point $\dot{x}_0 \in \Upsilon$ is given as follows

$$\vartheta(\dot{x}_n, \zeta) \leq \frac{s\lambda^n (1-\lambda^v)}{(1-\lambda)(1-s\lambda^v)} \vartheta(\dot{x}_1, \dot{x}_0) \leq \frac{s\lambda^n \vartheta(\dot{x}_1, \dot{x}_0)}{(1-\lambda)(1-s\lambda^v)}, \tag{5.3}$$

for all $n \in Z_+$.

Proof. The existence of the fixed point ζ of T is ensured by the previous theorem 49 or by theorem 2.1 in [85] as a particular case of theorem 49 for $\theta(\dot{x}, \dot{y}) = s \geq 1$ for all $\dot{x}, \dot{y} \in \Upsilon$. Assume $\dot{x}_0 \in \Upsilon$ be arbitrary and define the sequence $\{\dot{x}_n\}$ given by $\dot{x}_{n+1} = T\dot{x}_n$ for all $n \in \mathbb{Z}_+$. If $\dot{x}_{n+1} = \dot{x}_n$ for some integer n, then $\zeta = \dot{x}_n$ and $\vartheta(\dot{x}_n, \zeta) = 0$ verifies the inequality (5.3). Assume us assume that $\dot{x}_n \neq \dot{x}_{n+1}$ for all $n \in \mathbb{Z}_+$ and setting $\vartheta_n = \vartheta(\dot{x}_n, \dot{x}_{n+1})$. From (5.2), we get

$$\vartheta(\dot{x}_n, \dot{x}_{n+1}) = \vartheta(T\dot{x}_{n-1}, T\dot{x}_n) \leq \lambda d(\dot{x}_{n-1}, \dot{x}_n), \text{ i.e. } \vartheta_n \leq \lambda \vartheta_{n-1}.$$

By induction, we obtain

$$\vartheta_n \leq \lambda^n \vartheta_0. \tag{5.4}$$

For the sequence $\{\dot{x}_n\}$, Assume us consider $\vartheta(\dot{x}_n, \dot{x}_m)$ such that $m-1 = n+vq+p$, where $q \in \mathbb{N}$ and $p \in \overline{0, n-1}$. Using the contraction inequality (5.4) and from

(5.2), we obtain by induction

$$\vartheta(\dot{x}_n, \dot{x}_m) \leq s\left[\vartheta(\dot{x}_n, \dot{x}_{n+1}) + \vartheta(\dot{x}_{n+1}, \dot{x}_{n+2}) + \cdots + \vartheta(\dot{x}_{n+v-1}, \dot{x}_{n+v}) + \vartheta(\dot{x}_{n+v}, \dot{x}_m)\right]$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq s\sum_{i=0}^{v-1}\vartheta_{n+i} + s^2\left[\sum_{i=0}^{v-1}\vartheta_{n+v+i} + \vartheta(\dot{x}_{n+2v}, \dot{x}_m)\right]$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq s\sum_{i=0}^{v-1}\vartheta_{n+i} + s^2\sum_{i=0}^{v-1}\vartheta_{n+v+i} + s^3\left[\sum_{i=0}^{v-1}\vartheta_{n+2v+i} + \vartheta(\dot{x}_{n+3v}, \dot{x}_m)\right]$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq s\sum_{i=0}^{v-1}\vartheta_{n+i} + \cdots + s^{q-1}\sum_{i=0}^{v-1}\vartheta_{n+(q-2)v+i} + s^q\left[\sum_{i=0}^{v-1}\vartheta_{n+(q-1)v+i} + \vartheta(\dot{x}_{n+qv}, \dot{x}_m)\right]$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq \sum_{k=1}^{q-1}s^k\sum_{i=0}^{v-1}\vartheta_{n+(k-1)v+i} + s^q\sum_{i=0}^{v-1}\vartheta_{n+(q-1)v+i} + s^q\vartheta(\dot{x}_{n+vq}, \dot{x}_m)$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq \sum_{k=1}^{q-1}s^k\sum_{i=0}^{v-1}\lambda^{n+(k-1)v+i}\vartheta_0 + s^q\sum_{i=0}^{v-1}\lambda^{n+(q-1)v+i}\vartheta_0 + s^q\lambda^{n+qv}\vartheta(\dot{x}_0, \dot{x}_p)$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq \vartheta_0\sum_{k=1}^{q}s^k\sum_{i=0}^{v-1}\lambda^{n+(k-1)v+i} + (s\lambda^v)^q\lambda^n\vartheta(\dot{x}_0, \dot{x}_p)$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq \vartheta_0\left(\frac{1-\lambda^v}{1-\lambda}\right)\lambda^n s\sum_{k=1}^{q}(s\lambda^v)^{(k-1)} + (s\lambda^v)^q\lambda^n\vartheta(\dot{x}_0, \dot{x}_p)$$
$$\vartheta(\dot{x}_n, \dot{x}_m) \leq s\vartheta_0\lambda^n\left(\frac{1-\lambda^v}{1-\lambda}\right)\left(\frac{1-(s\lambda^v)^q}{1-s\lambda^v}\right) + (s\lambda^v)^q\lambda^n\vartheta(\dot{x}_0, \dot{x}_p). \tag{5.5}$$

