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Numerical Solution Of Fractional Differential Equations Using Iterative Reproducing Kernel Method

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Dedication

I dedicate this work:

- **To my dear father**, the dearest to my heart, who has helped and supported me all the time, and his blessed supplication had the greatest impact on steering the ship of my scientific life until it docked in this way.

- **To my dear mother**, who has supported me all the time, facilitated all difficulties for me, endured a lot and suffered for the sake of supporting me, and my standing in this place would not have happened without her constant encouragement and motivation

- **To my dear brothers**, the source of my strengths and pride, who have been waiting for this day for a long time and have had a great impact on assisting me to overcome several obstacles and difficulties

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List of Abbreviations

DEs	Differential Equations
FDEs	Fractional Differential Equations
Eq./Eqs.	Equation/ Equations.
C	Caputo fractional operator
RKHS	Reproducing Kernel Hilbert Space
RKA	Reproducing Kernel Algorithm
LDEs	Logistic Differential Equations
FLDEs	Fractional Logistic Differential Equations
FBDEs	Fractional Bernoulli Differential Equations
FRDEs	Fractional Riccati Differential Equations
IVP	Initial Value Problem
BVP	Boundary Value Problem
IC's	Initial Conditions
BC's	Boundary Conditions

List of Symbols

$\mathbb{C}$	Set of Complex Numbers
$\mathbb{R}$	Set of Real Numbers
$\mathbb{K}$	Set of Real or Complex Numbers
$L^p[a,b]$	Lebesgue Space
$\mathrm{W}_p^m[a,b]$	Sobolev Space
$\mathrm{R}_x(y)$	Reproducing Kernel Function
$\mathrm{K}_x(y)$	Reproducing Kernel Function
$\psi_i(x)$	Orthogonal Function System
$\widehat{\psi}_i(x)$	Orthonormal Function System
$\mathrm{L}u$	Linear Bounded Operator
$u(x)$	Analytical Solution
$u_n(x)$	Approximate Solution

ABSTRACT

In this work, an accurate numerical approximation algorithm based on the reproducing kernel Hilbert space (RKHS) approach has been proposed to solve a class of fractional differential equations within the framework of the Caputo sense. The analytical solution is presented as a convergent series with accurately computable structures in the reproducing kernel space. The n-term approximation has been obtained and proven to converge uniformly to the analytical solution. The main advantage of the RKHS approach is its direct application without requiring linearization or perturbation, thereby avoiding errors associated with discretization. Several numerical examples are provided to demonstrate the accuracy of the computations and the effectiveness of the proposed approach. The numerical results indicate that the RKHS method is a powerful tool for finding effective approximated solutions to such systems arising in applied mathematics, physics, and engineering.

Introduction

At the end of the nineteenth century, Liouville and Riemann introduced the first definition of fractional derivatives. However, the concept of fractional derivatives and integrals, as an extension of traditional integer-order calculus, was already mentioned in 1695 by Leibniz and L'Hôpital. In recent years, fractional differential equations (FDEs) have been discovered in various fields such as physics, chemistry, and engineering (Podlubny, 1999; Miller and Ross, 1993). Due to the lack of exact analytical solutions for most fractional differential equations, approximation and numerical techniques are employed. These techniques include Adomian's decomposition method (ADM) (Adomian, 1994), the variational iteration method (VIM) (Odibat and Momani, 2006), the differential transform method (DTM) (Ertürk and Momani, 2008), and the homotopy perturbation method (HPM) (Abdulaziz, et al., 2008).

The Reproducing Kernel Hilbert Space Method (RKHSM) stands as a prominent numerical approach for tackling fractional differential equations (FDEs). The foundational concept of reproducing kernels was initially introduced by three mathematicians in Berlin: (Szegö, 1921), (Bergman, 1922), and (Bochner, 1922). In 1935, E. Moore further delved into positive definite kernels within his general analysis, characterizing them as positive Hermitian matrices. It was not until 1950 that N. Aronszajn formally introduced the term "Reproducing Kernel Function" and established the existence and uniqueness of a reproducing kernel Hilbert space. Subsequently, in 1986, Cui demonstrated that $W_2^1[a, b]$ constitutes a Hilbert space with a reproducing kernel function expressible by finite terms, marking the inception of the application of reproducing kernel theory across diverse domains. The general theory of reproducing kernel Hilbert spaces and its myriad applications were elucidated by S. Saitoh in 1988.

In recent years, numerous researchers have harnessed the RKHS method to obtain analytical approximations for a wide array of problems, including regular and singular initial value problems (IVPs), regular and singular boundary value problems (BVPs), system of regular and singular IVPs and BVPs, regular and singular integral equations (IEs), partial differential equations (PDEs), and inverse problems in PDEs. Additionally, reproducing kernel theory finds significant applications in probability and statistics (Berlinet, A. and Thomas, C. 2004). Although relatively few papers have utilized the reproducing kernel method for solving FDEs, notable contributions include the algorithm proposed by (Geng, F., and Cui, M. 2012) for solving nonlocal fractional boundary problems, as well as the investigation conducted by (Zhang Y., Niu, J., and Lin, C.2012) on the three-point boundary value problems of FDEs.

In this work, we adapt the RKHS method to provide precise solutions for a class of fractional differential equations. Our numerical findings affirm the method's accuracy and efficiency in solving such equations. The analytical solution $u(x)$ is expressed as a series in the reproducing kernel space, and the approximate solution $u_n(x)$ is derived by truncating the series to n-terms.

This work follows a structured outline. In Chapter One, we delve into the fundamental principles and definitions of functional analysis and fractional calculus. Chapter Two provides an in-depth exploration of RKHS, including fundamental concepts, definitions, and theorems. Additionally, we redefine the inner product of a reproducing kernel space to derive the analytical approximate solution for a general form of ordinary differential equations . We also present an analysis of the RKHS method and introduce an efficient algorithm based on this method.

Moving to Chapter Three, we apply the RKHS method to approximate solutions for fractional logistic differential equations. Various numerical examples are presented to demonstrate the method's efficiency and accuracy.

CHAPTER 1

Review of Fundamental Mathematical Concepts.

This chapter aims to review essential concepts and necessary preliminaries to support this thesis, facilitating the reader's understanding of the content without the need to refer to additional sources. The chapter comprises two main sections: the first covers basic symbols and theories in functional analysis, while the second discusses fundamental concepts in fractional calculus.

1.1 Foundational Concepts in Functional Analysis.

Definition 1.1.1. *A norm on a vector space X is a function $\|.\| : X \to [0, \infty)$ that satisfiesthe following properties:*

(i) $\|x\| = 0$ *if and only if* $x = 0$*,*

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$*,*

(iii) $\|x + y\| \leq \|x\| + \|y\|$*,*

for all x,$y \in X$ and $\alpha \in \mathbb{F}$ (where $\mathbb{F}$ is a field, either real or complex).

Definition 1.1.2. *A normed space X is a vector space endowed with a norm defined on it.*

Definition 1.1.3. *A sequence (x_n) in a normed space X is said to be convergent if there exists an element $x \in X$ such that the limit of the norm of the difference between x_n and x as n approaches infinity is zero, i.e.*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

In such a case, we denote $x_n \to x$ and refer to x as the limit of the sequence (x_n).

Definition 1.1.4. *A sequence (x_n) in a normed space X is said to be Cauchy if, for every $\varepsilon > 0$, there exists a positive integer N such that the norm of the difference between any two terms x_m and x_n is less than ε for all $m, n > N$, i.e.*

$$\|x_m - x_n\| < \varepsilon \quad \text{for all } m, n > N.$$

Corollary 1.1.5. *A normed space X is complete if and only if every Cauchy sequence X converges in X.*

Remark 1.1.6. *A complete normed space is called a Banach space.*

Definition 1.1.7. *A vector space X is considered an inner product space if there exists a mapping $\langle .,. \rangle : X \times X \to \mathbb{F}$, satisfying the following properties for all $x, y, z \in X$ and $\alpha \in \mathbb{F}$:*

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

(iii) $\langle x, z \rangle = \overline{\langle z, x \rangle}$

(iv) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

Definition 1.1.8. *A Hilbert space is a complete inner product space.*

Definition 1.1.9. *An inner product space, also known as a pre-Hilbert space, denoted by* X, *is a vector space equipped with an inner product.*

Definition 1.1.10. ***(Cauchy-Schwartz inequality).*** *Let* X *be a pre-Hilbert space. Then*

$$\forall x, y \in \mathrm{X}, \;\; |\langle x, y \rangle| \leq \|x\| \|y\|.$$

When, x and y are linearly dependent, the inequality becomes an equality.

Definition 1.1.11. *Consider vector spaces X and Y over the same field $\mathbb{F}$, and let $\mathcal{D}(T)$ be a subspace of X. A mapping $T : \mathcal{D}(T) \subset X \to Y$, satisfying $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ is referred to as a linear operator. If $\mathcal{D}(T) = X$, then, and only then, we write $T : X \to Y$.*

Definition 1.1.12. *A linear operator $T : X \to Y$ between two normed spaces X and Y is considered bounded if there exists a real number $c > 0$ such that for all $x \in \mathcal{D}(T)$*

$$\|Tx\| \leq c\,\|x\|.$$

Definition 1.1.13. *Consider an operator $T : \mathcal{D}(T) \to Y$, not necessarily linear, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces the operator T is continuous at an $x_0 \in \mathcal{D}(T)$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|Tx - Tx_0\| < \varepsilon$ for any $x \in \mathcal{D}(T)$ satisfying $\|x - x_0\| < \delta$.*

Definition 1.1.14. *T is continuous if T is continuous at every $x \in \mathcal{D}(T)$.*

Theorem 1.1.15. *A linear operator $T : \mathcal{D}(T) \to Y$, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces, is continuous if and only if it is bounded.*

Definition 1.1.16. *A linear functional f is an operator that maps from a vector space X to the scalar field $\mathbb{F}$, denoted as $f : X \longrightarrow \mathbb{F}$. In other words, f is a linear operator whose domain is the vector space X and whose range is the scalar field $\mathbb{F}$.*

Definition 1.1.17. *A bounded linear functional f is a bounded linear operator. Thus there exists a real number $c > 0$ for such that all $x \in \mathcal{D}(f)$, $|f(x)| \leq c\,\|x\|$.*

Theorem 1.1.18. ***(Riesz representation Theorem)*** *Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z \rangle$ where z depends on f is uniquely determined by f and has norm $\|z\| = \|f\|$.*

Definition 1.1.19. *Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert-adjoint operator T^* of T is the operator $T^* : H_2 \longrightarrow H_1$ such that for all $x \in H_1$ and $y \in H_2$,*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

Note : An operator $T : H \longrightarrow H$ is self-adjoint if $T = T^*$, where T^* denotes the Hilbert-adjoint of T.

