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Theme

Mathematical study of certain epidemiological models

Presented by:
Achach Moussaab

Examination Committee:

Dr. Nabti Abderrazak	Larbi Tebessi University	Président
Dr. Bahi Mohamed Cherif	Larbi Tebessi University	Examiner
Dr. Bouaziz Khelifa	Larbi Tebessi University	Encadreur

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إهداء

أحمد الله عز وجل على منه و عونه لإتمام هذا البحث.

إلى الذي وهبني كل ما يملك حتى أحقق له آماله، إلى من كان يدفعني قدما نحو الأمام لنيل المبتغى،إلى الإنسان الذي إمتلك الإنسانية بكل قوة، إلى الذي سهر على تعليمي بتضحيات جسام مترجمة في تقديسه للعلم، إلى مدرستي الأولى في الحياة،

أبي الغالي على قلبي أطال الله في عمره؛

إلى التي وهبت فلذة كبدها كل العطاء و الحنان، إلى التي صبرت على كل شيء، التي رعتني حق الرّعاية و كانت سندي في الشدائد، و كانت دعواها لي بالتوفيق، تتبعتني خطوة خطوة في عملي، إلى من إرتحت كلما تذكرت إبتسامتها في وجهي نبع الحنان أمي أعز ملاك على القلب و العين جزاها الله عني خير الجزاء في الدارين؛

إليهما أهدي هذا العمل المتواضع لكي أُدخل على قلبهما شيئا من السعادة إلى إخوتي وأخي الذين تقاسموا معي عبء الحياة ؛

قال الله تعالى " : **إن الله لا يغير ما بقوم حتى يغيروا ما بأنفسهم**"....

الآية 11من سورة الرعد

و إلى كل من يؤمن بأن بذور نجاح التغيير هي في ذواتنا و في أنفسنا قبل أن تكون في أشياء أخرى...

إلى كل هؤلاء أهدي هذا العمل.

شكر و عرفان

قال رسول الله صلى الله عليه و سلم:

"من لم يشكر الناس لم يشكر الله"

صدق رسول الله صلى الله عليه و سلم

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هذه المذكرة لإيفائه حقه **بصبره الكبير علي**، ولتوجيهاته العلمية التي لا تقدر بثمن؛ و التي

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و أدخلني برحمتك في عبادك الصالحين".

Abstract:

The objective of this memoire is to examine the intricacies of a reaction-diffusion (susceptible-infectious-susceptible) SIS epidemic model featuring a nonlinear incidence rate, which characterizes the spread of a contagious illness among people. We demonstrate that, given a single condition, the suggested model has two steady states. We establish the local and global asymptotic stability of the non-negative constant steady states subject to the basic reproduction number being greater than unity and of the disease-free equilibrium subject to the basic reproduction number being smaller than or equal to unity in the ODE case by analyzing the eigenvalues, and using an appropriately constructed Lyapunov function. Through the application of a suitably constructed Lyapunov function, we determine the global stability condition in the PDE scenario. This is done by comparing R_0 with one, where in the case of $1 < R_0$ we found that the system accepts global stability in the vicinity of the point E^*but in the case of $1 > R_0$ the system accepts global stability in the vicinity of the point E_0 . Finally, we provide a few numerical examples that both illustrate and validate the analytical findings that have been made throughout the work.

Keywords: Reproductive number R_0, epidemiological, equilibrium points, disease-free equilibria , local and global stability , Lyapunov function ·

Résumé :

L'objectif de cette mémoire est d'analyser les subtilités d'un modèle épidémique de réaction-diffusion SIS (susceptible-infectieux-susceptible) avec une incidence non linéaire, qui caractérise la propagation d'une maladie contagieuse parmi les individus. On montre que, sous une seule condition, le modèle suggéré présente deux états stables. On établit la stabilité asymptotique locale et globale des systèmes constants non négatifs lorsque le nombre de reproduction basique est supérieur à un, ainsi que celle du système équilibré sans maladie lorsque le nombre de reproduction basique est inférieur à un, dans le cas de l'ODE, en analysant les valeurs propres et en utilisant une fonction Lyapunov appropriée. En utilisant une fonction Lyapunov appropriée, nous déterminons la condition de stabilité globale dans le scénario PDE Cela se fait en comparant R_0 avec un, où dans le cas de $1 < R_0$ nous avons constaté que le système accepte une stabilité globale au voisinage du point E^*, mais dans le cas de $1 > R_0$, le système accepte une stabilité globale au voisinage aux point. Enfin E_0, nous présentons des exemples numériques qui illustrent et confirment les conclusions analytiques qui ont été établies tout au long de l'action.

mots-clés: Nombre de reproduction R_0, épidémiologique, points d'équilibre, équilibres sans maladies, stabilité local et globale, fonction de Lyapunov ·

الملخص:

الهدف من هذه المذكرة هو دراسة نموذج الوباء (معرض – مصاب –معرض) SIS الذي يتميز بمعدل حدوث غير خطي، والذي يميز انتشار المرض المعدي بين الناس. لقد أثبتنا أنه، في ظل شرط واحد، فإن النموذج المقترح له حالتين ثابتتين. نحن نحدد الاستقرار المقارب المحلي والكلي للحالات الثابتة غير السالبة التي تخضع ألن يكون رقم التكاثر الأساسي أكبر من الوحدة وأن يكون التوازن الخالي من الأمراض خاضعًا ألن يكون رقم التكاثر الأساسي أصغر من أو يساوي الوحدة في حالة ODE من خلال تحليل القيم الذاتية، واستخدام دالة ليابونوف التي تم إنشاؤها بشكل مناسب. من خلال تطبيق دالة ليابونوف التي تم إنشاؤها بشكل مناسب، نحدد حالة الاستقرار العالمي في سيناريو PDE و ذلك من خلال مقارنةR_0 مع الواحد حيث في حالة $R_0 > 1$ وجدنا ان الدالة يقبل الاستقرار الكلي في جوار النقطةE^+ ،اما في حالة $R_0 < 1$ قبل الدالة الاستقرار الكلي في جوار النقطة E_0 . وأخيرا، نقدم بعض الأمثلة العددية التي توضح وتؤكد صحة النتائج التحليلية التي تم التوصل إليها طوال العمل.

الكلمات المفتاحية: رقم التكاثر الأساسي R_0، علم االوبئة ، نقاط التوازن، التوازن الخالي من الأمراض، الاستقرار المحلي والكلي، دالة ليابونوف.

Contents

General Introduction

Throughout history, many epidemics have had major implications for human society, from killing large proportions of the world's population to making humans think of a solution to reduce them, mathematical modeling has been the way to do so, by modeling problems and analyzing them mathematically using non-linear differential equation systems, these illnesses now pose fresh difficulties that need for mathematical modeling that combines medical and mathematical view points Epidemiology, medicine, biology, and mathematical cross in the field of mathematical modeling,which uses equations to show a condensed version of reality.

In recent years, scientists have worked to create mathematical models that are more and more realistic and answer ever-more-complex issues. The nature of the problems under investigation and the accessibility more detailed and the accurate data are the causes of this complexity.

In general, mathematical models provide a condensed understanding of reality by formalizing complicated occurrences and making it easier to examine numerous factors and their relationships.

Furthermore, mathematical models serve the primary purpose of predicting events across diverse situations, finding particular use in the field of communicable disease epidemiology through various models based on differential equation equations or probabilistic approaches epidemiological models play avital role in comprehending the spread of infectious diseases and predicting future outcomes. Analytical studies of epidemiological models, involving the analysis of disease transmission dynamics and considering factors such as population size and disease criteria, examine the mathematical behavior and characteristics of epidemics. Non-linear equations, Such as reaction-diffusion models, describe the interaction between epidemiological variables like the number of infected, susceptible, exposed, and recovered individuals.

The SIS epidemiology model has captured the interest of many researchers, so in this research, I conducted an analytical study of an SIS epidemiological model specifically, targeted at *HIV*. The study in modved involved a comprehensive analysis of the disease's dynamics within a population, wherein the SIS model represents individuals as susceptible to *HIV* infection or infected and capable of transmitting the virus.

This analytical approach allowed for the exploration of equilibrium points, determination of their stability, and prediction of long-term trends in *HIV* transmission.

SYMBOL

$\mathbb{R}$	:	Set of real numbers .
$\mathbb{R}_{\geq 0}$	:	Set of positive real numbers.
$\mathbb{R}^n$	:	Vectorial space of dimension of dimension $n \geq 1$.
$\mathbb{R}^2_{\geqslant 0}$	:	Set of positive real numbers of dimension $n = 2$.
Ω	:	Open bounded subset of $\mathbb{R}^n$.
$\bar{\Omega}$	:	Closing of Ω.
$\partial\Omega$	:	Smooth boundary.
$C\left(\bar{\Omega}\right)$	:	Continuous set of functions on $\bar{\Omega}$.
$C^1\left(\Omega,\ \mathbb{R}\right)$	:	Space of the functions continuously differentiable to ordre1in Ω in $\mathbb{R}$.
$H^1\left(\Omega\right), H^2\left(\Omega\right)$	:	Sobolev spaces.
$L^2\left(\Omega\right)$	:	Space of square integrable functions in Ω.
$\frac{\partial u(x,\,t)}{\partial t}$	:	Partial derivative of u with respect to time t.
Δ	:	Laplacian operator.
∇	:	Gradien operator.
L	:	Linearizing operator.
$\|\|.\|\|_2$	:	Norm in L^2.
$\|\|.\|\|_p$	:	Norm in L^p.
$\langle\ ,\ \rangle$	:	Euclidean scalar product.
R_0	:	Basic reproduction number.
$J\left(u\ ,\ v\right)$	:	Jacobian matrix.
tr	:	Trace of matrix.
$\det$	:	Determinant of matrix.
$ODEs$	:	Ordinary differential equations.
$PDEs$	:	Partial differential equations.
SIS	:	Susceptible-infection-susceptible.
HIV	:	Human Immunodeficiency Virus.
SIR	:	Susceptible-infection-recovered.
$SIRS$	:	Susceptible-infection-recovered-Susceptible.
N	:	Total population.
Λ	:	Force of infection.

Chapter 1

Some mathematical definitions

1.1 Space L^p

Definition 1.1 *[6]Let Ω be a domain in $\mathbb{R}^n$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable function U defined on Ω for wich*

$$\int_{\Omega} |U(t,x)|^P \, dx < \infty,$$

if $U \epsilon L^p$,we define its norm

$$\|U(t,x)\|_p = \left(\int_{\Omega} |U(t,x)|^P \, dx \right)^{\frac{1}{p}}.$$

Corollary 1.1 *$L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle U, U \rangle_{L^2} = \|U\|_{L^2}^2 = \int_{\Omega} U^2 dx.$$

1.1.1 *Soboleve space*

Definition 1.2 *[18]The Sobolev space $W^{k,p}(\Omega)$ consists of function $u \; \epsilon L^p(\Omega)$such that for every multi-index α with $|\alpha| \leq k$, the weak derivatives $D^\alpha u$ exists $D^\alpha u \; \epsilon L^p(\Omega)$. Thus*

$$W^{k,p}(\Omega) = \{u \; \epsilon L^p(\Omega), D^\alpha u \; \epsilon L^p(\Omega), |\alpha| \leq k\}.$$

Definition 1.3 *We call Sobolev space of order 1 on Ω the space*

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{ v \epsilon L^2(\Omega), \partial_{x_i} v \epsilon L^2(\Omega), 1 \leq i \leq d \right\}.$$

1.2 Basic theories

1.2.1 *Gronwall's Inequality*

Theorem 1.1 [2]*Let $N(t)$ be a continuous nonnegative function such that*

$$N(t) \leq \alpha + \int_{t_0}^{t} (\beta N(s) + \gamma) ds, \text{on } t \geqslant t_0 \,,$$

where $\alpha \geqslant 0, \beta \geqslant 0$ and $\gamma \geqslant 0$.Then for $t \geqslant t_0$.$N(t)$ satisfies

$$N(t) \leq \alpha \exp(\beta(t - t_0)) + \frac{\gamma}{\beta}(\exp(\beta(t - t_0)) - 1).$$

1.2.2 *Non-negativity of solutions*

Definition 1.4 [24] *Let $F :\subseteq \overline{\mathbb{R}^n_+} \to \mathbb{R}^n$.*
Then F is essentially nonnegative if $f_i(U) \geqslant 0$, for all $i = 1 \ldots n$ and $U \epsilon \overline{\mathbb{R}^n_+}$ such that $u_i = 0$.
where u_i denotes the i^{th} component of U.

Proposition 1.1 [?] *Suppose $I \subset \overline{\mathbb{R}^n_+}$ .Then $\overline{\mathbb{R}^n_+}$ is an invariant set with respect to $ODEs$ system if and only if F is essentially nonnegative.*

1.2.3 *Intermediate value theorem*

Theorem 1.2 [20] *Let h be a real-valued function which is continuous on the closed interval $[a, b]$.*
If k is any number between $h(a)$ and $h(b)$, then there exists at least one number $c \, \epsilon [a, b]$ such that $h(c) = k$.
The Intermediate value theorem can be used to determine whether there exists a solution to the equation $h(x) = k$ when $h(x)$ is a continuous function on a closed interval $[a, b]$.

Corollary 1.2 [20]*Let h be a real-valued function which is continuous on the closed interval $[a, b]$.*
If $h(a) \times h(b) < 0$, then there exists at least one number $c \, \epsilon [a, b]$ such that $h(c) = 0$.

Remark 1.1 [4] *if the function h is strictly monotonic and continuous on $[a, b]$ (i.e. strictly increasing or strictly decreasing) then the equation $h(x) = k$, has a unique solution.*

1.2.4 *Green Formula*

[1] Let u, v be two functions such that $u \epsilon H^2(\Omega)$ and $v \epsilon H^1(\Omega)$ then we have

$$\int_{\Omega} uvdx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} vds - \int \bigtriangledown u \bigtriangledown vdx.$$

1.2.5 Eigenfunction

Definition 1.5 *In mathematics, an eigenfunction of a linear operator P defined on some function space is any non-zero function Ψin that space that, when acted upon by P, is only multiplied by some scaling factor called an eigenvalue. As an equation, this condition can be written as*

$$P\Psi = \lambda\Psi.$$

for some scalar eigenvalue λ The solutions to this equation may also be subject to boundary conditions that limit the allowable eigenvalues and eigenfunctions.

