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Title

Dynamic analysis, stabilization and synchronization of novel chaotic fractional orders systems

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"Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas ? "

Edward Northon Lorenz

Prediction is difficult, especially of the future

Niels Bohr

ملخص

الهدف من هذه الأطروحة هو دراسة الإستقرار والمزامنة لجمل فوضوية كسرية جديدة، حيث قمنا باقتراح جمل فوضوية كسرية بمفهوم كابيتو. تم برهان أن هذه الجمل لها سلوك فوضوي باستعمال أسس ليابينوف وقمنا بتأكيد ذلك باستعمال الإختبار 0-1. أيضا تم دراسة الإستقرار والمزامنة لهذه الجمل المقترحة باستخدام طريقة المراقب المتكيف كون الطريقة تعتمد على تغيير وسائط الجملة الفوضوية عند كل لحظة. اعتمدنا خلال حلنا للمعادلات التفاضلية الكسرية على طريقة أدامز باشفورت مولتون. بهدف تأكيد الوجود الفيزيائي للجمل الكسرية المقترحة قمنا برسم الدوائر الإلكترونية الممثلة لها باستعمال برنامج ميلتيزيم ومقارنة النتائج المتحصل عليها مع تلك الناتجة عن المحاكاة العددية.

الكلمات المفتاحية : أنظمة ديناميكية فوضوية، رتبة كسرية، مزامنة متكيفة، أسس ليابينوف، دارة إلكترونية.

ABSTRACT

The objective of this thesis is to study the stabilization and synchronization of new chaotics fractionals systems, where we proposed chaotic fractional systems in Caputo sense. We have demonstrated that the behavior of these systems is chaotic using the exponents of Lyapunov and confirmed their behavior using the 0-1 test. The stability and synchronization of the proposed systems have also been studied using the adaptive control method which depends on the change of the chaotic system parameters at each moment. We adopted during our solution to the fractional differential equations on the Adams Bashfort Moulton method. In order to confirm the physical presence of the proposed chaotic systems, we drew the electronic circuits representing them using the Multisim program and compared the results obtained with those from numerical simulations.

Keywords: chaotic dynamical systems, fractional order, adaptive synchronization, lyapunov exponents, electronic circuit.

RÉSUMÉ

L'objectif de cette thèse est d'étudier la stabilité et la synchronisation de nouvelles systèmes fractionnaires chaotiques, où nous avons proposé des systèmes fractionnaires chaotiques aux sense de Caputo. Nous avons démontré que le comportement de ces systèmes est chaotique en utilisant les exposants de Lyapunov et confirmé leur comportement en utilisant le test 0-1. La stabilisation et la synchronisation des systèmes proposées ont également été étudiées en utilisant la méthode de control adaptatif qui dépend du changement des paramètres du système chaotique à chaque instant. Nous avons adopté lors de notre solutions des équations différentielles fractionnaires sur la méthode Adams Bashfort Moulton. Afin de confirmer la présence physique des systèmes chaotiques proposées, nous avons dessiné les circuits électroniques les représentant à l'aide du programme Multisim et comparé les résultats obtenus avec ceux issus de simulations numériques.

Mots clés: systèmes dynamiques chaotiques, ordre fractionnaire, synchronisation adaptative, exposants de lyapunov, circuit électronique.

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Rami AMIRA

DEDICATION

To everyone who is interested in this work

Rami AMIRA

LIST OF PUBLICATIONS

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2. Amira, Rami and Fareh Hannachi. "A Novel Fractional-Order Chaotic System and its Synchronization via Adaptive Control Method. "Journal of Nonlinear Dynamics and System Theory". 23.04 (2023): 359–366.

3. Amira, Rami, et al. "Nonlinear dynamic in a remanufacturing duopoly game: spectral entropy analysis and chaos control." AIMS Mathematics 9.3 (2024): 7711-7727.

CONTENTS

LIST OF FIGURES

GENERAL INTRODUCTION

Chaotic system is deterministic which means that is completely determined by the equations formed the states, its parameters, and initial conditions. At the same time, he have the property of sensitivity to initial conditions, the anti interception capability and the unpredictability in long term. It is therefore simple to produce and duplicate. The chaotic signal is a very interesting area in the study of chaos application because of its randomness, unpredictability, complexity, wide frequency band character, deterministic parameters and simple for the implementation. Furthermore, chaotic systems are one of the most powerful crypto-systems used in secure communications [15, 7], with researchers seeking to construct new chaotic systems that are constantly powerful in encryption, which has a large Kaplan-York dimension and a large bandwidth, the systems that have the largest dimension of Kaplan-York are the systems that have the more complex behavior than the others and are stronger for encryption [15]. Many chaotic systems have been proposed by researchers for use in data encryption, including [52, 5], researchers did not stop there, but rather used fractional calculus in secure communication and encryption and this for its strength compared to ordinary derivatives. Over the last three centuries, researchers interest in fractional calculus has increased considerably, this is for its different applications [42]. The most famous definition are Caputo and Riemman-liouville [43]. In 1990, Pecora and Carroll introduced the notion of chaotic synchronization [41], the secret to achieving chaotic security communication is the synchronization. The method of chaos synchronization proposed by Pecora and Carroll did not cease at chaotic systems with integer orders, but they extended to chaotic systems with fractional orders. different schemes of chaos synchronization have been devlloped for integer-integer order chaotic systems , fractional-integer order chaotic systems, fractional-fractional order chaotics ystems in same or with different dimensions. The synchronization depends on many control methods, among these methods we have: active control [27], passive control [54], adaptive control [2], fuzzy control[60], sliding mode control [24], state observer method [33] and backstepping control [57], each method is used in appropriate case as needed. In recent years, several fractional chaotic systems are studied and used in control and synchronization in different research papers. In our work we aims to enrish the list of fractional chaotic systems by some ones in order to use them in the future in the process of cryptography or the securisation of data. Indeed, we present two new chaotic systems in dimensien three and one hyperchaotic system in dimension four , a detailed dynamic analysis has been carried out of the first proposed chaotic system, the chaos in the system is detected throught the spectrum of lyapunov and also bifurcation diagrams. Additionally, in aim to see the advantage of the proposed system we have make a comparison with thirty other chaotic systems via Kaplan-York dimension. Also, adaptive synchronization is used to implement an identical synchronization. Furthermore, an extension to fractional cases of the proposed systems was performed by showing that the chaotic systems in the fractional case is also exhibit the chaotic behavior. Therefore it is ready for use in secure communications schemes. As an application of the three proposed fractional systems, we shall implement in Multisim the electronic circuits for each one of them to show that the proposed systems are physically realizable through a comparison of numerical simulation results and Multisim results [4, 5].

This thesis is organized as:

the first chapter is devoted to present some of the basic tools of fractional calculus including the special functions, fractional differential equation and numerical method to solve them. The second chapter is devoted to recall necessary definitions on fractional dynamical systems, stability, chaos detection in chaotic systems including Lyapunov exponents, 0-1 test and spectral entropy analysis. In the chapter three, we give some definitions about synchronization methods of fractional dynamical systems including complete synchronization, anti sytnchronization, projective synchronization, delayed synchronization, adaptive synchronization, generalized synchronization, Q-S synchronization, FSHPS and IFSHP synchronization. The chapters four are deals to study of integer and fractional chaotic dynamical systems with implementation of its electronic circuits design using Multisim software. A general conclusion and perspectives are given in the last.

CHAPTER 1

INTRODUCTORY NOTIONS ON FRACTIONAL DERIVATIVES

Fractional calculus was formed based on two functions which are gamma and Mittag-Luffler, we introduce in this chapter these two functions and we will give a brief introduction to fractional calculus. This chapter seeks also to define the fundamental fractional derivatives in the senses of Grunwald− Letnikov, Riemman−Liouville and the Caputo sense which will be taken in more detail than the others since it is the definition that we use in the application chapter. At the last Adamd Bashfort Moulton method is given to solve fractional differential equation.

1.1 Some special functions of fractional calculus

1.1.1 Gamma function

Gamma function is one of the basic functions in the fractional calculus that generalize the factorial ($m!$) from integer to take non-integer and also complex numbers.

We recall in the present section some properties of gamma function that are very important for demonstration or calculations [42, 43].

Definition 1.1.1. *For all $\zeta \in \mathbb{C}$ where $Re(\zeta) > 0$. Euler's Gamma function noted by $\Gamma(\zeta)$ is defined by:*

$$\Gamma(\zeta) = \int_0^{+\infty} e^{-t} t^{\zeta-1} dt. \tag{1.1}$$

Remark 1.1.1. *We take $Re(\zeta) > 0$ due to the integral in (1.1) is absolutely converge on the complex half-plan.*

Here we mention some properties of gamma function.

1) As a fundamental propertie, gamma function satisfies the equation:

$$\Gamma(\zeta + 1) = \zeta\, \Gamma(\zeta). \tag{1.2}$$

Proof. The equation (1.1) can be proven by integration by parts as follow:

$$\begin{aligned} \Gamma(\zeta + 1) &= \int_0^{+\infty} e^{-t} t^{\zeta} dt \\ &= [-e^{-t} t^{\zeta}]_0^{+\infty} + \zeta \int_0^{+\infty} e^{-t} t^{\zeta-1} dt \\ &= \zeta\, \Gamma(\zeta). \end{aligned} \tag{1.3}$$

■

2) Gamma is considered as a generalization of the factorial $m!$: it is clear that $\Gamma(1) = 1$ we use the relation (1.3), we can get by induction for $\zeta = \overline{1; m}$ that:

$$\begin{aligned} \Gamma(2) &= 1\, \Gamma(1) &&= 1!, \\ \Gamma(3) &= 2\, \Gamma(2) &&= 2\,(1!) = 2!, \\ \Gamma(4) &= 3\, \Gamma(3) &&= 3\,(2!) = 3!, \\ \Gamma(5) &= 4\, \Gamma(4) &&= 4\,(3!) = 4!, \\ &\vdots \\ \Gamma(m) &= (m-1)\, \Gamma(m-1) &&= (m-1)\,(m-2)! = (m-1)!, \\ \Gamma(m+1) &= m\, \Gamma(m) &&= m\,(m-1)! = m!. \end{aligned} \tag{1.4}$$

We can extend $\Gamma(\zeta)$ in case when ζ is negative. Indeed, by substitution in relation (1.1) we get:

$$\begin{aligned} \Gamma(\zeta) &= (\zeta-1)\Gamma(\zeta-1) \Longrightarrow & \Gamma(\zeta-1) &= \tfrac{\Gamma(\zeta)}{\zeta-1} & ,-1<\zeta-1<0, \\ \Gamma(\zeta-1) &= (\zeta-2)\Gamma(\zeta-2) \Longrightarrow & \Gamma(\zeta-2) &= \tfrac{\Gamma(\zeta-1)}{\zeta-2} & ,-2<\zeta-2<-1, \end{aligned} \tag{1.5}$$

by induction we obtain:

$$\Gamma(\zeta) = \frac{\Gamma(\zeta+1)}{\zeta} \qquad -m<\zeta<-(m-1), \tag{1.6}$$

as a consequence gamma function id defined by formula (1.1) for all negative values except $-m,\ m \in \mathbb{N}^*$.

3) At $\zeta = \frac{1}{2}$ the equation (1.1) give us a very useful value: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

To understand the gamma function we have the presentation in the plane as we see in figure (1.1).