Now one can observe that when n is fixed and m tends to infinity, it follows that q tends also to infinity. Hence, by making m tend towards infinity in the inequality (5.5), the estimate (5.3) follows immediately. □

Example 54. *Assume $\Upsilon = \{a, b, c, d, e, f\}$ and consider the generalized metric d : $\Upsilon \times \Upsilon \longrightarrow\longrightarrow R_+$ defined for every $\dot{x} \in \Upsilon$ as follows*

$$\vartheta(a, b) = 5$$
$$\vartheta(a, c) = \vartheta(a, d) = \vartheta(a, e) = \vartheta(b, c) = \vartheta(b, d) = \vartheta(b, e) = \vartheta(c, d) = \vartheta(c, e) = \vartheta(d, e) = 1$$
$$\vartheta(a, f) = \vartheta(b, f) = \vartheta(c, f) = \vartheta(d, f) = \vartheta(e, f) = 6$$

Assume us consider $T : \Upsilon \longrightarrow \Upsilon$ the mapping defined as follows

$$T\dot{x} = \begin{cases} c, & \text{if } \dot{x} \in \{a, b, c, d, e\}, \\ a, & \text{if } \dot{x} = f. \end{cases}$$

Then, (Υ, ϑ) is an hexagonal metric space ($b_v(s)$-metric space for $v = 4$ and $s = 1$). Moreover, for all $\dot{x}, \dot{y} \in \Upsilon$, we have

$$\vartheta\,(T\dot{x}, T\dot{y}) \leq \frac{1}{6}\vartheta(\dot{x}, \dot{y}).$$

It follows that T verifies the Banach contraction condition of theorem 53. Therefore, according to theorem 53, the mapping T has a unique fixed point, which is clearly $\dot{x} = c$. It is worth noting that (Υ, ϑ) is not a metric space or a rectangular metric space. Indeed, we have

$$5 = \vartheta(a,b) > \vartheta(a,c)+\vartheta(c,d)+\vartheta(d,e)+\vartheta(e,b) = 1+1+1+1 = 4 > \vartheta(a,c)+\vartheta(c,d)+\vartheta(d,b) = 3.$$

Now state for the extended $b_v(\theta)$-metric space, another variant fixed point result which is analogue to fixed point theorem obtained by Hicks and Rhoades [83] and extended recently by [11] for extended $b_1(\theta)$-metric spaces. Before this, we need the following definition given in [11] and corrected here, since there is a simple mistake of notation in the statement as it is presented in the previous mentioned paper [11, Definition 6].

Definition 55. *([11]) Assume $T : \Upsilon \longrightarrow \Upsilon$ and for some $\dot{x}_0 \in \Upsilon$, Assume $O(\dot{x}_0) = \left\{\dot{x}_0, T\dot{x}_0, T^2\dot{x}_0, \cdots\right\}$ be the orbit of $\dot{x}_0$. A function $G : \Upsilon \longrightarrow R$ is said to be T-orbitally lower semi-continuous at $\xi \in \Upsilon$ if $\{\dot{x}_n\} \subset O(\dot{x}_0)$ and $\dot{x}_n \longrightarrow \xi$ implies $G(\xi) \leq \liminf_{n\longrightarrow\infty} G(\dot{x}_n)$.*

Now we are ready to state our result.