1.3 Equilibrium points

An equilibrium point, or a steady state, is a point in a system where the system remains unchanged at the equilibrium point. In other words, the net change in the system's state variables is zero, many stability problems are naturally formulated with respect to equilibrium points [7]

Definition 1.6 *A state $x^*$is an equilibrium state (or equilibrium point) of the system if once $x(t)$ is equal to x^*, it remains equal to x^* for all future time. This means*

$$f(x^*) = 0.$$

After we solve this nonlinear algebraic equations we can found the equilibrium points. A linear time-invariant system

$$\dot{x} = Ax.$$

If A is nonsingular that is mean the system has a single equilibrium point. If A is singular, it has an infinity of equilibrium point. A nonlinear system can have several (or infinitely many) isolated equilibrium points.

1.3.1 *Disease-free equilibria*

It is a special state in which the entire population is free from the infectious disease under consideration, meaning that it refers to a state in which the disease is completely eradicated from the population, and there are no infected individuals present. It represents a stable equilibrium point at which the number of infected individuals is zero $(I = 0)$.

1.4 Some basic results about reaction-diffusion equations

In this section, we will present the most important ndings about reaction-diusion systems (2.9)-(2.10), represented in local existence & uniqueness of solution, comparaison principle, and positivity of solution.

Local existence of solutions

The following basic assumptions on system (2.9)-(2.10) are assumed to hold:
$f \ = (f_i)_{i=1}^m : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+^m \rightarrow \ \mathbb{R}^m$ is locally Lipschitz in each variable.
$u_0 = (u_{0i})_{i=1}^m \epsilon L^\infty(\Omega; \mathbb{R}^m)$.
Before declaring the main result in this subsection we will need this definition:

Definition 1.7 *A function* $u := (u_1, ..., u_m)$*is a classical solution to (2.9)-(2.10) on* $(0, T)$ *if, for* $i = 1, ..., m$*, we have:*

$$u_i \epsilon C^{2,1}(\overline{\Omega} \times (0,T)) \bigcap C\left([0,T)\,; L^\infty(\Omega)\right),$$

and u satisfies (2.9)-(2.10).

The local existence of solution of systems (2.9)-(2.10) follows from classical results, as follows:

Theorem 1.3 *[23]Suppose that (2.9)-(2.10) hold. Then the systems (2.9)-(2.10) admit a unique, classical solution on* $\Omega \times [0, T_{max})$*where*
$0 < T_{max} \leq \infty$*Moreover,*

$$\textit{if } T_{max} < +\infty, \textit{then } \lim_{t \to T_{max}} \sum_{i=1}^m \|u_i(.,t)\|_{L^\infty(\Omega)} = +\infty$$

Comparaison principle and positivity of solutions

First, we will present an important theorem that helps in obtaining initial estimates for solutions of reaction-diusion systems, known as the comparaison principle:

Theorem 1.4 *[3]Let $T > 0$, if the functions*

$$u := (u_1, ..., u_m), v := (v_1, ..., v_m) \epsilon \left[C^{2,1}(\overline{\Omega} \times (0,T))\right]^m,$$

satisfy

$$\begin{aligned} \frac{\partial u_i}{\partial t} - d_i \Delta u_i - f_i(x,t,u) &\leq \frac{\partial v_i}{\partial t} - d_i \Delta v_i - f_i(x,t,v), \text{ in } \Omega \times (0,T), i = 1,..,m, \\ u_{0i}(x) &\leq v_{0i}(x), \text{ in } \Omega, i = 1,..,m, \\ \frac{\partial u_i}{\partial \upsilon} &= \frac{\partial u_i}{\partial \upsilon} = 0, \text{on } \partial\Omega \times (0,T), i = 1,..,m. \end{aligned}$$

Then

$$u_i(x,t) \leq v_i(x,t), \text{in } \overline{\Omega} \times (0,T], i = 1,..,m.$$

Since the properties of chemical concentration, density of population, number of individuals, ... ect, are positive quantities (either in the initial stage or after a period of time), then we need to elicit this property in system(2.9)-(2.10). Before showing the positivity of solution of (2.9)-(2.10), we need the following precondition:

Definition 1.8 ***(Quasi-positive function)*** *The nonlinearity $f = (f_i)_{i=1}^m$ (in system (2.9)-(2.10)) is called quasi-positive if satisfies:*

$$\forall u = (u_1, ..., u_m) \epsilon \mathbb{R}_+^m, \forall i = 1, ..., m, f_i(u_1, .., u_{i-1}, 0, u_{i+1}, .., u_m) \geq 0.$$

According to the comparaison principle and the precondition stated above, we get the following result about positivity of solution for system (2.9)-(2.10).

Corollary 1.3 *[12]Suppose that (2.9)-(2.10)hold, moreover, the nonlinearity $f = (f_i)_{i=1}^m$ is quasi-positive. Then the systems (2.9)-(2.10) admit a unique, classical solution on $\Omega \times [0, Tmax)$ where, for $i = 1, ..., m$*

$$\forall x \,\epsilon\, \Omega, u_{0i}(x) \geq 0 \Rightarrow \forall (x,t) \,\epsilon\, \Omega \times [0, Tmax), u_i(x,t) \geq 0.$$

Global existence of solutions for reaction-dffiusion equations

Consider the initial-boundary value problem (semilinear parabolic equation):

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta u = f\ (x,t,u) \text{ in } \Omega \times \mathbb{R}_+^*, \\ \frac{\partial u}{\partial v}(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R}_+^*, \\ u(x,0) = u_0(x) \text{ in } \overline{\Omega}. \end{cases} \tag{1.1}$$

Where $\Omega \subset \mathbb{R}^{\mathbb{N}}$ (Ω is bounded, simply connected and smooth), and $d > 0$. Furthermore we assume:
$f : \Omega \times \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian function of all its arguments.
$u_0 \in C\left(\overline{\Omega}\right)$.
Under the above assumptions (2.1), The problem (2.1)admits a unique classical solution on $\Omega \times [0, Tmax)$ where $0 < Tmax \leq \infty$. Moreover, we have the same characterization of $Tmax$.

1.5 Local stabillity

1.5.1 Local stability in case ODEs

To understand the fundamental theories pertaining to the stability of the system ODEs. First you must find:
Jaccobian matrix, a matrix of partial derivatives of the first order. The approximation of non-linear systems around equilibrium points is a crucial task for the Jacobian matrix in the context of differential equations and dynamical systems. The localized behavior of a system of equations near equilibrium is characterized by the Jacobian matrix when it involves many variables.
If we have this system of equations:

$$\begin{cases} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ \qquad \cdots \\ f_n(x_1, x_2, ..., x_n) = 0. \end{cases}$$

Its Jacobian Matrix denoted by J is given by

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

By evaluating the Jacobian matrix at an equilibrium point, you can obtain valuable information about the stability properties of the system by examining the eigenvalues of the matrix.[8]

Theorem 1.5 *[21]The system Given is locally asymptotically stable at the equilibrium (u^* ,v^*) if and only if the trace of A is negative and its determinant is positive, i.e.*

$$\begin{cases} tr(J) = m < 0, \forall m \in \mathbb{R} \\ \det(J) = n > 0, \forall n \in \mathbb{R} \end{cases}$$

1.5.2 Local stability in case PDEs

A popular technique for examining the PDEs system's local asymptotic stability is the eigenfunction expansion method, It's crucial to review some of the theory around the Laplace operator's eigenvalues.[22]

The Eigenvalues of the Laplace Operator

We have $0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... < \lambda_i = +\infty$ where are indefinite sequence of postive eigenvalues for the Laplacien operator Δ over Ω, with Neuman boundary condition where each λ_i has, multipicty $m_i \geq 1$, Also let $(\Phi_{ip})_{p=\overline{1,m_i}}$ where $1 \leq p \leq m_i$, be the correspanding normalized eigenfunction, that is Φ_{ip} and λ_i satisfy -$\Delta\Phi_{ip} = \lambda_i$ in Ω with $\frac{\partial \Phi_{ip}}{\partial \Omega} = 0$ in $\partial\Omega$ and $\int_\Omega \Phi_{ip}^2 dx = 1$.

1.6 Global stability

Proving global stability[7]in epidemiology is often more challenging than demonstrating local stability. global stability refers to demonstrating that an equilibrium point in an epidemiological model is not only locally stable but also stable for all possible initial conditions and perturbations. It ensures that the disease prevalence will converge to the equilibrium state from any starting point in the system's state space. It usually involves the use of Lyapunov functions, which are scalar functions that measure the energy or "distance" of the system from the equilibrium. A Lyapunov function is defined such that it decreases over time, reaching a minimum at the equilibrium point.

So to establish global stability, the Lyapunov function must be definite positive (it is positive for all points in the state space except at the equilibrium point where it is zero), and its derivate with respect to time must be negative or zero.
We define this function as follow:

Theorem (Lyapunov function)

Theorem 1.6 [7] *Let x^* be an equilibrium solution of the equation.*

$$x' = f(x(t)).$$

Let Ω be a neighborhood of x^* contained in U, and let $V : \Omega \to \mathbb{R}$ be a C^1 class function such that:
$\triangleright V(x^*) = 0.$
$\triangleright \forall\ x \in \Omega \backslash \{x^*\}, V(x) > 0.$
$\triangleright \forall\ x \in \Omega, V'(x) \leq 0.$
Then, x^* is stable.
V named Lyapunov function.

Theorem (Strict Lyapunov Function)

Theorem 1.7 [16]*Let x^* be an equilibrium solution of the equation*

$$x' = f(x(t)).$$

Let Ω be a neighborhood of x^* included in U, and let $V : \Omega \to \mathbb{R}$ be a C^1 class function such that:
$\triangleright V(x^*) = 0.$
$\triangleright \forall\ x \in \Omega \backslash \{x^*\}, V(x) > 0.$
$\triangleright \forall\ x \in \Omega, V'(x) < 0.$
Then, x^* is asymptotically stable.
V named Strict Lyapunov function.

The Negative Criteria:

Among the methods used to establish the global asymptotic stability of solutions are Bendixson's and Dulac's criteria. Below is a summary of these criteria as described in [10]. It is important to note that $(proposition\ of\ Bendixson's\ criterion)$ is merely a special case of Dulac's $(Proposition\ Dulac's)$.

Proposition 1.2 (*Bendixson's criterion*)
Given the simply connected region $\sum$*, if the expression*

$$C = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$$

is not zero for all (x, y) *in* $\sum$ *and does not change sign in* $\sum$*, then there are no limit cycles in* $\sum$.

Proposition 1.3 *(Dulac's criterion)*
Given the simply connected region $\sum$*, if there exists a function* $B(x, y) \epsilon \mathbb{C}^1$ *such that*

$$C = \frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y}$$

is not zero for all (x, y) *in* $\sum$ *and does not change sign in* $\sum$*, then there are no limit cycles in* $\sum$.

The Poincare-Bendixson Theory

This theorem is based on the observation that two dimensional planes have some specific characteristics that may not exist in higher dimensions. Particularly, any trajectory may only have one of four limiting values: a critical point, a limit cycle, cycle graph, or infinite xy values. Furthermore, if the trajectory is bounded, then it may only approach a critical point or a cycle graph. This is basis of the Poincare-Bendixson theory, which states that if a certain trajectory is bounded for $t \geqslant t_0$ and does not tend to a singular point, then it either is a limit cycle or tends to a limit cycle. For more on the theory, see[10]
The following theorem summarizes the Poincare-Benidixson theory:

Theorem 1.8 *If an* $ODEs$ *system of the form*

$$\frac{du}{dt} = F(u),$$

where F *is locally Lipschitz in* u*, has a solution that is bounded for* $t \geqslant 0$*, then either*

▷ϕ *is periodic,*

▷ϕ *approaches a periodic solution,*

▷ϕ *gets close to an equilibrium point infinitely often.*

Theorem 1.9 *[9](La Salle invariance theorem)*
Let $V : \Omega \rightarrow \mathbb{R}^+$ be a function of $\mathbb{C}^1$ and suppose that $V(u) \leq 0$ for all $u \,\epsilon\, \Omega$. Define

$$E = \{u \,\epsilon\, \Omega : V(u) = 0\}.$$

Let L be the largest invariant set contained in E. Then, any bounded solution tends to L as the time goes to infinity. If, furthermore, L reduce to u^*, then $u^*$is asymptotically stable.

Definition 1.9 *(Global Stability)*
Function u is globally asymptotically stable on Ω if for all $u_0 \,\epsilon\, \Omega$, the solution u satisfies

$$\lim_{t\rightarrow\infty} \|u(t) - u^*\| = 0.$$

Chapter 2

Introduction to epidemiology

2.1 Introduction

A mathematical model is an abstract representation or interpretation of reality in different domains which is accessible to analysis and calculation based on a set of assumptions. Compartmental models are among the first mathematical models to have been used in epidemiology, which plays an important role in studying the evolution of infectious diseases and eradicating them and, at most, should make it possible to better understand epidemic phenomena and therefore better control them.

The idea has become to study demographic variations in societies. To model, it is first necessary to know the biology of the disease well [19] making a model of deterministic or stochastic compartments in discrete or continuous time depending on the disease to be studied, except the use of stochastic models is more complicated than the other. In this chapter, we are interested in this last type of model, which is based on two concepts: compartments and rules, compartments divide the population into various possible states by disease(*susceptible, infected, etc*.), rules specify the proportion of individuals moving from one compartment to another.

2.2 Concept of the epidemiology

2.2.1 Definition of epidemic

The term "epidemic" [16] is derived from the Greek words "epi", which means "on" or "among", and "demos", which means "people" or "population". The Oxford English Dictionary states that the word was first used in english in the late 16^{th} century, specifically in 1580.

The term epidemiology, which means "study of epidemics", was first used in 1830 by the french physician Louis-René Villermé. Later, Dr. Pierre Charles Alexandre Louis popularized the term, which was then gradually incorporated into the English language. While epidemiology is often associated with medicine, it encompasses much more.

2.2.2 Definition of epidemiology

The term epidemiology, which means "study of epidemics", was first used in 1830 by the french physician Louis-René Villermé. Later, Dr. Pierre Charles Alexandre Louis popularized the term, which was then gradually incorporated into the English language. While epidemiology is often associated with medicine, it encompasses much more.
It is a branch of medicine that studies the spread of diseases, their transmission factors, and their impact on the human population, including risk factors, prevention, and control measures.

