People's Democratic Republic of Algeria Ministry of higher education and Scientific Research
Larbi Tebessi University – Tebessa
Faculty of Exact Sciences and Natural and Life Sciences
***Departement :** Mathematics*

End of study dissertation for obtaining the master's degree
***Domain :** Mathematics and Informatics*
***Field :** Mathematics*
***Option :** PDE and Applications*

Theme:

Delayed Differential Equation in Epidemiology

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Academic year : 2023 - 2024

بسم الله الرحمن الرحيم

شكر وتقدير

اليوم تصل الرحلة لنهايتها، لقد كانت طويلة مليئة بالمطبات و صعبة في كثير من الأحيان، ولكنها أيضا كانت محاطة بالحب و مع ذلك فهي ليست سوى خطوة على درب الحياة.
أود في المقام الأول أن أشكر الله تعالى الذي منحني الصبر و الشجاعة لإكمال هذا العمل. كما أود ان أعرب عن خالص امتناني و شكري للأستاذ المشرف - **عبد الرزاق نابتي** - على مساعدته ونصائحه القيمة التي كانت ضرورية لإنجاز هذا العمل.

أتوجه بالشكر الجزيل إلى أعضاء لجنة التحكيم الذين وافقو على تحكيم مذكرتي كل من الأستاذ - **بوعزيز خليفة** - رئيسا للجنة و الأستاذ - **الميطة سمير** - ممتحن الذين لا شك أنهما سيفيضون علي بتوجيهاتهما القيمة و ملاحظتهما السديدة.

كما أتقدم بالشكر الجزيل إلى جميع أساتذتي و كل من ساهم بكلمة أو نصيحة أو نقد في توجيه تفكيري خلال مسيرتي الدراسية.
ولا أنسى شكر زملائي الطلبة و خاصة طلبة ماستر ٢ دفعة ٢٠٢٤، راجين من المولى العلي القدير كل التوفيق و الفلاح.
جزيل الشكر و الإمتنان للجميع.

* قتال وئام *

إهـــــداء

الحمد لله حبا و شكرا و امتنانا على البدء و الختام
﴿وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ﴾

وبكل حب أهدي ثمرة نجاحي و تخرجي إلى من أحمل اسمه بكل فخر إلى قدوتي في الحياة إلى من علمني أن الدنيا صراع وسلاحها العلم و المعرفة
والدي حفظه الله.

إلى المرأة التي جعلت مني فتاة طموحة و سهلت علي بدعائها الخفي إلى القلب الحنون
أمي حفظها الله.

إلى رفاق دربي و أصدقائي و قرة عيني و الأغلى على قلبي إخوتي : منال، هبة الله، نسرين، معتز، عبد السميع، عبد العليم
حفظهم الله.

إلى كل الأصدقاء و الزملاء الأعزاء التي جمعتني بهم الدراسة. خصوصا شريكات النجاح حبيبات قلبي : عويشات جيهان، بخوش رحمة، غانم رباب، وفقكم الله.

* قتال وئام *

ملخص

في هذه المذكرة نتناول موضوع التأخيرات الزمنية التي تظهر عادة في المعادلات التفاضلية العادية. الغرض من هذه المذكرة هو التطرق لأنواع المشاكل التي لا تحلها المعادلات التفاضلية العادية و احتاجت للمعادلات التفاضلية التأخيرية لحلها، و دراسة بعض النتائج الرئيسية للمعادلة التفاضلية ذات تأخير الزمني لما يوافقها من نتائج في المعادلات التفاضلية العادية، وعلى وجه الخصوص ينصب التركيز على المعادلات التفاضلية ذات التأخير الثابت. و كذلك سنتطرق لمفهوم الاستقرار في المعادلات التفاضلية ذات التأخير الزمني.

الكلمات المفتاحية: معادلات تفاضلية ذات تأخير زمني، (DDEs) ، استقرار، وجود ووحدانية الحل، تأخير ثابت.

Résumé

Dans ce mémoire, nous abordons le sujet des retards temporels qui apparaissent généralement dans les équations différentielles ordinaires.
L'objectif de ce mémoire est d'examiner les types de problèmes que les équations différentielles ordinaires ne parviennent pas à résoudre et qui ont nécessité l'utilisation d'équations différentielles à retardement pour les résoudre. Nous étudions également certains des principaux résultats de l'équation différentielle à retardement en les comparant aux résultats des équations différentielles ordinaires. En particulier, nous nous concentrons sur les équations différentielles à retardement constant. De plus, nous aborderons le concept de stabilité dans les équations différentielles à retardement temporel.

Mots-clés : Équations différentielles à retard, (EDRET), Stabilité, existence et unicité des solutions, Retard constant.

Abstract

In this thesis, we address the issue of time delays that commonly appear in ordinary differential equations (ODEs).
The purpose of this thesis is to explore the types of problems that cannot be solved by ODEs and require delay differential equations (DDEs) for their solution. We will also study some of the key results of DDEs that correspond to their counterparts in ODEs, with a particular focus on DDEs with constant delay.
Additionally, we will discuss the concept of stability in delayed differential equations DDEs.

Keywords : Differential equations with time delay, (DDEs), Stability, Existence and uniqueness of solutions, Constant delay.

Contents

Introduction

Ordinary differential equations (ODEs) and delay differential equations (DDEs) are used to describe a wide range of physical phenomena. While ODEs involve derivatives that depend on the solution of the equation at the current value of the independent variable "time", in DDEs the evolution of the system at a given time depends on the time history (past memory) [22]. In other words, DDEs take into account the effect of the past on predicting the future.

Delay differential equations (DDEs) are widely used for modeling and prediction in various fields of life sciences, including population dynamics, epidemiology, immunology, physiology, and neural networks [2] [19]. While delay is often neglected in some models, it has been shown to have a significant impact in many cases, and results that take delay into account are often more accurate and realistic.

These time delays can be associated with the duration of specific processes such as the time between cell infection and the production of new viruses, the time between the infectious period and the immune period, and others [20].

The importance of delay differential equations lies in understanding complex phenomena, predicting future behavior, and improving system models.

Delay Differential Equations (DDEs) stand out as an initial step towards generalizing Ordinary Differential Equation (ODE) Theory. These equations are characterized by the following formal expression: $\dot{x} = f(t, x(t), x(t - \tau))$.

Despite the apparent similarity between delay differential equations (DDEs) and ordinary differential equations (ODEs), the solutions of DDE problems can differ from ODE problems in fundamental and significant ways. Many of the concepts and tools used in (ODEs) can be applied to (DDEs), and this is the subject of this thesis with stability study.

In addition to an introduction and a references section, the thesis consists of three chapters:
Chapter 1: We will present some of the basic concepts and theories that we need to study the rest of the thesis.
Chapter 2: We begin by providing an overview of ordinary differential equations (ODEs). We then delve into the study of some key results of delay differential equations (DDEs) through seven section as follows:

- We will establish the existence and uniqueness of a solution to the equation

$$\begin{cases} \dot{x} = f\left(t, x_t\right) \\ x_{t_0} \equiv u. \end{cases} \tag{0.1}$$

  using the fixed-point theorem in Banach space.

- We study the existence and uniqueness of the maximal solutionof equation (0.1).

- Prove that the solution of equation (0.1) is continuous.

- We study the continuity and differentiability of the solution of Equation (0.1).

- Address the concept of the solution map.

- We will study the existence and uniqueness of the solution for the linear system of DDEs.

- We will introduce the different types of delay differential equations (DDEs), with a particular focus on DDEs with constant delay. We will discuss how to utilize the properties of ordinary differential equations (ODEs) to solve this type of DDE.

Chapter 3: In this chapter, we study the concept of stability of solutions in the context of differential equations, and apply Floquet theory to linear delay differential equations, and provide an example of analyzing the stability of the SIR mathematical model with delay.

The primary references used in this thesis are [10] [9] [11] [14] .

CHAPTER 1

Preliminary

In this chapter, we introduce some basic concepts and notions that will be essential for the following chapters of this master's thesis.

1.1 General definitions

Definition 1.1 *(**Continuous function**) ([18], page 109) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be **continuous** if for each $a \in \mathbb{R}$ and each positive number ε, there exists a positive real number δ such that $|x - a| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.*

Definition 1.2 *(**Uniformly continuous**) ([15], page 13) Let $X, Y \subseteq \mathbb{R}$, the function $f : X \to Y$ is **uniformly continuous** if it satisfies*

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that } \forall x, y \in X, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Definition 1.3 *(**Metric space**) ([21], page 123) Let X be a non-empty set and d a real valued function defined on $X \times X$ such that for $a, b \in X$*

- $d(a, b) \geq 0$,
- $d(a, b) = d(b, a)$,
- $d(a, c) \geq d(a, b) + d(b, c)$ *for all* a, b *and* $c \in X$.

Then d is said to be a metric on X, (X, d) is caled a metric space.

Theorem 1.1 *([15], page 13) Let X and Y be metric spaces, and $f : X \to Y$ a continuous function. If X is compact, then f is uniformly continuous.*

Theorem 1.2 *(**Fundamental Theorem of calculus**)([7], page 3) If $f : [a, b] \to \mathbb{R}$ is a continuous function $(-\infty < a < b < +\infty)$, and $F : [a, b] \to \mathbb{R}$ is an antiderivative of f on $[a, b]$, then*

$$\frac{d}{dx}\int_a^x f(u)du = f(x),$$

$$\int_a^b f(u)\,du = F(b) - F(a),$$

for each $x \in [a, b]$.

Definition 1.4 *(**Homomorphism**)([18], page 95) Let (X, τ) and (Y, τ_1) be topological space. Then they are said to be **homomorphic** if there exists a function $f : X \to Y$ which has the following proprities:*

1. *f is one-to-one (that is $f(x_1) = f(x_2)$ implies $x_1 = x_2$),*
2. *f is onto (that is, for any $y \in Y$ there exists an $x \in X$ such that $f(x) = y$),*
3. *for each $U \in \tau_1$, $f^{-1}(U) \in \tau$,*
4. *for each $V \in \tau$, $f(V) \in \tau_1$.*

Definition 1.5 *(**Compact space**) ([18], page 176) The subset A of a topological spase (X, τ) is said to be **compact** if every open covering of A has a finite subcovering, if the compact subset A equals X, then (X, τ) is said to be a **compact space**.*

Theorem 1.3 *([21], page 200) For a subset K of $\mathbb{R}^n$, K is closed and bounded $\Leftrightarrow$ K is compact.*

Proposition 1.1 *([21], page 200) Let f be a continuous mapping from a compact metric space X to a metric space Y. Then its image $f(X)$ also is compact.*

Definition 1.6 *(**Topological space**)([18], page 24) Let X be a non-empty set. A set τ of subsets of X is said to be a **topological** on X if:*

1. *X and $\emptyset$ belongs to τ,*
2. *The of any (finite or infinite) number of sets τ belongs to τ,*

3. *The intersection of any two sets in τ belongs to τ,*

The pair (X, τ) is called a topological space .

Definition 1.7 *(**Neighbourhood**)([18], page 80) Let (X, τ) be a topological space, N a subset of X and P a point in V. Then V is said to be a **neighbourhood** of the point p if there exists an open set U such that $p \in U \subseteq V \subseteq X$.*

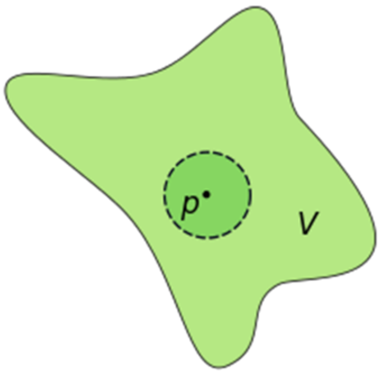

Figure 1.1: Neighborhood of point p.

Definition 1.8 *(**Closed ball**)([21], page 188) Let (X, d) be a metric space. For a point x in the metric space (X, d) and $r > 0$,the set*

$$\bar{B}(x, r) = \{y \in X \mid d(y, x) \leq r\},$$

*is called the **closed ball** centered at x of radius r.*

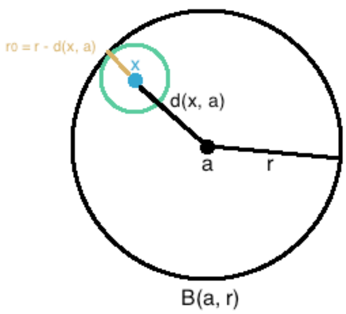

Figure 1.2: Closed ball (in $\mathbb{R}^2$).

Definition 1.9 *(**Closed set**)([18], page 33)*
*Let (X, τ) be a topological space. A subset S of X is said be a **closed set** in (X, τ) if its complement in X, namely X/S, is open in (X, τ).*

Proposition 1.2 *([18], page 141) Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A.*

Definition 1.10 *(**Equicontinuous**)([21], page 207) A collection F of real-valued functions on a metric space X is said to be **equicontinuous** at the point $x \in X$ provided for each $\varepsilon > 0$, there is $\delta > 0$ such that for every $f \in F$ and $y \in X$,if*

$$d(x, y) < \delta, \quad then \quad d(f(x), f(y)) < \varepsilon.$$

Definition 1.11 *(**Convex set**)([18], page 166) Let S be a subset of a real vector space V. The set S is said to be **convex**, if for each $x, y \in S$ and every real number $0 < \lambda < 1$, the points $\lambda x + (1 - \lambda)y$ is in S .*

Definition 1.12 *(**Convergent sequence**)([18], page 140) Let (X, d) be a metric space and $x_1, ...x_n...$ a sequence of points in X. Then the sequence is said to **converge to** $x \in X$ if given any $\varepsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0$, $d(x, x_n) < \varepsilon$.*

Definition 1.13 *(**Continuous with sequence**)([21], page 190) A mapping f from a metric space X to a metric space Y is said to be continuous at the point $x \in X$ provided for any sequence $\{x_n\}$ in Xif,*

$$\{x_n\} \to x, \quad then \quad \{f(x_n)\} \to f(x).$$

Theorem 1.4 *(**The Dominated Convergence Theorem**)([13], page 138) Let (X, S) mesurable space, f be a real-valued nonnegative mesurable function on X, and let $\{u_n\}$ be a sequence of on S. If $u_n(A) \to u(A)$ $\forall A \in S$ and $u_n < v$. for some mesurable u and v such that $\int f dv < \infty$ then*

$$\lim_{n \longrightarrow \infty} \int f du_n = \int f du.$$

Definition 1.14 *(**Lipschitz**)([23], page 22) Let $X \subset \mathbb{R}^m$. Suppose $x \longmapsto f(x)$ is a function from X to $\mathbb{R}^n$. The function f is said to be **Lipschitz continuous** on S if there exists a constant $C > 0$ such that*

$$\| f(x_1) - f(x_2) \|_{\mathbb{R}^n} \leq C \| x_1 - x_2 \|_{\mathbb{R}^m},$$

for all $x_1, x_2 \in X$.

Definition 1.15 ***(Locally Lipschitz)****([23], page 22) Let* $\Omega \subset \mathbb{R}^{n+1}$ *be an open set. A continuous function* $(t;x) \longmapsto f(t;x)$ *from* Ω *to* $\mathbb{R}^n$ *is said to be* ***Locally Lipschitz*** *continuous in* x *if for every compact set* $K \subset \Omega$ *there is a constant* $C_K > 0$ *such that*

$$\| f(t, x_1) - f(t, x_2) \| \leq C_K \| x_1 - x_2 \|,$$

for every (t, x_1);$(t, x_2) \in K$. *If there is a constant for which the inequality holds for all* (t, x_1) ;$(t, x_2)\in \Omega$, *then f is said to be Lipschitz continuous in* x.

Definition 1.16 ***(Banach space)****([23], page 73) A* ***Banach space*** *is acomplete normed vector space over* $\mathbb{R}$ *or* $\mathbb{C}$.

