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THESIS
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Resolution of Fractional Differential Equation

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Dedicate

This modest work is dedicated to my family,

Mohammed Matine, Mohammed Firas, Maissam, Kadar,

My Colleagues of University of Larbi Tebessi,

My best friends: Dr. Zaineb, Awatef, Assia.

ABSTRACT

Motivated by the immense success of application of fractional equation in the branch of mathematical physics, the thesis studies three general models of fractional order partial differential equations using different definitions : Hilfer- Hadamard, Caputo and Caputo-Fabrizio fractional derivatives. *In the first model,* the study finds the critical exponents p_c for which solutions cannot exist for all time in the subcritical case, whereas, in the critical and supercritical cases, global small data solutions exist. The discussion is based on the semi-group theory, fixed point theorem and the test function method. *In the second model,* the study shows that no solutions can exist for all time for certain values of p. Clearly, sufficient conditions for non-existence provide necessary conditions for existence of solutions.

In many cases is difficult to find an analytical solution. For this reasons, the study uses a novel finite difference discretization scheme to solve numerically *fractional-order's partial differential equation* involving a Caputo- Fabrizio fractional derivative supplemented with initial and boundary conditions (*the third model*).

Keywords: Critical exponent, nonexistence, novel finite difference, structural damping, time fractional derivatives.

ملخص

نظرا لأهمية تطبيق المعادلات التفاضلية ذات الاس الكسري في مجال الرياضيات التطبيقية قمنا بدراسة ثلاث نماذج باستعمال تعاريف مختلفة هلفر-هدمار, كبوتو و كبوتو فبريزو. **في النموذج الاول :** تم ايجاد نقطة الانفجار و التي يكون من اجلها الحل الشامل (الكلي) غير موجود في الجزء تحت نقطة الانفجار بينما في الجزء الاخر فالحل هو كلي بإضافة شروط ابتدائية. اعتمد في الدراسة علي نظريات النقطة الصامدة ...**النموذج الثاني:** تم ايجاد قيم المتغير p و التي تمثل شرط ضروري و غير كافي لوجود الحل الكلي .

في الكثير من الحالات يصعب ايجاد حل الدقيق للمعادلات التفاضلية لذلك نلجأ الي استخدام طرق عددية لتقريب الحل ولهذا قمنا باستخدام طريقة عددية لتمثيل الحل (**النموذج الثالث**).

الكلمات المفتاحية: نقطة الانفجار, عدم وجود الحل, التخميد الهيكلي ، المشتقات الزمنية الكسرية.

Abbreviations and Notation

$\mathbb{N}$	Natural numbers.
$\mathbb{R}$	Real numbers.
$[\alpha]$	The integer part of α.
$L^1(\Omega)$	Space of Lebesgue complex-valued measurable functions f, for which $\|f\|_{L^1(\Omega)} = \int_\Omega |f(s)|ds < \infty$.
$L^p(\Omega)$	Space of all measurable functions f on Ω, for which $|f|^p \in L^1(\Omega)$.
$C(\Omega)$	Banach space of all continuous functions from Ω.
$C_{\gamma,\log}[a,b]$	Space of all continuous functions f such that $\left(\log \frac{t}{a}\right)^\gamma f(t) \in C[a,b]$.
$\Gamma(.)$	Euler gamma function.
$B(.,.)$	Beta function.
$E_\alpha(.)$	Standard Mittag-Leffler function.
$E_{\alpha,\beta}(.)$	Mittag-Leffler function in two arguments α and β.
Δf	Laplacian operator of f.
I^α	Riemann-Liouville fractional integral of order α.
$\mathcal{I}^\alpha$	Hadamard fractional integral of order $\alpha > 0$ of a function $f \in L^q[a,b]$.
${}^{CF}I^\alpha$	Caputo-Fabrizio fractional integral of order α.
D^α	Riemann-Liouville fractional derivative of order α.
$\mathbf{D}^\alpha$	Caputo fractional derivative of order α.
$\mathcal{D}^\alpha$	Hadamard fractional derivative.
$\mathcal{D}^{\alpha,\beta}$	Hilfer- Hadamard fractional derivative of order $0 < \alpha < 1$ and type β.
$\mathbb{D}^\alpha$	Caputo-Fabrizio fractional derivative of order α.
PDE	Partial Differential Equation.
FDPE	Fractional-order's Partial Differential Equation.
R-L	Riemann-Liouville.
C-F	Caputo-Fabrizio.

INTRODUCTION

"If you wish to foresee the future of mathematics our proper course is to study the history and present condition of the science." **Henri Poincare.**

We know to define nth derivatives and integrals for $n \in \mathbb{N}$. How do we define derivatives and integrals to non-integer order?

In mathematics literature, we can play with symbols for example we write $\frac{1}{2}$ as $1 : 2$ that prompted L'Hospital to ask Leibiniz " what if n be $\frac{1}{2}$ in $\frac{d^n y}{dx^n}$" Leibiniz replied " you can see by that, Sir, that one can express by an infinite series a quantity such $d^{1/2}xy$. Although infinite series and Geometry are distant relations, they admit only the use of exponents which are positive and negative integers, and do not, as yet, know the use of fractional exponents. "

The subject of fractional calculus was not limited to Euler's attention. It has been developed progressively up to now. However, there are many of these definitions in the literature nowadays, but few of them are commonly used, including: Riemann-Liouville, Caputo, Hadamard and Caputo-Fabrizio fractional calculus which we will define carefully in a further part of this thesis. All these fractional derivatives definitions have their advantages and disadvantages.

Let us first recall some works related to our models.
The semilinear pseudo-parabolic equation

$$\begin{cases} u_t - k\Delta u_t - \Delta u = |u|^p, & (t,x) \in (0,\infty) \times \Omega, \quad p > 1, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

arises in many fields of science and engineering : the aggregation of population [33] and the non-stationary processes in semiconductors [26]. Eq. (1) is also called a Sobolev type equation; Sobolev Galpern type equation [2]. Many researchers have studied the existence and blow-up of solutions for problem (1) [12, 40], by using different methods, such as the potential well method and the Galerkin method combined with the compactness method. When $k = 0$, Eq. (1) reduces to the heat equation

$$\begin{cases} u_t - \Delta u = |u|^p, & (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

Fujita [15] has studied the global existence of mild solutions to (2), if $p > 1 + \frac{2}{N}$ and small initial data. In addition, he proved that the mild solution cannot exist globally when $1 < p < 1 + \frac{2}{N}$ and $u_0 > 0$.

In [42], Weissler proved that if $p = 1+\frac{2}{N}$ (critical case) and u_0 is small enough in $L^{q_c}(\mathbb{R}^N)$, $q_c = N(p-1)/2$, then the solution of (2) exists globally.

Xu and Su [43] showed that all non-trivial solutions u of the following problem

$$\begin{cases} u_t - \Delta u_t - \Delta u = |u|^p, & (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3}$$

where $1 < p < \infty$ if $N = 1,2$; $1 < p \leqslant \frac{N+2}{N-2}$ if $N \geqslant 3$, exist for all time under some conditions and they obtain sufficient conditions for non-existence of solutions.

In 1982, Chen and Russell [11] investigated the following linear elastic system described by

$$\begin{cases} u_{tt} + Bu_t + Au = 0, & t > 0, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{4}$$

where A: the elastic operator and B: the damping operator in a Hilbert space $\mathbb{H}$. They reduce the problem (4) to

$$\frac{d}{dt}\begin{pmatrix} A^{\frac{1}{2}}u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -B \end{pmatrix}\begin{pmatrix} A^{\frac{1}{2}}u \\ u_t \end{pmatrix}.$$

The problem (4) is equivalent to the first order equation in $\mathbb{H}$.

In 2013, Fan, Li and Chen [14] studied the existence and uniqueness of mild solutions for the semilinear elastic system with structural damping in Banach spaces $\mathbb{X}$

$$\begin{cases} u_{tt} + \rho A u_t + A^2 u = f(t, u(t)), & t > 0, \quad x \in \Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, & x \in \Omega, \end{cases} \tag{5}$$

where $A \in \mathbb{X}$ is a closed linear operator and $\rho \geqslant 2$, the function $f \in C([0,a] \times \mathbb{X}, \mathbb{X})$. For $f(t, u(t)) = 0$, Fan and Li studied the asymptotic stability of solutions and the analyticity.

In 2017, Luong and Tung [30] considered the following Cauchy problem

$$\begin{cases} u_{tt} + \rho A u_t + A^2 u(t) = f(t, u(t)) & t > 0, \quad x \in \Omega, \\ u(0) = u_0 + g(u), \quad u_t(0) + h(u) = u_1, & x \in \Omega, \end{cases} \tag{6}$$

where A is closed operator, they established the existence of decay mild solutions to (6) by using a suitable measure of non-compactness on the space of continuous functions.

Recently, fractional differential equations have interested in real-life phenomena. They describe diverse phenomena in sciences and engineering fields and appear naturally in viscoelasticity, porous media, chemistry, electromagnetism physics, mechanics and biology. Hence more applications have been found. The solution of non -integer order partial differential equations (PDEs) has an important property. It describes future, present and past states, but in many cases, it is difficult to find the solution. Therefore, few researchers have suggested numerical methods for studying PDE with fractional order: finite element methods [21], mixed finite element methods [27, 28], finite difference methods [38, 39] and finite volume methods. In 2015, Caputo and Fabrizio [9] proposed a new derivative. This derivative is a product of convolution of $f'(t)$ (derivative of function $f(t)$) and exponential function $(\mathrm{e}^{\frac{-\alpha}{1-\alpha}\mathrm{t}})$ where $0 < \alpha < 1$.

The fractional diffusion -wave equation plays an important role to modeling the diffusion and wave in fluid flow, oil strata... The fractional diffusion-wave equation obtained from the classical diffusion or wave equation by replacing the first-or second-order time derivative by a fractional derivative of order $\alpha > 0$. For $1 < \alpha < 2$, the fractional equation with initial and boundary value is expected to interpolate the diffusion equation and the wave equation, thus it is referred to as the time fractional diffusion-wave equation. In recent years, many eminent researchers innovated some numerical methods to study this kind of equations. In 2005, Sun and Wu [41] showed a novel finite difference discreet scheme for a diffusion-wave system. They proved the stability and L_∞ convergence by using the energy method.

The present thesis consists of four Chapters: Chapter 1 introduces some definitions about fractional calculus which will be used in the sequel. Also, all the important results for the properties of Riemann-Liouville, Caputo, Caputo-Fabrizio and Hilfer-Hadamard fractional derivatives are represented. This Chapter is finished by the notion of blow-up where we have introduce in particular what do authors mean by blow-up.

The rest of thesis contains the Chapters corresponding to the articles published or accepted during the work of thesis ([3], [4], [6], [7], [8]).

Chapter 2 devotes to fractional partial differential equations under Caputo sense, in Section 1, we study the semilinear equation with a time fractional structural damping

$$\mathbf{D}^{\beta}_{0|t}u(t,x) - 2\Delta\mathbf{D}^{\alpha}_{0|t}u(t,x) + \Delta^2 u(t,x) = |u(t,x)|^p \quad t>0,\ x\in\Omega,$$

we obtain the blow- up result under some positive data when $1<p<1+\frac{4\alpha}{N-4\alpha+2}$, whereas, if $p\geqslant 1+\frac{4\alpha}{N-4\alpha+2}$ and $\|u_0\|_{L^{q_c}(\Omega)}$, $q_c = N(p-1)/2$ is sufficiently small, we prove the existence of global solution. In Section 2, we consider the time fractional semilinear equation with a structural damping and a nonlinear memory

$$\mathbf{D}^{1+\alpha_1}_{0|t}u + (-\Delta)^{\sigma}u + (-\Delta)^{\delta}\mathbf{D}^{\alpha_2}_{0|t}u = I^{1-\gamma}_{0|t}|u|^p, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N,$$

we prove the non-existence of global solutions if

$$1<p\leqslant \frac{2(2+\alpha_1-\gamma)}{(\frac{\alpha_1+1}{\sigma}N+2\gamma-2\alpha_1-2)_+}+1,$$

for any space dimension $N\geqslant 1$. Then (Section 3), we extend our idea to the system of semilinear coupled equations

$$\mathbf{D}^{\alpha_1}_{0|t}u + (-\Delta)^{\delta_1}u = I^{1-\gamma_1}_{0|t}|v|^p, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N,$$

$$\mathbf{D}^{\alpha_2}_{0|t}v + (-\Delta)^{\delta_2}v = I^{1-\gamma_2}_{0|t}|u|^q, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N.$$

Chapter 3 consecrates to the following semilinear equation with Hilfer- Hadamard fractional derivative

$$\mathcal{D}^{\alpha_1,\beta}_{a^+}u - \Delta\mathcal{D}^{\alpha_2,\beta}_{a^+}u - \Delta u = |u|^p, \quad t>a>0, \quad x\in\Omega,$$

we establish the necessary conditions for the existence of global solutions.

In Chapter 4, we suggest a novel approximation of the Caputo-Fabrizio fractional derivative of order α $(1 < \alpha < 2)$. Our novel discretization is found by using discrete fractional derivative at $t = t_k$ with new coefficients. Section 1, we study the existence and uniqueness of solution. Section 2, we give the novel finite difference discretization scheme.

At the end of this thesis, there is an alphabetic list of the references used to prepare this thesis under the title Bibliography.

Chapter 1

Notations and Preliminaries

This Chapter mainly introduces definitions and basic properties of fractional calculus, including Riemann-Liouville, Caputo, Hilfer- Hadamard and Caputo-Fabrizio, which will all be at the core of this work.

1.1 Basic Fractional Integrals and Derivatives

A great number of researchers throughout history have defined fractional derivatives and integrals in many different ways. This thesis restricts the attention to the use of: Riemann-Liouville, Caputo, Hilfer- Hadamard, Caputo-Fabrizio. It also presents some results and basic properties of fractional calculus.

1.1.1 Riemann-Liouville Fractional Integrals and Derivatives

As it is the case of the majority of works on fractional calculus, we begin with a generalization of repeated integration, let f be a continuous function on the real line, then we can form the definite integral from a to t

$$I^1_{a|t}f(t) = \int_a^t f(\tau)\ d\tau.$$

Repeating this process gives

$$I^2_{a|t}f(t) = \int_a^t \left(\int_a^s f(\tau)\ d\tau \right) ds.$$

According to the Fubini theorem,

$$\begin{aligned} I^2_{a|t}f(t) &= \int_a^t f(\tau) \int_\tau^t ds\ d\tau \\ &= \int_a^t (t-\tau) f(\tau)\ d\tau. \end{aligned}$$

For the n-fold iterated integral, we obtain

$$\begin{aligned} I^n_{a|t} f(t) &= \int_a^t \int_a^{t_1} \dots \int_a^{t_{n-1}} f(t_n)\ dt_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\ d\tau,\ \ t > a. \end{aligned} \tag{1.1}$$

A proof is given by induction. Equation 1.1 can be generalized for non-integer n. Using the Gamma function $\Gamma(n) = (n-1)!$, to remove the discrete nature of the factorial, we obtain a fractional generalization of the integral

$$I^\alpha_{a|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\ d\tau,\ \ t > a.$$

Definition 1.1.1. Let $[a, b](-\infty < a < b < +\infty)$ be a finite interval of $\mathbb{R}$. The left and right Riemann- Liouville fractional integrals of order $\alpha > 0$ are defined by

$$I^\alpha_{a|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\ d\tau,\ \ t > a,\ \ \text{(left hand)}, \tag{1.2}$$

and

$$I^\alpha_{t|b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\ d\tau,\ \ t < b,\ \ \text{(right hand)}. \tag{1.3}$$

Lemma 1.1.2 (Fractional integration by parts)**.** *Let $f(t)$ and $g(t)$ be continuous functions on $[a, b]$, $\frac{1}{n} < \alpha < 1$, for $n \in \mathbb{N} \setminus \{0, 1\}$. Then*

$$\int_a^b g(t) I^\alpha_{a|t} f(t)\ dt = \int_a^b f(t) I^\alpha_{t|b} g(t)\ dt. \tag{1.4}$$

Proof. For $f, g \in C([a, b], \mathbb{R})$, then there exist $C_1 > 0$ and $C_2 > 0$ such that $|f(t)| \leqslant C_1$ and $|g(t)| \leqslant C_2$. Therefore

$$\begin{aligned} \left| \int_a^b g(t) I^\alpha_{a|t} f(t)\ dt \right| &\leqslant C_1 C_2 \int_a^b \int_a^t \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} d\tau\ dt \\ &\leqslant C_1 C_2 \int_a^b \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} d\tau\ dt. \end{aligned}$$

We observe that

1. if $\tau \leqslant t-1$, then $\log(t-\tau) \geqslant 0$ and $(t-\tau)^{\alpha-1} < 1$,

2. if $\tau > t-1$, then $\log(t-\tau) < 0$ and $(t-\tau)^{\alpha-1} < (t-\tau)^{\frac{1}{n}-1}$.

Thus

$$
\begin{aligned}
C_1C_2 \int_a^b \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} d\tau \ dt &\leqslant C_1C_2 \int_a^b \frac{1}{\Gamma(\alpha)} \left(\int_a^{t-1} d\tau + \int_{t-1}^t (t-\tau)^{\frac{1}{n}-1} d\tau \right) dt \\
&\leqslant C_1C_2 \int_a^b \left(1 + \frac{\alpha-1}{\alpha^2+1}\right)(t-a-1+n) \ dt \\
&\leqslant C_1C_2(b-a)\left(\frac{b+a}{2} - 1 - a + n\right) < \infty.
\end{aligned}
$$

Due to the following inequality[20]

$$
\Gamma(x+1) > \frac{x^2+1}{x+1}, \ x \in [0,1].
$$

Now, as the Fubini theorem is applied we get

$$
\begin{aligned}
\int_a^b g(t) I_{a|t}^{\alpha} f(t) \ dt &= \int_a^b \left(\int_a^t \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} g(t) f(\tau) \ d\tau \right) dt \\
&= \int_a^b f(\tau) \left(\int_\tau^b \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} g(t) \ dt \right) d\tau \\
&= \int_a^b f(\tau) I_{\tau|b}^{\alpha} g(\tau) \ d\tau.
\end{aligned}
$$

This completes the proof.