Definition 1.1.20. *Let Ω be a domain in $\mathbb{R}^n$ and let $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω, for which $\int_\Omega |u(x)|^p \, dx < \infty$.*
$L^p(\Omega)$ is Banach spaces with respect to the norms $\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}}$.

Definition 1.1.21. *A function $u : [a, b] \longrightarrow \mathbb{R}$ is called absolutely continuous (Abs. C), if for every positive ε, there exists a positive δ such that for any finite set of disjoint intervals $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$ $(\subset [a, b])$ with $\sum_{j=1}^{k} |y_j - x_j| < \delta$, then*

$$\sum_{j=1}^{k} |u(y_j) - u(x_j)| < \varepsilon.$$

Theorem 1.1.22. ***(Fundamental theorem of Lebesgue integral calculus)***
Let $u : [a, b] \longrightarrow \mathbb{R}$ be a function. Then u is absolutely continuous if and only if there is a function $v \in L^1[a, b]$ such that $u(x) = u(a) + \int_a^x v(t) \, dt$, $\forall x \in [a, b]$.

1.2 Foundational Concepts in Fractional Calculus

Fractional calculus extends traditional calculus by dealing with integrals and derivatives of non-integer orders. Its origins date back to the works of prominent mathematicians like Leibniz and Riemann, Liouville, Grünwald and Letnikov (Oldham, 1974; Podlubny, 1999). Fractional derivatives possess dynamic memory, enabling precise modeling of complex systems, unlike integer-order derivatives. They offer more accurate representations of real-world phenomena, making them invaluable in various fields. This section introduces basic tools and definitions of fractional derivatives with singular kernels, but prior knowledge of special functions is required for a deeper understanding.

1.2.1 Special Functions

1. **The Gamma Function**

   The Gamma function, attributed to Euler, is widely employed in fractional-order differential equations. Its definition, denoted by $\Gamma(x)$ can be expressed as follows:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}^+. \tag{1.2.1}$$

   Hence, the Gamma function exhibits the following properties:

   (a) $\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R}^+.$

   (b) $\Gamma(x) = (x-1)!, \quad x \in \mathbb{N}.$

   (c) $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$

   (d) $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

2. **The Beta Function**

The Beta function plays a crucial role in computing fractional derivatives of power functions. It is defined by the two-parameter integral as follows:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x, y \in \mathbb{R}^+. \tag{1.2.2}$$

The Beta function can also be defined in terms of the Gamma function:

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \;\; x, y \in \mathbb{R}^+.$$

The Beta function demonstrates the symmetric property: $B(x,y) = B(y,x)$, evident from its definition.

1.2.2 Fractional Integration and Differentiation

Definition 1.2.1. ***(Diethelm, 2010)*** *Let $\alpha \in \mathbb{R}^+$. The Riemann-Liouville fractional integral operator of order α, denoted by J_a^α, is defined on $L^1[a,b]$ as follows:*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{1.2.3}$$

for $a \le x \le b$.

Note that: For $\alpha = 0$, we set $J_a^0 = I$, the identity operator.

The integral operator shares a crucial property with fractional integration, which can be stated as follows:

Theorem 1.2.2. ***(Diethelm, 2010)*** *Let $\alpha, \beta \ge 0$ and $f \in L^1[a,b]$. Then,*

$$J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f = J_a^\beta J_a^\alpha f.$$

Holds almost everywhere on $[a, b]$. *If additionally* $f \in C[a, b]$ *or* $\alpha + \beta \geq 1$, *then the above identity holds everywhere on* $[a, b]$.

Example 1.2.3. *let* $f(x) = (x-a)^c$ *for some* $c > -1$ *and* $\alpha > 0$. *Then,*

$$J_a^\alpha f(x) = \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)}(x-a)^{\alpha+c}. \tag{1.2.4}$$

Proof : We perform a direct derivation:

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^c (x-t)^{\alpha-1}\, dt.$$

By substituting $t = a + s(x-a)$, we simplify to:

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha+c} \int_0^1 s^c (1-s)^{\alpha-1}\, ds \\ &= \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)}(x-a)^{\alpha+c}. \end{aligned}$$

This thorough derivation establishes the fractional integration of $J_a^\alpha f(x)$ with respect to x over the interval $[a, x]$.

□

Definition 1.2.4. ***(Diethelm, 2010)*** *Let* $\alpha \in \mathbb{R}^+$ *and* $n-1 < \alpha < n$. *The operator* D_a^α, *defined by*

$$D_a^\alpha f = D^n J_a^{n-\alpha} f = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt, \tag{1.2.5}$$

for $a \leq x \leq b$, *is called the Riemman-Liouville fractional differential operator of order* α.

Certainly:

- For $\alpha = 0$, we define $D_a^0 = I$, the identity operator.
- As a first consequence of this definition, we note that if $\alpha \in \mathbb{N}$ the operator D_a^α in Definition (1.2.4) coincides with the classical differential operator D^α.

Lemma 1.2.5. *Let $\alpha \in \mathbb{R}^+$ and let $n \in \mathbb{N}$ such that $n > \alpha$. Then,*

$$D_a^\alpha = D^n J_a^{n-\alpha}.$$

Example 1.2.6. *Let $f(x) = (x-a)^c$ for some $c > -1$ and $\alpha > 0$. Then,*

$$D_a^\alpha f(x) = \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha}. \tag{1.2.6}$$

Proof : Let $n \in \mathbb{N}$ such that $n = [\alpha]$. Thus,

$$\begin{aligned} D_a^\alpha f(x) &= D_a^\alpha (x-a)^c \\ &= D^n J_a^{n-\alpha} (x-a)^c \\ &= D^n \left(\frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} (x-a)^{n-\alpha+c} \right) \\ &= \frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} D^n (x-a)^{n-\alpha+c} \\ &= \frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} \frac{\Gamma(n-\alpha+c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} \\ &= \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} \end{aligned}$$

□

Remark 1.2.7. *The property $D_a^\alpha D_a^\beta f = D_a^{\alpha+\beta} f = D_a^\beta D_a^\alpha f$ is not satisfied in both equalities as seen in the following example.*

Example 1.2.8.

a. *Let* $f(x) = x^{-\frac{1}{2}}$ *and* $\alpha = \beta = \frac{1}{2}$. *Then,*

$$\begin{aligned} D_0^{\alpha} f(x) &= D_0^{\beta} f(x) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}+1-\frac{1}{2}\right)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} \\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(0)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} = 0. \end{aligned}$$

Hence, $D_0^{\alpha} D_0^{\beta} f(x) = 0$.

But,

$$D_0^{\alpha+\beta} f(x) = D^1 f(x) = \frac{-x^{-\frac{3}{2}}}{2}.$$

b. *Let* $f(x) = x^{\frac{1}{2}}$ *and* $\alpha = \frac{1}{2}$ *and* $\beta = \frac{3}{2}$. *Then,*

$$\begin{aligned} D_0^{\alpha} f(x) &= \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1-\frac{1}{2}\right)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)} = \frac{\sqrt{\pi}}{2}, \end{aligned}$$

and

$$D_0^{\beta} f(x) = 0,$$

this implies

$$D_0^{\alpha} D_0^{\beta} f(x) = 0.$$

But

$$D_0^{\beta} D_0^{\alpha} f(x) = \frac{-x^{-\frac{3}{2}}}{2} = D_0^{\alpha+\beta} f(x).$$

Now we state some relations between Riemann-Liouville integral and the differential operators:

Theorem 1.2.9. *Let* $\alpha \geq 0$. *Then for every* $f \in L^1[a,b]$,

$$D_a^\alpha J_a^\alpha f = f, \tag{1.2.7}$$

almost everywhere.

Proof : The case $\alpha = 0$ is trivial.

For $\alpha > 0$, let $n = [\alpha]$. Then,

$$D_a^\alpha J_a^\alpha f(x) = D^n J_a^{n-\alpha} J_a^\alpha f(x) = D^n J_a^n f(x) = f(x).$$

□

We have thus established that D_a^α serves as the left inverse of J_a^α. However, we cannot assert its role as the right inverse. Specifically, the following theorem provides clarification.

Theorem 1.2.10. ***(Diethelm, 2010)*** *Let* $\alpha \geq 0$ *and* $n-1 < \alpha \leq n$. *Then,*

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{z \to a^+} D^{n-k-1} J_a^{n-\alpha} f(z). \tag{1.2.8}$$

Specifically, for $0 < \alpha < 1$, *we have:*

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z \to a^+} J_a^{1-\alpha} f(z). \tag{1.2.9}$$

In 1967, M. Caputo introduced a groundbreaking concept: the Caputo fractional derivative. This novel approach not only redefined fractional derivatives but also shed light on their connection to the fractional Riemann-Liouville derivative. This linkage was pivotal, offering profound insights into fractional calculus and opening doors for its wide-ranging applications across diverse fields in science and engineering.

Definition 1.2.11. *Let $\alpha \in \mathbb{R}^+$ and $n-1 < \alpha \leq n$. The operator ${}^C D_a^\alpha$, defined by*

$$ {}^C D_a^\alpha f(x) = J_a^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t)\, dt \tag{1.2.10} $$

for $a \leq x \leq b$, is called the Caputo differential operator of order α

Example 1.2.12. *Let $\alpha \geq 0$ and $n-1 < \alpha \leq n$ and $f(x) = (x-a)^c$ for some $c \geq 0$. Then,*

$$ {}^C D_a^\alpha f(x) = \begin{cases} 0 & \text{if } c \in \{0,1,2,...,n-1\} \\ \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} & \text{if } c \in \mathbb{N} \text{ and } c \geq n \\ & \text{or } c \notin \mathbb{N} \text{ and } c > n-1 \end{cases} $$

Please refer to (Diethelm, K. 2010) for the detailed proof.

Theorem 1.2.13. *If f is continuous and $\alpha > 0$, then*

$$ {}^C D_a^\alpha J_a^\alpha f = f. $$

Moreover, it's worth noting that the Caputo derivative does not serve as the right inverse of the Riemann-Liouville integral.

Theorem 1.2.14. *Let $\alpha \geq 0$ and $n-1 < \alpha \leq n$ and. Then,*

$$ J_a^\alpha({}^C D_a^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{k!} (x-a)^k . \tag{1.2.11} $$

Proof : Please refer to (Diethelm, K. 2010) for the detailed proof. □

In this thesis, our objective is to solve the fractional differential equation (FDE) of the following form:

$$ {}^{C}D_{a}^{\alpha}u\left(x\right)=f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right),\quad a\leq x\leq b,\quad m-1<\alpha\leq m \qquad (1.2.12)$$

subject to conditions

$$\begin{aligned} u^{(i)}\left(a\right) &= c_i, \quad i=0,1,2,...,r-1, && (1.2.13)\\ u^{(i)}\left(b\right) &= d_i, \quad i=r,r+1,...,m-1. \end{aligned}$$

where ${}^{C}D_{a}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha, c_i, 0\leq i\leq r-1$ and $d_i,\ r\leq i\leq m-1$ are real constants, $u\left(x\right)$ is unknown function to be determined and $f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right)$ is a linear or nonlinear depending on the problem discussed.

