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Carleman estimates to study some controllability problems of hyperbolic PDES

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بسم الله الرحمن الرحيم

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Dedication

I dedicate this modest work to: My father "Mohamed" for his love, support, constant encouragement and patience with me, his advice and guidance that all my words are not enough to thank him, thank you for everything you have given me.

To my mother, who taught me steadfastness to the greatest and kindest heart, she has the credit for what I am now, may God prolong her life and give her health and well being.

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Abstract

In this memory,we present carleman estimates for some hyperbolic Partial Differential Equations and on the techniques of construction and application to solving controllability problems.

In the first chapter, we mention some important and comprehensive notion and theories that we need later.

In the second chapter, we introduced some methodes for creating carleman estimates of hyperbolic (PDES).

In the third chapter we present some applications to solve problems of controllability of hyperbolic equation.

Key words: Carleman estimates,hyperbolic equations,null controllability, weight functions

ملخص

في هذه المذكرة، نقدم متراجحات كارلمان لبعض المعادلات التفاضلية الجزئية الزائدية وعلى تقنيات بنائها و تطبيقها لحل مسائل قابلية التحكم.
في الفصل الاول نذكر بعض المفاهيم و النظريات المهمة و الشاملة التي نحتاجها لاحقا.
في القصل الثاني نقدم بعض الطرق لإنشاء متراجحات كارلمان للمعادلات التفاضلية الجزئية الزائدية.
في الفصل الثالث نقدم فيه بعض التطبيقات لحل مشاكل التحكم في المعادلة الزائدية.

كلمات مفتاحية : متراجحات كارلمان،معادلات زائدية، التحكم المعدوم، معادلة الوزن (الثقل).

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Notations & abbreviations

$\mathbb{R}$	Set of real numbers.
Ω	An open set in $\mathbb{R}^n$ with boundary $\partial\Omega$.
$\overline{\Omega}$	Closure of Ω respectively.
$\Gamma\ ou\ \partial\Omega$	A boundary of Ω.
$d\Gamma$	Lebesgue measure on boundary Γ.
H^1, H^2	Sobolev space.
$L^2(\Omega)$	The space of measurable functions of summable squares in Ω.
$L^2(0,T;X)$	The bounded linear operator space.
C^1	The class of functions with continuous first derivative.
$\lvert . \rvert_H$	A semi norm in H.
$\lVert . \rVert_H$	A norm in Hilbert space H.
$(.,.)_H$	A scalar product in Hilbert space H.
$\partial_i u$	Derivative for u.
$\frac{\partial u}{\partial \nu} = \nabla u.\nu$	The conormal derivative.
$\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$	The gradient operator.
$\mathcal{L}(\mathcal{V}, \mathcal{X})$	The space of linear bounded operators from $\mathcal{V}$ to $\mathcal{X}$.
$D(A)$	The domain of definition of the operator A.
PDE	Partial differential equation.
$\sup_{x\in\Omega} ess\lvert u(x)\rvert$	Sup essentail of function u.
iff	if only if.
a.e.	Almost every where.
$s.t$	Such That.

Introduction

In 1930s , carleman estimates were presented by the swedish mathematician Torsten carleman [**9**].These estimates give essential information about the internal behavior of the system, helping to complete the analysis and design of control strategies to acheive desired outcomes.

Carleman's works focused on the study of partial differential equations (PDES) and their solution. The basic idea of Carleman estimates is to use an integrated identity that includes a PDE solution and weight function. Choosing the weight function should be carefully to make sure that these estimates are found in good form [**7**].

One of the most important uses of the latter in mathematics is some applications that include inverse problems [**14**] where it has been used for certain types of inverse problems of partial differential equations, where it seeks to recover information about an unknown operator or function due to some of them Data and knowledge of the differential equation governing their behavior, as well as Integral Equations. Carleman estimates have been used in many theories, the most important of which is :

control theory (see for example [**22**] and [**16**])in the study of controllability and stability of systems governed by partial differential equations.

Spectral theory [**17**] was used to study the spectral properties of operators in Hilbert spaces.Descriptions of nonlinear analysis are also used to study the behavior of nonlinear systems, such as nonlinear Partial differential equations.

Finally, we must not forget also the geometric analysis of the study of geometry Properties of cubes and subfolders, such as curvature and volume.

So this work is organised as follows :

In the first chapter, we will give a reminder of some elements in functional analysis , firstly, the basic spaces as follows :Hilbert spaces, L^p spaces and Sobolev space, as well as some theorems that we will use later, secondly, some notions and definitions necessary for control, and finally we can consider some concepts of carleman estimates.

In the second chapter, we will introduce some techniques for creating carleman estimates for hyperbolic partial differential equations .These estimates were introduced by the Swedish mathematician Torsten carleman in 1922 who named them carleman's inequality.

The last chapter will be devoted to the proof of the null control of some hyperbolic equations , that is it is associated with the use of the already presented Carleman estimates Chapter 2,finally ,we will prove the equivalence between Controllability problem and Variational problem.

Chapter 1

Basics on the controllability of hyperbolic PDES

In this chapter, we will present a call on fundamental spaces in functional analysis which contains some essential notions that concern the L^p spaces, the spaces of Hilbert and Sobolev as well as a part of definitions and theories then, we will consider to deduce some concepts of controllability, and finally, we will familiarize ourselves with the concept of Carleman's estimates that they are necessary to know in order to approach the continuation of this memory.

1.1 Elements of Functional Analysis

1.1.1 Inner Product Spaces

Definition 1.1 [**11**] *Assume that E is real space , $(.,.)_E : E \times E \longrightarrow \mathbb{R}$ is an inner product on E iff the following condition are met :*

1. $\forall u, v \in E \qquad (u,v)_E = (v,u)_E \qquad (symetry)$.

2. $\begin{cases} \forall u \in E \qquad (u,u)_E \geqslant 0 \\ \forall u \in E \qquad (u,u)_E = 0 \Rightarrow u = 0 \end{cases} \qquad (positivity)$.

3. $\forall u, v \in E \qquad (\gamma u + \mu v, w)_E = \gamma(u,w)_E + \mu(v,w)_E \ \forall \gamma \ , \ \mu \ \in \mathbb{R}$ *(bilinearity)* .

1.1.2 Hilbert Spaces

Definition 1.2 [11] *A Hilbert space H is an complet inner product space (see Def 1.1), considering that $||.||_H$ be the norm associated with inner product $(.,.)_H$ such that $||.||_H = \sqrt{(.,.)_H}$.*

1.1.3 $L^p(\Omega)$ Spaces

Definition 1.3 [19] *Let Ω be an open of $\mathbb{R}^n$, and $p \in \mathbb{R}$ with $1 \leqslant p < \infty$, we define $L^p(\Omega)$ space as follow :*

$$L^p(\Omega) = \{\varphi : \Omega \longrightarrow \mathbb{R}, \varphi \; is \; mesurable \; and \; \int_\Omega |\varphi(x)|^p dx < \infty\} .$$

with the norm

$$||\varphi||_p = \left(\int_\Omega |\varphi(x)|^p dx\right)^{\frac{1}{p}} .$$

for $p = \infty$:

$$L^\infty(\Omega) = \{\varphi : \Omega \longrightarrow \mathbb{R}, \varphi \; is \; mesurable, \; \exists \, C > 0 \; s.t \; |\varphi(x)| \leqslant C \; a.e \; on \; \Omega\} .$$

is equipped with the norm

$$||\varphi||_\infty = supp \; ess \; (\varphi) = inf\{C : |\varphi(x)| \leqslant C \; a.e\} .$$

1.1.4 Sobolev Spaces

The space $H^1(\Omega)$

Definition 1.4 [18] *Let Ω be an open of $\mathbb{R}^n$ of boundary Γ, we call Sobolev space of order one on Ω the space*

$$H^1(\Omega) = \left\{u \in L^2(\Omega), \partial_{x_i} u \in L^2(\Omega), \forall i = 1, ..., n\right\} .$$

equipped with the norm

$$||u||^2_{H^1(\Omega)} = \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \ .$$

And the inner product given by

$$(u,v)_{H^1(\Omega)} = \int_\Omega uv \, dx + \int_\Omega \nabla u \nabla v \, dx \ .$$

The space $H^1(\Omega)$ is a Hilbert space.

The space $H^1_0(\Omega)$

Definition 1.5 [**1**] *We defined the space H^1_0 as follows*

$$H^1_0(\Omega) = \{u \in H^1(\Omega), u = 0 \ on \ \partial\Omega\} \ .$$

The norm in $H^1_0(\Omega)$ is given by

$$||u||^2_0 = \int_\Omega |\nabla u|^2 \, dx \ .$$

Remark 1.1 :[**14**] *We can characterize the structure of $\mathcal{V}$ as a subspace of a weighted sobolev space. Indeed, let $H_\rho(Q)$ be the weighted Hilbert space defined by*

$$H_\rho(Q) = \{u \in L^2(Q) \text{ such that } \int_Q \frac{1}{\rho^2}|u|^2 r dr dt < \infty\}.$$

endowed with the natural norm

$$||.||_{H_\rho(Q)} = \left(\int_Q \frac{1}{\rho^2}|.|^2 dx dt\right)^{1/2} .$$

1.1.5 $L^p(0,T,X)$ Spaces

Definition 1.6 [**20**] *Let X be a Banach space, we denote by $L^p(0,T,X)$ the space of measurable functions $u :]0;T[\longrightarrow X$ such that*

$$||u||_{L^p(0,T,X)} = \left(\int_0^T ||u(t)||^p_X dt\right)^{\frac{1}{p}} < \infty \ for \ 1 \leqslant p < \infty \ .$$

and for $p = \infty$ we have

$$||u||_{L^p(0,T,X)} = \sup_{t\in[0,T]} \ ess\ ||u(t)||_X < \infty\ .$$

The space $L^p(0,T,X)$ is a Banach space for all $1 \leqslant p < \infty$

If X is of Hilbert for the scalar product $(.,.)_X$, $L^2(0,T,X)$ is a Hilbert space for the Scalar product

$$(u,v)_{L^2(0,T,X)} = \int_0^T (u(t),v(t))_X dt\ .$$

1.1.6 Some fundamental theorems

Variational problem

L'ets consider the following problem, called variational problem, with $a = a(u,v)$ be a bilinear form in Hilbert space H :

$$\{\ Find\ u\ \in\ H\ such\ that\ a(u,v) = (F,v), \forall v \in\ H\} \tag{1.1}$$