2.2.3 Descriptive of epidemiology

In descriptive epidemiology, data is analyzed to describe the patterns of occurrence by person, place, and time, as well as the distribution of diseases or health-related events in communities. The goal of this area of epidemiology is to determine the population impacted, the location and timing of the incidents, and any patterns or contributing factors.[17]

2.3 Mathematical Epidemiology

Mathematical epidemiology is a branch of epidemiology that utilizes mathematical and statistical models and methods to study the spread of diseases in populations, analyze factors influencing this spread, and provide forecasts, preventive measures, and control strategies based on available data and mathematical models.
In this definition, we will mention some famous examples used in this field.

2.3.1 *The classic model in compartement*

The SI, SIS model

The SI model is one of the classic models created by W. Hamar and developed in 1906 where individuals can be divided into two compartments:

Figure 2.1: SI model diagram

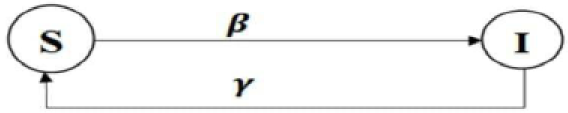

Figure 2.2: The SIS model

The compartment or box of susceptible (healthy) individuals are receptive to the infectious agent who are not contaminated but can catch the disease and become contagious noted (S).
The compartment of infected individuals noted (I) are those affected and who are therefore infectious.
The infection is spread by direct contact between the susceptible and the infected. We see that in the SI model, there are no cures and is only relevant in incurable diseases or if the phenomenon of acquired immunity can be neglected[25]. An individual changes state (either infected or susceptible... etc.) he therefore changes his compartment with outgoing or incoming flows which indicate the rate of transfer between them. On the other hand, as the change in the number of infected people occurs over time, compartment I includes $I(t)$ and the same for $S(t)$. By the assumption of the constant of the size of the population the model is formed as follows :

The system of differential equations is written :

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t). \\ \frac{dI(t)}{dt} = \beta I(t)S(t). \end{cases} \tag{2.1}$$

With :
β : is the infection rate per unit time.
γ : the rate of each infected heals.
$N(t)$:is total population.
With $N(t) = S(t) + I(t)$ is the total population and is constant through time t. There are cases where susceptible becomes infected and the infected are cured at the rate γ but do not develop immunity and become susceptible like the case of tuberculosis, the following graph concisely summarizes the model :

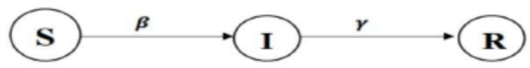

Figure 2.3: SIR model

The associated differential equations are :

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t) + \gamma I(t). \\ \frac{dI(t)}{dt} = \beta I(t)S(t) - \gamma I(t). \end{cases} \tag{2.2}$$

The SIR, SIRS model

The SIR model is the model proposed by kermack and MC Kendrick, consists of three categories of population : healthy people $S(t)$, infected people $I(t)$, recovered or cured people $R(t)$ who are conferred
a immunization against reinfection or death.
The following figure schematizes the transfers of individuals between each group.
Mathematically, the SIR model is given by the following system :

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t). \\ \frac{dI(t)}{dt} = \beta I(t)S(t) - \gamma I(t). \\ \frac{dR(t)}{dt} = \gamma I(t). \end{cases} \tag{2.3}$$

Where :
β is the transmission rate.
γ is the recovery rate.

The term $\beta I(t)S(t)$ represents the number of contacts between healthy and infected people. On the other hand, we only encounter diseases, the individual has not acquired permanent immunization,he los- es his immunity and returns to the S compartment at the rate η, this is the SIRS model schematizing as follows :
This model is formulated as follows :

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t) + \eta R(t). \\ \frac{dI(t)}{dt} = \beta I(t)S(t) - \gamma I(t). \\ \frac{dR(t)}{dt} = \gamma I(t) - \eta R(t). \end{cases} \tag{2.4}$$

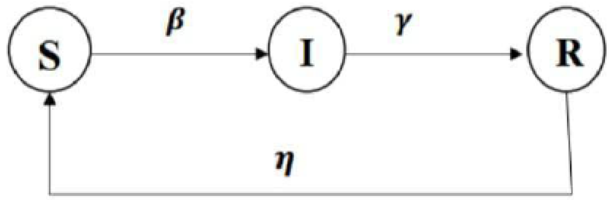

Figure 2.4: The SIRS model

Figure 2.5: SEI model

With :
β : is the transmission rate.
γ : the rate of each infected heals.
η : the rate of loss of immunity (each removed becomes healthy again).

The SEI, SEIR model

The constitution of these models are based on a subpopulation is already infected but not yet contagious (non-infectious) i.e. susceptible subpopulations before go to class I, it requires spending a period to make infectious s'calls the latency period or incubation at an intermediate compartment denoted E(exposed), taking into account β the incubation rate of a disease. Schemes and ED are developed as follows :

The model is translated as follows :

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t). \\ \frac{dE(t)}{dt} = \beta I(t)S(t) - \alpha E(t). \\ \frac{dI(t)}{dt} = \alpha E(t). \end{cases} \tag{2.5}$$

As well as :
The EDO system is elaborated to the following :

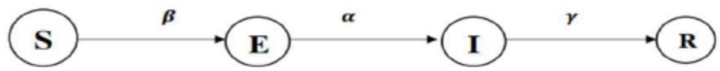

Figure 2.6: SEIR model

$$\begin{cases} \frac{dS(t)}{dt} = -\beta I(t)S(t). \\ \frac{dE(t)}{dt} = \beta I(t)S(t) - \alpha E(t). \\ \frac{dI(t)}{dt} = \alpha E(t) - \gamma I(t). \\ \frac{dR(t)}{dt} = \gamma I(t). \end{cases} \tag{2.6}$$

2.4 The force of infection

The key parameter in all epidemiological models of infectious diseases is the force of infection. The latter accounts for the contamination process by expressing the probability that a susceptible individual will contract the disease so that each infected encounters at the rate C, each of these encounters with a type C individual causes contamination with the probability P, we score $\beta = CP$. It is this force of infection which moves individuals from compartment S to compartment I in the previous figures and which can be written in two different ways :
Λ is force of infection.

$$\Lambda = \beta I. \tag{2.7}$$

If disease transmission increases with population N density like (the influenza virus).

$$\Lambda = \frac{\beta I}{N}. \tag{2.8}$$

If the transmission does not depend on it as (*HIV*).

2.5 Basic reproduction number R_0

One of the first questions the epidemiologist asks is whether there is going to be an epidemic or not. The answer to this question in a very simple way by examining the system of differential equations. The first step is to translate our question into mathematical form. Then we calculate a quantity that describes the average number of secondary cases, generated by a typical infectious

individual durin his period of infectivity, when he is introduced into a population consisting entirely of susceptible, this quantity is called basic reproduction number, noted by R_0, so the first idea of this number was by Théophile Lotz (1980) (Nichiura, Dietz, Eichner 2006). He finds that R_0 is a threshold and after Ross describes the first differential model and gives the threshold conditions as follows :
If $R_0 \leq 1$, then the disease-free point is globally stable, i.e. an individual infects on average less than one, which means that the disease disappears from the population. Conversely, if $R_0 > 1$, then the endemic point is globally stable, i.e. The disease can spread in the population. Note that is determined according to the parameters of the model and later, it is used in the equilibrium stability theorems of the disease of the population.[26]
Who want to calculate this number must follow this method It was created in the 1980s by
R. M. Anderson and R. M. May, and the discipline of infectious disease epidemiology makes extensive use of it.

2.5.1 *The Van Den Driessche and Watmough method*

Either F, V two matrices

$$F = \begin{pmatrix} f_1(S, U_1, U_2) \\ f_2(S, U_1, U_2) \end{pmatrix}.$$

$$V = \begin{pmatrix} v_1(U_1, U_2) \\ v_2(U_1, U_2) \end{pmatrix}.$$

Derivatives of f_1, f_2 with respect to U_1 and U_2 respectively:

$$F = \begin{pmatrix} \frac{\partial f_1(S,U_1,U_2)}{\partial U_1} & \frac{\partial f_1(S,U_1,U_2)}{\partial U_2} \\ \frac{\partial f_2(S,U_1,U_2)}{\partial U_1} & \frac{\partial f_2(S,U_1,U_2)}{\partial U_2} \end{pmatrix}$$

.Derivatives of v_1, v_2 with respect to U_1 and U_2 respectively:

$$V = \begin{pmatrix} \frac{\partial v_1(U_1,U_2)}{\partial U_1} & \frac{\partial v_1(U_1,U_2)}{\partial U_2} \\ \frac{\partial v_2(U_1,U_2)}{\partial U_1} & \frac{\partial v_2(U_1,U_2)}{\partial U_2} \end{pmatrix}.$$

Then we calculate the inverse matrix of V.

$$V^{-1} = \frac{1}{\frac{\partial v_1(U_1,U_2)}{\partial U_1}\frac{\partial v_2(U_1,U_2)}{\partial U_2} - \frac{\partial v_1(U_1,U_2)}{\partial U_2}\frac{\partial v_2(U_1,U_2)}{\partial U_1}} \begin{pmatrix} \frac{\partial v_2(U_1,U_2)}{\partial U_2} & -\frac{\partial v_1(U_1,U_2)}{\partial U_2} \\ -\frac{\partial v_2(U_1,U_2)}{\partial U_1} & \frac{\partial v_1(U_1,U_2)}{\partial U_1} \end{pmatrix}.$$

Then we calculate the matrix (FV^{-1}) .The basic reproductive number R_0 is then defined as the spectral radius of the Jacobian matrix. The spectral radius refers to the maximum absolute value of the eigenvalues of the matrix (σ is the eigenvalues of the matrix).

$$\begin{aligned}\rho\left(FV^{-1}\right) &= \max\left\{\left|\lambda, \lambda \in \sigma\left(FV^{-1}\right)\right|\right\}\\ \rho\left(FV^{-1}\right) &= R_0.\end{aligned}$$

In the context of epidemiology, the eigenvalues represent the growth rates of different infection states.

2.6 Reaction-diffusion

Reaction-diffusion systems of partial differential equations have long piqued the curiosity of scientists, including Alan Turing, because they may be used to simulate practical situations, in 1952. In essence, reaction-diffusion systems show how two processes result in the change of certain physical attributes in space and time. Reaction is the first process in which a quantity changes into another; diffusion is the second process in which the quantities spread out over space.

2.6.1 Model for reaction-diffusion

We study the population dynamic by merging both mechanisms after investigating different approaches to simulating dispersion and reproduction, either for infection, prey, or population in isolation. Our goal is to track the population size's temporal and spatial behavior while taking various growth models, such logistic and exponential growth, into account. We concentrate in diffusion-reaction systems, where populations can spread and expand at the same time
An example of a reaction-diffusion system in its most generic form is provided by

$$\frac{\partial}{\partial t}U(t,x) = D\Delta U(t,x) + F(U(t,x))x \,\epsilon\, \Omega, t \geqslant 0 \tag{2.9}$$

where

$$U(t,x) = (u_1(t,x), u_2(t,x), \ldots u_n(t,x))^T \tag{2.10}$$

is the unknown vector function, D is an $n \times n$ matrix of diffusion coefficients, and

$$F(u) = (f_1(u), f_2(u), \ldots f_n(u))^T \tag{2.11}$$

is a functional representing the interaction.

2.7 Stability in the general condition

In this section we will discuss an overview of stability.

2.7.1 Local stability generally

Equilibrium Point: An equilibrium point is a state where the system's behavior does not change over time. Mathematically, it's where the derivative of the system's state with respect to time is zero.

Local Stability: Local stability refers to how the system behaves near an equilibrium point. Specifically, it investigates whether small perturbations (changes) in the system's state lead to those perturbations eventually decreasing and the system returning to the equilibrium point, or if they diverge, causing instability.

Linearization: One common method to analyze local stability is through linearization. This involves approximating the behavior of the system near the equilibrium point using linear equations. Linear systems are often easier to analyze mathematically.

Eigenvalues: In linear systems, local stability is often determined by the eigenvalues of the system's *Jacobian matrix* evaluated at the equilibrium point. If all *eigenvalues* have negative real parts, the system is locally stable. If any eigenvalue has a positive real part, the system is unstable.

Lyapunov Stability: Another approach to analyzing stability is Lyapunov stability theory. This theory provides conditions under which a system remains close to an equilibrium point over time. It involves finding a Lyapunov function, which is a scalar function that decreases along trajectories of the system.

Applications: Local stability analysis is widely used in various fields, including control theory, physics, biology, and economics. It helps predict the behavior of systems ranging from mechanical systems to population dynamics.

2.7.2 Global stability

The notion of global stability encompasses not only the behavior of a system locally around equilibrium points, but also the behavior of the system over its whole state space. Generally speaking:

Equilibrium Points: Similar to local stability, global stability often begins with identifying equilibrium points of the system, where its dynamics are such that the state does not change over time.

Lyapunov Functions: Global stability is often analyzed using Lyapunov functions. A Lyapunov function is a scalar function of the system's state variables that is positive definite, meaning it's always positive except possibly at the equilibrium point, and it decreases along system trajectories away from the equilibrium point. If a Lyapunov function can be found for the system that decreases monotonically (or non-increasingly) along all trajectories, the system is globally stable.
LaSalle's Invariance Principle: This principle is a powerful tool for establishing global stability. It states that if there exists a Lyapunov function for a dynamical system, and if the derivative of that Lyapunov function along the trajectories of the system tends to zero as time approaches infinity, then the system's trajectories will converge to the largest invariant set contained within the region where the derivative of the Lyapunov function is zero. In simpler terms, trajectories of the system eventually converge to the set where the Lyapunov function doesn't change, even if it's not the entire state space.
Stability Analysis Techniques: Global stability analysis often involves more sophisticated techniques than local stability analysis. These can include using tools from differential equations, dynamical systems theory, nonlinear control theory, and optimization.
Applications: Global stability analysis is crucial for understanding the long-term behavior of systems. It's used in various fields including control theory, robotics, ecology, epidemiology, and economics, where it helps predict whether a system will reach a steady state, oscillate, or exhibit chaotic behavior across its entire state space.
Global stability analysis is often more challenging than local stability analysis because it requires considering the behavior of the system across its entire state space ?, rather than just in the neighborhood of equilibrium points. However, it provides deeper insights into the long-term behavior and robustness of dynamical systems.

