Poeple's Democratic Republic Algeria
Ministry of Higher Education and
Scientific Research
Larbi Tebessi University - Tebessa -
Faculty of Exactes Sciences and Sciences of
Nature and Life
Departement: Mathematics and Computer Science

Final dissertation
for the graduation of MASTER

Domaine: Mathematics and Computer Science
Field: Mathematics
Option: Partial differential Equations and Applications

THEME

Stability Results of a Coupled Wave Equations With Time Delay

Presented by:
Zerdoumi Nabila

Before the Jury:

Toualbia Abdelatif	MCB	Larbi Tebessi University	President
Lemita Samir	MCB	Larbi Tebessi University	Examiner
Boumaza Nouri	PROF	Larbi Tebessi University	Supervisor

Date of Dessertation : 09/06/2024

بسم الله الرحمن الرحيم

شكر و عرفان

نحمد الله و نشكره شكرا جزيلا إذ هو خالقنا، و معيننا

فهو الأولى بالشكر في كل الأوقات و الظروف.

نحمد الله عز و جل و نثني عليه الخير كله الذي وفقنا

لإتمام هذا العمل، و نسأله ان يجعل هذا كله خالصا لوجهه الكريم و أن ينفعنا به و ينتفع به من بعدنا.

اتقدم بكل إحترام و تقدير بشكر و عرفان للأستاذ و البروفيسور

الفاضل الذي كان موجهنا في البحث العلمي "بومعزة نوري"، الذي كان له الفضل الكبير في شق الطريق نحو النجاح

و على كل النصائح و التوجيهات.

كما اتقدم بالشكر لكل الأساتذة في تكويننا عبر مسيرتنا الدراسية

من الإبتدائية إلى الجامعة،

و إلى كل من قدم لنا يد المساعدة من قريبب أو من بعيد

إهـــــــــداء

إلى نـــور الهـــداية و مــــعلم البـــشرية الـــمبعوث رحـــمة للـــــعالمين
ســـــيدنا مــــحمد عــــليه أفـــضل الصــــلاة و الســـــلام.
إلى اللــــوالدين الكــــريمين
حبيبة قلبي وضياء دربي أمي الغالية التي غمرتني بعطفها وحنانها وسقتني بحبها حفظها
الله وأطال في عمرها .
ألى الذي غرس في حب العمل وظل ينمو وينمو إلى أن أثمر وتفتحت أزهاره وفاح
عبيره هو أبي العزيز رحمه الله وأسكنه فسيح جنانه
إلى كل من علمني حرفا و أنار لي طريق العلم.
الى اخوتي واخواتي وفقهم الله وانار دربهم
الى زوجي الكريم اطال الله في عمره
إلى كل من قدم لي يد المساعدة في إتمام هذا البحث المتواضع.

Résume

Le but de ce travail est de donner des résultats liés aux équations d'ondes couplées localement avec un amortissement viscoélastique localisé non lisse de type Kelvin-Voigt et un retard temporel localisé (étudié par Mohammad Akil et al [1]). La recherche vise à étudier l'existence et l'unicité de la solutions sous des hypothèses appropriées utilisant la théorie des semi-groupes. En utilisant un critère général d'Arendt-Batty, nous montrons la forte stabilité de notre système en l'absence de compacité de la résolvante.

Mots clés: Equation d'onde couplée, Amortissement de Kelvin-Voigt, retard temporel, stabilité forte, stabilité polynomiale, approche de domaine fréquentiel.

Abstract

The aim of this work is to give a results related to locally coupled wave equations with non-smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay (studied by Mohammad Akil et al[1]). The research aims to study the existence and uniqueness of solutions under appropriate assumptions using semigroup theory. Using a general criterion of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent.

Keywords: Coupled wave equation, delay term, Kelvin-Voigt damping, strong stability, polynomial stability, frequency domain approach.

ملخص

الهدف من هذا العمل هو إعطاء نتائج تتعلق بمعادلات الموجات المقترنة محليًا مع التخميد اللزج المرن الموضعي غير السلس من نوع Kelvin-Voigt والتأخير الزمني الموضعي (درسه محمد عقيل وآخرون[1]). يهدف البحث إلى دراسة وجود ووحدانية الحلول في ظل فرضيات مناسبة باستخدام نظرية أشباه الزمر. باستخدام المعيار العام أرندت-باتي ، نبين الاستقرار القوي لنظامنا في غياب تماسك المذيب

الكلمات المفتاحية: معادلة الموجة المقترنة، حد التأخير، تخميد كلفن فويغت، الاستقرار القوي، استقرار كثير الحدود، مقاربة مجال التردد.

Contents

Introduction

Elastic/viscoelasric coupled wave equations are among the most important equations in the fields of applied mathematics and engineering physics. They are essential for describing and understanding the behavior of materials that exhibit elastic and viscoelastic characteristics when subjected to external forces. In engineering, these equations model the response of different materials to stress and deformation, which aids in designing safer and more efficient structures and products. In the realm of applied physics, these equations elucidate various natural phenomena, such as the propagation of seismic waves and the transmission of sound waves through various media, thereby enhancing the comprehension and analysis of these phenomena.
Our thesis dedicated to the study of the stability of local coupled wave equations with singular localized viscoelastic damping of Kelvin-Voigt type and localized time delay, which is defined as follows [1]:

$$\begin{cases} u_{tt} - \left[a u_x + b\left(x\right)\left(k_1 u_{tx} + k_2 u_{tx}\left(x, t-\tau\right)\right)\right\}]_x + c\left(x\right) y_t = 0, & (x,t) \in (0,L) \times (0,\infty) \\ y_{tt} - y_{xx} - c\left(x\right) u_t = 0, & (x,t) \in (0,L) \times (0,\infty) \end{cases} \quad (1)$$

Under the boundary conditions:

$$u\left(0,t\right) = u\left(L,t\right) = y\left(0,t\right) = y\left(L,t\right) = 0 \qquad t > 0$$

And the intial conditions:

$$\begin{cases} \left(u\left(0,t\right), u_t\left(0,t\right)\right) = \left(u_0\left(x\right), u_1\left(x\right)\right) & x \in (0,L) \\ \left(y\left(x,0\right), y_t\left(x,0\right)\right) = \left(y_0\left(x\right), y_1\left(x\right)\right) & x \in (0,L) \\ \left(y\left(x,0\right), y_t\left(x,0\right)\right) = \left(y_0\left(x\right), y_1\left(x\right)\right) & x \in (0,L) \end{cases}$$

where L, τ, a and k_1 are positive real numbers, k_2 is a non-zero real number and $(u_0, u_1, y_0, y_1, f_0)$ belongs to a suitable space.
We suppose that there exists $0 < \alpha < \beta < \gamma < L$ and a non-zero constant c_0,
such that

$$b\left(x\right) = \begin{cases} 1, x \in (0,\beta) \\ 0, x \in (\beta, L) \end{cases}$$

and

$$c(x)=\begin{cases} c_{0}, x\in(\alpha,\gamma)\\ 0, x\in(0,\alpha)\cup(\gamma,L)\end{cases}$$

The system (1.1) consists of two wave equations. Where there is only one singular viscoelastic damping acting on the first equation, while the second equation undergoes indirect damping through a singular coupling between them. In this context, the presence of viscoelastic damping in the first equation implies the impact of elastic and viscous properties on the wave behavior in that equation.

On the other hand, the indirect damping of the second equation means that the damping effect transmitted through a specific coupling between the two equations, reflecting a complex interaction between the wave fields in the system

Many previous studies have addressed the stability of Elastic/viscoelastic coupled wave equations, employing various mathematical techniques to analyze these systems. However, research focusing on the impact of time delay on the stability of these equations remains limited.

The idea of indirect damping mechanisms presented by Russell in [46] has drawn the attention of many authors (see, for example, [15, 16,17 ,18,19, 14, 20, 21]). The examination of these systems is also prompted by various physical considerations, such as the Timo instance, [22, 23, 24, 25]). In fact, there are few results concerning the stability of coupled wave equations with local Kelvin-Voigt damping without time delay, especially in the absence of smoothness of the damping and coupling coefficients (see Subsection 1.2.1). The last motivates our interest to study the stabilization of system (1.1) in the present paper.

In the recent years, there has been increasing interest among researchers in problems involving this type of damping, with various types of stability bieng proposed, depending on the smoothnees of the damping coefficients (see[26,27,28,29,30,31,32,33,34]. Let us briefly recall some systems of wave equations Coupled wave equations with Kelvin-Voigt damping and without time delay, as represented in the previous literature.

In 2020, Hayek et all in [47] studied the stabilization of a system of weakly coupled wave equations with one or two locally internal Kelvin–Voigt damping and non-smooth coefficient at the interface.

Their research led to the establishment of various stability outcomes. Similarly, in 2021, Hassine and Souayeh in [4] studied the behavior of a system with coupled wave equations with a partial KelvinVoigt damping, by considering the following system.

$$\begin{cases} u_{tt} - \left(u_x + b_2\left(x\right) u_{tx}\right)_x + v_t = 0 \ , & (x,t) \in (-1,1) \times (0,\infty) \\ y_{tt} - cv_{xx} - u_t = 0, & (x,t) \in (-1,1) \times (0,\infty) \\ u\left(0,t\right) = v\left(0,t\right) = 0, \quad u\left(1,t\right) = v\left(1,t\right) = 0 & t > 0 \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), & x \in (-1,1) \\ v\left(x,0\right) = v_0\left(x\right), v_t\left(x,0\right) = v_1\left(x\right), & x \in (-1,1) \end{cases} \quad (1,2)$$

where $c > 0$, and $b_2 \in L^\infty(-1,1)$ is a non-negative function they posited that the damping coefficient follows a piecewise function, specifically suggesting that $b_2(x) = d1_{[0;1]}(x)$, where dd is a strictly positive constant. Consequently, they took the damping coefficient to be near the boundary with a global coupling coefficient. Their findings

revealed the lack of exponential stability, that the semigroup loses speed and it decays polynomially with a slower rate then given in [2], down to zero at least as $t^{\frac{-1}{12}}$.

In 2021, Akil, Issa, and Wehbe, as documented in [3], extended the findings of Hassine and Souayeh in [4] by demonstrating a polynomial decay rate of the form t-1, by considering the following system

$$\begin{cases} u_{tt} - \left(au_x + b\left(x\right) u_{tx}\right)_x + c\left(x\right) y_t = 0 \ , & (x,t) \in (0,L) \times (0,\infty) \\ y_{tt} - y_{xx} - c\left(x\right) u_t = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u\left(0,t\right) = u\left(L,t\right) = y\left(0,t\right) = y\left(L,t\right) = 0 & t > 0 \\ \left(u\left(0,t\right), u_t\left(0,t\right)\right) = \left(u_0\left(x\right), u_1\left(x\right)\right) & x \in (0,L) \\ \left(y\left(x,0\right), y_t\left(x,0\right)\right) = \left(y_0\left(x\right), y_1\left(x\right)\right) & x \in (0,L) \end{cases}$$

where

$$b\left(x\right) = \begin{cases} 1, \ x \in (\alpha_1, \alpha_2) \\ 0, \text{otherwise} \end{cases}$$

and

$$c\left(x\right) = \begin{cases} c_0, x \in (\alpha_2, \alpha_4) \\ 0, \text{otherwise} \end{cases}$$

where $a > 0, b_0 > 0, c_0 > 0$ and $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$.

They investigate the stabilization of a locally coupled wave equations with only one internal viscoelastic damping of Kelvin-Voigt type. A key innovation in their study lies in the fact that both the damping and coupling coefficients are non-smooth. Additionally, the control of partial differential equations with time delays have become common among scientists.Time delays have been

utilized in various applications, such as in physical, chemical, biological, and thermal phenomena, because they no longer rely solely on the current state but also on past events (see [36, 35]). This type of delay can lead to instances of instability (see [2, 12, 38, 39]). Let us briefly recall some systems of wave equations with time delay and without Kelvin-Voigt damping.
In 2006, Nicaise and Pignotti, as documented in [5], examined the multidimensional wave equation under two scenarios. The initial scenario involves a wave equation with boundary feedback and a delay term at the boundary:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \quad , & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma_D \times (0,\infty) \\ \frac{\partial u}{\partial v}(x,t) = 0 \quad , & (x,t) \in \Gamma_N \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \quad , & x \in \Omega \\ u_t(x,t) \ = f_0(x,t) \ , & (x,t) \in \Gamma_N \times (-\tau,0) \end{cases} \tag{1,4}$$

The second scenario pertains to a wave equation featuring internal feedback and a delayed velocity term, specifically an internal delay, alongside a mixed Dirichlet-Neumann boundary condition.