Example 1.1 *[23]* $C^1(F, \mathbb{R}^m)$ *is set the of function f frome* F *into* $\mathbb{R}^m$ *such that* $Df(x)$ *exists and is continuous. Define the norm*

$$\|f\|_{C^1} = \|f\|_\infty + \|Df\|_\infty \,,$$

thene

$$C_b^1(F, \mathbb{R}^m) = \{f \in C^1(F, \mathbb{R}^m) : \|f\|_{C^1} < \infty\},$$

is a banach space.

Definition 1.17 ***(Trajectories of solutions)****([5], page 36) The projection of an integral curve onto the t-axis in phase space is known as a* ***phase curve*** *or* ***trajectories*** *of solutions.*

Theorem 1.5 ***(Variation of Parameters)****([23], page 54) Let* $t \to A(t)$ *be a continuous map from* $\mathbb{R}$ *into the set of* $n \times n$ *matrices over* $\mathbb{R}$. *Let* $t \to F(t)$ *be a continuous map from* $\mathbb{R}$ *into* $\mathbb{R}^n$. *Then for ever* $y(t_0, x_0) \in \mathbb{R}^n$, *the initial value problem*

$$x^{'}(t) = A(t)x(t) + F(t), \qquad x(t_0) = x_0,$$

has a unique global solution $x(t, t_0, x_0)$, *given by the formula*

$$x(t, t_0, x_0) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)F(s)ds.$$

1.2 Generalists on the spectral theory of operators

Definition 1.18 *(**Point spectrum**)([3], page 52) Let $T \in \mathcal{L}(E)$ be a bounded operator on a Banach space E. An eigenvalue of T is a number $\lambda \in \mathbb{C}$ such that $\ker(T-\lambda) \neq \{0\}$. The set formed by the eigenvalues is called **Point spectrum**. It is denoted by $sp_p(T)$.*

Definition 1.19 *(**Linear Operators**)([1], page 40) Let X and Y be two Banach spaces. The operator $A : X \to Y$ is called **linear** if*

$$A(x+y) = A(x) + A(y) \quad and$$

$$A(\lambda x) = \lambda(Ax).$$

Definition 1.20 *(**Exponential map**)([5], page 18) An **exponential map** is the map defined as: $\exp : g \to G$, G being a normed linear space and $g \in G$. In series form, we can define the **exponential map** as:*

$$e^A = \sum_0^\infty \frac{A^n}{n!}.$$

Proposition 1.3 *(**of the exponential map**)([5], page 19) Suppose $A, B \in \mathcal{L}(\mathbb{R}^n)$:*

- $e^{-1} = (e^A)^{-1}$.
- $\| e^A \| \leq e^{\|A\|}$.
- $e^A \in \mathcal{L}(\mathbb{R}^n)$ *if* $A \in \mathcal{L}(\mathbb{R}^n)$.
- $B^{-1}e^AB = e^{B^{-1}AB}$ *if B is nonsingular.*

Definition 1.21 *(**Diagonal matrix**)([5], page 16)*
A diagonal matrix is a square matrix where the non-zero elements are specifically positioned along the principal diagonal.

Definition 1.22 *(**Eigenvalues and Eigenvectors**)([5], page 16) Let A be an $n \times n$ square matrix. Then a scalar λ is called an eigenvalue of A, if there exists a nonzero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In this case, the λ is called an eigenvalue of A and the v eigenvector associated with λ.*
The eigenvalues λ of A are also the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ with $p(\lambda)$ of degree$n$.

1.3 Generalities on stability

Definition 1.23 *(**Equilibrium point**)([23], page 40) Let $\Omega = \mathbb{R} \times \mathcal{O}$ for some open set $\mathcal{O} \subset \mathbb{R}^n$, and suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the hypotheses of the Picard Theorem. A point $x_\epsilon \in \mathcal{O}$ in a set $\mathcal{O}$ is called an **equilibrium point** of the differential equation $\dot{x} = f(t, x)$ if $f(t, x_\epsilon) = 0$ for any $t \in \mathbb{R}$.*

Consider the autonomous system of differential equations:

$$\dot{x} = f(x) \quad x \in M \subset \mathbb{R}^n, \tag{1.1}$$

where x is a vector in n-dimensional $\mathbb{R}^n$ and f is a continuous vector-valued function on M. Let x_ε an equiliberium point of eq(1.1).

Definition 1.24 *(**Stable**)([5], page 37) The equilibrium point x_ε of (1.1) is **stable** if for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that for any solution x of (1.1), if $\| x_\varepsilon - x(t_0) \| < \delta$, then the solution x exists $\forall t \geq t_0$ and $\| x_\varepsilon - x(t) \| < \varepsilon, \forall t > t_0$.*

Example 1.2 *The equation $x' = 1 - x$ is stable at the equilibrium point $x_\varepsilon = 1$.*

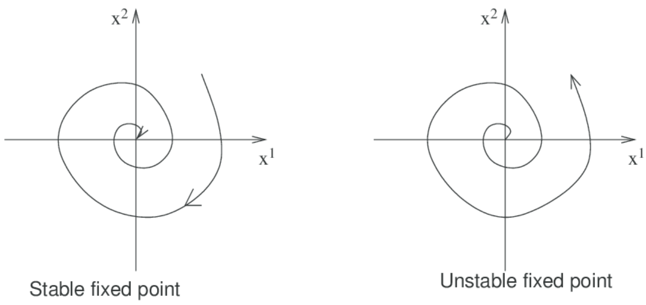

Figure 1.3: stable and unstable of fixed points

Definition 1.25 *(**Asymptotically stable**)([5], page 37) An equilibrium point x_ε of equation 1.1 is **asymptotically stable** if it is stable and $\exists \delta_0 > 0$ such that*

$$\lim_{t \to \infty} x(t) = x_\varepsilon \;\; if \quad \| x(t_0) - x_\varepsilon \| < \delta_0.$$

Example 1.3 *The equilibrium solution x_ε of the differential equation $x' = -ax,$ $where$ $a > 0$, is stable and asymptotically stable.*
Consider any solution x of $x' = -ax$ with the initial value $x(0) = 1$, this solution has the form

$x(t) = e^{-at}$ as time approaches $t \to +\infty$, the solution $x(t)$ approaches 0, which is the equilibrium solution x_ε.

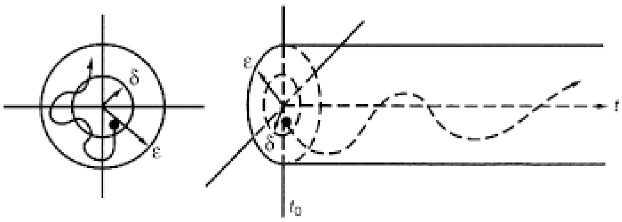

Figure 1.4: asymptotically stable point.

If the equilibrium solution x_ε is not stable, then it is **unstable**.

Example 1.4 *The equilibrium solution $x_\varepsilon = 0$ of $x' = x^2$ is unstable. For any initial condition $t_0, x_0 > 0$, the solution $x(t) = \frac{x_0}{1+x_0(t_0-t)}$ becomes undefined at the specific time $t = x_0^{-1} + t_0$.*

Case of linear system:
Consider the following linear system

$$x' = A(t)x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.2}$$

where $A(t)$ is a continuous matrix-valued function and X is a fundamental matrix of (1.2) satisfying $X(t_0) = I$.

Theorem 1.6 *(**Stable**)([5], page 39) All solutions of (1.2) are **stable** iff each solution is bounded.*

We now examine the ODE (1.2) when the matrix $A = A(t)$ is constant.

Definition 1.26 *(**Stable**)([5], page 39) A polynomial $p(\lambda)$ is **stable** when all solutions (roots) of the characteristic polynomial $p(\lambda)$ have negative real parts.*

Theorem 1.7 *(**Asymptotically stable**) ([5], page 40) Every solution of (1.2) is **asymptotically stable** if the characteristic polynomial of $A = A(t)$ is stable.*

Proposition 1.4 *(**Stability test using eigenvalues**)([5], page 41)*

- *If each real portion of the eigenvalue of A is strictly less than zero, the constant coefficient system $\dot{x} = Ax$ is said to be **asymptotically stable**.*

- $\dot{x} = Ax$ *is* ***stable*** *if all eigenvalues of A have real parts zero and each eigenvalue is less than or equal to zero.*

In order to assess the stability of x, it is essential to comprehend the behavior of the solution in the vicinity of x. This understanding is achieved through a technique known as **linearization**.

Definition 1.27 *(**Linearization**) ([5], page 40) **linearization** involves approximating a complicated nonlinear system to a linear one. The concept of **linearization** is to approximate a nonlinear map with one that is linear.*

Considering system (1.2), we denote the Jacobian matrix of function f at the equilibrium point $\bar{x}$ by $J_f(x_\epsilon) = \frac{\partial f}{\partial x}(x_\epsilon) = A$. We call system $\dot{x} = Ax$ the linear approximation of the nonlinear system (1.2) at the point $\bar{x}$.

Theorem 1.8 *([23], page 42) Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and let $f : \mathcal{O} \to \mathbb{R}^n$ be C^1.Suppose that $x_\epsilon \in \mathcal{O}$ is an equilibrium point of f and that the eigenvalues of $A = J_f(x_\epsilon)$ all satisfy $Re\lambda < 0$. Then $\bar{x}$ is asymptotically stable.*

Routh-Hurwitz Criterion

Routh-Hurwitz criterion is a method to show the system stability by taking the coefficients of an equation characteristic without counting the roots. Suppose the equation characteristic:

$$\mathrm{P}(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n,$$

with a_j is a coefficient in a real number, $j = 1, 2, \ldots n$.

Theorem 1.9 *(**Routh-Hurwitz Criterion**) [17] Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ is a polynomial with real coefficient, all the roots of $P(\lambda)$ have negative real parts if and only if all the principal minors of the Hurwitz matrix*

$$\Delta_i = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_n \end{vmatrix}, \ (i = 1, \ldots, n)$$

are strictly positive.

1.4 Floquet Theory on Banach Spaces

Nonautonomous Differential Equations On Banach Spaces

Let X be a Banach space, we now consider the Cauchy problem :

$$\begin{cases} y^{'} = A(t)y(t), \ \ t \geq s, \\ y\left(s\right) = y_s \in X, \end{cases} \tag{1.3}$$

where $A(t)$ is linear, bounded operators on X. The above kind of differential equations are called evolution equations. First, we want to show the existence of the solution of the Cauchy problem 1.3.

Theorem 1.10 ***(Existence and Uniqueness Theorem)*** *[1] Suppose the function $t \to A(t)$ is continuous on $\mathbb{R}$. Then, for every initial value $y_s \in X$ the Cauchy problem 1.3 has a unique solution.*

Thanks to the existence and uniqueness theorem, we can now define a family of bounded operators on X that characterizes the solutions of the IVP 1.3.

Definition 1.28 *[1] For each $y_0 \in X, U(t,s)y_0 := y(t)$, where $y(\cdot)$ is the unique solution of*

$$\begin{cases} y^{'}(t) = A(t)y(t), \\ y\left(s\right) = y_s \in X. \end{cases} \tag{1.4}$$

The family $\{U(t,s)\}_{t\geq s}$ is called an evolution family generated by family $\{A(t)\}_{t\in\mathbb{R}}$.

Here, we present the basic characteristics of an evolution family.

Theorem 1.11 *[1] For every t and s with $t > s$, $U(t,s)$ are linear and bounded operators on X.*

Theorem 1.12 *[1] Let $\{U(t,s)\}_{t\geq s}$ be an evolution family generated by $\{A(t)\}_{t\in\mathbb{R}}$. Then the following statements hold:*

1. *$U(t,t) = Id$, for all $t \in \mathbb{R}$.*

2. *$U(t,r)U(r,s) = U(t,s)$ for $s \leq r \leq t$.*

3. *For each $s \in \mathbb{R}$ and $y \in X$ the function $t \to U(t,s)y$ is continuously differentiable and*

$$\frac{d}{dt}U(t,s)y = A(t)U(t,s)y.$$

4. *For each $t \in \mathbb{R}$ and $y \in X$ the function $s \to U(t,s)y$ is continuously differentiable and*

$$\frac{d}{ds}U(t,s)y = -U(t,s)A(s)y.$$

5. *The solution of non-homogenous problem*

$$\begin{cases} y^{'}(t) = A(t)y(t) + f(t), \qquad t \geq s, \\ y\,(s) = y_s. \end{cases} \tag{1.5}$$

Stability of periodic solution

Definition 1.29 *(**Periodic solution**)([5], page 37) In the context of the ordinary differential equation (ODE) 1.1, a solution x is classified as a **periodic solution** if there exists a constant T such that $x(t+T) = x(t)$ for all t in the interval I.*

We now study the IVP 1.3, in which $A(t)$ is periodic with the period, i.e. $A(t+\omega) = A(t)$. That system is called a Floquet system on Banach spaces. We have the following observation on a Floquet system.

Theorem 1.13 *[1] Suppose $\{U(t,s)\}_{t\geq s}$ is an evolution family generated by $\{A(t)\}_{t\in\mathbb{R}}$ in a Floquet system. Then we have*

$$U(t+\omega, s+\omega) = U(t,s),$$

for all t and s with $t \geq s$.

We now proceed to define the following operator valued function:

$$P(t) = U(t+\omega, t),$$

and the operator

$$V = P(0) = U(\omega, 0),$$

we refer to the operator V as the monodromy of a Floquet system.

Theorem 1.14 *[1] We have*

1. *The function $P(t)$ is-periodic i.e. $P(t+\omega) = P(t)$.*

2. *The point spectrum set of $P(t)$ is independent of t. In other words,*

$$\sigma_p(p(t)) = \sigma_p(V), \quad \forall t \in \mathbb{R}.$$

Theorem 1.15 *[1] The number μ is an eigenvalue of $V = U(\omega, 0)$ if and only if the Floquet system*

$$y^{'}(t) = A(t)y(t) \tag{1.6}$$

has a nontrivial solution $y(t)$ with

$$y(t+\omega) = \mu y(t), \quad \forall t \in \mathbb{R}.$$

The following theorem shows that the Floquet multipliers determine the stability of the solutions of 1.6 :

Theorem 1.16 *[1] Assume that $\mu_1, \mu_2, \cdots, \mu_n$ are the Floquet multipliers of the Floquet system $y = A(t)y$. Then the solutions of the Floquet system are*

- *asymptotically stable on $[0, \infty)$ iff $\mid \mu_i \mid < 1, 1 \leq i \leq n$.*

- *stable on $[0, \infty)$ provided $\mid \mu_i \mid < 1, 1 \leq i \leq n$ and whenever $\mid \mu_i \mid = 1$, μ_i is a simple eigenvalue.*

- *unstable on $[0, \infty)$ provided there is an i_0, $1 \leq i \leq n$, such that $\mid \mu_{i_0} \mid > 1$.*

CHAPTER 2

Delay Differential Equations

Similar to ODEs, DDEs possess fundamental properties such as existence, uniqueness, and continuity ..., we will delve into these concepts in this chapter.

2.1 Ordinary differential equations

Ordinary differential equations (ODEs) are powerful mathematical tools for modeling natural and technological phenomena. They describe the relationship between a function and its derivatives, making them suitable for representing dynamic changes in systems.