Theorem 1.1 **(Lax-Miligram)**[**8**] *Let H be a Hilbert space,$a = a(u,v)$ be a bilinear form in H, there exist a unique solution $\widetilde{u}\ \in\ H$ of the problem (1.1) , iff :*

i) a is continuous, if there exist $\exists$ a constant $M > 0$ such that

$$|a(u,v)| \leqslant M\ ||u||_H ||v||_H\ \forall u,v\ \in\ H\ .$$

ii) a is coercive, if there exist $\exists$ a constant $\alpha > 0$ such that

$$a(u,u) \geqslant \alpha\ ||u||_H^2\ \forall v\ \in\ H\ .$$

Proof. given in [**11**] ■

Theorem 1.2 [**8**] *Consider u and $v \in H$, then, Schwarz's inequality given by :*

$$|(u,v)_H \leqslant ||u||_H ||v||_H\ .$$

Theorem 1.3 **(Integration by parts)**[7] *Let $\Omega \subset \mathbb{R}^n$ be a bounded spatial domain with smooth boundary $\partial\Omega$, where ν denote the external unit normal vector .*
Let u and v be any two real functions of class $C^1(\overline{\Omega})$, we have

$$\int_\Omega \nabla uv \; dx = \int_\Gamma uv.\nu \; d\Gamma - \int_\Omega u\nabla v \; dx \; .$$

Theorem 1.4 **(Green Formula)**[7] *Let $\Omega \subset \mathbb{R}^n$ be a bounded spatial domain with smooth boundary $\partial\Omega$, where ν denote the external unit normal vector .*
Let u and v be any two real functions of class $C^1(\overline{\Omega})$, we have

$$\int_\Omega (\Delta u)v \; dx = \int_\Gamma (\partial_\nu u)v \; \partial\Gamma - \int_\Omega \nabla u \nabla v \; dx \; .$$

1.2 The controllability of evolution equations

1.2.1 Some notions of linear controllability

Consider the following system with infinite dimensional

$$\begin{cases} x'(t) = Ax(t) + Bv(t), t \in [0,T] \\ x(0) = x_0 \in D(A) \end{cases} \tag{1.2}$$

where :
$(A, D(A))$ is the infinitesimal generator of C_0 semi-group $\{S(t)\}_{t\geqslant 0}$ in a Hilbert space H.
V is an Hilbert space and B is an operator in $\mathcal{L}(V, X)$.
The function $x(t) \in X$ and x_0 is the initial data , we assume that $v \in L^2(0, T, V)$ be the control.
The system (1.2) has the unique solution $x(t) \in X$ characterized by :

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bv(s)ds, \forall t \in [0,T] \; .$$

The principle of control is based on asking the following question is there exist a function v that allows to pass (in a sense to define precisely) from the state x_0 to

the state x_d in a specific time $T > 0$?
Our goal is controlling the system(1.2) to get x_d , now we will introduce various types of controllability :

1.2.2 Diffrent consepts of controllability

Exact controllability

Definition 1.7 [6] *the system (1.2) is called exactly controllable in X on $[0, T]$ if there existe $v \in L^2(0, T, V)$ such that the solution of (1. 2) satisfies $x(T) = x_d$.*

Null controllability

Definition 1.8 [4] *the system (1.2) is called null controllable in X on $[0, T]$ if there existe $v \in L^2(0, T, V)$ such that the solution of (1. 2) satisfies $x(T) = 0$.*

Approximate controllability

Definition 1.9 [10] *the system (1.2) is called approximate (weakly) controllable in X on $[0, T]$ if there existe $v \in L^2(0, T, V)$ such that the solution of (1. 2) satisfies*

$$||x(T) - x_d||_X < \varepsilon \ .$$

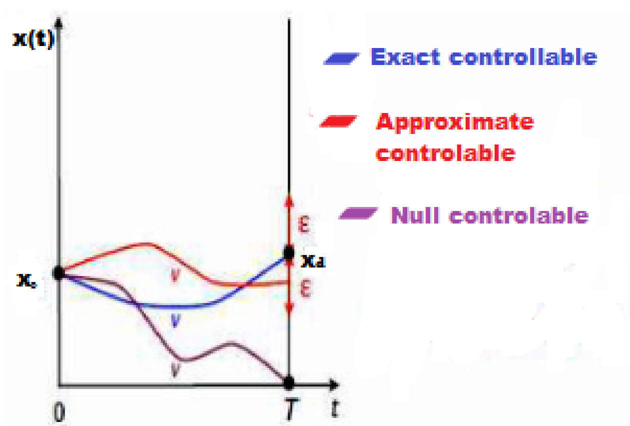

FIGURE 1.1 – Diffrent Concepets of controllability

The operator of controllability

For system (1.2) , let's introduce the operator defined by

$$\begin{cases} L_T : L^2(0, T, V) \longrightarrow X \\ v \longmapsto \int_0^T e^{A(T-s)} Bv(s)ds \end{cases}$$

Proposition 1.1 [**13**]

1. *The system (1.2) is exactly controllable , iff L_T is surjective then*

$$ImL_T = X \ .$$

2. *The system (1.2) is approximately controllable , iff*

$$\overline{ImL_T} = X \ .$$

Proof. See [**3**] ■

1.2.3 The wave equation with potential coefficient

Let $\Omega \subset \mathbb{R}^n$ be an open domain,we set $Q = \Omega \times [0, T]$ and $\Sigma = \Gamma \times [0, T]$. Consider the following problem search a function $y(x, t)$ satisfying

$$\frac{\partial^2 y}{\partial t^2} - \Delta y + py = f \ \ in \ \ Q \tag{1.3}$$

$$y(x, 0) = y'(x, 0) = 0 \ \ in \ \ \Omega \tag{1.4}$$

$$y(x, t) = 0 \ \ on \ \ \Sigma \tag{1.5}$$

with , $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, for all $i = \{0, ..., n\}$, $(n \in \mathbb{N})$.

1.2.4 Weight function

To create Carleman estimates,we use a special function called the weight function , we summarize the conditions under which this function can be chosen.
for $x_0 \notin \Omega, \lambda > 0$ and $\beta \in (0,1)$, and for any $(x,t) \in \Omega \times [0,T]$:

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0 \text{ and } \varphi(x,t) = e^{\lambda\psi(x,t)} .$$

Where $C_0 > 0$ is chosen so that $\psi \geqslant 1$ on $\Omega \times [0,T]$.

1.3 What is carleman estimates ?

Careleman estimates is a mathematical concept named after swedish mathematicien Torsten careleman in the early 20th century which he presented in 1939[**9**] , Carleman's estimates were from fundamental tools in the field of functional analysis and partial differential equations.
Careleman estimates are particularly used in the study of uniqueness and stability of solution,over time careleman estimate have found application in various areas of mathematics, including inverse problem,control theory and mathematical physics.
Careleman estimates are a tool used in the study of controllability for hyperbolic partial differential equations,They provide a quantitive measure of the controlability of a system by estimating the minimal amount of control necessery to drive the system from one state to another in specifique time.
Carleman's estimates can be presented as an inequality of the form :

$$||e^{s\varphi}u||_{L^2(\Omega)} \leqslant C \; ||e^{s\varphi}Pu||_{L^2(\Omega)}$$

Where φ is the weight function , s is a large positive parameter and u is any smooth compactly supported function in Ω and $C > 0$.

Chapter 2

Construction of Carleman estimates for some hyperbolic systems

In this chapter,we will present some methodes for creating Carleman estimates of hyperbolic (PDE), These estimates were introduced in 1922 by the mathematician Torsten Carlman and named them Carleman inequality.

2.1 Carleman estimates for the wave equation with potential coefficient

This section present a Carleman inequality for the wave equations given by (1.3)-(1.5) :
we define, for $m \geqslant 0$ the set :

$$L^{\infty}_{\leqslant m}(\Omega) = \{p \in L^{\infty}(\Omega)\ s.t\ ||p||_{L^{\infty}(\Omega)} \leqslant m\}.$$

Define the typical wave operator L as follows :

$$L = \partial_t^2 - \Delta\ .$$

Let $z \in \mathrm{L}^2(0, T, H_0^1(\Omega))$ be a function such that $Lz \in \mathrm{L}^2(0, T, L^2(\Omega))$ and satisfying $z(x, T) = z'(x, T) = 0$, $\forall x \in \Omega$.

Now let us define ,
for $x_0 \notin \Omega, \lambda > 0$ and $\beta \in (0,1)$, and for any $(x,t) \in \Omega \times [0,T]$:

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0 \text{ and } \varphi(x,t) = e^{\lambda\psi(x,t)} .$$

where it is determined that $C_0 > 0$ such that $\psi \geqslant 1$ on $\Omega \times [0,T]$. Also, we set for $s > 0$

$$w(x,t) = e^{s\varphi(x,t)} z(x,t) .$$

For $p \in L^\infty_{\leqslant m}(\Omega)$, we consider the operator L_p as :

$$L_p = \partial_t^2 - \Delta + p .$$

It's clear that $L_p z \in L^2(0,T,L^2(\Omega))$ if $Lz \in L^2(0,T,L^2(\Omega))$
First, let's introduce the Carleman estimate, which we shall formalize and calculate as follows :

$$Pw = e^{s\varphi} L(e^{-s\varphi} w) .$$

We set

$$\begin{aligned} Pw &= e^{s\varphi}(\partial_t^2 - \Delta)(e^{-s\varphi} w) \\ &= e^{s\varphi}[\partial_t^2(e^{-s\varphi} w) - \Delta(e^{-s\varphi} w)] . \end{aligned}$$

we obtain :

$$\begin{aligned} \partial_t^2(e^{-s\varphi(x,t)} w(x,t)) &= \partial_t(\partial_t(e^{-s\varphi(x,t)} w(x,t))) \\ &= \partial_t(e^{-s\varphi(x,t)} \partial_t w(x,t) + w(x,t)\partial_t e^{-s\varphi(x,t)}) \\ &= \partial_t(e^{-s\varphi(x,t)} \partial_t w(x,t) - s\partial_t\varphi(x,t) e^{-s\varphi(x,t)} w(x,t)) \qquad (2.1) \\ &= \partial_t^2 w(x,t) e^{-s\varphi(x,t)} + s^2(\partial_t\varphi(x,t))^2 e^{-s\varphi(x,t)} w(x,t) \\ &- 2s(\partial_t\varphi(x,t))(\partial_t w(x,t)) e^{-s\varphi(x,t)} . \end{aligned}$$

besides that, for all i=1 to n :