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x, t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma_D \times (0,\infty) \\ \frac{\partial u}{\partial v}(x,t) = 0, & (x,t) \in \Gamma_N \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f_0, & (x,t) \in \Gamma_N \times (-\tau,0) \end{cases} \tag{1,5}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with a boundary Γ of class C^2 and $\Gamma_1 = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$. Under the assumption $\mu_2 < \mu_2$, an exponential decay achieved for the both systems (1.4)-(1.5). In [6] Ait Benhassi et al studied the problem (1.5) in more general abstract setting . The scope of stability analyses for second-order evolution equations with delay was extended, enhancing the overall understanding of achieving stability in the analysis of dynamic systems with delays and guides future research in this field .
In 2010.Ammari et al (see [7] studied the wave equation with interior delay damping and dissipative undelayed boundary condition in an open domain Ω of $\mathbb{R}^N$, $N \geq 2$.The system is described by:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma_D \times (0,\infty) \\ \frac{\partial u}{\partial v}(x,t) = -k u_t(x,t), & (x,t) \in \Gamma_1 \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f_0(x,t), & (x,t) \in \Omega \times (-\tau,0) \end{cases} \quad (1,6)$$

Where $\tau > 0$, $a > 0$ and $k > 0$.Under the condition that Γ_1 satisfies the T-codition introduced in [8], they proved that system (1,6) is uniformly asymptotically stable wheneverthe delay coeficiently small .
In 2012, Pignotti, in [9], studied the following system

$$\begin{cases} u_{tt} - \Delta u + a\chi_w u_t + k u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f(x,t), & (x,t) \in \Omega \times (-\tau,0) \end{cases} \quad (1,7)$$

where $k \in \mathbb{R}$, $\tau > 0$, $a > 0$ and w is the intersection betwen an open neighborhood of the set $\Gamma_0 = \{x \in \Gamma, (x - x_0), v(x) > 0\}$ and Ω . Moreover ,χ_w is the characteristic function of w, which is

awave equation with intrernal distributed time delay and local damping in a bounded and smooth domain $\Omega \subset \mathbb{R}^N, N \geq 1$.They proved an exponential stability result under some Lions geometric condition. The proof of the main result is based on an identity with multipliers that allows to obtain a uniform decay estimatefor a suitable Lyapunov functional.
Several studies have been conducted on wave equations with time delay affecting the boundary, as evidenced by ([38, 40, 41, 42, 43, 44, 45]), and various types of stability have been demonstrated. There has also been significant interest from many researchers in studying wave equations with Kelvin-Voigt damping and time delay, among these studies :
In 2016, Messaoudi et al. in [10] considered the stabilization of the following wave equation with strong time delay:

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f_0(x,t), & (x,t) \in \Gamma_N \times (-\tau,0) \end{cases} \quad (1,8)$$

where $\mu_1 > 0$ and μ_2 is a non zero real number. The equation can be considered as a Kelvin-Voigt linear model for a viscoelastic material with a delayed response. Assuming $|\mu_2| < \mu_1$, they demonstrate well-posedness and establish an exponential decay result under appropriate assumptions regarding the damping and delay weights.

In 2016, Nicaise et al. in [11] studied the stabilization problem for the wave equation with localized Kelvin–Voigt damping and mixed boundary condition with time delay

$$\begin{cases} u_{tt}(x,t) - \Delta u \ \ (x,t) - \operatorname{div}(a(x)\nabla u_t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial v}(x,t) = -a(x)\frac{\partial u_t}{\partial v}(x,t) - k u_t(x,t-\tau), & (x,t) \in \Gamma_1 \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0) \end{cases} \quad (1,9)$$

where $\tau > 0$, $k \in \mathbb{R}$, $a(x) \in L^\infty(\Omega)$ and $a(x) \geq a_0 > 0$on w such that $w \subset \Omega$ is an open neighborhood of Γ.By using a frequency domain approach we show that, and under an appropriate geometric condition on Γ_1 and assuming that a $a \in C^{1,1}\left(\overline{\Omega}\right), \Delta a \in L^\infty(\Omega)$, an exponential stability result holds. In this sense, this extends the result of [12] where, in a more general setting, the case of distributed structural damping is considered.

In 2019, Anikushyn and al. in [13] considered an initial boundary value problem for a viscoelastic wave equation subjected to a strong time localized delay in a Kelvin-Voigt type. The system is given by the following:

$$\begin{cases} u_{tt} - c_1\Delta u \ - c_2\Delta u(x,t-\tau) - d_1\Delta u_t - d_1\Delta u_t(x,t-\tau), & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial v}(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,\infty) \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega \\ u_t(x,t) \ = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0) \end{cases} \quad (1,10)$$

The global exponential decay rate has been verified under appropriate conditions on the coefficients, and the stability region in the parameter space has been further examined using Lyapunov's indirect method. Additionally, they have finally presented a numerical example from a real-world application in biomechanics.

Our thesis is presented as follows: Firstly, it provides an introduction to the research topic, reviews relevant literature, and lays out the theoretical framework for the study. The second chapter is devoted to some preliminary notions, in which we define certain theorems and inequalities that are heavily used in our work. In the third chapter, we will calculate the energy for this model and

prove the well-posedness of our system using a semigroup approach based on the work of Mohammad Akil et al [1]. Next, in chapter 4, by employing a general criterion of Arendt-Batty, we demonstrate the strong stability of our system in the absence of compactness of the resolvent. Additionally, by utilizing a frequency domain approach combined with a specific multiplier method, we prove a polynomial energy decay rate of order t^{-1}. Finally, we conclude with a summary and a list of references used in this dissertation.
In this chapter we recall the main concepts that we will need, it devotes to the notions of the theory of functional spaces, theorems, formulas and very inequalitiesused in our memory, As we me ntion the theory of operators and semi group, because they are standard and known among readers as they can be found in many mathematics references

0.1 Functional spaces

0.1.1 normed spaces

Definition 0.1 *(Vector subspaces)*

Let E be a vector space over field $\Bbbk$, and let F be a subset of E . We say that F is a subspace of E if and only if

1. $F \neq \emptyset$
2. $\forall x \in F, \forall y \in F : x + y \in F$. In other words F is stable through addition
3. $\forall x \in F$.For $\lambda \in \Bbbk : \lambda x \in F$. in other words F is stable by scalair multiplication

Definition 0.2 *(Normed vector spaces)*

A linear vector space E is called a normalized space if for each elemt $u \in E$ there exists a real number denoted by $\|u\|$ verfying the axioms:

1) $\|u\| = 0 \Longleftrightarrow u = 0,$

2) $\|u + v\| \leq \|u\| + \|v\|, \forall u, v \in \Bbbk,$

3) $\|\lambda u\| = |\lambda| \|u\|, \forall u \in E, \forall \lambda \in \Bbbk.$

Definition 0.3 *(Cuchy suite)*

Let be $(E, \|.\|)$ normalized space and a sequence of elements of E , we say that the seqence . East a continuation of Cauchy $(u_n)_{n\in\mathbb{N}}$ if

$$\forall\varepsilon > 0, \exists n_0(\epsilon), \forall n, m \geq 0 \Longrightarrow \|u_n - u_m\| < \epsilon$$

0.1.2 Complet space

Definition 0.4 *Let E be a vector space , we say that E is a complet space if any sequence of Cauchy $(u_n)_{n\in\mathbb{N}}$ of space E converges to an elemnet u of E*

0.1.3 Banach spaces

Definition 0.5 *(Banach spaces)*

Let be $(E, \|.\|)$ a normalized space , we say that E is a Banach space if E is a complet space

0.1.4 Hilbert space

Definition 0.6 *(Scalar product)*

Let H be vector space , we call application of $H \times H$ in the body $K = C$
defined by $\langle ., . \rangle$ is a dot produit if :

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$,for all $u, v \in H$,
- $\langle \lambda u_1 + u_2, v \rangle = \lambda \langle u_1, v \rangle + \langle u_2, v \rangle$,for all $u, v \in H$,and $\lambda \in \mathbb{C}$,
- $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$, for all $\lambda \in \mathbb{C}$,
- $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Longleftrightarrow u = 0$.

Definition 0.7 *(Hilbert space)*

A Hilbert space is a Banach space $(H, \|.\|_H)$ complete normed space)
equipped with a scalar product for the associated norm

$$\|u\|_H = \langle u, u \rangle^{\frac{1}{2}} (i.e) \|u\|_H^2 = \langle u, u \rangle$$

0.1.5 The $L^p(\Omega)$ spaces

Definition 0.8 *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$*

define the standard lebesgue space

$$L^p(\Omega) \text{ by } L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \textbf{ is measurable and } \int_\Omega |u(x)|^p \, dx < \infty\}.$$

the standard is noted :

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}}$$

If $p = \infty$,we have
$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}$ is measurable and there exists a constant C suchthat $|u(x)| \leq C$ i.e $\in \Omega\}$
also , we denote by

$$\|u\|_\infty = \underset{x\in\Omega}{esssup}\, |u(x)| = \inf \{C, |u(x)| < C \text{ p.p on } \Omega\}$$

.

Proposition 0.1 *$L^p(\Omega)$ menu of its norm $\|\cdot\|_{L^p}$ is a Banach space for all $1 \leq p \leq \infty$.*

Definition 0.9 *We say that a function $u \to \mathbb{R}$ belongs to $L^1_{loc}(\Omega)$ for everything compact $K \subset \Omega$.*

Definition 0.10 *$L^2(\Omega)$ is a Hilbert space, with the scalar product*

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)\, v(x)\, dx, \text{for everything } u, v \in L^2(\Omega)$$

Space $L^p((0,T), E)$

Definition 0.11 *Let $p \in \mathbb{R}$ .and $1 < p \leq \infty$. we define the space of classes of functions $L^p(\Omega)$ with*

$$L^p(\Omega) = \left\{u : \Omega \to \mathbb{R}.\ u \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < +\infty\right\}$$

the standard is noted by

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}}$$

Lemma 0.1 *Let* $u \in L^p((0,T),E)$ *and* $\frac{\partial u}{\partial t} \in L^p((0,T),E),(1 \leq p \leq \infty)$ *then the function* u *is continuous of* $[0,T]$ *in* $E\,(i,e)\,u \in C^1((0,1),E)$.

0.1.6 Sobolev space

Weak derivative

Definition 0.12 *Either*$\in$ Ω *an open of* $\mathbb{R}^n$ $,1 \leq i \leq n$ *and* $u \in L^1_{loc}(\Omega)$ *a function has weak i-th derivative in* $L^1_{loc}(\Omega)$ *existe* $f_i \in L^1_{loc}(\Omega)$ *such as for everything* $\varphi \in C_0^\infty(\Omega)$ *we have*

$$\int_\Omega u(x)\partial_i\varphi(x)\,dx = -\int_\Omega f_i(x)\varphi(x)\,dx$$

This amounts to saying that f_i is the i-th derivative of u in the sense of distributions, we will write $\partial_i u = \frac{\partial u}{\partial x_i} = f_i$

space $W^{1,p}(\Omega)$

Definition 0.13 *Either* Ω *any open of* $\mathbb{R}^n$ *and* $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, *space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \text{suchat } \partial_i u \in L^p(\Omega)\}$$

or $\partial_i$is the i-thweak derivative of $u \in L^1_{loc}(\Omega)$

space $W^{1,m}(\Omega)$

Definition 0.14 *Either* Ω *an open of* $\mathbb{R}^n, m > 2$ *and* $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, *space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,m}(\Omega) = \{u \in L^p(\Omega), \text{ sach tat } D^\alpha u \in L^p(\Omega), \forall\alpha, |\alpha| \leq m\}$$

or $\alpha \in \mathbb{N}^n, |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, and $D^\alpha = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of $u \in L^1_{loc}(\Omega)$, space $W^{1,m}(\Omega)$ is provided by norme

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0<|\alpha \leq m|} \|D^\alpha u\|_{L^p(\Omega)}$$

Definition 0.15 *if* $p = 2$, *we note by* $W^{m,2}(\Omega) = H^m$ *and* $W^{m,2}(\Omega) = H_0^m(\Omega)$ *provided by the standard*

$$\|u\|_{H^{m,2}(\Omega)} = (\sum_{|\alpha|\leq m} (\|\partial^{\alpha}u\|_{L^2(\Omega)})^2)^{\frac{1}{2}}$$

such that $H^m(\Omega)$ Hilbert space , with the dot product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle D^{\alpha}u, D^{\alpha}v\rangle_{L^2(\Omega)} = \sum_{|\alpha|\leq m} \partial^{\alpha}u\partial^{\alpha}vdx, \text{foreverything } u, v \in H^m(\Omega)$$

1) The space $W^{1,p}(\Omega)$ are Banach spaces .

2) if $m = 0$ we have $W^{0,p}(\Omega) = L^p(\Omega)$.

0.2 Trace Theorem

Theorem 0.1 *(of trace)*

Either Ω a limited and regular open .We can define a linear and continouus application ,

$$\Phi : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$$

$$u \longrightarrow \Phi(u)$$

Extending the application trace for continuous functions on $\overline{\Omega}$ for everything

$$u \in H^1(\Omega) \cap C^0(\overline{\Omega}) : \Phi(u) = u_{\mid}\partial\Omega$$

The trac application is continuous of $H^1(\Omega)$ in $L^2(\partial\Omega)$,which means that there is a constant C_Ω such as

$$\|\Phi(u)\|_{L^2(\partial\Omega)} \leq C_\Omega \|u\|_{H^1(\Omega)}$$

0.3 Some useful formulas

Definition 0.16 *(Integration by part)*

Either $(u,v) \in H^1(\Omega)$, for everything $1 \leq i \leq n$ we have

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int_{\Omega} \frac{\partial v}{\partial x_i} u dx + \int_{\partial\Omega} uv\eta_i d\sigma.$$

or $\eta_i(x) = \cos(\eta_i, x_i)$ is the direction cosine of the angle between the exterior normal has $\partial\Omega$ at the point and the axis of x_i

0.4 Some useful inequalities

0.4.1 Teoreme (Cauchy schwartz inequality)

such as $u, v \in L^2(\Omega)$

$$\left| \int_{\Omega} uvdx \right| \leq \int_0^t |uv| \, dx \leq (\int_{\Omega} |u|^2)^{\frac{1}{2}} \left(\int_{\Omega} |v|^2 \, dx \right)^{\frac{1}{2}}$$

(i.e)

$$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

0.4.2 Teoreme (Young algebraic inequality)

such as $a, b \in \mathbb{R}_+$ we have :

$$|ab| \leq \delta |a|^2 + \frac{1}{4\delta} |b|^2 \text{, with } \delta > 0$$

0.4.3 Teoreme (Young inequality)

such as $(a, b) \in \mathbb{R}^2$ we have :

$$|ab| \leq \frac{1}{P} |a|^P + \frac{1}{q} |b|^q ,$$

or p, q strictly positive real numbers linked by the relation $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

0.4.4 Formula (Young inequality with ε)

such as $\varepsilon > 0$ so for everything $(a, b) \in \mathbb{R}^2$,we have

$$|ab| \leq \varepsilon |a|^P + c(\varepsilon) |b|^q ,$$

or p, q strictly positive real numbers linked by the relation $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, and $c(\varepsilon) = \frac{1}{P}(\varepsilon p)^{\frac{-q}{p}}$.