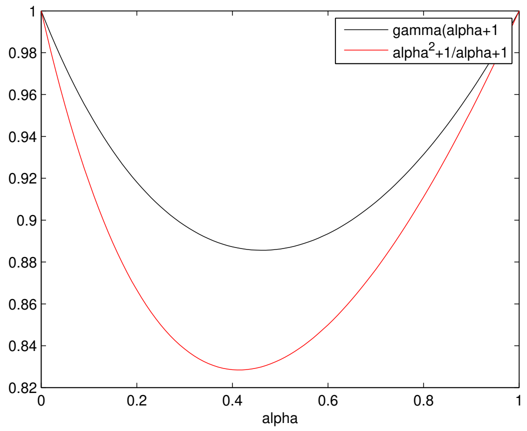

Figure 1.1 – Simulation of $\Gamma(\alpha+1)$ and $\frac{\alpha^2+1}{\alpha+1}$

$\square$

The Riemann-Liouville fractional derivatives based on the Riemann-Liouville fractional integrals.

Definition 1.1.3. The left and right Riemann-Liouville fractional derivatives $D^{\alpha}_{a|t}$ and $D^{\alpha}_{t|b}$ of order $\alpha > 0$ of function $f \in L^1(a,b)$ are defined by

$$\begin{aligned} D^{\alpha}_{a|t} f(t) &= \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{a|t} f(t), \ \ t > a \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \ \ \text{(left hand)}, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} D^{\alpha}_{t|b} f(t) &= (-1)^n \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{t|b} f(t), \ \ t < b \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau, \ \ \text{(right hand)}, \end{aligned} \tag{1.6}$$

where $n = [\alpha] + 1$ ($[\alpha]$ the integer part of α). In particular, when $\alpha = n \in \mathbb{N}^*$, then

$$D^{n}_{a|t} f(t) = \left(\frac{d}{dt}\right)^n f(t) \ \text{ and } \ D^{n}_{t|b} f(t) = (-1)^n \left(\frac{d}{dt}\right)^n f(t).$$

The aim of the study, now is to present some results that will be used in the coming sections.

Lemma 1.1.4. *1. Suppose $f \in C(0,\infty)$, $p \geqslant q > 0$, and $D^{p-q}_{t|T} f(t)$ exists, then*

$$D^{p}_{t|T}(I^{q}_{t|T} f)(t) = D^{p-q}_{t|T} f(t). \tag{1.7}$$

In particular, when $p = n$ we have[44]

$$D^{n}_{t|T}(I^{q}_{t|T} f)(t) = (-1)^n D^{n-q}_{t|T} f(t). \tag{1.8}$$

2. Let $n-1 \leqslant q < n$ and $q + p - m > 0$, then for every $t > 0$,

$$I^{m-p}_{t|T} D^{q}_{t|T} f(t) = D^{p+q-m}_{t|T} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{m-p-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)}. \tag{1.9}$$

Proof. 1. Let $m-1 \leqslant p < m$, $n-1 \leqslant p-q < n < m$ and by (1.8), we have

$$
\begin{aligned}
D^{p}_{t|T}(I^{q}_{t|T}f)(t) &= (-1)^m \left(\frac{d}{dt}\right)^n \left(\frac{d}{dt}\right)^{m-n} (I^{m-p+q}_{t|T} f)(t) \\
&= (-1)^n \left(\frac{d}{dt}\right)^n (I^{n-p+q}_{t|T} f)(t) \\
&= D^{p-q}_{t|T} f(t).
\end{aligned}
$$

2. Using (1.7), we can write

$$
\begin{aligned}
I^{m-p}_{t|T} D^{q}_{t|T} f(t) &= D^{q+p-m}_{t|T}\{I^{q}_{t|T} D^{q}_{t|T} f(t)\} \\
&= D^{q+p-m}_{t|T}\left\{ f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{q-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k+1)} \right\} \\
&= D^{q+p-m}_{t|T} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{m-p-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)},
\end{aligned}
$$

due to the following equality (see [22])

$$
I^{q}_{t|T} D^{q}_{t|T} f(t) = f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{q-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k+1)}.
$$

□

Lemma 1.1.5. *Let p and q be real numbers, if $m-1 \leqslant p < m$ and $n-1 \leqslant q < n$, then*

$$
D^{p}_{t|T} D^{q}_{t|T} f(t) = D^{p+q}_{t|T} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{-p-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k)\Gamma(1-k-p)(m-k-p+1)}. \tag{1.10}
$$

Proof. By the semigroup property of fractional integrals and (1.9), we can write

$$
\begin{aligned}
D^{p}_{t|T} D^{q}_{t|T} f(t) &= \left(\frac{d}{dt}\right)^m \left[(-1)^m I^{m-p}_{t|T} D^{q}_{t|T} f(t)\right] \\
&= \left(\frac{d}{dt}\right)^m (-1)^m \left[D^{p+q-m}_{t|T} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{m-p-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)} \right] \\
&= D^{p+q}_{t|T} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{-p-k} D^{q-k}_{t|T} f(T)}{\Gamma(q-k)\Gamma(1-k-p)(m-k-p+1)}.
\end{aligned}
$$

□

1.1.2 Caputo Fractional Derivatives

Defined via the Riemann-Liouville fractional derivatives.

Definition 1.1.6. The left and right -sided Caputo fractional derivatives of order $\alpha > 0$ of a function f, where $n = [\alpha] + 1$ are

$$\mathbf{D}^{\alpha}_{a|t} f(t) = D^{\alpha}_{a|t}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^k(a)}{k!}(t-a)^k \right), \text{ (left hand)}, \tag{1.11}$$

and

$$\mathbf{D}^{\alpha}_{t|b} f(t) = D^{\alpha}_{t|b}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^k(b)}{k!}(b-t)^k \right), \text{ (right hand)}. \tag{1.12}$$

The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives are connected with each other by the following relations.

If $\alpha \notin \mathbb{N}$ and $f(t)$ is function for which the Caputo fractional derivatives $\mathbf{D}^{\alpha}_{a|t}$ and $\mathbf{D}^{\alpha}_{t|b}$ exist together with the Riemann- Liouville fractional derivatives $D^{\alpha}_{a|t}$ and $D^{\alpha}_{t|b}$, then

$$\mathbf{D}^{\alpha}_{a|t} f(t) = D^{\alpha}_{a|t} f(t) - \sum_{k=0}^{[\alpha]} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \tag{1.13}$$

and

$$\mathbf{D}^{\alpha}_{t|b} f(t) = D^{\alpha}_{t|b} f(t) - \sum_{k=0}^{[\alpha]} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}. \tag{1.14}$$

Proposition 1.1.7. *If $f \in C^n([a,b],\mathbb{R})$ and $\alpha > 0$, then the Caputo derivatives given as*

$$\begin{aligned} \boldsymbol{D}^{\alpha}_{a|t} f(t) &= I^{n-\alpha}_{a|t} \circ \left(\frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau)\, d\tau, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned}\boldsymbol{D}^{\alpha}_{t|b} f(t) &= (-1)^n I^{n-\alpha}_{t|b} \circ \left(\frac{d}{dt}\right)^n f(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau)\ d\tau. \end{aligned} \tag{1.16}$$

Proposition 1.1.8. *For $0 < \alpha < 1$. Assume $f \in C[0,T]$, $g \in C^1([0,T])$ and $g(T) = 0$. Then, we have the following formula of integration by parts*

$$\int_0^T g(t) \boldsymbol{D}^{\alpha}_{0|t} f(t) dt = \int_0^T (f(t) - f(0)) \boldsymbol{D}^{\alpha}_{t|T} g(t) dt. \tag{1.17}$$

Proof. Definition 1.1.6 and Lemma 1.1.2 allow us to write

$$\begin{aligned}\int_0^T g(t) \mathbf{D}^{\alpha}_{0|t} f(t) dt &= \int_0^T g(t) D^{\alpha} \Big(f(t) - f(0) \Big) dt \\ &= \int_0^T g(t) \frac{d}{dt} I^{1-\alpha}_{0|t} \Big(f(t) - f(0) \Big) dt. \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}\int_0^T g(t) \frac{d}{dt} I^{1-\alpha}_{0|t} \Big(f(t) - f(0) \Big) dt &= I^{1-\alpha}_{0|t} \Big(f(t) - f(0) \Big) g(t) \Big|_0^T - \int_0^T I^{1-\alpha}_{0|t} \Big(f(t) - f(0) \Big) \frac{d}{dt} g(t) dt \\ &= - \left(I^{1-\alpha}_{0|t} \Big(f(t) - f(0) \Big) g(t) \right)_{t=0} + \int_0^T \Big(f(t) - f(0) \Big) \mathbf{D}^{\alpha}_{t|T} g(t) dt \\ &= \int_0^T (f(t) - f(0)) \mathbf{D}^{\alpha}_{t|T} g(t) dt. \end{aligned}$$

□

Proposition 1.1.9 ([19])**.** *Let $1 < \alpha + \beta < 2$. If $\frac{d}{dt} f$ is absolutely continuous and $\frac{d}{dt} f(0) = 0$, then*

$$\boldsymbol{D}^{\alpha}_{0|t} \boldsymbol{D}^{\beta}_{0|t} f(t) = \boldsymbol{D}^{\alpha+\beta}_{0|t} f(t).$$

1.1.3 Hilfer-Hadamard Fractional Integrals and Derivatives

In this part, some results and basic properties of Hilfer-Hadamard fractional calculus are represented ([1, 23, 24, 34]).

Definition 1.1.10. Let $[a,b]$ be a finite interval of the half-axis $\mathbb{R}^+$ and $0 \leqslant \gamma < 1$. We introduce the weighted spaces of continuous functions

$$C_{\gamma,\log}[a,b] = \left\{ f : [a,b] \to \mathbb{R} : \left(\log \frac{t}{a} \right)^{\gamma} f(t) \in C[a,b] \right\}, \tag{1.18}$$

$$C^{\gamma}_{1-\gamma,\log}[a,b] = \left\{ f \in C_{1-\gamma,\log}[a,b] : \mathcal{D}^{\gamma}_{a^+} f \in C_{1-\gamma,\log}[a,b] \right\}, \tag{1.19}$$

and

$$C^{n}_{\delta,\gamma}[a,b] = \left\{ f : [a,b] \to \mathbb{R} : \delta^k f \in C[a,b],\ 0 \leqslant k \leqslant n-1,\ \delta^n f \in C_{\gamma,\log}[a,b] \right\}, \tag{1.20}$$

where $\delta = t\frac{d}{dt}$ and $n \in \mathbb{N}$. In particular, when $n = 0$ we define

$$C^{0}_{\delta,\gamma}[a,b] = C_{\gamma,\log}[a,b].$$

In the space $C_{\gamma,\log}[a,b]$, we define the norm

$$\|f\|_{C_{\gamma,\log}[a,b]} = \sup_{t\in[a,b]} \left| \left(\log \frac{t}{a} \right)^{\gamma} f(t) \right|. \tag{1.21}$$

Definition 1.1.11. The Banach space $X^p_c(a,b)$ $(1 \leqslant p \leqslant \infty, c \in \mathbb{R})$ consists of those real-valued Lebesgue measurable functions $f : (a,b) \subset \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|f\|_{X^p_c} = \left(\int_a^b \left| t^c f(t) \right|^p \frac{dt}{t} \right)^{1/p} < \infty, \quad p < \infty, \tag{1.22}$$

$$\|f\|_{X^\infty_c} = \operatorname{ess}\sup_{a\leqslant t\leqslant b} \left| t^c f(t) \right| < \infty. \tag{1.23}$$

When $c = 1/p$, we see that $X^p_{1/p}(a,b) = L^p(a,b)$.

Definition 1.1.12. The Hadamard fractional integrals of order $\alpha > 0$ of a function f in $L^q[a,b]$ $(1 \leqslant q < \infty, 0 < a \leqslant b \leqslant +\infty)$, are defined by

$$\left(\mathcal{I}^{\alpha}_{a^+} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau)\, \frac{d\tau}{\tau}, \quad a < t < b, \text{ (left hand)}, \tag{1.24}$$

and

$$\left(\mathcal{I}^{\alpha}_{b^-} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{\tau}{t}\right)^{\alpha-1} f(\tau)\, \frac{d\tau}{\tau}, \quad a < t < b, \text{ (right hand)}. \tag{1.25}$$

Lemma 1.1.13. *Let $\alpha > 0$, $\beta > 0$ and $1 \leqslant p \leqslant +\infty$. Then, if $f \in X^p_c(a,b)$ the semigroup property holds*

$$\mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f = \mathcal{I}^{\alpha+\beta}_{a^+} f. \tag{1.26}$$

Proof. We prove (1.26) for sufficiently good function f. Applying Fubini theorem, we get

$$\begin{aligned} \mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\mu}\right)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_a^\mu \left(\log \frac{\mu}{\tau}\right)^{\beta-1} f(\tau) \frac{d\tau}{\tau} \frac{d\mu}{\mu} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau) \int_\tau^t \left(\log \frac{t}{\mu}\right)^{\alpha-1} \left(\log \frac{\mu}{\tau}\right)^{\beta-1} \frac{d\mu}{\mu} \frac{d\tau}{\tau}. \end{aligned} \tag{1.27}$$

The inner integral is evaluated by change of variable $y = \log \frac{\mu}{\tau} / \log \frac{t}{\tau}$. So

$$y \log \frac{t}{\mu} = (1-y) \log \frac{\mu}{\tau}.$$

By simple calculation, one has

$$\int_\tau^t \left(\log \frac{t}{\mu}\right)^{\alpha-1} \left(\log \frac{\mu}{\tau}\right)^{\beta-1} \frac{d\mu}{\mu} = \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Substituting this relation into (1.27), we have

$$\mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} f(\tau) \frac{d\tau}{\tau}.$$

□

Lemma 1.1.14. *Let $0 \leqslant \gamma < 1$ and $0 < \alpha$. If $\gamma \leqslant \alpha$, then the operator $\mathcal{I}_{a+}^{\alpha}$ is bounded from $C_{\gamma,\log}(a,b)$ into $C(a,b)$. In particular, it is bounded in $C_{\gamma,\log}(a,b)$.*

Proof. Using the definition of left Hadamard fractional integral

$$\begin{aligned}
\|\mathcal{I}_{a+}^{\alpha} f\|_{C(a,b)} &= \sup_{t\in(a,b)} \left| \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau} \right| \\
&= \sup_{t\in(a,b)} \left| \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{\tau}{a}\right)^{-\gamma} \left(\log \frac{\tau}{a}\right)^{\gamma} f(\tau)\frac{d\tau}{\tau} \right| \\
&\leqslant \left(\log \frac{b}{a}\right)^{\alpha-\gamma} \|f\|_{C_{\gamma,\log}(a,b)}.
\end{aligned}$$

□

Lemma 1.1.15 ([34])**.** *Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(a,b)$ and $g \in X^q_{-1/p}(a,b)$, then*

$$\int_a^b f(t)(\mathcal{I}_{a+}^{\alpha} g)(t)\frac{dt}{t} = \int_a^b (\mathcal{I}_{b-}^{\alpha} f)(t)g(t)\frac{dt}{t}. \tag{1.28}$$

Definition 1.1.16. Let $0 < a < t < b$ and $n-1 < \alpha < n$. The Hadamard fractional derivatives of order α for a function f are defined by

$$\begin{aligned}
\left(\mathcal{D}_{a+}^{\alpha} f\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} f(\tau)\ \frac{d\tau}{\tau} \\
&= \delta^n \left(I_{a^+}^{n-\alpha} f\right)(t), \ \text{(left hand)},
\end{aligned} \tag{1.29}$$

and

$$\begin{aligned}
\left(\mathcal{D}_{b-}^{\alpha} f\right)(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_t^b \left(\log\frac{\tau}{t}\right)^{n-\alpha-1} f(\tau)\ \frac{d\tau}{\tau} \\
&= (-1)^n\delta^n \left(I_{b^-}^{n-\alpha} f\right)(t), \ \text{(right hand)},
\end{aligned} \tag{1.30}$$

where $\delta = t\frac{d}{dt}$, $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α.

Definition 1.1.17. Let $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. The Hilfer- Hadamard fractional derivative of order α and type β is defined by

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} f\right)(t) = \left(\mathcal{I}_{a^+}^{\beta(1-\alpha)} \delta \mathcal{I}_{a^+}^{(1-\beta)(1-\alpha)} f\right)(t),$$

that is,

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} f\right)(t) = \mathcal{I}_{a^+}^{\beta(1-\alpha)} \left(t\frac{d}{dt}\right)\left(\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha)} f\right)(t). \tag{1.31}$$

Lemma 1.1.18. *Assume $f \in C^1_{\delta,\gamma}[a,b]$, for $a < t < b$, $0 < \gamma < 1$ and $0 < \alpha < 1$. Then $\mathcal{D}^{\alpha}_{a^+}$ exists on $(a,b]$ and $\mathcal{D}^{\alpha}_{b^-}$ on $[a,b)$ and can be represented as*

$$\left(\mathcal{D}^{\alpha}_{a^+} f\right)(t) = \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{-\alpha} f'(\tau)d\tau, \tag{1.32}$$

$$\left(\mathcal{D}^{\alpha}_{b^-} f\right)(t) = \frac{f(b)}{\Gamma(1-\alpha)}\left(\log\frac{b}{t}\right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)}\int_t^b \left(\log\frac{\tau}{t}\right)^{-\alpha} f'(\tau)d\tau, \tag{1.33}$$

respectively.

Proof. Since $f \in C^1_{\delta,\gamma}[a,b]$, we have

$$f(t) = \int_a^t f'(\tau)d\tau + f(a),$$

substituting the above relation into (1.29), we obtain

$$\begin{aligned}\left(\mathcal{D}^{\alpha}_{a^+} f\right)(t) = &\frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t \left(\log\frac{t}{\tau}\right)^{-\alpha}\int_a^{\tau} f'(y)\, dy \frac{d\tau}{\tau} \\ &+ \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t \left(\log\frac{t}{\tau}\right)^{-\alpha} f(a)\frac{d\tau}{\tau}.\end{aligned}$$

Interchanging the order of integration, we have

$$\int_a^t \left(\log\frac{t}{\tau}\right)^{-\alpha}\int_0^{\tau} f'(y)\, dy\frac{d\tau}{\tau} = \int_a^t f'(y)\int_y^t \left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau}dy.$$

The inner integral is evaluated by the change of variable $w = \log \frac{t}{\tau}$, we obtain

$$\begin{aligned}
\left(\mathcal{D}_{a^+}^{\alpha} f\right)(t) &= \frac{1}{\Gamma(2-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t f'(\tau)\left(\log\frac{t}{\tau}\right)^{1-\alpha} d\tau \\
&+ \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha} \\
&= \frac{1}{\Gamma(2-\alpha)}\int_a^t \left(t\frac{d}{dt}\right) f'(\tau)\left(\log\frac{t}{\tau}\right)^{1-\alpha} d\tau \\
&+ \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha}.
\end{aligned}$$

Thus Lemma 1.1.18 is proved. □

1.1.4 Caputo-Fabrizio Fractional Integral and Derivative

In this part certain relevant definitions and some properties of Caputo-Fabrizio fractional derivative and anti-derivative are represented.