$$
\begin{aligned}
\frac{\partial(e^{-s\varphi(x,t)}w(x,t))}{\partial x_i} &= -s\frac{\partial\varphi(x,t)}{\partial x_i}e^{-s\varphi(x,t)}w(x,t) + e^{-s\varphi(x,t)}\frac{\partial w(x,t)}{\partial x_i} . \\
\frac{\partial^2(e^{-s\varphi(x,t)}w(x,t))}{\partial x_i^2} &= -s\left[\frac{\partial}{\partial x_i}\left(\frac{\partial\varphi(x,t)}{\partial x_i}(e^{-s\varphi(x,t)}w(x,t))\right)\right] + \frac{\partial}{\partial x_i}\left(e^{-s\varphi(x,t)}\frac{\partial w(x,t)}{\partial x_i}\right) \\
&= -s\left[\frac{\partial^2\varphi(x,t)}{\partial x_i^2}(e^{-s\varphi(x,t)}w(x,t))) + \frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial}{\partial x_i}(e^{-s\varphi(x,t)}w(x,t))\right] \\
&+ \frac{\partial w(x,t)}{\partial x_i}\frac{\partial}{\partial x_i}(e^{-s\varphi(x,t)}) + \frac{\partial^2 w(x,t)}{\partial x_i^2}e^{-s\varphi(x,t)} \\
&= -s\left[\frac{\partial^2\varphi(x,t)}{\partial x_i^2}(e^{-s\varphi(x,t)}w(x,t)))\right] + s^2\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial\varphi(x,t)}{\partial x_i}e^{-s\varphi(x,t)} \\
&- s\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial w(x,t)}{\partial x_i}e^{-s\varphi(x,t)} - s\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial w(x,t)}{\partial x_i}e^{-s\varphi(x,t)} \\
&= -se^{-s\varphi(x,t)}w(x,t)\frac{\partial^2\varphi(x,t)}{\partial x_i^2} + s^2(\frac{\partial\varphi(x,t)}{\partial x_i})^2e^{-s\varphi(x,t)}w(x,t) \\
&- 2se^{-s\varphi(x,t)}\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial w(x,t)}{\partial x_i} + \frac{\partial^2 w(x,t)}{\partial x_i^2}e^{-s\varphi(x,t)} .
\end{aligned}
\tag{2.2}
$$

On the other hand , we have , $\varphi(x,t) = e^{\lambda\psi(x,t)}$.

Then , for all i :

$$
\frac{\partial\varphi(x,t)}{\partial x_i} = \lambda\frac{\partial\psi(x,t)}{\partial x_i}e^{\lambda\psi(x,t)} . \tag{2.3}
$$

and

$$
\frac{\partial^2\varphi(x,t)}{\partial x_i} = \lambda^2\left(\frac{\partial\psi(x,t)}{\partial x_i}\right)^2 e^{\lambda\psi(x,t)} + \lambda\left(\frac{\partial^2\psi(x,t)}{\partial x_i^2}\right)e^{\lambda\psi(x,t)} . \tag{2.4}
$$

When we replace both (2.3) and (2.4) in (2.2), we have this result

$$
\begin{aligned}
\frac{\partial^2(e^{-s\varphi(x,t)}w(x,t))}{\partial x_i^2} &= -s\left(\frac{\partial^2\varphi(x,t)}{\partial x_i^2}\right)e^{-s\varphi(x,t)}w(x,t)+s^2\left(\frac{\partial\varphi(x,t)}{\partial x_i}\right)^2 e^{-s\varphi(x,t)}w(x,t) \\
&-2se^{-s\varphi(x,t)}\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial w(x,t)}{\partial x_i} + \frac{\partial^2 w(x,t)}{\partial x_i^2}e^{-s\varphi(x,t)} .
\end{aligned}
\tag{2.5}
$$

Using the previous result (2.5) we conclude that

$$\Delta(e^{-s\varphi(x,t)}w(x,t)) = -se^{-s\varphi(x,t)}w(x,t)\Delta\varphi + s^2(\nabla\varphi)^2e^{-s\varphi(x,t)}w(x,t)$$
$$-2se^{-s\varphi(x,t)}\nabla w\nabla\varphi + \Delta w\ . \tag{2.6}$$

Then, using (2.1) and (2.6), we have this result

$$\begin{aligned}Pw &= e^{s\varphi(x,t)}[\partial_t^2 w(x,t)e^{-s\varphi(x,t)} - s^2(\partial_t\varphi(x,t))^2e^{-s\varphi(x,t)}w(x,t)\\ &- 2s(\partial_t\varphi(x,t))(\partial_t w(x,t))e^{-s\varphi(x,t)} - se^{-s\varphi(x,t)}w(x,t)\Delta\varphi\\ &+ s^2(\nabla\varphi)^2e^{-s\varphi(x,t)}w(x,t) - 2se^{-s\varphi(x,t)}\nabla w\nabla\varphi + \Delta w]\\ &= \partial_t^2 w - 2s\lambda\varphi(\partial_t w\partial_t\psi - \nabla w\nabla\psi) + s^2\lambda^2 w\varphi(|\partial_t\psi|^2 - |\nabla\psi|^2) - \Delta w\\ &- sw\varphi(\lambda\partial_t^2\psi + \lambda^2(\nabla\psi)^2)\end{aligned}$$

Also , we can write

$$Pw = \partial_t^2 w - 2s\lambda\varphi(\partial_t w\partial_t\psi - \nabla w.\nabla\psi) + s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2 - |\nabla\psi|^2) - \Delta w$$
$$-s\lambda\varphi w(\partial_t^2\psi - \Delta\psi) - s\lambda^2\varphi w(|\partial_t\psi|^2 - |\nabla\psi|^2)$$

If we consider the operators :

$$P_1 w = \partial_t^2 w - \Delta w + s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2 - |\nabla\psi|^2)$$

$$P_2 w = (\alpha-1)s\lambda\varphi w(\partial_t^2\psi-\Delta\psi)-s\lambda^2\varphi w(|\partial_t\psi|^2-|\nabla\psi|^2)-2s\lambda\varphi(\partial_t w\partial_t\psi-\nabla w.\nabla\psi) \tag{2.7}$$

$$Rw = -\alpha s\lambda\varphi w(\partial_t^2\psi - \Delta\psi)$$

Choosing α later so that $\frac{2\beta}{\beta+n} < \alpha < \frac{2}{\beta+n}$ (see[2]) , we obtain

$$P_1 w + P_2 w = Pw - Rw.$$

Now, we could state a Carleman inequality for the wave equations with potential coefficient under consideration

Theorem 2.1 [**2**] *Let us suppose that there exists $x_0 \notin \Omega$ such that*

$$\Gamma_0 \supset \{x \in \partial\Omega; (x - x_0).\nu(x) \geqslant 0\} \tag{2.8}$$

where $\nu(x)$ denote the external unit normal vector at x.
Then for every m >0, there exists $\lambda_0, s_0 > 0$ and a constant
$C = C(\Omega, T, m, \lambda_0, s_0, \beta, x_0)$

such that for all p $\in L^\infty_{\leqslant m}(\Omega)$,and for all $\lambda > \lambda_0, s > s_0$:

$$s\lambda \int_Q e^{2s\varphi}(|\partial_t z|^2 + |\nabla z|^2)dxdt + s^3\lambda^3 \int_Q e^{2s\varphi}|z|^2 dxdt + \int_Q |P_1(e^{s\varphi}z)|^2 dxdt$$

$$+ \int_Q |P_2(e^{s\varphi}z)|^2 dxdt \leq C \int_Q e^{2s\varphi}|L_p z|^2 dxdt + Cs\lambda \int_{\Sigma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \tag{2.9}$$

for all $z \in \mathrm{H}^1(0, T, H_0^1(\Omega))$ satisfying $Lz \in \mathrm{L}^2(Q)$ and $z(x, T) = z'(x, T) = 0$, $\forall x \in \Omega$.

2.2 Construction of Caleman estimates for the considered wave equations

In this section, we are interested to creating a Carleman estimate for the considered wave equation with Lz in the right hand side instead of $\mathrm{L}_p z$,Then we will see at the end that the result hold as well for L_p since $p \in L^\infty_{\leqslant m}(\Omega)$:
As we started writing, for $w(x,t) = e^{s\varphi(x,t)}z(x,t)$ we have $Pw = e^{s\varphi}L(e^{-s\varphi}w(x,t))$ and

$$\int_0^T \int_\Omega (|P_1 w|^2 + |P_2 w|^2)dxdt + 2\int_0^T \int_\Omega P_1 w P_2 w \; dxdt$$
$$= \int_0^T \int_\Omega |Pw - Rw|^2 dxdt. \tag{2.10}$$

Now we seek to find a lower estimate for

$$\int_0^T \int_\Omega P_1 w P_2 w \, dxdt \, .$$

The First step : some explicite calculation

First , we need to calculate both of :

$$\partial_x^2 \psi = 2, \quad \partial_x \partial_t \psi = 0, \quad \partial_t^2 \psi = -2\beta;$$
$$\partial_x \varphi = \lambda(\partial_x \psi)\varphi, \quad \partial_t \varphi = \lambda(\partial_t \psi)\varphi;$$
$$\partial_x^2 \varphi = (\lambda^2 (\partial_x \psi)^2 + \lambda \partial_x^2 \psi)\varphi;$$
$$\partial_t^2 \varphi = (\lambda^2 (\partial_t \psi)^2 + \lambda \partial_t^2 \psi)\varphi;$$

We denote

$$(P_1 w, P_2 w)_{L^2(Q)} = \sum_{i,j=1}^{n} I_{i,j} \, .$$

where $I_{i,j}$ is the integral of the ith-term in $P_1 w$ multiplied by the jth-term in $P_2 w$.