Now, apply the operator J_{a}^{α} of both sides of equation (1.2.12), then by using (1.2.11) we have:

$$u\left(x\right)-\sum_{k=0}^{m-1}u^{(k)}\left(a\right)\frac{\left(x-a\right)^{k}}{k!}=F\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right), \qquad (1.2.14)$$

where, $F\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right)=J_{a}^{\alpha}\left(f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right)\right).$

CHAPTER 2

Reproducing Kernel Hilbert Space

2.1 Reproducing Kernel Hilbert Space

Definition 2.1.1. *Let X be an arbitrary set and K be a kernel on X, defined as:*

$$K : X \times X \to \mathbb{C}$$

1. *The kernel K is called* ***Hermitian*** *if for any finite set of points $\{x_1, ..., x_n\} \subseteq X$ and any complex numbers $\alpha_1, ..., \alpha_n$ we have $\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j K(x_i, x_j) \in \mathbb{R}$.*

2. *The kernel K is called* ***positive definite*** *if $\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j K(x_i, x_j) \geq 0$.*

Definition 2.1.2. ***(Aronszajn, 1950)*** *Let H be a Hilbert space of functions $f : X \longrightarrow \mathbb{F}$ on a set X. A function $K : X \times X \longrightarrow \mathbb{C}$ is a reproducing kernel of H if the following are satisfied:*

1. *$K(., x) \in H$ for all $x \in X$.*

2. *$\langle f, K(., x)\rangle = f(x)$ for all $f \in H$ and for all $x \in X$.*

The last condition is called "The reproducing property", the value of the function f at the point x is reproduced by the inner product of f with $K(., x)$. Further, one can rewrite the first condition as: for all $x \in X$, $k_x(y) = k(x, y)$ as a function of y belongs to H, $y \in X$. So applying the reproducing property to the function k_x at y, we get:

$$k_x(y) = \langle k_x, k_y\rangle, \quad \text{for } x, y \in X.$$

Consequently, for all $x \in X$ we obtain $\|k_x\|^2 = \langle k_x, k_x\rangle = k\langle x, x\rangle$.

Definition 2.1.3. *A Hilbert space H of functions on a set X is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel k of H.*

Theorem 2.1.4. ***(Aronszajn, 1950)*** *If a Hilbert space H of functions on a set X admits a reproducing kernel, then the reproducing kernel $k(x,y)$ is uniquely determined by the Hilbert space H.*

Proof : Let $k(x,y)$ be a reproducing kernel of H. Suppose that there exists another kernel $R(x,y)$ of H. Then, for all $x \in X$, applying the reproducing property for k and R we get:

$$\begin{aligned}\|k_x - R_x\|^2 &= \langle k_x - R_x, k_x - R_x\rangle = \langle k_x - R_x, k_x\rangle - \langle k_x - R_x, R_x\rangle \\ &= (k_x - R_x)(x) - (k_x - R_x)(x) = 0.\end{aligned}$$

Hence $k_x = R_x$, that is, $k_x(y) = R_x(y)$, for all $y \in X$. This means that $k(x,y) = R(x,y)$ for all $x, y \in X$. □

Theorem 2.1.5. ***(Aronszajn, 1950)*** *For a Hilbert space H of functions on X, there exists a reproducing kernel K for H if and only if for every x of X, the Dirac functional $\delta_x : f \to f(x)$ is a bounded linear functional on H.*

Theorem 2.1.6. *The reproducing kernel $k(x,y)$ of a reproducing kernel Hilbert space H is a positive definite kernel.*

Proof : we have

$$0 \leq \left\|\sum_{i=1}^{n} \alpha_i k_{x_i}\right\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i k_{x_i}, \sum_{i=1}^{n} \alpha_i k_{x_i} \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle k_{x_i}, k_{x_j}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} k(x_i, x_j).$$

Hence, $\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}k\left(x_i,x_j\right)\geq 0$ □

Propertie 2.1.7. ***(Aronszajn, 1950)*** *Let H be a RKHS and its kernel $k\left(x,y\right)$ on X, then, for all $x,y\in X$, we have:*

(i) $\left|k\left(x,y\right)\right|^2\leq k\left(x,x\right)k\left(y,y\right)$.

(ii) *Let $x_0\in X$. Then the following are equivalent:*

- **(a)** $k\left(x_0,x_0\right)=0$.
- **(b)** $k\left(x_0,y\right)=0$ *for all* $y\in X$.
- **(c)** $f(x_0)=0$ *for all* $f\in H$.

Proof : (i) By Schwartz Inequality in H we have

$$\begin{aligned}\left|k\left(x,y\right)\right|^2 &= \left|\left\langle k_x,k_y\right\rangle\right|^2\leq\left(\left\|k_x\right\|\left\|k_y\right\|\right)^2=\left\|k_x\right\|^2\left\|k_y\right\|^2\\ &= \left\langle k_x,k_x\right\rangle\left\langle k_y,k_y\right\rangle=k\left(x,x\right)k\left(y,y\right).\end{aligned}$$

(ii) It follows from (i) that $\left|k\left(x_0,y\right)\right|^2\leq k\left(x_0,x_0\right)k\left(y,y\right)=0$. Hence $k\left(x_0,x_0\right)=0$ is equivalent with $k\left(x_0,y\right)=0$ for all $y\in X$ if and only if $f(x_0)=0$ for all $f\in H$.

□

Theorem 2.1.8. *For any positive definite kernel $k\left(x,y\right)$ on X, there exists a uniquely determined Hilbert space H of functions on X, admitting the reproducing kernel $k\left(x,y\right)$.*

Proof : see (Aronszajn, 1950). □

Theorem 2.1.9. *Every sequence of functions $\left\{f_n\left(x\right)\right\}_{n\geq 1}$ which converges strongly to a function f in H converges also in the pointwise sense, that is, $\lim_{n\to\infty}f_n\left(x\right)=f\left(x\right)$, for* *any point $x \in X$. Further, this convergence is uniform on every subset of X on which $x \longmapsto k(x,x)$ is bounded.*

Proof : For $x \in X$, using the reproducing property and Schwartz Inequality,

$$\begin{aligned} |f(x) - f_n(x)| &= |\langle f(x), k_x(x)\rangle - \langle f_n(x), k_x(x)\rangle| \\ &= |\langle f(x) - f_n(x), k_x(x)\rangle| \\ &\leq \|f - f_n\| \cdot \|k_x\| \\ &= \|f - f_n\| \, k(x,x)^{1/2}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} f_n(x) = f(x)$ for any point $x \in X$.

Moreover it is clear from the above inequality that this convergence is uniform on every subset of X on which $x \longmapsto k(x,x)$ is bounded. $\square$

Definition 2.1.10. *The function space $W_2^m[a,b]$ is defined as follows:*

$$W_2^m[a,b] = \left\{u : u^{(i)}, i = 1,2,...,m-1 \text{ are absolutely continuous on } [a,b], u^{(m)} \in L^2[a,b]\right\}. \tag{2.1.1}$$

The inner product and the norm in the function space $W_2^m[a,b]$ are defined as follows respectively; for any functions $u(x), v(x) \in W_2^m[a,b]$

$$\langle u, v\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\, v^{(i)}(a) + \int_a^b u^{(m)}(x)\, v^{(m)}(x)\, dx. \tag{2.1.2}$$

$$\|u\|_{W_2^m[a,b]} = \sqrt{\langle u, v\rangle_{W_2^m[a,b]}} \tag{2.1.3}$$

Theorem 2.1.11. *The function space $W_2^m[a,b]$ is a Hilbert Space.*

Proof : see (Zhang, 2012). $\square$

Theorem 2.1.12. *The function space* $W_2^m[a,b]$ *is a reproducing kernel space. That is, for each fixed* $x \in [a,b]$ *and any* $u(y) \in W_2^m[a,b]$, *there exists* $k_x(y) \in W_2^m[a,b]$, $y \in [a,b]$ *such that* $\langle u(y), k_x(y)\rangle = u(x)$, *and* $k_x(y)$ *is called the reproducing kernel function of space* $W_2^m[a,b]$.

Proof : see (Aronszajn, 1950). □

2.2 Reproducing Kernel Function

In this section, we will find out the expression forms of the reproducing kernel function in the space $W_2^m[a,b]$. These expression can be represented by piecewise polynomial of degree $2m-1$. The reproducing kernel function has a unique representation. Further, we will give some corollaries and remarks related to these kernel functions.

At the end of this section, several examples of such kernel functions are given in $W_2^1[a,b]$. Now, let's find out the expression form of the reproducing kernel function $k_x(y)$ in the space $W_2^m[a,b]$.

Suppose $k_x(y)$ is the reproducing kernel function of the space $W_2^m[a,b]$, then for each fixed $x \in [a,b]$ and any $u(y) \in W_2^m[a,b]$, $y \in [a,b]$ we have $\langle u(y), k_x(y)\rangle = u(x)$. Based on (2.1.2) and (2.1.3), we have:

$$\langle u(y), k_x(y)\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\, k_x^{(i)}(a) + \int_a^b u^{(m)}(y)\, k_x^{(m)}(y)\, dy. \qquad (2.2.1)$$

Applying the integration by parts for the second scheme of the right-hand of (2.2.1), we obtain

$$\int_a^b u^{(m)}(y)\, k_x^{(m)}(y)\, dy = \sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, k_x^{(m+i)}(y)\, |_{y=a}^{b} + \int_a^b (-1)^m\, u(y)\, k_x^{(2m)}(y)\, .$$

Let $j = m - i - 1$, the first term of the right-hand side of the above formula can be rewritten as

$$\sum_{i=0}^{m-1} (-1)^i u^{(m-i-1)}(y)\, k_x^{(m+i)}(y)\, |_{y=a}^b = \sum_{j=0}^{m-1} (-1)^{m-j-1} u^{(j)}(y)\, k_x^{(2m-j-1)}(y)\, |_{y=a}^b .$$

After some simplification, Equation (2.2.1) became

$$\begin{aligned}\langle u(y), k_x(y)\rangle_{W_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a) \left(k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a)\right) \\ &+ \sum_{i=0}^{m-1} (-1)^{m-i-1} u^{(i)}(b)\, k_x^{(2m-i-1)}(b) + \int_a^b (-1)^m u(y)\, k_x^{(2m)}(y)\, dy.\end{aligned}$$

Since $k_x(y), u(y) \in W_2^m[a,b]$, it follows that

$$k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0,\ k_x^{(2m-i-1)}(b) = 0,\ i = 0, 1, ..., m-1.$$

Then $\langle u(y), k_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y)\, ((-1)^m k_x^{(2m)}(y))dy$

Now, for each $x \in [a,b]$, if $k_x(y)$ satisfies $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$, where δ is dirace-delta function, then $\langle u(y), k_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y)\, \delta(x-y)\, dy = u(y)$ obviously, $k_x(y)$ is the reproducing kernel of the space $W_2^m[a,b]$.