Definition 2.1 *(**ODE**) ([23], page 1) The most general order ordinary differential equation (ODE) has the form*

$$F(t, y, y', \dots, y^n) = 0,$$

where F is a continuous function from some open set $\Omega \subset \mathbb{R}^{n+2}$ into $\mathbb{R}$. An n times continuously differentiable real-valued function $y(t)$ is a solution on an interval $I \subset \mathbb{R}$ if

$$F(t, y(t), y'(t), \dots, y^n(t)) = 0, \quad \forall t \in I.$$

Example 2.1 *[25] The **SIR** model is a simple mathematical model for disease transmission using*

ODEs of the form:

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI, \\ \dfrac{dI}{dt} = -\beta SI - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases}$$

where $\frac{dS}{dt}$ represents the rate of change in the number of susceptible individuals, $\frac{dI}{dt}$ represents the rate of change in the number of infected individuals and $\frac{dR}{dt}$ represents the rate of change in the number of recovered individuals. with the disease transmission rate $\beta > 0$ and the recovery rate $\gamma > 0$.

Definition 2.2 *We call the order of a differential equation the most order high of the derivative in this equation. So,*

$$y^{'} - 2xy^2 + 3 = 0$$

is a one order equation, and

$$x^{''} + (x^{'})^2 = 0$$

is a two-order equation, ets.

Linear Differential Equations

A differential equation of order n is linear if and only if it can be expressed in the form :

$$a_0(x)y + a_1(x)y^{'} + a_2(x)y^{''} + \ldots + a_n(x)y^n = f(x)$$

where $a_i(x)$ (for $i = 0$ to n) are continuous functions of the independent variable to x.
The differential equation

$$a_0(x)y + a_1(x)y^{'} + a_2(x)y^{''} + \ldots + a_n(x)y^n = 0$$

is called homogeneous equation.
A first-order linear differential equation (LDE) can be written in the form:

$$a(x)y^{'}(x) + b(x)y(x) = c(x).$$

Linear Differential systems

Consider the system of n linear differential equations involving n variables [1] :

$$\begin{cases} y_1' = a_{11}(t)\,y_1 + \ldots\ldots\ldots + a_{1n}(t)y_n + f_1(t)\,, \\ \vdots \\ y_n' = a_{n1}(t)\,y_1 + \ldots\ldots\ldots + a_{nn}(t)y_n + f_n(t)\,. \end{cases}$$

We can write this system in the following matrix form :

$$y' = A(t)y + f(t),$$

where

$$y = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, y' = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}, A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \textit{and} \;\; f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

If the initial condition is specified as $y(t_0) = y_0$, then an initial value problem (IVP) is established. Our aim is to first demonstrate the existence and uniqueness of a solution for this IVP. A detailed proof for the general case can be found in Theorem 2.2 [1].

Theorem 2.1 *[1] Assume that the $n \times n$ matrix-valued function $A(t)$ and the vector-valued function $f(t)$ are continuous for $t \geq 0$, then the initial value problem:*

$$(IVP) = \begin{cases} y'(t) = A(t)y(t) + f(t), \\ y(0) = y_0, \end{cases} \tag{2.1}$$

has a unique solution. If $f(t) = 0$ the differential equation equation :

$$y'(t) = A(t)y(t), \tag{2.2}$$

is called homogeneous equation.

Existence and uniqueness theorem

Let U be an open set in $\mathbb{R} \times \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a continuous function.

Definition 2.3 *(**Cauchy problem**) [12] Consider a first-order differential equation in the following normal form :*

$$x^{'}(t) = f(t, x),$$

for $(t, x(t)) \in U$, and $(t_0, x_0) \in U$, the corresponding Cauchy problem involves finding solutions $x = x(t)$ such that :

$$x(t_0) = x_0.$$

The Cauchy problem is denoted as follows :

$$\begin{cases} x^{'}(t) = f(t, x), \\ x(t_0) = x_0. \end{cases} \tag{2.3}$$

The following theorem proves the existence and uniqueness of a solution :

Theorem 2.2 *(**Cauchy Lipschitz**) [13] Let f a defined on an open set $U \subseteq I \times \mathbb{R}$, where I is an open interval of $\mathbb{R}$. Let $a, b \in \mathbb{R}$ constants, the set $D \subset U$ defined by :*

$$D = \{(x, y) \in I \times \mathbb{R}, \mid x - x_0 \mid \leq a; \mid y - y_0 \mid \leq b\}.$$

We say that the Cauchy problem 2.3 has a unique solution x defined on the closed interval $T[x_0 - d, x_0 + d]; d \leq a$. If and only if :

- *f continuous on the set D.*
- *f satisfies the lipshitz condition in y : i.e*

$$\mid f(x, y_1) - f(x, y_2) \mid \leq L \mid y_1 - y_2 \mid, \quad (L > 0 \ \ is \ \ a \ \ lipshitz \ \ constant).$$

Example 2.2 *We consider the following initial value problem:*

$$\begin{cases} y^{'}(x) = y(x), \\ y\left(x_0\right) = y_0, \end{cases}$$

has a solution $y(x) = y_0 e^{x-x_0}$*, and this solution is unique.*

Lemma 2.1 ***(Gronwall)****([23], page 26) Let* $f(t)$*,* $\varphi(t)$ *be nonnegative continuous functions on an open interval* $J = (\alpha, \beta)$ *containing the point* t_0*. Let* $c_0 \leq 0$*. If*

$$f(t) \leq c_0 + \left| \int_{t_0}^{t} \varphi(s) f(s) \, ds \right|,$$

for all $t \in J$*, then*

$$f(t) \leq c_0 + \exp \left| \int_{t_0}^{t} \varphi(s) \, ds \right|.$$

Banach fixed-point theorem

Definition 2.4 ***(Fixed point)*** *([18], page 160) Let* f *be a mapping of a set* X *into itself. Then a point* $x \in X$ *is said to be a fixed point of* f *if* $f(x) = x$*.*

Definition 2.5 ***(Contraction mapping)*** *([18], page 160) Let* (X, d) *be a metric space and* f *a mapping of* X *into itself. Then* f *is said to be a contraction mapping if there exists an* $r \in (0, 1)$ *such that*

$$d(f(x_1), f(x_2)) \leq r d(x_1, x_2), \quad for\ all \quad x_1, x_2 \in X.$$

Theorem 2.3 ***(Contraction Mapping Theorem)****([18], page 161) Let* (X, d) *be a complete metric space and f a contraction mapping of* (X, d) *into itself. Then* f *has precisely one fixed point.*

There some real life problem that cant treated and analyzed by ODE therefore we need using the to pass to DDE with the aim to deal with this kind of models and problems.

Example 2.3 *[16]* ***DDE Model of Angiogenesis :***
Angiogenesis is the physiological process through which new blood vessels form from pre-existing vessels. It plays a crucial role in various biological processes, including growth, development, wound healing, and granulation tissue formation. However, angiogenesis also contributes to the progression of certain diseases, particularly cancer.

$$\begin{cases} \frac{dN}{dt} = \alpha N(t)(1 - \frac{N(t)}{1+f_1(E(1-\tau_1))}), \\ \frac{dP}{dt} = f_2\left(E\left(t\right)\right) N\left(t\right) - \delta P\left(t\right), \\ \frac{dE}{dt} = f_3\left(P\left(t-\tau_2\right)\right) - \alpha \left(1 - \frac{N(t)E(t)}{1+f_1(E(t-\tau_1))}\right), \end{cases}$$

where N is the number of tumor cells, P is the quantity growth factors known to be involved in supplying the tumor, and E represents the vessel density, where $E = VN$, and V is the volume of blood vessels feeding the tumor. The functions $f1, f2, f3$, model tumor cell proliferation rate, the protein production rate, and the vessel growth rate, respectively, and are given by:

$$f_1(E) = \frac{b_1 E^n}{c_1 + E^n},\ f_2(E) = \frac{a_2 + c_2}{c_2 + E},\ f_3(P) = \frac{b_3\left(P^2 - m(t)^2\right)}{\frac{m(t)^2}{a_3} + p_2}.$$

Now we will move on to defining and characterizing delay differential equations.

Definition 2.6 *Suppose $t_0 \in \mathbb{R}$, $a \geq 0$, and $r \geq 0$. If $x \in \mathcal{C}([t_0 - r, t_0 + a], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + a]$, we define $\mathrm{C} := \mathcal{C}([0, r], \mathbb{R}^n)$ and $x_t \in \mathrm{C}$ by $x_t(\tau) = x(t - \tau)$.*
Now, consider Ω as a subset of $\mathbb{R} \times \mathrm{C}$ and $f : \Omega \to \mathbb{R}^n$ a function. A Delay Differential Equation on Ω is the relation :

$$\dot{x} = f(t, x_t), \tag{2.4}$$

where $\dot{x}$ represents the right-hand derivative of x.

If $r = 0$, then the DDE coincides with the ODE.

Remark 2.1 *The analysis can be extended to the m-dimensional case by defining $\Omega \subset \mathbb{R} \times \mathrm{C}^m$. The corresponding differential equation becomes:*

$$\dot{x} = f(t, x_t(\tau 1), ..., x_t(\tau m)).$$

Definition 2.7 *A map x is considered a solution of 2.4 on $[t_0 - r, t_0 + a)$ if there exist $t_0 \in \mathbb{R}$ and $a > 0$ satisfying :*

- $x \in \mathcal{C}([t_0 - r, t_0 + a), \mathbb{R}^n)$.
- $(t, x_t) \in \Omega$.
- *For any $t \in [t_0, t_0 + a)$, the equation $\dot{x} = f(t, x_t)$ holds.*

Definition 2.8 *If $(t_0, u) \in \Omega$, then $x(t_0, u)$ is defined as a solution of (2.4) with the initial condition u at t_0 when there exists $a > 0$ such that :*

- *$x(t_0, u)$ is a solution of 2.4 on $[t_0 - r, t_0 + a)$.*
- $x_{t_0}(t_0, u) \equiv u$.

Firstly, we have a result concerning the integral equation of the delay differential equation.

Lemma 2.2 *Given* $(t_0, u) \in \Omega$ *and* $f : \Omega \to \mathbb{R}^n$ *continuous, the task of finding a solution to the system*

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0} \equiv u. \end{cases}$$

is equivalent to solving the integral equation :

$$\begin{cases} x(t) = u(0) + \int_{t_0}^{t} f(s, x_t)\, ds, \quad for\ t \geq t_0, \\ x_{t_0} \equiv u. \end{cases}$$

Proof $\Rightarrow$) Assuming $x(t_0, u)$ is a solution to the initial value problem, we have :

$$\begin{cases} \dfrac{\partial x(t_0, u)}{\partial t}(t) = f(t, x_t(t_0, u)), \\ x_{t_0}(t_0, u) \equiv u. \end{cases}$$

By the Fundamental Theorem of Calculus, we get :

$$\begin{aligned} x(t_0, u)(t) &= u(0) + \int_{t_0}^{t} f(s, x_s(t_0, u))ds \\ x(t_0, u)(t) - u(0) &= \int_{t_0}^{t} f(s, x_s(t_0, u))ds \\ x(t_0, u)(t) - x_{t_0}(t_0, u)(0) &= \int_{t_0}^{t} f(s, x_s(t_0, u))ds \\ x(t_0, u)(t) - x(t_0, u)(t_0) &= \int_{t_0}^{t} f(s, x_s(t_0, u))ds \\ \int_{t_0}^{t} x'(t_0, u)(s)ds &= \int_{t_0}^{t} f(s, x_s(t_0, u))ds \\ x'(t_0, u)(t) &= f(t, x_t(t_0, u)). \end{aligned}$$

$\Leftarrow$) Similarly, applying the Fundamental Theorem of Calculus, we have :

$$\begin{aligned} &x'(t_0, u)(t) = \frac{d}{dt}(u(0) + \int_{t_0}^{t} f(s, x_t(t_0, u))), \\ &x'(t_0, u)(t) = f(s, x_t(t_0, u)). \end{aligned}$$

We will now demonstrate that any Initial Value Problem can be adjusted such that the initial time is t_0 :

Notation 2.1 *If* $(t_0, u) \in \mathbb{R} \times C$*, then let* $\tilde{u}$ *be in* $\mathcal{C}([t_0 - r, +\infty), \mathbb{R}^n)$ *defined as*

$$\widetilde{u}_{t_0} \equiv u,$$
$$\widetilde{u}\,(t_0 + t) = u\,(0)\,, \quad for\ all\ t \geq 0,$$

where the function $\widetilde{u}$ is considered an extension of the function u.

Lemma 2.3 *For* $(t_0, u) \in \Omega$ *and* $f : \Omega \to \mathbb{R}^n$ *being continuous, the systems:*

$$\begin{cases} \dot{x} = f\,(t, x_t)\,, \\ x_{t_0} \equiv u, \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = f\,(t_0 + t, \widetilde{u}_{t_0+t} + y_t)\,, \\ y_0 \equiv 0, \end{cases}$$

possess the same solutions. In other words, the equations :

$$\begin{cases} x\,(t) = u\,(0) + \int_{t_0}^{t} f\,(s, x_s)\,ds, \quad t \geq t_0, \\ x_{t_0} \equiv u. \end{cases} \qquad \begin{cases} y\,(t) = \int_{0}^{t} f\,(t_0 + s, \widetilde{u}_{t_0+s} + y_t)\,ds, \quad t \geq 0, \\ y_0 \equiv 0. \end{cases}$$

are equivalent.

Proof Utilizing Lemma 2.2, if $x(t)$ serves as a solution for the initial conditions (t_0, u), then we have :

$$\begin{cases} x(t) = u\,(0) + \int_{t_0}^{t} f\,(s, x_t)\,ds, \quad t \geq t_0, \\ x_{t_0} \equiv u. \end{cases}$$

Now, let's introduce the transformation $y(t) = x(t_0 + t) - \widetilde{u}(t_0 + t)\ for\ t \geq -r$.

For $t \geq 0$, we observe :

$$\begin{aligned} y(t) &= x(t_0 + t) - \widetilde{u}(t_0 + t) \\ &= x(t_0 + t) - u(0) = x(t_0 + t) - x(t_0) \\ &= \int_{t_0}^{t_0+t} f(s, x_s)ds = \int_0^t f(t_0 + s, x_{t_0+s})ds \\ &= \int_0^t f(t_0 + s, \widetilde{u}_{t_0+s} + y_s)ds. \end{aligned}$$

In the case where $-r \leq t \leq 0$, we find that

$$\begin{aligned} y_0(-t) = y(t) &= x(t_0+t) - \widetilde{u}(t_0+t) \\ &= x_{t_0}(-t) - \widetilde{u}_{t_0}(-t) = u(-t) - u(-t) = 0. \end{aligned}$$

Hence, $y(t)$ satisfies the integral equation :

$$\begin{cases} y\,(t) = \int_0^t f\,(t_0+s, \widetilde{u}_{t_0+s} + y_t)\, ds, \quad t \geq 0, \\ y_0 \equiv 0. \end{cases}$$

The outcome is deduced from Lemma 2.2.