Chapter 3

Analysis and inderstanding of the spread of an epidemic model

Compartmental models have been instrumental in studying disease, offering simple equations to assess outbreak impact and estimate susceptible population sizes locally and globally. Among these, the SIS model serves as a cornerstone in mathematical epidemiology, portraying the fundamental spread of infectious diseases within a population. Unlike more intricate models, individuals in the SIS model remain indefinitely infected without acquiring immunity or receiving treatment, thus perpetually interacting with susceptible populations. This chapter delves into an extended version of theSIS epidemic model proposed in [11], which introduces nonlinear incidence dynamics captured by the function $r\Psi\left(j\right)$.

In this chapter, we are studing a reaction-diffusion presented in [11], the system is a version of the SIS model and described as :

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \theta r - \xi r \Psi\left(j\right), \text{in } \mathbb{R}^+ \times \Omega. \\ \frac{\partial j}{\partial t} - d_2 \Delta j = -\alpha j + \xi r \Psi(j), \quad \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{3.1}$$

Where as the initial Condition:

$$r(x,0) = r_0(x), \qquad j(x,0) = j_0(x), \quad \text{in } \Omega. \tag{3.2}$$

And the homogeneous Neuman boundary conditions

$$\frac{\partial r}{\partial t} = \frac{\partial j}{\partial t} = 0 \quad \text{in } \mathbb{R}^+ \times \Omega. \tag{3.3}$$

Where. The symbol Δ is the laplacien operator on Ω and d_1, d_2 are postives constants. And $\beta, \theta, \alpha, \xi > 0$, constants parametres, we assume Ψ to be a continuously differentible function on $\mathbb{R}^+$satisfying

$$\Psi(0) = 0. \tag{3.4}$$

And

$$0 < j\Psi'(j) \leq \Psi(j) \text{ for all } j > 0. \tag{3.5}$$

The considered system (3.1)-(3.3) describ the transmission of a communicable disease between individuals such as *HIV/AIDS*.

3.1 Positivity of the solution

Let us assume that the initial condition $(r_0, j_0) \epsilon \mathbb{R}^2$.
We have

$$\begin{cases} F(0,j) = \beta - \theta(0) - \xi(0)\Psi(j) = \beta > 0. \\ G(r,0) = -\alpha(0) + \xi r \Psi(0) = 0. \end{cases}$$

Which makes the function $(F,G)^T$essentially nonnegative.
Hence, the nonnegative quadrant $\mathbb{R}^2 \geq 0$ is an invariant set in this section we cancel the spatial variable, In order for us to have a system of $(ODEs)$.

$$\begin{cases} \frac{\partial r}{\partial t} = F(r,j), \text{in } \mathbb{R}^+. \\ \frac{\partial j}{\partial t} = G(r,j), \text{in } \mathbb{R}^+. \end{cases} \tag{3.6}$$

With intial conditions

$$r_0(x) \geq 0, \quad j_0(x) \geq 0 \tag{3.7}$$

3.2 Invariant Regions

In this subsection we defined an invariant region for the system.
We let $N = r + j$ and $\alpha_0 = min(\alpha;\theta)$.

$$D = \left\{(r;j) : r;j \text{ and } r + j \leq \frac{\beta}{\alpha_0}\right\}.$$

D is an invariant region of system (3.6)-(3.7).

Proposition 3.1 *the region D is non-empty, attracting and positively invariant.*

Proof. we have the equation of system (3.6)-(3.7)

$$\begin{aligned}
N(t) &= r(t) + j(t) \\
\frac{d}{dt}(N(t)) &= \frac{d}{dt}(r(t) + j(t)) \\
\frac{d}{dt}(N(t)) &= \frac{d}{dt}(r(t)) + \frac{d}{dt}(j(t)) \\
&= \beta - \theta r - \xi r \Psi(j) - \alpha j + \xi r \Psi(j) \\
&= \beta - \theta r - \alpha j \\
&\leq \beta - \alpha_0(r, j) = \beta - \alpha_0(N(t)).
\end{aligned}$$

So we integrate both ends

$$\begin{aligned}
\int_0^t \frac{d}{ds} N(s) ds &\leq \int_{0(}^t \beta - \alpha_0 N(s)) ds \\
N(t) - N(0) &\leq \Lambda t - \sigma_0 \int_0^t N(s) ds \\
N(t) &\leq N(0) + \beta t - \alpha_0 \int_0^t N(s) ds
\end{aligned}$$

We apply of the *Gronowall's* inequality, we implies

$$N(t) \leq N_0 e^{-\alpha_0 t} - \frac{\beta}{\alpha_0}(e^{-\alpha_0 t} - 1).$$

So

$$(r + j)(t) \leq (r + j)(0) e^{-\alpha_0 t} + \frac{\beta}{\alpha_0}(1 - e^{-\alpha_0 t}) \text{ for } t \geq 0.$$

If the initial stales satisfy$(r + j)(0) \leq \frac{\beta}{\alpha_0}$then $(r + j)(t) \leq \frac{\beta}{\alpha_0}$ and

$$\lim_{t \to \infty} \sup N(t) \leq \frac{\beta}{\alpha_0}.$$

The region D is positively invariant and attracting . ■

3.3 Exictence of Equilibria

The aim of this section is study the solution of equation (3.1)-(3.2) and Extract the Basic repreduction R_0, we simplified the system of equition of $(ODEs)$.

$$\begin{cases} \frac{\partial r}{\partial t} = \beta - \theta r - \xi r \Psi(j), \text{in } \mathbb{R}^+ \times \Omega. \\ \frac{\partial j}{\partial t} = -\alpha j + \xi r \Psi(j), \quad \text{in } \mathbb{R}^+ \times \Omega. \end{cases}$$

$$r(0) = r_0(x) \geq 0, \quad j(0) = j_0(x) \geq 0.$$

1.If $R_0 > 1$ the system has equlibria pointsE_0.

2.If $R_0 < 1$ the system accepted two equlibria points E_0 and E^*.

Proof. the postive equlibria of the system (3.6)-(3.7)

$$\begin{cases} \beta - \theta r - \xi r \Psi(j)) = 0. \\ -\alpha j + \xi r \Psi(j) = 0. \end{cases}$$

If $r = 0$ the system has no equilibrium.

If $j = 0$ it implies.

$$\begin{cases} \beta - \theta r - \xi r \Psi(0) = 0. \\ \xi r \Psi(0) - \alpha(0) = 0. \end{cases} \tag{3.8}$$

$$\begin{cases} \beta = \theta r \text{ so } r = \frac{\beta}{\theta}. \\ E_0 = (\frac{\beta}{\theta}, 0). \end{cases}$$

Now, we study endemic steady state condition, from the second part of (3.8),we have

$$\begin{aligned} \xi r \Psi(j) - \alpha j &= 0. \\ \xi r \Psi(j) &= \alpha j. \\ r &= \frac{\alpha j}{\xi \Psi(j)}. \end{aligned}$$

From the first equation we make up r in the first equation in part "1" of (3.8),

$$\begin{aligned} \beta - \theta r - \xi r \Psi(j) &= \beta - \theta \frac{\alpha j}{\xi \Psi(j)} - \xi \frac{\alpha j}{\xi \Psi(j)} \Psi(j). \\ &= \beta - \theta \frac{\alpha j}{\xi \Psi(j)} - \alpha j. \\ &= \beta \frac{\xi \Psi(j)}{\theta \alpha j} - \frac{\theta \xi \Psi(j) j}{\theta \alpha j} - 1. \\ &= \beta \frac{\xi \Psi(j)}{\theta \alpha j} - \frac{\theta \xi \Psi(j)}{\theta \alpha} - 1. \end{aligned}$$

$$h(j) = 0 \text{ for all } j > 0, \tag{3.9}$$

$h(j)$ it is continuous for$j > 0$, by applying the theory of the mean values,

$$\begin{aligned}\lim_{j\to 0} h(j) &= \lim_{j\to 0}\left[\frac{\beta\xi}{\theta\alpha}\frac{\Psi(j)}{j} - \frac{\beta\xi}{\theta\alpha}\Psi(j) - 1\right].\\ &= \lim_{j\to 0}\left[\frac{\beta\xi}{\theta\alpha}\Psi'(j) - \frac{\beta\xi}{\theta\alpha}\Psi(j) - 1\right].\\ &= \frac{\beta\xi}{\theta\alpha}\Psi'(0) - 1 = R_0 - 1.\end{aligned}$$

And

$$\begin{aligned}\lim_{j\to\frac{\beta}{\alpha_0}} h(j) &= \lim_{j\to\frac{\beta}{\alpha_0}}\left[\frac{\beta\xi}{\theta\alpha}\frac{\alpha_0}{\beta}\Psi'(\frac{\beta}{\alpha_0}) - \frac{\beta\xi}{\theta\alpha}\Psi(\frac{\beta}{\alpha_0}) - 1\right].\\ &= h(\frac{\beta}{\alpha_0}).\\ &= \frac{\xi\left(\alpha_0-\alpha\right)}{\theta\alpha}\Psi\left(\frac{\beta}{\alpha_0}\right) - 1 < 0.\end{aligned}$$

Hence for $R_0 > 1$, we have

$$h(\frac{\beta}{\alpha_0})\lim_{j\to 0} h(j) = h(\frac{\beta}{\alpha_0})\left(R_0 - 1\right) < 0.$$

Then

$$\begin{aligned}\left[\frac{\xi\left(\alpha_0-\alpha\right)}{\theta\alpha}\Psi\left(\frac{\beta}{\alpha_0}\right) - 1\right](\frac{\beta\xi}{\theta\alpha}\Psi'(0) - 1) &= 0.\\ \left(\frac{\xi\left(\alpha_0-\alpha\right)}{\theta\alpha}\Psi\left(\frac{\beta}{\alpha_0}\right)\left(\frac{\beta\xi}{\theta\alpha}\Psi'(0)\right)\right) - \frac{\xi\left(\alpha_0-\alpha\right)}{\theta\alpha}\Psi\left(\frac{\beta}{\alpha_0}\right) - \frac{\beta\xi}{\theta\alpha}\Psi'(0) + 1 &= 0.\end{aligned}$$

If $\alpha_0 = \theta$ do $h(\frac{\beta}{\alpha_0})\left(R_0 - 1\right) < 0$. ■

By appling the intermediate value theorem , ther existe a real $j^*\epsilon(0, \frac{\beta}{\alpha_0})$ such that(3.9)holds, and if the derivating of h is negative for all valus larger then zero the function is monotonically decrea where using in (3.5)

$$\begin{aligned}\frac{dh(j)}{dj} &= \left(\left[\frac{\beta\xi}{\theta\alpha}\frac{\Psi(j)}{j} - \frac{\alpha\xi}{\theta\alpha}\Psi(j) - 1\right]\right)',\\ &= \frac{\beta\xi}{\theta\alpha}\left(\frac{\Psi^{'}(j)j - \Psi(j)}{j^2}\right) - \frac{\alpha\xi}{\theta\alpha}\Psi^{'}(j),\\ &= \frac{\beta\xi\left(\Psi^{'}(j)j - \Psi(j)\right) - \xi\alpha\Psi^{'}(j)}{\theta\alpha j^2} < 0.\end{aligned}$$

And, there existe a unique real j^*, confind between $(0, \frac{\beta}{\alpha_0})$ knoing that $h(j^*) = 0$, wich implies the existence of $r^* = \frac{\alpha j^*}{\xi\Psi(j^*)}$ between $\left(\frac{\beta}{\alpha_0}, +\infty\right)$the scond equation of (3.8) has no solution.
Because

$$\max_{x\epsilon\left(\frac{\beta}{\alpha_0},+\infty\right)} h(j) \le h(\frac{\beta}{\alpha_0}) < 0.$$

3.3.1 *Basic repreduction number R_0 of system*

The R_0 can be defined as the spectral racins of this operator SV^{-1} the system (3.6)-(3.7) may be rewritten in vector form

$$\begin{aligned}\begin{pmatrix} j_t \\ r_t \end{pmatrix} &= \begin{pmatrix} -\alpha j + \xi r\Psi(j) \\ \beta - \theta r - \xi r \Psi(j) \end{pmatrix}, \\ &= \begin{pmatrix} \xi r \Psi(j) \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha j \\ -\beta + \theta r + \xi r \Psi(j) \end{pmatrix}.\end{aligned}$$

The jacobian matrice identical to vector (1) and (2) at the (DFE) $E_0 = (\frac{\beta}{\theta}, 0)$ we proved by :

$$J_1(j,r) = \begin{pmatrix} \frac{\beta\xi}{\theta}\Psi'(0) & 0 \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = J_1(E_0),$$

and

$$J_2(E_0) = \begin{pmatrix} \alpha & 0 \\ \frac{\beta\xi}{\theta} & \Psi'(0) \end{pmatrix} := \begin{pmatrix} V & 0 \\ V_1 & V_2 \end{pmatrix}.$$

Calculating V^{-1}

$$\begin{aligned} V^{-1} &= \frac{1}{VV_2}\begin{pmatrix} V_2 & 0 \\ -V_1 & V \end{pmatrix}, \\ &= \begin{pmatrix} \frac{1}{V} & 0 \\ \frac{-V1}{VV_2} & \frac{1}{V_2} \end{pmatrix}. \end{aligned}$$

We implies $SV^{-1} = K$.

$$\begin{aligned} SV^{-1} &= K = \left(\frac{\beta\xi}{\theta}\Psi'(0)\right)(\alpha^{-1}) = \frac{\beta\xi}{\theta\alpha}\Psi'(0). \\ R_0 &= (SV^{-1}) = \frac{\beta\xi}{\theta\alpha}\Psi'(0). \end{aligned}$$

3.4 The local ODE stability

In this theorem study the local asymptotic stability of the previously defined E_0 and E^* points.