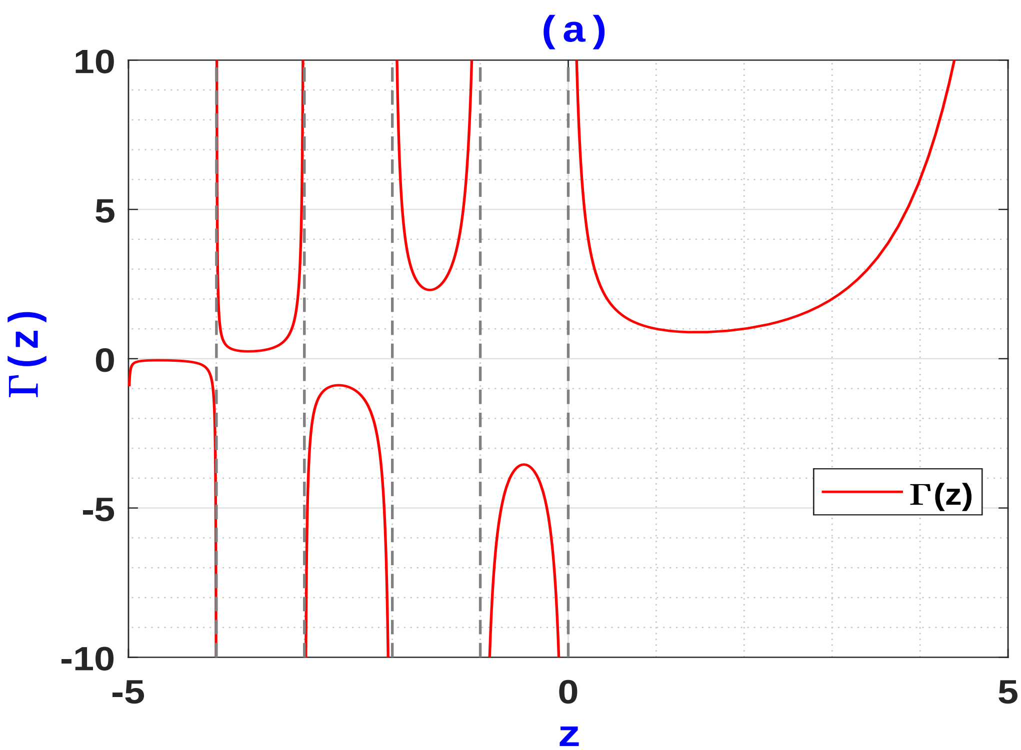

Figure 1.1: graphical representation of the function gamma

1.1.2 Beta function

Sometimes we encounter values that are combinations of the values of the gamma function, it is preferable to utilize the beta function instead of the complicated combination.

Definition 1.1.2. *[42, 43]*
The function β is defined by:

$$\beta(\zeta,\theta) = \int_0^1 t^{\zeta-1}(1-t)^{\theta-1}dt \qquad ,(Re(\zeta)>0, Re(\theta)>0). \tag{1.7}$$

1) Beta function have the symmetric propertie, ie: $\beta(\zeta,\theta)=\beta(\zeta,\theta)$.

2) There is a link between the two functions of euler gamma and beta, this link is given by:

$$\beta(\zeta,\theta) = \frac{\Gamma(\zeta)\Gamma(\theta)}{\Gamma(\zeta+\theta)} \ ,\forall\zeta,\theta\neq-1,-2,-3,\cdots. \tag{1.8}$$

1.1.3 Mittag-Leffler function

The exponential function $\zeta\longmapsto e^{\zeta}$ play an important role in resolution of ordinary differential equations. As a consequence, we need to generalize trigonometric and exponential functions and use them in the process of resolution, this is the main role of Mittag-Leffler function. This function is used to give an explicit expression of the solution. Also, this function has been introduced by Mittag-Leffler and it is considered as a generalization of one parameter of the exponential function.

Definition 1.1.3. *The function of Mittag-Leffler is defined by:*

$$E_\alpha(\zeta) = \sum_{k=0}^{\infty}\frac{\zeta^k}{\Gamma(\alpha k+1)} \qquad (\zeta\in\mathbb{C} \ ,Re(\alpha)>0). \tag{1.9}$$

Indeed, for $\alpha=1,2$, we have:

$$E_1(\zeta) = e^{\zeta}, \quad E_2(\zeta) = \cosh(\sqrt{\zeta}). \tag{1.10}$$

Definition 1.1.4. *The mettag-Leffler function is generalized in two parameters by the following formula:*

$$E_{\alpha,\beta}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha k + \beta)} \qquad (\zeta, \beta \in \mathbb{C}, Re(\alpha) > 0). \tag{1.11}$$

Indeed, if we put $\beta = 1$ in the formula (1.11), we obtain the formula (1.9) as follow:

$$E_{\alpha,1}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha k + 1)} = E_{\alpha}(\zeta) \quad , \qquad (\zeta \in \mathbb{C}, Re(\alpha) > 0). \tag{1.12}$$

Some particular cases can results from the formula (1.11) as follows:

$$\begin{aligned} E_{1,1}(\zeta) &= \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} = e^{\zeta}, \\ E_{1,2}(\zeta) &= \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{\zeta^k}{(k+1)!} = \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{(k+1)!} = \frac{e^{\zeta} - 1}{\zeta}, \\ E_{1,m}(\zeta) &= \frac{1}{\zeta^{m-1}} [e^{\zeta} - \sum_{k=0}^{m-2} \frac{\zeta^k}{k!}]. \end{aligned} \tag{1.13}$$

Also we have :

$$\begin{aligned} E_{2,2}(\zeta) &= \frac{\sinh(\zeta)}{\sqrt{\zeta}},, \\ E_{2,1}(\zeta^2) &= \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{2k!} = \cosh(\zeta), \\ E_{2,2}(\zeta^2) &= \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{1}{\zeta} \frac{\zeta^{2k+1}}{(2k+1)!} = \frac{\sinh(\zeta)}{\zeta}. \end{aligned} \tag{1.14}$$

We can see in the figure (1.2) the relevent behavior of the Mettag-Leffler function for one and for two parameters.

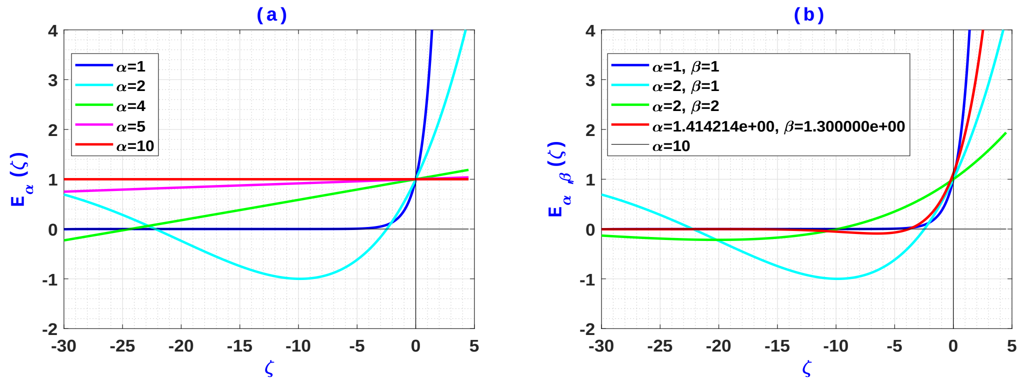

Figure 1.2: graphical representation of the function of Mitag-Luffler

1.2 The Laplace transform

It is simpler to solve linear differential equation with initial conditions when we transform them into algebraic equation through the Laplace transform. So, the Laplace transform involve changing the differential equation from the time domain to the Laplace domain, where operations like differentiation and integration become algebraic operations. Furthermore, in the circuit analysis when the circuit contain resistors, capacitors, inductors, and other components,the Laplace transform is widely used. It makes the differential equations arising from circuit analysis easier to solve.

Definition 1.2.1. *Let ϕ a function of real variable $t \in \mathbb{R}^+$ and of exponential order α, we mean by ϕ of exponentaial order α that:*

$$\exists M, T : e^{-\alpha}t|\phi(t)| \leq M \quad , \forall t > T, \tag{1.15}$$

in other words the equation (1.15) mean that ϕ must grow faster than a certain exponential function when $t \longrightarrow \infty$.
The Laplace Transform of the function ϕ is given by the function Φ as follow:

$$\Phi(s) \quad = \quad L\{\phi(t), s\} = \int_0^{+\infty} e^{-st}\phi(t)dt, \qquad , \ s \in \mathbb{C}, \tag{1.16}$$

note that the integral in equation (1.16) exist if ϕ is a of exponential order α.

We can switch between a function ϕ and its Laplace transform Φ with the inverse Laplace transform given by:

$$\phi(t) \quad = \quad L^{-1}\{\Phi(s), t\} = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\Phi(s)ds \qquad (\xi = Re(s) > \xi_0), \tag{1.17}$$

where ξ_0 is the index of the convergence of the integral in the equation (1.16).
To recall, the Laplace and its inverse transform are linear, indeed $\forall \alpha, \beta \in \mathbb{R}$:

$$L\{\alpha\phi(t) + \beta\psi(t), s\} = \alpha L\{\phi(t), s\} + \beta L\{\psi(t), s\} = \alpha\Phi(s) + \beta\Psi(s). \tag{1.18}$$

$$L^{-1}\{\alpha\Phi(s) + \beta\Psi(s), t\} = \alpha L^{-1}\{\Phi(s), t\} + \beta L^{-1}\{\Psi(s), t\} = \alpha\phi(t) + \beta\psi(t). \tag{1.19}$$

In some fractional integrals we need the convolution product of two function ϕ and ψ which is defined by:

$$\phi(t) * \psi(t) = \int_0^t \phi(t-\tau)\psi(\tau)d\tau = \int_0^t \phi(\tau)\psi(t-\tau)d\tau = \psi(t) * \phi(t). \tag{1.20}$$

Now, assume that $\Phi(s)$ and $\Psi(s)$ exists. We give The Laplace transform of the convolution product of ϕ and ψ as the form:

$$L\{\phi(t) * \psi(t)), s\} \quad = \quad \Phi(s)\Psi(s). \tag{1.21}$$

The derivative of an integer order m of a function ϕ has the Laplace transform defined by:

$$L\{\phi^{(m)}(t), s\} \quad = \quad s^m\Phi(s) - \sum_{k=0}^{m-1} s^k\phi^{(m-k-1)}(0). \tag{1.22}$$

The next part is devoted to a brief reminder on the theory of fractional calculations. By introducing some properties and definitions of fractional order integration and derivation operators. We focus more on fractional derivative in the sense of Caputo that will be used in the future chapters.

1.3 Definitions of some fractional order derivatives and integrals

The derivation and the integration of fractional order has several definitions, it is a very large domain, in this work we are not interested in the creation of a new definition or to develope one of the exists definitions, just we are using some exists definitions, the reader who wants to take a deep view and deep details is invited to consult the references [42, 43]. In the present section we will introduce the most used fractional derivatives with some proprieties. More recently, the derivative of Grunwald– Letnikov, the derivative of Riemman–Liouville and the derivative of Caputo that we will used in this work. The definition of fractional approach can be given in two main approaches, the first generalize the idea that differentiation and integration are limits of finite differences like the definition of Grünwald–Letnikov, a convolutional representation of repeated integration is used in the other generalized method like the Riemann –Liouville and Caputo approach.

Remark 1.3.1. *In this thesis we shoose the notation in Podluby Igor [43] as in the following:*

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & q > 0 \\ 1, & q = 0 \\ \int_a^t d\tau^{(-q)}, & q < 0, \end{cases} \tag{1.23}$$

where ${}_aD_t^q$ refers to the derivative operator of order q, a and t are the lower and upper limits of that operator, respectively.

1.3.1 Derivation and integration in the sense of Grünwald–Letnikov

Grünwald–Letnikov method involves expressing the iterated integral $(-q)$ times if q is negative and the integer derivative q if q is positive of a function ϕ. The definitions of integral and derivative in the context of Grünwald–Letnikov are provided in this section.

Let ϕ be a continuous function, so we can define the first derivative of ϕ by:

$$\phi'(t) = \frac{d\phi}{dt} = \lim_{h\to 0} \frac{\phi(t) - \phi(t-h)}{h}, \tag{1.24}$$

now, we apply this definition again but on ϕ' , we obtain the second order derivative:

$$\begin{aligned}\phi''(t) &= \frac{d^2\phi}{dt^2} = \lim_{h\to 0}\frac{\phi'(t)-\phi'(t-h)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\Big[\frac{\phi(t)-\phi(t-h)}{h} - \frac{\phi(t-h)-\phi(t-2h)}{h}\Big] \\ &= \lim_{h\to 0}\frac{\phi(t)-2\phi(t-h)+\phi(t-2h)}{h^2},\end{aligned} \tag{1.25}$$

from (1.24) et (1.25) we get:

$$\phi^{(3)}(t) = \frac{d^3\phi}{dt^3} = \lim_{h\to 0}\frac{\phi(t)-3\phi(t-h)+3\phi(t-2h)-\phi(t-3h)}{h^3}, \tag{1.26}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\phi^{(m)}(t) = \frac{d^m\phi}{dt^m} = \lim_{h\to 0}\frac{1}{h^m}\sum_{k=0}^{m}(-1)^k\begin{pmatrix} m \\ k \end{pmatrix}\phi(t-kh), \tag{1.27}$$

for a positive value of m, the binomial coefficients with alternating signs are defined as:

$$\begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}, \tag{1.28}$$

when m has a negative value, we have:

$$\begin{pmatrix} -m \\ k \end{pmatrix} = \frac{-m(-m-1)(-m-2)\cdots(-m-k+1)}{k!} = (-1)^k\frac{m(m+1)\cdots(m+k-1)}{k!} = (-1)^k\begin{bmatrix} m \\ k \end{bmatrix}, \tag{1.29}$$

if we replace $-m$ in (1.27) for m, we get:

$$\frac{d^{-m}}{dt^{-m}}\phi(t) \equiv \phi^{(-m)}(t) = \lim_{h\to 0}\frac{1}{h^m}\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\phi(t-kh),\ \ m\in\mathbb{N}, \tag{1.30}$$

based on equations (1.24)—(1.27), the definition of the fractional derivative of order q

where ($q \in \mathrm{R}^{+}$) can be expressed by the following equation:

$$D_t^q\phi(t) = \lim_{h\to 0}\frac{1}{h^q}\sum_{k=0}^{m}(-1)^k\begin{pmatrix} q \\ k \end{pmatrix}\phi(t-kh). \tag{1.31}$$

The relation between Euler's gamma function and factorial is used to calculate the binomial coefficients:

$$\begin{pmatrix} q \\ k \end{pmatrix} = \frac{q!}{k!(q-k)!} = \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)}, \tag{1.32}$$

and $\begin{pmatrix} \gamma \\ 0 \end{pmatrix} = 1.$

In equation (1.31), if $m = \dfrac{t-a}{h}, a \in \mathbb{R}$, so one can write the Grünwald-Letnikov fractional derivative of a function ϕ by:

$${}_aD_t^q\phi(t) = \lim_{h\to 0}\frac{1}{h^q}\sum_{k=0}^{\left[\frac{t-a}{h}\right]}(-1)^k\begin{pmatrix} q \\ k \end{pmatrix}\phi(t-kh), \tag{1.33}$$

where $\left[\dfrac{t-a}{h}\right]$ design the integer part of $\dfrac{t-a}{h}$.