Therefore, $k_x(y)$ is the solution of the following generalized differential equation:

$$\begin{cases} (-1)^m k_x^{(2m)}(y) = \delta(x-y) \\ k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, i = 0, 1, ..., m-1 \\ k_x^{(2m-i-1)}(b) = 0, i = 0, m-1. \end{cases} \quad (2.2.2)$$

while $x \neq y$

$$(-1)^m k_x^{2m}(y) = 0. \quad (2.2.3)$$

with the boundary conditions (BCs):

$$k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, k_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1. \qquad (2.2.4)$$

The characteristic equation of Equation (2.2.3) is $\lambda^{2m} = 0$, and their characteristic values are $\lambda = 0$ with $2m$ multiple roots. So, the general solution of Equation (2.2.3) is given by:

$$k_x(y) = \begin{cases} \sum_{i=0}^{2m-1} p_i(x) y^i, & y \leq x; \\ \sum_{i=0}^{2m-1} q_i(x) y^i, & y > x. \end{cases} \qquad (2.2.5)$$

On the other hand, since $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$, we have:

$$k_x^{(i)}(x+0) = k_x^{(i)}(x-0), i = 0, 1, ..., 2m-2. \qquad (2.2.6)$$

Integrating, $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $k_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m (k_x^{(2m-1)}(x+0) - k_x^{(2m-1)}(x-0)) = 1. \qquad (2.2.7)$$

Equations (2.2.6) and (2.2.7) provided $2m$ conditions for solving the coefficients $p_i(x)$ and $q_i(x), i = 0, 1, ..., 2m-2$, in equation (2.2.6). Further, equation (2.2.4) provided $2m$ BCs. So, we have $2m$ equations. It is easy to know that these $4m$ equations are linear equations with the variables $p_i(x)$ and $q_i(x)$, and the unknown coefficients $p_i(x)$ and $q_i(x)$ of equation (2.2.5) could be calculated out by using Mathematica 8.0 software package.

The following corollary gives some important properties of the reproducing kernel $k_x(y)$

Corollary 2.2.1. *The reproducing kernel $k_x(y)$ is symmetric, unique and $k_x(y) \geq 0$, for any fixed $x \in [a,b]$.*

Proof : By the reproducing property, we have $k_x(y) = \langle k_x(.), k_y(.)\rangle = \langle k_y(.), k_x(.)\rangle = k_y(x)$. Now, let $k_x(y)$ and $R_x(y)$ be all the reproducing kernel of the space $W_2^m[a,b]$, then $k_x(y) = \langle k_x(.), R_y(.)\rangle = \langle R_y(.), k_x(.)\rangle = R_y(x)$.By the symmetry of $R_x(y)$, we have the unique representation of $k_x(y)$. For the last condition, we note that

$k_x(x) = \langle k_x(.), k_x(.)\rangle = \|k_x(.)\|^2 \geq 0.$ □

Now we present some expressions of reproducing kernel function in the space $W_2^1[a,b]$ with respect to different norms by using the approaches proposed in this section,

Example 2.2.2. *Consider the space*

$W_2^1[a,b] = \{u : u(x) \text{ is absolutely continuous on } [a,b] \text{ and } u'(x) \in L^2[a,b]\}$.*The inner product and the norm in the space $W_2^!$ $[a,b]$ are given by,*

$$\langle u,v\rangle_{W_2^1} = u(a)\,v(a)+\int_a^b u'(y)\,v'(y)\,dy \quad \text{and} \quad \|u\|_{W_2^1} = \sqrt{\langle u,u\rangle}, \text{ where } u(x), v(x) \in W_2^1[a,b].$$

To find the reproducing kernel function $k_x(y)$, we apply integration by parts to see that

$$\langle u,k_x\rangle_{W_2^!} = u(a)\,k_x(a) + u(y)\,k_x^{'}(y)\,|_{y=a}^{b} - \int_a^b u(y)\,k_x^{''}(y)\,dy.$$

Since $u(y), k_x(y) \in W_2^1[a,b]$, we have $k_x(a) - k_x^{'}(a) = 0$ and $k_x^{'}(b) = 0$.Thus, we need to solve the BVP $-k_x^{''}(y) = \delta(x-y)$ subject to $k_x(a) - k_x^{'}(a) = 0$ and $k_x^{'}(b) = 0$.

The characteristic equation is $\lambda^2 = 0$, and the characteristic value is $\lambda = 0$ with 2 multiple *roots. So,*

$$k_x(y) = \begin{cases} p_1(x) + p_2(x)\,y, & y \le x, \\ q_1(x) + q_2(x)\,y, & y > x, \end{cases}$$

Also, by using equation (2.2.6) *and* (2.2.7) *we have* $k_x(x+0) = k_x(x-0)$ *and* $k_x'(x+0) - k_x(x-0) = -1$,*hence the unknown coefficients* $p_i(x)$ *and* $q_i(x)\,, i = 1, 2$,*can be obtained by solve the following equations*

1. $k_x(a) - k_x'(a) = 0.$
2. $k_x'(b) = 0.$
3. $k_x(x+0) = k_x(x-0).$
4. $k_x'(x+0) - k_x'(x-0) = -1.$

So the reproducing kernel function is given by

$$k_x(y) = \begin{cases} 1 - a + y, & y \le x, \\ 1 - a + x, & y > x, \end{cases}$$

Remark 2.2.3. *Yao (2008) proved that the space* $W_2^1[a,b]$ *in Example* (2.2.2) *is a complete reproducing kernel space and its reproducing kernel function is*

$$k_x(y) = \begin{cases} 1 + y, & y \le x, \\ 1 + x, & y > x, \end{cases}$$

Example 2.2.4. *Consider the space* $W_2^1[a,b] = \{u : u(x)$ *is absolutely continuous on* $[a,b]$ *and* $u'(x) \in L^2[a,b]$ *and* $u(a) = u(b) = 0\}$.

The inner product and the norm in the space $W_2^! [a, b]$ *are given, respectively, by*

$$\langle u, v\rangle_{W_2^!} = \int_a^b u'(y)\, v'(y)\, dy \text{ and } \|u\| = \sqrt{\langle u, u\rangle}, \text{ where } u(x), v(x) \in W_2^! [a, b].$$

Similarly, as in example (2.2.2) *we have:*

$$R_x(y) = \begin{cases} p_1(x) + p_2(x)\, y, & y \le x, \\ q_1(x) + q_2(x)\, y, & y > x, \end{cases}$$

the unknown coefficients $p_i(x)$ *and* $q_i(x), i = 1, 2,$*can be obtained by solve the following equations*

1. $R_x(a) = 0.$
2. $R_x(b) = 0.$
3. $R_x(x+0) = R_x(x-0).$
4. $R_x'(x+0) - R_x'(x-0) = -1.$

So the reproducing kernel function is given by

$$R_x(y) = \begin{cases} \frac{(b-x)(a-y)}{a-b}, & y \le x, \\ \frac{(a-x)(b-y)}{a-b}, & y > x, \end{cases}$$

Remark 2.2.5. *Paulsen (2009) proved that the space* $W_2^1 [a, b]$ *in Example* (2.2.4) *is a complete reproducing kernel space and its reproducing kernel function is*

$$R_x(y) = \begin{cases} (1-x)\, y, & y \le x, \\ (1-y)\, x, & y > x, \end{cases}$$

2.3 Description of Reproducing Kernel Method

In this section, we will establish an iterative method to construct and calculate the solution for the general m^{th}-order BVP.

Consider the general m^{th}-order BVP of the following type

$$u^{(m)}(x) + a_1(x) u^{(m-1)} + ... + a_{m-1}(x) u'(x) = F(x, u(x)), \ a \leq x \leq b, \tag{2.3.1}$$

subject to the BC's

$$\begin{aligned} u^{(i)}(a) &= c_i, \ i = 0, 1, 2, ..., r-1 \\ u^{(i)}(b) &= d_i, \ i = r, r+1, ..., m-1. \end{aligned} \tag{2.3.2}$$

where $a_i(x)$, $i = 1, 2, ..., m-1$, are continuous real-valued functions, $c_i, 0 \leq i \leq r-1$ and $d_i, r \leq i \leq m-1$ are real constants $u(x)$, is unknown function to be determined, $u^{(m)}(x)$ indicates the m^{th} derivative of $u(x)$, and $F(x, u(x))$ is a linear or nonlinear depending on the problem discussed.

In order to solve the BVP (2.3.1) and (2.3.2) using the RKHS method; First of all, we construct a reproducing kernel space $W_2^{m+1}[a, b]$ in which every function satisfies the homogeneous BC's of Equation (2.3.2) and then utilize the space $W_2^1[a, b]$. The inner product and the norm in the space $W_2^{m+1}[a, b]$ can be obtained as in Equations (2.1.2) and (2.1.3), respectively.

Let $k_x(y)$ and $R_x(y)$ be the reproducing kernel functions of the space $W_2^{m+1}[a, b]$ and $W_2^1[a, b]$, respectively. Define a differential operator $L : W_2^{m+1}[a, b] \longrightarrow W_2^1[a, b]$ such that $Lu(x) = u^{(m)}(x) + a_1(x) u^{(m-1)}(x) + ... + a_{m-1}(x) u'(x)$. Note that we can show that L is bounded operator by using the following lemma;

Lemma 2.3.1. *if* $u(x) \in W_2^{m+1}[a,b]$, *then* $|u(x)| \leq M_0 \|u(x)\|_{W_2^{m+1}}$.

Moreover, $\left|u^{(i)}(x)\right| \leq M_i \|u(x)\|_{W_2^{m+1}}$, *where* M_i *are constants,* $i = 1, 2, ..., m$.

Proof : By reproducing property of $k_x(y)$ and Schwartz inequality, also since $k_x^{(i)}(y)$, $i = 0, 1, 2, ..., m$ is uniformly bounded about x and y, we obtain

$$|u(x)| = \left|\langle u(y), k_x(y)\rangle_{W_2^{m+1}}\right| \leq \|k_x(y)\|_{W_2^{m+1}} \|u(y)\|_{W_2^{m+1}} \leq M_0 \|u(x)\|_{W_2^{m+1}}.$$

From the representation of $k_x(y)$, we can get

$$\left|u^{(i)}(x)\right| = \left|\left\langle u(y), k_x^{(i)}(y)\right\rangle_{W_2^{m+1}}\right| \leq \left\|k_x^{(i)}(y)\right\|_{W_2^{m+1}} \|u(y)\|_{W_2^{m+1}} \leq M_i \|u(x)\|_{W_2^{m+1}}.$$

□

Thus, after homogenization of the BC's (2.3.2), the BVP (2.3.1) and (2.3.2) can be converted into the equivalent form as follows:

$$Lu(x) = F(x, u(x)) \quad , a \leq x \leq b; \tag{2.3.3}$$

$$u^{(i)}(a) = 0, i = 0, 1, 2, ..., r-1; \;\; u^{(i)}(b) = 0, i = r, r+1, ..., m-1. \tag{2.3.4}$$

where $u(x) \in W_2^{m+1}[a,b]$ and $F(x,u) \in W_2^1[a,b]$.