Theorem 3.1 *for condition (3.6)-(3.7)*
▷ *if $R_0 < 1$, the disase-free equilibrium solution $E_0 = (\frac{\beta}{\theta}, 0)$, is the only steady state of the system and is locally asymptotically stable.*
▷*if $R_0 > 1$, E_0 is instable and $E^* = (r^*, j^*)$ is locally asymptotically stable.*

Proof. we prove the local asymptotic stability,we calculate the Jacobian matrix :

$$J(r,j) = \begin{pmatrix} -\xi\Psi(j) - \theta & -\xi r\Psi'(j) \\ \xi\Psi(j) & \xi r\Psi'(j) - \alpha \end{pmatrix},$$

we study the local stability asymptotic for $R_0 < 1$,then

$$J(E_0) = \begin{pmatrix} -\theta & -\xi\frac{1}{\theta}\Psi'(0) \\ 0 & \xi\frac{1}{\theta}\Psi'(0) - \alpha \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = -\theta, \lambda_2 = \xi\frac{1}{\theta}\Psi'(0) - \alpha$

$\triangleright$ $\lambda_1 > 0, \lambda_2 > 0$, so $R_0 < 1$, leading to the asymptotic stability.

$\triangleright$ $\lambda_1 < 0, \lambda_2 > 0$, so $R_0 > 1$, leading to instability .

and

$\triangleright$ $\lambda_1 = -\theta, \lambda_2 = 0$, so $R_0 = 1$, E_0 is asymptomatically stable.

We study E^* for $R_0 > 1$, we have

$$J(E^*) = J(r^*, j^*) = \begin{pmatrix} -\xi\Psi(j^*) - \theta & -\xi r^*\Psi'(j^*) \\ \xi\Psi(j^*) & \xi r^*\Psi'(j^*) - \alpha \end{pmatrix},$$

the determinant of the Jacobian matrix is

$$\det(J(E^*)) = \xi\alpha\Psi(j^*) + \theta\alpha - \theta\xi r^*\Psi'(j^*), \tag{3.10}$$

and the trace

$$tr(J(E^*)) = -(\xi\Psi(j^*) + \theta) + \left(\xi r^*\Psi'(j^*) - \alpha\right),$$

and we have

$$\begin{cases} \beta = \xi r^*\Psi(j^*) + \theta r^* \\ \alpha = \frac{\xi r^*\Psi(j^*)}{j^*}, \end{cases} \tag{3.11}$$

using (3.5) and (3.11), we found.

$$\begin{aligned} \det(J(E^*)) &= \xi\frac{\xi r^*\Psi(j^*)}{j^*}\Psi(j^*) + \theta\frac{\xi r^*\Psi(j^*)}{j^*} - \theta\xi r^*\Psi(j^*). \\ &= \frac{\xi^2 r^*(\Psi(j^*))^2}{j^*} + \theta\xi r^*\left[\frac{\Psi(j^*)}{j^*} - \Psi(j^*)\right], \end{aligned}$$

and

$$\begin{aligned} tr(J(E^*)) &= -\frac{(\xi\Psi(j^*)r^* + \theta r^*)}{r^*} - \frac{\xi\Psi(j^*)r^*}{j^*} + \xi\Psi'(j^*)r^*. \\ &= -\frac{\beta}{r^*} - \xi r^*\left[\frac{\Psi(j^*)}{j^*} - \Psi'(j^*)\right], \end{aligned}$$

from the condition (3.5) we obtain

$$\begin{aligned} \det\left(J\left(E^{*}\right)\right) &> 0, \\ \operatorname{tr}\left(J\left(E^{*}\right)\right) &< 0. \end{aligned}$$

Then the equilibrium E^{*} is locally asymptomatically stable . ■

3.5 Global existence of solution

Our goal in this section are to comput the fundamental reproduction and demonstrate the existence of equllirium solution for $(3.1)-(3.2)$

Lemma 3.1 *condition (3.5) implies*

$$0<\frac{\Psi\left(j\right)}{j}\leq\Psi^{'}\left(j\right) \text{ for all } j>0. \tag{3.12}$$

Proof. we have inequality

$$0<j\Psi^{'}\left(j\right)\leq\Psi\left(j\right) \text{ for all } j>0,$$

divide both sides by j

$$0<\Psi^{'}\left(j\right)\leq\frac{\Psi\left(j\right)}{j}.$$

We have $\Psi\left(0\right)=0$,$\Psi^{'}\left(0\right)$ existe, we use the mean value theorem because $\Psi\left(j\right)$ is continuously differentiable.

$$\begin{aligned} \Psi^{'}\left(0\right) &= \frac{\Psi\left(j\right)-\Psi\left(0\right)}{j-0} \\ \Psi\left(j\right) &= \Psi\left(0\right)+j\Psi^{'}\left(0\right), \end{aligned}$$

diveding by j, we have

$$\frac{\Psi\left(j\right)}{j}=\Psi^{'}\left(0\right),$$

then

$$0\leq\frac{\Psi\left(j\right)}{j}=\frac{\Psi\left(j\right)-\Psi\left(0\right)}{j-0},$$

we get

$$0<\Psi^{'}\left(j\right)\leq\frac{\Psi\left(j\right)}{j}\leq\Psi^{'}\left(0\right), \tag{3.13}$$

■

Proposition 3.2 *For any initial circumstances* $r_0, j_0 \in C(\bar{\Omega}) \times C(\bar{\Omega})$. *Then the solution* (r, j) *of systems* $(3.1 - 3.3)$ *existing both globally and uniquely across time. Furthermore, a positive constant is present.* $C(r_0, j_0, \beta, \theta, \alpha, \xi) > 0$, *such that* $\forall t > 0$.

$$\|r(\cdot, t)\|_{L^\infty(\Omega)} + \|j(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t > 0. \tag{3.14}$$

Furthermore, there exists a positive constant $\tilde{C}(\beta, \theta, \alpha, \xi)$ *such that for a large* $T > 0$,

$$\|r(\cdot, t)\|_{L^\infty(\Omega)} + \|(\cdot, t)\|_{L^\infty(\Omega)} \le \tilde{C} \quad \textit{for all } t > T. \tag{3.15}$$

Proof. Let use now consider the case $r(t, x)\epsilon(0, T\max) \times \Omega$, when given by:

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \theta r - \xi r \Psi(j) \text{ in } (0, \infty) \times \Omega \, . \\ \quad r(0, x) = r_0(x), \qquad \text{on} \quad \Omega. \\ \frac{\partial r}{\partial v} = 0 \qquad\qquad \text{on} \qquad \partial\Omega. \end{cases} \tag{3.16}$$

for any nonnegative fonction $j(t, x)$, there is an upper solution exists for (3.16), is provided by

$$N_1 := \max\left\{\frac{\beta}{\theta}, \|r_0\|_{c(\overline{\Omega})}\right\},$$

by using the comparison principle, we obtain $w(t, x) \le N_1$ in $[0, T_{\max}) \times \overline{\Omega}$, r is uniformly bounded in $[0, T_{\max}) \times \overline{\Omega}$ and by integration of *equation (1)* we attain

$$\begin{aligned} \frac{d}{dt} \int_\Omega (r(t, x) + j(t, x)) dx &= |\Omega| \beta - \int_\Omega (\theta r(t, x) + \alpha j(t, x)) dx. \\ &\le |\Omega| \beta - \alpha_0 \int_\Omega (r(t, x) + j(t, x)) dx. \end{aligned} \tag{3.17}$$

Using the well-know Gronowall' inequality, for $\alpha_0 = \min(\theta, \alpha)$ and $t = [0, T_{\max})$.

$$\int_\Omega (r(t, x) + j(t, x)) dx \le N_2, \tag{3.18}$$

where $N_2 > 0$, for $t = [0, T_{\max})$.

$$\int_\Omega j(t, x) \le N_2, \tag{3.19}$$

using the $J(t, x)$- equation, $\exists N_3 > 0$ depending on N_2 such that $j(t, x) \le N_3$ over $[0, T_{\max}) \times \overline{\Omega}$, j is uniformly bounded in $[0, T_{\max}) \times \overline{\Omega}$,
by using the standard theory of semilinear parabolic systems, we deduce $T_{\max} = +\infty$,
when $T_{\max} = +\infty$ the problem(3.16) implies.

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \theta r - \xi r \Psi(j) \le \beta - \theta r \text{ in } (0, \infty) \times \Omega \, . \\ \quad r(0, x) = r_0(x) \le \|r_0\|_{(\overline{\Omega})} \qquad \text{on } \Omega. \\ \qquad \frac{\partial r}{\partial v} = 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{cases} \tag{3.20}$$

By using the comparison principle , we obtain $r(t,x) \leq w(t)$ for $t\epsilon\,[0,\infty)$, where $w(t) = \|r_0\|_{\overline{\Omega}}\, e^{-\theta t} + \left(\frac{\beta}{\theta}\right)\left(1 - e^{-\theta t}\right)$ is the unique solution of the initial value problem :

$$\begin{cases} \frac{dw}{dt} = \beta - \theta w \quad t > 0. \\ w\,(0) = \|r_0\|_{\overline{\Omega}}. \end{cases} \tag{3.21}$$

Then , for $x\,\epsilon\,\overline{\Omega}$, we have

$$r(t,x) \leq w(t) \overset{t\to\infty}{\to} \frac{\beta}{\theta}.$$

Thus , we have an upper bound for $\|r\,(t,\cdot)\|$also bounded by a positive constant independent of the initial data for a large enough t. ∎

3.6 The local PDE stability

In this section, we exam the local stability of more general partial differential equation $(PDEs)$ car $(3.1) - (3.3)$.

Theorem 3.2 *for system* $(3.1) - (3.3)$.

1. if $R_0 < 1$,the (DFE) E_0 is locally asymptotically stable.
2.if $R_0 > 1$, the endemic equilibrium $E^*$is locally asymptotically stable.

Proof. we have the system in $PDEs$, given by :

$$\begin{cases} d_1\Delta r + \beta - \theta r - \xi r \Psi\,(j) = 0. \\ d_2\Delta j - \alpha j + \xi r \Psi(j) = 0. \end{cases}$$

The subject to the homogen Neuman boundry condition

$$\frac{\partial r}{\partial \upsilon} = \frac{\partial j}{\partial \upsilon} = 0, \text{in } \mathbb{R}^+ \times \partial\Omega.$$

We have $0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... < \lambda_i = +\infty$ wher are indefinite sequence of postive eigenvalues for the laplacien operator Δ over Ω, with Neuman boundary condition where each λ_i has ,multipicty $m_i \geq 1$, Also let $(\Phi_{ip})_{p=\overline{1,m_i}}$ where $1 \leq p \leq m_i$, be the correspanding normalized eigenfunction, that is Φ_{ip} and λ_i satisfy -$\Delta\Phi_{ip} = \lambda_i$ in Ω with $\frac{\partial \Phi_{ip}}{\partial \Omega} = 0$ in $\partial\Omega$ and $\int_\Omega \Phi_{ip}^2 dx = 1$.
Linearizing system $(3.1) - (3.3)$.
We start by the first equilibrum point E_0.

$$\frac{\partial r}{\partial t} = \begin{pmatrix} d_1\Delta r - \theta & -\xi\frac{\beta}{\theta}\Psi'(0) \\ 0 & d_2\Delta j + \xi\frac{\beta}{\theta}\Psi'(0) - \alpha \end{pmatrix} r,$$

then the linearizing operator is given by

$$L(E_0) = \begin{pmatrix} d_1\Delta r - \theta & -\xi\frac{\beta}{\theta}\Psi'(0) \\ 0 & d_2\Delta j + \xi\frac{\beta}{\theta}\Psi'(0) - \alpha \end{pmatrix},$$

let $(\Phi(x), \varphi(x))$ be an eigenfunction of L corssponding to an eigenvalues γ. By definition of eigenfunction in the First Chapter,
we have

$$L(\Phi(x), \varphi(x))^T = \gamma(\Phi(x), \varphi(x))^T,$$

leading to

$$(L - \gamma I)\begin{pmatrix}\Phi \\ \varphi\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

substiting for L yields

$$\begin{pmatrix} d_1\Delta r - \theta - \xi & -\xi\frac{\beta}{\theta}\Psi'(0) \\ 0 & d_2\Delta j + \xi\frac{\beta}{\theta}\Psi'(0) - \alpha - \xi \end{pmatrix}\begin{pmatrix}\Phi \\ \varphi\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

leading to :

$$\sum_{0 \le i \le \infty, 1 \le p \le m_i} (J_i - \gamma I)\begin{pmatrix}a_{ip} \\ b_{ip}\end{pmatrix}\Phi_{ip} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

Where

$$\Phi = \sum_{0 \le i \le \infty, 1 \le p \le m_i} a_{ip}\Phi_{ip} \quad , \quad \varphi = \sum_{0 \le i \le \infty, 1 \le p \le m_i} b_{ip}\varphi_{ip}.$$

And

$$J_i(E_0) = \begin{pmatrix} d_1\lambda_i - \theta & -\xi\frac{\beta}{\theta}\Psi'(0) \\ 0 & d_2\lambda_i + \xi\frac{\beta}{\theta}\Psi'(0) \end{pmatrix} \text{ for } i \ge 0,$$

the eigenvalues are

$$\begin{cases} n_{i1} = d_1\lambda_i - \beta. \\ n_{i2} = d_2\lambda_i + \xi\frac{\beta}{\theta}\Psi'(0). \end{cases}$$

Both n_{i1} and n_{i2} clearly have negative real parts for $R_0 < 1$.
Now , we study the scond equilibrium E^*, we get

$$L(E^*) = \begin{pmatrix} -d_1\Delta r - \theta - \xi r\Psi(j^*) & -\xi r^*\Psi'(j^*) \\ \xi\Psi(j^*) & d_2\Delta j - \alpha + \xi r^*\Psi'(j^*) \end{pmatrix}.$$

We defined $J_i(E^*)$

$$J_i(E^*) = \begin{pmatrix} -d_1\lambda_i - \theta - \xi r\Psi(j^*) & -\xi r^*\Psi'(j^*) \\ \xi\Psi(j^*) & -d_2\lambda_i - \alpha + \xi r^*\Psi'(j^*) \end{pmatrix},$$

trace of $J_i(E^*)$ is

$$tr(J_i(E^*)) = -d_1\lambda_i - \beta - \xi r\Psi(j^*) - d_2\lambda_i - \alpha + \xi r^*\Psi'(j^*),$$

so trace $(J_i(E^*)) < 0$ for all $i \geq 0$,
determinant of $J_i(E^*)$

$$\begin{aligned}\det(J_i(E^*)) &= (-d_1\lambda_i - \theta - \xi r\Psi(j^*))\left(-d_2\lambda_i - \alpha + \xi r^*\Psi'(j^*)\right) + \xi^2 r^*\Psi'(j^*)\Psi(j^*).\\ &= d_1d_2\lambda_i^2 + \lambda_i H_0 + \det(J(r^*, j^*)), \text{for all } i \geq 0.\end{aligned}$$

Where

$$H_0 = -d_1\xi r^*\Psi'(j^*) + d_1\alpha + d_2\xi\Psi(j^*) + \theta d_2,$$

using (3.5) and (3.11), we get

$$\begin{aligned}H_0 &\geq -d_1\xi r^*\frac{\Psi(j^*)}{j^*} + d_1\xi r^*\frac{\Psi(j^*)}{j^*} + d_2\xi\Psi(j^*) + \theta d_2.\\ &= d_2(\xi\Psi(j^*) + \theta).\\ &= d_2\frac{\beta}{r^*} > 0\end{aligned}$$

Which is leading to $\det(J_i(E^*)) > 0$.
Hence, E^* is locally asymptomatically stable. ■

3.7 Global asymptotic stability

In this section, we study the global asymptotic stabilty, using E_0 and E^* that depends on the reproduction R_0 wich is why we decided to address scenarios $R_0 > 1$ and $R_0 < 1$ spuratedy. First, "but ", only, let as necessary *lemma* that will aid with the proofs to come.