Remark 1.3.2. *It is clear that the derivative of integer order of a constant Cst is zero, unfortunately the derivative of Grünwald-Letnikov lose this important properties. Indeed:*

Let $\phi(t) = Cst$ and q not integer, we have $f^{(k)}(t) = 0$ for $k = \overline{1;m}$ but in the fractional case we have:

$$\begin{aligned} {}_a^GD_t^q\phi(t) &= \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q} + \frac{1}{\Gamma(m-q)}\int_a^t (t-\tau)^{n-q-1}\phi^{(m)}(\tau)d\tau \\ &= \frac{Cst(t-a)^{-q}}{\Gamma(1-q)} + \underbrace{\sum_{k=1}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}}_{=0} + \underbrace{\frac{1}{\Gamma(n-q)}\int_a^t (t-\tau)^{m-q-1}\phi^{(m)}(\tau)d\tau}_{=0} \\ &= \frac{Cst(t-a)^{-q}}{\Gamma(1-q)}. \end{aligned} \tag{1.34}$$

Laplace transform of the fractional derivative of Grünwald-Letnikov:

Let ϕ be a function with its Laplace transform $\Phi(s)$, for $0 \leq q \leq 1$ we have:

$$ {}_0^G D_t^q \phi(t) = \frac{\phi(0)t^{-q}}{\Gamma(1-q)} + \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} \phi'(\tau) d\tau, \tag{1.35} $$

then

$$ L[{}_0^G D_t^q \phi(t)](s) = \frac{\phi(0)}{s^{1-q}} + \frac{1}{s^{1-q}} [s\Phi(s) - \phi(0)] = s^q \Phi(s). \tag{1.36} $$

1.3.2 Derivation and integration in the sense of Riemann-Liouville

We can devide this approach into some parts, integral, derivative, properties, and Laplace transform, we start by the definition of the integral as follows:

1) **Fractional order integral:**

In order to extend the notion of $m-$uple integration of an integrable function ϕ to non-integer values m, we can start from the Cauchy formula:

$$ {}_a D_t^{-m} \phi(t) = \frac{1}{(m-1)!} \int_a^t (t-\tau)^{m-1} \phi(\tau) d\tau, \tag{1.37} $$

we replace the integer m with a real q we get:

$$ {}_a^R D_t^{-q} \phi(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{q-1} \phi(\tau) d\tau. \tag{1.38} $$

The integer m in (1.37) must verifie $m \geq 1$, this condition becomes weak for m in (1.38), for the existing integral of Riemann-Liouville (1.38) one must have $q > 0$.

Proposition 1.3.1. *Let ϕ a continuous function and $q > 0$, we have:*

$$ \lim_{q \to 0} ({}_a^R D_t^{-\gamma} \phi(t)) \quad = \quad \phi(t). \tag{1.39} $$

Proposition 1.3.2. *Let $\phi \in C^0([a,b]), q_1 > 0, q_2 > 0$ then the integral of Riemann-Liouville has the property of the semi-group as follows:*

$$ {}_a^R D_t^{-q_1} ({}_a^R D_t^{-q_2} \phi(t)) \quad = \quad {}_a^R D_t^{-(q_1+q_2)} \phi(t) = {}_a^R D_t^{-q_2} ({}_a^R D_t^{-q_1} \phi(t)). \tag{1.40} $$

Proof.

By definition we have:

$$\begin{aligned} {}_a^R D_t^{-q_1}({}_a^R D_t^{-q_2}\phi(t)) &= \frac{1}{\Gamma(q_1)}\int_a^t (t-\tau)^{q_1-1}({}_a^R D_\tau^{-q_2} f(\tau))d\tau \\ &= \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t (t-\tau)^{q_1-1}d\tau(\int_a^\tau (\tau-x)^{q_2-1}\phi(x)dx), \end{aligned} \tag{1.41}$$

using Fubini theorem, we get:

$${}_a^R D_t^{-q_1}({}_a^R D_t^{-q_2}\phi(t)) = \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t \phi(x)dx\int_x^t (t-\tau)^{q_1-1}(\tau-x)^{q_2-1}d\tau, \tag{1.42}$$

in order to calculate the integral from x to t, we can put $\tau = x + y(t-x)$,

thus $t-\tau = (t-x)(1-y)$, then:

$$\begin{aligned} \int_x^t (t-\tau)^{q_1-1}(\tau-x)^{\delta-1}d\tau &= (t-x)^{q_1+q_2-1}\int_0^1 (1-y)^{q_1-1}y^{q_2-1}dy \\ &= (t-x)^{q_1+q_2-1}\beta(q_1,q_2) \\ &= (t-x)^{q_1+q_2-1}\frac{\Gamma(q_1)\Gamma(q_2)}{\Gamma(q_1+q_2)}, \end{aligned} \tag{1.43}$$

then

$${}_a^R D_t^{-q_1}({}_a^R D_t^{-q_2}\phi(t)) = \frac{1}{\Gamma(q_1+q_2)}\int_a^t (t-x)^{q_1+q_2-1}\phi(x)dx = {}_a^R D_t^{-q_1-q_2}\phi(t), \tag{1.44}$$

we can interchange q_1 and q_2 we obtain:

$${}_a^R D_t^{-q_1}({}_a^R D_t^{-q_2}\phi(t)) = {}_a^R D_t^{-(q_1+q_2)} = {}_a^R D_t^{-q_2}({}_a^R D_t^{-q_1}\phi(t)). \tag{1.45}$$

■

2) **The fractional derivative in the sense of Riemann-Liouville:**

The fractional derivative in the sense of Riemann - Liouville of order $q > 0$ of an integrable function ϕ is given for all $m-1 \leq \gamma \leq m$ by:

$$\begin{aligned} {}_a^R D_t^{q}\phi(t) &= \frac{d^m}{dt^m}({}_a^R D_t^{-(m-q)}\phi(t)) \\ &= \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_a^t (t-\tau)^{(m-q-1)}\phi(\tau)d\tau. \end{aligned} \tag{1.46}$$

• If $q = m-1$ then we have a conventional order derivative $m-1$:

$$\begin{aligned} {}_a^RD_t^{m-1}\phi(t) &= \frac{d^m}{dt^m}({}_a^RD_t^{-(m-(m-1))}\phi(t)) \\ &= \frac{d^m}{dt^m}({}_a^RD_t^{-1}\phi(t)) \\ &= \phi^{(m-1)}(t). \end{aligned} \tag{1.47}$$

3) **Some properties of derivative of Riemann-Liouville:**

1. For $q > 0$ and $t < a$ we have:

$${}_a^RD_t^q({}_a^RD_t^{-q}\phi(t)) = \phi(t). \tag{1.48}$$

2. If ${}_a^RD_t^p\phi(t), (n-1 \leq p < n)$ of $\phi(t)$ is integrable then:

$${}_a^RD_t^{-q}({}_a^RD_t^q\phi(t)) = \phi(t) - \sum_{k=1}^{m}[{}_a^RD_t^{q-k}\phi(t)]_{t=a}\frac{(t-a)^{q-k}}{\Gamma(q-k+1)}, q > 0, t > a. \tag{1.49}$$

3. The derivative of order m in the sense of fractional Riemann-Liouville of order q for any $m \in \mathbb{N}^*$ is given by:

$$\frac{d^m}{dt^m}({}_a^RD_t^q\phi(t)) = {}_a^RD_t^{m+q}\phi(t), \tag{1.50}$$

but the fractional derivative of ordre q of the derivative of order m of a function ϕ is given by:

$${}_a^RD_t^q(\frac{d^m}{dt^m}\phi(t)) = {}_a^RD_t^{m+q}\phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)(t-a)^{k-q-m}}{\Gamma(k+1-q-m)}, \tag{1.51}$$

so the fractional derivative operator ${}_a^RD_t^q$ of Riemann-Liouville commutes with $\frac{d^m}{dt^m}$ if and only if

$$\phi^{(k)}(a) = 0, \qquad (k = \overline{0; m-1}). \tag{1.52}$$

Remark 1.3.3. *Similarly with the fractional derivative of Grünwald-Letnikov, the derivative of a constant function $\phi(t) = Cst$ in the sense of Riemann-Liouville is not zero and also not constant. Hence, we have:*

$$ {}_a^R D_t^q \phi(t) = \frac{Cst(t-a)^{-q}}{\Gamma(1-q)}. \tag{1.53} $$

4) **Laplace transform of the derivative in the sense of Riemann-Liouville:**

In particular, we can express the fractional integral of Riemann-Liouville as the convolution product of the function $\psi(t) = \frac{t^{q-1}}{\Gamma(q)}$ and $\phi(t)$,

we have

$$ \begin{aligned} {}_0^R D_t^q \phi(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \phi(\tau) d\tau \\ &= \frac{1}{\Gamma(q)} (t^{q-1} * \phi(t)), \end{aligned} \tag{1.54} $$

remember that the Laplace transform of $\psi(t) = \frac{t^{q-1}}{\Gamma(q)}$ is given as follow:

$$ \Psi(s) = L\{\frac{t^{q-1}}{\Gamma(q)}, s\} = s^{-q}, \tag{1.55} $$

using the Laplace transform of the convolution product of two functions given in (1.21), we get the Laplace transform of the fractional integral in the sense of Riemann-Liouville by:

$$ L\{{}_a^R D_t^q \phi(t), s\} = s^{-q}\Phi(s), \tag{1.56} $$

also we can get the Laplace transform of the fractional derivative of the sense of Riemann-Liouville of the function $\phi(t)$, putting ${}_a^R D_t^q \phi(t) = \psi^{(m)}(t)$ then:

$$ \psi(t) = {}_0^R D_t^{-(m-q)} \phi(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-\tau)^{m-q-1} \phi(\tau) d\tau, \quad (m-1 \leq q < m), \tag{1.57} $$

we can use the formula in (1.22), we obtain:

$$ L\{{}_0^R D_t^q \phi(t), s\} = s^m \Psi(s) - \sum_{k=0}^{m-1} s^k \psi^{(m-k-1)}(0), \tag{1.58} $$

where

$$\Psi(s) \quad = \quad s^{-(m-q)}\Phi(s), \tag{1.59}$$

from the definition of fractional derivative of Riemann-Liouville, we get:

$$\psi^{(m-k-1)}(t) = \frac{d^{m-k-1}}{dt^{m-k-1}}\ {}_0D_t^{-(m-q)}\phi(t) = {}_0^RD_t^{q-k-1}\phi(t), \tag{1.60}$$

substituting (1.59) and (1.60) in (1.58). We can get the final expression of the Laplace transform of Riemann Liouville fractional derivative as:

$$L\{{}_0^RD_t^q\phi(t), s\} = s^{\gamma}\Phi(s) - \sum_{k=0}^{m-1} s^k[{}_0^RD_t^{(q-k-1)}\phi(t)]_{t=0}, \quad m-1 \leq q < m. \tag{1.61}$$

5) **Relation between the fractional derivative of Riemann-liouville and the fractional derivative of Grünwald-Letnikov:**

Assume that $\phi \in C^m$, then by making integrations by parts and repeated differentiations we get:

$${}_a^RD_t^q\phi(t) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q} + \frac{1}{\Gamma(m-q)}\int_a^t (t-\tau)^{m-q-1}\phi^{(m)}(\tau)d\tau = {}_a^{\Phi}D_t^q\phi(t), \tag{1.62}$$

in this case the Grünwald-Letnikov approach and the Riemann-Liouville approach are equivalent.