Theorem 56. *Assume $(\Upsilon, \vartheta_\theta)$ be a complete extended $b_v(\theta)$-metric space so that ϑ_θ is a continuous functional. Assume us consider the mapping $T : \Upsilon \longrightarrow T$ and assume there exists $\dot{x}_0 \in \Upsilon$ such that*

$$\vartheta_\theta(T^2 z, Tz) \leq \lambda\, \vartheta_\theta(Tz, z) \text{ for ever } z \in O(\dot{x}_0), \tag{5.6}$$

where $\lambda \in [0, 1)$ be a fixed real number such that

$$\lim_{n,m\longrightarrow\infty} \theta(T^n\dot{x}_0, T^m\dot{x}_0) < \frac{1}{\lambda^v}.$$

Then, $\lim_{n\longrightarrow\infty} T^n\dot{x}_0 = \zeta \in X$. Moreover, ζ is a fixed point if and only if $G(\dot{x} = \vartheta_\theta(T\dot{x}, \dot{x}$ is T-orbitally lower semi continuous at ζ.

Proof. For $\dot{x}_0 \in \Upsilon$, Assume us consider the sequence $\{\dot{x}_n\}$ such that $\dot{x}_{n+1} = T\dot{x}_n$ for all $n \in \mathbb{Z}_+$. Following the same procedure as in the proof of the previous theorem 49, we derive that $\{\dot{x}_n\}$ is a Cauchy sequence. Since $(\dot{x}, \vartheta_\theta)$ is a complete space, it follows that $T^n \longrightarrow \zeta \in \Upsilon$. On the other hand, by successively applying inequality (5.6) for $z = \dot{x}_0 \in O(\dot{x}_0)$, we get for all $n \in \mathbb{N}$

$$\vartheta_\theta(T^{n+1}z, T^n z) \leq \lambda \vartheta_\theta(T^n \dot{x}_0, T^{n-1}\dot{x}_0) \leq \cdots \leq \lambda^n \vartheta_\theta(\dot{x}_1, \dot{x}_0).$$

Now Assume us consider the function $G : \dot{x} \longrightarrow \mathbb{R}_+$ defined by $G(\dot{x} = \vartheta_\theta(T\dot{x}, \dot{x}$ and assume that ζ is a fixed point of T such that $\lim\limits_{n\longrightarrow\infty} \dot{x}_n = \zeta$ with $\dot{x}_n \in O(\dot{x}_0)$. Since $\liminf\limits_{n\longrightarrow\infty} \vartheta_\theta(T^{n+1}\dot{x}_0, T^n\dot{x}_0) \geq 0$, then we get

$$G(\zeta) = \vartheta_\theta(T\zeta, \zeta) = 0 \leq \liminf_{n\longrightarrow\infty} G(\dot{x}_n).$$

Conversely, assume that the function G is T-orbitally lower semi continuous at ζ. Then, we have

$$\begin{aligned} \vartheta_\theta(T\zeta, \zeta) = G(\zeta) &\leq \liminf_{n\longrightarrow\infty} G(\dot{x}_n) = \liminf_{n\longrightarrow\infty} \vartheta_\theta(T^{n+1}\dot{x}_0, T^n\dot{x}_0) \\ &\leq \liminf_{n\longrightarrow\infty} \lambda^n \vartheta_\theta(\dot{x}_1, \dot{x}_0) = 0. \end{aligned}$$

Hence, $\vartheta_\theta(T\zeta, \zeta) = 0 \Longleftrightarrow T\zeta = \zeta$, which shows that ζ is a fixed point of T. □

Remark 57. *If $\theta(\dot{x}, \dot{y}) = 1$ is a constant function, then theorem 56 corresponds to [83, Theorem 1]. Furthermore, when we deal with the extended b-metric space, i.e. $v = 1$, then theorem 56 reduces to the main result of Kamran et al [11, Theorem 3]. It follows that theorem 56 extends and generalizes [83, Theorem 1] and [11, Theorem 3] for more general spaces.*