Definition 1.1.19. Let $f \in C^1(0,\infty)$ and $\alpha \in (0,1)$ then, the definition of the new Caputo derivative is given as

$$\mathbb{D}_{0|t}^{\alpha} f(t) = \frac{1}{1-\alpha}\int_0^t \mathrm{e}^{-\frac{\alpha(\mathrm{t}-\tau)}{1-\alpha}} f'(\tau)\, d\tau. \tag{1.34}$$

The definition above of the fractional derivative operator is called the Caputo-Fabrizio derivative.

If $n \geqslant 1$ and $0 < \alpha < 1$ the fractional operator $\mathbb{D}_{0|t}^{n+\alpha}$ of order $n+\alpha$ is defined by

$$\mathbb{D}_{0|t}^{n+\alpha} f(t) = \mathbb{D}_{0|t}^{\alpha}\Big(D^n f(t)\Big). \tag{1.35}$$

The anti- derivative of the Caputo- Fabrizio derivative is recalled as

Definition 1.1.20. Let $0 < \alpha < 1$. The fractional integral of a function f is given as

$${}^{CF}I_{0|t}^{\alpha} f(t) = (1-\alpha)f(t) + \alpha\int_0^t f(\tau)\, d\tau. \tag{1.36}$$

Proposition 1.1.21. *The fractional differential and integral operators given by the (1.34) and (1.36) satisfy the following useful relation*

$$^{CF}I^{\alpha}_{0|t}\mathbb{D}^{\alpha}_{0|t}f(t) = f(t) - f(0). \tag{1.37}$$

Proof. By the Caputo-Fabrizio fractional calculus, we obtain

$$\begin{aligned}
^{CF}I^{\alpha}_{0|t}\mathbb{D}^{\alpha}_{0|t}f(t) &= (1-\alpha)\mathbb{D}^{\alpha}_{0|t}f(t) + \alpha\int_0^t \mathbb{D}^{\alpha}_{0|\tau}f(\tau)d\tau \\
&= (1-\alpha)\mathbb{D}^{\alpha}_{0|t}f(t) + \frac{\alpha}{1-\alpha}\int_0^t\int_0^\tau \mathrm{e}^{-\frac{\alpha(\tau-\mathrm{y})}{1-\alpha}}f'(y)\ dyd\tau \\
&= (1-\alpha)\mathbb{D}^{\alpha}_{0|t}f(t) + \frac{\alpha}{1-\alpha}\int_0^t f'(y)\int_y^t \mathrm{e}^{-\frac{\alpha(\tau-\mathrm{y})}{1-\alpha}}d\tau dy \\
&= (1-\alpha)\mathbb{D}^{\alpha}_{0|t}f(t) - \int_0^t f'(\tau)\left(\mathrm{e}^{-\frac{\alpha(\mathrm{t}-\tau)}{1-\alpha}} - 1\right)d\tau.
\end{aligned}$$

□

1.2 Fundamental Examples

Example 1: Consider $f(t) = t$, then by Definition

$$I^{\alpha}_{0|t}t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau d\tau.$$

We introduce the following scaled variable $\tau = st$ and by definition of the Beta function, we get

$$\begin{aligned}
I^{\alpha}_{0|t}t &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau\ d\tau \\
&= \frac{t^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s\ ds \\
&= \frac{t^{\alpha+1}}{\Gamma(\alpha)}B(\alpha, 2).
\end{aligned}$$

Therefore

$$\boxed{I^{\alpha}_{0|t}t = \frac{1}{\Gamma(\alpha+2)}t^{\alpha+1}.} \tag{1.38}$$

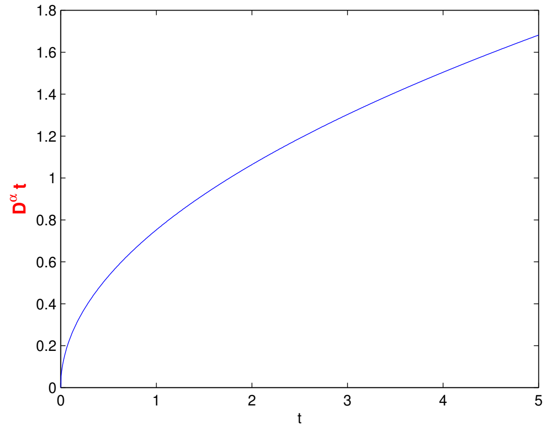

Figure 1.2 – Simulation of R-L integral, with $\alpha = 0.5$.

Using (1.7), we can re-write

$$\begin{aligned} D^{\alpha}_{0|t} t = D^n (I^{n-\alpha}_{0|t} t) &= D^n \left[\frac{1}{\Gamma(\alpha+2)} t^{n-\alpha+1} \right] \\ &= \frac{1}{\Gamma(\alpha+2)} D^n [t^{n-\alpha+1}], \end{aligned}$$

for $n-1 < \alpha < n$. Therefore

$$\boxed{D^{\alpha}_{0|t} t = \frac{1}{\Gamma(n-\alpha+2)} t^{-\alpha+1}.} \tag{1.39}$$

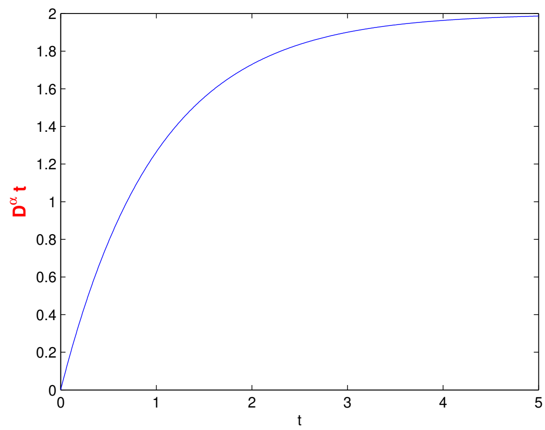

Figure 1.3 – Simulation of R-L derivative, with $\alpha = 0.5$.

The Caputo-Fabrizio of function f is given by

$$\begin{aligned}\mathbb{D}^{\alpha}_{0|t}f(t) &= \frac{1}{1-\alpha}\,\mathrm{e}^{\frac{-\alpha}{1-\alpha}\mathrm{t}}\int_0^t \mathrm{e}^{\frac{\alpha}{1-\alpha}\tau}\,\mathrm{d}\tau \\ &= \frac{1}{\alpha}\,\mathrm{e}^{\frac{-\alpha}{1-\alpha}\mathrm{t}}\left(\mathrm{e}^{\frac{\alpha}{1-\alpha}\mathrm{t}}-1\right), \quad 0<\alpha<1.\end{aligned}$$

So

$$\boxed{\mathbb{D}^{\alpha}_{0|t}t = \tfrac{1}{\alpha}\left(1-\mathrm{e}^{\frac{-\alpha}{1-\alpha}\mathrm{t}}\right).} \tag{1.40}$$

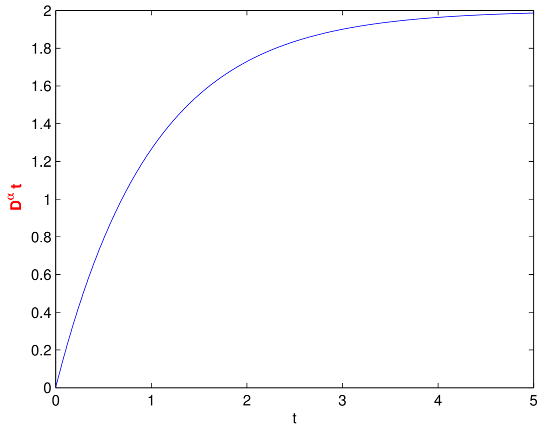

Figure 1.4 – Simulation of C-F derivative, with $\alpha = 0.5$**.**

Example 2: Let $f \in C^{\infty}([0,\infty))$ satisfying

$$f(t) = \begin{cases} (1-\frac{t}{T})^{\eta}, & 0 < t \leqslant T, \quad \eta > 1, \\ 0, & t \geqslant T. \end{cases} \tag{1.41}$$

Definition of Riemann-Liouville derivative allows us to write

$$D^{\alpha}_{t|T}f(t) = \frac{-1}{\Gamma(1-\alpha)}D\int_t^T(\tau-t)^{-\alpha}\left(1-\frac{\tau}{T}\right)^{\eta}d\tau, \quad 0<\alpha<1.$$

Using the Euler change of variable

$$y = \frac{\tau - t}{T-t},$$

since we have

$$\begin{aligned} D^{\alpha}_{t|T} f(t) &= \frac{-1}{\Gamma(1-\alpha)} D\left[\int_0^1 (T-t)^{1-\alpha} y^{-\alpha} \left(1 - \frac{y(T-t)+t}{T}\right)^{\eta} dy \right] \\ &= \frac{-B(1-\alpha, \eta+1)}{\Gamma(1-\alpha)} D\left[(T-t)^{\eta+1-\alpha} T^{-\eta} \right]. \end{aligned}$$

Using the following Beta formula

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.42}$$

we obtain

$$\boxed{I^{\alpha}_{t|T} f(t)_1(t) = \tfrac{(1-\alpha+\eta)B(1-\alpha;\eta-1)}{\Gamma(1-\alpha)} T^{1-\alpha}(1-\tfrac{t}{T})^{\eta-\alpha+1},} \tag{1.43}$$

and

$$\boxed{D^{\alpha}_{t|T} f(t) = \tfrac{\eta\Gamma(\eta-\alpha)}{\Gamma(1+\eta-2\alpha)} T^{-\alpha}(1-\tfrac{t}{T})^{\eta-\alpha}.} \tag{1.44}$$

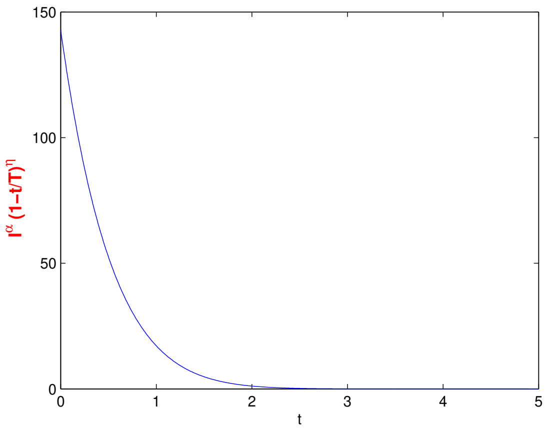

Figure 1.5 – Approximation of function $(1-\frac{t}{T})^{\eta}$ via Riemann-Liouville integral, $\alpha = 0.5$, $\eta = 10, T = 5$.

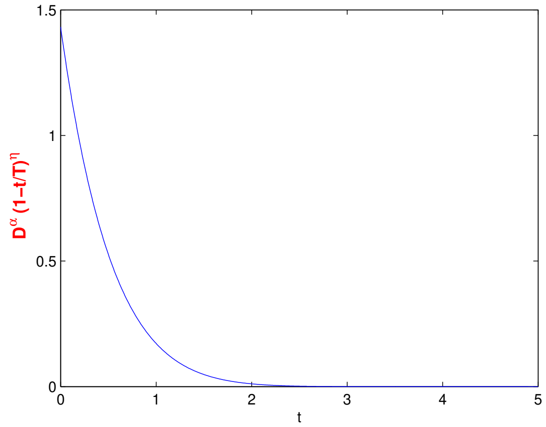

Figure 1.6 – Approximation of function $(1-\frac{t}{T})^{\eta}$ via Caputo derivative $\alpha = 0.5, \eta = 10$.

Lemma 1.2.1 ([17]). *Let f as in* (1.41)*, for* $0 < \alpha < 1$ *and* $\eta > \frac{p}{p-1}\alpha - 1$

$$\int_0^T D^{\alpha}_{t|T} f(t) = C_1 T^{1-\alpha},$$

and

$$\int_0^T f(t)^{-p'/p} |D^{\alpha}_{t|T} f(t)|^{p'} = C_2 T^{1-p'\alpha},$$

where

$$C_1 = \frac{\eta\Gamma(\eta-\alpha)}{(\eta+1-\alpha)\Gamma(\eta+1-2\alpha)} \quad \text{and} \quad C_2 = \frac{\eta^{p'}}{\eta+1-p'\alpha}\left[\frac{\Gamma(\eta-\alpha))}{\Gamma(\eta+1-2\alpha)}\right]^{p'}.$$

1.3 Notion of Blow-up

It occurs often that the solution to the nonlinear evolution equation is not able to extend after some time. This phenomenon is called the blow-up or explosion (in Latin languages) of the solution. The blow-up is a general term that refers to the fact that some solutions in a Banach space tend to infinity in norm as t approaches some finite explosion T which depends on the solution. We recall the famous example on ODE is

$$\begin{cases} u_t = u^2(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.45}$$

it is immediate that if $u_0 = \frac{1}{T}$ for some $T > 0$ then, there exists a unique solution $u(t) = \frac{1}{T-t}$ in the interval $(0, T)$, we notice that it is smooth function $t < T$ and also that $u(t) \longrightarrow +\infty$ as $t \longrightarrow T^-$.

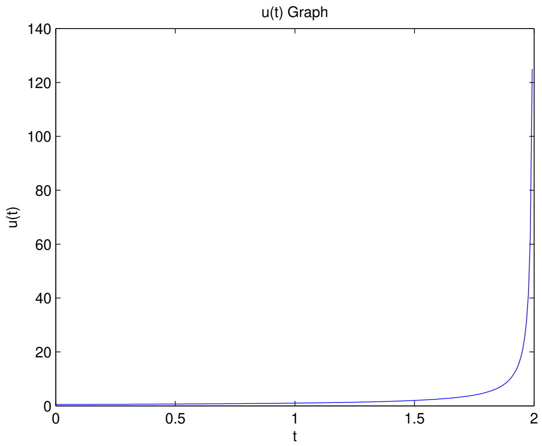

Figure 1.7 – Simulation of function $u(t) = \frac{1}{T-t}$ when $t \longrightarrow T^-$.

Definition 1.3.1. Let $A \subset \mathbb{R}^N$ and $u(t, x)$ be a solution of a given evolution PDE on the set $\Omega := [0, T] \times A$. We say that $u(t, x)$ blows up in finite time T if such that

$$\lim_{t \to T} |u(t, x)| = +\infty.$$

In this case one has

$$\sup_{x \in \Omega} |u(t, x)| = +\infty,$$

and T is called the time of blow-up.

Chapter 2

Fractional Partial Differential Equations using Caputo's Definitions

The fractional calculus is one of the best tools to describe a short memory. In this Chapter 2, we study the fractional partial differential equations under Caputo operators, the discussion is based on some classical analysis such as: *the semi-group theory, fixed point theorem and the test function method.*

2.1 On Generalized Fractional Elastic System with Fractional Damping

In Section 2.1([4]), we study the problem

$$\begin{cases} \mathbf{D}^{\beta}_{0|t}u - 2\Delta \mathbf{D}^{\alpha}_{0|t}u + \Delta^2 u = |u|^p & (t,x) \in (0,\infty)\times\Omega, \\ \Delta u(t,x) = u(t,x) = 0 & (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0, \quad u_t(t,x)|_{t=0} = 0 & x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 1$, $\frac{1}{2} < \alpha < 1$, $1 < \beta < 2$ and Δ denotes the Laplacian operator with respect to the x variable. The operator $\mathbf{D}^{\alpha}_{0|t}u = D^{\alpha}_{0|t}(u(t) - u(0))$. The term $\Delta\mathbf{D}^{\alpha}_{0|t}u$ represents a generalized structural damping.

Our target is to find the critical exponent p_c for which solutions cannot exist for all time in the sub-critical case, whereas, in the critical and super-critical cases, global small data solutions exist.

Throughout this Section, we take $\beta = 2\alpha$.

2.1.1 Existence and Uniqueness of Solutions

In this part, we discuss the existence of mild solutions of the semilinear equation with a time fractional structural damping (problem 2.1).

Definition of Mild Solutions

Definition 2.1.1. The Wright function $\Phi_\alpha(\theta)$ is defined by

$$\Phi_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \quad \theta \in \mathbb{R}.$$

(Mainardi G- Forenflo, 2010 [31]).

Let $X = L^2(\Omega)$ be a Banach space, $A = \Delta : D(A) \subset X \to X$ is the infinitesimal generator of C_0 semi-group $T(t)(t > 0)$.

Definition 2.1.2 ([45])**.** Let $u_0 \in X$, $P_\alpha(t)$ and $S_\alpha(t)$ two operators defined as follow

$$P_\alpha(t)u_0 = \int_0^\infty \Phi_\alpha(\theta)T(t^\alpha\theta)u_0 \, d\theta, \tag{2.2}$$

and

$$S_\alpha(t)u_0 = \alpha \int_0^\infty \theta\Phi_\alpha(\theta)T(t^\alpha\theta)u_0 \, d\theta. \tag{2.3}$$

Lemma 2.1.3 ([45])**.** *The operators $P_\alpha(t)$ and $S_\alpha(t)$ satisfy the following properties*

(1) Let $1 < p \leqslant q < \infty$, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{2}{N}$, then

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leqslant (4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(1 - N/2r)}{\Gamma(1 - \alpha N/2r)} \|u_0\|_{L^p(\mathbb{R}^N)}. \tag{2.4}$$

(2) Let $1 < p \leqslant q \leqslant \infty$, if $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{4}{N}$, then

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leqslant \alpha(4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(2 - N/2r)}{\Gamma(1 + \alpha - \alpha N/2r)} \|u_0\|_{L^p(\mathbb{R}^N)}. \tag{2.5}$$

Lemma 2.1.4 ([45])**.** *Assume $f \in C((0, T), L^2(\Omega))$, let*

$$w(t) = \int_0^t (t - s)^{\alpha - 1} S_\alpha(t - s) f(s) ds,$$

then, $w \in C([0,T], L^2(\Omega))$.

Considering the following in-homogeneous equation corresponding to (2.1)

$$\begin{cases} \mathbf{D}^{2\alpha}_{0|t}u - 2\Delta\mathbf{D}^{\alpha}_{0|t}u + \Delta^2 u = f(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ \Delta u(t,x) = u(t,x) = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(t,x)|_{t=0} = u_1(x) = 0, & x \in \Omega. \end{cases} \tag{2.6}$$

First, we present the following Lemma that will be used to give the definition of a mild solution to the problem we study.