1.3.3 Derivation and integration in the sense of Caputo

The definition of Caputo for the fractional derivative is widely used in the literature because of its interesting properties headed by the derivatives of a constant that is cenvenable with the integr case, this propertie make this aproach faster to aplicate in mathematical modeling. We give here the definition, some properties, laplace transforme, relation between caputo and Riemman–liouvill aproach.

Definition 1.3.1. *Let $\phi \in C^m([a,b]), q > 0$; the fractional derivatiove of a function ϕ in the sense of Caputo is defined by:*

$$\begin{aligned} {}^{C}_{a}D^{q}_{t}\phi(t) &= {}_{a}D^{-(m-q)}_{t}(\frac{d^m}{dt^m}\phi(t)) \\ &= \frac{1}{\Gamma(m-q)}\int_a^t \frac{\phi^{(m)}(\tau)}{(t-\tau)^{q-m+1}}d\tau, \end{aligned} \tag{1.63}$$

where $m-1 < q < m$ and $t > a$.

1) **Relation between the fractional derivative of Caputo and the fractional derivative of Riemann-Liouville :**

The fractional derivative of Caputo and that of Riemann-Liouville are related by the formula:

$$ {}^{C}_{a}D^{q}_{t}\phi(t) = {}^{R}_{a}D^{q}_{t}\phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}. \tag{1.64}$$

2) **Some properties of the fractional derivative in the sense of Caputo:**

We give in the following some properities of fractional Caputo aproach.

Properties 1.3.1. *Contrary to the fractional derivative of Grünwald–Letnikov and Riemann - Liouville, the derivative of a constant function $\phi(t) = Cst$ of Caputo is null. This property is a strength for the fractional derivative to the sense of Caputo since the other definitions of fractional derivatives lose this property.*

Proof. if $\phi(t) = Cst$ then $\phi^{(m)}(t) = 0$ then

$$ {}^{C}_{a}D^{q}_{t}Cst = \frac{1}{\Gamma(m-q)}\int_a^t \frac{\phi^{(m)}(\tau)}{(t-\tau)^{q-m+1}}d\tau = 0. \tag{1.65}$$

■

Properties 1.3.2. *For all $m \in N^*, 0 < m-1 < q < m$, we have:*

$$\lim_{q\to m}({}^{C}_{a}D^{q}_{t}\phi(t)) = \phi^{(m)}(t). \tag{1.66}$$

Proof.

$$\begin{aligned}
\lim_{q\to m} {}^C_aD^q_t\phi(t) &= \lim_{q\to m}[{}^R_aD^q_t\phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}] \\
&= \lim_{q\to m}\frac{\phi^{(m)}(a)(t-a)^{m-q}}{\Gamma(m-q+1)} + \frac{1}{\Gamma(-q+m+1)}\int_a^t (t-\tau)^{m-q}\phi^{(m+1)}(\tau)d\tau \\
&= \phi^{(m)}(t). \qquad (1.67)
\end{aligned}$$

■

Properties 1.3.3. *For all $r \in N^*$ and $m-1 < q < m$, we have:*

$$ {}^C_aD^q_t({}_aD^r_t\phi(t)) = {}^C_aD^{q+r}_t\phi(t). \qquad (1.68)$$

Proof.

$$\begin{aligned}
{}^C_aD^q_t({}_aD^r_t\phi(t)) &= {}_aD^{-(m-q)}_t\,{}_aD^m_t({}_aD^r_t\phi(t)) \\
&= {}_aD^{-(m-q)}_t\,{}_aD^{m+r}_t\phi(t) \\
&= {}^C_aD^{r+q}_t\phi(t). \qquad (1.69)
\end{aligned}$$

■

Properties 1.3.4. *For all $r \in N^*$ and $m-1 < q < m$, we have:*

$$ {}_aD^r_t({}^C_aD^q_t\phi(t)) = {}^C_aD^{q+r}_t\phi(t) + \sum_{k=m}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)}, \qquad (1.70)$$

particularly, if $\phi^{(k)}(a) = 0$ for $k = m, m+1, \cdots, m+q-1$ it comes:

$$ {}_aD^r_t({}^C_aD^q_t\phi(t)) = {}^C_aD^q_t({}_aD^r_t\phi(t)) = {}^C_aD^{r+q}_t\phi(t). \qquad (1.71)$$

Proof.

$$\begin{aligned}
{}_aD^r_t({}^C_aD^q_t\phi(t)) &= {}_aD^r_t[{}^R_aD^q_t f(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}] \\
&= {}^R_aD^{r+q}_t f(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-q+1)}{}_aD^r_t(t-a)^{k-q}
\end{aligned}$$

Then:

$$
\begin{aligned}
{}_aD_t^r({}_a^CD_t^q\phi(t)) &= {}_a^RD_t^{r+q}f\phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)} \\
&= {}_a^RD_t^{r+q}\phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(r+q)+1)}(t-a)^{k-(q+r)} \\
&- \sum_{k=m}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)} \\
&+ \sum_{k=m}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)} \\
&= {}_a^RD_t^{r+q}\phi(t) - \sum_{k=0}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)} \\
&+ \sum_{k=m}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)} \\
&= {}_a^CD_t^{r+q}\phi(t) + \sum_{k=m}^{r+m-1}\frac{\phi^{(k)}(a)}{\Gamma(k-(q+r)+1)}(t-a)^{k-(q+r)}, \qquad (1.72)
\end{aligned}
$$

then

$$
{}_aD_t^r({}_a^CD_t^q\phi(t)) = {}_a^CD_t^{r+q}\phi(t), \qquad (1.73)
$$

if and only if $\phi^{(k)}(a) = 0$ for all $k = m, m+1\cdots, m+r-1$ ■

3) **Laplace transform of the fractional derivative in the sense of Caputo:**

The Laplace transform of the fractional derivative sense of Caputo is given by:

$$
L\{{}_0^CD_t^q\phi(t), s\} = s^q\Phi(s) - \sum_{k=0}^{m-1}s^{q-k-1}\phi^{(k)}(0), \quad m-1 \le \gamma \le m. \qquad (1.74)
$$

1.4 Some general properties of fractional integrals and derivatives

According to Oldham and Spanier, fractional integrals and derivatives have the following primary characteristics [42]:

1. Let $\phi(t)$ an analytical function of t, then the fractional derivative ${}_0D_t^q\phi(t)$ give us an analytical function of t and γ.

2. If m is an integer then for $q = m$, the operation ${}_0D_t^q\phi(t)$ provides the same result as the classical differentiation with integer order m.

3. If we take $q = 0$, the operation ${}_0D_t^q\phi(t)$ give us the identity operator:

$$ {}_0D_t^0\phi(t) = \phi(t). \tag{1.75}$$

4. Similarly with integer order differentiation, fractional integration and differentiation are linear operations :

$$ {}_aD_t^q(\lambda\phi(t) + \mu\psi(t)) = \lambda{}_aD_t^q\phi(t) + \mu{}_aD_t^\psi(t). \tag{1.76}$$

5. Semigroup property (the additive index law) holds if the function $\phi(t)$ satisfies some reasonable constraints.

$$ {}_0D_t^{q_1}{}_0D_t^{q_2}\phi(t) = {}_0D_t^{q_2}{}_0D_t^{q_1}\phi(t) = {}_0D_t^{q_1+q_2}\phi(t). \tag{1.77}$$

6. The integer derivative commute with the fractional derivative (the operators ${}_aD_t^q$ and $\frac{d^m}{dt^m}$ commute) .

$$\frac{d^m}{dt^m}\left({}_aD_t^q\phi(t)\right) = {}_aD_t^q\left(\frac{d^m\phi(t)}{dt^m}\right) = {}_aD_t^{q+m}\phi(t), \tag{1.78}$$

if $t = a$ we have $\phi^{(k)}(a) = 0, (k = 0, 1, 2, \ldots, m-1)$.

7. If $\phi_1(t)$ and $\phi_2(t)$ and all their derivatives are continuous in the interval $[a, t]$. So the Leibniz's rule for fractional differentiation is:

$$ {}_aD_t^m(\phi_1(t)\phi_2(t)) = \sum_{k=0}^{\infty}\left(\begin{array}{c} m \\ k \end{array}\right)\phi_1^{(k)}(t){}_aD_t^{m-k}\phi_2(t), \tag{1.79}$$

1.5 Fractional differential equations in the sence of Caputo

In this part of the chapter we introduce the basic notions for the fractional order differential equations including existence and uniqueness, resolution and finally we present a numerical methods of solving these fractional differential equations.

1.5.1 Existence and uniqueness of solutions

Consider the fractional differential equation of order γ with initial value problems as follows:

$$\begin{cases} {}^{C}D^{q}y(t) = \phi(t, y(t)) \\ {}^{C}D^{i}y(0) = y_0^{(i)}, \quad i = 0, 1, \ldots, m-1, \quad \text{where } m = \lceil q \rceil. \end{cases}, \tag{1.80}$$

where $m - 1 < q < m$ and ${}_{t}^{C}D^{q}$ denote the Caputo fractional operator.

The next theorem gives us the existence and the uniqueness of (1.80).

Theorem 1.5.1. *Let $q > 0, m \notin \mathbb{N}$ and $m = \lceil \gamma \rceil$. Furthermore, let $K > 0, h^* > 0$, and $a_0, \ldots, a_{m-1} \in \mathbb{R}$. Define*

$$\Phi := [0, h^*] \times [a_0 - K, a_0 + K]\,, \tag{1.81}$$

let now $\phi : \Phi \to \mathbb{R}$ be continuous a function. Then, there exists some $h > 0$ and a function $y \in C[0, h]$ solving the fractional differential equation of Caputo type with initials conditions in (1.80). For $q \in (0, 1)$ the h is defined by

$$h := \min\left\{h^*, (K\Gamma(q+1)/M)^{1/q}\right\}, \text{ with } \quad M := \sup_{(x,z)\in\Phi} |\phi(x, z)|, \tag{1.82}$$

moreover, if ϕ satisfies a Lipschitz condition with respect to the second variable, i.e.

$$\left|\phi(x, y_1) - \phi(x, y_2)\right| \leq L\left|y_1 - y_2\right|. \tag{1.83}$$

with some constant $L > 0$ independent of x, y_1, and y_2, the function $y \in C[0, h]$ is unique.

To prove this theorem we go through the integral equation of Lotka volterra, so first we define the integral equation of Lotka volterra.

Theorem 1.5.2. *Under the assumptions of the previous theorem the function* $y \in C[0,h]$ *is a solution to the fractional differential equation of Caputo type with initials conditions in (1.80), if and only if it is a solution of the Volterra integral equation of the second kind*

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} b_k + \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} \phi(t, y(t)) dt. \tag{1.84}$$

Proof. Assume that y is a solution of (1.84). So, y can be written as:

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + {}_0D_t^{-q} \phi(t, y(t)), \tag{1.85}$$

applying ${}_0^C D_t^q$ on both sides of (1.85), we get:

$$\begin{aligned} {}_0^c D_t^q y(t) &= \sum_{k=0}^{m-1} \frac{y_0^{(k)} {}_0^c D_t^q t^k}{k!} + {}_0^c D_t^q {}_0D_t^{-q} \phi(t, y(t)) \\ &= \sum_{k=0}^{m-1} \frac{y_0^{(k)} {}_0^c D_t^q t^k}{k!} + \phi(t, y(t)), \end{aligned} \tag{1.86}$$

since $k < q$, so ${}_0^C D_t^q t^k = 0$. Hence y is a solution of (1.80),

we shall now to proof that: ${}_0^C D_t^j y(0) = y_0^{(j)}$. Applying the differentiation operator ${}_0^C D_t^j, 0 \leq i \leq m-1$ on (1.84), we obtain:

$$\begin{aligned} {}_0^c D_t^i y(t) &= \sum_{k=0}^{m-1} \frac{y_0^{(k)} {}_0^c D_t^i t^k}{k!} + {}_0^c D_t^i {}_0D_t^{-i} {}_0D_t^{-(\gamma-i)} \phi(t, y(t)) \\ &= \sum_{k=0}^{m-1} \frac{y_0^{(k)} {}_0^c D_t^i t^k}{k!} + {}_0D_t^{-(q-i)} \phi(t, y(t)), \end{aligned} \tag{1.87}$$

to recall, we have the following fractional derivative:

$${}_0^c D_t^i t^k = \begin{cases} 0, & \text{if } i > k, \\ \Gamma(k+1), & \text{if } i = k, \\ \frac{\Gamma(k+1)}{\Gamma(k-i+1)} t^{k-i}, & \text{if } i < k, \end{cases} \tag{1.88}$$

the equation (1.88) implies:

$$ {}_0^cD_t^i t^k\Big|_{t=0} = \begin{cases} 0, & \text{if } i > k, \\ \Gamma(k+1), & \text{if } i = k, \\ 0, & \text{if } i < k, \end{cases} \tag{1.89}$$

we have also $\gamma - i \geq 1$, then l'intégrale ${}_0D_t^{-(q-i)}\phi(t, y(t))\Big|_{t=0} = 0$. as a consequence ${}_0^cD_t^i y(0) = y_0^{(i)}$, Assume that y is a solution of the initial value problems (1.80) and let's show that y is the solution of Volterra integral equation (1.84).