Using integration by parts[**Theorem 1.4**] in time

$$\begin{aligned} I_{11} &= \int_0^T \int_\Omega \partial_t^2 w((\alpha - 1)s\lambda\varphi w(\partial_t^2 \psi - \Delta\psi))dxdt \, . \\ &= (1-\alpha)s\lambda \int_0^T \int_\Omega \partial_t w[\varphi(\partial_t w)(\partial_t^2 \psi - \Delta\psi) \\ &\quad -(\partial_t \varphi)w(\partial_t^2 \psi - \Delta\psi) - \varphi w \partial_t(\partial_t^2 \psi - \Delta\psi)]dxdt \, . \end{aligned}$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-(1-\alpha)s\lambda \int_0^T \int_\Omega w(\partial_t w)(\partial_t\varphi)(\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega 2w(\partial_t w)(\partial_t\varphi)(\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega \partial_t(w^2)(\partial_t\varphi)(\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega |w|^2(\partial_t^2\varphi)(\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega |w|^2\varphi[\lambda^2|\partial_t\psi|^2 + \lambda\partial_t^2\psi](\partial_t^2\psi - \Delta\psi)dxdt \ .$$

$$= (1-\alpha)s\lambda \int_0^T \int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda^2 \int_0^T \int_\Omega \varphi|w|^2\partial_t^2\psi(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda^3 \int_0^T \int_\Omega \varphi|w|^2|\partial_t\psi|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

In a similar way, we get

$$I_{12} = \int_0^T \int_\Omega \partial_t^2 w(-s\lambda^2\varphi w(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt \ .$$

$$= -s\lambda^2 \int_0^T \int_\Omega \partial_t w[(\partial_t\varphi)w(|\partial_t\psi|^2 - |\nabla\psi)|^2)$$

$$+\varphi(\partial_t w)(|\partial_t\psi|^2 - |\nabla\psi)|^2) + \varphi w\partial_t(|\partial_t\psi|^2 - |\nabla\psi)|^2)]dxdt\ .$$

$$= -s\lambda^2\int_0^T\int_\Omega \varphi|\partial_t w|^2(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-s\lambda^2\int_0^T\int_\Omega(\partial_t w)w(\partial_t\varphi)(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-s\lambda^2\int_0^T\int_\Omega(\partial_t w)w\varphi\partial_t(|\partial_t\psi|^2)dxdt\ .$$

$$= -s\lambda^2\int_0^T\int_\Omega \varphi|\partial_t w|^2(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2(\partial_t^2\varphi)(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt\ .$$

$$= -s\lambda^2\int_0^T\int_\Omega \varphi|\partial_t w|^2(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega w(\partial_t w)\varphi(2(\partial_t\psi)(\partial_t^2\psi)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2[\lambda^2|\partial_t\psi|^2 + \lambda\partial_t^2\psi](|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt\ .$$

$$= s\lambda^2\int_0^T\int_\Omega \varphi|\partial_t w|^2(|\partial_t\psi|^2 - |\nabla\psi)|^2) - s\lambda^2\int_0^T\int_\Omega \varphi|w|^2|\partial_t^2\psi|^2dxdt$$

$$+\frac{s\lambda^3}{2}\int_0^T\int_\Omega \varphi|w|^2|\nabla\psi|^2\partial_t^2\psi dxdt$$

$$-(2+\frac{1}{2})s\lambda^3\int_0^T\int_\Omega \varphi|w|^2|\partial_t\psi|^2(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^4}{2}\int_0^T\int_\Omega \varphi|w|^2|\partial_t\psi|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

Moreover, by employing integrations by part in the space variable, we obtain

$$I_{13} = \int_0^T\int_\Omega \partial_t^2 w(-2s\lambda\varphi(\partial_t w\partial_t\psi - \nabla w.\nabla\psi))dxdt\ .$$

$$= \int_0^T\int_\Omega(\partial_t w)\partial_t(-2s\lambda\varphi(\partial_t w\partial_t\psi - \nabla w.\nabla\psi))dxdt\ .$$

$$= -2s\lambda \int_0^T \int_\Omega (\partial_t w)\lambda(\partial_t \psi)\varphi(\partial_t w \partial_t \psi - \nabla w.\nabla \psi))dxdt$$

$$-2s\lambda \int_0^T \int_\Omega (\partial_t w)\varphi \partial_t(\partial_t w \partial_t \psi)dxdt + 2s\lambda \int_0^T \int_\Omega (\partial_t w)\varphi \partial_t(\nabla w.\nabla \psi)dxdt \ .$$

$$= -2s\lambda^2 \int_0^T \int_\Omega (\partial_t w)(\partial_t \psi)\varphi(\partial_t w \partial_t \psi - \nabla w.\nabla \psi))dxdt$$

$$-2s\lambda \int_0^T \int_\Omega (\partial_t^2 w)(\partial_t w)\partial_t \psi dxdt - 2s\lambda \int_0^T \int_\Omega (\partial_t w)\varphi(\partial_t w)(\partial_t^2 \psi)dxdt$$

$$+2s\lambda \int_0^T \int_\Omega (\partial_t w)\varphi \partial_t(\nabla w).\nabla \psi dxdt + 2s\lambda \int_0^T \int_\Omega (\partial_t w)\varphi \nabla w.\partial_t(\nabla \psi)dxdt \ .$$

$$= s\lambda^2 \int_0^T \int_\Omega |\partial_t w|^2 |\partial_t \psi|^2 dxdt - 2s\lambda^2 \int_0^T \int_\Omega (\partial_t w)(\partial_t \psi)\nabla w.\nabla \psi dxdt$$

$$+s\lambda \int_0^T \int_\Omega \varphi |\partial_t w|^2 \partial_t^2 \psi dxdt + s\lambda \int_0^T \int_\Omega |\partial_t w|^2 (\nabla \varphi)(\nabla \psi)dxdt$$

$$+s\lambda \int_0^T \int_\Omega |\partial_t w|^2 \Delta \psi dxdt \ .$$

$$= s\lambda \int_0^T \int_\Omega \varphi |\partial_t w|^2 \partial_t^2 \psi dxdt + s\lambda^2 \int_0^T \int_\Omega \varphi |\partial_t w|^2 |\partial_t \psi|^2 dxdt$$

$$+s\lambda \int_0^T \int_\Omega \varphi |\partial_t w|^2 \Delta \psi dxdt + s\lambda^2 \int_0^T \int_\Omega \varphi |\partial_t w|^2 |\nabla \psi|^2 dxdt$$

$$-2s\lambda^2 \int_0^T \int_\Omega \varphi \partial_t w \partial_t \psi \nabla w.\nabla \psi dxdt \ .$$

We use the Green Formula [**Theorem 1.5**] and integration by part

$$I_{21} = \int_0^T \int_\Omega -\Delta w((\alpha - 1)s\lambda \varphi w(\partial_t^2 \psi - \Delta \psi))dxdt \ .$$

$$= \int_0^T \int_\Omega -\nabla w \nabla((\alpha - 1)s\lambda \varphi w(\partial_t^2 \psi - \Delta \psi))dxdt \ .$$

$$= -(1-\alpha)s\lambda \int_0^T \int_\Omega -\nabla w[\nabla \varphi(w(\partial_t^2 \psi - \Delta \psi))$$

$$+\varphi(\nabla w)(\partial_t^2 \psi - \Delta \psi) + \varphi w \nabla(\partial_t^2 \psi - \Delta \psi)]dxdt \ .$$

$$= -(1-\alpha)s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-(1-\alpha)s\lambda \int_0^T \int_\Omega \nabla w \nabla\varphi w(\partial_t^2\psi - \Delta\psi)dxdt \;.$$

$$= -(1-\alpha)s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega |w|^2 \nabla(\nabla\varphi)(\partial_t^2\psi - \Delta\psi)dxdt \;.$$

$$= -(1-\alpha)s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda \int_0^T \int_\Omega |w|^2[\lambda^2|\nabla\psi|^2 + \lambda\Delta\psi](\partial_t^2\psi - \Delta\psi)dxdt \;.$$

$$= -(1-\alpha)s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi(\partial_t^2\psi - \Delta\psi)dxdt \;.$$

$$-\frac{(1-\alpha)}{2}s\lambda^3 \int_0^T \int_\Omega |w|^2|\nabla\psi|^2(\partial_t^2\psi - \Delta\psi)dxdt$$

$$-\frac{(1-\alpha)}{2}s\lambda^2 \int_0^T \int_\Omega |w|^2\Delta\psi(\partial_t^2\psi - \Delta\psi)dxdt \;.$$

In the same manner, we have

$$I_{22} = \int_0^T \int_\Omega -\Delta w(-s\lambda^2\varphi w(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt \;.$$

$$= \int_0^T \int_\Omega -(\nabla w)\nabla(-s\lambda^2\varphi w(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt \;.$$

$$= -s\lambda^2 \int_0^T \int_\Omega (\nabla w)(\nabla\varphi)w(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-s\lambda^2 \int_0^T \int_\Omega (\nabla w)(\nabla w)\varphi(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-s\lambda^2 \int_0^T \int_\Omega (\nabla w)(w)\varphi\nabla(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt \;.$$

$$= -s\lambda^2 \int_0^T \int_\Omega |\nabla w|^2\varphi(|\partial_t\psi|^2 - |\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega \nabla(w^2)\nabla\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega \nabla(w^2)\varphi(\nabla|\partial_t\psi|^2)dxdt+\frac{s\lambda^2}{2}\int_0^T\int_\Omega \nabla(w^2)\varphi(\nabla|\nabla\psi|^2)dxdt\ .$$

$$=-s\lambda^2\int_0^T\int_\Omega |\nabla w|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt-\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2\Delta\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2\nabla\varphi(\nabla|\partial_t\psi|^2)dxdt+\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2\nabla\varphi(\nabla|\nabla\psi|^2)dxdt\ .$$

$$=-s\lambda^2\int_0^T\int_\Omega |\nabla w|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^2}{2}\int_0^T\int_\Omega |w|^2\varphi[\lambda^2|\nabla\psi|^2+\lambda\Delta\psi](|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\nabla\psi\varphi(|\partial_t\psi|^2)dxdt-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\nabla\psi\varphi\nabla(|\partial_t\psi|^2)dxdt$$

$$+\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\nabla\psi\varphi\nabla(|\nabla\psi|^2)dxdt-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\nabla\psi\varphi\nabla(|\nabla\psi|^2)dxdt\ .$$

$$=-s\lambda^2\int_0^T\int_\Omega |\nabla w|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt+\frac{s\lambda^4}{2}|\nabla\psi|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$+\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\nabla\psi)|^2)dxdt$$

$$-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\partial_t\psi)|^2)dxdt-\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\nabla\psi)|^2)dxdt\ .$$

$$=-s\lambda^2\int_0^T\int_\Omega |\nabla w|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt+\frac{s\lambda^4}{2}|\nabla\psi|^2\varphi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt$$

$$+\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\partial_t\psi|^2-|\nabla\psi)|^2)dxdt+\frac{s\lambda^3}{2}\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\partial_t\psi|^2)dxdt$$

$$-s\lambda^3\int_0^T\int_\Omega |w|^2\varphi\Delta\psi(|\nabla\psi|^2dxdt\ .$$

By using the notion that $w|_{\partial\Omega\times(0,T)}=0$, we obtain, on $\partial\Omega\times(0,T)$,$\nabla w=(\partial_\nu w)\nu$, which results in $|w|^2=|\partial_\nu w|^2$. Consequently, we have the ability to obtain

$$I_{23} = \int_0^T \int_\Omega -\Delta w(-2s\lambda\varphi(\partial_t w \partial_t \psi - \nabla w.\nabla\psi))dxdt \; .$$