2.2 Existence and uniqueness

Lemma 2.4 *states that if $x \in \mathcal{C}([t_0 - r, t_0 + a], \mathbb{R}^n)$, then x_t is a continuous function of t for $t \in [t_0, t_0 + a]$.*

Proof Given that x is continuous, it is uniformly continuous on the interval $I = [t_0 - r, t_0 + a]$, thus

$$\forall \varepsilon > 0, \exists \delta > 0; \forall t, s \in I, |t - s| < \delta \Rightarrow |x(t) - x(s)| < \varepsilon.$$

For any $t, s \in [t_0, t_0 + a]$ and for all $\tau \in [0, r]$, then

$$|t - \tau - s + \tau| < \delta \Rightarrow |t - s| < \delta, \quad \text{and}$$

$$|x(t - \tau) - x(s - \tau)| = |x_t(\tau) - x_s(\tau)| < \varepsilon.$$

Notation 2.2 *Consider positive real numbers a, b, and r, where :*

$$\begin{gathered} \bar{\mathrm{I}}_a = [0, a], \\ \bar{B}_b = \{v \in \mathrm{C} : |v| \leq b\}, \\ A(a, b) = \{v \in \mathcal{C}([-r, a], \mathbb{R}^n) : v_0 \equiv 0, v_t \in \bar{\mathrm{B}}_b, t \in \bar{\mathrm{I}}_a\}. \end{gathered}$$

Lemma 2.5 *In an open set $\Omega \subset \mathcal{C} \times \mathbb{R}$, where $K \subset \Omega$ is a compact and $f : \Omega \to \mathbb{R}^n$ is continuous, the the following hold :*

- *There exists a neighborhood V of K in Ω such that $f\lceil V \in \mathcal{B}(V, \mathbb{R}^n)$.*

- *There exists a neighborhood U of f in $\mathcal{B}(V, \mathbb{R}^n)$ and positive constants M, a, and b such that :*

$$|g(t, v)| < M, \quad for\ all\ (t, v) \in V\ and\ for\ allall\ g \in U. \tag{2.5}$$

Additionally, for each (t_0, u) in K, it holds that

$$(t_0 + t, \widetilde{u}_{t_0+t} + v_t) \in V, \quad \forall t \in \bar{\mathrm{I}}_a \text{ and } \forall v \in A(a, b).$$

Proof Since K is compact and the function f is continuous, there exists $M > 0$ such that

$$|f(t_0, u)| < M, \quad \forall (t_0, u) \in K.$$

Furthermore, due to compactness, there exist positive values α, β, and ε such that

$$|f(t_0 + t, u + v)| < M - \varepsilon, \quad \forall (t_0, u) \in K \text{ and } \forall (t, v) \in \bar{I}_\alpha \times \bar{B}_\beta.$$

By defining V as $V = \{(t_0 + t, u + v) : (t_0, u) \in K\ and\ (t, v) \in \bar{I}_\alpha \times \bar{B}_\beta\}$, we ensure that f is bounded in V. Additionally, there exists a neighborhood U of $f \in \mathrm{B}(V, \mathbb{R}^n)$ satisfying condition (2.5). Considering the compactness of K, we select $a < \alpha$ and $0 < b < \beta$ such that

$$\|\widetilde{u}_{t_0+t} - u\| < \beta - b, \quad \forall (t_0, u) \in K \text{ and } \forall t \in \bar{\mathrm{I}}_a.$$

Consequently, based on the construction of V, it follows that

$$\| v_t + \widetilde{u}_{t_0+t} - u \| < b + \beta - b = \beta, \quad \forall v \in A(a, b).$$

Lemma 2.6 *States that for positive real numbers M and b, the set*

$$W = \{v \in \mathcal{C}(\mathrm{I}, \mathbb{R}^n) : \|v\| \leq b \text{ and } |v(t) - v(s)| \leq M|t - s| \ \ \forall t, x \in \mathrm{I}\},$$

is compact in $\mathcal{C}(\mathrm{I}, \mathbb{R}^n)$ for any compact subset I of $\mathbb{R}^m$.

Before giving the proof, we would like to present the following Corollary.

Corollary 2.1 *Every uniformly bounded and equicontinuous sequence of continuous functions from a compact metric space X to a Banach space Y possesses a uniformly convergent subsequence on X.*

Proof (of lemma 2.6) Let's apply the Corollary 2.1.

- Closed. Any sequence $(v^k) \subset W$ that converges to v must have v as an element of W.

$$\| v \| < \varepsilon + b \quad \text{and} \quad |v(t) - v(s)| < 2\varepsilon + M|t - s|.$$

  So, We have $\| v \| \leq b$ and $|v(t) - v(s)| \leq M|t - s|$ as $\varepsilon \to 0$.

- Uniformly bounded. It is clear from the fact that $\|v\| \leq b$.

- Equicontinuous. It is clear that

$$\forall \varepsilon > 0, \exists \delta > 0; \forall v \in W \text{ and } \forall x, y \in \mathrm{I}, |x - y| < \delta \Rightarrow |v(x) - v(y)| < \varepsilon.$$

  So given $\varepsilon > 0$, we take $\delta < \frac{\varepsilon}{M}$.

Therefore, W is compact. The remaining task is to prove convexity, i.e, show that $(1-\lambda)u + \lambda v \in W$,

$$|(1-\lambda)(u(t) - u(s)) + \lambda(v(t) - v(s))| \leq (1-\lambda)|u(t) - u(s)| + \lambda|v(t) - v(s)| \leq M|t - s|, \text{ and}$$

$$|(1-\lambda)u + \lambda v| \leq (1-\lambda)|u| + \lambda|v| \leq (1-\lambda)b + \lambda b = b.$$

Lemma 2.7 *States that for an open set Ω, a compact subset $K \subset \Omega$ and a continuous function $f : \Omega \to \mathbb{R}^n$, with neighborhoods U and V and positive constants M, a, and b as obtained from Lemma 2.5. The map*

$$T : K \times U \times A(a, b) \to \mathcal{C}([-r, a], \mathbb{R}^n)$$

defined by

$$T(t_0, u, g, v)(t) = \begin{cases} 0 & \text{if } t[-r, 0], \\ \int_0^t g(t_0 + s, \widetilde{u}_{t_0+s} + v_s)ds & \text{if } t \in \bar{\mathrm{I}}_a. \end{cases}$$

is continuous. Additionally, there exists a compact set $W \in \mathcal{C}([-r, a], \mathbb{R}^n)$ such that

$$T : K \times U \times A(a, b) \to W.$$

Furthermore, if $Ma \leq b$, then

$$T : K \times U \times A(a,b) \to A(a,b).$$

Proof The map $T : K \times U \times A(a,b) \to \mathcal{C}([-r,a],\mathbb{R}^n)$ is well-defined, meaning that for every $(t_0,u,g,v) \in K \times U \times A(a,b)$, the output $T(t_0,u,g,v)$ belongs to $\mathcal{C}([-r,a],\mathbb{R}^n)$.
Condition 2.5 states that for any $t,s \in \bar{\mathrm{I}}_a$,

$$|T(t_0,u,g,v)(t) - T(t_0,u,g,v)(s)| \leq M|t-s|$$
$$|T(t_0,u,g,v)(t)| \leq Ma.$$

Let us assume

$$W = \{v \in \mathcal{C}([-r,a],\mathbb{R}^n) : |v(t)-v(s)| \leq M|t-s| \text{ and } |v(t)| \leq Ma\}.$$

Lemma 2.6 ensures that W is compact. Consequently, $T : K \times U \times A(a,b) \to W$.
Assuming $Ma \leq b$, then $W \subset A(a,b)$, and the map $T : K \times U \times A(a,b) \to A(a,b)$ is well-defined. Intuitively,$v \in A(a,b)$ if, and only if, for each $\tau \in [0,r]$ and for each $t \in [0,a]$

$$v(\tau) = 0 \quad \text{and} \quad |v(t-\tau)| \leq b.$$

In simpler terms,$|v(t)| \leq b$ for all $t \in [-r,a]$. This condition is guaranteed if $v \in K$ and $Ma \leq b$.
Finally, to establish the continuity of T, we consider a sequence of points $((t^k,u^k,g^k,v^k))_k \in K \times U \times A(a,b)$ that converges to (t_0,u,g,v) as $k \to \infty$, where $(t_0,u,g,v) \in K \times U \times A(a,b)$.
Since W is compact and $T(t^k,u^k,g^k,v^k) \in W$ for all k, there exists a subsequence of $\{T(t^k,u^k,g^k,v^k)\}$ that converges. We denote this subsequence with the same index k, such that

$$T(t^k,u^k,g^k,v^k) \to h \quad \text{as } k \to \infty.$$

We have established that $h \in W$, Lemma 2.5 ensures that all functions g^k and g are uniformly bounded. Additionally, as $k \to \infty$,

$$g^k(t^k+s, \widetilde{u}^k_{t^k+s} + v^k_s) \to g(t_0+s, \widetilde{u}_{t_0+s} + v_s),$$

for any $t \in \bar{\mathrm{I}}_a$. By the Dominated Convergence Theorem, this convergence implies the following equality for all $t \in \bar{\mathrm{I}}_a$,

$$h(t) = \lim_{k \longrightarrow \infty} \int_0^t g^k(t^k + s, \widetilde{u}^k_{t^k+s} + v^k_s)ds = \int_0^t g(t_0 + s, \widetilde{u}_{t_0+s} + v_s)ds = T\left(t_0, u, g, v\right)(t).$$

We have proven that for any convergent subsequence of $\{T(t^k, u^k, g^k, v^k)\}$, its limit is independent of the chosen subsequence. This implies that $\{T(t^k, u^k, g^k, v^k)\}$ itself converges. So, T is continuous.

Lemma 2.8 *The set $A(a, b)$ is a closed, bounded, and convex set in $\mathcal{C}([-r, a], \mathbb{R}^n)$.*

Proof

- Closed. Consider a sequence $(v^k) \subset A(a, b)$ that converges to v as $k \to \infty$. Our objective is to demonstrate that v is an element of $A(a, b)$. Specifically :
  For all $0 \leq \tau \leq r$ and every $\varepsilon > 0$, we have

  $$|v(\tau)| \leq |v(\tau) - v_k(\tau)| + |v^k(\tau)| < \varepsilon.$$

  This implies $v_0 = 0$.
  For all $0 \leq t \leq a$, all $0 \leq \tau \leq r$, and every $\varepsilon > 0$, we observe

  $$|v(t - \tau)| \leq |v(t - \tau) - v^k(t - \tau)| + |v^k(t - \tau)| < \varepsilon + b.$$

  Consequently, $v_t \in \bar{\mathrm{B}}_b$ for any $t \in \bar{\mathrm{I}}_a$.

- Uniformly bounded. For any $v \in A(a, b)$,

  $$\sup_{0 \leq \tau \leq r} |v(t - \tau)| \leq b \quad \forall\, 0 \leq t \leq a.$$

  Hence, $A(a, b)$ is uniformly bounded.

- Convex. We need to show that $\forall u, v \in W$ and any $0 < \lambda < 1$, $(1 - \lambda)u + \lambda v \in A(a, b)$

  $$\begin{aligned} |(1 - \lambda)u + \lambda v| &\leq |(1 - \lambda)||v| + |\lambda||u| \\ &\leq (1 - \lambda)b + \lambda b \leq b. \end{aligned}$$

Theorem 2.4 *[8] Let $\Omega \subset \mathbb{R} \times \mathrm{C}$ be open set and $f : \Omega \to \mathbb{R}^n$continuous. If $K \subset \Omega$ is compact, there are*

- *$V \subset \Omega$ neighbourhood of K such that $f(V)$ is contained in $\mathcal{B}(V, \mathbb{R}^n)$.*
- *$U \subset \mathcal{B}(V, \mathbb{R}^n)$ neighbourhood of $f(V)$ in $\mathcal{B}(V, \mathbb{R}^n)$.*
- *a positive real number.*

such that for any $(t_0, u) \in K$ and any $g \in U$, there is a solution $x(t; t_0, u, g)$ of

$$\begin{cases} \dot{x} = g(t, x_t), \\ x_{t_0} \equiv u. \end{cases}$$

that exists on $[t_0 - r, t_0 + a]$. Moreover, if $g(t, v)$ is Lipschitz in v in each compact subset in Ω, the solution is unique.

It is important to present the following corollary to prove Theorem 2.4.

Corollary 2.2 *Suppose K is a closed, bounded, and convex subset of a real Banach space X. If u is a compact map from K to itself, then u has a fixed point.*

Proof Let $g \in U$ be a fixed value, we define the set $W = \{(t_0, u)\}$. By Lemma 2.7, the map $T(t_0, u, g, .)$ has a fixed point in the closed, bounded, and convex set $A(a, b)$ according to Corollary (2.2). By Lemmas 2.2 and 2.3, we conclude that a solution to the given differential equation exists on $[t_0 - r, t_0 + a]$.
If x and y are solutions on $[t_0 - r, t_0 + a]$. Then, Lemma 2.2 implies that,

$$\begin{cases} x_{t_0} - y_{t_0} \equiv 0, \\ x(t) - y(t) = \int_{t_0}^{t} (g(s, x_s) - g(s, y_s))ds, \qquad t \geq t_0. \end{cases}$$

If L is the Lipschitz constant of $g(t, v)$ within any compact subset of Ω containing the trajectories $\{(t, x_t)\}$ and $\{(t, y_t)\}$, where $t \in \bar{\mathrm{I}}_a$, we can choose α such that $0 < (\alpha - t_0)L < 1$. Therefore, for all $t \in \bar{\mathrm{I}}_\alpha$,

$$|x(t) - y(t)| \leq L \int_{t_0}^{t} \parallel x_x - y_s \parallel ds \leq (\alpha - t_0)L \sup_{t_0 \leq s \leq t} |x_s - y_s|.$$

It can be observed that

$$\sup_{t_0 \leq s \leq t} |x_s - y_s| = \sup_{t_0 \leq s \leq t} \sup_{0 \leq \tau \leq r} |x\,(s-\tau) - y\,(s-\tau)|$$
$$= \leq \sup_{t_0 - r \leq s \leq t} |x\,(s) - y\,(s)| \leq \sup_{t_0 - r \leq s \leq \alpha} |x\,(s) - y\,(s)|\,.$$

As a result, we have demonstrated that the mapping:

$$\mathcal{C}([t_0 - r, t_0 + a], \mathbb{R}^n) \longrightarrow \mathcal{C}([t_0 - r, t_0 + a], \mathbb{R}^n)$$

$$x\,(t) \longmapsto \begin{cases} u\,(t_0 - t)\,, & t \in [t_0 - r, t_0]\,, \\ u\,(0) + \int_{t_0}^{t} g\,(s, x_s)\,ds, & t \in [t_0, t_0 + a]\,. \end{cases}$$

is a contraction on $\bar{\mathrm{I}}_\alpha$. Therefore, $x(t) = y(t)$ for all $t \in \bar{\mathrm{I}}_\alpha$.

result :

In $\mathbb{R}^n$, maps are locally Lipschitz if and only if they're Lipschitz on every compact subset. This might not be true in Banach spaces due to closed balls not always being compact. We'll prove an implication in Proposition 2.1 to show that locally Lipschitz maps are Lipschitz on every compact subset in Banach spaces.
Therefore, a locally Lipschitz map in a Banach space is Lipschitz on every compact subset. Conversely, in $\mathbb{R}^n$, we know that a C^1 map on an open set Ω is locally Lipschitz within Ω. However, in Proposition 2.2, we will demonstrate that a C^1 map on an open subset of a real Banach space is Lipschitz on all compact subsets of that open set.
In summary, an initial value problem $\dot{x} = f(t, x_t)$ possesses a unique solution if f is C^1 with respect to the second variable.

Proposition 2.1 *If $f : X \to Y$ is a locally Lipschitz mapping between real Banach spaces and X is compact, then f is Lipschitz.*

Proof (See [8], page 8).

Proposition 2.2 *If Ω is an open set in a real Banach space, and $f : \Omega \to \mathbb{R}^n$ is continuously differentiable (C^1), then f is Lipschitz on every compact subset within Ω.*

Proof (See [8], page 9).

Theorem 2.5 (***Globally uniqueness***) [8] *Consider an open set $\Omega \subset \mathbb{R} \times \mathrm{C}$, with $(t_0, u) \in \Omega$, and a continuous function $f : \Omega \to \mathbb{R}^n$ that is Lipschitz in each compact subset with respect to the second variable. If $x : [t_0 - r, a] \to \mathbb{R}^n$ and $y : [t_0 - r, b] \to \mathbb{R}^n$ are solutions of:*

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0} \equiv u. \end{cases} \tag{2.6}$$

Then $x_t \equiv y_t$ for all $t \in [t_0, c]$ where $c = \min\{a, b\}$.