Lemma 2.1.5. *Let* $\frac{1}{2} < \alpha < 1$, $u_0 \in L^2(\Omega)$ *and* $v_0 = (\mathbf{D}^{\alpha}_{0|t}u\Big|_{t=0} - \Delta u_0) \in L^2(\Omega)$. *Then the problem (2.6) admits a unique mild solution* $u \in C([0,T], L^2(\Omega))$ *given by*

$$\begin{aligned} u(t,x) = &P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)P_\alpha(s)v_0 ds \\ &+ \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-1}S_\alpha(s-\tau)f(\tau,x)d\tau ds, \end{aligned} \tag{2.7}$$

where $P_\alpha(t)$ *and* $S_\alpha(t)$ *were defined as in (2.2) and (2.3) respectively.*

Proof. By Proposition 1.1.9, the problem (2.6) is re-written into two abstract Cauchy problems

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t}v - \Delta v = f(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ v(t,x) = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t}u - \Delta u = v(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{2.9}$$

which means

$$v_0(x) = \mathbf{D}^{\alpha}_{0|t}u\Big|_{t=0} - \Delta u_0. \tag{2.10}$$

If $f \in C([0,T], L^2(\Omega))$ and $v_0 \in L^2(\Omega)$, then by [45] the problem (3.15) has a unique mild solution $v \in C([0,T], L^2(\Omega))$ is given by

$$v(t,x) = P_\alpha(t)v_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,x) ds. \tag{2.11}$$

Similarly, if $v \in C([0,T], L^2(\Omega))$, then the mild solution of problem (2.9) is expressed by

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) v(s,x) ds. \tag{2.12}$$

Substituting (2.11) into (2.12), we get

$$\begin{aligned} u(t,x) =& P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(s) v_0 ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau,x) d\tau ds. \end{aligned} \tag{2.13}$$

□

Definition 2.1.6. Let $\frac{1}{2} < \alpha < 1$, $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$. We say that u is a mild solution of (2.1), if $u \in C([0,T], L^2(\Omega))$ and satisfies

$$\begin{aligned} u(t,x) =& P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(s) v_0 ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u(\tau,x)|^p d\tau ds, \end{aligned} \tag{2.14}$$

where $P_\alpha(t)$, $S_\alpha(t)$ were defined as (2.2), (2.3) and v_0 was specified in (2.10).

Theorem 2.1.7. *Let $\frac{1}{2} < \alpha < 1$ and $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$. Then there exists $T_{\max} > 0$ such problem (2.1) has a unique mild solution $u \in C([0, T_{\max}), L^2(\Omega))$.*

Proof. We apply the Banach fixed point theorem to prove the local existence of a unique mild solution. Let

$$E = C([0,T), L^2(\Omega)).$$

For $T > 0$, E is a Banach space endowed with the norm

$$\|u\|_E = \sup_{t \in (0,T)} \|u(t)\|_{L^2(\Omega)}.$$

Define the operator G as

$$
\begin{aligned}
Gu(t) =& P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(t) v_0(x) ds \\
&+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u|^p(\tau, x) d\tau ds.
\end{aligned}
\tag{2.15}
$$

For each $u(t) \in B_E(R)$, where $R = 2(\|u_0\|_{L^2(\Omega)} + T^\alpha \|v_0\|_{L^2(\Omega)})$. Then $G(u) \in C([0,T), L^2(\Omega))$.

First, we prove G maps $B_E(R)$ into itself. By using (2.4) and (2.5), we have

$$
\begin{aligned}
\|G(u)(t)\|_{L^2(\Omega)} =& \Big\| P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(t) v_0(x) ds \\
&+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u|^p(\tau) d\tau ds \Big\|_{L^2(\Omega)} \\
\leqslant& \Big\| P_\alpha(t) u_0 \Big\|_{L^2(\Omega)} + \int_0^t (t-s)^{\alpha-1} \Big\| S_\alpha(t-s) P_\alpha(t) v_0(x) \Big\|_{L^2(\Omega)} ds \\
&+ \int_0^t (t-s)^{\alpha-1} \Big\| S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u|^p(\tau) \Big\|_{L^2(\Omega)} d\tau ds \\
&\leqslant \|u_0\|_{L^2(\Omega)} + \frac{1}{\Gamma(1+\alpha)} T^\alpha \|v_0\|_{L^2(\Omega)} \\
&+ \frac{1}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1} \| \, |u(\tau)|^p \|_{L^2(\Omega)} d\tau ds \\
&\leqslant \frac{R}{2} + \frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^p.
\end{aligned}
$$

We choose T small enough such that

$$
\frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^{p-1} \leqslant \frac{1}{2}.
$$

Second, we show that G is a contraction map. For $u, v \in B_E(R)$, we have

$$
\begin{aligned}
\|G(u)(t) - G(v)(t)\|_{L^2(\Omega)} =& \Big\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) \Big(|u|^p(\tau) - |v|^p(\tau) \Big) \Big\| \\
\leqslant& \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1} \Big\| S_\alpha(s-\tau) \Big(|u|^p(\tau) - |v|^p(\tau) \Big) \Big\|_{L^2(\Omega)} d\tau ds \\
\leqslant& \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1} \Big\| \Big(|u|^p(\tau) - |v|^p(\tau) \Big) d\tau \Big\|_{L^1(\Omega)} ds \\
\leqslant& \frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^{p-1} \|u - v\|_E.
\end{aligned}
$$

Due to the following inequality

$$\big|\,|u(t)|^p - |v(t)|^p\,\big| \leqslant C(p)|u(t)-v(t)|(|u(t)|^{p-1} + |v(t)|^{p-1}).$$

We choose T such that

$$\frac{1}{\Gamma(2\alpha+1)}T^{2\alpha}R^{p-1} < 1.$$

Therefore, G is a strict contraction on $B_E(R)$. According to the Banach fixed point theorem. Then problem (2.1) admits a unique mild solution $u \in C([0,T_{\max}), L^2(\Omega))$, where

$$T_{\max} = \sup\left\{T > 0 \;\;|\;\; \text{there exists a mild solution } u \in C([0,T), L^2(\Omega)) \text{ to (2.1)}\right\}.$$

□

2.1.2 Blow-up and Global Existence

We investigate under what conditions the solutions of (2.1) blow-up or exist globally.

Lemma 2.1.8 ([30]). *Let $B_R(0) = \left\{x \in \mathbb{R}^N : \;\; |x| < R\right\}$ and $\Omega_R = \Omega \cap B_R(0)$ for large R. We introduce φ_2 the first eigenfunction of $-\Delta$ with λ the first eigenvalue on Ω_R*

$$\begin{cases} -\Delta\varphi_2(x) = \lambda\varphi_2(x), & x \in \Omega_R, \\ \varphi_2(x) > 0, & x \in \Omega_R, \\ \|\varphi_2\|_{L^\infty(\Omega_R)} = 1, \end{cases} \tag{2.16}$$

there exist C_1 and C_2 independent of R such that

$$C_1R^{-2} \leqslant \lambda \leqslant C_2R^{-2}. \tag{2.17}$$

The weak solution of the problem (2.1) is defined as follows:

Definition 2.1.9. Let $T > 0$ and $\frac{1}{2} < \alpha < 1$. A weak solution for the Cauchy problem (2.1) is a function $u \in L^p((0,T), L^2(\Omega))$ satisfies

$$\begin{aligned}
&\int_0^T \int_\Omega u \mathbf{D}_{t|T}^{2\alpha}\varphi - 2\int_0^T \int_\Omega u\Delta \mathbf{D}_{t|T}^{\alpha}\varphi + \int_0^T \int_\Omega u\Delta^2\varphi \\
&= \int_0^T \int_\Omega |u|^p\varphi + \int_0^T \int_\Omega u_0 \mathbf{D}_{t|T}^{2\alpha}\varphi - 2\int_0^T \int_\Omega u_0\Delta \mathbf{D}_{t|T}^{\alpha}\varphi,
\end{aligned} \tag{2.18}$$

for each $\varphi \in C_{t,x}^{2,4}([0,T]\times\Omega)$ compactly supported and $\varphi(T,.) = \varphi_t(T,.) = 0$.

Theorem 2.1.10. *Assume $u_0 \in L^2(\Omega)$ and $u_0(x) \geqslant 0$. If*

$$1 < p < 1 + \frac{4\alpha}{N - 4\alpha + 2},$$

then any solution to (2.1) blows-up in a finite time.

Proof. By contradiction. We assume that the solution u is globally if (2.18) holds for any $T > 0$. Let

$$\varphi(t,x) = \varphi_1(t)\varphi_2(x),$$

where φ_2 is the first eigenfunction of $-\Delta$ and $\varphi_1(t) = \left(1 - \frac{t}{T^2}\right)^\eta$ where $\eta > \frac{p}{p-1}2\alpha - 1$.

Equality (2.18) actually reads

$$\int_0^T \int_{\Omega_R} u\varphi_2 \mathbf{D}_{t|T}^{2\alpha}\varphi_1 - 2\int_0^T \int_{\Omega_R} u\Delta\varphi_2 \mathbf{D}_{t|T}^{\alpha}\varphi_1 + \int_0^T \int_{\Omega_R} u\varphi_1\Delta^2\varphi_2 = \int_0^T \int_{\Omega_R} |u|^p\varphi + \mathcal{I} + \mathcal{J}, \tag{2.19}$$

where

$$\mathcal{I} = \int_0^T \int_{\Omega_R} u_0\varphi_2 \mathbf{D}_{t|T}^{2\alpha}\varphi_1 = CT^{1-2\alpha}\int_{\Omega_R} u_0\varphi_2,$$

and

$$\mathcal{J} = 2\int_0^T \int_{\Omega_R} u_0(-\Delta)\varphi_2 \mathbf{D}_{t|T}^{\alpha}\varphi_1 = \lambda C T^{1-\alpha}\int_{\Omega_R} u_0\varphi_2.$$

Under the condition $u_0 \geqslant 0$, the Eq.(2.19) becomes

$$\begin{aligned}\int_0^T\int_{\Omega_R}|u|^p\varphi &\leqslant \int_0^T\int_{\Omega_R}u\varphi_2\mathbf{D}_{t|T}^{2\alpha}\varphi_1+2\lambda\int_0^T\int_{\Omega_R}u\varphi_2\mathbf{D}_{t|T}^{\alpha}\varphi_1+\lambda^2\int_0^T\int_{\Omega_R}u\varphi_2\varphi_1\\ &=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3. \end{aligned}\tag{2.20}$$

Using the Young inequality with parameters p and $p'=\frac{p}{p-1}$, we have

$$\begin{aligned}\mathcal{I}_1 &\leqslant \int_0^T\int_{\Omega_R}|u|\varphi^{1/p}\varphi^{-1/p}\varphi_2\mathbf{D}_{t|T}^{2\alpha}\varphi_1\\ &\leqslant \frac{1}{6p}\int_0^T\int_{\Omega_R}|u|^p\varphi+\frac{6^{p'-1}}{p'}\int_0^T\int_{\Omega_R}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}_{t|T}^{2\alpha}\varphi_1\right|^{p'}, \end{aligned}\tag{2.21}$$

$$\begin{aligned}\mathcal{I}_2 &\leqslant 2\lambda\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}\varphi^{-1/p}\varphi_2\mathbf{D}_{t|T}^{\alpha}\varphi_1\\ &\leqslant CR^{-2}\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}\varphi^{-1/p}\varphi_2\mathbf{D}_{t|T}^{\alpha}\varphi_1\\ &=\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}CR^{-2}\varphi^{-1/p}\varphi_2\mathbf{D}_{t|T}^{\alpha}\varphi_1\\ &\leqslant \frac{1}{6p}\int_0^T\int_{\Omega_R}|u|^p\varphi+C\frac{6^{p'-1}}{p'}R^{-2p'}\int_0^T\int_{\Omega_R}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}_{t|T}^{\alpha}\varphi_1\right|^{p'}, \end{aligned}\tag{2.22}$$

and

$$\begin{aligned}\mathcal{I}_3 &\leqslant \lambda^2\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}\varphi^{-1/p}\varphi_2\varphi_1\\ &\leqslant CR^{-4}\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}\varphi^{-1/p}\varphi_2\varphi_1\\ &=\int_0^T\int_{\Omega_R}|u|\varphi^{1/p}CR^{-4}\varphi^{-1/p}\varphi_2\varphi_1\\ &\leqslant \frac{1}{6p}\int_0^T\int_{\Omega_R}|u|^p\varphi+C\frac{6^{p'-1}}{p'}R^{-4p'}\int_0^T\int_{\Omega_R}\varphi_2\varphi_1. \end{aligned}\tag{2.23}$$

Taking into account the above relations (2.21), (2.22) and (2.23) in (2.20), we find

$$\begin{aligned}\left(1-\frac{1}{2p}\right)\int_0^T\int_\Omega |u|^p\varphi &\leqslant C\int_0^T\int_{\Omega_R}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{2\alpha}_{t|T}\varphi_1\right|^{p'}+R^{-2p'}\int_0^T\int_{\Omega_R}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{\alpha}_{t|T}\varphi_1\right|^{p'}\\ &\quad+R^{-4p'}\int_0^T\int_{\Omega_R}\varphi_2\varphi_1\\ &\leqslant C\int_0^T\int_{\Omega}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{2\alpha}_{t|T}\varphi_1\right|^{p'}+R^{-2p'}\int_0^T\int_{\Omega}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{\alpha}_{t|T}\varphi_1\right|^{p'}\\ &\quad+R^{-4p'}\int_0^T\int_{\Omega}\varphi_2\varphi_1.\end{aligned}\tag{2.24}$$

Note that, for $0<\alpha<1$ and $1<2\alpha<2$, then

$$\begin{cases}\mathbf{D}^{2\alpha}_{t|T}\varphi_1(t) &= CT^{-4\alpha}\left(1-\frac{t}{T^2}\right)^{\eta-2\alpha},\\ \mathbf{D}^{\alpha}_{t|T}\varphi_1(t) &= CT^{-2\alpha}\left(1-\frac{t}{T^2}\right)^{\eta-\alpha}.\end{cases}$$

We take $R=T$ and we introduce the following scaled variables

$$\tau=\frac{t}{T^2}\ \text{ and }\ \xi=\frac{|x|}{T},\ \ T\gg 1.$$

It appears that

$$\begin{aligned}\int_0^T\int_\Omega |u|^p\varphi &\leqslant \int_0^{T^2}\int_{\Omega}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{2\alpha}_{t|T}\varphi_1\right|^{p'}+T^{-2p'}\int_0^{T^2}\int_{\Omega}\varphi_2\varphi_1^{\frac{-p'}{p}}\left|\mathbf{D}^{\alpha}_{t|T}\varphi_1\right|^{p'}\\ &\quad+T^{-4p'}\int_0^{T^2}\int_{\Omega}\varphi_2\varphi_1\\ &\leqslant CT^{2-4\alpha p'+N}+CT^{2-(2+2\alpha)p'+N}+CT^{2-4p'+N}.\end{aligned}\tag{2.25}$$

Therefore, if a solution of (2.1) exists globally, then taking $T\to+\infty$, we get

$$\lim_{T\to\infty}\int_0^T\int_\Omega |u|^p\varphi=0.$$

Consequently, $u\equiv 0$. This leads to a contradiction. □

We are now in a position to state and prove the global existence of solutions of (2.1).

Theorem 2.1.11. *Let $\frac{1}{2}<\alpha<1$. If $p\geqslant 1+\frac{4\alpha}{N-4\alpha+2}$ and $\|u_0\|_{L^{q_c}(\Omega)}$ sufficiently small, where $q_c=\frac{N(p-1)}{2}$, then the mild solution of (2.1) exists globally.*

Proof. We apply the contraction mapping principle to prove the global solution of (2.1).

Clearly, from $p \geqslant 1 + \frac{4}{N-2}$, we see that

$$\frac{1}{p-1} - 1 \leqslant \frac{N}{4}, \tag{2.26}$$

for $\frac{1}{r} = \frac{1}{q_c} - \frac{1}{2} < \frac{2}{N}$. We can deduce

$$\|P_\alpha(t)u_0\|_{L^2(\Omega)} \leqslant (4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(1-N/2r)}{\Gamma(1-\alpha N/2r)} \|u_0\|_{L^{q_c}(\Omega)} < \infty.$$

Let

$$Y = \left\{ u \in C((0,\infty), L^2(\Omega)) : \quad \sup_{t>0} \|u(t)\|_{L^2(\Omega)} \leqslant R \right\}.$$

We define the operator G as

$$\begin{aligned} G(u)(t) =& P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(s) v_0 ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u(s,x)|^p ds, \end{aligned}$$

for each $u \in Y$. It is easy to notice that the operator G is well defined on Y. According to (2.5) we get

$$\begin{aligned} &\Big\| G(u)(t) - G(v)(t) \Big\|_{L^2(\Omega)} \\ &= \Big\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) \big[|u|^p(\tau) - |v|^p(\tau) \big] d\tau ds \Big\|_{L^2(\Omega)} \\ &\leqslant \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1} \Big\| S_\alpha(s-\tau) \big[|u|^p(\tau) - |v|^p(\tau) \big] \Big\|_{L^2(\Omega)} d\tau ds \\ &\leqslant C \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1-\alpha\frac{N}{4}} \Big\| |u|^p(\tau) - |v|^p(\tau) \Big\|_{L^1(\Omega)} d\tau ds \\ &\leqslant C \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1-\alpha\frac{N}{4}} \Big(\|u(\tau)\|^{p-1}_{L^2(\Omega)} + \|v(\tau)\|^{p-1}_{L^2(\Omega)} \Big) \Big\| u(\tau) - v(\tau) \Big\|_{L^2(\Omega)} \\ &\leqslant C R^{p-1} \int_0^t (t-s)^{\alpha-1} s^{\alpha-\alpha\frac{N}{4}} \Big\| u(\tau) - v(\tau) \Big\|_{L^2(\Omega)} ds \\ &\leqslant C R^{p-1} t^{2\alpha-\alpha\frac{N}{4}} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-\alpha\frac{N}{4}} dw \Big\| u(\tau) - v(\tau) \Big\|_{L^2(\Omega)} \\ &\leqslant C R^{p-1} t^{2\alpha-\alpha\frac{N}{4}} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-\frac{\alpha}{2}} dw \Big\| u(\tau) - v(\tau) \Big\|_{L^2(\Omega)} \end{aligned}$$

$$\left\|G(u)(t)-G(v)(t)\right\|_{L^2(\Omega)} \leqslant CR^{p-1}\frac{\Gamma(\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(3\alpha/2+1)}\left\|u-v\right\|_Y.$$

If we choose R small enough such that $CR^{p-1}<\frac{1}{2}$, then we get

$$\left\|G(u)-G(v)\right\|_Y < \frac{1}{2}\left\|u-v\right\|_Y.$$

□

2.2 Fractional Wave Equation with Structural Damping

This part based on the paper of Kirane and Laskri ([25]). Consider the following Cauchy problem

$$\begin{cases}\mathbf{D}_{0|t}^{1+\alpha_1}u+(-\Delta)^{\sigma}u+(-\Delta)^{\delta}(\mathbf{D}_{0|t}^{\alpha_2}u)=I_{0|t}^{1-\gamma}|u|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ u(0,x)=u_0,\ u_t(0,x)=u_1(x), \quad x\in\mathbb{R}^N,\end{cases} \tag{2.27}$$

where $u=u(t,x), p>1, 0<\gamma<\alpha_2\leq\alpha_1\leq 1, 0<\sigma<\delta<1$ and $I_{0|t}^{1-\alpha}$ is the Riemann-Liouville fractional integral of order $1-\alpha$. The fractional Laplacien operator is defined as

$$(-\Delta)^{\delta}u(t,x)=\frac{C(N,\delta)}{2}\int_{\mathbb{R}^N}\frac{u(t,x)-u(t,y)}{|x-y|^{N+2\delta}}dy,$$

where $C(N,\delta)$ is a positive normalizing constant depending on N and δ. The term $(-\Delta)^{\sigma}\mathbf{D}_{0|t}^{\alpha_2}u$ represents a generalized structural damping.