$$= \int_0^T \int_\Omega -\nabla w \nabla(\partial_t w \partial_t \psi - \nabla w.\nabla\psi))dxdt \; .$$

$$= 2s\lambda \int_0^T \int_\Omega (\nabla w)(\nabla\varphi)(\partial_t w \partial_t \psi - \nabla w.\nabla\psi))dxdt$$

$$+2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi\nabla(\partial_t w \partial_t \psi)dxdt - 2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi\nabla(\nabla w.\nabla\psi)dxdt \; .$$

$$= 2s\lambda^2 \int_0^T \int_\Omega (\nabla w)\varphi\nabla\psi(\partial_t w \partial_t \psi - \nabla w.\nabla\psi))dxdt$$

$$-2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi\nabla(\nabla w)\nabla\psi dxdt + 2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi\nabla(\partial_t w)\partial_t \psi dxdt$$

$$-2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi(\partial_t w)\nabla(\partial_t \psi)dxdt - 2s\lambda \int_0^T \int_\Omega (\nabla w)\varphi(\nabla w)\nabla(\nabla\psi)dxdt \; .$$

$$= 2s\lambda^2 \int_0^T \int_\Omega (\nabla w)\varphi\nabla\psi(\partial_t w \partial_t \psi)dxdt$$

$$-2s\lambda^2 \int_0^T \int_\Omega (\nabla w)\varphi\nabla\psi(\nabla w.\nabla\psi)dxdt + s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi\Delta\psi dxdt$$

$$-2s\lambda^2 \int_0^T \int_\Omega |\nabla w|^2 \varphi\nabla\psi dxdt + 2s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi\partial_t^2\psi dxdt$$

$$-2s\lambda \int_0^T \int_\Omega |\nabla w|^2 \varphi\Delta\psi dxdt - 2s\lambda^2 \int_0^T \int_\Omega |\nabla w|^2 \varphi|\nabla\psi|^2 dxdt \; .$$

$$= s\lambda \int_0^T \int_\Omega \varphi|\nabla w|^2(\partial_t^2\psi - \Delta\psi))dxdt$$

$$+2s\lambda^2 \int_0^T \int_\Omega |\nabla\psi.\nabla w|^2 dxdt - 2s\lambda^2 \int_0^T \int_\Omega \varphi\partial_t w \partial_t \psi \nabla w.\nabla\psi dxdt$$

$$+s\lambda^2 \int_0^T \int_\Omega \varphi|\nabla w|^2(|\partial_t\psi|^2 - |\nabla\psi|^2)dxdt - s\lambda \int_0^T \int_{\Gamma_0} \varphi|\partial_\nu w|^2 \nabla\psi.\nu(x)d\Gamma dt$$

$$+2s\lambda \int_0^T \int_\Omega \varphi D^2\psi|\nabla w|^2 dxdt \; .$$

where the matrix $D^2\psi$ is symetric

As for the following, we can easily write

$$I_{31}=\int_0^T\int_\Omega s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2-|\nabla\psi)|^2((\alpha-1)s\lambda\varphi w(\partial_t^2\psi-\Delta\psi))dxdt$$
$$=(\alpha-1)s^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2(\partial_t^2\psi-\Delta\psi)(|\partial_t\psi|^2-|\nabla\psi)|^2dxdt\ .$$

as well

$$I_{32}=\int_0^T\int_\Omega s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2-|\nabla\psi|^2)(-s\lambda^2\varphi w(|\partial_t\psi|^2-|\nabla\psi|^2)dxdt$$
$$=-s^3\lambda^4\int_0^T\int_\Omega\varphi^3|w|^2(|\partial_t\psi|^2-|\nabla\psi|^2)dxdt\ .$$

In conclusion,using integrations by parts we have

$$I_{33}=\int_0^T\int_\Omega s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2-|\nabla\psi|^2)(-2s\lambda\varphi(\partial_t w\partial_t\psi-\nabla w.\nabla\psi))dxdt\ .$$
$$=-2s^3\lambda^3\int_0^T\int_\Omega\varphi^3 w|\partial_t\psi|^2(\partial_t w\partial_t\psi)dxdt+2s^3\lambda^3\int_0^T\int_\Omega\varphi^3 w|\partial_t\psi|^2(\nabla w.\nabla\psi)dxdt$$
$$+2s^3\lambda^3\int_0^T\int_\Omega\varphi^3 w|\nabla\psi|^2(\partial_t w\partial_t\psi)dxdt-2s^3\lambda^3\int_0^T\int_\Omega\varphi^3 w|\nabla\psi|^2(\nabla w.\nabla\psi)dxdt\ .$$
$$=-s^3\lambda^3\int_0^T\int_\Omega\varphi^3\partial_t(w^2)\partial_t\psi|\partial_t\psi|^2dxdt+s^3\lambda^3\int_0^T\int_\Omega\varphi^3\nabla(w^2)(\nabla\psi)|\partial_t\psi|^2dxdt$$
$$+s^3\lambda^3\int_0^T\int_\Omega\varphi^3\partial_t(w^2)\partial_t\psi|\nabla\psi|^2dxdt-s^3\lambda^3\int_0^T\int_\Omega\varphi^3\nabla(w^2)(\nabla\psi)|\nabla\psi|^2dxdt\ .$$
$$=-s^3\lambda^3\int_0^T\int_\Omega|w|^2\partial_t(\varphi^3\partial_t\psi|\partial_t\psi|^2)dxdt+s^3\lambda^3\int_0^T\int_\Omega|w|^2\nabla(\varphi^3(\nabla\psi)|\partial_t\psi|^2)dxdt$$
$$+s^3\lambda^3\int_0^T\int_\Omega|w|^2\partial_t(\varphi^3\partial_t\psi|\nabla\psi|^2)dxdt-s^3\lambda^3\int_0^T\int_\Omega|w|^2\nabla(\varphi^3(\nabla\psi)|\nabla\psi|^2)dxdt\ .$$
$$=s^3\lambda^3\int_0^T\int_\Omega|w|^2 3\varphi^2(\partial_t\varphi)\partial_t\psi|\partial_t\psi|^2dxdt+s^3\lambda^3\int_0^T\int_\Omega|w|^2\partial_t^2\psi|\partial_t\psi|^2dxdt$$
$$+s^3\lambda^3\int_0^T\int_\Omega|w|^2\varphi^3(\partial_t\psi)2(\partial_t\psi)\partial_t^2\psi dxdt-s^3\lambda^3\int_0^T\int_\Omega|w|^2 3\varphi^2(\nabla\varphi)\nabla\psi|\partial_t\psi|^2dxdt$$
$$-s^3\lambda^3\int_0^T\int_\Omega|w|^2\varphi^3\nabla(\nabla\psi)|\partial_t\psi|^2dxdt-s^3\lambda^3\int_0^T\int_\Omega|w|^2\varphi^3(\nabla\psi)2\nabla(\partial_t\psi)\partial_t\psi dxdt$$

$$-s^3\lambda^3\int_0^T\int_\Omega |w|^2 3\varphi^2(\partial_t\varphi)\partial_t\psi|\nabla\psi|^2dxdt + s^3\lambda^3\int_0^T\int_\Omega |w|^2\partial_t^2\psi|\nabla\psi|^2dxdt$$

$$-s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3(\nabla\psi)2(\partial_t\psi)\partial_t(\nabla\psi)dxdt + s^3\lambda^3\int_0^T\int_\Omega |w|^2 3\varphi^2(\nabla\varphi)\nabla\psi|\nabla\psi|^2dxdt$$

$$+s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3\nabla(\nabla\psi)|\nabla\psi|^2dxdt + s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3(\nabla\psi)2\nabla\psi(\Delta\psi)dxdt\ .$$

$$= 3s^3\lambda^4\int_0^T\int_\Omega |w|^2\varphi^3|\partial_t\psi|^2|\partial_t\psi|^2dxdt + s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3\partial_t^2\psi|\partial_t\psi|^2dxdt$$

$$+2s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3\partial_t^2\psi|\partial_t\psi|^2dxdt - 3s^3\lambda^4\int_0^T\int_\Omega |w|^2\varphi^3|\nabla\psi|^2|\partial_t\psi|^2dxdt$$

$$-s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3(\Delta\psi)|\partial_t\psi|^2dxdt - 3s^3\lambda^4\int_0^T\int_\Omega |w|^2\varphi^3|\partial_t\psi|^2|\nabla\psi|^2dxdt$$

$$-s^3\lambda^3\int_0^T\int_\Omega |w|^2\partial_t^2\psi|\nabla\psi|^2dxdt + 3s^3\lambda^4\int_0^T\int_\Omega |w|^2\varphi^3|\nabla\psi|^2|\nabla\psi|^2dxdt$$

$$+s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3\Delta\psi|\nabla\psi|^2dxdt + 2s^3\lambda^3\int_0^T\int_\Omega |w|^2\varphi^3\Delta\psi|\nabla\psi|^2dxdt\ .$$

$$= s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2(\partial_t^2\psi-\Delta\psi)(|\partial_t\psi|^2-|\nabla\psi|^2)dxdt$$

$$+s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2(2\partial_t^2\psi|\partial_t\psi|^2)+\nabla\psi\nabla(|\nabla\psi|^2))dxdt$$

$$+3s^3\lambda^4\int_0^T\int_\Omega \varphi^3|w|^2(|\partial_t\psi|^2-|\nabla\psi|^2)^2dxdt\ .$$

Adding together all of the calculated terms, we obtain

$$\int_0^T\int_\Omega P_1wP_2wdxdt$$

$$= 2s\lambda\int_0^T\int_\Omega \varphi|\partial_tw|^2\partial_t^2\psi dxdt - \alpha s\lambda\int_0^T\int_\Omega \varphi|\partial_tw|^2(\partial_t^2\psi-\Delta\psi)dxdt$$

$$+2s\lambda^2\int_0^T\int_\Omega \varphi(|\partial_tw|^2|\partial_t\psi|^2 - 2\partial_tw\partial_t\psi\nabla w\nabla\psi + |\nabla w\nabla\psi|^2)dxdt$$

$$-s\lambda\int_0^T\int_\Omega \varphi|\partial_\nu w|^2\nabla\psi.\nu(x)d\Gamma dt + 2s\lambda\int_0^T\int_\Omega \varphi D^2\psi|\nabla w|^2dxdt \qquad (2.11)$$

$$+\alpha s\lambda\int_0^T\int_\Omega\varphi|\nabla w|^2(\partial_t^2\psi-\Delta\psi)dxdt+2s^3\lambda^4\int_0^T\int_\Omega\varphi^3|w|^2(|\partial_t\psi|^2-|\nabla\psi)|^2)$$
$$+s^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2(2\partial_t^2\psi|\partial_t\psi|^2)+\nabla\psi\nabla(|\nabla\psi)|^2))dxdt$$
$$+\alpha s^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2(\partial_t^2\psi-\Delta\psi)(|\partial_t\psi|^2-|\nabla\psi)|)dxdt$$
$$+X_1\ .$$

Where X_1 is the sum of the remaining termes. then, utilizing the regularity of ψ and that $\psi\geqslant 1$ means $\lambda\leqslant e^{\lambda\psi}=\varphi$.

$$|X_1|\leqslant Cs\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2dxdt \tag{2.12}$$

with $C>0$ denotes a generic constant that is independent of s and λ but at least dependent on T and Ω .