Lemma 3.2 *Given that Ψ satisfies criterion (3.5) and*

$$L(x) = x - 1 - ln(x), \textit{forall } x > 0. \tag{3.22}$$

The inequality

$$L\left(\frac{\Psi(j)}{\Psi(j^*)}\right) \leq L\left(\frac{j}{j^*}\right), \tag{3.23}$$

where j^* is the second component of the equilibrium point E^*, holds.
Proof. *For $j > 0$, the function $\frac{\Psi(j)}{j}$ is a decreasing according to condition (3.5) this proof is separate into two region.* ■

first region : *suppose* $j \geq j^*$ *, we have*

$$\frac{\Psi(j)}{\Psi(j^*)} \leq \frac{j}{j^*},$$

by condition (3.5) Ψ *is non decreasing*

$$\Psi(j) \geq \Psi(j^*),$$

then ,

$$1 \leq \frac{\Psi(j)}{\Psi(j^*)} \leq \frac{j}{j^*},$$

we have $L(x)$ *is increasing for* $x > 1$ *, because* $L(x)$ *is postive in* $[1, \infty)$
then ,

$$L\left(\frac{\Psi(j)}{\Psi(j^*)}\right) \leq L\left(\frac{j}{j^*}\right), \text{ for all } j \geq j^*.$$

second region : *suppose* $0 < j < j^*$ *, we have*
$\frac{\Psi(j)}{\Psi(j^*)} > \frac{j}{j^*},$
and given the non decreasing nature of Ψ*, we end up with*

$$\Psi(j) < \Psi(j^*),$$

then

$$1 > \frac{\Psi(j)}{\Psi(j^*)} > \frac{j}{j^*} > 0,$$

we have $L(x)$ *is deceasing for* $0 < x < 1$.

$$L\left(\frac{\Psi(j)}{\Psi(j^*)}\right) < L\left(\frac{j}{j^*}\right) \text{ , for all } j > j^*.$$

3.7.1 *Global asymptotic stability with* $R_0 < 1$

We conseder $V_\delta(t)$ the Lyapunov function. To construction the global asymptotic of E_0 (DFE). Then,

$$V_\delta(t) = \int_\Omega \left[rj + \frac{\delta}{2}\left(r - \frac{\beta}{\theta}\right)^2 + \frac{1}{2}j^2 + \frac{\beta}{\alpha}j \right] dx, \text{ where } \delta > 0.$$

Theorem 3.3 *If* $R_0 < 1$*,* E_0 *is a globally asymptotically stable disease-free steady state for system* $(3.1) - (3.3)$
under the assumption

$$\Psi'(0) \leq \frac{\theta + \alpha}{\xi\left(\delta\frac{\beta}{\theta} + \frac{\beta}{\alpha}\right)}, \tag{3.24}$$

with

$$\delta \geq \frac{(d_1 + d_2)^2}{4d_1 d_2}. \tag{3.25}$$

Proof. we prove $V_\delta(t)$ is a lyapunov function . in ordre to verfi E_0 is Global asymptotic stable.
We have $V_\delta(t) > 0$ and $V_\delta(t) = 0$.
We prove that $\frac{dV_\delta(t)}{dt} \leq 0$, with resito time

$$\begin{aligned}
\frac{dV_\delta(t)}{dt} &= \frac{d}{dt}\left(\int_\Omega \left[rj + \frac{\delta}{2}\left(r - \frac{\beta}{\theta}\right)^2 + \frac{1}{2}j^2 + \frac{\beta}{\alpha}j\right] dx\right). \\
&= \int_\Omega \left(\frac{\partial r}{\partial t}j + r\frac{\partial j}{\partial t}\right) dx + \delta \int_\Omega \left(r - \frac{\beta}{\theta}\right)\frac{\partial r}{\partial t} + \int_\Omega j\frac{\partial j}{\partial t}dx + \frac{\beta}{\alpha}\int_\Omega \frac{\partial j}{\partial t}dx. \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By replacing the values of the partial derivative $\frac{\partial r}{\partial t}$ and $\frac{\partial j}{\partial t}$ with their corresponding valus from equation (3.1), we can obtain the following expressing :

$$\begin{aligned}
I_1 &= \int_\Omega \left(\frac{\partial r}{\partial t}j + r\frac{\partial j}{\partial t}\right) dx. \\
&= \int_\Omega \left(\left[d_1\Delta r + \beta - \theta r - \xi r\Psi\left(j\right)\right]j + r\left[d_2\Delta j - \alpha j + \xi r\Psi(j)\right]\right) dx. \\
&= d_1\int_\Omega \Delta r j dx + \beta\int_\Omega j dx - \theta\int_\Omega r j dx - \xi\int_\Omega r j\Psi\left(j\right)dx + d_2\int_\Omega r\Delta j dx \\
&\quad -\alpha\int_\Omega r j dx + \xi\int_\Omega r^2\Psi(j)dx,
\end{aligned}$$

we apply Green formula:

$$\begin{aligned}
= &-\left(d_1 + d_2\right)\int_\Omega \nabla r\nabla j dx + \beta\int_\Omega j dx - \xi\int_\Omega r j\Psi\left(j\right)dx + \xi\int_\Omega r^2\Psi(j)dx \\
&-\left(\theta + \alpha\right)\int_\Omega r j dx.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \delta\int_\Omega \left(r - \frac{\beta}{\theta}\right)\frac{\partial r}{\partial t}dx. \\
&= \delta\int_\Omega \left(r - \frac{\beta}{\theta}\right)\left[d_1\Delta r + \beta - \theta r - \xi r\Psi\left(j\right)\right]dx. \\
&= \delta d_1\int_\Omega r\Delta r dx + \delta\beta\int_\Omega r dx - \delta\theta\int_\Omega r^2 dx - \delta\xi\int_\Omega r^2\Psi\left(j\right)dx. \\
&\quad -\delta\frac{\beta}{\theta}d_1\int_\Omega \Delta r dx - \delta\frac{\beta^2}{\theta}\int_\Omega dx + \delta\beta\int_\Omega r dx + \delta\frac{\beta}{\theta}\xi\int_\Omega r\Psi\left(j\right)dx,
\end{aligned}$$

$$= -\delta d_1\int_\Omega \left|\nabla r\right|^2 dx - \theta\delta\int_\Omega \left(r - \frac{\beta}{\theta}\right)^2 dx - \xi\int_\Omega r^2\Psi\left(j\right)dx + \delta\frac{\beta}{\theta}\xi\int_\Omega r\Psi\left(j\right)dx.$$

$$\begin{aligned}
I_3 &= \int_\Omega j\frac{\partial j}{\partial t}dx + \frac{\beta}{\alpha}\int_\Omega \frac{\partial j}{\partial t}dx. \\
&= d_2\int_\Omega j\Delta j dx - \alpha\int_\Omega j^2 dx + \xi\int_\Omega rj\Psi(j)\,dx + \frac{\beta}{\alpha}d_2\int_\Omega \Delta j dx - \beta\int_\Omega j dx \\
&\quad + \frac{\beta}{\alpha}\xi\int_\Omega r\Psi(j)\,dx,
\end{aligned}$$

$$= -d_2\int_\Omega |\triangledown j|^2 dx + \xi\int_\Omega rj\Psi(j)\,dx - \alpha\int_\Omega j^2 dx + \frac{\beta}{\alpha}\xi\int_\Omega r\Psi(j)\,dx - \beta\int_\Omega j dx.$$

we implies :

$$\begin{aligned}
\frac{dV_\delta(t)}{dt} &= -\delta d_1\int_\Omega |\nabla r|^2 dx - (d_1+d_2)\int_\Omega \nabla r\nabla j dx - d_2\int_\Omega |\triangledown j|^2 dx + \xi(\beta-\delta)\int_\Omega r^2\Psi(j)\,dx \\
&\quad -(\alpha+\theta)\int_\Omega rj dx - \theta\delta\int_\Omega\left(r-\frac{\beta}{\theta}\right)^2 dx - \alpha\int_\Omega j^2 dx \\
&\quad +\xi\left(\delta\frac{\beta}{\theta}-\frac{\beta}{\alpha}\right)\int_\Omega r\Psi(j)\,dx \\
&= I + J.
\end{aligned}$$

First part is:

$$I = -\delta d_1\int_\Omega |\nabla r|^2 dx - (d_1+d_2)\int_\Omega \nabla r\nabla j dx - d_2\int_\Omega |\triangledown j|^2 dx.$$

Second part is:

$$\begin{aligned}
J &= \xi(\beta-\delta)\int_\Omega r^2\Psi(j)\,dx - (\alpha+\theta)\int_\Omega rj dx - \theta\delta\int_\Omega\left(r-\frac{\beta}{\theta}\right)^2 dx \\
&\quad -\alpha\int_\Omega j^2 dx + \xi\left(\delta\frac{\beta}{\theta}-\frac{\beta}{\alpha}\right)\int_\Omega r\Psi(j)\,dx.
\end{aligned}$$

We writing the first part

$$\begin{aligned}
I &= -\int_\Omega Q(\nabla r,\nabla j)\,dx. \\
&= -\int_\Omega \left(\delta d_1|\nabla r|^2 + (d_1+d_2)\nabla r\nabla j + d_2|\triangledown j|^2\right)dx. \\
Q(\nabla r,\nabla j) &= \left(\delta d_1|\nabla r|^2 + (d_1+d_2)\nabla r\nabla j + d_2|\triangledown j|^2\right).
\end{aligned}$$

Where $Q(\nabla r,\nabla j)$ is a quadratic form.
As we know Q is postive because δ, d_1 and d_2 are satisfying the condition δd_1 and $\delta \geq \frac{(d_1+d_2)^2}{4d_1d_2}$ so:

$$I \leq 0.$$

Now, the second part by using the inequality $\delta \geq \frac{(d_1+d_2)^2}{4d_1d_2} \geq 1$, we have,

$$\begin{aligned} J &= \xi(\beta-\delta)\int_\Omega r^2\Psi(j)\,dx - (\alpha+\theta)\int_\Omega rj dx - \theta\delta\int_\Omega \left(r-\frac{\beta}{\theta}\right)^2 dx \\ &\quad -\alpha\int_\Omega j^2 dx + \xi\left(\delta\frac{\beta}{\theta}-\frac{\beta}{\alpha}\right)\int_\Omega r\Psi(j)\,dx, \end{aligned}$$

we appling *(lemma 3.1)* yields.

$$J \leq \int_\Omega \left[\xi\left(\delta\frac{\beta}{\theta}-\frac{\beta}{\alpha}\right)\Psi'(0) - (\theta+\alpha)\right] rj dx - \theta\delta\int_\Omega\left(r-\frac{\beta}{\theta}\right)^2 dx - \alpha\int_\Omega j^2 dx \leq 0. \tag{3.26}$$

Hence,$V_\delta(t)$ is a Lyapunov functional and E_0 is globally asymptotically stable . ■

3.7.2 *Global asymptotic stability with* $R_0 > 1$

Theorem 3.4 *If* $R_0 > 1$ *.Then* E^* *is a globally asymptotic stable endemic steady-state for system* $(3.1)-(3.3)$.