2.2.1 Main Results

Lemma 2.2.1 ([13]). *Let $\psi\in C^2(\mathbb{R}^N)$ be a function defined as*

$$\psi(x)=\begin{cases}1 & |x|\leqslant 1,\\ (1+(|x|-1)^4)^{1/4} & |x|>1.\end{cases} \tag{2.28}$$

Then, for all $x \in \mathbb{R}^N$

$$|(-\Delta)^\sigma \psi(x)| \leqslant \psi(x). \tag{2.29}$$

Firstly give the definition of a weak solution of (2.27).

Definition 2.2.2. Let $0 < \alpha_i < 1$, $p > 1$. For $u_0, u_1 \in L^p_{loc}(\mathbb{R}^N)$, the function $u \in L^p_{loc}(Q_T)$ is a weak solution of problem (2.27) if

$$\begin{aligned}&\int_{Q_T} u D^{1+\alpha_1}_{t|T}\phi + \int_{Q_T} u(-\Delta)^\sigma \phi + \int_{Q_T} u(-\Delta)^\delta D^{\alpha_2}_{t|T}\phi \\ &= \int_{Q_T} I^{1-\gamma}_{0|t}|u|^p\phi + \int_{\mathbb{R}^{\mathbb{N}}} u_0 D^{\alpha_1}_{t|T}\phi(0) + \int_{Q_T} u_1 D^{\alpha_1}_{t|T}\phi + \int_{Q_T} u_0(-\Delta)^\delta D^{\alpha_2}_{t|T}\phi,\end{aligned} \tag{2.30}$$

for $Q_T := [0,T] \times \mathbb{R}^{\mathbb{N}}$, $\phi > 0$, $\phi \in C^\infty_c([0,T]\times\mathbb{R}^{\mathbb{N}})$ with $\phi(T,.) = 0$.

Theorem 2.2.3. *Assume that $u_0 = 0$, $u_1 \in L^1(\mathbb{R}^N)$ and $u_1 \geqslant 0$. If*

$$1 < p \leqslant p* := \frac{2(2+\alpha_1-\gamma)}{(\frac{\alpha_1+1}{\sigma}N + 2\gamma - 2\alpha_1 - 2)_+} + 1,$$

then any solution to (2.27) blows up in a finite time.

Proof. We assume the contrary. Let

$$\phi(t,x) = D^{1-\gamma}_{t|T}\tilde{\varphi}(t,x) = D^{1-\gamma}_{t|T}\left(\varphi_1(t)\varphi_2(x)\right),$$

where $\varphi_1(t)$ and $\varphi_2(x) = \psi(T^{-\theta/2}x)$ are defined as in (1.41) and (2.29). According to (2.30) and Lemma 1.1.4, we have

$$\begin{aligned}&\int_{Q_T} |u|^p\tilde{\varphi} + \int_{Q_T} u_1\varphi_2 D^{1+\alpha_1-\gamma}_{t|T}\varphi_1 \\ &= \int_{Q_T} u\varphi_2 D^{2+\alpha_1-\gamma}_{t|T}\varphi_1 + \int_{Q_T} u(-\Delta)^\delta\varphi_2 D^{1+\alpha_2-\gamma}_{t|T}\varphi_1 + \int_{Q_T} u(-\Delta)^\sigma\varphi_2 D^{1-\gamma}_{t|T}\varphi_1.\end{aligned} \tag{2.31}$$

Therefore, by Lemma 1.2.1, we get

$$\begin{aligned}&\int_{Q_T} |u|^p\tilde{\varphi} + CT^{-(\alpha_1-\gamma)}\int_{\mathbb{R}^N} u_1(x)\varphi_2(x)dx \\ &\leqslant \int_{Q_T} |u|\left(\varphi_2 D^{2+\alpha_1-\gamma}_{t|T}\varphi_1 + D^{1+\alpha_2-\gamma}_{t|T}\varphi_1|(-\Delta)^\delta\varphi_2| + D^{1-\gamma}_{t|T}\varphi_1|(-\Delta)^\sigma\varphi_2|\right).\end{aligned}$$

Using the Young inequality with parameters p and $p' = \frac{p}{p-1}$, we obtain

$$\begin{aligned}
&\int_{Q_T} |u|^p \tilde{\varphi} + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_2(x)dx \\
&\leqslant \frac{1}{6p}\int_{Q_T} |u|^p \tilde{\varphi} + \frac{6^{p'-1}}{p'}\int_{Q_T} \varphi_2 \varphi_1^{-p'/p} |D_{t|T}^{2+\alpha_1-\gamma}\varphi_1|^{p'} \\
&+ \frac{1}{6p}\int_{Q_T} |u|^p \tilde{\varphi} + \frac{6^{p'-1}}{p'}\int_{Q_T} \varphi_2^{-p'/p} \varphi_1^{-p'/p} |(-\Delta)^{\delta}\varphi_2|^{p'} |D_{t|T}^{1+\alpha_2-\gamma}\varphi_1|^{p'} \\
&+ \frac{1}{6p}\int_{Q_T} |u|^p \tilde{\varphi} + \frac{6^{p'-1}}{p'}\int_{Q_T} \varphi_2^{-p'/p} \varphi_1^{-p'/p} |(-\Delta)^{\sigma}\varphi_2|^{p'} |D_{t|T}^{1-\gamma}\varphi_1|^{p'}.
\end{aligned} \tag{2.32}$$

Using Lemma 2.2.1, it holds

$$\int_{\mathbb{R}^N} \varphi_2(x)^{-p'/p} |(-\Delta)^{\delta}\varphi_2|^{p'} \leqslant T^{-\theta p' + \frac{\theta N}{2}}.$$

Passing to the scaled variables

$$\tau = \frac{t}{T}\ , \ \ \xi = \frac{|x|}{T^{\theta/2}}, \ \ \theta = \frac{\alpha_1 + 1}{\sigma} \ \text{ and } \ \ T \gg 1.$$

Hence

$$\begin{aligned}
&\left(1 - \frac{1}{p}\right) \int_{Q_T} |u|^p \tilde{\varphi} + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_2(x)dx \\
&\leqslant C\left(T^{-p'(2+\alpha_1-\gamma)+\frac{\theta N}{2}+1} + T^{-p'(\theta\delta+1+\alpha_2-\gamma)+\frac{\theta N}{2}+1} + T^{-p'(\theta\sigma+1-\gamma)+\frac{\theta N}{2}+1}\right) \\
&\leqslant C\left(T^{-p'(2+\alpha_1-\gamma)+\frac{\theta N}{2}+1} + 2T^{-p'(\theta\sigma+1-\gamma)+\frac{\theta N}{2}+1}\right) \\
&\leqslant CT^{-p'(2+\alpha_1-\gamma)+\frac{\theta N}{2}+1}.
\end{aligned} \tag{2.33}$$

Under the condition $u_1(x) \geqslant 0$, we obtain

$$\frac{1}{p'}\int_{Q_T} |u|^p \tilde{\varphi} \leqslant CT^{-p'(2+\alpha_1-\gamma)+\frac{\theta N}{2}+1}. \tag{2.34}$$

Since

$$p \leqslant p* = \frac{2(2+\alpha_1-\gamma)}{(\frac{\alpha_1+1}{\sigma}N + 2\gamma - 2\alpha_1 - 2)_+} + 1,$$

we have to distinguish two cases.

In case $p < p*$: if a solution of (2.27) exists globally, then taking $T \to +\infty$, we get

$$\lim_{T\to\infty} \int_{Q_T} |u|^p \tilde{\varphi} < 0.$$

Contradiction the fact that $\int_0^\infty \int_{\mathbb{R}^N} |u|^p \tilde{\varphi} \geqslant 0$.

In case $p = p*$: we repeat the same calculation as above by taking $\varphi_2(x) = \psi(\frac{|x|}{B^{-1}T^{\theta/2}})$, where $1 \ll B < T$ and when T goes to infinity we don't have B goes to infinity at the same time, employing the Hölder's inequality instead of Young's, we obtain

$$\int_{Q_T} |u|\varphi_2 D_{t|T}^{2+\alpha_1-\gamma}\varphi_1 \leqslant B^{-\frac{N}{p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p}. \tag{2.35}$$

Using Lemma 2.2.1 and the Hölder inequality, we have

$$\int_{Q_T} |u(-\Delta)^\delta \varphi_2| D_{t|T}^{1+\alpha_2-\gamma}\varphi_1 \leqslant CT^{-(\theta(\delta-\sigma)+\alpha_2)} B^{2\delta-\frac{N}{p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p}, \tag{2.36}$$

and

$$\int_{Q_T} |u(-\Delta)^\sigma \varphi_2| D_{t|T}^{1-\gamma}\varphi_1 \leqslant B^{2\sigma-\frac{N}{p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p}. \tag{2.37}$$

Combining (2.35), (2.36) with (2.37), we get

$$\begin{aligned} \int_{Q_T} |u|^p \tilde{\varphi} + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_2(x)dx \leqslant CB^{-\frac{N}{\beta p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p} \\ + CT^{-(\theta(\delta-\sigma)+\alpha_2)} B^{2\delta-\frac{N}{p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p} \\ + CB^{2\sigma-\frac{N}{p'}} \left(\int_{Q_T} |u|^p \tilde{\varphi} \right)^{1/p}. \end{aligned}$$

Thus, passing the limit as $T \longrightarrow +\infty$ and then where $B \longrightarrow +\infty$, with $\sigma < \frac{N}{2}(1 - \frac{1}{p*})$ we have

$$\lim_{T\to\infty} \int_{Q_T} |u|^p \tilde{\varphi} < 0.$$

This leads to a contradiction.

□

2.3 Time–Space Fractional Diffusion System

This part is concerned with the study of the following system ([7])

$$\begin{cases} \mathbf{D}^{\alpha_1}_{0|t} u + (-\Delta)^{\delta_1} u = I^{1-\gamma_1}_{0|t} |v|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ \\ \mathbf{D}^{\alpha_2}_{0|t} v + (-\Delta)^{\delta_2} v = I^{1-\gamma_2}_{0|t} |u|^q, & (t,x) \in (0,\infty) \times \mathbb{R}^N, \end{cases} \tag{2.38}$$

supplemented with the initial conditions

$$\begin{cases} u(0,x) = u_0(x), & x \in \mathbb{R}^N, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.39}$$

where $p > 1$, $q > 1$, $0 < \alpha_i < 1$ and $0 < \delta_i < 1$. The operator $(-\Delta)^\delta$ is defined as a power of Laplacian operator $-\Delta$.

2.3.1 Nonexistence Results

Theorem 2.3.1. *Assume that* $u_0 > 0,\ v_0 > 0$. *If*

$$N/2 < \max \left\{ \frac{\frac{1}{q}\Theta_2 + \Theta_1 - (1 - \frac{1}{pq})}{\frac{\alpha_2}{\delta_2 q p'} + \frac{1}{q'}}, \frac{\frac{1}{p}\Theta_1 + \Theta_2 - (1 - \frac{1}{pq})}{\frac{\alpha_1}{\delta_1 q' p} + \frac{1}{p'}} \right\},$$

where $\Theta_i = 1 + \alpha_i - \gamma_i$. *Then system* (2.38)−(2.39) *admit no global nontrivial weak solutions.*

Proof. The proof proceeds by contradiction. Let

$$\phi(t,x) = D^{1-\gamma_i}_{t|T} \tilde{\varphi}(t,x) = D^{1-\gamma_i}_{t|T} \left(\varphi_1(t)\varphi_2(x)\right), \quad i = 1, 2,$$

where φ_1 is defined as in (1.41) however with condition $\eta > \left\{ \frac{p}{p-1}\Theta_i \right\}$ and $\varphi_2(x) = \psi\left(\frac{|x|}{T^{\theta_i/2}}\right)$ is defined above.

The weak solutions to system (2.38) − (2.39) reads as

$$\int_{Q_T} |v|^p \tilde{\varphi} + \int_{Q_T} u_0 \varphi_2 \mathbf{D}^{\alpha_1+1-\gamma_1} \varphi_1 = \int_{Q_T} u \varphi_2 \mathbf{D}^{\alpha_1+1-\gamma_1} \varphi_1 + \int_{Q_T} u (-\Delta)^{\delta_1} \varphi_2 \mathbf{D}^{1-\gamma_1} \varphi_1, \tag{2.40}$$

and

$$\int_{Q_T} |u|^q \tilde{\varphi} + \int_{Q_T} v_0 \varphi_2 \mathbf{D}^{\alpha_2+1-\gamma_2} \varphi_1 = \int_{Q_T} v \varphi_2 \mathbf{D}^{\alpha_2+1-\gamma_2} \varphi_1 + \int_{Q_T} v(-\Delta)^{\delta_2} \varphi_2 \mathbf{D}^{1-\gamma_2} \varphi_1. \quad (2.41)$$

Using the Hölder inequality, we obtain

$$\int_{Q_T} u \varphi_2 \mathbf{D}^{\alpha_1+1-\gamma_1} \varphi_1 \leqslant \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/q} \left(\int_{Q_T} \varphi_2 \varphi_1^{\frac{-1}{q-1}} |\mathbf{D}^{\alpha_1+1-\gamma_1} \varphi_1|^{q'} \right)^{1/q'}, \quad (2.42)$$

and

$$\int_{Q_T} u(-\Delta)^{\delta_1} \varphi_2 \mathbf{D}^{1-\gamma_1} \varphi_1 \leqslant \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/q} \left(\int_{Q_T} \varphi_2^{-p'/p} |(-\Delta)^{\delta_1} \varphi_2|^{q'} \varphi_1^{\frac{-1}{q-1}} |\mathbf{D}^{1-\gamma_1} \varphi_1|^{q'} \right)^{1/q'}. \quad (2.43)$$

Taking into account the above relation (2.42) and (2.43), we find

$$cT^{\alpha_1-\gamma_1} \int_{\mathbb{R}^N} u_0 \varphi_2 + \int_{Q_T} |v|^p \tilde{\varphi} \leqslant \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/q} \mathcal{A}, \quad (2.44)$$

we have set

$$\mathcal{A} = \left(\int_{Q_T} \varphi_2 \varphi_1^{\frac{-1}{q-1}} |\mathbf{D}^{\alpha_1+1-\gamma_1} \varphi_1|^{q'} \right)^{1/q'} + \left(\int_{Q_T} \varphi_2^{-q'/q} |(-\Delta)^{\delta_1} \varphi_2|^{q'} \varphi_1^{\frac{-1}{q-1}} |\mathbf{D}^{1-\gamma_1} \varphi_1|^{q'} \right)^{1/q'}.$$

Similarly, we get

$$cT^{\alpha_2-\gamma_2} \int_{\mathbb{R}^N} v_0 \varphi_2 + \int_{Q_T} |u|^q \tilde{\varphi} \leqslant \left(\int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/p} \mathcal{B}, \quad (2.45)$$

with

$$\mathcal{B} = \left(\int_{Q_T} \varphi_2 \varphi_1^{\frac{-1}{p-1}} |\mathbf{D}^{\alpha_2+1-\gamma_2} \varphi_1|^{p'} \right)^{1/p'} + \left(\int_{Q_T} \varphi_2^{-p'/p} |(-\Delta)^{\delta_2} \varphi_2|^{p'} \varphi_1^{\frac{-1}{p-1}} |\mathbf{D}^{1-\gamma_2} \varphi_1|^{p'} \right)^{1/p'}.$$

Therefore, as $u_0, v_0 \geqslant 0$, we obtain

$$\int_{Q_T} |v|^p \tilde{\varphi} \leqslant \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/q} \mathcal{A}, \quad (2.46)$$

and

$$\int_{Q_T} |u|^q \tilde{\varphi} \leqslant \left(\int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/p} \mathcal{B}. \tag{2.47}$$

Now, combining (2.46) and (2.47), we write

$$\begin{cases} \left(\int_{Q_T} |v|^p \tilde{\varphi} \right)^{1-\frac{1}{pq}} \leqslant \mathcal{B}^{\frac{1}{q}} \mathcal{A}, \\ \\ \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1-\frac{1}{pq}} \leqslant \mathcal{A}^{\frac{1}{p}} \mathcal{B}. \end{cases} \tag{2.48}$$

Using Lemma 1.2.1 and making the change of variables

$$x = \xi T^{\frac{\theta_1}{2}} \text{ with } \theta_1 = \frac{\alpha_1}{\delta_1} \text{ in } \mathcal{A},$$

$$x = \xi T^{\frac{\theta_2}{2}} \text{ with } \theta_2 = \frac{\alpha_2}{\delta_2} \text{ in } \mathcal{B}.$$

We obtain the estimates

$$\begin{cases} \left(\int_{Q_T} |v|^p \tilde{\varphi} \right)^{1-\frac{1}{pq}} \leqslant T^{l_1}, \\ \\ \left(\int_{Q_T} |u|^q \tilde{\varphi} \right)^{1-\frac{1}{pq}} \leqslant T^{l_2}, \end{cases} \tag{2.49}$$

where

$$l_1 = \left(-(1+\alpha_2-\gamma_2) + \frac{1}{p'}(\frac{\theta_2 N}{2}+1) \right) \frac{1}{q} - (1+\alpha_1-\gamma_1) + \frac{1}{q'}(\frac{\theta_1 N}{2}+1),$$

and

$$l_2 = \left(-(1+\alpha_1-\gamma_1) + \frac{1}{q'}(\frac{\theta_1 N}{2}+1) \right) \frac{1}{p} - (1+\alpha_2-\gamma_2) + \frac{1}{p'}(\frac{\theta_2 N}{2}+1).$$

Hence, by taking the limit as $T \to \infty$ in (2.49), we obtain

$$\begin{cases} \int_0^\infty \int_{\mathbb{R}^N} |v|^p \tilde{\varphi} < 0, \\ \\ \int_0^\infty \int_{\mathbb{R}^N} |u|^q \tilde{\varphi} < 0, \end{cases}$$

which is a contradiction. Then (u, v) cannot be a global solution. $\square$

CHAPTER 3

FRACTIONAL SOBOLEV TYPE EQUATION USING HILFER- HADAMARD'S DEFINITIONS

In Chapter 3 ([8]), we mainly consider semilinear equation with Hilfer- Hadamard fractional derivative

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha_1,\beta}u - \Delta\mathcal{D}_{a^+}^{\alpha_2,\beta}u - \Delta u = f(u), & t > a > 0, \quad x \in \Omega, \\ u(t,x) = 0, & t > a > 0, \quad x \in \partial\Omega, \\ \left(\mathcal{D}_{a^+}^{(\beta-1)(1-\alpha_1)}u\right)(a,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where Ω be an open bounded set with sufficiently smooth boundary $\partial\Omega$ and $0 < \alpha_2 \leqslant \alpha_1 < 1$. $\mathcal{D}_{a^+}^{\alpha_i,\beta}$ $(i = 1, 2)$ is the Hilfer- Hadamard fractional derivative of order α_i and of type β. For simplicity, we formulate most of our assertions for the model case $f(u) = |u|^p$, $p > 1$. The equation (3.1) is a generalization of the well-known pseudo-parabolic equation of first order. The integer derivative is replaced by a fractional derivative in the sense of Hilfer-Hadamard. The second Hilfer-Hadamard fractional derivative of the Laplacian is allowed to be different from the first one.