The Second step : Bounding each terme from below

First, as we can see that

$$2s\lambda^2\int_0^T\int_\Omega\varphi(|\partial_t w|^2|\partial_t\psi|^2-2\partial_t w\partial_t\psi\nabla w\nabla\psi+|\nabla w\nabla\psi|^2)dxdt$$
$$=2s\lambda^2\int_0^T\int_\Omega\varphi(\partial_t w\partial_t\psi-\nabla w\nabla\psi)^2dxdt\geqslant 0 \tag{2.13}$$

Furthermore, taking the terms in $s\lambda$ which now has to give the dominant terms in $|\partial_t w|^2$ and $|\nabla w|^2$ and so they must be strictly positive, One may assume that we need

$$2\partial_t^2\psi-\alpha(\partial_t^2\psi-\Delta\psi)>0\ \ and\ \ 2D^2\psi+\alpha(\partial_t^2\psi-\Delta\psi)>0 \tag{2.14}$$

In the definition (2.7) of P_2w, this will limit the value of the undefined constant $\alpha>0$. Through explicit computations, where $\beta\in(0,1)$ and that α must satisfy

$$\frac{2\beta}{\beta+n}<\alpha<\frac{2}{\beta+n} \tag{2.15}$$

Consequently, we are able to write

$$
\begin{aligned}
&2s\lambda\int_0^T\int_\Omega \varphi|\partial_t w|^2\partial_t^2\psi dxdt-\alpha s\lambda\int_0^T\int_\Omega \varphi|\partial_t w|^2(\partial_t^2\psi-\Delta\psi)dxdt\\
&+2s\lambda\int_0^T\int_\Omega \varphi D^2\psi|\nabla w|^2dxdt+\alpha s\lambda\int_0^T\int_\Omega \varphi|\nabla w|^2(\partial_t^2\psi-\Delta\psi)dxdt\\
&\qquad\geqslant Cs\lambda\int_0^T\int_\Omega \varphi|\partial_t w|^2dxdt+Cs\lambda\int_0^T\int_\Omega\varphi|\nabla w|^2dxdt \qquad (2.16)
\end{aligned}
$$

we can note that

$$
\begin{aligned}
&2s^3\lambda^4\int_0^T\int_\Omega \varphi^3|w|^2(|\partial_t\psi|^2-|\nabla\psi)|^2)^2dxdt\\
&+s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2(2\partial_t^2\psi|\partial_t\psi|^2+\nabla\psi\nabla(|\nabla\psi)|^2))dxdt\\
&+\alpha s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2(\partial_t^2\psi-\Delta\psi)(|\partial_t\psi|^2-|\nabla\psi)|)dxdt\\
&=s^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2G_\lambda(\psi)dxdt\ .
\end{aligned}
$$

such that

$$
\begin{aligned}
G_\lambda(\psi)&=2\lambda(|\partial_t\psi|^2-|\nabla\psi)|^2)^2+(2\partial_t^2\psi|\partial_t\psi|^2+\nabla\psi\nabla(|\nabla\psi)|^2))+\alpha(\partial_t^2\psi-\Delta\psi)(|\partial_t\psi|^2-|\nabla\psi)|^2)\\
&=32\lambda(\beta^2t^2-|x-x_0|^2)^2-16(\beta^3t^2-|x-x_0|^2)-8\alpha(\beta+n)(\beta^2t^2-|x-x_0|^2)\\
&=32\lambda(\beta^2t^2-|x-x_0|^2)^2-8(\alpha(\beta+n)+2\beta)(\beta^2t^2-|x-x_0|^2)+16(1-\beta)|x-x_0|^2\ .
\end{aligned}
$$

Since $x_0\notin\Omega$, we have $16(1-\beta)|x-x_0|^2\geqslant c*>0$. Therefore, Let's consider a polynome $P(X)=32\lambda X^2-8(\alpha(\beta+n)+2\beta)X+c*$ and $\lambda>0$ large enough, the minimum of P will be strictly positive. Thus,

$$
s^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2G_\lambda(\psi)dxdt\geqslant Cs^3\lambda^3\int_0^T\int_\Omega\varphi^3|w|^2dxdt \qquad (2.17)
$$

Consequently, by entering (2.13), (2.16), and (2.17) into equation (2.11), we get

$$\int_0^T\int_\Omega (P_1wP_2w)dxdt + 2s\lambda\int_0^T\int_{\Gamma_0}\varphi|\partial_\nu w|^2(x-x_0).\nu(x)d\Gamma dt - X_1$$
$$\geqslant Cs\lambda\int_0^T\int_\Omega \varphi(|\partial_t w|^2+|\nabla w)|^2)dxdt + Cs^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2dxdt\ .$$

also, we have easily

$$\int_0^T\int_\Omega |Pw-Rw|^2dxdt \leqslant 2\int_0^T\int_\Omega |Pw|^2dxdt + 2\int_0^T\int_\Omega |Rw|^2dxdt$$
$$\leqslant C\int_0^T\int_\Omega |Pw|^2dxdt + Cs^2\lambda^2\int_0^T\int_\Omega \varphi^2|w|^2dxdt\ .$$

using (2.10) and (2.12) we obtain

$$s\lambda\int_0^T\int_\Omega \varphi(|\partial_t w|^2+|\nabla w)|^2)dxdt + s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2dxdt$$
$$+\int_0^T\int_\Omega (|P_1w|^2+|P_2w|^2)dxdt$$
$$\leqslant C\int_0^T\int_\Omega |Pw|^2dxdt + Cs\lambda\int_0^T\int_{\Gamma_0}\varphi|\partial_\nu w|^2(x-x_0).\nu(x)d\Gamma dt$$
$$+Cs\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2dxdt + Cs^2\lambda^2\int_0^T\int_\Omega \varphi^2|w|^2dxdt\ .$$

We now take s_0 large enough such that the terms (taken from X_1 and $|Rw|^2$) in the final line are absorbed as soon as $s > s_0$ by the dominant term in $s^3\lambda^3|w|^2\varphi^3$. Finally, we get for some positive constant $C = C(s_0, \lambda_0, m, \Omega, \beta, x)$ by using the condition (2.8) on Γ_0 as well.

$$\begin{aligned} &s\lambda\int_0^T\int_\Omega \varphi(|\partial_t w|^2+|\nabla w)|^2)dxdt + s^3\lambda^3\int_0^T\int_\Omega \varphi^3|w|^2dxdt \\ &+\int_0^T\int_\Omega |P_1w|^2dxdt + \int_0^T\int_\Omega |P_2w|^2dxdt \\ &\leqslant C\int_0^T\int_\Omega |Pw|^2dxdt + Cs\lambda\int_0^T\int_{\Gamma_0}\varphi|\partial_\nu w|^2dxdt \end{aligned} \tag{2.18}$$

$\forall s > s_0, \forall\lambda > \lambda_0.$

The Third step : Ruturn to the variable z

Setting $w = ze^{s\varphi}$ allows for all $x \in \Omega$ and $t \in (0,T)$.

$$\partial_t w = (\partial_t z)e^{s\varphi} + s(\partial_t \varphi)ze^{s\varphi} \, .$$

$$|\partial_t w|^2 = |(\partial_t z)e^{s\varphi} + s(\partial_t \varphi)ze^{s\varphi}|^2 \, .$$

$$|\partial_t w|^2 \geqslant |\partial_t z|^2 e^{2s\varphi} + s^2|\partial_t \varphi|^2|z|^2 e^{2s\varphi} \, .$$

$$|\partial_t z|^2 e^{2s\varphi} \leqslant |\partial_t w|^2 + s^2|\partial_t \varphi|^2|z|^2 e^{2s\varphi} \, .$$

$$|\partial_t z|^2 e^{2s\varphi} \leqslant |\partial_t w|^2 + s^2|\partial_t \varphi|^2|w|^2 \, .$$

$$|\partial_t z|^2 e^{2s\varphi} \leqslant 2|\partial_t w|^2 + 2s^2|\partial_t \varphi|^2|w|^2 \, .$$

In the same manner we have

$$|\nabla z|^2 e^{2s\varphi} \leqslant 2|\nabla w|^2 + 2s^2|\nabla \varphi|^2|w|^2 \, .$$

and on $\partial\Omega$

$$|\partial_\nu z|^2 e^{2s\varphi} = |\partial_\nu w|^2 \, .$$

Considering therefore we can return to the variable z in (2.18) and determine that there exists some positive constant $C = C(s_0, \lambda_0, m, \Omega, \beta, x)$ such that for any $\forall s > s_0$ and $\forall \lambda > \lambda_0$. by construction $Pw = e^{s\varphi}Lz$.

$$\begin{aligned} s\lambda \int_0^T \int_\Omega e^{2s\varphi}(|\partial_t z|^2 + |\nabla z)|^2) dxdt + s^3\lambda^3 \int_0^T \int_\Omega |z|^2 e^{2s\varphi} dxdt \\ + \int_0^T \int_\Omega (|P_1(ze^{s\varphi})|^2) dxdt + \int_0^T \int_\Omega (|P_2(ze^{s\varphi})|^2) dxdt \\ \leqslant C \int_0^T \int_\Omega e^{2s\varphi}|Lz|^2 dxdt + Cs\lambda \int_0^T \int_{\Gamma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 dxdt \end{aligned} \tag{2.19}$$

For the operator $L = \partial_t^2 - \Delta$, it completes the demonstration of a Carleman estimate.