Now, we construct an orthogonal function system of the space $W_2^{m+1}[a,b]$. For a countable dense set $\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, let $\varphi_i(x) = R_{x_i}(x)$, where $R_x(y)$ is the reproducing kernel of $W_2^1[a,b]$. So, from the properties of $R_x(y)$ for every $u(x) \in W_2^1[a,b]$, it follows that $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = \langle u(x), R_{x_i}(x)\rangle_{W_2^1} = u(x_i)$. Additionally, let $\psi_i(x) = L^*\varphi_i(x)$, where

L^* is the adjoint operator of L.

Obviously, $\psi_i(x) \in W_2^{m+1}[a,b]$. In terms of the properties of $k_x(y)$, we have:

$$\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i), \text{ where } i = 1, 2, ...$$

Lemma 2.3.2. *$\psi_i(x)$ can be expressed in the form $\psi_i(x) = L_y k_x(y)|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y.*

Proof : From the above assumption, it is clear that

$$\begin{aligned} \psi_i(x) &= L^*\varphi_i(x) = \langle L^*\varphi_i(y), k_x(y)\rangle_{W_2^{m+1}} \\ &= \langle \varphi_i(y), Lk_x(y)\rangle_{W_2^1} = L_y k_x|_{y=x_i}. \end{aligned}$$

□

Theorem 2.3.3. *Suppose that the inverse operator L^{-1}in Equation (2.3.3) exist. Thus, if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^{m+1}[a,b]$.*

Proof : For each fixed $u(x) \in W_2^{m+1}[a,b]$, let $\langle u(x), \psi_i(x)\rangle = 0, i = 1, 2, ...$, that is

$$\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(y)\rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i) = 0.$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, therefore, $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1} and the continuity of $u(x)$. □

Now, we will form an orthonormal function $\left\{\widehat{\psi}_i(x)\right\}_{i=1}^{\infty}$ of the space $W_2^{m+1}[a,b]$ by

Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \ i = 1, 2, ... \tag{2.3.5}$$

where β_{ik} are orthogonalization coefficients and are given by

$$\beta_{11} = \frac{1}{\| \psi_1 \|}, \quad \beta_{ii} = \frac{1}{\sqrt{\| \psi_i \|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\| \psi_i \|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad j < i, \tag{2.3.6}$$

where $c_{ik} = \langle \psi_i, \psi_k \rangle_{W_2^{m+1}}$.

Theorem 2.3.4. *For each $u(x) \in W_2^{m+1}[a,b]$, the series $\sum_{i=1}^{\infty} \left\langle u(x), \widehat{\psi}_i(x) \right\rangle \widehat{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^{m+1}[a,b]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$ then the solution of the BVP (2.3.1) and (2.3.2)is unique and given by*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k)) \widehat{\psi}_i(x), \tag{2.3.7}$$

Proof : Applying Theorem (2.3.3), it is easy to see that $\left\{\widehat{\psi}_i(x)\right\}_{i=1}^{\infty}$ is the complete orthonormal basis of the space $W_2^{m+1}[a,b]$. Thus, $u(x)$ can be expanded in the Fourier series about the orthonormal system $\left\{\widehat{\psi}_i(x)\right\}_{i=1}^{\infty}$ as $u(x) = \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\psi}_i(x) \right\rangle \psi_i(x)$. Moreover, the space $W_2^{m+1}[a,b]$ is Hilbert space, then the series $\sum_{i=1}^{\infty} \left\langle u(x), \widehat{\psi}_i(x) \right\rangle \widehat{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^{m+1}[a,b]$. Since $\langle v(x), \varphi_i(x) \rangle = v(x_i)$ for each

$v(x) \in W_2^1[a,b]$, we have:

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\psi}_i(x) \right\rangle_{W_2^{m+1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \left\langle u(x), \sum_{k=1}^{i} \beta_{ik}\psi_k(x) \right\rangle_{W_2^{m+1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), \psi_k(x) \right\rangle_{W_2^{m+1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), L^*\varphi_k(x) \right\rangle_{W_2^{m+1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle Lu(x), \varphi_k(x) \right\rangle_{W_2^{1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle F(x, u(x)), \varphi_k(x) \right\rangle_{W_2^{1}} \widehat{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k)) \widehat{\psi}_i(x).
\end{aligned}$$

We denote the n-term approximate solution to $u(x)$ by □

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k)) \widehat{\psi}_i(x), \tag{2.3.8}$$

Theorem 2.3.5. *For any $u(x) \in W_2^{m+1}[a,b]$, $u_n^{(i)}(x)$ are uniformly convergent to $u^{(i)}(x)$, $i = 0, 1, ..., m$.*

Proof : By using Lemma (2.3.1), for any $x \in [a,b]$, we get

$$\begin{aligned}
\left|u_n^{(i)}(x) - u^{(i)}(x)\right| &= \left|\left\langle u_n^{(i)}(x) - u^{(i)}(x), k_x(x)\right\rangle_{W_2^{m+1}}\right| = \left|\left\langle u_n(x) - u(x), k_x^{(i)}(x)\right\rangle_{W_2^{m+1}}\right| \\
&\leq \left\|k_x^{(i)}(x)\right\|_{W_2^{m+1}} \left\|u_n(x) - u(x)\right\|_{W_2^{m+1}} \\
&\leq M_i \left\|u_n(x) - u(x)\right\|_{W_2^{m+1}} \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Thus, the approximate solution $u_n(x)$ and $u_n^{(i)}(x)$ converge uniformly to $u(x)$ and its derivative $u^{(i)}(x)$, respectively. $\square$

CHAPTER 3

RKHS method for Solving Fractional Logistic Models

Fractional calculus is the study of derivatives and integrals of non-integer order that provides an attractive mechanism to explain memory and hereditary characteristics of complex systems. Recently, it has been widely used in modeling real-world models due to its accuracy in providing and neglecting the influence of external forces as in physics, engineering, mechanics, biology, medicine and economics (Kilbas, et al., 2006) and (Mainardi F, 2010). One model that has benefited from the development of fractional calculus is the logistic model, which is an attempt to describe several phenomena that possess growth data such as population growth, the spread of bacteria, infectious diseases and social media. See (Bacaër, 2011) and the references cited therein. The actual formula of this model was first proposed by the Belgian mathematician Verhulst in 1835 , who stated that the virtual increase of population is limited by the size and the fertility of the country. As a result, the population gets closer and closer to a steady state. He introduced the nonlinear first-order ordinary DE in the following quadratic form :

$$\frac{du(x)}{dx} = \lambda u(x)\left(1 - \frac{u(x)}{\delta}\right),$$

where $u(x)$, λ, and δ are related to the population, growth rate parameter, and carrying capacity, respectively. This model known as the Verhulst model, assumes that the growth rate λ is linearly increasing and decreasing in terms of $u(x)$. Furthermore, by substituting $u(x) = \frac{u(x)}{\delta}$, the classical logistic differential equation (LDE) is described by:

$$\frac{du(x)}{dx} = \lambda u(x)(1 - u(x)),$$

which has the exact solution as follows:

$$u(x) = \frac{u_0}{u_0 + (1 - u_0)e^{-\lambda x}},$$

where $u_0 = u(0)$ is related to the initial population data.

In terms of methodology, the approximate methods have been known as powerful mathematical tools for dealing with many complex natural models with local and non-local operators, which arise at studies of physics, engineering, biology, chemistry, and other sciences. These advanced methods are used when the classical analytical techniques accomplished fail. It is also being continuously improved to keep pace with the rapid developments taking place in the universe and the emergence of new global economic, biological, chemical, and astronomical models. However, with respect to the proposed model, some modern numerical approaches have been applied to solve the quadratic and cubic fractional Logistic equations, including the spectral Laguerre collocation method (Khader and Babatin ,2013), Decomposition method (Momani and Qaralleh, 2007), operational matrices of Bernstein polynomials (Al-Bar. 2015), and the Residual power series method (Alshammari, et al., 2019).

The motivation of the current study is to expand the applications of the RKHS method for obtaining approximate solutions for a class of modified quadratic fractional Logistic differential equations (FLDEs) in the frame of Caputo fractional derivative. At any rate, implementations of the method on a nonlinear FLDEs of quadratic type:

$$^{C}D_a^{\alpha} u(x) = \lambda u(x)(1 - u(x)), \;\; u(a) = u_0, \tag{3.0.1}$$

In which, $0 < \alpha \leq 1$, $x \in [a, b]$; $\lambda > 0$. Here, note that $^{C}D_a^{\alpha}$ indicates the Caputo fractional derivative and $u(x)$ indicates smooth solution to be determined in the desirable space $\mathrm{W}_2^2[a,b]$. These equations are generalized by applying the fractional derivative rather than the classical integer order derivative to standard LDEs to contribute to improving model accuracy.

3.1 Quadratic Fractional Logistic Differential Equation

In this section, a modified technique based of the reproducing kernel concept is proposed to solve a class of FLDEs of type quadratic in RKHS $\mathrm{W}_2^2[a,b]$. A new reproducing kernel function is constructed to create an orthogonal system and to calculate the analytical and approximate solutions in the desirable space $\mathrm{W}_2^2[a,b]$. The convergence, and complexity of the proposed approach are discussed. The main motivation for using the proposed technique is high accuracy and low computational cost compared to other existing methods especially when involving fractional differentiation operators. In this orientation, the effectiveness, applicability, and feasibility of this technique are verified by numerical examples. From a numerical viewpoint, the obtained results indicate that the suggested intelligent method has many advantages in accuracy and stability using the Caputo fractional operator.

3.1.1 Introduction

We consider the quadratic FLDE of the following form:

$$^{C}D_a^{\alpha}u(x) = \lambda u(x)(1 - u(x)), \ \ x \geq a, \ \ 0 < \alpha \leq 1, \tag{3.1.1}$$

subject to the IC:

$$u(a) = u_0, \tag{3.1.2}$$

where $\lambda > 0$, ${}^C D_a^\alpha$ indicates the Caputo fractional derivative, while $u(x)$ indicates smooth solution to be determined in the desirable space $\mathrm{W}_2^2[a,b]$. We suppose that FLDE (3.1.1) have a unique solution.