0.4.5 Formula (Minkowski inquality)

such as $1 \leq p \leq \infty$, we have

$$\|u + v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p}$$

0.5 The operators

Let E and F be two Banach space , let us note $\|.\|$ the standard with which they are provided .

Definition 0.17 *(Linear operator)*

Let E and F be two Banach space , A linear operator is a linear application

$$A : D(A) \in E \longrightarrow F$$

(i.e)

$$\forall (u, v) \in D(A)^2 , A(u, v) = Au + Av.$$

$$\forall \lambda \in \mathbb{C}, A(\lambda u) = \lambda A u$$

Definition 0.18 *(Domen)*

A linear operator A of E in F is a linear application A defined by on a subspace vector $D(A)$ of E called domain of A such that

$$D(A) = \{u, Au \in F\}.$$

$$A : D(A) \subset E \longrightarrow F$$

we say that A is bounded if there exists $C \geq 0$ such that

$$\forall u \in D(A), \|Au\|_F \leq C \|u\|_E.$$

Other wise , A is said to be unbounded .

Definition 0.19 *(Graphe\Nayau\Image)*

The graph of A is the vector subspace of $E \times F$ denoted $Gr(A)$ defined by

$$Gr(A) = \{(u, Au), u \in D(A)\}.$$

We call nayau of A the subspace of E denoted $\ker(A)$ defined by:

$$\ker(A) = \{u \in D(A), Au = 0\}.$$

and Image of A the subspace of Fnoted $\text{Im}(A)$ defined by:

$$\text{Im}(A) = A(D(A)) = \{u \in D(A), Au = 0\}$$

.

We say that A is injective if $\ker(A) = \{0\}$,and that A is surjective if $\text{Im}(A) = F$, The operator is ijective and surjective.

Definition 0.20 *(Invertible operator)*

We say that an operator A of domain $D(A)$ is invertible if

$$A : D(A) \subset E \longrightarrow F$$

Is bijective and has an inverse,

$$A^{-1} : F \longrightarrow D(A) \subset E,$$

bounded (as operator of F in E).

Definition 0.21 *(Resolvante)*

Let A be a linear operator (not necessarily continuous) defined on a Banach .For everything complex number λ such that $(\lambda I - A)^{-1}$, existe and is continuous ,we define the resolvent of A by

$$R_\lambda = (\lambda I - A)^{-1}$$

Set of values of λ for which the resolvent exists and called the resolvan set note, $p(A)$.
The spectrum $\sigma(A)$ is the complement of the resolvon set :

$$\sigma(A) = \mathbb{C} / \rho(A).$$

0.5.1 Dissipative operator

Definition 0.22 *(Dissipative operator)*

A linear operator A in E is said to be dissipative if we have :

$$\forall x \in D(A), \forall \lambda > 0, \|\lambda x - Ax\|_E \geq \lambda \|x\|_E.$$

A is said to be m-dissipative if A is dissipative and for all $\lambda > 0$,The operator $\lambda I - A$ is surjective , (i.e),

$$\forall y \in X, \forall \lambda > 0, \exists x \in D(A), \lambda x - Ax = y$$

Theorem 0.2 *If A is m-dissipative then for all $\lambda > 0$, the operator $(\lambda I - A)$ admits an inverse, $(\lambda I - A)^{-1} y$ belongs to $D(A)$ for everything $y \in X$,and $(\lambda I - A)^{-1}$ is a linear operator bounded on X checking*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Theorem 0.3 *Let $(A, D(A))$ be an unbounded dissipative operator in X . The operator A is m-dissipative if and only if:*

$$\exists \lambda_0 > 0, \forall y \in X, \exists x \in D(A), \lambda_0 x - Ax = y$$

.

Theorem 0.4 *A operator $(A, D(A))$, linear unbounded in H, is dissipative if and only if:*

$$\exists x \in D(A) : \langle Ax, x \rangle \leq 0.$$

0.5.2 Monotonic maximal operators

Definition 0.23 *Let $A : D(A) \subset H \longrightarrow H$ a operator linear unbounded .We say that A is monotinic*

if

$$(Av, v) \geq 0 \qquad \forall v \in D(A)$$

A is maximal monotonic if in addition $R(I + A) = H$ i.e,

$$\forall f \in H, \exists u \in D(A) \text{ for everything } u + Au = f.$$

Proposition 0.2 *Let A a operator maximal monotonic .So*

- $D(A)$ is dense in H,
- A is closed,
- for everything $\lambda > 0, (I + \lambda A)$ is bijictive of $D(A)$ on $H, (I + \lambda A)^{-1}$ is a bounded operator and.

$$\left\|(I + \lambda A)^{-1}\right\|_{L(H)} \leq 1.$$

Remark 0.1 *Some authors say that A is accretive or that A is dissipative.*

Definition 0.24 *The operator A is lipchitz continuous if there exists $M > 0$ such that*

$$\|Au - Av\|_H \leq M \|u - v\|_H \, \forall u, v \in H$$

0.6 Strongly continuus semigroup

the roughout this section $(E, \|.\|)$, will denote a Banach space

Definition 0.25 *(Strongly continuus semigroup)*

A family of opertors $(S(t))_{t \geq 0}$ of $\pounds(E)$ is a strongly continuous semigroup on E when the following conditions are met

1) $S(0) = I$, (I is the identity operator on E),

2) $S(t + s) = S(t) S(s), t, s \geq 0$, (semigroup property),

3) for each $x \in X, S(t)x$ is continuous t on $[0, \infty)$.

This type of semigroup will simply be called a C_0-semi group

Definition 0.26 *A semigroup of bounded linear operators is said to be*

1) Uniformly continuous if:

$$\lim_{t \to 0^+} \|S(t) - I\| = 0.$$

2) Strongly continuous or class C_0 if:

$$\lim_{t \to s} S(t)x - x = 0 \text{ , } \forall x \in E$$

3) Class contraction semigroup C_0 he's classy C_0 and:

$$\|S(t)\| \leq 1 \text{ , } \forall t \geq 0.$$

Remark 0.2 *If $(S(t))_{t \geq 0}$ is a uniformly continuous semi group , then*

$$\lim_{t \longrightarrow s} \|S(t) - S(s)\| = 0.$$

0.6.1 Infinitesimal generator

Definition 0.27 *The infinitesimal generator of $S(t)$ is the linear operator A of domain*

$$D(A) = \left\{ x \in E \lim_{t \to 0^+} \frac{S(t)x - x}{t}, \text{ existe} \right\}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}, \quad u \in D(A)$$

Theorem 0.5 *Let $(A, D(A))$ be the infinitesimal generator of a semigroup $(S(t))_{t \geq 0}$ strongly continuous on E : for all $x_0 \in D(A)$, $x(t) = S(t)x_0$ is the unique solution of the problem*

$$x \in C([0,\infty)), D(A) \cap C^1([0,\infty)), E$$

$$x'(t) = Ax(t)$$

0.6.2 Hille-Yosida

Theorem 0.6 *An unbounded linear operator $(A, D(A))$ on X is the infinitesimal generator of a C_0.semigroup of contractions $(S(t))_{t\geq 0}$ if and only if.*

- A is closed and $\overline{D(A)} = X$,
- The resolvent set $p(A)$ of A contains $\mathbb{R}^+$,and for all $\lambda > 0$.

$$\left\|(\lambda I - A)^{-1}\right\|_{\pounds(X)} \leq \lambda^{-1}$$

0.6.3 Lummer-Phillips

Theorem 0.7 *Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of aC_0-semigroup of contractions if and only if it is a m-dissipative operator.*

Theorem 0.8 *Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with, $R(I - A) = X$ and X is reflexive then $\overline{D(A)} = X$*

Corollary 0.1 *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is a m-dissipative operator.*

Theorem 0.9 *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H.If A is dissipative and $0 \in p(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H*

Theorem 0.10 *.Let $(A,$ For $U0 \in D(A),)$ be an unbounded linear operator on H . Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$*

1) For$U_0 \in D(A)$, the problem admits a unique strong solution,

$$U\left(t\right)=S\left(t\right)U_{0}\in C^{0}\left(\mathbb{R}^{+},D(A)\right)\cap C^{1}\left(\mathbb{R}^{+},H\right)$$

2) or$U_0 \in D\left(A\right)$, the problem admits a unique strong solution.

$$U\left(t\right)\in C^{0}\left(\mathbb{R}^{+},H\right)$$

0.6.4 Lax-Milgram

Definition 0.28 *We say a bilinear form $a\left(u,v\right):H\times H\longrightarrow\mathbb{R}$:*

i) Continues if there exists a constant C such that:

$$\left|a\left(u,v\right)\right|\leq c\left|u\right|\left|v\right|\qquad\forall u,v\in H$$

ii) Coercive : if there is a constant $\alpha>0$ such that:

$$a\left(v,v\right)\geq\alpha\left|v\right|^{2}\qquad\forall v\in H$$

0.7 Stability of semigroup

Let $(X,\|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $\langle .,.\rangle_H$ and the induced norm $\|.\|_H$.

Definition 0.29 *Assume that A is the generator of a strongly continuous semigroup of contractions $\left(S\left(t\right)\right)_{t\geq0}$ on X.*

We say that the C0-semigroup $\left(S\left(t\right)\right)_{t\geq0}$ is

- Strongly stable if:

$$\lim_{t\rightarrow+\infty}\left\|S\left(t\right)u\right\|_{X}=0\qquad\forall u\in X.$$

- Uniformly stable if:

$$\lim_{t\to+\infty} \|S(t)u\|_{\mathcal{L}(X)} = 0.$$

- Exponentially stable if there exist two positive constants M and ε such that:

$$\|S(t)u\|_X \leq M\exp(-\varepsilon t)\,, \qquad \forall t \geq 0,\ \forall u \in X.$$

- Polynomially stable if there exist two positive constants C and ■ such that:

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X \quad \forall t \geq 0,\ \forall u \in X.$$

Chapter 1

Existence and Uniqueness of the solution

In this chapter we will calculate the energy for this model and demonstrate the local existence and uniqueness of the solution, using semigroup theory Westudy thefollowing Problem

1.1 Statement of problem

$$\begin{cases} u_{tt} - [au_x + b(x)(k_1 u_{tx} + k_2 u_{tx}(x, t-\tau))\}]_x + c(x) y_t = 0 & (x,t) \in (0,L) \times (0,\infty) \\ y_{tt} - y_{xx} - c(x) u_t = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0 & t > 0 \\ (u(0,t), u_t(0,t)) = (u_0(x), u_1(x)) & x \in (0,L) \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)) & x \in (0,L) \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)) & x \in (0,L) \end{cases}$$

where k_1 and are L, τ, a positive real numbers, k_2 is a non-zero real number and $(u_0, u_1, y_0, y_1, f_0)$ belongs t_o a suitable space [1]. We suppose that there exists $0 < \alpha < \beta < \gamma < L$ and a non-zero constant c_0, such that:

$$b(x) = \begin{cases} 1, x \in (0,\beta) \\ 0, x \in (\beta, L) \end{cases}$$

and

$$c(x) = \begin{cases} c_0, x \in (\alpha_2, \alpha_4) \\ 0, \text{otherwise} \end{cases}$$

In order to prove the existence and unity of the solution, we will change a new variable as following [5]:

$$\eta(x,\rho,t) := u_t(x, t-\rho\tau), \;\; x \in (0,\beta) \quad \rho \in (0,1), t > 0 \tag{2,1}$$

Then, system (1.1) becomes

$$u_{tt} - (S_b(u,u_t,\eta))_x + c(x)\, y_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty) \tag{2,2}$$

$$y_{tt} - y_{xx} - c(x)\, u_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty) \tag{2,3}$$

$$\tau\eta_t(x,\rho,t) + \eta_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \;\in (0,\beta) \times (0,1) \;\times (0,\infty) \tag{2,4}$$

where $S_b(u,u_t,\eta) := au_x + b(x)(k_1 u_{tx} + k_2 u_{tx}(x, t-\tau))$. Moreover, from the definition of $b(.)$, we have

$$(S_b(u,u_t,\eta)) := \tag{2,5}$$

$$\begin{cases} S_1(u,u_t,\eta) := au_x + k_1 u_{tx} + k_2 \eta_x(0,1,t) & x \;\in (0,\beta) \\ au_x, & x \in (\beta, L) \end{cases}$$

With the following boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0 \\ \eta(0,\rho,t) := 0, & (\rho,t) \quad \times (0,1) \;\times (0,\infty) \end{cases} \tag{2,6}$$

and the following initial conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L) \\ y_0(x,0) = y_0(x), y_1(x,0) = y_1(x), x \in (0,L) \\ \eta(x,\rho,0) = f_0(x, -\rho\tau), (x,\rho) \in (0,\beta) \times (0,1) \end{cases} \tag{2,7}$$

1.1.1 Preliminaries and Assumptions

Throughout this works,we use the space

$$V = \left\{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_1\right\}$$

the scalar products:

$$(u,v) = \int_\Omega u(x)\, v(x)\, dx, (u,v)_{\Gamma_0} = u(x)\, v(x)\, ds$$

and the norms:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right), \|u\|_{L^p\left(\Gamma_0\right)} = \left(\int_\Omega |u|^p \, ds\right)^{\frac{1}{p}}$$

1.2 Energy of system

In this section we will calculate the energy for this model, the energy of system (2.2)-(2.7) is given by

$$\begin{cases} \mathbf{E}_1(t) = \frac{1}{2}\int_0^L \left(|u_t|^2 + a\|u_x\|^2\right) \mathbf{dx}, & \mathbf{E}_2(t) = \frac{1}{2}\int_0^L \left(\|y_t\|^2 + \|y_x\|^2\right) \mathbf{dx} \\ \mathbf{E}_3(t) = \frac{\tau|K_2|}{2}\int_0^\beta \int_0^1 |\eta_x(.,\rho,t)|^2 \, d\rho dx \end{cases}$$