Proof Assume there exists $t_0 < t_1$ such that $x_{t_1} \not\equiv y_{t_1}$. Let's define $t_* = \inf\{t \in [t_0, c] : x_t \neq y_t\}$. Therefore, $x_t \equiv y_t$ for any $t \in [t_0, t_*]$. Let $v \equiv x_{t_*} \equiv y_{t_*}$. According to Theorem (2.4), at $(t_*, v) \in \Omega$, there exists a solution $z : I_* \to \mathbb{R}^n$ of the initial value problem (2.6). This contradicts the choice of t_*.

2.3 Maximal solution

Definition 2.9 *In an open set $\Omega \subset \mathbb{R} \times \mathrm{C}$, where $f : \Omega \to \mathbb{R}^n$ is continuous function, let x be a solution on $[t_0, a)$ of :*

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0} \equiv u. \end{cases}$$

The solution x is considered maximal when for any other solution y on $[t_0 - r, b)$ with $a < b$, if $y \upharpoonright [t_0, a) = x$, then $a = b$.

Theorem 2.6 (***Existence and uniqueness of maximal solutions***) [8] *In an open set $\Omega \subset \mathbb{R} \times \mathrm{C}$, where $f : \Omega \to \mathbb{R}^n$ is continuous function and Lipschitz in each compact subset with respect to the second variable, if $(t_0, u) \in \Omega$, then there exists a maximal solution $x : \mathrm{I}(t_0, u) \to \mathbb{R}^n$ of the delay differential equation :*

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0} \equiv u. \end{cases} \tag{2.7}$$

Furthermore, $\mathrm{I}(t_0, u) = [t_0 - r, a)$ with $t_0 < a$.

Proof Consider $\mathcal{S}(t_0, u) = \{\mathrm{I}_y \xrightarrow{y} \mathbb{R}^n : y \ \ is\ solution\ of\ (2.7)\}$. Define $\mathrm{I}(t_0, u) = \bigcup_{y \in \mathcal{S}(t_0,u)} \mathrm{I}_y$, and let $x : \mathrm{I}(t_0, u) \to \mathbb{R}^n$ be defined as $x(t) = y(t)$ if $; t \in \mathrm{I}_y$. Based on Theorem (2.5), x is well-defined as it does not rely on the specific choice of solution y. It is evident that x is maximal.
If $I(t_0, u) = [t_0 - r, a]$, then $(a, x(a)) \in \Omega$, and there exists a solution with initial condition $(a, x(a))$ which serves as a left extension of x. This contradicts the maximality of x.

2.4 Continuation of solutions

Theorem 2.7 *[8] Let $\Omega \subset \mathbb{R} \times \mathrm{C}$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a continuous function. If x is a maximal solution on $[t_0 - r, a)$ of the delay differential equation $\dot{x} = f(t, x_t)$, then for every compact subset $K \subset \Omega$, there exists $t_K \in [t_0 - r, a)$ such that for all $t \in [K_t, a)$, the point $(t, x_t) \notin K$.*

Proof

- If $a = +\infty$, the result is evidently valid.
- For $r = 0$, it corresponds to the scenario of an Ordinary Differential Equation.
- If the conclusion is not true for $r > 0$, there exists a sequence $t_k \to a^-$ as $k \to \infty$ and $v \in \mathrm{C}$ such that $(t_k, x_{t_k}) \to (a, v)$ as $k \to \infty$ with $(t_k, x_{t_k}) \in W$. Consequently, for any $\varepsilon > 0$, $\sup_{\tau \in [\varepsilon, r]} |x_{t_k}(\tau) - v(\tau)| \to 0$ as $k \to \infty$. Thus, $x(a - \tau) = v(\tau)$ for $0 < \tau \leq r$. Therefore, x can be continuously extended as follows:

$$\hat{x}\,(t) = \begin{cases} x\,(t) & t \in [t_0 - r, a), \\ v\,(0) & \quad t = a. \end{cases}$$

- Now, since $(a, \hat{x}_a) \in \Omega$, one can discover solutions through $(a, \hat{x}_a)$ to the right of a. However, this contradicts the maximality assumption of x.

2.5 Continuity and differentiability of solutions

Theorem 2.8 (***Continuity of initial conditions***) [8] *In an open set $\Omega \subset \mathbb{R} \times C$, where $f : \Omega \to \mathbb{R}^n$ is continuous, the solution $x(t_0, u, f)$ of the Delay Differential Equation*

$$\begin{cases} \dot{x} = f\left(t, x_t\right), \\ x_{t_0} \equiv u, \end{cases}$$

remains continuous concerning t_0, u, and f.

Theorem 2.9 (***Differentiability of initial conditions***) [8] *For an open set $\Omega \subset \mathbb{R} \times C$ and $f \in \mathcal{C}^P(\Omega, \mathbb{R}^n)$ with $p \geq 1$, the solution $x(t_0, u, f)$ of the Delay Differential Equation*

$$\begin{cases} \dot{x} = f\left(t, x_t\right), \\ x_{t_0} \equiv u, \end{cases}$$

is unique and $\mathcal{C}^P$ concerning u and f for t within any compact set in the domain of $x(t_0, u, f)$.

In order to prove the content of the previous theorem, we need the following theorem:

Theorem 2.10 [8] *Consider U as a subset of a Banach space and V as a closed subset of another Banach space. If $u : U \times V \to V$ is a uniform contraction and $g(x)$ represents the unique fixed point of the mapping $u(x, \cdot) : V \to V$, then :*

i. g is continuous.

ii. If u is continuously differentiable ($\mathcal{C}^1$), then g is locally Lipschitz.

iii. If u is $\mathcal{C}^p$, then g is $\mathcal{C}^p$. Additionally, the derivative of g, $Dg(x)$, is given by

$$(id - D_2u(x, g(x))) - 1 \circ D_1u(x, g(x)).$$

Note: The conditions $\mathcal{C}^1$ and $\mathcal{C}^p$ apply to open sets with closures contained in $U \times V$.

Proof (of Theorem 2.9) Utilizing Theorem 2.4 and Proposition 2.2, the uniqueness of the solution for equation 2.9 is established. Considering the maximal interval of existence of

$x(t_0, u, f)$ as $[t_0 - r, t_0 + \beta)$.
The goal is to demonstrate that $x(t_0, u, f)$ is $\mathcal{C}^1$ with respect to u on $[t_0 - r, t_0 + \alpha]$, where $\alpha < \beta$. An open neighborhood U of u is chosen such that $x(t_0, v, f)$ is well-defined for any $v \in U$ on $[t_0 - r, t_0 + \alpha]$. Defining $K = \{(t, x_t) : t \in [t_0, t_0 + \beta)\}$, which is compact, and applying Lemma 2.5, parameters M, a, b, U and V are obtained. Selecting a such that

$$Ma \leq b \quad \text{and} \quad 0 < La < 1, \tag{2.8}$$

where La is a bound of the derivative of f with respect to u on Ω.
The solution transformation from Lemma 2.3, $y\left(t\right) = x\left(t_0 + t\right) - \widetilde{u}\left(t_0 + t\right) \;\; for \; t \in \bar{\mathrm{I}}_a$, and the map $T(t_0, u, f)$ from Lemma 2.7 are considered. By Lemma 2.3, $y(t)$ represents a fixed point of $T(t_0, u, f)$. The constraint 2.8 on a and b implies that $T(t_0, u, f)$ maps $A(a, b)$ into itself for each a, b, acting as a contraction.
Furthermore, the contraction constant remains independent of $(t_0, u, f) \in V \times U$. As $T(t_0, u, f)$ is $\mathcal{C}^P$ in Ω, according to Theorem 2.10, the fixed point $y(t_0, u, f)$ is $\mathcal{C}^p$ in Ω. A similar proof demonstrates that $x(t_0, u, f)(t)$ is $\mathcal{C}^P$ in f for $t \in [t_0, t_0 + a]$.

2.6 The solution map

Definition 2.10 *In an open set $\Omega \subset \mathbb{R} \times \mathrm{C}$, where $f : \Omega \to \mathbb{R}^n$ is continuous, the solution map is defined as follows :*

$$T\left(t, t_0\right) : \mathrm{C} \longrightarrow \mathrm{C}, \qquad u \longrightarrow x_t\left(t_0, u\right),$$

where $x_t(t_0, u)$ represents the solution of the Delay Differential Equation with a unique solution given by :

$$\begin{cases} \dot{x} = f\left(t, x_t\right), \\ x_{t_0} \equiv u. \end{cases}$$

By Theorem 2.8, T is continuous.

In the context of Ordinary Differential Equations, the solution map establishes a homeomorphism. Nevertheless, this assertion does not necessarily hold for Delay Differential Equations. Here, we present a concise overview of the properties of the solution map $T(t, t_0)$ that hold true for any equation where a unique solution exists for any specified initial condition.

Firstly, let us start with general facts :

Definition 2.11
- *In a metric space, a bounded set is defined as a subset that fits within a ball.*
- *A mapping between metric spaces is considered bounded if it transforms closed bounded sets into bounded sets.*
- *When a mapping from a topological space to a metric space maps some neighborhood of each point into a bounded set, it is termed locally bounded.*
- *For a mapping from a metric space to a topological space, compactness is achieved when it maps bounded sets to relatively compact sets.*
- *A mapping from a metric space to a topological space is locally compact if it maps a bounded neighborhood of each point into a relatively compact set.*

Corollary 2.3 *For $t \geq t_0$, the map $T(t, t_0)$ is locally bounded.*

Lemma 2.9 *For $t \geq t_0 + r$, the map $T(t, t_0)$ is locally compact.*

Proof (See [8], page 13).

Next, we will demonstrate that under certain additional conditions, the solution map becomes compact.

Definition 2.12
- *A mapping $T(\lambda) : X \to Y$ between metric spaces that depends on a parameter in a metric space Λ is considered to be uniformly bounded on compact sets of Λ if for every compact subset $\Lambda_0 \subset \Lambda$ and every bounded subset $U \subset X$, there exists a bounded subset $V \subset Y$ such that for every $\lambda \in \Lambda_0$, $T(\lambda)(U) \subset V$.*
- *A mapping $T(t, t_0) : X \to Y$ from a topological space to a metric space, where $t \geq t_0$, is termed conditionally compact when $T(t, t_0)(u)$ is continuous in (t, t_0, u), and for every bounded subset $V \geq Y$, there exists a compact subset $K \subset Y$ such that for all $t_0 \leq s \leq t$, if $T(s, t_0)(u) \in V$, then $T(t, t_0)(u) \in K$.*

Lemma 2.10 *If $T(t, t_0) : X \to Y$ is a mapping between metric spaces defined for $t \geq t_0$ and is uniformly bounded on compact sets of $[t_0, \infty)$, and if $T(t, t_0)$ is conditionally compact, then it is compact for $t \geq t_0$.*

Proof For a bounded subset $U \subset X$ and $t \geq t_0$, there exists a bounded subset $V \subset Y$ such that

$$T(s, t_0)(U) \subset V \quad for \quad t \geq s \geq t_0.$$

Additionally, there exists a compact subset $K \subset Y$ such that $T(t, t_0)(U) \subset K$. Therefore, $\overline{T(t, t_0)(U)}$ is compact.

Theorem 2.11 ***(Representation of the Solution Map)*** *Consider a bounded continuous map $f : \omega \to \mathbb{R}$. The solution map can be expressed as :*

$$T(t, t_0) = \phi(t - t_0) + \psi(t, t_0), \; for \; t \geq t_0,$$

where $\phi(t - t_0)$: $\mathrm{C} \to \mathrm{C}$ is defined as :

$$u(\tau) \mapsto \begin{cases} u(t - \tau) - u(0) & if \;\; t - \tau < 0, \\ 0 & if \;\; t - \tau \geq 0, \end{cases}$$

and $\psi(t, t_0) : \mathrm{C} \to \mathrm{C}$ is conditionally compact. Consequently, $T(t, t_0)$ acts as a contraction for $t > t_0$ and is conditionally compact for $t \geq t_0 + r$.

Proof (See [8], page 13).

The following corollary can be easily derived by first applying Theorem 2.11 and then utilizing Lemma 2.10.

Corollary 2.4 *states that if $f : \Omega \to \mathbb{R}^n$ is a bounded continuous map and if the solution map $T(t, t_0) : \mathrm{C} \to \mathrm{C}$ for $t \geq t_0$ is uniformly bounded on compact sets of $[t_0, +\infty)$, then :*

- *For $t \geq t_0 + r$, the map $\Psi(t, t_0)$ from Theorem 2.11 is compact.*
- *For $t \geq t_0 + r$, the map $T(t, t_0)$ is compact.*

2.7 Linear systems

A specific instance of a Delay Differential Equation is the linear Delay Differential Equation, for $(t_0, u) \in \mathbb{R} \times \mathrm{C}$, consider the equation :

$$\begin{cases} \dot{x}(t) = A(t)\, x_t + h(t), & for\ t \geq t_0, \\ x_{t_0} \equiv u. \end{cases} \tag{2.9}$$

Here, h is a continuous, and A is a linear and continuous function.

Theorem 2.12 *[8] states that there is a sole solution $x(t_0, u)$ for equation 2.9, which is defined on the interval $[t_0 - r, +\infty)$.*

Proof Due to the linearity and continuity of $A(t)$, it is Lipschitz, ensuring a locally unique solution as per Theorem 2.4. Considering x as a maximal solution of 2.9 on $[t_0 - r, +\infty)$, upon integrating the system, we have :

$$|x(t)| \leq |u(0)| + \int_{t_0}^{t} |A(s)\, x_s|\, ds + \int_{t_0}^{t} |h(s)|\, ds,$$

for any $t \in [t_0, a)$. This leads to :

$$\|x_t\| \leq \|u\| + \int_{t_0}^{t} \|A(s)\|\, \|x_s\|\, ds + \int_{t_0}^{t} |h(s)|\, ds.$$

By applying Gronwall's Lemma, we get :

$$\|x_t\| \leq \left(\|u\| + \int_{t_0}^{t} |h(s)|\, ds \right) \exp \int_{t_0}^{t} \|A(s)\|\, ds,$$

for any $t \in [t_0, a)$. The right-hand side of the above inequality is bounded locally for $t \in [t_0, +\infty)$, resulting in :

$$\sup_{t_0 \leq t < a} \|x_t\| = M < +\infty.$$

Furthermore, x is uniformly continuous on $[t_0, a)$ by the inequality

$$\left| x(t) - x\left(t'\right) \right| \leq M \int_{t}^{t'} \|A(s)\|\, ds + \int_{t}^{t'} |h(s)|\, ds \quad t_0 \leq t < t' < a.$$

So, $\{(t, x_t) : t_0 \leq t < a\}$ falls within a compact set in $\mathbb{R} \times \mathrm{C}$, contradicting Theorem 2.7.

Corollary 2.5 *states that if $x(t_0, u, h)$ is the solution of 2.9, then it can be expressed as $x(t_0, u, h) = x(t_0, u, 0) + x(t_0, 0, h)$, where the mappings*

$$x(t_0, \cdot, 0) : C \longrightarrow \mathcal{C}([t_0 - r, +\infty), \mathbb{R}^n), \qquad x(t_0, 0, \cdot) : \mathcal{C}([0, t], \mathbb{R}^n) \longrightarrow \mathcal{C}([t_0 - r, +\infty), \mathbb{R}^n),$$
$$u \longmapsto x(t_0, u, 0), \qquad h \longrightarrow x(t_0, 0, h),$$

are linear and continuous. Additionally, for $t \geq t_0$, we have :

$$|x(t_0, u, 0)(t)| \leq \|u\| \exp \int_{t_0}^{t} \|A(s)\| \, ds,$$

and

$$|x(t_0, 0, h)(t)| \leq \int_{t_0}^{t} |h(s)| \, ds \exp \int_{t_0}^{t} \|A(s)\| \, ds.$$

2.8 Types of Delay Differential Equations

Discrete delay DDEs : [4] In discrete delay DDEs, the delays occur at specific, discrete time intervals. For example, a discrete delay DDE could model the population of a city, where the birth rate and death rate are both affected by the population size at previous times. Its equation is of the form

$$y'(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_1), \ldots, y(t - \tau_n)).$$

Example 2.4 *In a population growth model, a discrete delay equation is represented by*

$$\frac{dx(t)}{dt} = rx(t - \tau),$$

where τ can represent the time it takes for the population to adapt to changes in available resources.