The fourth step : Wave operator with potential

As we put at the beginning that $L = \partial_t^2 - \Delta$ and $L_p = \partial_t^2 - \Delta + p$ with $p \in L^\infty_{\leqslant m}(\Omega)$ we can conclude $L_p z = Lz + p$, So we can say that

$$|Lz|^2 \leqslant |L_p z - pz|^2 \leqslant |L_p z|^2 + |p|^2|z|^2 .$$

$$|Lz|^2 \leqslant |L_p z|^2 + ||p||^2_{L^\infty_{\leqslant m}}|z|^2 \leqslant 2|L_p z|^2 + 2||p||^2_{L^\infty_{\leqslant m}}|z|^2 .$$

The Carleman estimate (2.9) for the operator

$$|Lz|^2 \leqslant 2|L_p z|^2 + 2||p||_{L^\infty_{\leqslant m}(\Omega)}|z|^2 \leqslant 2|L_p z|^2 + 2m|z|^2 .$$

In fact, if one chooses s_0 (or λ_0) large enough,one can have the term

$$2Cm \int_0^T \int_\Omega |z|^2 e^{2s\varphi} dxdt .$$

on the left side of (2.19) and obtain (2.9), using somewhat different constants.Finaly, Theorem 2.1 proof is now complete.

Chapter 3

Application to some controllability of hyperbolic PDES

This last chapter will be devoted to the proof of the null control of some hyperbolic equations , that is it is associated with the use of the already presented Carlman estimates Chapter 2 in the proof of coercivity , finally ,we will prove the equivalence between Controllability problem and Variational problem.

3.1 Null controllability of linear wave equation with mixed boundary condition

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open domain with a smooth border Γ,Γ_0 a subset that is non-empty of Γ, denote $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $\Sigma_0 = \Gamma_0 \times (0, T)$ and $T > 0.$

Consider the following wave equation given by :

$$\begin{cases} y'' - \Delta y + py = f & in\ Q, \\ y(x,0) = 0\ ,\ y'(x,0) = 0 & in\ \Omega, \\ y(x,t) = 0 & on\ \Sigma, \end{cases} \tag{3.1}$$

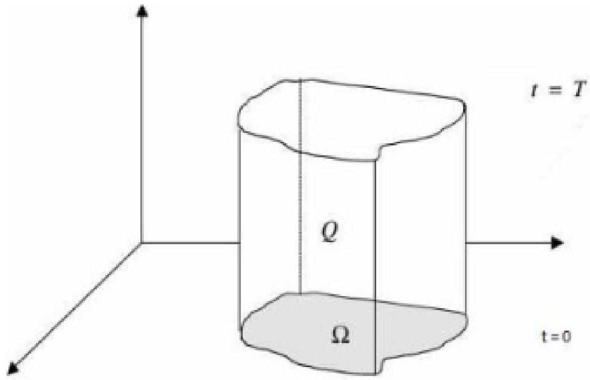

FIGURE 3.1 – The space-time cylinder

Theorem 3.1 [**14**] *Denote the operator* $L_p = (\partial^2/\partial t^2) - \Delta + p$ *in the distribution sense, there exists a* C^2 *weighted positive function* ρ *defined on* Q *such that* $1/\rho$ *is bounded in* Q *and* $C = C(\Omega, T, \Gamma_0, \rho) > 0$ *such that :*

$$\int_0^T \int_\Omega \frac{1}{\rho^2}|z|^2 dxdt \leqslant C \left[\int_0^T \int_\Omega \frac{1}{\rho^2}|L_p z|^2 dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2} \left| \frac{\partial z}{\partial \eta} \right|^2 d\Gamma dt \right] \tag{3.2}$$

And this is the space that we will be working in :

$$\mathcal{V} = \{u \in L^2(0,T,H_0^1)(\Omega), L_p u \in L^2(Q), \frac{\partial u}{\partial \eta}|_{\Sigma_0} = 0\} .$$

Taking [**Theorem 2.1**] in the following form, which represents the Carleman estimates

$$s\lambda \int_Q e^{2s\varphi}(|\partial_t z|^2 + |\nabla z|^2)dxdt + s^3\lambda^3 \int_Q e^{2s\varphi}|z|^2 dxdt + \int_Q |P_1(e^{s\varphi}z)|^2 dxdt$$

$$+ \int_Q |P_2(e^{s\varphi}z)|^2 dxdt$$

$$\leq C \int_Q e^{2s\varphi}|L_p z|^2 dxdt + Cs\lambda \int_{\Sigma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \ .$$

we notice that all sides are positives so we can write :

$$s^3\lambda^3 \int_Q e^{2s\varphi}|z|^2 dxdt \leqslant C \int_Q e^{2s\varphi}|L_p z|^2 dxdt + Cs\lambda \int_{\Sigma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \ .$$

For now, it's enough to note that $1/\rho^2 = e^{2s\varphi}$,we take

$$s\lambda \int_{\Sigma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \leqslant C_1 \int_{\Sigma_0} e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \ .$$

because $s \geqslant s_0 > 0$ and $C_1 > 0$ and By taking $C = max(C, C_1)$ we obtain

$$Cs\lambda \int_{\Sigma_0} \varphi e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \leqslant C \int_{\Sigma_0} e^{2s\varphi}|\partial_\nu z|^2 d\Gamma dt \ .$$

then

$$\int_0^T \int_\Omega \frac{1}{\rho^2}|z|^2 dxdt \leqslant C \left(\int_0^T \int_\Omega \frac{1}{\rho^2}|L_p z|^2 dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2}|\frac{\partial z}{\partial \eta}|^2 d\Gamma dt \right) \ .$$

On the other hand let's consider a null controllability problem for the following wave equation :

$$\begin{cases} y'' - \Delta y + py = f & in\ Q, \\ y(x,0) = 0, y'(x,0) = g & in\ \Omega, \\ y(x,t) = \begin{cases} v & on\ \Sigma_0, \\ 0 & on\ \Sigma/\Sigma_0, \end{cases} \end{cases} \tag{3.3}$$

or :

$$L_p y = f \qquad in\ Q \tag{3.4}$$

with $y(x,T) = y'(x,T) = 0$ in Ω .

Where $v \in \mathrm{L}^2(\Sigma_0)$ boundary control .

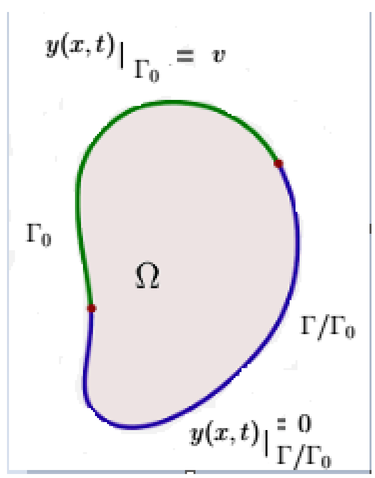

FIGURE 3.2 – Boundary control action

Let's start by multiplying the preceding equation in the function $\phi \in \mathcal{V}$ to produce the weak formulation of the subsequent equation (3.4) then integrate on Q :

$$\int_0^T \int_\Omega L_p y.\phi \, dxdt = \int_0^T \int_\Omega f.\phi \, dxdt \qquad in \; Q \tag{3.5}$$

All of this demonstrates the controllability property by condition

$$y(x,T) = y'(x,T) = 0 \;\; on \;\; \Omega \tag{3.6}$$

Using integration by parts and the Green Formula in(3.5),we have

$$\int_0^T \int_\Omega (y'' - \Delta y + py).\phi \, dxdt = \int_0^T \int_\Omega f.\phi \, dxdt \; .$$

$$\int_0^T \int_\Omega y''.\phi \, dxdt - \int_0^T \int_\Omega \Delta y.\phi \, dxdt + \int_0^T \int_\Omega py.\phi \, dxdt = \int_0^T \int_\Omega f.\phi \, dxdt \; .$$

$$\int_0^T \int_\Omega y\frac{\partial^2 \phi}{\partial t^2} \, dxdt + \left[\int_\Omega \frac{\partial y}{\partial t}\phi \, dx - \int_\Omega y\frac{\partial \phi}{\partial t}\phi \, dx\right]_0^T - \int_0^T \int_\Omega y.\Delta\phi \, dxdt + \int_0^T \int_\Gamma \frac{\partial y}{\partial \eta}\phi \, d\Gamma dt$$

$$+\int_0^T \int_\Gamma y\frac{\partial \phi}{\partial \eta} \, d\Gamma dt + \int_0^T \int_\Omega py.\phi \, dxdt = \int_0^T \int_\Omega f.\phi \, dxdt \; .$$

$$\int_0^T \int_\Omega y\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi + p\phi\right) \, dxdt + \int_\Omega \frac{\partial y}{\partial t}(x,T)\phi(x,T) \, dx - \int_\Omega \frac{\partial y}{\partial t}(x,0)\phi(x,0) \, dx$$

$$-\int_\Omega y(x,T)\frac{\partial \phi}{\partial t}(x,T) \, dx + \int_\Omega y(x,0)\frac{\partial \phi}{\partial t}(x,0) \, dx + \int_0^T \int_\Gamma \frac{\partial y}{\partial \eta}\phi \, d\Gamma dt + \int_0^T \int_\Gamma y\frac{\partial \phi}{\partial \eta} \, d\Gamma dt$$

$$= \int_0^T \int_\Omega f.\phi dxdt \ .$$

$$\int_0^T \int_\Omega y.L_p\phi \ dxdt - \int_\Omega g.\phi(x,0) \ dx + \int_0^T \int_{\Gamma_0} \frac{\partial y}{\partial \eta}\phi \ d\Gamma dt - \int_0^T \int_{\Gamma/\Gamma_0} \frac{\partial y}{\partial \eta}\phi \ d\Gamma dt$$

$$+ \int_0^T \int_\Gamma y\frac{\partial \phi}{\partial \eta} \ d\Gamma dt = \int_0^T \int_\Omega f.\phi dxdt \ .$$

$$\int_0^T \int_\Omega y.L_p\phi \ dxdt + \int_0^T \int_{\Gamma_0} v\frac{\partial \phi}{\partial \eta} \ d\Gamma dt = \int_0^T \int_\Omega f.\phi \ dxdt + \int_\Omega g.\phi(x,0) \ dx \ .$$

We consider that y=$1/\rho^2 Lr$ and v=$1/\rho^2(\partial r/\partial \Gamma)$ with $r \in \mathcal{V}$,we obtain

$$\int_0^T \int_\Omega \frac{1}{\rho^2}L_pr.L_p\phi \ dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2}\frac{\partial r}{\partial \eta}\frac{\partial \phi}{\partial \eta} \ d\Gamma dt = \int_0^T \int_\Omega f.\phi \ dxdt + \int_\Omega g.\phi(x,0) \ dx \ .$$

we take

$$a(r,\phi) = \int_0^T \int_\Omega \frac{1}{\rho^2}L_pr.L_p\phi \ dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2}\frac{\partial r}{\partial \eta}\frac{\partial \phi}{\partial \eta} \ d\Gamma dt \ .$$

and

$$l(\phi) = \int_0^T \int_\Omega f.\phi \ dxdt + \int_\Omega g.\phi(x,0) \ dx \ .$$

Our problem with null controllability turns into

$$a(r,\phi) = l(\phi) \ , \forall \ \phi \in V \tag{3.7}$$

We treat the following subspace V as a Hilbert space for the associated norm and the scaler product $a(r, \phi)$.

$$\phi \to \|\phi\|_V = \sqrt{a(\phi;\phi)}\ .$$

and V be the completion of $\mathcal{V}$.