Lemma 1.1 *Let* (u, u_t, y, y_t, η) *be a regular solution of system (2.2)-(2.7). Then, the energy E(t) satisfes the following estimation*

$$\frac{dE(t)}{dt} \leq -(k_1 - |k_2|)\int_0^\beta |u_{tx}|^2 \, dx \tag{2,9}$$

Proof. By multiplying equation (2.2) by $\overline{u_t}$ and integrating over the $(0, L)$, ■

$$\int_0^L \left[u_{tt} - \left(\left(S_b(u, u_t, \eta)\right)_x + c(x)\, y\right] \overline{u_t} dx$$

then we tak the real part, we find

$$\Re\left\{\int_0^L \mathbf{u}_{tt}\overline{u_t}\mathbf{dx}\right\} - \Re\left\{\int_0^L S_b(u, u_t, \eta)\, \overline{u_t} dx\right\} + \Re\left\{\int_0^L c(x)\, y_t\right\} \overline{u_t} dx = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L |u_t|^2 \, dx - \Re\left\{\int_0^L S_b(u, u_t, \eta)\, \overline{u_t} dx\right\} + \Re\left\{\int_0^L c(x)\, y_t\right\} \overline{u_t}\mathbf{dx} = \mathbf{0}$$

using integration by parts and substituting the terms (2.6),(A-B) from the definition

$$c(x) = \begin{cases} c_{0,} x \in (\alpha, \gamma) \\ 0, x \in (0, \alpha) \cup (\gamma, L) \end{cases}$$

From the above equation and the defnition of $S_b(u, u_t, \eta)$ and $c(.)$,and integration by part with (2,6) we obtien

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 \, dx + \Re\left\{\int_0^L S_b(u, u_t, \eta)\, \overline{u_{xt}} dx\right\} + \Re\left\{c_0 \int_\alpha^\gamma y_t \overline{u_t} dx\right\} = 0$$

from the definition of $S_b(u, u_t, \eta)$,we obtien

$$\frac{1}{2}\frac{d}{dt}\int_0^L \|u_t\|^2 dx + +\Re\left\{a\int_0^L u_x\overline{u_{xt}}dx\right\} + \Re\left\{\int_0^L b(x)\, k_1 u_{tx}\overline{u}_{tx}dx\right\}$$

$$+\Re\left\{k_2\int_0^L \eta_x(.,1,t)\,\overline{u_{tx}}dx\right\} + \Re\left\{c_0\int_0^L y_t\overline{u_t}dx\right\} = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left\{\|v_t\|^2 + a\|u_x\|^2\right\} dx = -k_1\int_0^\beta |u_{tx}|^2 dx$$

$$-\Re\left\{k_2\int_0^\beta \eta_x(.,1,t)\,\overline{u_{tx}}dx\right\} - \Re\left\{c_0\int_\alpha^\gamma y_t\overline{u_t}dx\right\} = 0$$

Using Young's inequality in the above equation, we get

$$\frac{1}{dt}\mathbf{E}_1(t) \le -\left(k_1 - \frac{|k_2|}{2}\right)\int_0^\beta |u_{tx}|^2 dx \tag{1.1}$$

$$+\frac{|k_2|}{2}\int_0^\beta |\eta_x(.,1,t)|^2 dx - \Re\left\{c_0\int_\alpha^\gamma y_t\overline{u_t}dx\right\}$$

Then

$$E_1(t) = \frac{1}{2}\int_0^L \left(|u_t|^2 + a\|u_x\|^2\right) dx \tag{2,10}$$

Now, multiplying (2.3) by $\overline{y_t}$, integrating over$(0, L)$, using the definition of $c(.)$, then taking the real part, we get

$$\int_0^L \left[y_{tt} - y_{xx} - c(x)\, u_t\right]\overline{y_t}\, dx$$

$$\int_0^L y_{tt}\overline{y_t}\, dx - \Re\left\{\int_0^L y_{xx}\overline{y_t}\, dx\right\} - \Re\left\{\int_0^L c(x)u_t\overline{y_t}\, dx\right\} = 0$$

Using the integration by part and with the definition of $c(.)$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^L \left(\|y_t\|^2 + \|y_x\|^2\right) dx\right\} = \Re\left\{c_0\int_\alpha^\beta u_t\overline{y_t}dx\right\}$$

$$\frac{d}{dt}E_2(t) = \Re\left\{c_0\int_\alpha^\gamma u_t\overline{y_t}dx\right\} \tag{1.2}$$

Then

$$E_2(t) = \frac{1}{2}\int_0^L \left(\|y_t\|^2 + \|y_x\|^2\right) dx \tag{2,11}$$

We have

$$\boldsymbol{\tau\eta}_t(.,\rho,t)+\boldsymbol{\eta}_\rho(.,\rho,t)=\mathbf{0}$$

Deriving (2.4) with respect to x, we obtain

$$\tau\eta_{xt}(.,\rho,t)+\eta_{x\rho}(.,\rho,t)=0 \tag{2,12}$$

Multiplying (2.12) by $|k_2|\,\overline{\eta_x}(.,\rho,t)$, integrating over $(0,\beta)\times(0,1)$, then taking the real part, we get :

$$\int_0^\beta\int_0^1 \tau\eta(.,\rho,t)\,|k_2|\,\overline{\eta_x}(.,\rho,t)\,d\rho dx$$
$$+\int_0^\beta\int_0^1 \eta_{x\rho}(.,\rho,t)\,|k_2|\,\overline{\eta_x}(.,\rho,t)\,d\rho dx=0$$

$$\frac{\tau\,|k_2|}{2}\frac{d}{dt}\int_0^\beta\int_0^1 |\eta_x(.,\rho,t)|^2\,d\rho dx$$
$$+\frac{|k_2|}{2}\frac{d}{d\rho}\int_0^\beta\int_0^1 |\eta_x(.,\rho,t)|^2\,d\rho dx=0$$

$$\frac{d}{dt}\mathbf{E}_3(t)=-\frac{|k_2|}{2}[\int_0^\beta |\eta_x(.,\rho,t)|^2\,dx]_0^1$$

$$-\frac{|k_2|}{2}\left[\int_0^\beta |\eta_x(.,1,t)|^2-\int_0^\beta |\eta_x(.,0,t)|^2\,dx\right]$$

Using that fact and taking such as $\eta_x(.,0,t)=u_{tx}$ the part real we get

$$\mathbf{E}_3(t)=\frac{\tau\,|K_2|}{2}\int_0^\beta\int_0^1 |\eta_x(.,\rho,t)|^2\,d\rho dx \tag{2,13}$$

Finally, adding (2.10), (2.11) and (2.13), we obtain (2.9). The proof is thus complete

$$\frac{d}{dt}E(t)=\frac{d}{dt}(E_1(t)+E_2(t)+E_3(t))$$
$$\leq-\left(k_1-\frac{|k_2|}{2}\right)\int_0^\beta |u_{tx}|^2$$

$$+\frac{|k_2|}{2}\int_0^\beta |\eta_x(.,1,t)|^2\,dx-\Re\left\{c_0\int_\alpha^\gamma \overline{u}_t y_t dx\right\}$$
$$+\Re\left\{c_0\int_\alpha^\gamma u_t\overline{y_t}dx\right\}-\frac{|k_2|}{2}\int_0^\beta |\eta_x(.,1,t)|^2\,dx+\frac{|k_2|}{2}\int_0^\beta |u_{tx}|^2$$

Then

$$\frac{d}{dt}E(t) \leq -\left(k_1 - \frac{|k_2|}{2}\right)\int_0^\beta |u_{tx}|^2$$

In the sequel, the assumption on k_1 and k_2 will ensure that

$$k_1 > 0, k_2 \in \mathbb{R}^*, |k_2| < k_1 \tag{h}$$

Under the hypothesis (h) and from Lemma 2.1, the system (2.2)-(2.7) is dissipative in the sense that its energy is non-increasing with respect to time (i.e.$E_0(t) \leq 0$). Let us de ne the Hilbert space H by

$$H := \left(H_0^1(0,L) \times L^2(0,1)\right)^2 \times W$$

where

$$W := L^2(0,1); H_L^1(0,\beta) \quad \text{and } H_L^1(0,\beta) := \{\widetilde{\eta} \in H(0,\beta) \backslash\} \ \widetilde{\eta}(0) = 0.$$

The space W is an Hilbert space of $H_L^1(0,\beta)$ valued functions on $(0,1)$, equipped with the following inner product

$$\left(\eta^1,\eta^2\right) W := \int_0^\beta \int_0^1 \eta_x^1 \overline{\eta_x^2} d\rho dx \qquad \forall \eta^1,\eta^2 \in W.$$

The Hilbert space H is equipped with the following inner product

$$\left(U,U^1\right)_H = \int_0^L \left(au_x\overline{u_x^1} + v\overline{v^1} + y_x\overline{y_x^1} + z\overline{z^1}\right)dx \tag{2,14}$$

$$+\tau|k_2|\int_0^\beta\int_0^1 \eta_x(.,\rho)_x \overline{\eta_x^1}(.,\rho)\, d\rho dx$$

where $U = \left(u,v,y,z,\eta^1(.,\rho)\right)^T$, $U^1 = \left(u^1,v^1,y^1,z^1,\eta^1(.,\rho)\right)^T \in H$ Now, we de ne the linear unbounded operator

$A : D(A) \subset H \longrightarrow H$

with the domene

$$D(A) = \begin{cases} U = \left(u,v,y,z,\eta^1(.,\rho)\right)^T \in H \backslash y \in H_0^2(0,L) \cap H_0^1(0,L) \quad ,v,z \in H_0^1(0,L) \\ \qquad (S_b(u,u_t,\eta))_x \in L^2(0,1), \eta_\rho(.,\rho) \in W, \eta(.,0) = v(.) \text{ in } (0,\beta) \end{cases}$$

We have the system (2,4),(2,2)

$$\begin{cases} u_{tt} - (S_b(u,u_t,\eta))_x + c(x)\, y_t = 0 \\ y_{tt} - y_{xx} - c(x)\, u_t = 0 \\ \qquad \tau\eta_t(x,\rho,t) + \eta_\rho(x,\rho,t) = 0 \end{cases}$$

and from him

$$\begin{cases} u_{tt} = (S_b(u, u_t, \eta))_x - c(x) y_t \\ y_{tt} = y_{xx} + c(x) u_t \\ \tau \eta_t(x, \rho, t) = -\eta_\rho(x, \rho, t) \end{cases}$$

and from him

$$\begin{cases} u_{tt} = (S_b(u, u_t, \eta))_x - c(x) y_t \\ y_{tt} = y_{xx} + c(x) u_t \\ \eta_t(x, \rho, t) = -\tau^{-1} \eta_\rho(x, \rho, t) \end{cases}$$

We pose $v = u_t$ and $y_t = z$

So

$$\begin{cases} v_t = (S_b(u, v, \eta))_x - c(x) y_t \\ z_t = y_{xx} + c(x) v \\ \eta_t(., \rho) = -\tau \eta_\rho(., \rho) \end{cases} \tag{*}$$

We pose

$$U = (u, u_t, y, y_t, \eta), U = (u, v, y, z, \eta)$$

We transform the system (*) to Cauchy system

$$U_t = AU$$

$$\begin{Bmatrix} u_t \\ v_t \\ y_t \\ z_t \\ \eta_t \end{Bmatrix} = \begin{pmatrix} v \\ (S_b(u, v, \eta))_x - c(.) y_t \\ z \\ y_{xx} + c(.) v \\ -\tau \eta_\rho(., \rho) \end{pmatrix} \tag{2,15}$$

for all

$$U = (u, v, y, z, \eta(., \rho))^T \in D(A)$$

Now, if $U = (u, v, y, z, \eta(., \rho))^T$, then system (2.2)-(2.7) can be written as the following firrst order evolution equation

$$U_t = AU, \qquad U(0) = U_0 \tag{2,16}$$

where

$$U_0 = (u_0, u_1, y_0, f_0(., \rho))^T \in H$$

Remark 1.1 *The linear unbounded operator A is injective (i.e.*$\ker(A) = \{0\}$*. Indeed, if $U \in D(A)$ such that $AU = 0$, then $v = z = \eta(.,\rho) = 0$ and since $\eta(.,0) = v(.)$, we get $\eta(.,\rho) = 0$ Consequently, $(S_b(u, u_t, \eta))_x = au_{xx} = 0$ and $y_{xx} = 0$. Now, since $u(0) = u(L) = y(0) = y(L) = 0$, then $u = y = 0$. Thus $U = (u, v, y, z, \eta(.,\rho))^T = 0$*

1.3 Local Existence

In this section we will demonstrate the local existence and uniqueness of solution , using semi-group theory. Wea the solvability of the problem (2,14) (2,15) is ensured by the following proposition.