Example 58. *Consider the same $b_v(\theta)$-metric space $(\Upsilon, \vartheta_\theta)$ given in example 51 and we also define the same family of mapping $T_\delta\dot{x} = \sin(\delta\dot{x}$ for all $\dot{x} \in Y = \left[0, \frac{\pi}{2}\right]$, with the parameter $\delta \in]0, \gamma[$, where $\gamma = \min\left(\frac{2}{\pi}, \sqrt[pv]{(v+1)^{1-p}}\right)$. Assume us check the contraction condition* (5.6) *of theorem 56. In fact, according to Lagrange mean value theorem, for all $\dot{x} \in \Upsilon$, we get*

$$\begin{aligned} \vartheta_\theta(T_\delta^2\dot{x}, T_\delta\dot{x} &= \left|T_\delta^2\dot{x} - T_\delta\dot{x}\right|^p = |\sin(\delta\sin(\delta\dot{x}) - \sin(\delta\dot{x}|^p \\ &\leq |\delta\sin(\delta\dot{x} - \delta\dot{x}|^p = \delta^p\,|\sin(\delta\dot{x} - \dot{x}|^p = \delta^p\vartheta_\theta(T\dot{x}, \dot{x}). \end{aligned}$$

Consequently, for every $\delta \in]0, \gamma[$*, there exists a fixed point for the function* T_δ*. However, and unlike theorem 49, theorem 56 does not allows us to deduce the uniqueness of the fixed point for the mapping* T *on* $\Upsilon = \left[0, \frac{\pi}{2}\right]$.

5.4 Application

In this section, we derive the conditions under which we can ensure either the existence of a solution of the Fredholm integral equation or a local solution of the ordinary differential equations in the $b_v(\theta)$-metrics spaces and in particular for the $b_v(s)$-metric spaces. First, consider the following Fredholm integral equation given as follows:

$$\dot{x}(t) = \int_a^b K(t, u, \dot{x}(u))du + h(t) \text{ for } t, u \in [a, b], \tag{5.7}$$

where $K, h \in C([a, b], (0, \infty))$. Assume us define the function $\vartheta : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ by

$$\vartheta(\dot{x}, \dot{y}) = \sup_{t \in [a,b]} |\dot{x}(t) - \dot{y}(t)|^n, \text{ for } n \geq 3 \text{ is a fixed integer.}$$

Consequently, the pair (Υ, ϑ) is a complete $b_v(s)$-metric space with $v = n$ and $s = n^{n-1} > 1$. Applying either theorem 49 with $\theta(\dot{x}, \dot{y}) = s = n^{n-1} > 1$ for $\dot{x}, \dot{y} \in X$ or theorem 53 with $s = n^{n-1} > 1$, we obtain the following result.

Theorem 59. *Assume that for all* $\dot{x}, \dot{y} \in \Upsilon$ *and* $\alpha > 1$ *a fixed real number, we have*

$$|K(t, u, \dot{x}(u)) - K(t, u, \dot{y}(u))| \leq \frac{1}{\alpha(b-a)} |\dot{x}(u) - \dot{y}(u)|, \tag{5.8}$$

for all $t, u \in [a, b]$*. Then, the integral equation* (5.7) *has a unique solution.*

Proof. First, we define the mapping $T : \Upsilon \longrightarrow \Upsilon$ as follows: for all $t \in [a, b]$

$$T\dot{x}(t) = \int_a^b K(t, u, \dot{x}(u)du + h(t).$$

Then, x is a fixed point of T if and only if it is a solution of the integral equation

(5.7). On the other side, for all $\dot{x}, \dot{y} \in [a, b]$, we have

$$
\begin{aligned}
|T\dot{x}(t) - T\dot{y}(t)|^n &\leq \left(\int_a^b |K(t, u, \dot{x}(u)) - K(t, u, \dot{y}(u))| \, du \right)^n \\
&\leq \left(\frac{1}{\alpha(b-a)} |\dot{x}(u) - \dot{y}(u)| \, du \right)^n \\
&\leq \frac{1}{(\alpha(b-a))^n} \sup_{t \in [a,b]} |\dot{x}(u) - \dot{y}(u)|^n \left(\int_a^b du \right)^n \\
&\leq \frac{1}{\alpha^n} \vartheta(\dot{x}, \dot{y}).
\end{aligned}
$$