In order to solve FLDE (3.1.1) in the frame of Caouto fractional derivative, we construct two reproducing kernel functions. The RKHS $\mathrm{W}_2^2[a,b]$ is determined as

$$\mathrm{W}_2^2[a,b] = \{u(x) : u, u' \text{ is absolutely continuous on } [a,b],\ \ u'' \in L^2[a,b],\ \ x \in [a,b],\ \ u(a) = 0\}.$$

The standard inner product and the norm associated with the RKHS $\mathrm{W}_2^2[a,b]$ are given, respectively, by:

$$\langle u_1, u_2 \rangle_{\mathrm{W}_2^2} = \sum_{i=0}^{1} u_1^{(i)}(a) u_2^{(i)}(a) + \int_a^b u_1^{(2)}(x) u_2^{(2)}(x) dx,\ \ u_1, u_2 \in \mathrm{W}_2^2[a,b], \tag{3.1.3}$$

and

$$\|u\|_{\mathrm{W}_2^2} = \sqrt{\langle u, u \rangle_{\mathrm{W}_2^2}},\ \ u \in \mathrm{W}_2^2[a,b]. \tag{3.1.4}$$

Theorem 3.1.1. *The unique representation of a reproducing kernel functions* $\mathrm{K}_x(y)$ *of the RKHS* $\mathrm{W}_2^2[a,b]$ *is given by*

$$\mathrm{K}_x(y) = \begin{cases} \rho(x,y), & y < x, \\ \rho(y,x), & x \le y, \end{cases} \tag{3.1.5}$$

where $\rho(x,y) = \frac{1}{6}(y-a)(2a^2 - y^2 + 3x(2-y) - a(6+3x+y))$.

Proof : To find out the expression form of $\mathrm{K}_x(y)$ for the RKHS $\mathrm{W}_2^2[a,b]$, we have to do the following: Through several integrations by parts for Eq. (3.1.3), we have

$$\langle u(y), \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2} = \sum_{i=0}^{1} u^{(i)}(y)(\partial_y^i \mathrm{K}_x(a) + (-1)^i \partial_y^{3-i}\mathrm{K}_x(a)) + \sum_{i=0}^{1}(-1)^{i-1}u^{(i)}(b)\partial_y^{3-i}\mathrm{K}_x(b))$$

$$+ \int_a^b u(y)\partial_y^4 \mathrm{K}_x(y)dy. \tag{3.1.6}$$

Since $\mathrm{K}_x(y) \in \mathrm{W}_2^2[a,b]$, it follows that $\mathrm{K}_x(a) = 0$. Further, since $u(x) \in \mathrm{W}_2^2[a,b]$, one gets $u(a) = 0$.

If $\partial_y^i \mathrm{K}_x(b) = 0,\ \ i = 2,3$ and $\partial_y^1 \mathrm{K}_x(a) - \partial_s^2 \mathrm{K}_x(a) = 0$, then

$$\langle u(y), \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2} = \int_a^b u(y)\partial_s^4 \mathrm{K}_x(y)dy. \tag{3.1.7}$$

Now, for each $x \in [a,b]$, if $\mathrm{K}_x(y) \in \mathrm{W}_2^2[a,b]$ also satisfies

$$\partial_y^4 \mathrm{K}_x(y) = \delta(x-y), \tag{3.1.8}$$

where $\delta(x)$ is the Dirac-Delta function, then

$$\langle u(y), \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2} = u(x). \tag{3.1.9}$$

The characteristic equation of Eq.(3.1.8) is $r^4 = 0$ and its characteristic value $r = 0$ with multiplicity root 4. So, let

$$\mathrm{K}_x(y) = \begin{cases} \sum_{i=1}^{4} a_i(x)y^{i-1}, & y < x, \\ \sum_{i=1}^{4} b_i(x)y^{i-1}, & x \le y. \end{cases} \tag{3.1.10}$$

Moreover, for Eq. (3.1.8), let $\mathrm{K}_x(y)$ satisfies the following linear equations:

$$\lim_{y\to x^-} \partial_y^i \mathrm{K}_x(y) = \lim_{y\to x^+} \partial_y^i \mathrm{K}_x(y), \quad i = 0, 1, 2,$$

and integrating Eq. (3.1.8) from $x - \epsilon$ into $x + \epsilon$ with respect to y and let $\epsilon \to 0$, we have the jump degree of $\partial_y^3 \mathrm{K}_x(y)$ at $y = x$

$$\lim_{y\to x^-} \partial_y^3 \mathrm{K}_x(y) - \lim_{y\to x^+} \partial_y^3 \mathrm{K}_x(y) = 1.$$

From the last descriptions and by using Mathematica software 12 package the unknown coefficients $a_i(x)$ and $b_i(x)$, i=1,2,3,4 can be obtained. □

Note that, $\mathrm{K}_x(y)$ is symmetric, unique, and semi d.p., i.e., $\mathrm{K}_x(y) \geq 0$, for any fixed $x \in [a, b]$.

Definition 3.1.2. *The RKHS* $\mathrm{W}_2^1[a, b]$ *is determined as*

$$\mathrm{W}_2^1[a, b] = \{u(x) : u \textit{ is absolutely continuous on } [a, b], \;\; u' \in L^2[a, b], \;\; x \in [a, b]\}.$$

The inner product and norm are attached, respectively, by

$$\langle u_1, u_2\rangle_{\mathrm{W}_2^1} = \int_a^b \big(u_1(x)u_2(x) + u_1'(x)u_2'(x)\big)dx, \;\; u_1, u_2 \in \mathrm{W}_2^1[a, b], \tag{3.1.11}$$

and

$$\|u\|_{\mathrm{W}_2^1} = \sqrt{\langle u, u\rangle_{\mathrm{W}_2^1}}, \;\; u \in \mathrm{W}_2^1[a, b]. \tag{3.1.12}$$

It's easy to proof that the space $\mathrm{W}_2^1[a, b]$ is a complete RKHS and its reproducing kernel function $R_x(y)$ given by

$$R_x(y) = \frac{1}{2\sinh(b-a)}\big[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)\big]. \tag{3.1.13}$$

From the definition of the RKHSs $W_2^2[a,b]$ and $W_2^1[a,b]$, we obtain $W_2^2[a,b] \hookrightarrow W_2^1[a,b]$, i.e., $\exists\, c > 0$ such that $\|u\|_{W_2^1} \leq c\|u\|_{W_2^2}$.

3.1.2 Implements of the Modified RKHS Method

In this section, we demonstrate how to implement the modified RKHS method to solve a class of quadratic FLDEs in the frame of Caputo fractional derivative in RKHS $W_2^2[a,b]$. To perform this, we must homogenize the IC using the simple transformation $u(x) = u(x) - u_0$. Thus, the equivalent form of the quadratic FLDE (3.1.1) is given by

$${}^{C}D_a^{\alpha}u(x) = \lambda(u(x)+u_0)(1-u(x)-u_0), \;\; 0 < \alpha \leq 1, \tag{3.1.14}$$

subject to the initial condition

$$u(a) = 0. \tag{3.1.15}$$

After that, we characterize the differential linear operator as follows

$$\begin{cases} L : W_2^2[a,b] \longrightarrow W_2^1[a,b]; \\ Lu(x) =^{C} D_a^{\alpha}u(x). \end{cases} \tag{3.1.16}$$

Lemma 3.1.3. *The fractional operator* L *from* $W_2^2[a,b]$ *into* $W_2^1[a,b]$ *is bounded and linear.*

Proof : We need to prove the presence of a positive constant $\mathfrak{K}$ such that $\|Lu(x)\|^2_{W_2^1} \leq$

$\mathfrak{K}\|u(x)\|^2_{\mathrm{W}_2^2}$. To do this, the reproducing property of $\mathrm{K}_x(y)$ are applied together with Cauchy-Schwartz inequality, we obtain ||

$$\begin{aligned} \left|\left(\mathrm{L}u\right)^{(i)}(x)\right| &= \left|\left\langle u(x), \left(\mathrm{LK}_x\right)^{(i)}(x)\right\rangle_{\mathrm{W}_2^2}\right| \\ &\leq \left\|\left(\mathrm{LK}_x\right)^{(i)}(x)\right\|_{\mathrm{W}_2^2} \|u\|_{\mathrm{W}_2^2} \\ &\leq \mathfrak{K}^{\{i\}}\|u(x)\|^2_{\mathrm{W}_2^2}, \;\; i = 0, 1. \end{aligned}$$

Here, $\mathfrak{K}^{\{i\}} = \left\|\left(\mathrm{LK}_x\right)^{(i)}(x)\right\|_{\mathrm{W}_2^2} = \left\|\left({}^C D_a^\alpha \mathrm{K}_x\right)^{(i)}(x)\right\|_{\mathrm{W}_2^2}$ in the indices $0 \leq i \leq 1$. Since $\mathrm{K}_x(y)$ is uniformly bounded about x and y, we have $\left({}^C D_a^\alpha \mathrm{K}_x\right)^{(i)}(x)$ is uniformly bounded about x. Now, by using the norm over the space $\mathrm{W}_2^1[a, b]$, we have

$$\|\mathrm{L}u(x)\|^2_{\mathrm{W}_2^1} = \|{}^C D_a^\alpha u(x)\|^2_{\mathrm{W}_2^1} = \int_a^b \left[\left({}^C D_a^{\alpha+1} u(x)\right)^2 + \left({}^C D_a^{\alpha+2} u(x)\right)^2\right] dx \leq \mathfrak{K}\|u(x)\|^2_{\mathrm{W}_2^2},$$

in which, $\mathfrak{K} = (b - a)\left((\mathfrak{K}^{\{1\}})^2 + (\mathfrak{K}^{\{2\}})^2\right)$. □

The next step is how to create an orthogonal function system of $W_2^2[a, b]$. To see this, we put $\varphi_i(x) = \mathrm{R}_{x_i}(x)$ and $\psi_i(x) = \mathrm{L}^\star \varphi_i(x)$, i=1,2,..., such that $\{x_i\}_{i=0}^\infty$ is dense countable sub set in $[a, b]$ and $\mathrm{L}^\star$ indicated the adjoint operator of L. Look this, from the properties of reproducing kernel $\mathrm{R}_x(y)$, for every $u(x) \in W_2^1[a, b]$, it follows that $\langle u(x), \varphi_i(x))\rangle_{W_2^1} = \langle u(x), \mathrm{R}_{x_i}(t)\rangle_{W_2^1} = u(x_i)$. Additionally, In terms of the properties of reproducing kernel $\mathrm{K}_x(y)$, one gets

$$\langle u(x), \psi_i(x)\rangle_{W_2^2} = \langle u(x), \mathrm{L}^\star \varphi_i(x)\rangle_{W_2^2} = \langle \mathrm{L}u(x), \varphi_i(x)\rangle_{W_2^1} = \mathrm{L}u(x_i), \;\; i = 1, 2, \ldots.$$

To derive the normal function basis $\{\widehat{\psi}_i(x)\}_{i=1}^\infty$ of the RKHS $W_2^2[a, b]$ from $\{\psi_i(x)\}_{i=1}^\infty$. we need to use the well-known Gram-Schmidt orthogonalization process, which can be used as follows:

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \varrho_{ik}\psi_k(x), \quad i = 1, 2, \dots, \tag{3.1.17}$$

where, ϱ_{ik} is the orthogonalization coefficients of $\{\psi_i(x)\}_{i=1}^{\infty}$ such that $\varrho_{ii} > 0$.