Our objective is to find the range of p for which non-trivial solutions cannot exist all the time. This leads us to shed some light on the interaction of the nonlinear source term with $\Delta\mathcal{D}_{a^+}^{\alpha_2,\beta}u$. The analysis is based mainly on *the test function method [32]*.

3.1 Blow-up of Solutions

First, we give the definition of a weak solution of (3.1), then we prove the non-existence of non-trivial solutions.

Definition 3.1.1. Let $u_0 \in C_0(\Omega)$ $0 < \alpha_2 < \alpha_1 < 1$ and $\delta = \alpha_1 + \beta - \alpha_1\beta$ The function $u \in C^{\gamma}_{1-\gamma,\log}([a,b], C_0(\Omega))$ is a weak solution for the problem (3.1), if

$$\int_{\Omega}\int_a^T \tilde{\varphi}\mathcal{D}^{\alpha_1,\beta}_{a^+}udtdx - \int_{\Omega}\int_a^T \Delta\tilde{\varphi}\mathcal{D}^{\alpha_2,\beta}_{a^+}udtdx - \int_{\Omega}\int_a^T \Delta\tilde{\varphi}udtdx = \int_{\Omega}\int_a^T |u|^p\tilde{\varphi}dtdx, \quad (3.2)$$

for all compactly supported test function $\tilde{\varphi} \in C^{1,2}_{t,x}([a,T]\times\Omega)$.

Theorem 3.1.2. *Let $u_0 \in C_0(\Omega)$ and $u_0 \geqslant 0$. If*

$$1 < p < \frac{\alpha_2 N + 1}{(\alpha_2 N + 1 - 2\alpha_2)},$$

then the problem (3.1) does not admit global non-trivial solutions in the space $C^{\gamma}_{1-\gamma,\log}([a,b], C_0(\Omega))$.

Proof. We assume the contrary. Let $\Phi \in C_0^{\infty}([0,\infty))$ be a decreasing function satisfying

$$\Phi(\sigma) = \begin{cases} 1, & 0 \leqslant \sigma \leqslant 1, \\ 0, & \sigma \geqslant 2. \end{cases}$$

We define the function $\tilde{\varphi}(t,x)$ as follows

$$\tilde{\varphi}(t,x) = \frac{\varphi_1(t)}{t}\varphi_2(x), \quad (3.3)$$

with $\varphi_1(t) \in C^1([a,\infty))$, $\varphi_1(t) \geqslant 0$ and $\varphi_1(t)$ is non-increasing such that

$$\varphi_1(t) = \begin{cases} 1, & 0 < a \leqslant t \leqslant \theta T, \quad 0 < \theta < 1, \\ 0, & t \geqslant T, \end{cases} \quad (3.4)$$

for $T > a > 0$ and we choose

$$\varphi_2(x) = \left[\Phi\left(\frac{\|x\|}{T^{\alpha_2}}\right)\right]^{\mu}, \quad \mu \geqslant \frac{2p}{p-1}. \quad (3.5)$$

Equality (3.2) actually reads

$$\int_{\Omega_1}\int_a^T \varphi_2(x)\varphi_1(t)\mathcal{D}^{\alpha_1,\beta}_{a^+}u\frac{dt}{t}dx - \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)\mathcal{D}^{\alpha_2,\beta}_{a^+}u\frac{dt}{t}dx - \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)u\frac{dt}{t}dx$$
$$= \int_{\Omega_1}\int_a^T |u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx. \quad (3.6)$$

where $\Omega_1 := \{x \in \Omega : \ \|x\| \leqslant 2T^{\alpha_2}\}$. From the definition of $\mathcal{D}^{\alpha,\beta}_{a^+}u$, we can re-write the above equation as

$$\int_{\Omega_1}\int_a^T \varphi_2(x)\varphi_1(t)\mathcal{I}^{\beta(1-\alpha_1)}_{a^+}\left(t\frac{d}{dt}\right)\left(\mathcal{I}^{(1-\beta)(1-\alpha_1)}_{a^+}u\right)\frac{dt}{t}dx$$
$$-\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)\mathcal{I}^{\beta(1-\alpha_2)}_{a^+}\left(t\frac{d}{dt}\right)\left(\mathcal{I}^{(1-\beta)(1-\alpha_2)}_{a^+}u\right)\frac{dt}{t}dx - \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)u\frac{dt}{t}dx$$
$$= \int_{\Omega_1}\int_a^T |u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx. \tag{3.7}$$

By Definition 1.1.10, we have $\left(\log\frac{t}{a}\right)^{1-\gamma}\mathcal{D}^{\gamma}_{a^+}u$ is continuous on $[a,T]$ implies that

$$\left|\left(\log\frac{t}{a}\right)^{1-\gamma}\mathcal{D}^{\gamma}_{a^+}u\right| \leqslant M, \quad \forall t \in [a,T],$$

for some positive constant M (the constant M will be a generic constant which may change at different places). Therefore

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathcal{D}^{\gamma}_{a^+}u\right)(t)\right|^{p'}\frac{dt}{t} &\leqslant M^{p'}\int_a^T t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}\\ &\leqslant M^{p'}\int_a^\infty t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. We introduce the following scaled variable

$$w = (p'-1)\log\big(t/a\big).$$

Then

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathcal{D}^{\gamma}_{a^+}u\right)(t)\right|^{p'}\frac{dt}{t} &\leqslant \frac{M^{p'}a^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\int_0^\infty w^{-p'(1-\gamma)}\mathrm{e}^{-\mathrm{w}}\mathrm{dw}\\ &\leqslant \frac{M^{p'}a^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\Gamma(1-p'(1-\gamma)) < \infty.\end{aligned} \tag{3.8}$$

Consequently, $\left(\mathcal{D}^{\gamma}_{a^+}u\right)(t) \in X^{p'}_{-1/p}$. Thus it follows from Lemma 1.1.15 that

$$
\begin{aligned}
&\int_{\Omega_1}\int_a^T \varphi_2(x)\mathcal{I}^{\beta(1-\alpha_1)}_{T^-}\varphi_1(t)\frac{d}{dt}\mathcal{I}^{(1-\beta)(1-\alpha_1)}_{a^+}u dt dx\\
&-\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\mathcal{I}^{\beta(1-\alpha_2)}_{T^-}\varphi_1(t)\frac{d}{dt}\mathcal{I}^{(1-\beta)(1-\alpha_2)}_{a^+}u dt dx-\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)u\frac{dt}{t}dx\\
&=\int_{\Omega_1}\int_a^T |u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx.
\end{aligned}
\tag{3.9}
$$

Using integration by parts in (3.9), we obtain

$$
\begin{aligned}
&\int_{\Omega_1}\varphi_2(x)\left[(\mathcal{I}^{\beta(1-\alpha_1)}_{T^-}\varphi_1)(t)(\mathcal{I}^{(1-\beta)(1-\alpha_1)}_{a^+}u)(t,x)\right]_{t=a}^{T}dx\\
&-\int_{\Omega_1}\int_a^T \varphi_2(x)\frac{d}{dt}\mathcal{I}^{\beta(1-\alpha_1)}_{T^-}\varphi_1(t)\mathcal{I}^{(1-\beta)(1-\alpha_1)}_{a^+}u dt dx\\
&-\int_{\Omega_1}\Delta\varphi_2(x)\left[(\mathcal{I}^{\beta(1-\alpha_2)}_{T^-}\varphi_1)(t)(\mathcal{I}^{(1-\beta)(1-\alpha_2)}_{a^+}u)(t,x)\right]_{t=a}^{T}\\
&+\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\frac{d}{dt}\mathcal{I}^{\beta(1-\alpha_2)}_{T^-}\varphi_1(t)\mathcal{I}^{(1-\beta)(1-\alpha_2)}_{a^+}u dt dx-\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\varphi_1(t)u\frac{dt}{t}dx\\
&=\int_{\Omega_1}\int_a^T |u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx.
\end{aligned}
\tag{3.10}
$$

Since $\varphi_1 \in C^1[a,b]$, then there exists a constant $M > 0$ such that $|\varphi_1(t)| \leqslant M$. Hence

$$
\begin{aligned}
|\mathcal{I}^{\beta(1-\alpha_i)}_{T^-}\varphi_1(t)| &\leqslant \frac{M}{\Gamma(\beta(1-\alpha_i))}\int_t^T\left(\log\frac{\tau}{t}\right)^{\beta(1-\alpha_i)-1}\frac{d\tau}{\tau}\\
&\leqslant \frac{M}{\Gamma(\beta(1-\alpha_i)+1)}\left(\log\frac{T}{t}\right)^{\beta(1-\alpha_i)},
\end{aligned}
$$

where $i = 1, 2$. We notice that

$$
(\mathcal{I}^{\beta(1-\alpha_i)}_{T^-}\varphi_1)(T) = \lim_{t\longrightarrow T}\mathcal{I}^{\beta(1-\alpha_i)}_{T^-}\varphi_1(t) = 0, \tag{3.11}
$$

and

$$
\left(\mathcal{I}^{(1-\beta)(1-\alpha_1)}_{a^+}u\right)(a^+,x) = \left(\mathcal{D}^{(\beta-1)(1-\alpha_1)}_{a^+}u\right)(a^+,x) = u_0(x). \tag{3.12}
$$

It is noticed from Lemma 1.1.14 that

$$\left|\left(\log\frac{t}{a}\right)^{1-\gamma}u(t,.)\right|\leqslant M,$$

for $1-\gamma<(1-\beta)(1-\alpha_2)$. We can deduce

$$\begin{aligned}\left|\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\right| &\leqslant \frac{1}{\Gamma((1-\beta)(1-\alpha_2))}\int_a^t\left|\left(\log\frac{t}{\tau}\right)^{(1-\beta)(1-\alpha_2)-1}u\frac{d\tau}{\tau}\right|\\ &\leqslant M\left(\log\frac{t}{a}\right)^{\gamma-1}\int_a^t\left(\log\frac{t}{\tau}\right)^{(1-\beta)(1-\alpha_2)-1}\frac{d\tau}{\tau}\\ &\leqslant M\left(\log\frac{t}{a}\right)^{\gamma-1+(1-\beta)(1-\alpha_2)}. \end{aligned}\tag{3.13}$$

Therefore

$$\left(\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\right)(a,x)=0.\tag{3.14}$$

Taking into account the above relations (3.12) and (3.14) in (3.10), we find

$$\begin{aligned}&-\int_{\Omega_1}\int_a^T\varphi_2(x)\frac{d}{dt}\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}udtdx\\ &+\int_{\Omega_1}\int_a^T\Delta\varphi_2(x)\frac{d}{dt}\mathcal{I}_{T^-}^{\beta(1-\alpha_2)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}udtdx-\int_{\Omega_1}\int_a^T\Delta\varphi_2(x)\varphi_1(t)u\frac{dt}{t}dx\\ &=\int_{\Omega_1}\int_a^T|u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx+\int_{\Omega_1}\varphi_2(x)(\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\varphi_1)(a)u_0(x)dx.\end{aligned}\tag{3.15}$$

Let

$$\mathcal{A}_1=-\int_{\Omega_1}\int_a^T\varphi_2(x)\frac{d}{dt}\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}udtdx,$$

and

$$\mathcal{A}_2=\int_{\Omega_1}\int_a^T\Delta\varphi_2(x)\frac{d}{dt}\mathcal{I}_{T^-}^{\beta(1-\alpha_2)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}udtdx.$$

Multiplying $\mathcal{A}_1$ and $\mathcal{A}_2$ by t/t, we notice that

$$\mathcal{A}_1 = \int_{\Omega_1}\int_a^T \varphi_2(x)\Big(-t\frac{d}{dt}\Big)\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}u\frac{dt}{t}dx, \tag{3.16}$$

and

$$\mathcal{A}_2 = -\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\Big(-t\frac{d}{dt}\Big)\mathcal{I}_{T^-}^{\beta(1-\alpha_2)}\varphi_1(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\frac{dt}{t}dx. \tag{3.17}$$

Definition 1.1.16 allows us to write

$$\mathcal{A}_1 = \int_{\Omega_1}\int_a^T \varphi_2(x)\Big(\mathcal{D}_{T^-}^{1-\beta(1-\alpha_1)}\varphi_1\Big)(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}u\frac{dt}{t}dx, \tag{3.18}$$

and

$$\mathcal{A}_2 = -\int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\Big(\mathcal{D}_{T^-}^{1-\beta(1-\alpha_2)}\varphi_1\Big)(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\frac{dt}{t}dx. \tag{3.19}$$

According to Lemma 1.1.18 and (3.4), we get

$$\begin{aligned}\Big(\mathcal{D}_{T^-}^{1-\beta(1-\alpha_i)}\varphi_1\Big)(t) &= \frac{-1}{\Gamma(\beta(1-\alpha_i))}\int_t^T\left(\log\frac{s}{t}\right)^{\beta(1-\alpha_i)-1}\varphi_1'(s)ds\\ &= -\Big(\mathcal{I}_{T^-}^{\beta(1-\alpha_i)}\delta\varphi_1\Big)(t), \quad i=1,2. \end{aligned}\tag{3.20}$$

In regards to (3.20), we obtain

$$\mathcal{A}_1 = -\int_{\Omega_1}\int_a^T \varphi_2(x)\Big(\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\delta\varphi_1\Big)(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}u\frac{dt}{t}dx, \tag{3.21}$$

and

$$\mathcal{A}_2 = \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\Big(\mathcal{I}_{T^-}^{\beta(1-\alpha_2)}\delta\varphi_1\Big)(t)\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\frac{dt}{t}dx. \tag{3.22}$$

Note that $\delta\varphi_1 \in L^p([a,T])$ and by the same arguments as in the proof of

$$\Big(\mathcal{D}_{a^+}^{\gamma}u\Big)(t) \in X_{-1/p}^{p'},$$

we may show that

$$\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_i)}u \in X_{-1/p}^{p'},$$

since $\mathcal{I}_{a^+}^{1-\gamma}u \in C_{1-\gamma,\log}[a,T]$.