By putting $\lambda = \frac{1}{\alpha^n} \in [0, 1)$, it follows either from theorem 49 or theorem 53 that T has a unique solution. □

Remark 60. *It is worth noting that if $\alpha(b-a) < 1$, then the kernel $K(t, s, \dot{x}(s))$ is not necessarily a contraction function with respect to the variable x. In this case, the condition* (5.8) *is weaker than the usual working condition of contraction used in many papers (see e.g. [11]).*

Now Assume us define the metric $\vartheta_\theta : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ given by

$$\vartheta_\theta(\dot{x}, \dot{y}) = \sup_{t \in [a,b]} |\dot{x}(t) - \dot{y}(t)|^n \text{ for } n \geq 2 \text{ is a fixed integer.}$$

We also consider the function $\theta : \Upsilon \times \Upsilon \longrightarrow [1, \infty)$ defined as follows:

$$\theta(\dot{x}, \dot{y}) = \sup_{t \in [a,b]} \left(n^{n-1} + |\dot{x}| + |\dot{y}| \right) \text{ for all } \dot{x}, \dot{y} \in \dot{x} \text{ and } n \geq 2.$$

Then, $(\Upsilon, \vartheta_\theta)$ is a complete extended polygonal b-metric space ($b_v(\theta)$-metric space for $v = n$). Hence the following result derives from theorem 56.

Theorem 61. *Assume that for all $\dot{x}, \dot{y} \in \Upsilon$ and $\alpha > 1$ a fixed real number, we have*

$$|K(t, u, T\dot{x}(u)) - K(t, u, \dot{x}(u))| \leq \frac{1}{\alpha(b-a)} |T\dot{x}(u) - \dot{x}(u)| \text{ for ever } u, t \in [a, b] \text{ and } \dot{x} \in X.$$

Then, the integral equation (5.7) *has at least one solution.*

Proof. We have

$$\begin{aligned}\left|T^2\dot{x}(t) - T\dot{x}(t)\right|^n = |T(T\dot{x}(t)) - T\dot{x}(t)|^n &\leq \left(\int_a^b |K(t,u,T\dot{x}(u)) - K(t,u,\dot{x}(u))|\, du\right)^n \\ &\leq \left(\int_a^b \left(\frac{1}{\alpha(b-a)}\right) |\dot{x}(u) - \dot{y}(u)|\, du\right)^n \\ &\leq \left(\frac{1}{\alpha(b-a)}\right)^n \sup_{t\in[a,b]} |\dot{x}(t) - \dot{y}(t)|^n \left(\int_a^b du\right)^n \\ &\leq \frac{1}{\alpha^n} \sup_{t\in[a,b]} |\dot{x}(t) - \dot{y}(t)|^n \\ &\leq \frac{1}{\alpha^n} \vartheta_\theta(\dot{x}, \dot{y}).\end{aligned}$$

By putting $\lambda = \frac{1}{\alpha^n} \in [0,1)$, it follows from theorem 56 that T has a solution. □

The same observation of remark 60 may be done.

Now, we apply theorem 56 for the resolution of the differential equation of first order $\dot{x} = f(t,\dot{x})$, where $f : D = R \times E \longrightarrow E$ are bounded mapping, i.e., it maps bounded sets in D to bounded sets in E, where $E \subset R$. More precisely, we seek sufficient conditions for the existence of local solutions $\dot{x} \in C^1(I,E)$, where $I \subset E$ an interval such that $\dot{x}(t_0) = \dot{x}_0$ where $t_0 \in I$. By integrating both sides, any function x satisfying the differential equation must also satisfy the integral equation