Let $z(t) = \phi(t, z(t))$, then $z \in [0, h]$, then we can write :

$$\begin{aligned} z(t) = (t, y(t)) = {}_0^CD_t^q y(t) &= {}_0^RD_t^q y(t) - \sum_{k=0}^{m-1} \frac{y_0^{(k)} t^{k-q}}{\Gamma(k-q+1)} \\ &= {}_0^RD_t^q y(t) - {}_0^RD_t^q \sum_{k=0}^{m-1} \frac{y_0^{(k)} t^k}{k!} \\ &= {}_0^RD_t^q \left(y(t) - \sum_{k=0}^{m-1} \frac{y_0^{(k)} t^k}{k!} \right) \\ &= {}_0^RD_t^q \left(y - T_{m-1}[y, 0]\right)(t) \\ &= {}_0D_t^m {}_0D_t^{-(m-q)} \left(y - T_{m-1}[y, 0]\right)(t), \end{aligned} \tag{1.90}$$

where $T_{m-1}[y, 0](t) = \sum_{k=0}^{m-1} \frac{e^k}{k!} y_0^{(k)}$, represente the Taylor polynomial of degree $m-1$. By applying ${}_0D_t^{-m}$ on the two members of (1.90), we get:

$${}_0D_t^{-m} z(t) = {}_0D_t^{-(m-q)} \left(y - T_{m-1}[y, 0]\right)(t) + p(t), \tag{1.91}$$

where p represent a polynom of degree $\leq m-1$.

Since the function z is continuous, the function ${}_0D_t^{-m} z$ has a zero of order at least m at the origin.

In addition to that, the difference $y - T_{m-1}[y, 0]$ having the same property by construction.

Then the function ${}_0D_t^{-(m-\gamma)} \left(y - T_{m-1}[y, 0]\right)$ must also have a zero of order m.

Consequently the polynomial p having the same property, but as it is of degree $\leq m-1$, it results that $q = 0$.

Therefore:

$$ {}_0D_t^{-m} z(t) = {}_0D_t^{-(m-a)} \left(y - T_{m-1}[y,0]\right)(t), \tag{1.92}$$

which implies:

$$\begin{aligned} y(t) &= T_{m-1}[y,0](t) + {}_0D_t^{-q} z(t) \\ &= \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \phi(\tau, y(\tau)) d\tau. \end{aligned} \tag{1.93}$$

■

1.6 Analytical solution of linear fractional order differential equations

In order to solve a linear fractional order differential equation, the Mittag Luffler function E_q is used to give an explicit expression of the solution.

1.6.1 One dimentional linear cases with application

Theorem 1.6.1. *[13] Let $q > 0, m = [q]$ and $\lambda \in \mathbb{R}$, the general solution of the next problem (1.94):*

$$\begin{cases} {}_0^C D_t^q y(t) &= \lambda y(t) + p(t), \\ y^{(k)}(0) &= y_0^{(k)} \qquad (k = 0,1,...,m-1), \end{cases}, \tag{1.94}$$

where $p \in C[0,h]$ is in the form :

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} u_k(t) + \tilde{y}(t). \tag{1.95}$$

with

$$\tilde{y}(t) = \begin{cases} {}_0D_t^{-q} p(t) & \text{if} \quad \lambda = 0, \\ \frac{1}{\lambda} \int_0^t p(t-\tau) u_0'(\tau) d\tau & \text{if} \quad \lambda \neq 0, \end{cases} \tag{1.96}$$

where

$$u_k(t) = D^{-k} e_q(t) \quad \text{such that} \quad e_q(t) = E_q(\lambda t^q), (k = 0,1,...,m-1). \tag{1.97}$$

Proof. 1) If $\lambda = 0$ then the probleme (1.94) become:

$$\begin{cases} {}_0^C D_t^q y(t) = p(t), \\ y^{(k)}(0) = y_0^{(k)} \qquad (k = 0, 1, ..., m-1), \end{cases} \tag{1.98}$$

we have $e_q(t) = E_q(0) = 1$, then $u_k(t) = \dfrac{t^k}{k!}$ for all k.
Using the relation between Riemman-Liouville and caputo, we get:

$${}_0^C D_t^q y(t) = {}_0^R D_t^\alpha y(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k-q+1)} t^{k-q} = p(t), \tag{1.99}$$

then

$${}_0^R D_t^q y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k-q+1)} t^{k-q} + p(t), \tag{1.100}$$

applying the intégrale of Riemann-Liouville of ordre q in the both sides :

$$\begin{aligned} {}_0D_t^{-\alpha} \; {}_0^R D_t^q y(t) &= \sum_{k=0}^{m-1} \frac{y^{(k)}(0) {}_0D_t^q t^{k-q}}{\Gamma(k-q+1)} + {}_0D_t^q p(t), \\ &= \sum_{k=0}^{m-1} \frac{y^{(k)}(0) t^k}{k!} + {}_0D_t^{-\alpha} p(t), \end{aligned} \tag{1.101}$$

then

$$y(t) = \sum_{k=0}^{m-1} y^{(k)}(0) u_k(t) + \tilde{y}(t), \qquad \text{where} \qquad \tilde{y}(t) = {}_0D_t^q p(t). \tag{1.102}$$

2) If $\lambda \neq 0$: we can divide the proof on two step (a) and (b) :

a) The function u_k satisfies the homogeneous differential equation, i.e :
${}_0^C D_t^\alpha u_k = \lambda u_k, \forall k = 1, ..., m-1$ and satisfie the initial conditions $u_k^{(j)}(0) = \delta_{kj}$ (delta of Kronecker) for $j, k = 0, ..., m-1$.

b) The function $\tilde{y}$ is a solution of the nonhomogeneous differential equation with homogeneous initial conditions.

We start with (a), we know that:

$$\epsilon_q(t) = E_q(\lambda t^\alpha) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{qj}}{\Gamma(qj+1)}, \tag{1.103}$$

then

$$u_k = D^{-k}\epsilon_q(t) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{qj+k}}{\Gamma(qj+1+k)}. \tag{1.104}$$

Let us now show that u_k is a solution of the homogeneous differential equation.

$$\begin{aligned}
{}_0^C D_t^q u_k(t) &= {}_0^C D_t^q \sum_{j=0}^{\infty} \frac{\lambda^j t^{qj+k}}{\Gamma(qj+1+k)} \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(qj+1+k)} {}_0^C D_t^q t^{qj+k} \\
&= \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(q(j-1)+1+k)} t^{q(j-1)+k} \\
&= \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(qj+1+k)} t^{qj+k} \\
&= \lambda u_k(t),
\end{aligned} \tag{1.105}$$

then u_k is a solution of the homogeneous equation.

For $j = k$ we have :

$$u_k^{(k)}(0) = D^k D^{-k} \epsilon_q(0) = \epsilon_q(0) = 1. \tag{1.106}$$

For $j < k$ we have :

$$u_k^{(j)}(0) = D^j D^{-k} e_q(0) = D^{-(k-j)} e_q(0) = 0. \tag{1.107}$$

Because ϵ_q is a continuous function.

For $j > k$ we have :

$$u_k^{(j)}(0) = D^j D^{-k} \epsilon_q(0) = D^{(j-k)} \epsilon_q(0) = 0, \tag{1.108}$$

we have

$$\begin{aligned}
\tilde{y}(t) &= \frac{1}{\lambda} \int_0^t p(t-\tau) u_0'(\tau) d\tau \\
&= \frac{1}{\lambda} \int_0^t p(t-\tau) \epsilon_q'(\tau) d\tau \\
&= \frac{1}{\lambda} \int_0^t p(t) \epsilon_q'(t-\tau) d\tau,
\end{aligned} \tag{1.109}$$

this integral exists whatever t, because q is a continuous function and ϵ'_q integrable, and $\tilde{y}(0) = 0$. Moreover, for $q > 1$ (i.e $m \geq 2$) according to the standard rule of the differentiation of an integral that depends on a parameter we have:

$$D\tilde{y}(t) = \frac{1}{\lambda}\int_0^t p(t)\epsilon_q^{''}(t-\tau)d\tau + \frac{1}{\lambda}p(t)\underbrace{\epsilon_q^{'}(0)}_{=0}, \tag{1.110}$$

in the same way as above, the continuity of p and the weak singularity of ϵ_p'' we see that $\tilde{y}^{'}(0) = 0$.

By the same way:

$$D^k\tilde{y}(t) = \frac{1}{\lambda}\int_0^t p(\tau)\epsilon_q^{(k+1)}(t-\tau)d\tau \qquad \text{pour} \qquad k = 0, ..., m-1, \tag{1.111}$$

then $D^k\tilde{y}(0) = 0$.

Then $\tilde{y}$ satisfies all homogeneous initial conditions, and it remains to be shown that $\tilde{y}$ solve the nonhomogeneous differential equation. For this purpose it is written:

$$\epsilon_q^{'}(u) = \frac{d}{du}\epsilon_q(u) = \sum_{j=1}^{\infty}\frac{\lambda^j u^{qj-1}}{\Gamma(qj)}, \tag{1.112}$$

then

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{\lambda}\int_0^t q(\tau)\epsilon_q^{'}(t-\tau)d\tau, = \frac{1}{\lambda}\int_0^t p(\tau)\sum_{j=1}^{\infty}\frac{\lambda^j(t-\tau)^{qj-1}}{\Gamma(qj)}d\tau, \\ &= \sum_{j=1}^{\infty}\lambda^{j-1}\frac{1}{\Gamma(qj)}\int_0^t p(\tau)(t-\tau)^{qj-1}d\tau = \sum_{j=1}^{\infty}\lambda^{(j-1)}\ {}_0D_t^{-qj}p(t), \end{aligned} \tag{1.113}$$

then

$$\begin{aligned} {}_0^CD_t^q\tilde{y}(t) &= \sum_{j=1}^{\infty}\lambda^{j-1}\ {}_0^CD_t^q\ {}_0D_t^{-qj}p(t) = \sum_{j=1}^{\infty}\lambda^{j-1}\ {}_0D_t^{-q(j-1)}p(t), \\ &= \sum_{j=0}^{\infty}\lambda^j\ {}_0D_t^{-qj}p(t) = p(t) + \sum_{j=1}^{\infty}\lambda^j\ {}_0D_t^{-qj}p(t), \\ &= q(t) + \lambda\tilde{y}(t). \end{aligned} \tag{1.114}$$

Example 1.6.1.

Consider the following problem:

$$\begin{cases} {}_0^C D_t^q y(t) = 5y(t) - 3, \\ y(0) = 0, y^{'}(0) = 0, \end{cases} \tag{1.115}$$

We have : $\lambda = 5$, $p(t) = -3$ *and* $y(t) = \sum_{k=0}^{1} y_0^{(k)} u_k(t) + \tilde{y}(t)$ *such that:*

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{\lambda} \int_0^t p(t-\tau) u_0^{'}(\tau) d\tau \\ &= \frac{1}{5} \int_0^t -3u_0^{'}(\tau) d\tau \\ &= \frac{-3}{5} [E_q(5\tau^q)]_0^t \\ &= \frac{-3}{5} [E_q(5t^q) - 1] \\ &= \frac{-3}{5} E_q(5t^q) + \frac{3}{5}, \end{aligned} \tag{1.116}$$

and

$$\sum_{k=0}^{1} y_0^{(k)} u_k(t) = y(0) E_q(5t^q) + y^{'}(0) \int_0^t E_q(5\tau^q) d\tau = 0, \tag{1.117}$$

finally, the general solution is given by:

$$y(t) = \frac{3}{5} - \frac{3}{5} E_q(5t^q). \tag{1.118}$$

1.6.2 Multidimentional linear cases with application

We consider the fractional differential equation [13, 39]

$$ {}_0^C D_t^q y(t) = Ay(t) + p(t). \tag{1.119}$$

with $0 < q < 1$, $A \in \mathbb{M}_m(\mathbb{R})$, $y(t) \in \mathbb{R}^m$ and $p : [0, h] \longrightarrow \mathbb{R}^m$.