Theorem 3.1.4. *If $\{x_i\}_{i=1}^{\infty}$ is dense countable sub set on $[a, b]$, then $\{\widehat{\psi}_i(x)\}_{i=1}^{\infty}$ is complete function system in $\mathrm{W}_2^2[a, b]$ and $\psi_i(x) = \mathrm{L}_y\mathrm{K}_x(y) \mid_{y=x_i}$.*

Proof : Note that

$$\psi_i(x) = \mathrm{L}^*\varphi_i(x) = \langle \mathrm{L}^*\varphi_i(y), \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2} = \langle \mathrm{L}\mathrm{K}_x(y), \varphi_i(y)\rangle_{\mathrm{W}_2^1} = \mathrm{L}_y\mathrm{K}_x(y) \mid_{y=x_i} .$$

To show the completeness, let $\langle u(x), \psi_i(x)\rangle = 0$. This mean that

$$\langle u(x), \psi_i(x)\rangle_{\mathrm{W}_2^2} = \langle u(x), \mathrm{L}^*\varphi_i(x)\rangle_{\mathrm{W}_2^2} = \langle \mathrm{L}u(x), \varphi_i(x)\rangle_{\mathrm{W}_2^1} = \mathrm{L}u(x_i) = 0.$$

By the density of the sequence $\{x_i\}_{i=1}^{\infty}$ on $[a, b]$, we have $\mathrm{L}u(x) = 0$. Further, from the existence of the inverse operator L^{-1}, we conclude that the $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete. □

Theorem 3.1.5. *Let $u(x) \in \mathrm{W}_2^2[a, b]$ be a unique solution of the quadratic FLDEs* (3.1.16)*. Thus, its analytical solution has the following form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \lambda\varrho_{ik}(u(x) + u_0)(1 - u(x) - u_0)\widehat{\psi}_i(x), \tag{3.1.18}$$

where ϱ_{ik} are the orthonormalization coefficients.

Proof : Since, $\{\widehat{\psi}_i(x)\}_{i=1}^{n}$ is a normal function basis of $\mathrm{W}_2^2[a, b]$, $u(x)$ can be written in the Fourier series expansion $u(x) = \sum_{i=1}^{\infty}\langle u(x), \widehat{\psi}_i(x)\rangle_{\mathrm{W}_2^2}\widehat{\psi}_i(x)$.

Additionally, since $\mathrm{W}_2^2[a, b]$ is a Hilbert space, then the series $\sum_{i=1}^{\infty}\langle u(x), \widehat{\psi}_i(x)\rangle_{\mathrm{W}_2^2}\widehat{\psi}_i(x)$

is convergent in sense of the norm $\|\cdot\|_{\mathrm{W}_2^2}$ of $\mathrm{W}_2^2[a,b]$. On the other hand, we have

$$\begin{aligned} u(x) &= \textstyle\sum_{i=1}^{\infty}\langle u(x), \widehat{\psi}_i(x)\rangle_{\mathrm{W}_2^2}\widehat{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \varrho_{ik}\langle u(x), \psi_i(x)\rangle_{\mathrm{W}_2^2}\widehat{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \varrho_{ik}\langle u(x), \mathrm{L}^{\star}\varphi_k(x)\rangle_{\mathrm{W}_2^2}\widehat{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \varrho_{ik}\langle \mathrm{L}u(x), \varphi_k(x)\rangle_{\mathrm{W}_2^1}\widehat{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \lambda\varrho_{ik}(u(x_k)+u_0)(1-u(x_k)-u_0)\widehat{\psi}_i(x). \end{aligned} \tag{3.1.19}$$

□

Lemma 3.1.6. *If $u(x) \in \mathrm{W}_2^2[a,b]$, then there exists a positive number $\mathfrak{K}$ such that*

$$\|u^{(i)}(x)\|_C \leq \mathfrak{K}\|u^{(i)}(x)\|_{\mathrm{W}_2^2}, \ \ i = 0, 1,$$

where $\|u^{(i)}(x)\|_C = \max\limits_{x\in[a,b]}|u(x)|$.

Proof : For any $y, x \in [a,b]$, we have $u^{(i)}(x) = \langle u(y), \partial_y^i \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2}$, in the indices $0 \leq i \leq 1$. By the expression formula of $\partial_y^i \mathrm{K}_x(y)$, it follows that $\|\partial_y^i \mathrm{K}_x(\cdot)\|_{\mathrm{W}_2^2} < \mathfrak{K}_i$, in the indices $0 \leq i \leq 1$. Thus, by Cauchy-Schwartz inequality, we obtain

$$|u^{(i)}(x)| = |\langle u(y), \partial_y^i \mathrm{K}_x(y)\rangle_{\mathrm{W}_2^2}| \leq \|\partial_y^i \mathrm{K}_x(y)\|_{\mathrm{W}_2^2}\|u(y)\|_{\mathrm{W}_2^2} \leq \mathfrak{K}_i\|u(x)\|_{\mathrm{W}_2^2}, \ \ i = 0, 1.$$

As result, $\|u^{(i)}(x)\|_C \leq \mathfrak{K}\|u^{(i)}(x)\|_{\mathrm{W}_2^2}$, such that $\mathfrak{K} = \max\{\mathfrak{K}_1, \mathfrak{K}_2\}$. □

By the direct application of Lemma (3.1.6) ,we get

$$\begin{aligned} |u_n^{(i)}(x) - u^{(i)}(x)| &= |\langle u_n^{(i)}(x) - u^{(i)}(x), \partial_s^i \mathrm{K}_y(x)\rangle_{\mathrm{W}_2^2}| \\ &\leq \|\partial_y^i \mathrm{K}_y(x)\|_{\mathrm{W}_2^2} \|u_n^{(i)}(x) - u^{(i)}(x)\|_{\mathrm{W}_2^2} \\ &\leq \mathfrak{K}_i \|u_n^{(i)}(x) - u^{(i)}(x)\|_{\mathrm{W}_2^2}, \ \ i = 0, 1. \end{aligned}$$

Hence, $|u_n^{(i)}(x) - u^{(i)}(x)| \leq \mathfrak{K}_i \|u_n^{(i)}(x) - u^{(i)}(x)\|_{\mathrm{W}_2^2} \to 0$ as $n \to \infty$, $i = 0, 1$. Thus, $u_n(x)$ and $u_n'(x)$ are uniformly convergent to $u(x)$ and $u'(x)$, respectively. Note that if the studied problem is linear, we can easily determine the approximate solution from the following truncated equation:

Note that, If the studied problem is linear, then we can easily determine directly the approximate solution by the following truncated equation:

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \lambda \varrho_{ik} (u(x_k) + u_0)(1 - u(x_k) - u_0) \widehat{\psi}_i(x).$$

Here, because the studied problem is nonlinear DE then the analytical and approximate solutions of FLDE (3.1.1) can be given using the iterative process that described in Section (2.3).

Consequently, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, and by using the iterative process ,

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \varrho_{ik} \lambda (u_{k-1}(x_k) + u_0)(1 - u_{k-1}(x_k) - u_0) \widehat{\psi}_i(x). \tag{3.1.20}$$

3.1.3 Computational Simulations for Quadratic FLDEs

In this section, the efficiency and accuracy of the modified RKHS method are demonstrated by including two numerical examples for quadratic FLDEs in the frame of Caputo fractional operator. The results obtained are compared to the exact solution at the integer-order $\alpha = 1$, and with each other at different values of arbitrary-order α. In this regard, the obtained results show that the proposed method provides a convenient methodology for controlling the convergence of the approximate solution.

Example 3.1.7. ***(Kumar, et al., 2017)*** *Consider the following quadratic FLDE in the frame Caputo fractional derivative:*

$$\begin{cases} {}^{C}D_0^{\alpha}u(x) = \frac{1}{2}u(x)(1-u(x)), \ \ 0 < \alpha \leq 1; \\ u(0) = \frac{1}{4}. \end{cases} \tag{3.1.21}$$

The exact solution at $\alpha = 1$ *is* $u(x) = \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}+3}$.

By using the modified RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, .., n$, *with the reproducing kernel function* $\mathrm{K}_x(y)$ *on the interval* $[a, b] = [0, 1]$, *the approximate solutions* $u_n(x)$ *at the different values of fractional order* α *are computed by Eq.* (3.1.20).

The numerical outcomes of the proposed method are listed in the form of tables and graphical representations as follows: Numerical approximations for Example (3.1.7) *compared with the exact solutions at* $\alpha = 1$ *are given in Table* (3.1) *over* $[a, b] = [0, 1]$ *with step size* 0.1. *Figure* (3.1) *exhibits a comparison among the behavior curves of exact and approximate solutions at* $\alpha = 1$. *While in Table* (3.2), *the numerical results at different values of fractional order* α *are summarized in the frame of Caputo fractional derivatives. In Figure* (3.2), *the behavior of the approximate solutions are presented in the frame of*

Caputo fractional derivatives with different values of α.

In addition, we compute the absolute error of n-order approximate function at a particular point x which is adopted from

$$e_n = |u(x) - u_n(x)|, \tag{3.1.22}$$

where, $x \in [0, 1]$, $u(x)$ is the exact solution and $u_n(x)$ is the approximate solution. Also, the relative error at a particular point t is calculated:

$$r_n = \frac{|u(x) - u_n(x)|}{u(x)}. \tag{3.1.23}$$

Table 3.1: Numerical results in Example (3.1.7): Approximate solution, absolute errors, relative errors.

x_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.	0.25	0.25	0.	0.
0.1	0.259492	0.259491	3.16557×10^{-7}	1.21991×10^{-6}
0.2	0.269214	0.269214	6.91116×10^{-7}	2.56716×10^{-6}
0.3	0.279164	0.279163	1.06151×10^{-6}	3.80248×10^{-6}
0.4	0.289336	0.289334	1.42619×10^{-6}	4.92920×10^{-6}
0.5	0.299724	0.299722	1.78378×10^{-6}	5.95141×10^{-6}
0.6	0.310322	0.31032	2.13312×10^{-6}	6.87388×10^{-6}
0.7	0.321124	0.321121	2.47329×10^{-6}	7.70200×10^{-6}
0.8	0.332120	0.332117	2.80365×10^{-6}	8.44169×10^{-6}
0.9	0.343302	0.343299	3.12380×10^{-6}	9.09929×10^{-6}
1.	0.354661	0.354658	3.43363×10^{-6}	9.68144×10^{-6}

Example 3.1.8. ***(Kumar, et al., 2017)*** *Consider the following quadratic FLDE within Caputo fractional derivative:*

$$\begin{cases} {}^{C}D_0^{\alpha} u(x) = \frac{1}{2} u(x)(1 - u(x)), \quad 0 < \alpha \leq 1; \\ u(0) = \frac{1}{2}. \end{cases} \tag{3.1.24}$$

The exact solution of Eq. (3.1.24) when $\alpha = 1$ is $u(x) = \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}} + 1}$.