Proposition 1.1 *Under the hypothesis (h), the unbounded linear operator A is m-dissipative in the energy space H.*

Proof. For all ,$U = (u, v, y, z, \eta(.,\rho))^T \in D(A)$ from (2.14) and (2.15), and taking the part real we define the scalar product on the energy space H as follows

$$\Re(AU, U)_H = \Re\left\{\left(\begin{array}{c} v \\ (S_b(u, v, \eta))_x - c(.)\, y_t \\ z \\ y_{xx} + c(.)\, v \\ -\tau\eta_\rho(.,\rho) \end{array}\right), \left\{\begin{array}{c} u \\ v \\ y \\ z \\ \eta \end{array}\right\}\right\}$$

■

$$\Re(AU, U)_H = \Re\left\{\int_0^L av_x\overline{u_x}dx\right\} - \Re\left\{\int_0^L (S_b(u, u_t, \eta))_x \overline{v}dx\right\}$$
$$+\Re\left\{\int_0^L z_x\overline{y_x}dx\right\} - \Re\{y_{xx}\overline{z}dx\} - \Re\left\{\frac{|k_2|}{2}\int_0^\beta\int_0^1 \frac{d}{d\rho}|\eta_x(.,\rho)|^2 d\rho dx\right\}$$

we apply integration by part with respect to x on Ω

$$\Re(AU, U)_H = \Re\left\{\int_0^L av_x\overline{u_x}dx\right\} - \Re\left\{\int_0^L a\overline{v}_x u_x dx\right\}$$
$$= \Re\left\{\int_0^L av_x\overline{u_x}dx\right\} + \Re\left\{\int_0^L (S_b(u, u_t, \eta))_x \overline{v}dx\right\}$$
$$+\Re\left\{\int_0^L z_x\overline{y_x}dx\right\} + \Re\{y_{xx}\overline{z}dx\} - \Re\left\{|k_2|\int_0^\beta\int_0^1 \eta_{x\rho}(.,\rho)\,\overline{\eta_x}(.,\rho)\, d\rho dx\right\}$$

Using integration by parts to the second and fourth terms in the above equation, then using the definition of $S_b(u, u_t, \eta)$ and the fact that $U \in D(A)$, we get

$$\Re (AU, U)_H = -\Re\left\{k_1 \int_0^\beta v_x \overline{v}_x dx\right\} - \Re\left\{k_2 \int_0^\beta \eta_x(., 1)\, \overline{v_x} dx\right\}$$
$$+\Re\left\{\int_0^L z_x \overline{y_x} dx\right\} - \Re\left\{\int_0^L \overline{z}_x y_x dx\right\} - \frac{k_2}{2}\int_0^\beta \int_0^1 \frac{d}{d\rho} |\eta_x(., \rho)|^2 d\rho dx$$

we find

$$\Re (AU, U)_H = -k_1 \int_0^\beta |v_x|^2 dx - \Re\left\{k_2 \int_0^\beta \eta_x(., 1)\, \overline{v_x} dx\right\}$$
$$-\frac{k_2}{2}\int_0^\beta \int_0^1 \frac{d}{d\rho} |\eta_x(., \rho)|^2 d\rho dx \qquad (*)$$

the fact that $\eta(., 0) = v(.)$ in $(0, \beta)$, implies that

$$\Re (AU, U)_H = -k_1 \int_0^\beta |v_x|^2 dx - \Re\left\{k_2 \int_0^\beta \eta_x(., 1)\, \overline{v_x} dx\right\} - \frac{k_2}{2}\int_0^\beta \int_0^1 \frac{d}{d\rho} |\eta_x(., \rho)|^2 d\rho dx$$

we find

$$\Re (AU, U)_H = -k_1 \int_0^\beta |v_x|^2 dx - \Re\left\{k_2 \int_0^\beta \eta_x(., 1)\, \overline{v_x} dx - \frac{k_2}{2}\int_0^\beta |\eta_x(., \rho)|^2 dx\right\} - |\eta_x(., 0)|^2 dx$$

then (*) becoms

$$\Re (AU, U)_H = -\left(k_1 - \frac{|k_2|}{2}\right)\int_0^\beta |v_x|^2 dx - \frac{|k_2|}{2}\int_0^\beta |\eta_x(., 1)|^2 .$$
$$-\Re\left\{k_2 \int_0^\beta \eta_x(., 1)\, \overline{v_x} dx\right\}$$

Using Young's inequality in the above equation and the hypothesis (h), we obtain

$$\Re (AU, U)_H \leq -(k_1 - |k_2|)\int_0^\beta |v_x|^2 dx \qquad (2,17)$$

from this conclude that

$$\Re (AU, U)_H \leq 0$$

which implies that A is dissipative. Now, let us prove that$A$ is maximal. For this aim, let $F = \left(f^1, f^2, f^3, f^4, f^5(., \rho)\right)^T \in H$,
we look for $U = (u, v, y, z, \eta(., \rho))^T \in D(A)$ unique solution of

$$-AU = F \tag{2,18}$$

Equivalently, we have the following system

$$-v = f^1 \tag{2,19}$$

$$-\left(S_b\left(u, u_t, \eta\right)\right)_x + c\left(.\right) z = f^2 \tag{2,20}$$

$$-z = f^3 \tag{2,21}$$

$$-y_{xx} - c\left(.\right) v = f^4 \tag{2,22}$$

$$-\tau^{-1}\eta_\rho\left(., \rho\right) = f^5\left(., \rho\right) \tag{2,23}$$

with the following boundary conditions

$$u\left(0\right) = u\left(L\right) = y\left(0\right) = y\left(L\right) = 0, \eta\left(0, \rho\right) = 0 \tag{2,24}$$

and $\eta\left(., 0\right) = v\left(.\right)$ in $\left(0, \beta\right)$

From (2.19), (2.23) and (2.24), we get

$$\boldsymbol{\tau}^{-1}\boldsymbol{\eta}_\rho\left(., \rho\right) = \mathbf{f}^5\left(., \rho\right)$$

$$\int_0^\rho \boldsymbol{\eta}_\rho\left(\cdot, \rho\right) ds = \boldsymbol{\tau}\int_{\mathbf{0}}^{\boldsymbol{\rho}} \mathbf{f}^5\left(\cdot, \rho\right) ds$$

$$\eta\left(x, \rho\right) - \boldsymbol{\eta}\left(0, \rho\right) = \boldsymbol{\tau}\int_{\mathbf{0}}^{\boldsymbol{\rho}} \mathbf{f}^5\left(\cdot, \rho\right) ds$$

$$\eta\left(x, \rho\right) = \boldsymbol{\tau}\int_{\mathbf{0}}^{\boldsymbol{\rho}} \mathbf{f}^5\left(\cdot, \rho\right) ds + v\left(\cdot\right) / v\left(\cdot\right) = -\mathbf{f}^1$$

$$\eta\left(x, \rho\right) = \tau\int_0^\rho f^5\left(x, s\right) ds - f^1, \qquad \left(x, \rho\right) \in \left(0, \beta\right) \times \left(0, 1\right) \tag{2,25}$$

Since, $f^1 \in H_0^1\left(0, L\right)$ and $f^5\left(., \rho\right) \in W$. Then, from (2.23) and (2.25), we get $\eta_\rho\left(., \rho\right), \eta\left(., \rho\right) \in W$. Now, see the de nition of $S_b\left(u, u_t, \eta\right)$, substituting (2.19), (2.21) and (2.25) in (2.20) and (2.22), we get the following system

$$\left[S_b\left(u,f^1,\tau\int_0^1 f^5(x,s)\,ds-f^1\right)\right]_x+c(.)\,f^3=-f^2 \tag{2,26}$$

$$-y_{xx}-c(.)\,f^1=-f^4 \tag{2,27}$$

$$u(0)=u(L)=y(0)=y(L)=0 \tag{2,28}$$

where

$$S_b\left(u,-f^1,\tau\int_0^1 f^5(x,s)\,ds-f^1\right)$$

$$=\begin{cases} au_x-(k_1+k_2)\,f_x^1+\tau k_2\int_0^1 f_x^5(.,s)\,ds, x\in(0,\beta)\\ au_x, \qquad x\in(\beta,L)\end{cases}$$

Let $(\phi,\psi)\in H_0^1(0,L)\times H_0^1(0,L)$ .Multiplying (2.26) and (2.27) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating over $(0,L)$,

$$\begin{cases} \int_0^L\left[S_b\left(u,f^1,\tau\int_0^1 f^5(x,s)\,ds-f^1\right)\right]_x\overline{\phi}dx+\int_0^L c(.)\,f^3\overline{\phi}dx=-\int_0^L f^2\overline{\phi}dx\\ -\int_0^L \mathbf{y}_{xx}\overline{\psi}dx-\int_0^L \mathbf{c}(.)\,\mathbf{f}^1\overline{\psi}dx=-\int_0^L \mathbf{f}^4\overline{\psi}dx\end{cases}$$

then using formal integrations by parts, we obtain

$$\begin{cases} -\int_0^L S_b\left(u,f^1,\tau\int_0^1 f^5(x,s)\,ds-f^1\right)\overline{\phi}_xdx+\int_0^L c(.)\,f^3\overline{\phi}dx=-\int_0^L f^2\overline{\phi}dx\\ -\int_0^L \mathbf{y}_x\overline{\psi}_xdx-\int_0^L \mathbf{c}(.)\,\mathbf{f}^1\overline{\psi}dx=-\int_0^L \mathbf{f}^4\overline{\psi}dx\end{cases}$$

$$\begin{cases} a\int_0^L u_x\overline{\phi}_xdx-\int_0^\beta(k_1+k_2)\,f_x^1\overline{\phi}dx+\int_0^\beta \tau k_2\int_0^1 f_x^5(\cdot,s)\,ds\overline{\phi}dx-\int_0^L c(.)\,f^3\overline{\phi}dx=\int_0^L f^2\overline{\phi}dx\\ \int_0^L \mathbf{y}_x\overline{\psi}_xdx=\int_0^L \mathbf{f}^4\overline{\psi}\mathbf{dx}-\mathbf{c}_0\int_\alpha^\gamma \mathbf{f}^1\overline{\psi}\mathbf{dx}\end{cases}$$

$$\begin{aligned} a\int_0^L u_x\overline{\phi}_xdx&=\int_0^L f^2\overline{\phi}dx+c_0\int_\alpha^\gamma f^3\overline{\phi}dx \\ &+(k_1+k_2)\int_0^\beta f_x^1\overline{\phi}_xdx-\tau k_2\int_0^\beta\left(\int_0^1 f^5(.,s)\,ds\right)\overline{\phi}_xdx\end{aligned} \tag{2,29}$$

and

$$\int_0^L y_x\overline{\psi}_xdx=\int_0^L f^4\overline{\psi}dx-c_0\int_\alpha^\gamma f^1\overline{\psi}dx \tag{2,30}$$

Adding (2.29) and (2.30), we obtain

$$B\left(\left(u,y\right),\left(\phi,\psi\right)\right)=\pounds\left(\phi,\psi\right),\forall\left(\phi,\psi\right)\in H_0^1\left(0,L\right)\times H_0^1\left(0,L\right) \tag{2,31}$$

where

$$B\left(\left(u,y\right),\left(\phi,\psi\right)\right)=a\int_0^L u_x\overline{\phi}_x dx+\int_0^L y_x\overline{\psi}_x dx$$

and

$$\pounds\left(\phi,\psi\right)=\int_0^L\left(f^2\overline{\phi}+f^4\overline{\psi}\right)dx+c_0\int_\alpha^\gamma\left(f^3\overline{\phi}-f^1\overline{\psi}\right)dx$$
$$-\tau k_2\int_0^\beta\left(\int_0^1 f^5\left(.,s\right)ds\right)\overline{\phi}_x dx+\left(k_1+k_2\right)\int_0^\beta f_x^1\overline{\phi}_x dx$$

It is easy to see that,B is a sesquilinear, continuous and coercive form on $H_0^1\left(0,L\right)\times\left(H_0^1\left(0,L\right)\right)^2$, and $\pounds$ is a linear and continuous form on $H_0^1\left(0,L\right)\times H_0^1\left(0,L\right)$ .Then,
it follows by Lax-Milgram theorem that (2.31) admits a unique solution $\left(u,y\right)\in H_0^1\left(0,L\right)\times H_0^1\left(0,L\right)$.
By using the classical elliptic regularity, we deduce that system (2.26)-(2.28) admits a unique solution $\left(u,y\right)\in\times H_0^1\left(0,L\right)\times\left(H_0^2\left(0,L\right)\cap H_0^1\left(0,L\right)\right)$ such that $\left(S_b\left(u,v,\eta\right)\right)_x\in L^2\left(0,1\right)$
and sinc $\ker\left(A\right)=\{0\}$, we get $U=\left(u,-f^{-1},y,-f^3,\tau\int_0^\rho\left(.,s\right)ds-f^{-1}\right)\in D\left(A\right)$ is a unique solution of (2.18).
Then ,A is an isomorphism and since $\rho(A)$ is open set of $\mathbb{C}$ we easily get $R\left(\lambda I-A\right)=H$ for a su ciently smal $\lambda>0$ This, together with the dissipativeness of A, imply that $D(A)$ is dense in H and that A is m-dissipative in H.
According to Lumer-Phillips theorem Proposition 2.1 implies that the well-posedness of (2.16). Then, we have the following result:

Theorem 1.1 *Under the hypothesis (h), for all $U_0\in H$, system (2.16) admits a unique weak solution:*

$$U\left(x,\rho,t\right)=\exp\left(At\right)U_0\left(x,\rho\right)\in C^0\left(\mathbb{R}^+,H\right).$$

Moreover, U_0 if $D(A)$, then system (2.16) admits a unique strong solution$U\left(x,\rho,t\right)=\exp\left(At\right)U_0\left(x,\rho\right)\in C^0\left(\mathbb{R}^+,D\left(A\right)\right)\cap C^1\mathbb{R}^+$

$$U\left(x,\rho,t\right)=\exp\left(At\right)U_0\left(x,\rho\right)\in C^0\left(\mathbb{R}^+,H\right)\cap C^1\left(\mathbb{R}^+,D\left(A\right)\right).$$

Chapter 2

stability

2.1 Strong stability

In this section , we will prove the strong stability of systeme (2,2)-(2,7)

$$u_{tt} - (S_b(u,v,\eta))_x + c(.)\, y_t = 0, \quad (x,t) \in (0,L) \times (0,\infty)$$