To solve the problem (1.119) we start with the corresponding homogeneous problem ($i.e \;\; p(t) = 0 \;\; \forall t \in [0, h]$).

Then

$$ {}_0^C D_t^q y(t) = Ay(t). \tag{1.120}$$

1) If A admits simple eigenvalues

Let $\lambda_1, \lambda_2, ..., \lambda_m$ the eigenvalue of A and $v_1, v_2, ..., v_m$ it's eigenvectors. Then the solution of (1.120) is in the form

$$y(t) = \sum_{k=1}^{m} c_k v_k E_q(\lambda_k t^q), \tag{1.121}$$

where $c_k \in \mathbb{R}, \forall k = 1, ..., m$.

2) If A admits multiple eigenvalues, for example λ of degree of multiplicity k so there are two cases:

• If the number of linearly independent eigenvectors associated with λ is equal to k in this case the solution of (1.120) is of the form (1.121).

• If the number of linearly independent eigenvectors associated with λ is equal to n (where $n < k$) in this case the other $(k - n)$ solutions that are linearly independent are given by :

$$y^{(i)}(t) = \sum_{j=n}^{i} u^{(j)} t^{(i-j)q} E_q^{(i-j)}(\lambda t^q), \qquad \text{pour} \quad i = n+1, ..., k, \tag{1.122}$$

such that the eigenvectors $u(j)$ are the solutions of the nonhomogeneous linear system

$$(A - \lambda I)u^{(j+1)} = u^{(j)}. \tag{1.123}$$

Remark 1.6.1. *Let $(y_1(t), y_2(t), ..., y_m(t))T$ be the solution of the homogeneous problem (1.120), then the solution of the problem not homogeneous (1.119) with the initial condition $y(0) = y_0$ is $(Y_1(t), Y_2(t), ..., Y_m(t))^T$ such that:*

$$Y_i(t) = y_i(t) + \int_0^t y_i(t-\tau)p_i(\tau)d\tau \qquad \forall i = 1, ..., m. \tag{1.124}$$

Example 1.6.2.

1) *Consider the following system:*
${}_0^C D_t^q y(t) = Ay(t)$, *such that* $y(t) \in \mathbb{R}^2$ *and* $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$,
• *the eigenvalue of* A *are:*$\lambda_1 = 2$ *and* $\lambda_2 = 1$.
•*The eigenvectors of* A *are :*$v_1 = (1,1)^T$ *associated to* $\lambda_1 = 2$ *and* $v_2 = (2,1)^T$ *associated to* $\lambda_2 = 1$.
• *The général solution of the system is:*

$$y(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_q(2t^q) + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} E_q(t^q), \tag{1.125}$$

if $y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *then* $c_1 = 2$ *and* $c_2 = -1$,
implie

$$y(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_q(2t^q) - \begin{pmatrix} 2 \\ 1 \end{pmatrix} E_q(t^q). \tag{1.126}$$

2) *Consider the following system:* ${}_0^C D_t^q y(t) = Ay(t)$ *such that:* $A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}$

• *The eigenvecors of* A *are:*$\lambda_1 = 3(double)$ *and* $\lambda_2 = -3, (simple)$
• *The eigenvecors of* A *are:* $v_1 = (-2,1,1)^T$ *associated with* $\lambda_1 = -3$ *and* $v_2 = (\frac{1}{2},1,0)^T, v_3 = (\frac{1}{2},0,1)^T$ *associated with* $\lambda_2 = 3$,
and since the number of linearly independent eigenvectors associated with $\lambda_1 = 3$ *is equal to the degree of multiplicity* 2, *in this case the solution is of the form :*

$$y(t) = c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} E_q(-3t^q) + c_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} E_q(3t^q) + c_3 \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} E_q(3t^q). \tag{1.127}$$

1.7 Numerical solution of fractional differential equations

The process of solving nonlinear ordinary differential equations often may not succeed using the methods that already exists in the literature. For this reason we make numerical approximations to exact solutions. It's the same problem with fractional-order differential equations. Several works give numerical methods for resolution of fractional differential equations, we cite among these methods, the Grunwald letnikov's method which is widely used to solve linear fractional differential equations, this method is based directly on the definition of Grunwald Letnikov [55]. Another methode is called VIM which is based on the determination of Lagrange multiplier in optimally way through variational theory [30]. Furthermore, Adomain Decomposition Methode (ADM) which based on the construction of a solution of the Abel- Volterra equation in the form of a series [16]. The frequency domain approximation method which is based on the approximation of the fractional order system in the frequency domain [40], this last method malevolently can lead us to fake chaos, where Tavazoei has shown the weakness of the method in his publication in [48].
Another interesting method is the Adams Bashfort Moulton (ABM) method [42], There is also the Preictor-Correcdtor method [32] which considered as a generalilization of the ADM method. This method is given in the next subsection.

1.7.1 Adams Bashfort Moulton algorithm (ABM) with aplication

In our work we opt to use the ABM in order to solve fractional differential equation of Caputo, this method is based on the fractional formulation of the classic ABM method. The ABM is presented by the following algorithm:
Consider the initial value problems (1.80),

$$\begin{cases} {}^{C}D^{q}y(t) = \phi(t, y(t)), \\ {}^{C}D^{i}y(0) = y_0^{(i)}, \quad i = 0, 1, \ldots, m-1, \quad \text{where } m = \lceil q \rceil, \end{cases} \tag{1.128}$$

where $m-1<q<m$ and ${}_t^C D^q$ denote the Caputo fractional operator.

The initial value problem (1.128) is equivalent to the lotka volterra equation under the theorem (1.84). For $j=\overline{1,k}$ in $[0;T]$, it is assumed that $y(j)$ is an approximation of $y(t_j)$. The next step is to obtain $y(t_{j+1})$, in this step we replace the integral in the Lotka volterra equation (1.128) by using the product trapezoidal quadrature formula, in other way we employ the nodes $t_j, j=\overline{1,k+1}$ then we interpret the function $t_{k+1-.}^{m-1}$ as a weight function for the integral. i.e. we apply the following approximation:

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} B(z)\mathrm{d}z \approx \int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} \tilde{B}_{k+1}(z)\mathrm{d}z, \tag{1.129}$$

where the knots are chosen at $t_{j,j=\overline{1,k+1}}$ and $\tilde{B}_{k+1}$ represent the piecewise linear interpolant for B with nodes. We notice that the integral on the right-hand side of (1.130) can be written as:

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} \tilde{B}_{k+1}(z)\mathrm{d}z = \sum_{j=0}^{k+1} a_{j,k+1} B\left(t_j\right). \tag{1.130}$$

Where

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1}-z)^{m-1} \phi_{j,k+1}(z)\mathrm{d}z, \tag{1.131}$$

and

$$\phi_{j,k+1}(z) = \begin{cases} \left(z-t_{j-1}\right)/\left(t_j-t_{j-1}\right) & \text{if } t_{j-1}<z\leq t_j, \\ \left(t_{j+1}-z\right)/\left(t_{j+1}-t_j\right) & \text{if } t_j<z<t_{j+1}, \\ 0 & \text{else.} \end{cases} \tag{1.132}$$

It is clear due to the functions $\phi_{j,k+1}$ satisfy the formula:

$$\phi_{j,k+1}\left(t_\mu\right) = \begin{cases} 0 & \text{if } j\neq\mu, \\ 1 & \text{if } j=\mu, \end{cases} \tag{1.133}$$

and that the functions $\phi_{j,k+1}$ are piecewise linear and continuous with breakpoints at the t_μ, so that they must be integrated exactly by 1.130. We choose arbitrary t_j, from (1.131) and (1.132), we can get:

$$a_{0,k+1} = \frac{(t_{k+1}-t_1)^{m+1} + t_{k+1}^{m}\left[mt_1 + t_1 - t_{k+1}\right]}{t_1 m(m+1)}, \tag{1.134}$$

$$\begin{aligned} a_{j,k+1} =& \frac{\left(t_{k+1}-t_{j-1}\right)^{m+1} + \left(t_{k+1}-t_j\right)^{m}\left[m\left(t_{j-1}-t_j\right) + t_{j-1} - t_{k+1}\right]}{\left(t_j - t_{j-1}\right)m(m+1)} \\ &+ \frac{\left(t_{k+1}-t_{j+1}\right)^{m+1} - \left(t_{k+1}-t_j\right)^{m}\left[m\left(t_j-t_{j+1}\right) - t_{j+1} + t_{k+1}\right]}{\left(t_{j+1} - t_j\right)m(m+1)}, \end{aligned} \tag{1.135}$$

if $1 \leq j \leq k$, and

$$a_{k+1,k+1} = \frac{(t_{k+1}-t_k)^n}{n(n+1)}, \tag{1.136}$$

when the nodes are equispaced .i.e. $\left(t_j = jh\right.$ with some fixed h), these relations reduce to:

$$a_{j,k+1} = \begin{cases} \frac{h^m}{m(m+1)}\left(k^{m+1} - (k-m)(k+1)^n\right) & \text{if } j = 0, \\ \frac{h^m}{m(m+1)}\left((k-j+2)^{m+1} + (k-j)^{m+1}\right. & \\ \left.-2(k-j+1)^{m+1}\right) & \text{if } 1 \leq j \leq k, \\ \frac{h^m}{m(m+1)} & \text{if } j = k+1. \end{cases} \tag{1.137}$$

Next, we derive the fractional version of the one-step Adams-Moulton technique (also known as the corrector formula), which is provided by:

$$y_{k+1} = \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)}\left(\sum_{j=0}^{k} a_{j,k+1} f\left(t_j, y_j\right) + a_{k+1,k+1} f\left(t_{k+1}, y_{k+1}^{\mathrm{P}}\right)\right), \tag{1.138}$$

where, y_{k+1}^{P} represent the prediction term.
Now, we need to calculate the value of y_{k+1}^{P} in order to determine the predictor formula. The aims now is to generalize the one-step ABM method. By replacing the integral on the lotka volterra equation (1.84) using the product rectangle rule given by:

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} B(z)\mathrm{d}z \approx \sum_{j=0}^{k} b_{j,k+1} B\left(t_j\right), \tag{1.139}$$

where

$$b_{j,k+1} = \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{n-1} \, \mathrm{d}z = \frac{\left(t_{k+1} - t_j\right)^m - \left(t_{k+1} - t_{j+1}\right)^m}{m}. \tag{1.140}$$

by replacing the functions ϕ_{kj} by functions being of constant value *one* on $\left[t_j, t_{j+1}\right]$ and *zero* on the remaining parts of $[0, t_{k+1}]$. Once more, in the equispaced case, we have:

$$b_{j,k+1} = \frac{h^m}{m} \left((k+1-j)^m - (k-j)^m\right), \tag{1.141}$$

finally, the predictor y_{k+1}^{P} is obtained by the ABM method

$$y_{k+1}^{\mathrm{P}} = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(m)} \sum_{j=0}^{k} b_{j,k+1} f\left(t_j, y_j\right). \tag{1.142}$$

Hence, the fractional version of ABM algorithm is described by (1.138) and (1.138) where (1.137) and (1.142) are used to defines the weights $a_{j,k+1}$ and $b_{j,k+1}$, respectively.

Remark 1.7.1. *The readers are invited to discover more details about the error analysis of the above method in reference [12].*

Example 1.7.1.

Consider the following fractional system [5]:

$$\begin{cases} {}_0D_t^{\alpha} x = a(y - x), \\ {}_0D_t^{\beta} y = cx - y - xz, \\ {}_0D_t^{\gamma} z = bxyz - y - bz, \end{cases} \tag{1.143}$$

whith $0 \leq \alpha, \beta, \gamma \leq 1$ *et* $a, b, c > 0$.