Distributed delay DDEs : [4] These are delay equations that deal with models in which the delay can vary randomly. Its equation is of the form

$$x'(t) = ax(t - \Delta),$$

where Δ represents the delay that varies randomly.

Example 2.5 *Let's assume we have a heat conduction model. The distribution of temperature in the* *rod can be represented by the following distributed delay equation :*

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u\left(x,t\right)}{\partial x^2} q\left(x, t - \tau\left(x\right)\right),$$

where $u(x,t)$ is the temperature at position x and time t, and a is the thermal diffusivity of the rod, and $q(x,t)$ is the internal heat source function, $\tau(x)$ is the delay kernel, which varies depending on the position x.

Variable-delay DDEs : [24] A Delay Differential Equation (DDE) with variable delay is a differential equation that involves the current value of a function as well as its values at variable time lags in the past. These time lags can be represented by a function $\tau(t)$, where t is the current time. Mathematically, a DDE with variable delay can be expressed as :

$$x^{'}(t) = f(t, x(t), x(t - \tau(t))).$$

Example 2.6 *Disease spread model with variable infection rate :*

$$x^{'}(t) = -\beta(x(t - \tau(t)))x(t),$$

where $x(t)$ is number of infected individuals at time t, $\beta(x)$ infection rate as a function of current number of infected individuals x, and $\tau(t)$ delay in effect, which may depend on current number of infected individuals or time.

2.8.1 Delay Differential Equations with a constant delay

In the preceding sections, we have examined the Banach space $\mathrm{C} = \mathcal{C}([0,r], \mathbb{R}^n)$ equipped with a uniform topology, where $\Omega \subset \mathbb{R} \times \mathrm{C}^2$ is an open subset and $f : \Omega \to \mathbb{R}^n$ is a continuous mapping. Consequently, an initial value problem is given by :

$$\begin{cases} \dot{x}\left(t\right) = f\left(t, x_t\left(\tau_1\right), x_t\left(\tau_2\right)\right), \\ x_{t_0} \equiv u, \end{cases} \tag{2.10}$$

where $x_t \in \mathrm{C}$ is defined $x_t(\tau) = x(t - \tau)$.

Now, our focus shifts to a specific instance of equation 2.10, namely a Delay Differential Equation with a constant delay. To achieve this, we consider the following differential equation :

$$\begin{cases} \dot{x}(t) = f(t, x_t(\tau_1(t)), x_t(\tau_2(t))), \\ \dot{\tau}_1 = 0, \\ \dot{\tau}_2 = 0, \\ \tau_1(0) = 0, \\ \tau_2(0) = 1. \end{cases}$$

This simplifies to :

$$\dot{x}(t) = f(t, x(t), x(t-1)). \tag{2.11}$$

The differential equation 2.11 can be interpreted as an Ordinary Differential Equation (ODE) if expressed as $\dot{x}(t) = f(t, x(t), \varphi(t))$. Consequently, the results established in ODE theory can be readily applied to each interval $[t_0 + k - 1, t_0 + k]$ with $k \geq 0$ being an integer. Specifically, if f is of class $\mathcal{C}^p$, then the solution is also of class $\mathcal{C}^p$ within each interval $(t_0 + k - 1, t_0 + k)$. Furthermore, at each point $t = t_0 + k$, we gain an additional level of differentiability up to $\mathcal{C}^p$.

Example 2.7 *One of the simplest examples is :*

$$\begin{cases} \dot{x}(t) = x(t-1), \\ x_{t_0} \equiv 1. \end{cases}$$

This initial value problem can be explicitly solved in each interval $[k-1, k]$ with $k \geq 0$. Specifically :

$$\begin{aligned} &t[-1,0], \quad x(t) = 1, \\ &t[-1,0], \quad x(t) = t + 1, \\ &t[-1,0], \quad x(t) = \frac{t^2}{2} + t + \frac{1}{2}, \\ &t[-1,0], \quad x(t) = \frac{t^3}{6} + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{6}, \\ &t[-1,0], \quad x(t) = \frac{t^4}{24} + \frac{t^3}{6} + \frac{t^2}{4} + \frac{t}{6} + \frac{1}{24}. \\ &\qquad\qquad \vdots \end{aligned}$$

In Figure 2.1, the cases of $\dot{x}(t) = x(t-1)$ and $\dot{x}(t) = x(t)$ are compared.

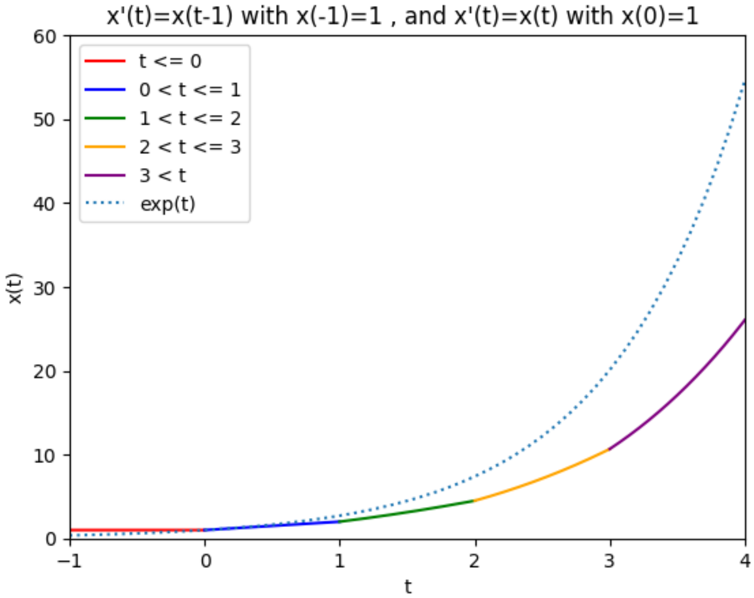

Figure 2.1: Graph of a DDE and its associated ODE.

Linear differential equation with a constant delay

The equation 2.9 with a constant delay transforms into :

$$\dot{x}(t) = Ax(t) + Bx(t-1) + h(t), \tag{2.12}$$

where A, B, and $\tau \geq 0$ are constants, and h is a continuous map provided. An immediate implication of Theorem 2.12 is :

Theorem 2.13 *[8] States that if u is a continuous function on the interval $[0,1]$, there exists a unique mapping $x(u,h)$ defined on the interval $[-1,+\infty)$ that matches u on $[-1,0]$ and satisfies Equation 2.12 for $t \geq 0$. Additionally, $x(u,h)(t)$ is continuously differentiable $\mathcal{C}^1$ for all $t > 0$ and is $\mathcal{C}^1$ at $t = 0$ if and only if $u(\tau)$ has a derivative at $\tau = 0$ such that*

$$\dot{u}(0) = Au(0) + Bu(-1) + h(0).$$

If the function h possesses derivatives of all orders, then $x(u,h)$ becomes increasingly smoother as t grows.

Proof If x is a solution of Equation 2.12 that aligns with u on the interval $[-1,0]$, then according to the ordinary variation-of-constants formula, x must adhere to the following conditions :

$$\begin{cases} x(t) = u(t), \quad t[-1,0], \\ x(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}(Bx(s-1) + h(s))\,ds, \quad t \geq 0. \end{cases} \tag{2.13}$$

Moreover, if x satisfies Equation 2.13, it must also satisfy Equation 2.12. The uniqueness aspect stems from its singularity in each interval $[k, k+1]$ for any integer $k \geq 0$. The remaining assertions can be deduced from Theorem 2.9.

Characteristic equation of a homogeneous linear differential equation with a constant delay :

Consider the linear Delay Differential Equation with a constant delay $\tau \geq 0$ given by :

$$\dot{x}(t) = Ax(t) + Bx(t-\tau). \tag{2.14}$$

This equation possesses a non-trivial solution $e^{\lambda t}c$ if, and only if, the following condition holds :

$$\lambda - A - Be^{-\lambda\tau} = 0.$$

The function $h(\lambda) = \lambda - A - Be^{-\lambda\tau}$ is referred to as the characteristic map of Equation 2.14. For any solution λ, the relationship $|\lambda - A| =| B | e^{-\tau Re\lambda}$ holds. Consequently, as $| \lambda |$ approaches infinity, $e^{-\tau Re\lambda}$ tends towards infinity.

Lemma 2.11 *States that for the linear Delay Differential Equation given $\dot{x}(t) = Ax(t) + Bx(t-\tau)$, a non-trivial solution $e^{\lambda t}c$ exists if, and only if, the equation*

$$\lambda - A - Be^{-\lambda\tau} = 0, \tag{2.15}$$

holds. If there exists a sequence (λ_j) of solutions such that $| \lambda_j | \to +\infty$ as $j \to \infty$, then

$$Re\lambda_j \to -\infty \quad as\ j \to \infty.$$

Consequently, there exists a real number α such that all solutions of 2.15 satisfy $Re\lambda < \alpha$, and only a finite number of solutions exist in any vertical strip in the complex plane.

Theorem 2.14 *[8] For the linear Delay Differential Equation given by $\dot{x}(t) = Ax(t) + Bx(t-\tau)$,* *where λ is a root of multiplicity m of the characteristic equation $h(\lambda) = \lambda - A - Be^{-\lambda\tau} = 0$, the solutions $t^k e^{\lambda t}$ with $k = 0, \cdots, m-1$ are valid solutions of the differential equation. As the equation is linear, any finite sum of such solutions remains a solution, and infinite sums are also solutions provided suitable conditions are met to guarantee convergence.*

Proof Assuming $x(t) = t^k e^{\lambda t}$, we have :

$$\begin{aligned} e^{-\lambda t}\left(\dot{x}\left(t\right) - Ax\left(t\right) - Bx\left(t-\tau\right)\right) &= t^k\lambda + kt^{k-1} - At^k - B\left(t-\tau\right)^k e^{-\lambda\tau} \\ &= t^k\lambda + kt^{k-1} - At^k - Be^{-\lambda\tau}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix} t^{k-j}\tau^j \\ &= \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix} t^{k-j}h^j\left(\lambda\right). \end{aligned}$$

If λ is a zero of $h(\lambda)$ with multiplicity m, then $h^{(j)}(\lambda) = 0$ for $j = 0, \cdots, m-1$. Thus, $x(t) = t^K e^{\lambda t}$ is a solution for $k = 0, \cdots, m-1$.

CHAPTER 3

Stability and Floquet Theory

This chapter provides a comprehensive introduction to the concepts of stability and Floquet theory in the context of delay differential equations.

3.1 Stability of solutions

Definition 3.1 *Let $\Omega \subset \mathbb{R} \times C$ be open, $f : \Omega \to \mathbb{R}^n$ continuous and $x = 0$ so that*

$$f(t, 0) = 0 \qquad \forall t.$$

Let $x(t_0, u)$ represent the solution to the initial value problem of $\dot{x} = f(t, x_t)$. The following classifications are defined:

- *$x = 0$ is stable if for all t_0 and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \geq t_0$, $\| u \| < \delta$ implies $\| x_t(t_0, u) \| < \varepsilon$.*
- *$x = 0$ is uniformly stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all t_0 and $t \geq t_0$, $\| u \| < \delta$ implies $\| x_t(t_0, u) \| < \varepsilon$.*
- *$x = 0$ is asymptotically stable if it is stable and for every t_0, there exists $\delta > 0$ such that $\| u \| < \delta$ implies that $x(t_0, u)(t)$ approaches 0 as $t \to +\infty$.*
- *$x = 0$ is uniformly asymptotically stable if it is asymptotically stable and for every t_0, there exists δ such that for every $\eta > 0$, there exists t_1 such that for all $t \geq t_0 + t_1$, $\| u \| \leq \delta$ implies*

$\| x_t(t_0, u) \| \leq \eta$.

Proposition 3.1 *Within an open set $\Omega \subset \mathbb{R} \times C$, where $f : \Omega \to \mathbb{R}^n$ is continuous and $\omega > 0$ such that*

$$f(t + \omega, v) = f(t, v), \quad \forall\ (t, v) \in \Omega.$$

If the differential equation $\dot{x} = f(t, x_t)$ has unique solutions, then the solution $x = 0$ is stable if and only if it is uniformly stable.

Proof $\Rightarrow$) Utilizing periodicity, for any $t \geq t_0$, any integer k, and any $u \in C$, we have :

$$x_t(t_0, u) = x_{t+k\omega}(t_0 + k\omega, u).$$

Specifically, $x_{t+k\omega}(t_0 + k\omega, u)$ satisfies :

$$\begin{aligned} \frac{\partial x\left(t_0 + k\omega, u\right)}{\partial t}\left(t + k\omega\right) &= f\left(t + k\omega, x_{t+k\omega}\left(t_0 + k\omega, u\right)\right) \\ &= f\left(t, x_{t+k\omega}\left(t_0 + k\omega, u\right)\right), \end{aligned}$$

and $x_{t_0+\omega} \equiv u$. Due to the uniqueness of solutions, $x_{t+t_0}(t_0, u) = x_{t+t_0+k\omega}(t_0+k\omega, u)$. Therefore, it is enough to show that

$$\forall \varepsilon > 0, \exists \delta > 0; \forall t_0 \in [0, \omega], \forall t \geq t_0, \| u \| < \delta \Rightarrow \| x_t(t_0, u) \| < \varepsilon.$$

For $0 \leq t_0 \leq \omega$ and $t \geq t_0$, $x_{t+\omega}(t_0, u) = x_{t+\omega}(\omega, x_\omega(t_0, u))$.
$\Leftarrow$) This is evident, Because uniformly stable define for all t_0.

3.2 Floquet Theory for Delay Differential Equations

Let's revisit the Floquet representation of a homogeneous linear periodic ordinary differential equation. In the case of the homogeneous linear ω-periodic system

$$\dot{x} = A(t)x, \quad \textit{and} \quad A(t + \omega) = A(t), \tag{3.1}$$

where $A(t)$ is a continuous n-by-n real or complex matrix function of t, each fundamental matrix solution $X(t)$ of 3.1 can be written as $X(t) = P(t)e^{Bt}$, with $P(t)$ and B being n-by-n matrices. Here, $P(t + \omega) = P(t)$ holds for all t, and B remains constant.

It is important to note that in functional differential equations, a complete Floquet theory does not exist. However, there are instances where a Floquet representation can be established.

Indexed families on a Banach space.

To utilize the Spectral Theory of a linear and continuous map, we introduce the concept of a periodic family of a Banach space.

Definition 3.2 *Consider X as a Banach space. An indexed family by $\mathbb{R} \times \mathbb{R}$ of linear and continuous maps $T(t, t_0) : X \to X$, where $t \geq t_0$, is defined as a family on X satisfying :*

- $T(t_0, t_0) = id$.
- $T(t, t_0) \circ T(t_0, s) = T(t, s)$ *for all* $t \geq t_0 \geq s$.