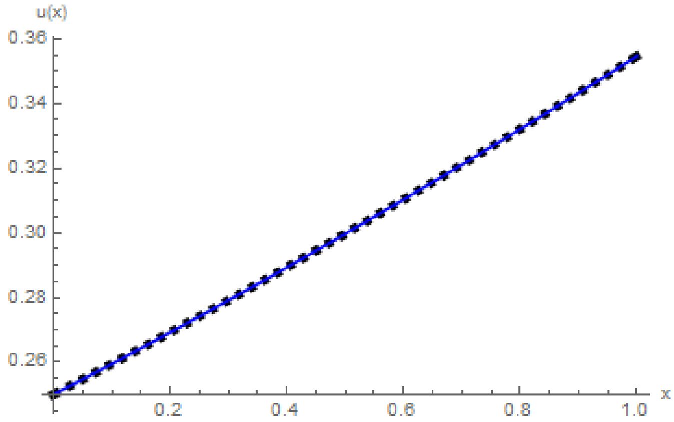


Figure 3.1: Solution curves of Example (3.1.7) at $\alpha = 1$: Solid line is exact; Dotted line is approximate solution

Table 3.2: Approximate solutions in the frame of Caputo fractional derivative of Example (3.1.7).

Caputo fractional derivative				
x_i	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.55$
0.0	0.2500000000	0.2500000000	0.2500000000	0.2500000000
0.1	0.2594908709	0.2608938185	0.2687497944	0.3817621873
0.2	0.2692124735	0.2713333169	0.2820584730	0.2972463094
0.3	0.2791610379	0.2817634954	0.2941860352	0.3102060310
0.4	0.2893317276	0.2922625543	0.3056488759	0.3216472469
0.5	0.2997189801	0.3028615396	0.3166810139	0.3321037761
0.6	0.3103163716	0.3135739971	0.3274046963	0.3418446428
0.7	0.3211166644	0.3244050442	0.3378919084	0.3510304015
0.8	0.3321119242	0.3353549334	0.3481882159	0.3597658954
0.9	0.3432933646	0.3464206999	0.3583238532	0.3681242085
1.0	0.3546513763	0.3575976995	0.3683187474	0.3761571814

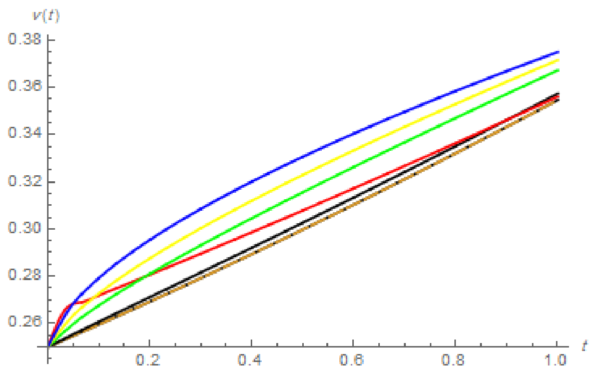


Figure 3.2: Graphical results of Example (3.1.7) in the frame Caputo fractional derivatives with different values of α: Blue line $\alpha = 0.55$; Brown line $\alpha = 0.65$; Green line $\alpha = 0.75$; Red line $\alpha = 0.85$; Gray line $\alpha = 0.95$; Black line $\alpha = 1$.

By taking $x_i = \frac{i-1}{n-1}$, $i = 1, \ldots, n$ *and* $n = 30$, *the numerical outcomes of the solutions using the RKHSM are summarized in the form of tables and graphs representations as* *follows: The absolute errors of Example (3.1.8) at $\alpha = 1$ are shown in Table (3.3). Table (3.4) shows the numerical results at different values of fractional order α in the frame of Caputo fractional concepts. The approximate solution curves for different values of α in the frame of Caputo fracional operators are presented in Figure (3.3).*

Table 3.3: Numerical results in Example (3.1.8): Approximate solution, absolute errors, relative errors.

x_i	Exact solution	Approximate solution	Absolute Error	Relative Error
0.	0.5	0.5	0.	0.
0.1	0.512497	0.512497	1.58773×10^{-7}	3.09803×10^{-7}
0.2	0.524979	0.524979	3.15903×10^{-7}	6.01745×10^{-7}
0.3	0.537430	0.537429	4.73191×10^{-7}	8.80470×10^{-7}
0.4	0.549834	0.549833	6.32423×10^{-7}	1.15020×10^{-6}
0.5	0.562177	0.562176	7.95317×10^{-7}	1.41471×10^{-6}
0.6	0.574443	0.574442	9.63465×10^{-7}	1.67721×10^{-6}
0.7	0.586618	0.586616	1.13828×10^{-6}	1.94041×10^{-6}
0.8	0.598688	0.598686	1.32095×10^{-6}	2.20641×10^{-6}
0.9	0.610639	0.610638	1.51241×10^{-6}	2.47677×10^{-6}
1.	0.622459	0.622458	1.71330×10^{-6}	2.75248×10^{-6}

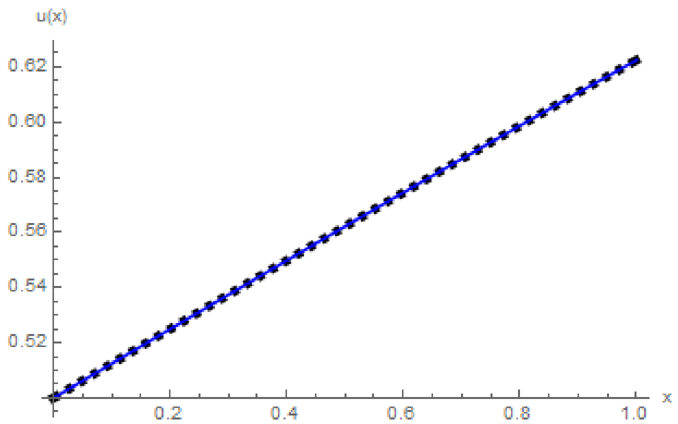
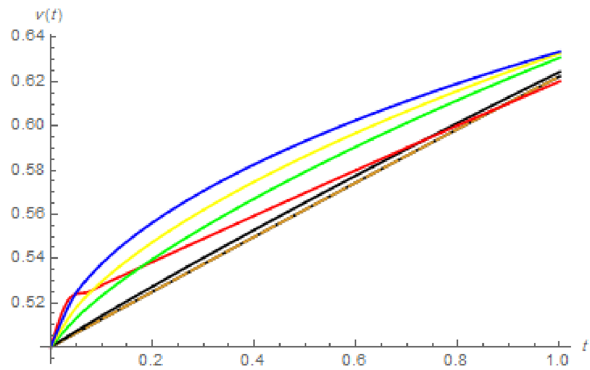


Figure 3.3: Solution curves of Example (3.1.8) at $\alpha = 1$: Solid line is exact; Dotted line is approximate solution

Table 3.4: Approximate solutions in the frame of Caputo fractional derivative of Example (3.1.8).

	Caputo fractional derivative				
x_i	$\alpha = 0.95$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.65$	$\alpha = 0.55$
0.	0.5	0.5	0.5	0.500000	0.5
0.1	0.514050	0.527931	0.523075	0.529519	0.537602
0.2	0.527374	0.538477	0.539677	0.547419	0.556275
0.3	0.540307	0.548963	0.554062	0.562023	0.570592
0.4	0.552965	0.559389	0.567134	0.574793	0.582632
0.5	0.565390	0.569746	0.579270	0.586315	0.593188
0.6	0.577604	0.580026	0.590673	0.596897	0.602669
0.7	0.589615	0.590218	0.601470	0.606729	0.611318
0.8	0.601428	0.600316	0.611746	0.615940	0.619298
0.9	0.613041	0.610311	0.621562	0.624619	0.626720
1.	0.624454	0.620196	0.630964	0.632834	0.633668



(a) Caputo fractional derivative

Figure 3.4: Graphical results of Example (3.1.8) in the frame of Caputo fractional derivatives with different values of α: Blue line $\alpha = 0.55$; Brown line $\alpha = 0.65$; Green line $\alpha = 0.75$; Red line $\alpha = 0.85$; Gray line $\alpha = 0.95$; Black line $\alpha = 1$.

CHAPTER 4

Conclusions

4.0.1 Conclusions

To conclude, in this work, a precise numerical approximation algorithm based on the reproducing kernel Hilbert space (RKHS) approach has been developed to address a specific class of fractional differential equations within the framework of the Caputo sense. The analytical solution is formulated as a convergent series with accurately computable structures in the reproducing kernel space. A finite-term approximation has been derived and proven to uniformly converge to the analytical solution. The primary advantage of the RKHS approach lies in its direct applicability without the need for linearization or perturbation, thereby circumventing errors associated with discretization. Several numerical examples have been provided to demonstrate the accuracy of the computations and the efficacy of the proposed approach. The numerical results indicate that the RKHS method serves as a robust tool for obtaining effective approximate solutions to such systems arising in applied mathematics, physics, and engineering.

For future endeavors, further research could concentrate on formulating novel reproducing kernel functions and different RKHS formulations to tackle a wider array of fractional differential equations with non-classical initial and boundary conditions.

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الحل العددي للمعادلات التفاضلية الكسرية باستخدام طريقة النواة المستنسخة التكرارية

اعداد

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المشرف

الدكتور نذير جدي

الملخص

تقترح هذه الرسالة خوارزمية تقريب عددي دقيقة، تستخدم منهج فضاء هلبرت ذو النواة المستنسخة (RKHS)، لمعالجة فئة محددة من المعادلات التفاضلية الكسرية ضمن إطار كابوتو. تعتمد منهجية الحل بشكل أساسي على بناء نواة استنساخ فضاء هيلبرت $W_2^m[a,b]$ تفي بشروط تحديد حل المعادلة التفاضلية الكسرية من ناحية. من ناحية أخرى، يعتمد على استخدام خاصية استنساخ النواة للحصول على الصيغة العامة للحلول التحليلية والتقريبية على فضاء هيلبرت $W_2^m[a,b]$ التي تم إنشاؤها سابقًا. يتم تمثيل الحلول التحليلية والتقريبية في شكل متسلسلة فورييه في فضاء هيليرت $W_2^m[a,b]$. الحلول التقريبية وجميع مشتقاتها تتقارب بشكل موحد مع الحلول التحليلية وجميع مشتاتها، على التوالي. يتم التحقق من فعالية النهج المقترح من خلال عدة أمثلة عددية، مما يشير إلى قوته في حل الأنظمة غير الخطية الناشئة في مجالات الرياضيات التطبيقية، والفيزياء، والهندسة.