Appliying ABM on the system (1.143) we get:

$$\begin{cases} x_{k+1} = x_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)}(a(y_{k+1}^p - x_{k+1}^p)) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} a_{1,j,k+1}(a(y_j - x_j)), \\ y_{k+1} = y_0 + \frac{h^{\beta}}{\Gamma(\beta+2)}(cx_{k+1}^p - y_{k+1}^p - x_{k+1}^p z_{k+1}^p) + \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k} a_{2,j,k+1}(cx_j - y_j - x_j z_j), \\ z_{k+1} = z_0 + \frac{h^{\gamma}}{\Gamma(\gamma+2)}(\beta x_{k+1}^p y_{k+1}^p z_{k+1}^p - y_{k+1}^p - \beta z_{k+1}^p) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{k} a_{3,j,k+1}(x_j y_j z_j - y_j - \beta z_j), \end{cases} \tag{1.144}$$

whith

$$\begin{cases} x_{k+1}^{p} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{1,j,k+1}(a(y_j - x_j)), \\ y_{k+1}^{p} = y_0 + \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k} b_{2,j,k+1}(cx_j - y_j - x_j z_j), \\ z_{k+1}^{p} = z_0 + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{k} b_{3,j,k+1}(x_j y_j z_j) y_j - \beta z_j), \end{cases} \tag{1.145}$$

$$\begin{cases} b_{1,j,k+1} = \frac{h^{\alpha}}{\alpha}((k+1-j)^{\alpha} - (k-j)^{\alpha}), \\ b_{2,j,k+1} = \frac{h^{\beta}}{\beta}((k+1-j)^{\beta} - (k-j)^{\beta}), \\ b_{3,j,k+1} = \frac{h^{\gamma}}{\gamma}((k+1-j)^{\gamma} - (k-j)^{\gamma}), \end{cases} \tag{1.146}$$

$$\begin{cases} a_{1,j,k+1} = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)}(k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}) & if \quad j = 0, \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & if \quad 1 \leq j \leq k, \end{cases} \\ a_{2,j,k+1} = \begin{cases} \frac{h^{\beta}}{\beta(\beta+1)}(k^{\beta+1} - (k-\beta)(k+1)^{\beta}) & if \quad j = 0, \\ \frac{h^{\beta}}{\beta(\beta+1)}((k-j+2)^{\beta+1} + (k-j)^{\beta+1} - 2(k-j+1)^{\beta+1}) & if \quad 1 \leq j \leq k, \end{cases} \\ a_{3,j,k+1} = \begin{cases} \frac{h^{\gamma}}{\gamma(\gamma+1)}(k^{\gamma+1} - (k-\gamma)(k+1)^{\gamma}) & if \quad j = 0, \\ \frac{h^{\gamma}}{\gamma(\gamma+1)}((k-j+2)^{\gamma+1} + (k-j)^{\gamma+1} - 2(k-j+1)^{\gamma+1}) & if \quad 1 \leq j \leq k, \end{cases} \end{cases} \tag{1.147}$$

The following figure (1.3) represent the attractor of the system (1.143) in the space .

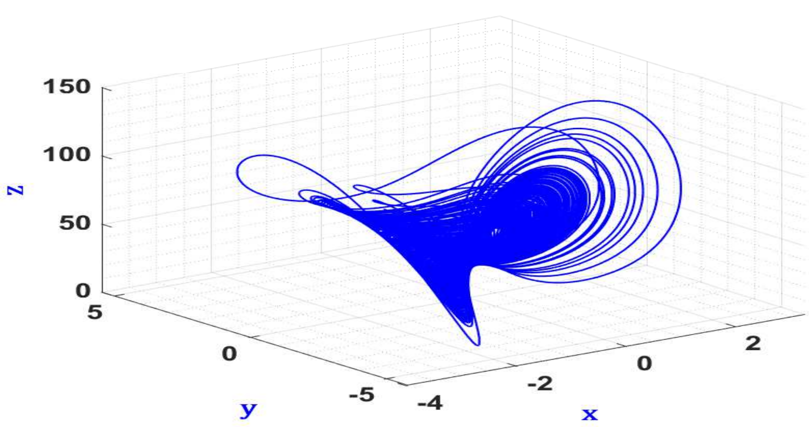

Figure 1.3: Strange attractor in the plane of (1.143).

Conclusion 1.1.

The fundamentals of fractional calculus, which builds on classical calculus to handle derivatives and integrals of arbitrary (non-integer) order, have been established in this chapter. We looked at several definitions of fractional derivatives and integrals, including the Riemann-Liouville, Caputo, and Grünwald-Letnikov approaches, as we investigated these basic ideas. The chapter also emphasized the numerical solution of fractional differential equations where we give the methode of Adams Bashfort Moulton algorithm.

CHAPTER 2

NOTIONS OF FRACTIONAL DYNAMICAL SYSTEMS AND ITS CHAOS DETECTION

As an intuitive definition, we consider a set of objects in interaction between them in the time, this interaction is called a Dynamical System (DS). Indeed, we can describe these interaction by: ordinary differential equations (ODE), difference equations, partial differential equations, integro-differential equations, stochastic equations and others. In our work we study the dynamical systems defined by the ODE or its generalization to the Fractional Differential Equation (FDE). In view of the considered time, the most types of modelisations is consider the continuous dynamical systems or the discret dynamical systems. Also, there is another type called hybrid dynamical systems where the fenomena is described by a combinaison between them. If the entire future and entire past of a dynamical system are uniquely determined by its state at the present time, then is called in this case a deterministic dynamical system. Otherwise, the dynamical system is called nondeterministic.

2.1 Definition of dynamical system

Let $t \in \mathbb{I} \subseteq \mathbb{R}^m$ and $x = x(t) \in \mathbb{R}^m$ be the vector representing the dynamic of a continuous dynamical system. As a mathematical definition, it can be defined by:

Definition 2.1.1. *[34]*

$$\dot{x} = \frac{dx}{dt} = \phi(x, t), \tag{2.1}$$

here, $\phi(x, t)$ is a sufficiently smooth function which defined on some subset $\Omega \subset \mathbb{R}^m \times \mathbb{R}$. Schematically, this can be presented as:

$$\underset{state\ space}{\mathbb{R}^m} \times \underset{times}{\mathbb{R}} = \underset{space\ of\ motions}{\mathbb{R}^{m+1}}. \tag{2.2}$$

*The usual interpretation of the variable **t** is the time. The nature of the DS is depend on the nature of $\phi(x, t)$; if it is linear then the DS is linear and the DS is nonlinear if it is nonlinear. Also, the time interval may be infinite, finite or semi finite.*

Secondly, the discret DS is defined by:

Definition 2.1.2.

$$x_{k+1} = \phi(x_k); \ \ k \in \mathbb{N}. \tag{2.3}$$

When ϕ depend explicitly on time, (2.1) is said nonautonomous DS. In the contrary case, (2.1) is called autonomous DS. In all our work, we consider uniquely the autonomous cases.

By using the fractional derivative definition, we may generalize the definition (2.1) to (2.1.3) as:

Definition 2.1.3.

$$\frac{d^q x}{d^q t} = \phi(x, t); \ \ \ q \in \mathbb{R}^+. \tag{2.4}$$

In all the rests of this work, we consider an autonomous system of fractional differential equations in the sense of Caputo (FDEC) which mean a system formed by fractional differential equations in the sence of Caputo.

So, let $x = (x_1(t), x_2(t), \cdots, x_m(t))$ and $\phi : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^m$. We consider the following vector representation of a system of FDEC:

$$^{C}D^{q_j}x(t) = \phi(\ x(t)\ );\ \ 0 < q_j < 1;\ \ j = \overline{1, m};\ \ m \in \mathbb{N}. \tag{2.5}$$

When we talk about the analysis of a dynamical system, we need to clarify the purpose of the analysis, originally when we want to solve a mathematical problem expressed in a FDE or in a system of FDE, it is better to look for an explicit solution, but this is not available in most cases because the functions are not linears generally and even constitute a difficulty. When it is difficult to find a solution, we implement a qualitative study uniquely which means we study how to calculate their fixed points, study their bifurcations and study their nature if it is regular or has a chaotic behavior or other type of qualitave study. We will provide some definitions that will help us to analyze dynamical systems in the following sections.

We destinguish two type of FOS, The commensurate and the incommensurate FOS, we give in the following the definitions of these notions.

Definition 2.1.4. *If the order of all equations forme the system (2.5) are equals* $i.e : q_1 = q_2 = \cdots = q_m$, *the system (2.5) is called a "commensurate " fractional order system (FOS). In the other hand, if there exist at least* j *and* i *where* $q_j \neq q_i$, *the system is called "incommensurate" FOS.*

In addition, $\sum_{j=1}^{m} q_j$ *is called the effective dimension of the equation (2.5).*

2.2 Stability of equilibrium points

Among the basic qualitativ analysis of a dynamical system is to find the fixed points, we announce the next definition for an equilibrium point.

Definition 2.2.1. *In order to calculate the equilibrium points of (2.5), we solve the following equation:*

$$^{C}D^{q}x(t) = 0, \tag{2.6}$$

all solutions of (2.6) is called an equilibrium points. We note an equilibrium point by x^{eq}.

Remark 2.2.1. *If $x^{eq} = 0$ is an equilibrium point of (2.6), in addition if there exist $\tau_1 \geq 0$ satisfying $x(\tau_1) = 0$, then $x(\tau) = 0$ for all $\tau \geq \tau_1$.*

2.2.1 Stability of linear system of FDE

When we talk about the stability of dynamical systems described by ODE, the region of stability and instability are divided on two half planes by the vertical axis, this region will change in FDE where we see that the stability region increases when the order q is between zero and one. In other words the FOS are more stable than integer order system. This difference in the stability region is because of the nature of the FOS that have the qualitative "memory systems" which are generally more stable than their counterparts which have low memory. In the other hand, the stability region decrease when the order is between one and two. The following stability theorem is announced.

Theorem 2.2.1. *[37] Let $x \in \mathbb{R}^m$ and $L \in \mathbb{R}^m \times \mathbb{R}^m$, the following commensurate fractional order linear system:*

$$\begin{cases} {}^{C}D^{q}x = \ L\ x\ ;\ \ 0 < q < 1, \\ x(0) = x_0, \end{cases} \tag{2.7}$$

is assyptotically stable if and only if $|\arg(eig(L))| \ > \ q\frac{\pi}{2}$.
Moreover, the system (2.7) is stable if and only if $|\arg(eig(L))| \ \geq \ q\frac{\pi}{2}$ and the critical eigenvalues which satisfy $|\arg(eig(L))| \ = \ q\frac{\pi}{2}$ have geometric multiplicity one, $eig(L)$ indicates the eigenvalues of the matrix L.

The figure (2.1) illustrate differente zone of stability in view of order q.

Remark 2.2.2. *It is necessary to pay attention that the stability region of FOS is different with the case of integer order systems.*

2.2.2 Stability of nonlinear system of FDE

In the case when the system is nonlinear of the following form:

$$^{C}D^{q}x(t) = \phi(\ x(t)\ );\ \ 0 < q < 1, x \in \mathbb{R}^{m},\ \ m \in \mathbb{N}, \tag{2.8}$$

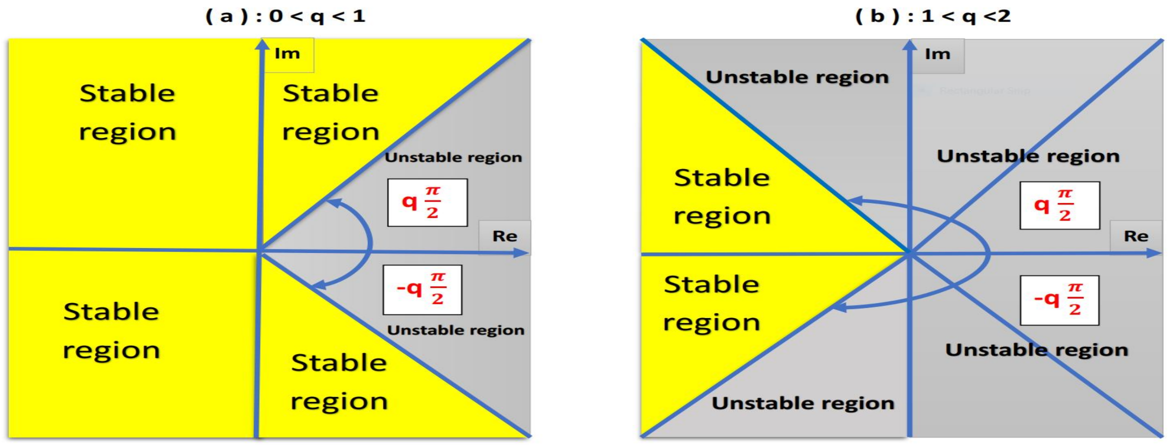

Figure 2.1: Stability region of FOS

it is the same with integer order, we use the linearization method (indirect method), the system (2.8) can be linearized around the equilibrium point as:

$$^{C}D^{q}x(t) = L\, x(t)\ ;\ \ 0 < q < 1,\ x \in \mathbb{R}^{m},\ m \in \mathbb{N}, \tag{2.9}$$

where L is the jacobian matrix associated with ϕ. Then, after the linearization we apply the theorem (2.2.1).