Therefor, we conclude that Lemma 1.1.15 is satisfied

$$\mathcal{A}_1 = -\int_{\Omega_1}\int_a^T \varphi_2(x)\delta\varphi_1(t)\left(\mathcal{I}_{a^+}^{\beta(1-\alpha_1)}\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_1)}u\right)(t,x)\frac{dt}{t}dx, \tag{3.23}$$

$$\mathcal{A}_2 = \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\delta\varphi_1(t)\left(\mathcal{I}_{a^+}^{\beta(1-\alpha_2)}\mathcal{I}_{a^+}^{(1-\beta)(1-\alpha_2)}u\right)(t,x)\frac{dt}{t}dx. \tag{3.24}$$

Lemma 1.1.13 yields

$$\mathcal{A}_1 = -\int_{\Omega_1}\int_a^T \varphi_2(x)\delta\varphi_1(t)\left(\mathcal{I}_{a^+}^{1-\alpha_1}u\right)(t,x)\frac{dt}{t}dx, \tag{3.25}$$

$$\mathcal{A}_2 = \int_{\Omega_1}\int_a^T \Delta\varphi_2(x)\delta\varphi_1(t)\left(\mathcal{I}_{a^+}^{1-\alpha_2}u\right)(t,x)\frac{dt}{t}dx. \tag{3.26}$$

By Definition 1.1.12 and the property of φ_1, we have

$$\begin{aligned}\mathcal{A}_1 &\leqslant \frac{1}{\Gamma(1-\alpha_1)}\int_{\Omega_1}\int_a^T \varphi_2(x)|\delta\varphi_1(t)|\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha_1}\frac{|u(s,x)|}{s}ds\frac{dt}{t}dx\\ &\leqslant \frac{1}{\Gamma(1-\alpha_1)}\int_{\Omega_1}\int_{\theta T}^T \varphi_2(x)\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha_1}\frac{|u(s,x)|\varphi_1^{1/p}(s)}{s}ds\frac{dt}{t}dx\\ &\leqslant \int_{\Omega_1}\int_{\theta T}^T \varphi_2(x)\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_1}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx.\end{aligned} \tag{3.27}$$

A similar analysis for $\mathcal{A}_2$, we get

$$\begin{aligned}\mathcal{A}_2 &\leqslant \frac{1}{\Gamma(1-\alpha_2)}\int_{\Delta\Omega_1}\int_{\theta T}^T |\Delta\varphi_2(x)|\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha_2}\frac{|u(s,x)|\varphi_1^{1/p}(s)}{s}ds\frac{dt}{t}dx\\ &\leqslant \int_{\Delta\Omega_1}\int_{\theta T}^T |\Delta\varphi_2(x)|\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_2}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx,\end{aligned} \tag{3.28}$$

where

$$\Delta\Omega_1 := \{x \in \Omega : \ \ T^{\alpha_2} \leqslant \|x\| \leqslant 2T^{\alpha_2}\}.$$

We obtain from (3.15), (3.27) and (3.28)

$$\begin{aligned}
&\int_{\Omega_1}\int_a^T |u|^p \varphi_2(x)\varphi_1(t)\frac{dt}{t}dx + \int_{\Omega_1} \varphi_2(x)(\mathcal{I}_{T^-}^{\beta(1-\alpha_1)}\varphi_1)(a)u_0(x)dx \\
&\leqslant \int_{\Omega_1}\int_{\theta T}^T \varphi_2(x)\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_1}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx \\
&+ \int_{\Delta\Omega_1}\int_{\theta T}^T |\Delta\varphi_2(x)|\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_2}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx + \int_{\Delta\Omega_1}\int_a^T |\Delta\varphi_2(x)|\varphi_1(t)u\frac{dt}{t}dx.
\end{aligned}$$

The condition $u_0 \geqslant 0$ yields

$$\begin{aligned}
&\int_{\Omega_1}\int_a^T |u|^p \varphi_2(x)\varphi_1(t)\frac{dt}{t}dx \\
&\leqslant \int_{\Omega_1}\int_{\theta T}^T \varphi_2(x)\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_1}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx \\
&+ \int_{\Delta\Omega_1}\int_{\theta T}^T |\Delta\varphi_2(x)|\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\left(\mathcal{I}_{a^+}^{1-\alpha_2}|u|\varphi_1^{1/p}\right)(t,x)\frac{dt}{t}dx + \int_{\Delta\Omega_1}\int_a^T |\Delta\varphi_2(x)|\varphi_1(t)u\frac{dt}{t}dx.
\end{aligned} \tag{3.29}$$

It is easy to prove that

$$\left|u\varphi_1^{1/p}\right| \in X_{-1/p}^{p'},$$

since $u(t,.) \in C_{1-\gamma,\log}[a,T]$. Thus, we can apply Lemma 1.1.15 to obtain

$$\begin{aligned}
&\int_{\Omega_1}\int_a^T |u|^p \varphi_2(x)\varphi_1(t)\frac{dt}{t}dx \\
&\leqslant \int_{\Omega_1}\int_{\theta T}^T \varphi_2(x)\left(\mathcal{I}_{T^-}^{1-\alpha_1}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right)(t)|u|\varphi_1^{1/p}\frac{dt}{t}dx \\
&+ \int_{\Delta\Omega_1}\int_{\theta T}^T |\Delta\varphi_2(x)|\left(\mathcal{I}_{T^-}^{1-\alpha_2}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right)(t)|u|\varphi_1^{1/p}\frac{dt}{t}dx + \int_{\Delta\Omega_1}\int_a^T |\Delta\varphi_2(x)|\varphi_1(t)u\frac{dt}{t}dx.
\end{aligned} \tag{3.30}$$

Using Young inequality with parameters p and $p' = \frac{p}{p-1}$, we have

$$\begin{aligned}
&\int_{\Omega_1}\int_{\theta T}^{T}\varphi_2(x)\left(\mathcal{I}_{T^-}^{1-\alpha_1}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right)(t)|u|\varphi_1^{1/p}\frac{dt}{t}dx\\
&\leqslant \frac{1}{6p}\int_{\Omega_1}\int_{\theta T}^{T}|u|^p\varphi_1(t)\varphi_2(x)\frac{dt}{t}dx+\frac{6^{p'-1}}{p'}\int_{\Omega_1}\int_{\theta T}^{T}\varphi_2(x)\left|\mathcal{I}_{T^-}^{1-\alpha_1}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx\\
&\leqslant \frac{1}{6p}\int_{\Omega_1}\int_{a}^{T}|u|^p\varphi_1(t)\varphi_2(x)\frac{dt}{t}dx+\frac{6^{p'-1}}{p'}\int_{\Omega_1}\int_{\theta T}^{T}\varphi_2(x)\left|\mathcal{I}_{T^-}^{1-\alpha_1}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx, \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
&\int_{\Delta\Omega_1}\int_{\theta T}^{T}|\Delta\varphi_2(x)|\left(\mathcal{I}_{T^-}^{1-\alpha_2}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right)(t)|u|\varphi_1^{1/p}\frac{dt}{t}dx\\
&\leqslant \frac{1}{6p}\int_{\Omega_1}\int_{\theta T}^{T}|u|^p\varphi_1(t)\varphi_2(x)\frac{dt}{t}dx+\frac{6^{p'-1}}{p'}\int_{\Delta\Omega_1}\int_{\theta T}^{T}\varphi_2(x)^{-p'/p}|\Delta\varphi_2(x)|^{p'}\left|\mathcal{I}_{T^-}^{1-\alpha_2}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx\\
&\leqslant \frac{1}{6p}\int_{\Omega_1}\int_{a}^{T}|u|^p\varphi_1(t)\varphi_2(x)\frac{dt}{t}dx+\frac{6^{p'-1}}{p'}\int_{\Delta\Omega_1}\int_{\theta T}^{T}\varphi_2(x)^{-p'/p}|\Delta\varphi_2(x)|^{p'}\left|\mathcal{I}_{T^-}^{1-\alpha_2}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx, \qquad (3.32)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Delta\Omega_1}\int_{a}^{T}|\Delta\varphi_2(x)|\varphi_1(t)u\frac{dt}{t}dx\\
&\leqslant \frac{1}{6p}\int_{\Omega_1}\int_{a}^{T}|u|^p\varphi_1(t)\varphi_2(x)\frac{dt}{t}dx+\frac{6^{p'-1}}{p'}\int_{\Delta\Omega_1}\int_{a}^{T}\varphi_2(x)^{-p'/p}|\Delta\varphi_2(x)|^{p'}\varphi_1(t)\frac{dt}{t}dx. \qquad (3.33)
\end{aligned}$$

Using inequalities (3.30), (3.31), (3.32) and (3.33), we obtain the inequality

$$\begin{aligned}
&\left(1-\frac{1}{2p}\right)\int_{\Omega_1}\int_{a}^{T}|u|^p\varphi_2(x)\varphi_1(t)\frac{dt}{t}dx\\
&\leqslant \frac{6^{p'-1}}{p'}\int_{\Omega_1}\int_{\theta T}^{T}\varphi_2(x)\left|\mathcal{I}_{T^-}^{1-\alpha_1}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx\\
&+\frac{6^{p'-1}}{p'}\int_{\Delta\Omega_1}\int_{\theta T}^{T}\varphi_2(x)^{-p'/p}|\Delta\varphi_2(x)|^{p'}\left|\mathcal{I}_{T^-}^{1-\alpha_2}\frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)}\right|^{p'}\frac{dt}{t}dx\\
&+\frac{6^{p'-1}}{p'}\int_{\Delta\Omega_1}\int_{a}^{T}\varphi_2(x)^{-p'/p}|\Delta\varphi_2(x)|^{p'}\varphi_1(t)\frac{dt}{t}dx. \qquad (3.34)
\end{aligned}$$

We introduce the following scaled variable

$$\tau = \frac{t}{T}, \ \ T \gg 1.$$

It shows that

$$\begin{aligned}\int_{\theta T}^{T} \left| \mathcal{I}_{T^-}^{1-\alpha_i} \frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)} \right|^{p'} \frac{dt}{t} &= \frac{1}{\Gamma^{p'}(1-\alpha_i)} \int_{\theta T}^{T} \left(\int_{t}^{T} \left(\log \frac{s}{t} \right)^{-\alpha_i} \frac{|\delta\varphi_1(s)|}{\varphi_1^{1/p}(s)} \frac{ds}{s} \right)^{p'} \frac{dt}{t} \\ &= \frac{1}{\Gamma^{p'}(1-\alpha_i)} \int_{\theta}^{1} \left(\int_{\tau T}^{T} \left(\log \frac{s}{\tau T} \right)^{-\alpha_i} \frac{|\varphi_1'(s)|}{\varphi_1^{1/p}(s)} ds \right)^{p'} \frac{d\tau}{\tau},\end{aligned}$$

for $i = 1, 2$. Another change of variable $r = \frac{s}{T}$ yields

$$\int_{\theta}^{1} \left(\int_{\tau T}^{T} \left(\log \frac{s}{\tau T} \right)^{-\alpha_i} \frac{|\varphi_1'(s)|}{\varphi_1^{1/p}(s)} ds \right)^{p'} \frac{d\tau}{\tau} = \int_{\theta}^{1} \left(\int_{\tau}^{1} \left(\log \frac{r}{\tau} \right)^{-\alpha_i} \frac{|\varphi_1'(r)|}{\varphi_1^{1/p}(r)} dr \right)^{p'} \frac{d\tau}{\tau}. \tag{3.35}$$

Since $\varphi_1 \in C^1[a, \infty)$, we assume without loss of generality that

$$\int_{\tau}^{1} \left(\log \frac{r}{\tau} \right)^{-\alpha_i} \frac{|\varphi_1'(r)|}{\varphi_1^{1/p}(r)} dr \leqslant M,$$

for $M > 0$. Then Eq (3.35) becomes

$$\frac{1}{\Gamma^{p'}(1-\alpha_i)} \int_{\theta}^{1} \left(\int_{\tau}^{1} \left(\log \frac{r}{\tau} \right)^{-\alpha_i} \frac{|\varphi_1'(r)|}{\varphi_1^{1/p}(r)} dr \right)^{p'} \frac{d\tau}{\tau} < C \int_{\theta}^{1} d\tau.$$

Putting $\theta = 1 - \mathrm{e}^{-T}$ with $T > a > 0$, we obtain

$$\int_{\theta T}^{T} \left| \mathcal{I}_{T^-}^{1-\alpha_i} \frac{|\delta\varphi_1(t)|}{\varphi_1^{1/p}(t)} \right|^{p'} \frac{dt}{t} \leqslant C\mathrm{e}^{-T}, \tag{3.36}$$

for some positive C independent of T.

Next, using the change of variable $y = \frac{\|x\|}{T^{\alpha_2}}$, we get

$$\int_{\Delta\Omega_1} \varphi_2(x)^{\frac{-p'}{p}} \left|\Delta\varphi_2(x)\right|^{p'} dx = T^{\alpha_2 N - 2\alpha_2 p'} \int_{1 \leqslant \|y\| \leqslant 2} \left[\Phi(y)\right]^{-\frac{\mu}{p-1}} \left|\Delta\left[\Phi(y)\right]^{\mu}\right|^{\frac{p}{p-1}} dy$$

$$\int_{\Delta\Omega_1} \varphi_2(x)^{\frac{-p'}{p}} \Big|\Delta\varphi_2(x)\Big|^{p'} dx \leqslant T^{\alpha_2 N - 2\alpha_2 p'} \int_{1 \leqslant \|y\| \leqslant 2} \Big[\Phi(y)\Big]^{\frac{\mu(p-1)-2p}{p-1}} dy$$
$$\leqslant T^{\alpha_2 N - 2\alpha_2 p'}. \tag{3.37}$$

Combining (3.34), (3.36) and (3.37), we get

$$\frac{1}{2p} \int_{\Omega_1} \int_a^T |u|^p \varphi_2(x)\varphi_1(t) \frac{dt}{t} dx < C\mathrm{e}^{-T} T^{\alpha_2 N} + C T^{\alpha_2 N - 2\alpha_2 p'} + C T^{\alpha_2 N - 2\alpha_2 p' + 1}, \tag{3.38}$$

when $T \longrightarrow +\infty$, we obtain

$$\lim_{T \longrightarrow +\infty} \mathrm{e}^{-T} T^{\alpha_2 N} = 0,$$

and

$$\lim_{T \longrightarrow +\infty} T^{\alpha_2 N - 2\alpha_2 p' + 1} = 0.$$

Therefore

$$\lim_{T \to \infty} \int_{\Omega_1} \int_a^T |u|^p \varphi_1(t)\varphi_2(x) \frac{dt}{t} dx = 0. \tag{3.39}$$

This leads to a contradiction. $\square$

Chapter 4

Fractional Diffusion-Wave Equation using Caputo-Fabrizio's Definitions

In 2015, Caputo and Fabrizio [9] have proposed new definition of fractional derivative, in order to eliminate the singular kernel in the fractional derivative. In 2016, they presented some applications of the new derivative [10]. The corresponding fractional integral has been defined by Losada and Nieto (2015) [29].

Fundamental solution of diffusion-wave equation time-fractional derivative have been studied by various authors. For example: Fujita [16] proved the existence and the uniqueness of the solution of the following problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta u}{\partial x^\beta}, \quad 1 \leqslant \alpha, \beta \leqslant 2.$$

When $\alpha = 1$ and $\beta = 2$, the above equation reduces to the heat equation and the wave equation if $\alpha = \beta = 2$. So, the results obtained offer an interpretation of both phenomena. In 2006, Sun and all [41] presented the numerical solution to a fractional diffusion-wave. They proved stability and L_∞ convergence by the energy method. In this part, we suggest a novel approximation of the Caputo-Fabrizio fractional derivative of order α $(1 < \alpha < 2)$. Our novel discretization is found by using discrete fractional derivative at $t = t_k$ with a new coefficients.

4.1 Existence of Solutions

In this Section, we apply Picard-Lindelof method to prove the existence and the uniqueness of the solution.

We consider the following time-fractional diffusion-wave equation

$$\mathbb{D}^{\alpha}_{0|t}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t). \tag{4.1}$$

Over region $\Omega = [0,L] \times [0,T]$, $1 < \alpha < 2$ with the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \tag{4.2}$$

and homogeneous boundary conditions

$$u(0,t) = u(L,t) = 0. \tag{4.3}$$

Obviously, the Caputo-Fabrizio operator $\mathbb{D}^{\alpha}_{0|t}$ is the composition of $\mathbb{D}^{\alpha-1}_{0|t}$ and $\frac{\partial}{\partial t}$, i.e.

$$\mathbb{D}^{\alpha}_{0|t}u(x,t) = \mathbb{D}^{\alpha-1}_{0|t}\frac{\partial}{\partial t}u(x,t).$$

Setting $v = \frac{\partial}{\partial t}u$, we have the following formulation

$$\begin{cases} \mathbb{D}^{\alpha-1}_{0|t}v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), \\ v(x,t) = \frac{\partial}{\partial t}u(x,t), \quad v(x,0) = 0. \end{cases} \tag{4.4}$$

By applying the anti- derivative operator ${}^{CF}I^{\alpha}_{0|t}$ on the both side of Eq.(4.4), we get

$$v(x,t) = (2-\alpha) \times \left\{\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t)\right\} + (\alpha-1) \times \int_0^t \left\{\frac{\partial^2 u(x,y)}{\partial x^2} + q(x,y)\right\}dy. \tag{4.5}$$

For simplicity, we consider the following projected function as the space variable is being neglected

$$u(x,t) = F(t), \quad q(x,t) = G(t).$$

Then, Equation (4.5) can be re-write as

$$v(x,t) = (2-\alpha) \times \left\{\frac{\partial^2 F(t)}{\partial x^2} + G(t)\right\} + (\alpha-1) \times \int_0^t \left\{\frac{\partial^2 F(y)}{\partial x^2} + G(y)\right\}dy. \tag{4.6}$$

For more simplicity, we defined the operator H as following

$$H(F,t) = \frac{\partial^2 F(t)}{\partial x^2} + G(t).$$

Let

$$C[c,v] = [t_0 - c, t_0 + c] \times [F_0 - v, F_0 + v], \quad L = \sup_{C[c,v]} \|H(F,t)\|$$
$$\|F(t)\| = \sup_{t \in [t_0 - c, t_0 + c]} |F(t)|. \tag{4.7}$$

We define the Picard's operator $P : C[c,v] \to C[c,v]$ as

$$P(D_t F(t)) = (2-\alpha)H(F,t) + (\alpha - 1)\int_0^t H(F,y)dy.$$

First, we prove P is well posed. By using (4.7) we have

$$\begin{aligned}\|P(D_t F(t))\| &\leqslant (2-\alpha)\|H(F,t)\| + (\alpha-1)\int_0^t \|H(F,y)\|dy \\ &\leqslant (2-\alpha)L + (\alpha-1)cL.\end{aligned}$$

We choose c small enough such that

$$(2-\alpha)L + (\alpha-1)cL \leqslant L.$$

Second, we show that operator P defines a contraction. We have the following relation

$$\begin{aligned}\Big\|P(D_t F(t)) - P(D_t G(t))\Big\| &= \Big\|(2-\alpha)\{H(F,t) - H(G,t)\} + (\alpha-1)\int_0^t \{H(F,y) - H(G,y)\}dy\Big\| \\ &\leqslant (2-\alpha)\Big\|H(F,t) - H(G,t)\Big\| + (\alpha-1)\int_0^t \Big\|H(F,y) - H(G,y)\Big\|dy \\ &\leqslant M\Big\{(2-\alpha) + (\alpha-1)c\Big\}\|F - G\|.\end{aligned}$$

Due to the following inequality

$$\|H(F,t) - H(G,t)\| \leqslant M\|F - G\|.$$

We choose c such that

$$M\Big\{(2-\alpha)+(\alpha-1)c\Big\}<1.$$

Therefore, P is a strict contraction on $C[c,v]$. According to the Banach fixed point theorem, then problem $(4.1)-(4.3)$ admits a unique solution.

4.2 An Application

In this section, we investigate the approximate numerical solution of problem (4.1), using implicit finite differences. To achieve this aim, we need to numerically approximate to the Caputo-Fabrizio derivative.