Remark 3.1 : We can characterize the structure of $\mathcal{V}$ as a subspace of a weighted sobolev space.
Indeed, let $H_\rho(Q)$ be the weighted Hilbert space defined by

$$H_\rho(Q) = \{u \in L^2(Q) \text{ such that } \int_Q \frac{1}{\rho^2}|u|^2 r dr dt < \infty\}.$$

endowed with the natural norm

$$||.||_{H_\rho(Q)} = \left(\int_Q \frac{1}{\rho^2}|.|^2 dx dt\right)^{1/2}.$$

This shows that V is embedded continuously in $H_\rho(Q)$ as :

$$\exists C > 0 : ||u||_{H_\rho(Q)} \leq C||u||_V \text{ for every } u \in V. \tag{3.8}$$

By the boundedness of $1/p^2$ on Q, we also see that $L^2(Q)$ is continuously embedded in $H_\rho(Q)$.

By applying Lax-Miligram Theorem (see[**Theorem 1.2**]) in the form (3.7),we check

1. $\forall r,\phi\in V, \exists M>0 : |a(r,\phi)|\leqslant M||r||_V||\phi||_V$.

$$|a(r,\phi)| = \left|\int_0^T\int_\Omega \frac{1}{\rho^2}L_p r.L_p\phi\ dxdt + \int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\frac{\partial r}{\partial\eta}\frac{\partial\phi}{\partial\eta}d\Gamma dt\right|$$
$$\leqslant \left|\int_0^T\int_\Omega \frac{1}{\rho^2}L_p r.L_p\phi\ dxdt\right| + \left|\int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\frac{\partial r}{\partial\eta}\frac{\partial s}{\partial\eta}\ d\Gamma dt\right|$$
$$\leqslant (r,\phi)_V \leqslant^{Cauchy-shwartz} ||r||_V||\phi||_V.$$

With

$$(r,\phi)_V = \int_0^T\int_\Omega \frac{1}{\rho^2}L_p r.L_p\phi\ dxdt + \int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\frac{\partial r}{\partial\eta}\frac{\partial\phi}{\partial\eta}\ d\Gamma dt\ .$$

the inner product in space V. Then $a(r,s)$ is continous and M=1 .

2. $\forall r\in V,\exists\ \alpha>0 : |a(r,r)|\geqslant\alpha||r||_V^2$.

$$|a(r,r)| = \left|\int_0^T\int_\Omega \frac{1}{\rho^2}(Lr)^2 dxdt + \int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\left(\frac{\partial r}{\partial\eta}\right)^2 d\Gamma dt\right|$$
$$= \int_0^T\int_\Omega \frac{1}{\rho^2}|Lr|^2 dxdt + \int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\left|\frac{\partial r}{\partial\eta}\right|^2 d\Gamma dt\ .$$

Using the Carleman estimates [**Theorem 3.1**] we get :

$$\int_0^T\int_\Omega \frac{1}{\rho^2}|Lr|^2 dxdt + \int_0^T\int_{\Gamma_0}\frac{1}{\rho^2}\left|\frac{\partial r}{\partial\eta}\right|^2 d\Gamma dt \geqslant \frac{1}{C}\left(\int_0^T\int_\Omega\frac{1}{\rho^2}|r|^2 dxdt\right)\ .$$

and the norm in space V takes the following form :

$$||r||_V^2 = \int_0^T \int_\Omega \frac{1}{\rho^2} |r|^2 dxdt \ .$$

From this we conclude

$$a(r,r) \geqslant \frac{1}{C} ||r||_V^2 \ .$$

Then $a(r,r)$ coercive .

Consequently, the condition of continuity is realized by using the Lax-Miligram ([**Theorem 1.2**]) with the use of the Carleman inequality (3.2), and there is then a unique and weak solution $r \in V$.

Finaly , equation (3.6) is null controllable by the Carleman estimate.

3.2 Equivalente between Controllability problem and Variational problem

In this section , We will prove the equivalence between Controllability problem and Variational problem.

First , let's start by introducing the Variational problem :

$$a(r,\phi) = l(\phi) \ , \forall \ \phi \in V \tag{3.9}$$

where

$$a(r,\phi) = \int_0^T \int_\Omega \frac{1}{\rho^2} L_p r . L_p \phi \ dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2} \frac{\partial r}{\partial \eta} \frac{\partial \phi}{\partial \eta} d\Gamma dt \tag{3.10}$$

and

$$l(\phi) = \int_0^T \int_\Omega f.\phi \ dxdt + \int_\Omega g.\phi(x,0) \ dx \tag{3.11}$$

Considering that $y = 1/\rho^2 Lr$ and $v = 1/\rho^2(\partial r/\partial\Gamma)$ with $r \in \mathcal{V}$,we have :

$$\int_0^T \int_\Omega y.L_p\phi \; dxdt + \int_0^T \int_{\Gamma_0} \frac{1}{\rho^2}\frac{\partial r}{\partial \eta}\frac{\partial \phi}{\partial \eta} d\Gamma dt = \int_0^T \int_\Omega f.\phi \; dxdt + \int_\Omega g.\phi(x,0) \; dx.$$

$$\int_0^T \int_\Omega (\phi'' - \Delta\phi + p\phi).y + \int_0^T \int_{\Gamma_0} v\frac{\partial \phi}{\partial \eta} d\Gamma dt = \int_0^T \int_\Omega f.\phi \; dxdt + \int_\Omega g.\phi(x,0) \; dx.$$

$$\int_0^T \int_\Omega \phi''.y \; dxdt - \int_0^T \int_\Omega \Delta\phi.y \; dxdt + \int_0^T \int_\Omega p\phi.y \; dxdt + \int_0^T \int_{\Gamma_0} v\frac{\partial \phi}{\partial \eta} d\Gamma dt$$

$$= \int_0^T \int_\Omega f.\phi \; dxdt + \int_\Omega g.\phi(x,0) \; dx.$$

by integration by parts (see [**Theorem 1.3**]),and the Green Formula (see [**Theorem 1.4**]).We get :

$$\int_0^T \int_\Omega \phi\frac{\partial^2 y}{\partial t^2} \; dxdt + \left[\int_\Omega \frac{\partial \phi}{\partial t} y \; dx - \int_\Omega \frac{\partial y}{\partial t}\phi \; dx\right]_0^T - \int_0^T \int_\Omega \phi.\Delta y \; dxdt$$

$$+ \int_0^T \int_\Gamma \frac{\partial \phi}{\partial \eta} y \; d\Gamma dt + \int_0^T \int_\Gamma \phi\frac{\partial y}{\partial \eta} \; d\Gamma dt + \int_0^T \int_\Omega p\phi.y \; dxdt + \int_0^T \int_{\Gamma_0} v\frac{\partial \phi}{\partial \eta} d\Gamma dt$$
$$= \int_0^T \int_\Omega f.\phi \; dxdt + \int_\Omega g.\phi(x,0) \; dx.$$

$$\int_0^T \int_\Omega \phi\left(\frac{\partial^2 y}{\partial t^2} - \Delta y + py\right) \; dxdt + \int_\Omega \frac{\partial \phi}{\partial t}(x,T)y(x,T) \; dx - \int_\Omega \frac{\partial \phi}{\partial t}(x,0)y(x,0) \; dx$$

$$- \int_\Omega \phi(x,T)\frac{\partial y}{\partial t}(x,T) \; dx + \int_\Omega \phi(x,0)\frac{\partial y}{\partial t}(x,0) \; dx + \int_0^T \int_\Gamma \frac{\partial \phi}{\partial \eta} y \; d\Gamma dt$$

$$+ \int_0^T \int_\Gamma \phi\frac{\partial y}{\partial \eta} \; d\Gamma dt + \int_0^T \int_{\Gamma_0} v\frac{\partial \phi}{\partial \eta} d\Gamma dt$$

$$= \int_0^T \int_\Omega f.\phi \; dxdt + \int_\Omega g.\phi(x,0) \; dx.$$

$$\int_0^T \int_\Omega \phi \left(\frac{\partial^2 y}{\partial t^2} - \Delta y + py - f \right) \; dxdt + \int_\Omega \frac{\partial \phi}{\partial t}(x,T) y(x,T) \; dx - \int_\Omega \phi(x,T) \frac{\partial y}{\partial t}(x,T) \; dx$$

$$- \int_\Omega \frac{\partial \phi}{\partial t}(x,0) y(x,0) \; dx + \int_0^T \int_\Gamma \frac{\partial \phi}{\partial \eta}(v - y) \; d\Gamma dt + \int_0^T \int_\Gamma \phi \frac{\partial y}{\partial \eta} \; d\Gamma dt$$

$$+ \int_\Omega \left(\frac{\partial y}{\partial t} - g \right) .\phi(x,0) \; dx = 0.$$

Since $\phi \in \mathcal{V}$ dense space,we conclude that :

$$\frac{\partial^2 y}{\partial t^2} - \Delta y + py - f = 0 \;\; a.e \;\; in \; Q.$$

In the same way, we can conclude that both :

$$\frac{\partial y}{\partial t} = g \quad in \; \Omega.$$

$$y(x,0) = 0, \frac{\partial y}{\partial t} = g \quad in \; \Omega.$$

and

$$y(x,t) = v \quad on \; \Gamma_0.$$

$$y(x,t) = 0 \quad on \; \Gamma / \Gamma_0$$

As well as $y(x,T) = y'(x,T) = 0 \; in \; \Omega$.Which brings us to the Controllability problem (3.3).

Biography of T. Carleman

FIGURE 3.3 – T. Carleman, 1892-1949

Tage Gillis Torsten carleman was born in 1892 in Sweden and died on January 11, 1949 in Stockholm , he completed his studies at Uppsala university2 ,first studied in 1923-1924 at Lund University, then in 1924 he was appointed professor at Stockholm university2, 3.

Among his most important achievements :

Carleman inequality, Carleman matrix, Denjoy Carleman–Ahlfors theorem, Carleman equation,Carleman condition.

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