(2,7) the initial conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L) \\ y_0(x,0) = y_0(x), y_1(x,0) = y_1(x), x \in (0,L) \\ \eta(x,\rho,0) = f_0(x,-\rho\tau), (x,\rho) \in (0,\beta) \times (0,1) \end{cases}$$

the main result of this section is the following theorem

Theorem 2.1 *Assume that (h) is true. Then, the C_0-semigroup of contraction $(\exp(At))_{t\geq 0}$ is strongly stable in H; i.e, for al $U_0 \in H$,*

the solution of (2.16) satisfies :

$$\lim_{t\to+\infty} \|\exp(At)\, U_0\|_H = 0$$

Proposition 2.1 *Under the hypothesis (h), we have*

$$i\mathbb{R} \subset \rho(A) \qquad (3,1)$$

We will prove Proposition 3.1 by contradiction argument. Remark that, it has been proved in Proposition 2.1 that $0 \in \rho(A)$ Now, suppose that (3.1) is false, then there exists $w \in \mathbb{R}^*$such tha $iw \notin \rho(A)$, let $\left\{\lambda^n, U^n := \left(u^n, v^n, y^n, z^n, \eta^n\left(., \rho\right)\right)^T\right\}_{n\geq 1} \subset \mathbb{R}^* \times D(A)$, with

$$\lambda^n \longrightarrow w \text{ as } n \longrightarrow \infty \text{ and } |\lambda^n| < |w| \tag{3,2}$$

and

$$\|U^n\|_H = \left\|\left(u^n, v^n, y^n, z^n, \eta^n\left(., \rho\right)\right)^T\right\|_H = 1 \tag{3,3}$$

such that

$$\left(i\lambda^n - A\right)U^n = F^n := \left(f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}\left(., \rho\right)\right)^T \longrightarrow 0 \text{ in } H \tag{3,4}$$

Equivalently, we have

$$i\lambda^n U^n - U^n = f^{1,n} \longrightarrow 0 \text{ in } H_0^1\left(0, L\right) \tag{3,5}$$

$$i\lambda^n U^n - \left(S_b\left(u, u_t, \eta\right)\right)_x + c\left(.\right)z^n = f^{2,n} \longrightarrow 0 \text{ in } L^2\left(0, L\right) \tag{3,6}$$

$$i\lambda^n U^n - z^n = f^{3,n} \longrightarrow 0 \text{ in } \longrightarrow 0 \text{ in } H_0^1\left(0, L\right) \tag{3,7}$$

$$i\lambda^n z^n - y^n_{xx} - c\left(.\right)v^n = f^{4,n} \longrightarrow 0 \text{ in } L^2\left(0, L\right) \tag{3,8}$$

$$i\lambda^n z^n \eta^n_\rho\left(., \rho\right) + \tau^{-1}\eta^n_\rho\left(., \rho\right) = f^{5,n}\left(., \rho\right) \longrightarrow 0 \text{ in } W \tag{3,9}$$

Then , we will proof condition (3,2) by finding a contraction with (3,3) such as $\|U^n\|_H \to 0$, the proof proposition (3,1) has been divided into several

Lemma 2.1 *Under the hypothesis (h), the solution* $U^n := \left(u^n, v^n, y^n, z^n, \eta^n\left(., \rho\right)\right)^T \in D(A)$

of system (3.5)-(3.9) satis es the following limits

$$\lim_{n\to\infty}\int_0^\beta |v^n_x|^2\,dx = 0 \tag{3,10}$$

$$\lim_{n\to\infty}\int_0^\beta |v^n|^2\,dx = 0 \tag{3,11}$$

$$\lim_{n\to\infty}\int_0^\beta |u^n_x|^2\,dx = 0 \tag{3,12}$$

$$\lim_{n\to\infty}\int_0^\beta\int_0^1 \left|\eta^n_\rho\left(., \rho\right)\right|^2 d\rho dx = 0 \tag{3,13}$$

$$\lim_{n\to\infty}\int_0^\beta \left|\eta^n_\rho\left(., 1\right)\right|^2 dx = 0 \tag{3,14}$$

$$\lim_{n\to\infty}\int_0^\beta \left|S_b\left(u^n, u^n_t, \eta^n\right)\right|^2 dx = 0 \tag{3,15}$$

Proof. First, taking the inner product of (3.4) with U^n in H and using (2.17) with the help of hypothesis (h), ■
we obtain

$$\Re\left(AU^n, U^n\right) \leq -\left(k_1 - |k_2|\right)\int_0^\beta \left|v_x^n\right|^2 dx$$

$$\int_0^\beta \left|v_x^n\right|^2 dx \leq -\frac{1}{k_1-|k_2|}\Re\left(AU^n, U^n\right)_H \tag{3,16}$$

$$= \frac{1}{k_1-|k_2|}\Re\left(F^n, U^n\right)_H \leq \frac{1}{k_1-|k_2|}\left\|F^n\right\|_H \left\|U^n\right\|_H \to 0$$

$$\lim_{n\to\infty}\int_0^\beta \left|v_x^n\right|^2 dx = 0$$

$$\Re\left(AU^n, U^n\right) = \Re\left(\left(i\lambda^n I - A\right)U^n, U^n\right) = \Re\left(-AU^n, U^n\right) = \Re\left(F^n, U^n\right)$$

so

$$-AU = F^n \Longrightarrow AU = F^n$$

Passing to the limit in (3.16), then using the fact that $\left\|U^n\right\|_H = 1$ and $\left\|F^n\right\|_H \longrightarrow 0$ we obtain (3.10). Now,
since $v^n \in H_0^1\left(0, L\right)$,
then it follows from Poincare inequality that there exists a constant $C_\rho > 0$ such that

$$\left\|v^n\right\|_{L^2(0,\beta)} \leq C_\rho \left\|v_x^n\right\|_{L^2(0,\beta)} \tag{3,17}$$

Thus, from (3.10) R and (3.17), we obtain (3.11). Next, from (3.5) and the fact that according to (3,11)

$$\lim_{n\to\infty}\int_0^\beta \left|v^n\right|^2 dx = 0$$

$$\int_0^\beta \left|f_x^{1,n}\right|^2 dx \leq \int_0^L \left|f_x^{1,n}\right|^2 dx \leq \frac{1}{\alpha}\left\|F^n\right\|_H^2 \text{ ,we deduce that} \tag{2.1}$$

$$\int_0^\beta \left|u_x^n\right|^2 dx \leq \frac{2}{\left(\lambda^n\right)^2}\int_0^\beta \left|v_x^n\right|^2 dx + \frac{2}{\left(\lambda^n\right)^2}\int_0^\beta \left|f_x^{1,n}\right|^2$$

$$\leq \frac{2}{\left(\lambda^n\right)^2}\int_0^\beta \left|v_x^n\right|^2 dx + \frac{2}{\left(\lambda^n\right)^2}\left\|F^n\right\|_H^2 \tag{3,18}$$

Passing to the limit in (3.18), then using (3.2), (3.10) and the fact that $\left\|F^n\right\|_H \longrightarrow 0$ we obtain (3.12). Moreover, from (3.9) and the fact that $\eta^n\left(.,0\right) = v^n\left(.\right)$

in $(0,\beta)$,we deduce that

$$\eta^{n}(x,\rho)=v^{n}\exp(-i\lambda^{n}\tau\rho)+\tau\int_{0}^{\rho}\exp(-i\lambda^{n}\tau(s-\rho))f^{5,n}(x,s)\,ds, \tag{3,19}$$

$$(x,\rho)\in(0,\beta)\times(0,1)$$

From (3.19), and the fact that $\rho\in(0,1)$ and$\int_0^\beta\int_0^1|f^{5,n}(.,s)|\,dsdx\leq\frac{1}{\tau|k_2|}\|F^n\|_H$.
we obtain

$$\int_0^\beta\int_0^1|\eta_x^n(.,\rho)|^2\,d\rho dx\leq 2\int_0^\beta|v_x^n|^2\,dx\int_0^\beta\int_0^1\int_0^\rho\rho\left|f^{5,n}(.,s)\right|^2dsd\rho dx$$

$$\leq 2\int_0^\beta|v_x^n|^2\,dx+2\tau^2\int_0^\beta\int_0^1\int_0^1\rho\left|f^{5,n}(.,s)\right|^2dsd\rho dx$$

$$=2\int_0^\beta|v_x^n|^2\,dx+2\tau^2\left(\int_0^1\rho d\rho\right)\int_0^\beta\int_0^1\left|f^{5,n}(.,s)\right|^2dsdx$$

$$=2\int_0^\beta|v_x^n|^2\,dx+\tau^2\int_0^\beta\int_0^1\left|f^{5,n}(.,s)\right|^2dsdx$$

$$\leq 2\int_0^\beta|v_x^n|^2\,dx+\tau\,|k_2|^{-1}\,\|F^n\|_H^2$$

Passing to the limit in the above inequality, then using (3.10) and the fact that $\|F^n\|_H\longrightarrow 0$,
we obtain (3.13). On the other hand, from (3.19), we have

$$\eta_x^n(.,1)=v_x^n\exp(-i\lambda^n\tau)+\tau\int_0^1\exp(-i\lambda^n\tau(s-1))f^{5,n}(.,s)\,ds$$

consequently, by using the same argument as proof of (3.13), we obtain (3.14). Next, it is clear to see that

$$\int_0^\beta|S_1(u^n,u_t^n,\eta^n)|^2=\int_0^\beta|au_x^n+k_1v_x^n+k_2\eta_x^n(.,1)|^2\,dx$$

$$\leq 3a^2\int_0^\beta|u_x^n|^2\,dx+3k_2^2\int_0^\beta|\eta_x^n(.,\rho)|^2\,dx$$

Finally, passing to the limit in the above inequality, then using (3.10), (3.12) and (3.14),
we obtain (3.15). The proof is thus complete. Now, we x a function $g\in C^1([\alpha,\beta])$ such that

$$g(\alpha)=-g(\beta)=1\text{and } set\max_{x\in[\alpha,\beta]}|g(x)|=M_g \tag{3,20}$$

$$\text{and}\max_{x\in[\alpha,\beta]}|g'(x)|=M_{g'}$$

Lemma 2.2 *Under the hypothesis (h), the solution* $U^n:=(u^n,v^n,y^n,z^n,\eta^n(.,\rho))^T\in D(A)$

of system (3.5)-(3.9) satis es the following inequalities

$$|z^n(\beta)|^2 + |z^n(\alpha)|^2 \leq M_{g'} \int_0^\beta |z^n|^2 \, dx \tag{3,21}$$

$$+2|\lambda^n| M_g \left(\int_\alpha^\beta |z^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g \|F^n\|_H$$

$$|y_x^n(\beta)|^2 + |y_x^n(\alpha)|^2 \leq M_{g'} \int_0^\beta |y_x^n|^2 \, dx \tag{3,22}$$

$$+2(|\lambda^n| + C_0) M_g \left(\int_\alpha^\beta |Y_x^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g \|F^n\|_H$$

and the following limits

$$\lim_{n\to\infty} |v^n(\alpha)| = 0 \quad \text{and} \quad \lim_{n\to\infty} |v^n(\beta)| = 0 \tag{3,23}$$

$$\lim_{n\to\infty} |(S_b(u^n, u_t^n, \eta^n))(\alpha)| = 0 \text{ and } \lim_{n\to\infty} |(S_b(u^n, u_t^n, \eta^n))(\beta)| = 0 \tag{3,24}$$

Proof. from (3.7),we deduce that

$$i\lambda^n y_x^n - z_x^n = f_x^{3,n} \tag{3,25}$$

■

Multiplying (3.25) and (3.8) by $2g\,\overline{z^n}$ and $2g\,\overline{y_x^n}$ respectively, integrating over (α, β), using the definition of $c(.)$, then taking the real part, we get

$$\Re\left\{2i\lambda^n \int_\alpha^\beta g y_x^n \overline{z^n} dx\right\} - \int_\alpha^\beta g\left(|z^n|^2\right)_x dx = \Re\left\{2\int_\alpha^\beta g f_x^{3,n} \overline{z^n} dx\right\} \tag{3,26}$$

and

$$\Re\left\{2i\lambda^n \int_\alpha^\beta g y_x^n \overline{z^n} dx\right\} - \int_\alpha^\beta g\left(|y_x^n|^2\right)_x dx \tag{3,27}$$

$$-\Re\left\{2c_0 \int_\alpha^\beta g v^n \overline{y_x^n} dx\right\} = \Re\left\{2\int_\alpha^\beta g f^{4,n} \overline{y_x^n}\right\}$$

Using integration by parts in (3.26) and (3.27), we obtain

$$\left[-g|z^n|^2\right]_\alpha^\beta = -\int_\alpha^\beta g' |z^n|^2 \, dx - \Re\left\{2i\lambda^n \int_\alpha^\beta g y_x^n \overline{z^n} dx\right\}$$

$$+\Re\left\{2\int_{\alpha}^{\beta} g f_x^{3,n}\overline{z^n}dx\right\}$$

and

$$\left[-g\left|y_x^n\right|^2\right]_{\alpha}^{\beta}=-\int_{\alpha}^{\beta} g'\left|y_x^n\right|^2dx-\Re\left\{2i\lambda^n\int_{\alpha}^{\beta} gz^n\overline{y_x^n}dx\right\}$$
$$+\Re\left\{2c_0\int_{\alpha}^{\beta} gv^n\overline{y_x^n}dx\right\}+\Re\left\{2\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} g f_x^{4,n}\overline{y_x^n}dx\right\}$$

Using the definition of g and Cauchy-Schwarz inequality in the above equations, we obtain

$$\left|z^n(\beta)\right|^2+\left|z^n(\alpha)\right|^2\leq M_{g'}\int_{\alpha}^{\beta}\left|z^n\right|^2dx$$
$$+2\left|\lambda^n\right|M_g\left(\int_{\alpha}^{\beta}\left|y_x^n\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}\left|z^n\right|^2dx\right)^{\frac{1}{2}}$$
$$+2M_g\left(\int_{\alpha}^{\beta}\left|f_x^{3,n}\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}\left|z^n\right|^2dx\right)^{\frac{1}{2}}$$

and

$$\left|y_x^n(\beta)\right|^2+\left|y_x^n(\alpha)\right|^2\leq M_{g'}\int_{\alpha}^{\beta}\left|y_x^n\right|^2dx$$
$$+2\left|\lambda^n\right|M_g\left(\int_{\alpha}^{\beta}\left|y_x^n\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}\left|z^n\right|^2dx\right)^{\frac{1}{2}}$$
$$+2\left|c_0\right|M_g\left(\int_{\alpha}^{\beta}\left|y_x^n\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}\left|v^n\right|^2dx\right)^{\frac{1}{2}}$$
$$+2M_g\left(\int_{\alpha}^{\beta}\left|f_x^{4,n}\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}\left|y_x^n\right|^2dx\right)^{\frac{1}{2}}$$