For some positive integers N, M, the gird sizes in time for finite difference technique is defined by $K=\frac{1}{M}$, the grid points in the time interval $[0,T]$ are labeled $t_j=jK, j=0...TM$, while the grid points in the space interval $[0,L]$ are numbers $x_i=ih$ where $h=\frac{1}{N}$ it is grid sizes in the space. Denotes u_i^j the approximate value of $u(x_i,t_j)$ and f^i is the value of $f(x_i)$. Define

$$\frac{\partial}{\partial t}u^n=\frac{u^n-u^{n-1}}{K}.$$

The standard central difference scheme

$$v_i^{k+\frac{1}{2}}=\frac{u_i^{k+1}-u_i^k}{K}+O(K^2). \tag{4.8}$$

The approximate numerical of Caputo-Fabrizio derivative $\mathbb{D}_{0|t}^{\alpha-1}v(x,t)$ obtained by the following formula

$$\begin{aligned}
\mathbb{D}_{0|t}^{\alpha-1}v(x_i,t_k) &= \frac{1}{2-\alpha}\int_0^{t_k}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial\tau}d\tau\\
&= \frac{1}{2-\alpha}\Bigg[\int_{t_{k-\frac{1}{2}}}^{t_k}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial\tau}d\tau+\int_{t_{\frac{1}{2}}}^{t_{k-\frac{1}{2}}}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial\tau}d\tau\\
&\quad+\int_0^{t_{\frac{1}{2}}}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial\tau}d\tau\Bigg]
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}^{\alpha-1}_{0|t}v(x_i,t_k) &= \frac{1}{2-\alpha}\Bigg[\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial \tau}d\tau + \sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial \tau}d\tau \\
&\quad - \int_{t_{\frac{-1}{2}}}^{0} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\frac{\partial v(x_i,\tau)}{\partial \tau}d\tau\Bigg] \\
&= \frac{1}{2-\alpha}\Bigg[\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left[\frac{v_i^{k-\frac{1}{2}}-v^{k-\frac{3}{2}}}{K}+O(K)\right]d\tau \\
&\quad - \int_{t_{\frac{-1}{2}}}^{0} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left[\frac{v_i^{0}-v_i^{-\frac{1}{2}}}{K}+O(K)\right]d\tau\Bigg] \\
&\quad + \frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left[\frac{v_i^{m+\frac{1}{2}}-v_i^{m-\frac{1}{2}}}{K}+\left(\frac{\partial v(x_i,\tau)}{\partial\tau}-\frac{v_i^{m+\frac{1}{2}}-v_i^{m-\frac{1}{2}}}{K}\right)d\tau\right].
\end{aligned} \tag{4.9}$$

Denote that $u_i^{-1}=u_i^0-Kv_i^0$ for $i\geqslant 0$. Then

$$v_i^{\frac{-1}{2}} = \frac{u_i^0-u_i^{-1}}{K}+O(K^2)=v_i^0+O(K^2). \tag{4.10}$$

Substituting (4.8) and (4.10) into (4.9), we get

$$\begin{aligned}
\mathbb{D}^{\alpha}_{0|t}u(x_i,t_k) &= \frac{1}{2-\alpha}\left[\frac{u_i^k-2u_i^{k-1}+u_i^{k-2}}{K^2}\right]\int_{t_{k-\frac{1}{2}}}^{t_k}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}d\tau - \frac{1}{2-\alpha}\int_{t_{\frac{-1}{2}}}^{0}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}O(K)d\tau \\
&\quad + \frac{1}{2-\alpha}\sum_{m=0}^{k-1}\left(\frac{u_i^{m+1}-2u_i^m+u^{m-1}}{K^2}\right)\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}d\tau \\
&\quad + \frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left(\frac{\partial v(x_i,\tau)}{\partial\tau}-\frac{v_i^{m+\frac{1}{2}}-v_i^{m-\frac{1}{2}}}{K}\right)d\tau.
\end{aligned} \tag{4.11}$$

Setting

$$\varsigma_K=\frac{1}{(\alpha-1)K^2},\quad w_{k,\alpha}=\left(1-\mathrm{e}^{-\frac{(\alpha-1)(t_k-t_{k-\frac{1}{2}})}{2-\alpha}}\right),\quad d_{m,\alpha}=\left(\mathrm{e}^{-\frac{(\alpha-1)(t_k-t_{m+\frac{1}{2}})}{2-\alpha}}-\mathrm{e}^{-\frac{(\alpha-1)(t_k-t_{m-\frac{1}{2}})}{2-\alpha}}\right),$$

$$R=\frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left(\frac{\partial v(x_i,\tau)}{\partial\tau}-\frac{v_i^{m+\frac{1}{2}}-v_i^{m-\frac{1}{2}}}{K}\right)d\tau.$$

Therefore

$$\mathbb{D}^{\alpha}_{0|t}u(x_i,t_k) = \varsigma_K\Big(u_i^k - 2u_i^{k-1} + u_i^{k-2}\Big)w_{k,\alpha} + \varsigma_K\sum_{m=0}^{k-1}\Big(u_i^{m+1} - 2u_i^m + u_i^{m-1}\Big)d_{m,\alpha} + O(K^3) + R. \tag{4.12}$$

Also, the second partial derivative with respect to x at the grid point (i,k) given as

$$\frac{\partial^2 u(x_i,t_k)}{\partial x^2} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + O(h^2). \tag{4.13}$$

Using (4.12) and (4.13) to discretize problem (4.1) at point (x_i,t_k) as

$$\varsigma_K\Big(u_i^k - 2u_i^{k-1} + u_i^{k-2}\Big)w_{k,\alpha} + \varsigma_K\sum_{m=0}^{k-1}\Big(u_i^{m+1} - 2u_i^m + u_i^{m-1}\Big)d_{m,\alpha} + R = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + q_i^k + O(K^3+h^2). \tag{4.14}$$

The first initial condition, can be written as

$$u(x_i,0) = f(x_i) = f_i \quad i = 0...N. \tag{4.15}$$

Approximating the second initial condition, we obtain

$$u_t(x_i,t_0) \simeq \frac{u_i^0 - u_i^{-1}}{K} = 0, \qquad i = 0...N. \tag{4.16}$$

By means of the similar method used in [41]. It is easy to proof that $|R| = O(K^3)$.

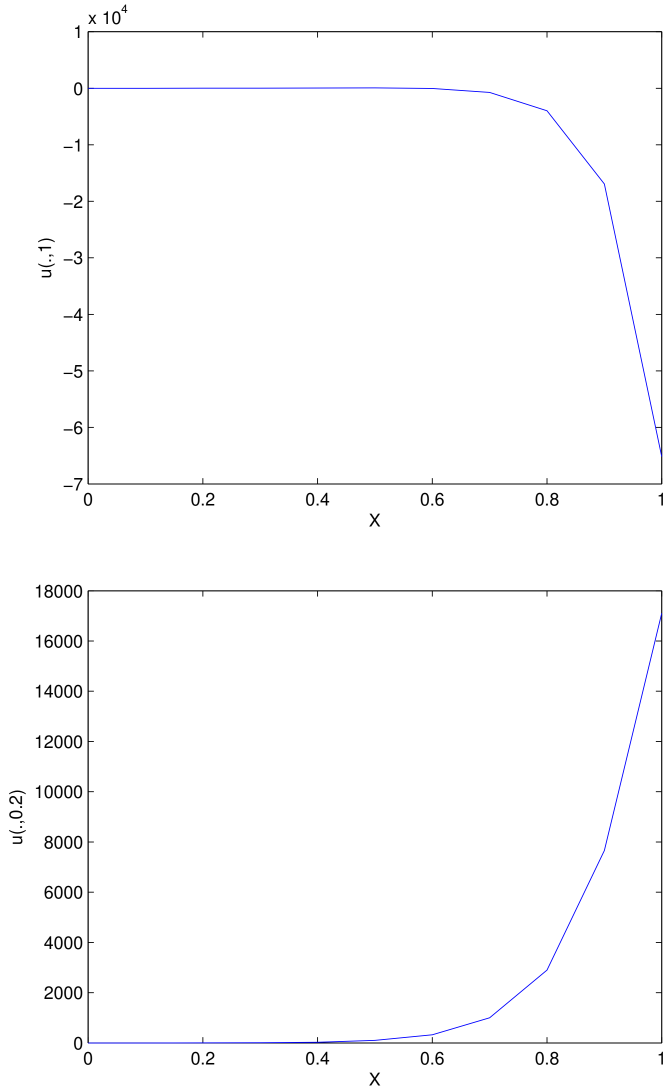

Figure 4.1 – Numerical simulations of Equation (4.1) for $t \in [0, 1]$, $\alpha = 1.5$ $u_0(x) = (1 - x)^2$ and $q(x, t) = 0$.

Conclusions and Perspectives

" In most sciences one generation overlaps what another has build and what one has established another undoes. In mathematics alone each generation adds new story to the old structure." **Hermann Hankel**

As seen in previous Chapters, the question coming to the mind is :

What is a fractional calculus?

This question has been asked trying to define *criteria* for which functional operators should be called fractional derivatives.

I- **Mathematically Speaking**, this is a valid question. Any term used in maths should have a precise definition. Also, we can define something more general than individual formulae

$$I^{\alpha}_{0|t} f(t) = \int_0^t K(t,\tau) f(\tau) d\tau. \tag{4.17}$$

If $K(t,\tau)$ is

- Power functions

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

  It represents Riemann–Liouville.

- Scale logarithmic function

$$I^{\alpha}_{0|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \, \frac{d\tau}{\tau}.$$

  It represents Hadamard fractional integral.

- One parameter Mittag-Leffler function

$$I^{\alpha}_{0|t}f(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)f(\tau)d\tau.$$

  Atangana–Baleanu fractional integral.

  The criteria selected of definition must be approached by convenience and applications, most researchers use Caputo fractional derivatives, but each researcher can apply his/her criteria and technique he/she prefers in class. Each class can also cover many different systems and processes. Despite these developments, we can not say that this definition is more suitable than this one.

II- **Applications**, the overall structure is not yet clear.

Perspectives

The fractional calculus is an immense field of study, it develops every day. In fact, the existence and uniqueness of solution of the direct problem is not a major problem. In future our field of interest is to study fractional inverse problems because the results of this one are totally different from the problems of the classical case.

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PUBLICATIONS and CONFERENCES

PUBLICATIONS

”On nonexistence of global solutions for a semilinear equation with Hilfer-Hadamard fractional derivative”, **K. Bouguetof and N-e. Tatar**, arXiv preprint arXiv:2003.01986 (2020).

”Blowing-up solutions of a time-space fractional semi-linear equation with a structural damping and a nonlocal in time nonlinearity”, **K. Bouguetof**, arXiv preprint arXiv:2002.09704 (2020).

”On local existence and blow-up solutions for a time-space fractional variable order superdiffusion equation with exponential nonlinearity”, **K. Bouguetof and D. Foukrach**, PanAmerican Mathematical Journal Vol 30(2020),N.3, 21 - 34.

”A novel finite difference scheme for time fractional diffusion -wave equation with singular kernel”, **K. Bouguetof and H. Kamel**, Turkic World Mathematical Society journal of applied and engineering mathematics. Accepted in November 2019

”Nonexistence results of global solutions for fractional order integral equations and systems” **K. Bouguetof and H. Kamel and Rebiai Belgacem** , Italian Journal of Pure and Applied Mathematics–(2020) N. 44–2020, 621–631.

CONFERENCES PRESENTATIONS

2019 - International Istanbul summer school in applied mathematics

2018 - Congrés des Mathématiciens Algériens
”New fractional derivatives with nonlocal kernel ”

2018-Journées d’accompagnement des doctorants

2018 - International conference on fractional differentiation and its applications
”The critical exponent for a fractional differential equations ”

2018 - Workshop
”Scientific paper: How to succeed its redaction and publication ”

2018 - Public speaking Workshop

2017 - The first international conference on the “Evolution of Contemporary Mathematics and their Impact in Sciences and Technology”
”Non existence for the Laplace equation of fractional type”

2016 - The first national conference of Mathematics

Bibliography

[1] B. Ahmad, A. Alsaedi, S. Ntouyas, and J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*, Springer, 2017.

[2] A. B. Al'shin, M. O. Korpusov, and A. G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations*, Walter de Gruyter, 2011.

[3] K. Bouguetof, *Blowing-up solutions of a time-space fractional semi-linear equation with a structural damping and a nonlocal in time nonlinearity*, arXiv preprint arXiv:2002.09704, (2020).

[4] K. Bouguetof, *The critical exponent to Cauchy problem for a time fractional semi-linear equation with a structural damping*, NSP Applied Mathematics Information Sciences, (Accepted in October 2020).

[5] K. Bouguetof and D. Foukrach, *On local existence and blow-up solutions for a time-space fractional variable order superdiffusion equation with exponential nonlinearity*, PanAmerican Mathematical Journal, 30 (2020), pp. 21–34.

[6] K. Bouguetof and K. Haouam, *A novel finite difference scheme for time fractional diffusion-wave equation with singular kernel*, TWMS Journal of Applied and Engineering Mathematics, (will be appear in the next volume).

[7] K. Bouguetof, K. Haouam, and B. Rebiai, *Nonexistence results of global solutions for fractional order integral equations and systems*, Italian Journal of Pure and Applied Mathematics, 44 (2020), pp. 621–631.

[8] K. Bouguetof and N.-e. Tatar, *On nonexistence of global solutions for a semilinear equation with Hilfer-Hadamard fractional derivative*, arXiv preprint arXiv:2003.01986, (2020).

[9] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, 1 (2015), pp. 1–13.

[10] ——, *Applications of new time and spatial fractional derivatives with exponential kernels*, Progress in Fractional Differentiation and Applications, 2 (2016), pp. 1–11.

[11] G. Chen and D. Russell, *A mathematical model for linear elastic systems with structural damping*, Quarterly of Applied Mathematics, 39 (1982), pp. 433–454.

[12] J. Dang, Q. Hu, and H. Zhang, *Asymptotic stability and blow-up of solutions for the generalized Boussinesq equation with nonlinear boundary condition*, Journal of Mathematical Analysis, 2 (2018), pp. 93–113.

[13] T. A. Dao and A. Z. Fino, *Blow up results for semi-linear structural damped wave model with nonlinear memory*, arXiv preprint arXiv:2002.06582, (2020).

[14] H. Fan, H. Li, and P. Chen, *Existence of mild solutions for the elastic systems with structural damping in Banach spaces*, in Abstract and Applied Analysis, vol. 2013, 2013, pp. 1–6.

[15] Y. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \triangle u + |u|^{\alpha+1}$*, Journal of the Faculty of Science, University of Tokyo. Sect. 1, Mathematics, Astronomy, Physics, 13 (1966), pp. 109–124.

[16] ——, *Cauchy problems of fractional order and stable processes*, Japan Journal of Applied Mathematics, 7 (1990), p. 459.

[17] K. Furati and M. Kirane, *Necessary conditions for the existence of global solutions to systems of fractional differential equations*, Fractional Calculus and Applied Analysis, 11 (2008), pp. 281–298.

[18] B. Guo, X. Pu, and F. Huang, *Fractional partial differential equations and their numerical solutions*, World Scientific, 2015.

[19] M. A. Herzallah, *Notes on some fractional calculus operators and their properties*, Journal of Fractional Calculus and Applications, 5 (2014), pp. 1–10.

[20] P. Ivady, *A note on a gamma function inequality*, Journal of Mathematical Inequalities, 3 (2009), pp. 227–236.

[21] Y. Jiang and J. Ma, *High-order finite element methods for time-fractional partial differential equations*, Journal of Computational and Applied Mathematics, 235 (2011), pp. 3285–3290.

[22] A. KILBAS, O. MARICHEV, AND S. SAMKO, *Fractional integral and derivatives (theory and applications)*, Gordon and Breach, Switzerland, 1993.

[23] A. A. KILBAS, B. BONILLA, AND J. J. TRUJILLO, *Existence and uniqueness theorems for nonlinear fractional differential equations*, Demonstratio Mathematica, 33 (2000), pp. 583–602.

[24] A. A. KILBAS AND S. A. MARZAN, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differential Equations, 41 (2005), pp. 84–89.

[25] M. KIRANE AND Y. LASKRI, *Nonexistence of global solutions to a hyperbolic equation with a space–time fractional damping*, Applied Mathematics and Computation, 167 (2005), pp. 1304–1310.

[26] M. O. KORPUSOV AND A. G. SVESHNIKOV, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics*, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 43 (2003), pp. 1835–1869.

[27] Y. LIU, Y. DU, H. LI, AND J. WANG, *An H^1 -Galerkin mixed finite element method for time fractional reaction–diffusion equation*, Journal of Applied Mathematics and Computing, 47 (2015), pp. 103–117.

[28] Y. LIU, Z. FANG, H. LI, AND S. HE, *A mixed finite element method for a time-fractional fourth-order partial differential equation*, Applied Mathematics and Computation, 243 (2014), pp. 703–717.

[29] J. LOSADA AND J. J. NIETO, *Properties of a new fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, 1 (2015), pp. 87–92.

[30] V. LUONG AND N. TUNG, *Decay mild solutions for elastic systems with structural damping involving nonlocal conditions*, Vestnik St. Petersburg University, Mathematics, 50 (2017), pp. 55–67.

[31] F. MAINARDI, *Concerning an equation in the theory of combustion*, Fractals and Fractional Calculus in Continuum Mechanics, 88 (2010), pp. 48–56.

[32] E. MITIDIERI AND S. POKHOZHAEV, *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Proceedings of the Steklov Institute of Mathematics, 234 (2001), pp. 1–383.

[33] V. Padrón, *Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation*, Transactions of the American Mathematical Society, 356 (2004), pp. 2739–2756.

[34] M. Qassim, K. Furati, and N.-e. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative*, in Abstract and Applied Analysis, vol. 2012, 2012, pp. 1–17.

[35] B. Rebiai, S. Benzeghari, and K. Haouam, *Necessary conditions of existence for a nonlinear reaction-diffusion system with fractional derivatives*, Novi Sad Journal of Mathematics, (2016), pp. 1–9.

[36] B. Rebiai and K. Haouam, *Nonexistence of global solutions to a nonlinear fractional reaction-diffusion system*, IAENG International Journal of Applied Mathematics, 45 (2015), pp. 259–262.

[37] B. Rebiai, S. Rouar, and K. Haouam, *Critical exponents for a nonlinear reaction-diffusion system with fractional derivatives*, Global Journal of Pure and Applied Mathematics, 6 (2016), pp. 5343–5351.

[38] E. Sousa, *A second order explicit finite difference method for the fractional advection diffusion equation*, Computers & Mathematics with Applications, 64 (2012), pp. 3141–3152.

[39] E. Sousa, *An explicit high order method for fractional advection diffusion equations*, Journal of Computational Physics, 278 (2014), pp. 257–274.

[40] F. Sun, L. Liu, and Y. Wu, *Global existence and finite time blow-up of solutions for the semilinear pseudo-parabolic equation with a memory term*, Applicable Analysis, 98 (2019), pp. 735–755.

[41] Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Applied Numerical Mathematics, 56 (2006), pp. 193–209.

[42] F. B. Weissler, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel Journal of Mathematics, 38 (1981), pp. 29–40.

[43] R. Xu and J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, Journal of Functional Analysis, 264 (2013), pp. 2732–2763.

[44] Z. Yong, W. Jinrong, and Z. Lu, *Basic theory of fractional differential equations*, World Scientific, 2016.

[45] Q. Zhang and H. Sun, *The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation*, Topological Methods in Nonlinear Analysis, 46 (2015), pp. 69–92.