This family is termed as an ω-periodic family on X if there exists $\omega > 0$ such that :

- $T(t + \omega, t_0 + \omega) = T(t, t_0)$ *for all* $t \geq t_0$.
- *There exists $M > 0$ such that for all $0 \leq t_0 \leq \omega$ and $t_0 \leq t \leq t_0 + \omega$,*

$$\| T(t, t_0) \| \leq M.$$

Definition 3.3 *Consider an ω-periodic family $\{T(t, t_0)\}$ for $t \geq t_0$. The period map is defined as*

$$\begin{aligned} P(t_0) :& X \longrightarrow X \\ & u \longmapsto T(t_0 + \omega, t_0)(u). \end{aligned} \tag{3.2}$$

Lemma 3.1 *Given $P(t_0)$ as the period map of an ω-period family $\{T(t, t_0)\}$ for $t \geq t_0$, the following properties hold:*

- $P(t_0)$ *is linear and continuous.*
- $P(t_0 + \omega) = P(t_0)$.
- $P^k(t_0) = T(t_0 + k\omega, t_0)$.
- *The composition $T(t, t_0) \circ P^k(t_0)$ is equal to $P^k(t) \circ T(t, t_0)$.*

Proof i. We have $P(t) = T(t_0 + \omega, t_0)$ is both linear and continuous.

ii. $P(t_0 + \omega) = T(t_0 + \omega + \omega, t_0 + \omega) = T(t_0 + \omega, t_0) = P(t_0)$.

iii. By induction on k :

- For $k = 1$, it follows directly from the definition.
- For $k > 1$, $P^k(t_0) = T(t_0 + k\omega, t_0)$, then

$$\begin{aligned} T(t_0 + (k+1)\omega, t_0) &= T(t_0 + k\omega + \omega, t_0) \\ &= T(t_0 + k\omega + \omega, t_0 + \omega) \circ T(t_0 + \omega, t_0) \\ &= P^k(t_0 + \omega) \circ P(t_0) \\ &= P^k(t_0) \circ P\left(t_0\right) = P^{k+1}\left(t_0\right). \end{aligned}$$

iv. We find that

$$\begin{aligned} T\left(t, t_0\right) \circ P^k\left(t_0\right) &= T\left(t, t_0\right) \circ T\left(t_0 + k\omega, t_0\right) \\ &= T\left(t + k\omega, t_0 + k\omega\right) \circ T\left(t_0 + k\omega, t_0\right) \\ &= T\left(t + k\omega, t_0\right). \end{aligned}$$

and

$$\begin{aligned} P^k\left(t\right) \circ T\left(t, t_0\right) &= T\left(t + k\omega, t\right) \circ T\left(t, t_0\right) \\ &= T\left(t + k\omega, t_0\right). \end{aligned}$$

Proposition 3.2 *If $P(t_0)$ is the period map of an ω-period family $\{T(t, t_0)\}$ for $t \geq t_0$, then the non-zero point spectrum is invariant with respect to t_0. Specifically, the dimension of the eigenspace corresponding to a non-zero eigenvalue remains constant regardless of t_0.*

Proof Considering $\sigma_P^*(P(t_0))$ as the point spectrum excluding 0, we aim to demonstrate $\sigma_P^*(P(t_0)) = \sigma_P^*(P(t))$. This involves two inclusions :

$\subseteq$) For any non-zero eigenvalue λ and vector $u \neq 0$ such that $P(t_0)(u) = \lambda u$, using Lemma (3.1), we have :

$$0 = (P(t_0) - id\lambda)(u) = (T(t, t_0) \circ (P(t_0) - \lambda.id))(u) = ((p(t) - \lambda.id) \circ T(t, t_0))(u),$$

for $t \geq t_0$. Notably, $T(t,t_0)(u)$ remains an eigenvalue λ for $t \geq t_0$. If it were zero, there exists $k \neq 0$ such that $t + k\omega \geq t_0$, leading to :

$$0 = (T(t+k\omega,t) \circ T(t,t_0))(u) = T(t+k\omega,t_0)(u) = P^k(t_0)(u) = \lambda^k u.$$

This contradicts the assumptions $\lambda \neq 0$ or $u \neq 0$.

$\supseteq$) When $t_0 + k\omega > t$, by Lemma 3.1, we have

$$\sigma_P^*(P(t)) \subseteq \sigma_P^*(P(t_0+k\omega))$$

and

$$P(t_0+k\omega) = P(t_0)$$

then, $\sigma_p^*(P(t)) \subseteq \sigma_P^*(P(t_0+k\omega)) \subseteq \sigma_P^*(P(t_0)$.

Definition 3.4 *Given $P(t_0)$ as the period map of an ω-period family $\{T(t,t_0)\}$ for $t \geq t_0$:*

- *The non-zero point spectrum is represented as $\sigma_P^*(P)$. The elements within are referred to as characteristic multipliers or Floquet multipliers.*
- *A λ is termed a characteristic exponent if $e^{\lambda\omega}$ is a characteristic multiplier.*

Corollary 3.1 *λ is classified as a characteristic multiplier if there exists a non-zero vector u such that for all $t \geq t_0$, $T(t+\omega,t_0)u = \lambda T(t,t_0)u$.*

Proof $\Rightarrow$) Assuming $\lambda \in \sigma_P^*(P)$, there exists a non-zero vector u such that $P(t_0)u = \lambda u$. Consequently, we have :

$$T(t+\omega,t_0)u = T(t+\omega,t_0+\omega)T(t_0+\omega,t_0)u = T(t,t_0)P(t_0)u = \lambda T(t,t_0)u.$$

$\Leftarrow$) Setting $t = t_0$, and utilizing proposition (3.2). If $\lambda \in \sigma_P^*(P)$ then $P(t_0)y = \lambda y$ for some $y \neq 0$, take $y = T(t,t_0)u$ then

$$\begin{aligned}\lambda T(t,t_0)u &= P(t_0)T(t,t_0)u = P(t_0)T(t_0,t)u \\ &= T(t_0+\omega,t_0)T(t_0,t)u = T(t_0+\omega,t).\end{aligned}$$

Lemma 3.2 *If A is a square matrix with a non-zero and distinct eigenvalue, then there exists a matrix B such that $A = e^B$.*

Proof Consider $\lambda \neq 0$ as the eigenvalue, and let $\log \lambda$ be a complex value such that $e^{log\lambda} = \lambda$. The matrix $C = (log\lambda)Id$ satisfies :

$$e^C = \sum_{k\geq 0} \frac{C^k}{k!} = \sum_{k\geq 0} \frac{log^k\lambda}{k!} Id = e^{log\lambda} Id = \lambda Id.$$

Given that A is diagonalizable,if U is a nonsingular we have :

$$A = U(\lambda Id)U^{-1} = Ue^C U^{-1} = e^{UCU^{-1}} = e^B.$$

Theorem 3.1 *[8] Given $P(t_0)$ as a period map of an ω-period family $\{T(t,t_0)\}$ for $t \geq t_0$ in a Banach space X, if $P(t_0)$ is compact, then for any characteristic multiplier λ, there exist $\varphi_1 \ldots \varphi_d$ in X, a constant d-by-d matrix B, and a d-row vector $\varphi(t)$ in X such that :*

i. $\sigma(e^{B\omega}) = \{\lambda\}$.

ii. $\varphi(t_0) = (\varphi_1, \ldots, \varphi_d)$.

iii. $\varphi(t+\omega) = \varphi(t)$, $\forall t \in \mathbb{R}$.

iv. *For all $t \geq t_0$,*

$$T(t,t_0)\varphi(t_0) = \varphi(t)e^{B(t-t_0)}.$$

If ψ is any d-vector, then

$$T(t,t_0)\varphi(t_0)\psi = \varphi(t)e^{B(t-t_0)}\psi.$$

Furthermore, the generalized eigenspace of $P(t)$ for λ has the same rank $k \geq 1$ and basis $\varphi(t)$, with its dimension being independent of $t \in \mathbb{R}$.

Proof (See [8], p 22).

Floquet representation for linear Delay Differential Equations.

Proposition 3.3 *Suppose $A(t) : \mathrm{C} \to \mathbb{R}^n$ is a linear and continuous map with $\omega > 0$ such that $A(t+\omega) = A(t)$ for all t. The family of maps indexed by*

$$\begin{aligned} T\,(t,t_0) :&\mathrm{C} \longrightarrow \mathrm{C}, \quad t \geq, t_0 \\ &u \longrightarrow x_t\,(t_0,u)\,. \end{aligned} \tag{3.3}$$

forms an ω-periodic family on C. Here, $x_t(t_0, u)$ represents the solution defined on $[t_0 - r, +\infty)$ of the differential equation

$$\begin{cases} \dot{x} = A\left(t\right) x_t, \\ x_{t_0} \equiv u. \end{cases}$$

Notably, its period map

$$P\left(t_0\right): \mathrm{C} \longrightarrow \mathrm{C}$$
$$u \longrightarrow x_{t_0+\omega}\left(t_0, u\right).$$

is ω-periodic concerning t_0 and exhibits linearity and continuity properties.

Proof As per Theorem 2.12, for any $t_0 \in \mathbb{R}$ and $u \in \mathrm{C}$, there exists a solution$x(t_0, u)$ of 3.3 defined on $[t_0 - r, +\infty)$. Additionally, based on Theorem 2.8, $x(t_0, u)$ is continuous, and by Lemma 2.4, $x(t_0, u)$ is also continuous. By Corollary 2.5, $x(t_0, .)$ is linear, implying that $x_t(t_0, .)$ is linear as well. Consequently, $T(t, t_0)$ is linear and continuous whenever $t \geq t_0$. To verify the other conditions of Definition 3.2 :

i. $T(t_0, t_0)(u) = x_{t_0}(t_0, u) \equiv u$, indicating that $T(t_0, t_0) = \mathrm{id}$.

ii. $T(t, t_0) \circ T(t_0, s) = T(t, s)$. Indeed, by the uniqueness of solutions,

$$x_t(t_0, u) \circ x_{t_0}(s, u) = x_t(t_0, x_{t_0}(s, u)) = x_0(s, u) \quad for\ t \geq t_0 \geq s.$$

iii. $T(t + \omega, t_0 + \omega) = T(t, t_0)$. Suppose $x(t_0 + \omega, u)$ and $x(t_0, u)$ are solutions. Then,

$$\frac{\partial x\left(t_0 + \omega, u\right)}{\partial t}\left(t + \omega\right) = A(t + \omega)x_{t+\omega}\left(t_0 + \omega, u\right) = A\left(t\right) x_{t+\omega}\left(t_0 + \omega, u\right),$$
$$\frac{\partial x\left(t_0, u\right)}{\partial t}\left(t + \omega\right) = A(t + \omega)x_{t+\omega}\left(t_0, u\right) = A(t)x_{t+\omega}(t_0, u).$$

and

$$\frac{\partial x\left(t_0, u\right)}{\partial t}\left(t\right) = A\left(t\right) x_t(t_0, u).$$

By uniqueness, $x_{t+\omega}(t_0 + \omega, u) = x_{t+\omega}(t_0, u) = x_t(t_0, u)$ for all u.

iv. This conclusion follows from Corollary 2.5.

The period map is derived from Definition 3.3 and Lemma 3.1.

Thanks to Theorem 3.1, we are able to state the following Theorem 3.2.

Theorem 3.2 *[8] Consider a linear and continuous map $A(t) : \mathrm{C} \to \mathbb{R}^n$ with $\omega > 0$ such that for any t, $A(t+\omega) = A(t)$. Let $P(t_0)$ be its period map. For any characteristic multiplier λ, there exist a d-dimensional basis $\varphi_1, \ldots, \varphi_d$ of a $P(t_0)$-invariant vector subspace, a constant d-by-d matrix B, and an n-by-d matrix function $\phi(t)$ on C satisfying :*

i. $\sigma(e^{B\omega}) = \{\lambda\}$.

ii. $\varphi(t_0) = (\varphi_1, \ldots, \varphi_d)$.

iii. $\varphi(t+\omega) = \varphi(t)$ *for all* t *in* $\mathbb{R}$.

iv. For $t \geq t_0$,$x_t(t_0, \phi(t_0)) = \phi(t)e^{B(t-t_0)}$.

If ψ is any d-vector, then

$$x_t(t_0, \phi(t_0)\psi) = \phi(t)e^{B(t-t_0)}\psi.$$

Moreover,

v. The characteristic multiplier remains constant regardless of t_0.

vi. The generalized eigenspace of $P(t)$ for λ has a consistent rank of $k \geq 1$ and is based on $\phi(t)$, with its dimension being invariant with respect to $t \in \mathbb{R}$.

vii. $\lambda = e^{\mu\omega}$ serves as a characteristic multiplier when there exists a non-zero solution to the Delay Differential Equation $\dot{x} = A(t)x_t$ in the form of

$$x(t) = p(t)e^{\mu t},$$

where $p(t+\omega) = p(t)$.

Proof Given $\omega > 0$, there exists an integer $m > 0$ such that $m\omega \geq r$. Consequently, $P^m(t_0) = T(t_0 + m\omega, t_0)$ by Lemma 3.1. Hence, according to Corollary 2.4, $P^m(t_0)$ is compact. If we replace ω with $m\omega$, then the implications [i], [ii], [iii], [iv], and [vi] can be deduced from Theorem 3.1. Proposition 3.2 establishes the validity of [v]. Therefore, the only remaining task is to demonstrate [vii].

vii. Assuming $t_0 = 0$, we have :

$$x_t(0,u)(\tau) = x(0,u)(t-\tau) = x_{t-\tau}(0,u)(0),$$

for $0 \leq \tau \leq r$. If $u \neq 0$ is an eigenvector with eigenvalue λ for all $0 \leq \tau \leq r$, then: $\phi(t)(\tau) = \phi(t-\tau)(0)e^{B\tau}$. Letting $\tilde{\phi}(t-\tau) = \phi(t-\tau)(0)$ and $u = \phi(0)v$ for some v, we have for all t in $\mathbb{R}$:

$$x(0,\varphi(0)v)(t) = \tilde{\varphi}(t)e^{Bt}v.$$

$\Rightarrow$) As $\sigma(e^{B\omega}) = \{\lambda\} = \{e^{\mu\omega}\}$, it follows that $\sigma(B) = \{\mu\}$. Hence,

$$x(0,\varphi(0)v)(t) = \tilde{\varphi}(t)e^{\mu t}v.$$

$\Leftarrow$) To establish that $\lambda = e^{\mu\omega}$ is a characteristic multiplier, we observe:

$$P(0)(u)(0) = x_\omega(0,u)(0) = x(0,u)(\omega) = p(\omega)e^{\mu\omega} = p(0)\lambda.$$

Given that $p(0) \neq 0$ by assumption, we conclude that λ is indeed a characteristic multiplier.

Corollary 3.2 *Consider $A(t) : \mathrm{C} \to \mathbb{R}^n$ as a linear, continuous, and ω-periodic map, with $P(t_0)$ being its periodic map having λ as a characteristic multiplier.*

- *If $|\lambda| < 1$ for all λ, then the solution $x = 0$ is uniformly asymptotically stable.*
- *If $|\lambda| \leq 1$ for all λ, then the solution $x = 0$ is uniformly stable.*

3.3 Application in the epidemiologY

The Mathematical Infection Disease Model

[6] We study an (SI) model for infectious diseases that contains equations describing the interaction of susceptible (S) and infected (I) individuals of the form

$$\begin{cases} \frac{dS}{dt} = S\alpha(S) - \beta(S,I), \\ \frac{dI}{dt} = \beta(S,I) - \delta I, \end{cases} \tag{3.4}$$

with $S(t_0) = S_0 > 0,\ I(t_0) = I_0 > 0$, where $\alpha(S)$ represents the susceptible intrinsic growth rate of the susceptible population, $\beta(S, I)$represents the transmission rate of the disease expressing nonlinear mass action and δ is the addition of the death rate due to the presence of the disease μ and the natural death rate γ i.e,$\delta = \mu + \gamma$.
Equation 3.4 can be developed by adding new variables such as the recovered population R, becoming of the form

$$\begin{cases} \frac{dS}{dt} = \Pi - \beta SI - \mu S, \\ \frac{dI}{dt} = \beta SI - (\gamma + \mu) I, \\ \frac{dR}{dt} = -\mu R. \end{cases} \tag{3.5}$$

On defining the delay term, τ the associated delay differential equation becomes

$$\begin{cases} \frac{dS}{dt} = \Pi - \beta SI - \mu S, \\ \frac{dI}{dt} = \beta S(t-\tau) I - (\gamma + \mu) I, \\ \frac{dR}{dt} = -\mu R, \end{cases} \tag{3.6}$$

with $S(t_0) = S_0 > 0,\ I(t_0) = I_0 > 0,\ R(t_0) = R_0 > 0\ and\ t_0\,[0, \tau]$, where Π is the recruitment into the population, and τ is a discrete disease transmission time delay describing the time between infection and the time the susceptible individual gets infected.