The refore, from the above inequalities and the fact that $\int_{\alpha}^{\beta}\left|\xi_1^n\right|^2dx\leq\int_0^L\left|\xi_1^n\right|^2dx\leq\left\|U^n\right\|_H^2=1$ with $\xi_1^n\in\{v^n,y_x^n,z^n\}$
and $\int_{\alpha}^{\beta}\left|\xi_1^n\right|^2dx\leq\int_0^L\left|\xi_2^n\right|^2dx\leq\left\|F^n\right\|_H^2$ with $\xi_2^n\in\{f_x^{3,n},f_x^{4,n}\}$, we obtain (3.21) and (3.22). On the other hand, from (3.5),
we deduce that

$$i\lambda^n u_x^n-v_x^n=f_x^{1,n} \tag{3,28}$$

Multiplying (3.28) and (3.6) by $2g\overline{v_n}$ and $2gS_1\left(u^n,u_t^n,\eta^n\right)$ respectively, integrating over (α,β), using the definition of $c(.)$ and $S_b\left(u^n,u_t^n,\eta^n\right)$, then taking the real part,
we get

$$\Re\left\{2i\lambda^n\int_{\alpha}^{\beta} gu_x^n\overline{v_n}\right\}dx-\int_{\alpha}^{\beta} g\left(\left|v^n\right|\right)_x dx \tag{3,29}$$

and

$$= \Re\left\{2\int_{\alpha}^{\beta} g f_x^{1,n}\overline{v_n}dx\right\}\Re\left\{2i\lambda^n\int_{\alpha}^{\beta} g v^n\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\} \tag{3,30}$$

$$-\int_{\alpha}^{\beta} g\left(\left|S_1\left(u^n,u_t^n,\eta^n\right)\right|\right)_x dx$$

$$+\Re\left\{2c_0\int_{\alpha}^{\beta} g z^n\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\}$$

$$= \Re\left\{2\int_{\alpha}^{\beta} g f_x^{2,n}\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\}$$

Using integration by parts in (3.29) and (3.30), we get

$$\left[-g\left|v^n\right|^2\right]_{\alpha}^{\beta} = -\int_{\alpha}^{\beta} g'\left|v^n\right|^2 dx - \Re\left\{2i\lambda^n\int_{\alpha}^{\beta} g u_x^n\overline{v^n}dx\right\}$$

$$+\Re\left\{2\int_{\alpha}^{\beta} g f_x^{1,n}\overline{v_n}dx\right\}$$

and

$$\left[-g\left|S_1\left(u^n,u_t^n,\eta^n\right)\right|^2\right]_{\alpha}^{\beta}$$

$$= -\int_{\alpha}^{\beta} g'\left|S_1\left(u^n,u_t^n,\eta^n\right)\right|^2 dx$$

$$-\Re\left\{2i\lambda^n\int_{\alpha}^{\beta} g v^n\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\}$$

$$-\Re\left\{2c_0\int_{\alpha}^{\beta} g z^n\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\}$$

$$+\Re\left\{2\int_{\alpha}^{\beta} g f_x^{2,n}\overline{S_1}\left(u^n,u_t^n,\eta^n\right)dx\right\}$$

Using the definition of g and Cauchy-Schwarz inequality in the above equations, then using the fact that

$$\left\{\begin{array}{c}\int_{\alpha}^{\beta}\left|z^n\right|^2 dx \le \int_0^L\left|z^n\right|^2 dx \le \left\|U^n\right\|_H^2 = 1, \int_{\alpha}^{\beta}\left|f_x^{1,n}\right|^2 dx \le \int_0^L\left|f_x^{1,n}\right|^2 dx \le \frac{1}{a}\left\|F^n\right\|_H^2 \\ \text{and } \int_{\alpha}^{\beta}\left|f_x^{2,n}\right|^2 dx \le \int_0^L\left|f_x^{2,n}\right|^2 dx \le \left\|F^n\right\|_H^2\end{array}\right.$$

we obtain

$$\left|v^n(\beta)\right|^2 + \left|v^n(\alpha)\right|^2 \le Mg'\int_{\alpha}^{\beta}\left|v^n\right|^2 dx$$

$$+2\left|\lambda^n\right| Mg\left(\int_\alpha^\beta \left|u_x^n\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\alpha^\beta \left|v^n\right|^2 dx\right)^{\frac{1}{2}}$$

$$+\frac{2}{\sqrt{a}} Mg\left(\int_\alpha^\beta \left|v^n\right|^2 dx\right)^{\frac{1}{2}} \left\|F^n\right\|_H^2$$

and

$$\left|\left(S_1\left(u^n, u_t^n, \eta^n\right)\right)\left(\beta^-\right)\right|^2+\left|\left(S_1\left(u^n, u_t^n, \eta^n\right)\right)\left(\alpha^-\right)\right|^2$$

$$\leq Mg' \int_\alpha^\beta \left|S_1\left(u^n, u_t^n, \eta^n\right)\right|^2 dx+2\left|\lambda^n\right| Mg\left(\int_\alpha^\beta \left|S_1\left(u^n, u_t^n, \eta^n\right)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\alpha^\beta \left|v^n\right|^2 dx\right)^{\frac{1}{2}}$$

$$+2c_0 Mg\left(\int_\alpha^\beta \left|S_1\left(u^n, u_t^n, \eta^n\right)\right|^2 dx\right)^{\frac{1}{2}}+Mg\left(\int_\alpha^\beta \left|S_1\left(u^n, u_t^n, \eta^n\right)\right|^2 dx\right)^{\frac{1}{2}} \left\|F^n\right\|_H$$

Finally, passing to limit in the above inequalities, then using (3.2), Lemma 3.1 and the fact that $\left\|F^n\right\|_H^2 \rightarrow 0$, we obtain (3.23) and (3.24). The proof is thus complete.

From (3.2), (3.21), (3.22), and the fact that $\left|U^n\right|_H=1$ and $\left\|F^n\right\|_H \rightarrow 0$, we obtain

$$\left|z^n(\alpha)\right|, \left|z^n(\beta)\right|, \left|y_x^n(\alpha)\right|, \left|y_x^n(\beta)\right| \text{ are bounded} \tag{3,32}$$

The solution $U^n:=\left(u^n, v^n, y^n, z^n, \eta^n(., \rho)\right)^T \in D(A)$ of system (3.5)-(3.8) satis es the following limits

$$\lim_{n\rightarrow\infty} \int_\alpha^\beta \left|z^n\right|^2 dx=0 \quad \text{ and } \quad \lim_{n\rightarrow\infty} \int_\alpha^\beta \left|y_x^n\right|^2 dx=0 \tag{3,33}$$

multiplying (3.6) by $\overline{z^n}$, integrating over (α, β), using the definition of $c(.)$ and $S_n\left(u^n, u_t^n, \eta^n\right)$, then taking the real part, we get

$$\Re\left\{i\lambda^n \int_\alpha^\beta v^n \overline{z^n} dx\right\}-\Re\left\{\int_\alpha^\beta S_1\left(u^n, u_t^n, \eta^n\right)_x \overline{z^n} dx\right\} \tag{3,34}$$

$$+c_0 \int_\alpha^\beta \left|z^n\right|^2 dx=\Re\left\{\int_\alpha^\beta f_x^{2,n} \overline{z^n} dx\right\}$$

From (3.7), we deduce that

$$\overline{z_x^n}=-i\lambda^n \overline{y_x^n}-\overline{f_x^{3,n}} \tag{3,35}$$

Using integration by parts to the second term in (3.34), then using (3.35), we get

$$c_0 \int_\alpha^\beta |z^n|^2 \, dx \tag{3,36}$$

$$\Re\left\{ i\lambda^n \int_\alpha^\beta S_1(u^n, u_t^n, \eta^n)\, \overline{y_x^n} dx \right\}$$

$$+\Re\left\{ \int_\alpha^\beta S_1(u^n, u_t^n, \eta^n)\, \overline{f_x^{3,n}} dx \right\} + \Re\left\{ \left[S\left({}_1 u^n, u_t^n, \eta^n\right) \overline{z^n} \right]_\alpha^\beta \right\}$$

$$+\Re\left\{ \int_\alpha^\beta f_x^{2,n} \overline{z^n} dx \right\} - \Re\left\{ i\lambda^n \int_\alpha^\beta v^n \overline{z^n} dx \right\}$$

Using Cauchy-Schwarz inequality in the above equation and the fact that
$\int_\alpha^\beta |\xi_1^n|^2 \, dx \leq \int_0^L |\xi_1^n|^2 \, dx \leq \|U^n\|_H^2 = 1$
with $\xi_1^n \in \{y_x^n, z^n\}$ and $\int_\alpha^\beta |\xi_2^n|^2 \, dx \leq \int_0^L |\xi_2^n|^2 \, dx \leq \|F^n\|_H^2$ with $\xi_2^n \in \{f_x^{2,n}, f_x^{3,n}\}$,
we obtain

$$\left| c_0 \int_\alpha^\beta |z^n|^2 \, dx \right| \leq \left(|\lambda^n| \, \|F^n\|_H \right) \left(\int_\alpha^\beta |S_1(u^n, u_t^n, \eta^n)|^2 \, dx \right)^{\frac{1}{2}} \tag{3,37}$$

$$+ |\lambda^n| \left(\int_\alpha^\beta |v^n|^2 \, dx \right)^{\frac{1}{2}} + \left| \left(S_1(u^n, u_t^n, \eta^n)\left(\beta^-\right) \right) \right| |z^n(\beta)|$$

$$+ \left| \left(S_1(u^n, u_t^n, \eta^n)\left(\alpha^-\right) \right) \right| |z^n(\alpha)| + \|F^n\|_H$$

Passing to the limit in the above inequality, then using (3.2), (3.32), (3.24),and the fact that $\|F^n\|_H \to 0$,
we obtain the rst limit in (3.33). On the other hand, multiplying (3.8) by $-\overline{z^n}(\lambda^n)^{-1}$,
using the definition of $c(.)$, then taking the real part, we get

$$-\int_\alpha^\beta |z^n|^2 \, dx + \Im\left\{ (\lambda^n)^{-1} \int_\alpha^\beta y_{xx}^n \overline{z^n} dx \right\}$$

$$+\Im\left\{ c_0 (\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} dx \right\} = -\Im\left\{ (\lambda^n)^{-1} \int_\alpha^\beta f_x^{4,n} \overline{z^n} dx \right\}$$

Using integration by parts to the second term in the above equation, then using (3.35),
we obtain

$$\int_\alpha^\beta |y_x^n|^2 \, dx = \int_\alpha^\beta |z^n|^2 \, dx - \Im\left\{ (\lambda^n)^{-1} \int_\alpha^\beta \overline{f_x^{3,n}} y_x^n dx \right\} - \Im\left\{ (\lambda^n)^{-1} \left[y_x^n \overline{z^n} \right]_\alpha^\beta \right\}$$

$$= -\Im\left\{ c_0 (\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} dx \right\} - \Im\left\{ (\lambda^n)^{-1} \int_\alpha^\beta f_x^{3,n} \overline{z^n} dx \right\}$$

Using Cauchy-Schwarz inequality in the above equation and the fact that $\|U^n\|_H = 1$, we get

$$\int_\alpha^\beta |y_x^n|^2 \, dx \leq \int_\alpha^\beta |z^n|^2 \, dx + c_0 \, |\lambda^n|^{-1} \left(\int_\alpha^\beta |v^n|^2 \, dx \right)^{\frac{1}{2}} \tag{3,38}$$

$$+2 \, |\lambda^n|^{-1} \, \|F^n\|_H + |\lambda^n|^{-1} \, |y_x^n(\beta)| \, |z^n(\beta)| + |\lambda^n|^{-1} \, |y_x^n(\alpha)| \, |z^n(\alpha)|$$

Passing to the limit in (3.21), then using (3.2), the rst limit in (3.33) and the fact that $\|F^n\|_H \to 0$, we get

$$\lim_{n\to\infty} |z^n(\alpha)| = 0 \quad \text{and} \quad \lim_{n\to\infty} |z^n(\beta)| = 0 \tag{3,39}$$

passing to the limit in (3.38), then using (3.2), (3.11), (3.32), the rst limit in (3.33), (3.39), and the fact that $\|F^n\|_H \to 0$,
we obtain the second limit in (3.33). The proof is thus complete.
Under the hypothesis (h), the solution $U^n := (u^n, v^n, y^n, z^n, \eta^n(., \rho))^T \in D(A)$ of system (3.5)-(3.9) satis es
the following estimations

$$\lim_{n\to\infty} |u^n(\beta)|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} |y^n(\beta)|^2 = 0 \tag{3,40}$$

$$\lim_{n\to\infty} |u_x^n(\beta)|^2 = 0 \quad \text{and} \lim_{n\to\infty} |y_x^n(\beta)|^2 = 0 \tag{3,41}$$

$$\lim_{n\to\infty} \left(\int_\beta^\gamma |u^n|^2 \, dx + \int_\beta^\gamma |u_x^n|^2 \, dx + \int_\beta^\gamma |y^n|^2 \, dx + \int_\beta^\gamma |y_x^n|^2 \, dx \right) = 0 \tag{3,42}$$

$$\lim_{n\to\infty} \int_\beta^\gamma |v^n|^2 \, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\beta^\gamma |z^n|^2 \, dx = 0 \tag{3,43}$$

From (3.5) and (3.7), we get

$$|u^n(\beta)|^2 \leq 2 (\lambda^n)^{-1} |v_n(\beta)|^2 + 2 (\lambda^n)^{-1} \left| f_x^{1,n}(\beta) \right|$$

and

$$|y^n(\beta)|^2 \leq 2 (\lambda^n)^{-1} |z_n(\beta)|^2 + 2 (\lambda^n)^{-1} \left| f_x^{3,n}(\beta) \right|$$