$$\dot{x}(t) = \dot{x}_0 + \int_{t_0}^t f(u,\dot{x}(u))du. \tag{5.9}$$

Assume us define the mapping $T : \Upsilon \longrightarrow \Upsilon$ as follows: $T\dot{x}(t) = \int_{t_0}^t f(u,\dot{x}(u))du$ for all $t \in I$. Then, x is a fixed point of T if and only if it is a solution of the integral equation (5.9). For this, we use the same previous metric $\vartheta_\theta(\dot{x},\dot{y}) = \sup_{t\in[a,b]} |\dot{x}(t) - \dot{y}(t)|^n$ for $n \geq 2$ and the function $\theta : \Upsilon \times \Upsilon \longrightarrow [1,\infty)$ defined as follows: $\theta(\dot{x},\dot{y}) = \sup_{t\in[a,b]} \left(n^{n-1} + |\dot{x}| + |\dot{y}|\right)$ for all $\dot{x},\dot{y} \in \Upsilon$, where $\Upsilon = C^1(\mathbb{R},E)$ is the set of all differentiable functions on some interval J of R. So we shall use theorem 56 to establish existence of local solution of this initial value problem.

Theorem 62. *Assume $\dot{x} \in \Upsilon$ be the constant function such that $\dot{x}(t) = \dot{x}_0$ for all $t \in R$*

and Assume us consider the following initial value problem

$$\begin{cases} \dot{x} = f(t, \dot{x}) \\ \dot{x}(t_0) = \dot{x}_0. \end{cases} \tag{5.10}$$

Where $f : D = R \times E \longrightarrow E$ *are bounded mapping and* $E \subset R$. *Assume that for every function* $\varphi \in \Upsilon$ *such that* $\varphi(t_0) = \dot{x}_0$, *we have*

$$|f(t, T\varphi(u)) - f(t, \varphi(u))| \leq \lambda\, |T\varphi(u) - \varphi(u)| \text{ for every } t, u \in I$$

where $\lambda < 1$ *is a real positive constant and* I *some interval such that* $t_0 \in I$. *Then, the problem* (5.10) *has at least a solution on some interval* $J \subseteq I$ *containing* t_0.

Proof. For all $\varphi \in \Upsilon$ such that $\varphi(t_0) = \dot{x}_0$, we have

$$\begin{aligned} \left|T^2\varphi(t) - T\varphi(t)\right|^n &\leq \left(\int_{t_0}^{t} |f(u, T\varphi(u)) - f(u, \varphi(u))|\, du\right)^n \leq \left(\int_{t_0}^{t} \lambda\, |T\varphi(u) - \varphi(u)|\, du\right)^n \\ &\leq (\lambda(t - t_0))^n \sup_{u \in I} |T\varphi(u) - \varphi(u)|^n . \end{aligned}$$

Hence, $\vartheta_\theta(T^2\varphi, T\varphi) \leq (\lambda(t - t_0))^n\, \vartheta_\theta(T\varphi, \varphi)$ for $n \geq 2$ a fixed integer. In order to apply theorem 56, we must have $(\lambda(t - t_0))^n < 1$, which leads us to impose the requirement $t \in \left]t_0 - \lambda^{-1}, t_0 + \lambda^{-1}\right[$. From theorem 56, we conclude that the integral equation have a solution $\dot{x} \in X$ on the interval $J = \left]t_0 - \lambda^{-1}, t_0 + \lambda^{-1}\right[$. This proves that the initial value problem admits a local solution on the interval J. □

Remark 63. *It is clear that if we impose the well known Cauchy-lipschitz condition for the function* f *with respect to the metric* ϑ, *then we can use similarly either theorem 49 for* $\theta(\dot{x}, \dot{y}) = s = n^{n-1}$ *or theorem 53 for* $s = n^{n-1}$ *to establish the existence and the uniqueness of a local solution of the initial value problem* (5.9) *in the polygonal b-metric space.*

Conclusion and perspectives

5.5 Conclusion

The study of single-valued and multi-valued fixed points for various types of contractions in different classes of metric spaces are the focus of this thesis. In addition to our newly developed class of metrics spaces, $b_v(s)$ and $b_v(\theta)$, furnished with numerous examples. We have examined the existence of an integral inclusion's solution as an application, such as initial value issues, voletra, type and Fredholm type integral equations.

5.6 Perspectives

We'll examine the following problems as examples in the future:

- We will try to apply the obtained results to study the existence of a fixed point in complex valued polygonal metric spaces
- We intend to explore alternative pairings of the different contractions in order to enhance our findings and discover fresh ones.

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