Theorem 3.3 *[6] For initial data that is positive, the solutions of the equations 3.6 remain positive for all values $t \geq 0$.*

Proof From the first equation in system 3.6,

$$\frac{dS}{dt} = \Pi - \beta SI - \mu S.$$

We assert that $S(t) > 0$ holds for all $t > 0$. Suppose not, there exist $t_1 > 0$ and $\varepsilon_1 > 0$ such that $S(t) > 0$ for $t < t_1, S(t) = 0$ for $S(t) < 0$ when $t \in [t_1, t_1 + \varepsilon_1)$. Thus

$$\begin{aligned} \frac{dS}{dt} &= \Pi - \beta S(t_1) I(t_1) - \mu S(t_1) \\ &= \Pi > 0. \end{aligned}$$

This leads to a contradiction. Therefore, $S(t)$ is positive for all $t > 0$.

From the second equation of the system 3.5,

$$\frac{dI}{dt} = \beta S(t-\tau) I - (\gamma+\mu) I,$$
$$\frac{dI}{I} = [\beta S(t-\tau) - (\gamma+\mu)] dt,$$

Upon integration and considering the initial value, we obtain

$$I(t) = I_0(t) e^{\int_0^t [\beta S(\nu-\tau)-(\gamma+\mu)]d\nu},$$

this implies $I(t) > 0$ for all $t > 0$ since $I_0(t) > 0$. Hence $I(t)$ is positive for all $t > 0$.

Also, from the third equation of equations 3.5, $R'(t) = -\mu R$, $\frac{dR}{R} = -\mu dt$,and on integrating based on the initial value, we have $R(t) = R_0 e^{-\mu t}$.

This implies $R(t) > 0$ for all $t > 0$. Hence $R(t)$ is positive for all $t > 0$.

Theorem 3.4 [6] *The solutions of model 3.6 are ultimately bounded.*

Proof Let $N = S + I + R$.

$$\frac{dN(t)}{dt} = \Pi - \mu S - (\gamma+\mu) I - \mu R + \beta I (S(t-\tau) - S(t)).$$

Since coefficients of $\beta S(t-\tau)$ and $\beta S(t)$ are equal, solutions need not approach the origin but a positive limit.

Thus, $\frac{dN(t)}{dt} \leq \Pi - \mu S - (\gamma+\mu) I - \mu R$.

Choose a constant $\xi > 0$ such that

$$\frac{dN(t)}{dt} + \xi N \leq \Pi - \mu(S-\xi) - (\gamma+\mu)(I-\xi) - \mu(R-\xi).$$

Define $\xi = \min\{\mu, \gamma+\mu, \gamma\}$ such that

$$\frac{dN(t)}{dt} + \xi N \leq \Pi. \text{ Therefore, } N e^{\xi t} \leq \int \Pi e^{\xi t} dt + C \text{ yields}$$

$$N(t) = \frac{\Pi}{\xi} + \left(N_0 - \frac{\Pi}{\xi}\right) e^{-\xi t} \text{ and } \lim_{t\to\infty} N(t) \leq \frac{\Pi}{\xi}.$$

Hence the result.

Stability Analysis of an Infectious Disease Model

In order to determine the behaviours of the system, system is rewritten with the following expression

$$\begin{cases} \frac{dS}{dt} = \mathit{\Pi} - \beta SI - \mu S = \psi(S, I, R), \\ \frac{dI}{dt} = \beta S\left(t-\tau\right)I - \left(\gamma+\mu\right)I = \varphi(S, I, R), \\ \frac{dR}{dt} = -\mu R = \phi(S, I, R). \end{cases} \tag{3.7}$$

The equilibrium of system 3.6 can be obtained by solving the equations

$$\frac{dS}{dt} = 0, \frac{dI}{dt} = 0 \text{ and } \frac{dR}{dt} = 0. \tag{3.8}$$

Stability Analysis of an Infectious Disease Equilibrium with Delay

For stability analysis of positive equilibrium point at $E^* = (S^*, I^*, R^*)$, we solve system 3.7 for when $I \neq 0$ and obtained the equilibrium point $E^* = (S^*, I^*, R^*) = (\frac{\gamma+\mu}{\beta}, \frac{\Pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0)$. The linearised system of 3.6 at $E^* = (S^*, I^*, R^*) = (\frac{\gamma+\mu}{\beta}, \frac{\Pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0)$ yields

$$\begin{cases} \frac{d\psi}{dt} = -\beta I^* S - \mu S - \beta S^* I, \\ \frac{d\varphi}{dt} = \beta I^* S\left(t-\tau\right) + \beta\left(S^* - \left(\gamma-\mu\right)\right)I, \\ \frac{d\phi}{dt} = -\mu R. \end{cases} \tag{3.9}$$

Since linearisation in DDE is a variational equation, the $n \times n$ matrices evaluated at the steady state for endemic equilibrium $E^* = (S^*, I^*, R^*) = (\frac{\gamma+\mu}{\beta}, \frac{\Pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0)$ of equation 3.7 yields

$$\frac{d}{dt}\begin{pmatrix} \psi\left(t\right) \\ \varphi\left(t\right) \\ \phi\left(t\right) \end{pmatrix} = J_A^* \begin{pmatrix} \psi\left(t\right) \\ \varphi\left(t\right) \\ \phi\left(t\right) \end{pmatrix} + J_B^* \begin{pmatrix} \psi\left(t-\tau\right) \\ \varphi\left(t-\tau\right) \\ \phi\left(t-\tau\right) \end{pmatrix} \tag{3.10}$$

where J_A^* and J_B^* are partitioned matrices of order 3 and the associated non-zero reliable Jacobian matrix of the linearised system 3.10 is given by

$$J_A^* = \begin{pmatrix} -\left(\frac{(\Pi\beta-\mu(\gamma+\mu))}{(\gamma+\mu)}+\mu\right) & -(\gamma+\mu) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mu \end{pmatrix} \quad and \quad J_A^* = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\Pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For stability analysis of positive equilibrium of equations 3.7 at $E^* = (S^*, I^*, R^*)$, we consider the associated reliable Jacobian matrix of the characteristic equation of the form

$$\left|\lambda I - J_A^* - e^{-\lambda\tau}\right| = 0. \tag{3.11}$$

The characteristic equation of the form

$$\begin{vmatrix} \lambda + \left(\frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)}+\mu\right) & (\gamma+\mu) & 0 \\ -\left(\frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)}\right)e^{-\lambda\tau} & \lambda & 0 \\ 0 & 0 & \lambda+\mu \end{vmatrix} = 0 \tag{3.12}$$

$$\Longrightarrow \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0, \tag{3.13}$$

where,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \mu + \frac{\Pi\beta}{(\gamma+\mu)} \\ \frac{\mu\Pi\beta}{(\gamma+\mu)} \\ \Pi\beta - \mu\gamma - \mu^2 \\ \mu\Pi\beta - \mu^2\gamma - \mu^3 \end{pmatrix}. \tag{3.14}$$

Some known general tests can applied for negative root test, for stability analysis of equation 3.13, we assume that conditions for asymptotic stability for $\tau = 0$ are satisfied. That is at $\tau = 0$ equation 3.13 becomes

$$\lambda^3 + a_1\lambda^2 + (a_1 + a_3)\lambda + a_4 = 0.$$

The derived Routh-Hurwitz criterion for negative root test is given by

$$a_1 > 0, a_2 - \frac{a_3}{a_1} > 0, a_3 > 0.$$

For positive root test, we let $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$, $(\omega \in \mathbb{R})$ be the eigenvalues of equation 3.13, where $\eta(\tau)$ and $i\omega(\tau)$ depend on the intracellular delay. Since the positive equilibrium of equation 3.6 is stable when $\tau = 0$, it follows that $\eta(0) < 0$ when $\tau = 0$. By continuity, if $\tau > 0$, is sufficiently small we have $\eta(\tau) < 0$ and the positive steady state E^* remains stable. If $\eta(\tau_0) = 0$ for $\tau_0 > 0$ such that $\lambda(\tau_0)$ is a purely imaginary root of 3.14, then the positive steady state E^* losses its stability and eventually becomes unstable where $\eta(\tau)$ is positive. If $\lambda(\tau) = \eta(\tau_0) + i\omega(\tau_0)$ is the continuation of the root of $i\lambda$, it is necessary to confirm that the root continue into the positive half plane as τ increases past τ_k. By the procedure for positive root test, we analyse the positive equilibrium by replacing $\lambda = i\omega(\tau)$ in equation 3.13 to get :

$$(i\omega)^3 + a_1(i\omega)^2 + a_2(i\omega) + a_3(i\omega)e^{-(i\omega)\tau} + a_4e^{-(i\omega)\tau} = 0,$$

which yields,

$$-i\omega^3 - a_1\omega^2 + ia_2\omega + ia_3\omega(\cos\omega\tau - i\sin\omega\tau) + a_4(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{3.15}$$

Separating equation 3.15 above into real and imaginary parts, we have

$$a_1\omega^2 = a_3\omega\sin\omega\tau + a_4\cos\omega\tau, \tag{3.16}$$

$$-\omega^3 + a_2\omega = a_4\sin\omega\tau - a_3\omega\cos\omega\tau, \tag{3.17}$$

and adding the squares of 3.16 and 3.17 yields

$$\omega^6 + (a_1^2 - 2a_2)\omega^4 + (a_2^2 - a_3^2)\omega^2 - a_4^2 = 0. \tag{3.18}$$

By using the Intermediate Value Theorem, Equation 3.18 has a solution.

Now we calculate the value of τ_0 that transforms the equilibrium point E^* from a stable state to an unstable state.

we let $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ be the eigenvalues of equation 3.13 near $\tau = \tau_k$ satisfying $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. From equations 3.16 and 3.17, we have

$$a_3\cos\omega_0\tau_0 = a_4\left[\frac{a_1\omega_0}{a_3} - \frac{a_4\cos\omega_0\tau_k}{a_3\omega_0}\right] + \omega_0^3 - a_2\omega_0$$

$$a_3 \cos \omega_0 \tau_k + \frac{a_4^2 \cos \omega_0 \tau_k}{a_3 \omega_0} = \frac{a_1 a_4 \omega_0}{a_3} + \omega_0^3 - a_2 \omega_0$$

$$\tau_k = \frac{1}{\omega_0} \left[\arccos \left(\frac{a_1 a_4 \omega_0^2 + a_3 \omega_0^4 - a_2 a_3 \omega_0^2}{a_3^2 \omega_0^2 + a_4^2} \right) \right] + \frac{2k\Pi}{\omega_0}, \quad k = 0, 1, 2, 3, \ldots$$

at $\tau = \tau_0$, we have

$$\tau_0 = \frac{1}{\omega_0} \left[\frac{a_1 a_4 \omega_0^2 + a_3 \omega_0^4 - a_2 a_3 \omega_0^2}{a_3^2 \omega_0^2 + a_4^2} \right].$$

From the equation 3.14 and the given parameter values for $\beta = 0.31, \gamma = 0.6$ and $\mu = 0.1$ (Egbetade et al., 2018) we obtain

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1.491 \\ 0,139 \\ 0.904 \\ 0.0904 \end{pmatrix}.$$

Upon substitution and simplification of equation 3.13, equation 3.17 results in

$$\omega^6 + \left(a_1^2 - 2a_2\right) \omega^4 + \left(a_2^2 - a_3^2\right) \omega^2 - a_4^2 = 0,$$

and with the parameter values of

$$\alpha = a_1^2 - 2a_2 = 1.9448, \beta = a_2^2 - a_3^2 = 0.7978, \gamma = a_4^2 = 0.00817.$$

we have $\omega^6 + 1.9448\omega^4 + 0.7978\omega^2 - 0.00817 = 0.$

Substituting the parameter values into equation 3.13 results in the obtained value of $\omega_0 = 0.5972 \in \mathbb{R}_+$ and $\tau_0 = 1.9510$. The data is graphically represented in figures 1-3 below :

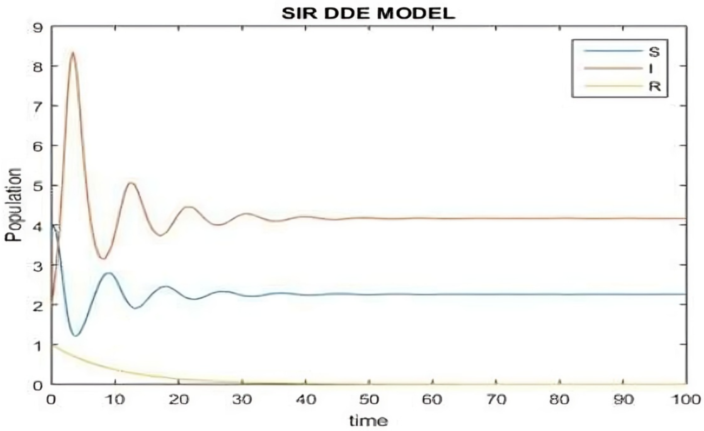

Figure 3.1: The Plot shows the stability of the Delay model when $\tau < \tau_0$.

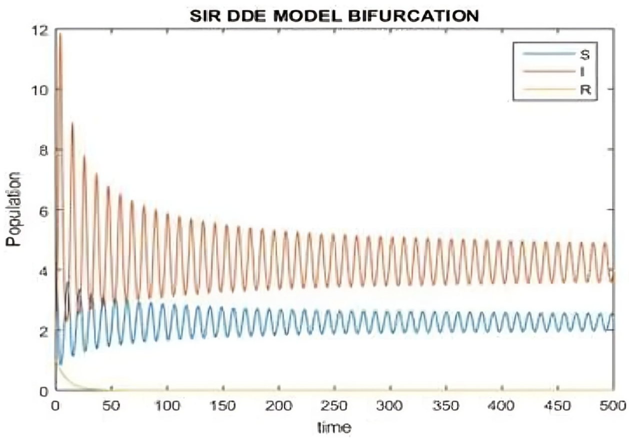

Figure 3.2: The Plot showing the bifurcation of the delay model at when $\tau = \tau_0$.

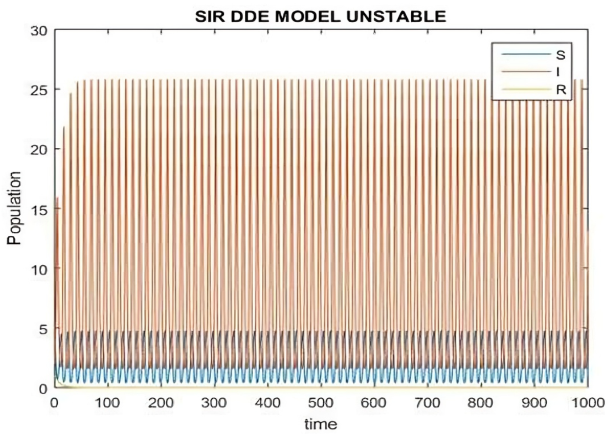

Figure 3.3: Plot showing the instability of the Delay model when $\tau > \tau_0$.

Conclusion

This thesis aims to shed light on an important field of advanced mathematics by addressing some of the challenges posed by time delays in dynamical systems, particularly in the context of delay differential equations. We have studied some of the main results and stability of differential equations with time delay. We have also observed that there are problems that cannot be solved using ordinary differential equations, and delay differential equations have emerged as a solution to these problems.

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