Using the fact that $\left| f_x^{1,n}(\beta) \right|^2 \leq \beta \int_0^\beta \left| f_x^{1,n} \right|^2 dx \leq \frac{\beta}{\alpha} \|F^n\|_H^2$ and $\left| f_x^{3,n}(\beta) \right|^2 \leq \beta \int_0^\beta \left| f_x^{3,n} \right|^2 dx \leq \beta \|F^n\|_H^2$ in the above inequalities,
we obtain

$$|u^n(\beta)|^2 \leq 2 (\lambda^n)^{-1} |v_n(\beta)|^2 + 2\beta a^{-1} (\lambda^n)^{-1} \|F^n\|_H^2$$

and

$$|y^n(\beta)|^2 \leq 2(\lambda^n)^{-1}|z_n(\beta)|^2 + 2\beta(\lambda^n)^{-1}\|F^n\|_H^2$$

Passing to the limit in the above inequalities, then using (3.2), (3.23), (3.39) and the fact that $\|F^n\|_H \to 0$,
we obtain (3.40). Secondly, since $S_b(u^n, u_t^n, \eta^n) \in H_0^1(0,L) \subset C([0,L])$,
then we deduce that

$$\left|S_1(u^n, u_t^n, \eta^n)(\beta^-)\right|^2 = \left|au_x^n(\beta^+)\right|^2 \tag{3,44}$$

Thus, from (3.24) and (3.44), we obtain the rst limit in (3.41). Moreover, passing to the limit in inequality (3.22), then using (3.2), the second limit in (3.33) and the fact that, we obtain the second limit in (3.41).
On the other hand, (3.5)-(3.8) can be written in (β,γ) as the following form

$$(\lambda^n)^2 u^n + au_{xx}^n - i\lambda^n c_0 y^n = G^{1,n} \quad \text{in} \quad (\beta,\gamma) \tag{3,45}$$

$$(\lambda^n)^2 y^n + ay_{xx}^n - i\lambda^n c_0 u^n = G^{2,n} \quad \text{in} \quad (\beta,\gamma) \tag{3,46}$$

where

$$G^{1,n} = -f_x^{2,n} - i\lambda^n f_x^{1,n} - c_0 f_x^{3,n} \text{ and } G^{2,n} = -f_x^{4,n} - i\lambda^n f_x^{3,n} - c_0 f_x^{1,n} \tag{3,47}$$

Let $V^n = (u^n, u_x^n, y^n, y_x^n)^T$, then (3.45)-(3.46) can be written as the following

$$V_x^n = B^n V^n + G^n \tag{3,48}$$

where

$$B^n = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ -a^{-1} & 0 & a^{-1}i\lambda^n c_0 & 0 \\ 0 & 0 & 0 & 1 \\ i\lambda^n c_0 & 0 & -(\lambda^n)^2 & 0 \end{Bmatrix} = (b_{ij})_{1\leq i,j\leq 4}$$

and

$$G^n = \begin{Bmatrix} 0 \\ a^{-1}G^{1,n} \\ 0 \\ G^{2,n} \end{Bmatrix}$$

The solution of the di erential equation (3.48) is given by

$$V^n(x) = \exp(B^n(x-\beta))V^n(\beta^+) + \int_\beta^x \exp(B^n(x-s))G^n(s)\,ds \tag{3,49}$$

where $\exp(B^n(x-\beta)) = (c_{ij})_{1\leq i,j\leq 4}$ and $\exp(B^n(x-s)) = (d_{ij})_{1\leq i,j\leq 4}$ are denoted by the exponential of the matrices $\exp(B^n(x-\beta))$
and $\exp(B^n(x-\beta))$ respectively. Now, from (3.2), the entries $b_{i,j}$ are bounded for all $_{1\leq i,j\leq 4}$ and consequently, the entries
$b_{i,j}(x-\beta)$ and $b_{i,j}(x-s)$ are bounded. In addition, from the de nition of the exponential of a square matrix, we obtain

$$\exp(B^n\zeta) = \sum_{k=0}^{\infty}\left(\frac{(B^n\zeta)^K}{k!}\right) \quad \text{for} \quad \zeta \in \{x-\beta, s-x\}. \tag{2.2}$$

The entries $c_{i,j}$ and d_{ij} are also bounded for all $_{1\leq i,j\leq 4}$ and consequently $\exp(B^n(x-\beta))$ and $\exp(B^n(x-s))$ are two bounded matrices. From (3.40) and (3.41) , we directly obtain

$$V^n(\beta) \to 0 \text{ in } \quad \left(L^2(\beta,\gamma)\right)^4, \quad \text{as} \quad n\to\infty \tag{3,50}$$

From (3.47), we deduce that

$$\int_\beta^\gamma \left| G^{1,n}\right|^2 dx \leq 3\int_0^L \left|f_x^{2,n}\right|^2 dx + 3(\lambda^n)^2 \int_0^L \left|f_x^{1,n}\right|^2 dx \tag{3,51}$$

$$+3c_0^2\int_0^L \left|f_x^{3,n}\right|^2 dx$$

and

$$\int_\beta^\gamma \left| G^{2,n}\right|^2 dx \leq 3\int_0^L \left|f_x^{4,n}\right|^2 dx + 3(\lambda^n)^2 \int_0^L \left|f_x^{3,n}\right|^2 dx \tag{3,52}$$

$$+3c_0^2\int_0^L \left|f_x^{1,n}\right|^2 dx$$

since $f_x^{1,n}, f_x^{4,n} \in H_0^1(0,L)$,then it follows from Poincar e inequality that there exist two constants $C_1>0$,and $C_2>0$
such that

$$\left\|f_x^{1,n}\right\|_{L^2(0,L)} \leq C_1\left\|f_x^{1,n}\right\|_{L^2(0,L)} \quad \text{and} \tag{3,53}$$

$$\left\|f_x^{3,n}\right\|_{L^2(0,L)} \leq C_2\left\|f_x^{3,n}\right\|_{L^2(0,L)}$$

From (3.51), (3.52) and (3.53), we get

$$\int_\beta^\gamma \left| G^{1,n}\right|^2 dx \leq 3\left(1+a^{-1}(\lambda^n C_1)^2 + (c_0C_2)^2\right)\|F^n\|_H^2 \tag{3,54}$$

and

$$\int_{\beta}^{\gamma} \left| G^{2,n} \right|^2 dx \leq 3 \left(1 + (\lambda^n C_1)^2 + (c_0 C_2)^2\right) \|F^n\|_H^2 \tag{3,55}$$

from (3.2), (3.54), (3.55) and the fact that$\|F^n\|_H \to 0$, we obtain

$$G^n \to 0 \text{ in } \left(L^2(\beta,\gamma)\right)^4,\ n \to \infty \tag{3,56}$$

from (3.49), (3.50), (3.56) and as $\exp(B^n(x-\beta)), \exp(B^n(x-s))$ are two bounded matrices, we get $V^n \to 0$ in $(L^2(\beta,\gamma))^4$ and consequently, we obtain (3.42) from (3.5) , (3.7) and (3.53), we deduce that

$$\int_{\beta}^{\gamma} |v^n|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |u^n|^2 dx + 2\int_{\beta}^{\gamma} \left|f_x^{1,n}\right|^2 dx$$
$$\leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |u^n|^2 dx + \frac{2C_1}{a} \|F^n\|_H^2$$

$$\int_{\beta}^{\gamma} |z^n|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |y^n|^2 dx + 2\int_{\beta}^{\gamma} \left|f_x^{3,n}\right|^2 dx$$
$$\leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |y^n|^2 dx + \frac{2C_1}{a} \|F^n\|_H^2$$

passing to the limit in the above inequalities, then using (3.2), (3.42) and the fact that $\|F^n\|_H \to 0$,we obtain (3.43). The proof is thus complete.

Lemma 2.3 *Let $h \in C^1([0,L])$ be a function. Under the hypothesis (h), the solution $U^n = (u^n, v^n, y^n, z^n, \eta(.,$ $D(A)$ of system (3.5)-(3.9) satis es the following equality*

$$\int_0^L h'\left(\frac{1}{a}|S_b(u^n,u_t^n,\eta^n)|^2 + |v^n|^2 + |z^n|^2 + |y_x^n|^2\right) dx$$
$$-\left[h\left(\frac{1}{a}|S_b(u^n,u_t^n,\eta^n)|^2\right)\right]_0^L - \Re\left\{2\int_0^L c(.)\, h v^n \overline{y_x^n} dx\right\}$$
$$+\Re\left\{\frac{2}{a}\int_0^L c(.)\, h z^n \overline{S_b}(u^n,u_t^n,\eta^n)\, dx\right\}$$
$$+\Re\left\{\frac{2i\lambda^n}{a}\int_0^\beta h v^n \left(k_1\overline{v_x^n} + k_2\eta_x^n(.,1)\right) dx\right\}$$
$$= \Re\left\{2\int_0^L h\overline{f_x^{1,n}} v^n dx\right\} + \Re\left\{\frac{2}{a}\int_0^L h f_x^{2,n}\overline{S_b}(u^n,u_t^n,\eta^n)\, dx\right\}$$
$$+\Re\left\{2\int_0^L h\overline{f_x^{3,n}} z^n dx\right\} + \Re\left\{\int_0^L h f_x^{4,n}\overline{y_x^n} dx\right\}$$

multiplying (3.6) and (3.8) by $2a^{-1}\overline{S_b}(u^n, u_t^n, \eta^n)$ and $2h\overline{y_x^n}$ respectively,integrating over $(0;L)$, then taking the real part, we get

$$\Re\left\{\frac{2i\lambda^n}{a}\int_0^L hv^n\overline{S_b}(u^n, u_t^n, \eta^n)\,dx\right\} \tag{3,57}$$

$$-\frac{1}{a}\int_0^L h\left(|S_b(u^n, u_t^n, \eta^n)|\right)_x dx$$

$$+\Re\left\{\frac{2}{a}\int_0^L c(.)\,hz^n\overline{S_b}(u^n, u_t^n, \eta^n)\,dx\right\}$$

$$= \Re\left\{\frac{2}{a}\int_0^L hf_x^{2,n}\overline{S_b}(u^n, u_t^n, \eta^n)\,dx\right\}$$

and

$$\Re\left\{2i\lambda^n\int_0^L hz^n\overline{y_x^n}dx\right\} - \int_0^L h\left(|y_x^n|\right)_x dx \tag{3,58}$$

$$-\Re\left\{2\int_0^L c(.)\,hz^n y_x^n dx\right\} = 2\Re\left\{\int_0^L hf_x^{4,n}\overline{y_x^n}dx\right\}$$

From (3.5) and (3.7), we deduce that

$$i\lambda^n\overline{u_x^n} = -\overline{v_x^n} - \overline{f_x^{1,n}} \tag{3,59}$$

$$i\lambda^n\overline{y_x^n} = -\overline{z_x^n} - \overline{f_x^{3,n}} \tag{3,60}$$

from (3.59) and the de nition $S_b(u^n, u_t^n, \eta^n)$, we have

$$i\lambda^n\overline{S_b}(u^n, u_t^n, \eta^n) = \begin{cases} -a\left(\overline{v_x^n} + \overline{f_x^{1,n}}\right) + i\lambda^n\left(k_1\overline{v_x^n} + k_2\overline{\eta_x^n}(.,1)\right), x \in (0,\beta) \\ -a\left(\overline{v_x^n} + \overline{f_x^{1,n}}\right), x \in (\beta, L) \end{cases} \tag{3,61}$$

Substituting (3.61) and (3.60) in (3.57) and (3.58) respectively, we obtain

$$-\int_0^L h\left(|v^n|^2 + \frac{1}{a}|S_b(u^n, u_t^n, \eta^n)|^2\right)_x dx$$

$$+\Re\left\{\frac{2i\lambda^n}{a}\int_0^\beta hv^n k_1\overline{v_x^n} + k_2\overline{\eta_x^n}(.,1)\,dx\right\}$$

$$+\Re\left\{\frac{2}{a}\int_0^L c(.)\,hz^n\overline{S_b}(u^n, u_t^n, \eta^n)\,dx\right\}$$

$$= \Re\left\{2\int_0^L h\overline{f_x^{1,n}}v^n dx\right\} + \Re\left\{\frac{2}{a}\int_0^L hf_x^{2,n}\overline{S_b}(u^n, u_t^n, \eta^n)\,dx\right\}$$

and

$$
\begin{aligned}
&-\int_0^L h\left(|z^n|^2+|y_x^n|^2\right)_x dx-\Re\left\{2\int_0^L c\left(.\right)hv^n y_x^n dx\right\}\\
&=\Re\left\{2\int_0^L hf_x^{4,n}\overline{y_x^n}dx\right\}+\Re\left\{2\int_0^L hf_x^{3,n}\overline{y_x^n}dx\right\}
\end{aligned}
$$

adding the above equations, then using integration by parts and the fact that $v^n\left(0\right)=v^n\left(L\right)=0$ and $z^n\left(0\right)=z^n\left(L\right)=0$, we obtain the desired result. The proof is thus complete

Conclusion

In conclusion, the study of coupled wave equations with singular viscoelastic\elastic damping with Time Delay singular couplings tability behaviors under certain conditions presents a complex and intriguing research area. Understanding the interplay between different damping mechanisms and their effects on wave behavior is crucial for various applications in mathematics, physics, and engineering, The stability of these dynamic systems opens promising prospects for practical applications, such as dynamic system control and signal transmission. These advances help to enrich our understanding of dynamic phenomena and stimulate technological innovation.This summary highlights the important advances made in the study of coupled wave equations with time delay and their relevance for various scientific and technological fields. Future research in this field may focus on exploring more sophisticated damping models, investigating stability properties under different conditions, and extending the analysis to higher-dimensional systems. By delving deeper into these topics, researchers can enhance our understanding of wave dynamics and contribute to the development of advanced mathematical models for practical applications.

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