Proof. to prove that we use Lyapunov function

$$V(t) = \int_\Omega \left[r^* L\left(\frac{r}{r^*}\right) + j^* L\left(\frac{j}{j^*}\right)\right] dx. \tag{3.27}$$

Where it is positive and continously differentiable function, we have

$$L\left(\frac{r}{r^*}\right) = \frac{r}{r^*} - 1 - \ln\left(\frac{r}{r^*}\right).$$

So

$$\frac{dL}{dt}\left(\frac{r}{r^*}\right) = \frac{1}{r^*}\frac{dr}{dt} - \frac{1}{r^*}\frac{dr}{dt}\frac{r^*}{r} = \frac{1}{r^*}\left(1-\frac{r^*}{r}\right)\frac{dr}{dt},$$

we conclude

$$\frac{dL}{dt}\left(\frac{j}{j^*}\right) = \frac{1}{j^*}\left(1-\frac{j^*}{j}\right)\frac{dj}{dt}.$$

Now, we differentiate $V(t)$ with respect to time

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_\Omega r^*\frac{dL}{dt}\left(\frac{r}{r^*}\right)dx + \int_\Omega j^*\frac{dL}{dt}\left(\frac{j}{j^*}\right)dx, \\ &= \int_\Omega r^*\frac{1}{r^*}\left(1-\frac{r^*}{r}\right)\frac{dr}{dt}dx + \int_\Omega j^*\frac{1}{j^*}\left(1-\frac{j^*}{j}\right)\frac{dj}{dt}dx, \\ &= \int_\Omega\left(1-\frac{r^*}{r}\right)\frac{dr}{dt}dx + \int_\Omega\left(1-\frac{j^*}{j}\right)\frac{dj}{dt}dx \end{aligned}$$

by replacing the values of partial derivatives $\frac{dr}{dt}$ and $\frac{dj}{dt}$ from equation (3.1):

$$\begin{aligned}
\frac{dV(t)}{dt} &= \int_\Omega \left(1-\frac{r^*}{r}\right)\left[d_1\Delta r+\beta-\theta r-\xi r\Psi(j)\right]dx \\
&+\int_\Omega \left(1-\frac{j^*}{j}\right)\left[d_2\Delta j-\alpha j+\xi r\Psi(j)\right]dx. \\
&= d_1\int_\Omega \left(1-\frac{r^*}{r}\right)\Delta r dx+\beta\int_\Omega\left(1-\frac{r^*}{r}\right)dx \\
&-\theta\int_\Omega r\left(1-\frac{r^*}{r}\right)dx-\xi\int_\Omega\left(1-\frac{r^*}{r}\right)r\Psi(j)\,dx \\
&+d_2\int_\Omega\left(1-\frac{j^*}{j}\right)\Delta j dx-\alpha\int_\Omega\left(1-\frac{j^*}{j}\right)jdx+\xi\int_\Omega\left(1-\frac{j^*}{j}\right)r\Psi(j)dx,
\end{aligned}$$

we use the Green formula and Neuman boundaries

$$\begin{aligned}
\frac{dV(t)}{dt} &= -d_1\int_\Omega \bigtriangledown\left(1-\frac{r^*}{r}\right)\bigtriangledown r dx+\beta\int_\Omega\left(1-\frac{r^*}{r}\right)dx-\xi\int_\Omega\left(1-\frac{r^*}{r}\right)r\Psi(j)\,dx \\
&-\theta\int_\Omega r\left(1-\frac{r^*}{r}\right)dx \\
&-d_2\int_\Omega\bigtriangledown\left(1-\frac{j^*}{j}\right)\bigtriangledown j dx+\xi\int_\Omega\left(1-\frac{j^*}{j}\right)r\Psi(j)dx-\alpha\int_\Omega\left(1-\frac{j^*}{j}\right)jdx \\
&= I+J.
\end{aligned}$$

Where

$$\begin{aligned}
I &= -d_1\int_\Omega\bigtriangledown\left(1-\frac{r^*}{r}\right)\bigtriangledown r dx-d_2\int_\Omega\bigtriangledown\left(1-\frac{j^*}{j}\right)\bigtriangledown j dx, \\
&= -\int_\Omega\left[d_1\frac{r^*}{r}\left|\bigtriangledown r\right|^2+d_2\frac{j^*}{j}\left|\bigtriangledown j\right|^2\right]dx\le 0,
\end{aligned} \tag{3.28}$$

and

$$J=\int_\Omega\left(1-\frac{r^*}{r}\right)\left[\beta-\theta r-\xi r\Psi(j)\right]dx+\int_\Omega\left(1-\frac{j^*}{j}\right)\left[-\alpha j+\xi r\Psi(j)\right]dx. \tag{3.29}$$

We simplifying the resulting equation, after using (3.11).

$$\begin{aligned}
J &= \int_\Omega\left(1-\frac{r^*}{r}\right)\xi r^*\Psi(j^*)\,dx+\int_\Omega\left(1-\frac{r^*}{r}\right)\theta j^* dx-\int_\Omega\left(1-\frac{r^*}{r}\right)\xi r\Psi(j)\,dx \\
&-\int_\Omega\left(1-\frac{r^*}{r}\right)\theta r dx+\int_\Omega\left(1-\frac{j^*}{j}\right)\xi r\Psi(j)\,dx-\int_\Omega\frac{\xi r^*\Psi(j^*)}{j^*}j\left(1-\frac{j^*}{j}\right)dx, \\
&= \int_\Omega\left(1-\frac{r^*}{r}\right)\left[1-\frac{r\Psi(j)}{r^*\Psi(j^*)}\right]\xi r^*\Psi(j^*)\,dx+\int_\Omega\left[\left(1-\frac{r^*}{r}\right)\left(1-\frac{r}{r^*}\right)\right]\theta r^* dx \\
&+\int_\Omega\xi r^*\Psi(j*)\left(1-\frac{j^*}{j}\right)\left[\frac{r\Psi(j)}{r^*\Psi(j^*)}-\frac{j}{j^*}\right]dx.
\end{aligned}$$

$$J=\int_\Omega\left|\theta r^* J_1+\xi r^*\Psi(j^*)\,J_2\right|. \tag{3.30}$$

where

$$\begin{aligned} J_1 &= \left(1-\frac{r^*}{r}\right)\left(1-\frac{r}{r^*}\right) \\ &= 1-\frac{r}{r^*}-\frac{r^*}{r}+1 \\ &= 1-\frac{r}{r^*}+\ln\left(\frac{r}{r^*}\right)-\ln\left(\frac{r}{r^*}\right)+1-\frac{r^*}{r}+\ln\left(\frac{r^*}{r}\right)-\ln\left(\frac{r^*}{r}\right) \\ &= -L\left(\frac{r}{r^*}\right)-L\left(\frac{r^*}{r}\right). \end{aligned}$$

Then

$$\begin{aligned} J_2 &= \left[\frac{r\Psi(j)}{r^*\Psi(j^*)}-\frac{j}{j^*}\right]\left(1-\frac{j^*}{j}\right)+\left(1-\frac{r^*}{r}\right)\left[1-\frac{r\Psi(j)}{r^*\Psi(j^*)}\right]. \\ &= -\frac{r\Psi(j)}{r^*\Psi(j^*)}-\frac{j}{j^*}-\frac{r^*}{r}+\frac{\Psi(j)}{\Psi(j^*)}+2. \\ &= 1-\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}+\ln\left(\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}\right)-\ln\left(\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}\right)+1-\frac{j}{j^*}+\ln\left(\frac{j}{j^*}\right) \\ &\quad -1+1-\frac{r^*}{r}+\ln\left(\frac{r^*}{r}\right)-\ln\left(\frac{r^*}{r}\right)-1+1+\frac{\Psi(j)}{\Psi(j^*)}-\ln\left(\frac{\Psi(j)}{\Psi(j^*)}\right) \\ &\quad +\ln\left(\frac{\Psi(j)}{\Psi(j^*)}\right). \\ &= -L\left(\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}\right)-L\left(\frac{j}{j^*}\right)-L\left(\frac{r^*}{r}\right)+L\left(\frac{\Psi(j)}{\Psi(j^*)}\right). \end{aligned}$$

We remplacing in J we given

$$\begin{aligned} J &= -\theta r^*\int_\Omega L\left(\frac{r^*}{r}\right)+L\left(\frac{r}{r^*}\right)dx+\xi r^*\Psi(j^*)\int_\Omega\left(-L\left(\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}\right)\right)-L\left(\frac{j}{j^*}\right) \\ &\quad -L\left(\frac{r^*}{r}\right)+L\left(\frac{\Psi(j)}{\Psi(j^*)}\right)dx \\ &= -\theta r^*\int_\Omega L\left(\frac{r^*}{r}\right)+L\left(\frac{r}{r^*}\right)dx-\xi r^*\Psi(j^*)\int_\Omega\left[L\left(\frac{r\Psi(j)\,j^*}{r^*\Psi(j^*)\,j}\right)+L\left(\frac{j}{j^*}\right)\right]dx \\ &\quad +\xi r^*\Psi(j^*)\int_\Omega\left[L\left(\frac{\Psi(j)}{\Psi(j^*)}\right)-L\left(\frac{j}{j^*}\right)\right]dx. \end{aligned}$$

We have the postivity of L and applying *(lemma 3.2)* , thus $J \leq 0$ we prove $\frac{d}{dt}V(t) \leq 0$. Hence, $V_\delta(t)$ is a lyapunov functional and E^* is globally asymptotically stable. ■

Finally, our choice of a function $r\Psi(j)$ was correct, and this has been proven throughout this chapter.

3.8 Numerical part

Example 3.1 *we consider the function* $\Psi(j) = nj$, *and we obtain*

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \xi rnj - \theta r & \text{in } (0,+\infty)\times\Omega. \\ \frac{\partial j}{\partial t} - d_2 \Delta j = -\alpha j + \xi rnj & \text{in } (0,+\infty)\times\Omega. \\ r_0(x) = r(x,0), j_0(x) = j(x,0) & \text{in } \Omega\,. \\ \frac{\partial r}{\partial \nu} = \frac{\partial j}{\partial \nu} = 0 & \text{in } (0,+\infty)\times\Omega. \end{cases} \tag{3.31}$$

The imposed conditions may be verified as follows:

$$\begin{cases} \Psi(0) = 0, \Psi'(j) = n = 1. \\ j\Psi'(j) = nj \le nj = \Psi(j)\,. \end{cases}$$

The steady states of system (3.31) *are given by* $E_0 = \left(\frac{\beta}{\theta}, 0\right)$ *and* $E^* = \left(\frac{\alpha}{\xi n}, \theta\alpha(R_0 - 1)\right)$ *with the reproductive number* $R_0 = \frac{\xi\beta}{\theta\alpha} n > 1$. *In the table bollow, we use different sets of parameters to obtain numerical solutions in the ODE .*

Table: *Simulation parameters for the Example 3.1*

Set	r_0	j_0	d_1	d_2	ξ	α	θ	β
ODE set 1	2.5	7	-	-	0.8	0.3	0.4	0.2
ODE set 2	6	2.5	-	-	0.8	0.6	0.1	0.3

The following is a description of the results:

Figure 1 *: shows the solutions in the ODE case subject to set 1, with* $R_0 = 1.33$. *In this case, as* $R_0 > 1, E^* = (0.18, 0.19)$ *is globally asymptotically stable.*

Figure 2 *: shows the solutions in the ODE case subject to set 2, with* $R_0 = 4$. *In this case, as* $R_0 > 1, E^* = (0.63, 0.52)$ *is globally asymptotically stable.*

Example 3.2 *we consider the function* $\Psi(j) = \frac{nrj}{1+kj}$, *and we obtain*

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \xi \frac{nrj}{1+kj} - \theta r & \text{in } (0,+\infty)\times\Omega. \\ \frac{\partial j}{\partial t} - d_2 \Delta j = -\alpha j + \xi \frac{nrj}{1+kj} & \text{in } (0,+\infty)\times\Omega. \\ r_0(x) = r(x,0), j_0(x) = j(x,0) & \text{in } \Omega. \\ \frac{\partial r}{\partial \nu} = \frac{\partial j}{\partial \nu} = 0 & \text{in } (0,+\infty)\times\Omega. \end{cases} \tag{3.32}$$

Verifying the function $\Psi(j)$ *satisfies the requirements (3.4) and (3.5)*

$$\begin{cases} \Psi(0) = 0. \\ \Psi(J) > 0. \\ \Psi'(J) = \frac{n}{(1+kj)^2}, n > 0, \Psi'(0) = n. \end{cases}$$

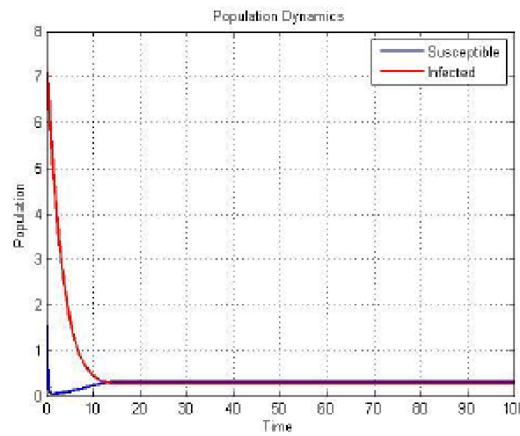

Figure 3.1: Numerical solutions ofsystem (3.31) (ODE case) subject to the first set

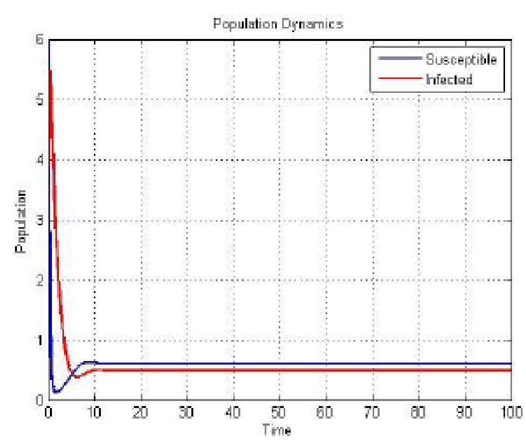

Figure 3.2: Figure 2: Numerical solutions of system (3.31) (ODE case) subject to the scond set

determining the system's stable states (3.32)

▷ *if* $r = 0$*, the system* (3.32) *has no equilibrium.*

▷ *if* $j = 0, E_0 = \left(\frac{\beta}{\theta}, 0\right)$ *is equilibrium.*

Now, we find E^*

$$-\alpha j^* + \xi \frac{nr^* j^*}{1 + kj^*} = 0 \,,$$

implies that

$$r^* = \frac{\alpha \left(1 + kj^*\right)}{\xi n},$$

we have

$$\beta - \theta r^* - \alpha j^* = 0,$$

we replace r^* *this equation*

$$\alpha j^* = \beta - \theta \left(\frac{\alpha \left(1 + kj^*\right)}{\xi n} \right),$$

we find

$$j^* = \frac{\theta \left(\frac{n\xi\beta - 1}{\theta\alpha} - 1\right)}{n\xi + \theta k} \textit{implies that } j^* = \frac{\theta \left(R_0 - 1\right)}{n\xi + \theta k},$$

So

$$E^* = \left(\frac{\alpha \left(1 + kj^*\right)}{\xi n}, \frac{\theta \left(R_0 - 1\right)}{n\xi + \theta k} \right).$$

▷E^* *exists and it is globally asymptotically stable provided that the reproduction number* $R_0 > 1$.

▷E_0 *is globally asymptotically stable when* $R_0 < 1$ *with* $\frac{(d_1+d_2)^2}{4d_1 d_2} \leq \gamma \leq \frac{\theta}{\beta}\left(\frac{\theta+\alpha}{\xi n} - \frac{\beta}{\alpha}\right)$ *when* $d_1 \neq d_2$.

Table: *Simulation parameters for the Example 3.2*

Set	r_0	j_0	d_1	d_2	ξ	n	α	β	θ	k
ODE set 1	0.1	3.2	-	-	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{7}{4}$	$\frac{22}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
ODE set 2	0.1	3.2	-	-	3	$\frac{6}{5}$	2	7	3	7
PDE set 1	$0.2 + \frac{\cos\frac{\pi}{3}}{10}$	$4.3 + \frac{\cos\frac{\pi}{3}}{10}$	3	2	$\frac{7}{12}$	$\frac{13}{4}$	$\frac{9}{4}$	$\frac{33}{4}$	$\frac{5}{4}$	$\frac{1}{2}$
PDE set 2	$0.2 + \frac{\cos\frac{\pi}{3}}{10}$	$4.3 + \frac{\cos\frac{\pi}{3}}{10}$	$\frac{7}{2}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{6}{5}$	2	7	3	7

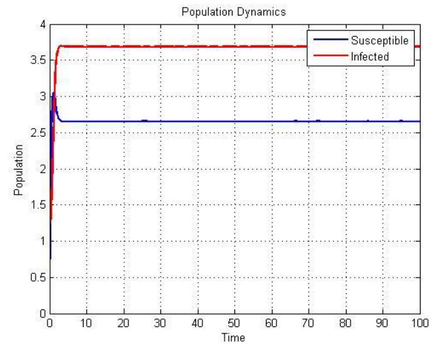

Figure 3: Numerical solutions of system (3.32) (ODE case) subject to the first

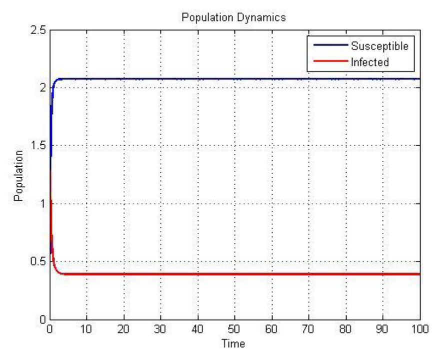

Figure 4 :Numerical solutions of system (3.32) (ODE case) subject to the scond

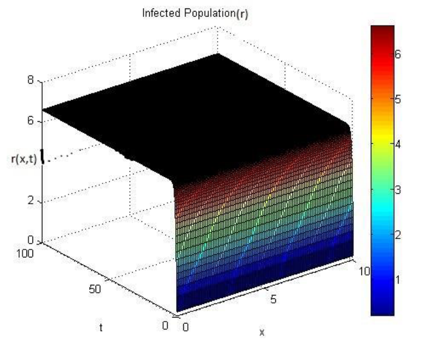

Figure 5: Numerical solutions of system (3.32) susceptible population (PDE case)

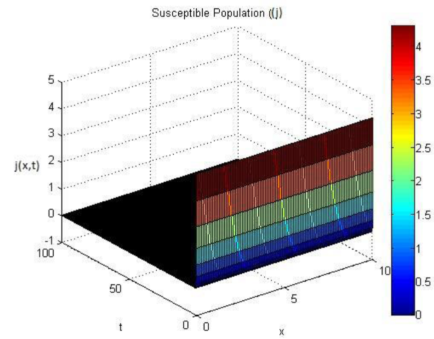

Figure 6: Numerical solutions of system (3.32) susceptible population (PDE case)

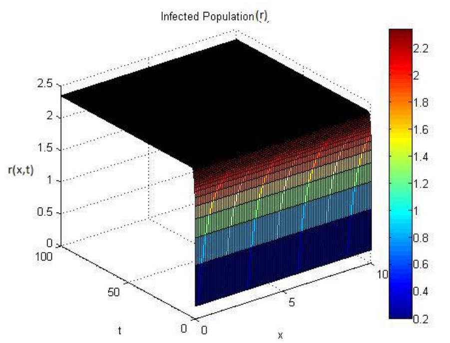

Figure 7: Numerical solutions of system (3.32) susceptible population (PDE case)

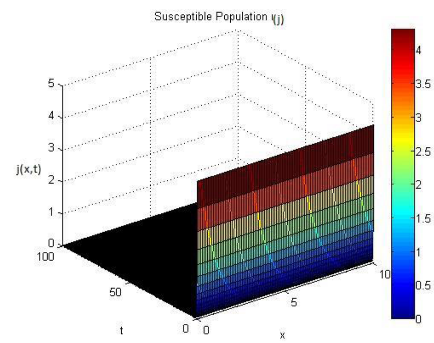

Figure 8: Numerical solutions of system (3.32) susceptible population (PDE case)

Figure 3: *shows the solutions in the ODE case subject to set 1, with* $R_0 = 22.08$. *In this case, as* $R_0 > 1, E^* = (2.696, 3.71)$ *is globally asymptotically stable.*
Figure 4 : *shows the solutions in the ODE case subject to set 2, with* $R_0 = 6.722$. *In this case, as* $R_0 > 1, E^* = (2.13, 0.4878)$ *is globally asymptotically stable.*
Figure 5,6 : *depicts the solution in the PDE case subject to parameter set* 1*, where* $R_0 = 5.205$, *means that* $E^* = (0.13, 4.151)$ *is globally asymptotically stable.*
Figure 7,8: *depicts the solution in the PDE case subject to parameter set* 2*, where* $R_0 = 1.933$*, which by Theorem (3-4) means that* $E^* = (0.23, 4.2)$ *is globally asymptotically stable.*

Example 3.3 *The last example we consider the function* $\Psi(j) = \frac{kj}{1+\left(\frac{j}{n}\right)}$, *with* $k = \frac{5}{6}, n = 1$, *The* *resulting system is given by*

$$\begin{cases} \frac{\partial r}{\partial t} - d_1 \Delta r = \beta - \xi \frac{kj}{1+\left(\frac{j}{n}\right)} r - \theta r & \textit{in } (0,+\infty) \times \Omega. \\ \frac{\partial j}{\partial t} - d_2 \Delta j = -\alpha j + \xi \frac{kj}{1+\left(\frac{j}{n}\right)} r & \textit{in } (0,+\infty) \times \Omega. \\ r_0(x) = r(x,0), j_0(x) = j(x,0) & \textit{in } \Omega. \\ \frac{\partial r}{\partial \nu} = \frac{\partial j}{\partial \nu} = 0 & \textit{in } (0,+\infty) \times \Omega. \end{cases} \tag{3.33}$$

Verifying the function $\Psi(j)$ *satisfies the requirements (3.4)and (3.5)*

$$\begin{cases} \Psi(0) = 0. \\ \Psi(J) > 0. \\ \Psi'(J) = \frac{k}{\left(1+\left(\frac{j}{n}\right)\right)^2} > 0, \Psi'(0) = k. \\ j\Psi'(J) = j\frac{k}{\left(1+\left(\frac{j}{n}\right)\right)^2} \leq \Psi(J) = \frac{kj}{1+\left(\frac{j}{n}\right)}. \end{cases}$$

Determining the system's stable states (3.33)

$$\begin{cases} \beta - \xi \frac{kj^*}{1+\left(\frac{j^*}{n}\right)} r^* - \theta r^* & = 0. \\ -\alpha j^* + \xi \frac{kj^*}{1+\left(\frac{j^*}{n}\right)} r^* & = 0. \end{cases} \tag{3.34}$$

$\triangleright$ *if* $r = 0$,*the system* (3.33) *has no equilibrium .*

$\triangleright$ *if* $j = 0, E_0 = \left(\frac{\beta}{\theta}, 0\right)$ *is equilibrium .*

Now, we find E^*.

We derive from the second equation

$$r^* = \frac{\alpha(n+j^*)}{\xi nk},$$

The first and second equations of (3.34) *are added up.*

$$\beta - \theta r^* - \alpha j^* = 0,$$

the value of r^*

$$\beta - \theta \frac{\alpha(n+j^*)}{\xi nk} - \alpha j^* = 0,$$

we find

$$j^* = \frac{\theta n\left(\frac{k\xi\beta}{\theta\alpha} - 1\right)}{(n\xi k + \theta)},$$

we have $R_0 = k\frac{\xi\beta}{\theta\alpha}$ *so*

$$j^* = \frac{\theta n(R_0 - 1)}{(n\xi k + \theta)}.$$

Two of the system's stable states (3.33) *are provided by*

$$\begin{cases} E_0 = \left(\frac{\beta}{\theta}, 0\right). \\ E^* = \left(\frac{\alpha(n+j^*)}{\xi n k}, \frac{\theta n(R_0 - 1)}{(n\xi k + \theta)}\right). \end{cases}$$

▷E^**exists and is globally asymptotically stable .*
▷E_0 *is globally asymptotically stable if* $\frac{(d_1+d_2)^2}{4d_1d_2} \leq \gamma \leq \frac{\theta}{\beta}\left(\frac{\theta+\alpha}{\xi n} - \frac{\beta}{\alpha}\right)$ *when* $d_1 \neq d_2$.
Table: *Simulation parameters for the Example 3.3*

Set	r_0	j_0	d_1	d_2	ξ	α	β	θ	n	k
ODE set 1	2	8	-	-	$\frac{3}{2}$	0.33	0.35	$\frac{1}{2}$	1	$\frac{5}{6}$
ODE set 2	6	$\frac{3}{2}$	-	-	3	$\frac{7}{100}$	0.7	$\frac{2}{100}$	1	$\frac{5}{6}$
PDE set 1	$0.3 + \frac{\cos\frac{\pi}{3}}{9}$	$1.5 + \frac{\sin\frac{\pi}{3}}{10}$	$\frac{4}{7}$	3	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{7}{4}$	$\frac{14}{9}$	$\frac{5}{6}$	$\frac{5}{6}$
PDE set 2	$0.3 + \frac{\cos\frac{\pi}{3}}{9}$	$0.2 + \frac{\sin\frac{\pi}{3}}{12}$	8	$\frac{15}{7}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$

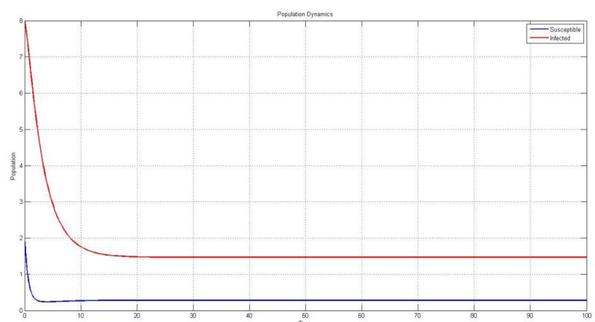

Figure 9: Numerical solutions of system (3.33) (ODE case) subject to the first set

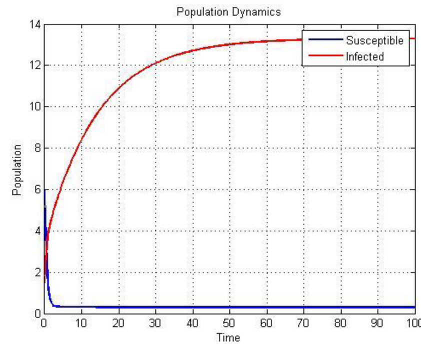

Figure 10: Numerical solutions of system (3.33) (ODE case) subject to the scond set

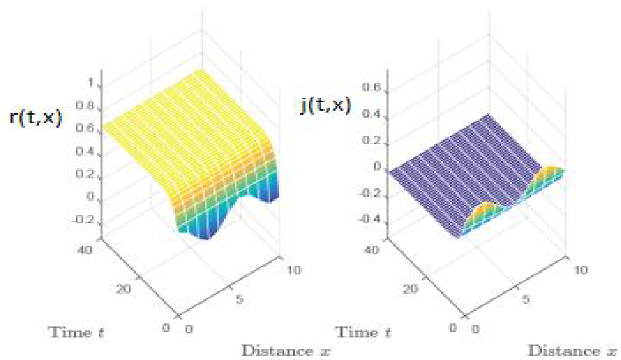

Figure 11: Numerical solutions of system (3.33) susceptible population (PDE case)

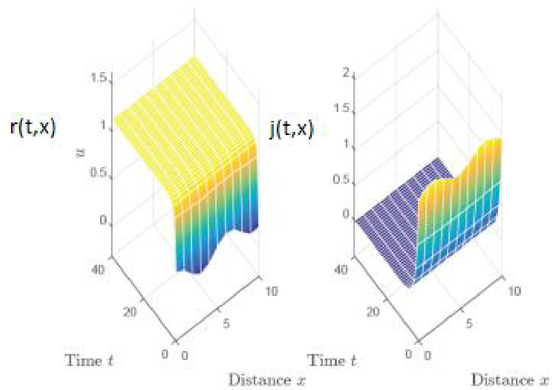

Figure 12: Numerical solutions of system (3.33) susceptible population (PDE case)

***Figure 9**: shows the solutions in the ODE case subject to set 1, with $R_0 = 2.6515$. In this case, as $R_0 > 1, E^* = (1.1885, 2.4918)$ is globally asymptotically stable.*
***Figure 10**: shows the solutions in the ODE case subject to set 2, with $R_0 = 1250$. In this case, as $R_0 > 1, E^* = (0.42, 13.31)$ is globally asymptotically stable.*
***Figure 11**: depicts the solution in the PDE case subject to parameter set 1, where $R_0 = 0.2813$. By Theorem (3-3) and given $\theta > \frac{625}{336}, E_0 = (0.667, 0)$ is globally asymptotically stable.*
***Figure 12 :** depicts the solution in the PDE case subject to parameter set 2, where $R_0 = 0.2$, which by Theorem (3-3) and given $\theta > \frac{5041}{3360}$,means that $E_0 = (1.1250, 0)$ is globally asymptotically stable.*

conclusion

We have reached the culmination of our mathematical scientific investigation, focusing on the stability analysis of an epidemic reaction-diffusion system. Our journey commenced by delving into foundational concepts and theories pertaining to global and local asymptotic stability. We then proceeded to introduce the most generalized form of a reaction-diffusion system before delving into a specific model addressing the dynamics of an epidemiological system (comprising susceptible and infectious populations) featuring a non-linear incidence function under the conditions $(3.4)$and (3.5). Central to our discourse was the determination of R_0, the basic reproduction number, which served as the focal point for our discussions. In the realm of Ordinary Differential Equations ($ODEs$), we established that the disease-free equilibrium attains asymptotic stability when R_0 is less than unity, whereas the endemic equilibrium achieves asymptotic stability when

R_0 exceeds unity. By applying the *Lyapunov function*, we extended our analysis to Partial Differential Equations ($PDEs$) to ascertain the global stability of the system, subsequently corroborating our findings through numerical simulations.
And. If R_0 is equal to one, it requires studying in other ways.

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