Theorem 2.2.2. *[42]*

Let consider the incommensurate FOS and assume that M is the LCM of the denominators v_i's of q_j's, where $q_j = \frac{u_j}{v_j}, u_j, v_j \in \mathbb{Z}^{+}$ for $j = \overline{1,m}$ and putting $\eta = 1/M$.
The system (2.8) is asymptotically stable if:

$$|\arg(\lambda)| > \eta\frac{\pi}{2} \tag{2.10}$$

for all roots λ of the next equation:

$$\det\left(\operatorname{diag}\left(\left[\lambda^{Mq_1}\lambda^{Mq_2}\ldots\lambda^{Mq_m}\right]\right) - Jac\right) = 0. \tag{2.11}$$

Example 2.2.1. *In this example we discus two cases of fractional nolinear Chen system,* *we discuss firstly the commensurate FO case:*

$$\begin{cases} {}^{C}D_t^{q_1} v_1(t) = 35(v_2 - v_1), \\ {}^{C}D_t^{q_2} v_2(t) = -7v_1 + 28v_2 - v_1 v_3, \\ {}^{C}D_t^{q_3} v_3(t) = v_1 v_2 - 3v_3, \end{cases} \tag{2.12}$$

where D^q *is the Caputo derivative operator;* v_1, v_2, v_3 *are the state variables and* $q_3 = q_1 = q_2 = 0.98$.

The system (2.12) have three equilibrium points: $p_1 = (0; 0; 0)$ *and* $p_{2,3} = (\pm 7.94; \pm 7.94; 21)$. *The linearization of (2.12) is given by:*

$$Jac(\pm 7.94; \pm 7.94; 21) = \begin{pmatrix} -35 & 35 & 0 \\ -7-21 & 28 & \pm 7.94 \\ \pm 7.94 & \pm 7.94 & -3 \end{pmatrix}, \tag{2.13}$$

we obtain the eigenvalues:

$$\lambda_{1,2} = 4.2155 \pm 14.888i, \lambda_3 = -18.431. \tag{2.14}$$

When $q = 0.98$, *we get:*

$$|arg(\lambda_{1,2})| = 1.5706 > \frac{0.98\pi}{2} = 1.5386, \tag{2.15}$$

and

$$|arg(\lambda_3)| = 3.1416 > \frac{0.98\pi}{2} = 0.49 \ , \tag{2.16}$$

in view of theorem (2.2.1), $\lambda_{1,2,3}$ *satisfy the condition* $|\arg(eig(Jac))| > q\frac{\pi}{2}$ *, which mean that* $p_{2,3}$ *are asymptotiqually stable equilibrium points.*

For the zero equilibrium p_1*:*

$$Jac(0; 0; 0) = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \tag{2.17}$$

we obtain the eigenvalues:

$$\lambda_1 = 23.836, \lambda_2 = -30.836, \lambda_3 = -3, \tag{2.18}$$

when $q = 0.98$*, there exist* $\lambda_1 = 23.836$ *which not satisfy the condition* $|\arg(eig(Jac))| > q\frac{\pi}{2}$ *, indeed:*

$$|arg(\lambda_1)| = 0 < \frac{0.98\pi}{2} = 0.49, \tag{2.19}$$

then the zero equilibrium p_1 *is instable equilibrium point.*

Example 2.2.2. *Secondly, we discuss the incommensurate fractional order case of (2.12) with* $(q_1; q_2; q_3) = (0.8; 1; 0.9)$.
We calculate the caracteristic equation (2.11) for $p_{2,3}$*, we get:*

$$\lambda^{27} + 35\lambda^{19} + 3\lambda^{18} - 28\lambda^{17} + 105\lambda^{10} - 21\lambda^8 + 4410 = 0, \tag{2.20}$$

all roots ot equation (2.20) satisfy the condition of assymptotic stability (2.10) except the followings roots:

$$|\arg(\lambda)| = |\arg(1.2928 \pm 0.20330i)| = 0.1560 < \eta\frac{\pi}{2} = 0.1570, \tag{2.21}$$

then $p_{2,3}$ *are unstable equilibrium points.*
For the zero equilibrium p_1*, we get the following caracteristic equation:*

$$\lambda^{27} + 35\lambda^{19} + 3\lambda^{18} + 35\lambda^{17} + 105\lambda^{10} + 1470\lambda^9 + 105\lambda^8 + 4410 = 0, \tag{2.22}$$

all root of the equation (2.22) satisfy the condition of assymptotic stability (2.10), then $p_1 = (0; 0; 0)$ *is assymptotically stable.*

Remark 2.2.3. *A necessary stability condition for FOS (2.8) to remain chaotic is to keep at least one eigenvalue λ in the unstable region [49]. Suppose that a chaotic system in dimension three has only three equilibrium points. Then, if the system contain a double-scroll attractor, in this case we have two saddle-focus points surrounded by scrolls and one additional saddle point in the attractor.*

Definition 2.2.2. *Assume that the unstable eigenvalues of scroll focus points are: $\lambda_{1,2} = \eta_{1,2} \pm j\theta_{1,2}$. The necessary condition to display double-scroll attractor of (2.8) is the eigenvalues $\lambda_{1,2}$ keeping in the unstable region [50].*
The condition in the commensurate order is given by:

$$q > \frac{2}{\pi} \operatorname{atan}\left(\frac{|\theta_j|}{\eta_j}\right), j = 1, 2. \tag{2.23}$$

A necessary condition to exhibit chaos for the example (2.12) is $q > 0.1629$.
We can use the condition in (2.23) to establish the minimum order at which chaos may be produced by a nonlinear DS (2.8), i.e. the system cannot be chaotic when the instability measure $\frac{\pi}{2M} - \min(|\arg(\lambda)|)$ is negative.
The characteristic equation (2.22) has $\lambda_{1,2} = 1.2928 \pm 0.2032i$ implie $|\lambda_{1,2}| = 0.1560$ which are unstable roots. Consequently, the system (2.12) satisfies the necessary condition to display a double scroll attractor. Indeed, the instability measure is given by :

$$\frac{\pi}{2M} - \min(|\arg(\lambda)|) = 0.0012. \tag{2.24}$$

Then, the instability mesure in not negative.

Definition 2.2.3. *In a nonlinear DS of dimension three, a "saddle point" is called " of index one " if one of the eigenvalues is unstable and the others are stables. Also, a "saddle point" is called " of index two" if two of the eigenvalues are unstable and one eigenvalue is stable.*

Remark 2.2.4. *When a three dimensional chaotic DS generates a double scroll attractor, so this DS has two saddle points of index two encircled by scrolls, the fixed points of index one are responsible only for connecting the scrolls.*

2.3 Generalized Mittag–Leffler stability

The Lyapunov function is a sufficient condition for stability of nonlinear system, it has the ability to study the stability in entire interval other than indirect method which give us just local stability. Furthermore, it give the stability without explicitly solving the differential equations. Unfortunately, there is no standard method to determine a Lyapunov function expect in the jerk (mechanical) systems when we put the energies as a function of Lyapunov [36]. Firstly, let us first define stability in Mittag Leffler's sense. Let the fractional nonlinear system in the sense of Caputo:

$$^{C}D^{q}x(t) = \phi(\ x(t)\ );\ \ 0 < q < 1. \tag{2.25}$$

Definition 2.3.1. *(Mittag-Leffler Stability)*
The solution of (2.25) is said to be Mittag Leffler stable if:

$$\|x(t)\| \leq \left\{\mu\left[x\left(t_0\right)\right] E_q\left(-\lambda\left(t-t_0\right)^q\right)\right\}^b, \tag{2.26}$$

here $b > 0, \lambda \geq 0$, $\mu(0) = 0, \mu(x) \geq 0$, t_0 is the initial time and $\mu(x)$ is locally Lipschitz on $x \in \mathbb{A} \in \mathbb{R}^m$ where μ_0 is a lipschitz constant.

Definition 2.3.2. *(Generalized Mittag-Leffler Stability)*
The solution of (2.25) is said to be generalized Mittag Leffler stable if:

$$\|x = (t)\| \leq \left\{\mu\left[x\left(t_0\right)\right]\left(t-t_0\right)^{-\gamma} E_{q,1-\gamma}\left(-\lambda\left(t-t_0\right)^q\right)\right\}^b, \tag{2.27}$$

here t_0 is the initial time, $\lambda \geq 0, b > 0$, $-q < \gamma \leq 1-q, \mu(0) = 0, \mu(x) \geq 0$, and $\mu(x)$ is locally Lipschitz on $x \in \mathbb{A} \in \mathbb{R}^m$ where μ_0 is a lipschitz constant.

Remark 2.3.1. *Asymptotic stability is implied by both generalized Mittag Leffler stability and Mittag Leffler stability..*

Secondly, we give the following theorem which is considered as the extention of the direct method of Lyapunov to the fractional case. This method leads to the stability of Mittag–Leffler.

Theorem 2.3.1. *Let $x^{eq} = 0$ be an equilibrium point for (2.25) and $\Omega \subset \mathbb{R}^m$ be a domain which containing the origin. Let $V(t, x(t)) : [0, \infty) \times \Omega \to \mathbb{R}$ be a continuously differentiable function and locally lipschitz with respect to x such that*

$$\begin{aligned} & q_1 \|x\|^a \le V(t, x(t)) \le q_2 \|x\|^{ab}, \\ & {}_0^C D_t^\beta V(t, x(t)) \le -q_3 \|x\|^{ab}, \end{aligned} \tag{2.28}$$

where q_1, q_2, q_3, a and b are arbitrary positive constants, $\beta \in (0, 1)$,$t \ge 0, x \in \Omega$. So $x^{eq} = 0$ is Mittag Leffler stable. Also, $x^{eq} = 0$ is globally Mittag-Leffler stable if the assumptions hold globally on $\mathbb{R}^m$.

Lemma 2.3.1. *Let $\beta \in (0, 1)$ and $M(0)$ be an arbitrary nonnegative constant, then:*

$${}_0^C D_t^\beta M(t) \le {}_0^R D_t^\beta M(t), \tag{2.29}$$

where ${}^R D$ and ${}^C D$ are the Riemann-Liouville and the Caputo fractional operators, respectively.

Proof. Through the use of the following propertie of the Riemann–Liouville fractional operator:

$${}_a^R D_t^{q_1} \left({}_a^R D_t^{q_2} \phi(t) \right) = {}_a^R D_t^{q_1+q_2} \phi(t) - \sum_{j=1}^{m} \left[{}_a^R D_t^{q_2-j} \phi(t) \right]_{t=a} \frac{(t-a)^{-q_1-j}}{\Gamma(1-q_1-j)}, \; q_1, q_2 \in \mathbb{R}, m \in \mathbb{Z}, m-1 \le q_1 < m. \tag{2.30}$$

Then, we have

$${}_0^C D_t^\beta M(t) = {}_0^R D_t^{\beta-1} \frac{\mathrm{d}}{\mathrm{d}t} M(t) = {}_0^R D_t^\beta M(t) - \frac{M(0) t^{-\beta}}{\Gamma(1-\beta)}. \tag{2.31}$$

Because $\beta \in (0,1)$ and $M(0) \geq 0$,

$$ {}_0^C D_t^\beta M(t) \leq {}_0^R D_t^\beta M(t). \tag{2.32}$$

■

Remark 2.3.2. *Lyapunov function is a sufficient condition for stability of nonlinear DS, which means that the DS may still be stable even one cannot find a Lyapunov function to conclude it's stability property.*

2.4 Fractional Lyapunov direct method using the class-$\mathcal{K}$ functions

In order to analyze the fractional Lyapunov direct method, we aims in this section to apply the class-$\mathcal{K}$ functions.

Definition 2.4.1. *[26]*

A continuous function $\beta : [0,t) \to [0,\infty)$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\beta(0) = 0$.

Lemma 2.4.1. *(Fractional Comparison Principle)*

If ${}_0^C D_t^q z(t) \geq {}_0^C D_t^q y(t)$ for all $q \in (0,1)$ and if $z(0) = y(0)$, then $z(t) \geq y(t)$.

Proof. It follows from ${}_0^C D_t^q z(t) \geq {}_0^C D_t^q y(t)$ that a nonnegative function $p(t)$ exists and that it satisfying:

$$ {}_0^C D_t^q z(t) = p(t) + {}_0^C D_t^q y(t), \tag{2.33}$$

the application of the Laplace transform on equation (2.33) yields:

$$ s^q Z(s) - s^{q-1} Z(0) = P(s) + s^q Y(s) - s^{q-1} y(0), \tag{2.34}$$

from $z(0) = y(0)$ we get:

$$Z(s) = s^{-q}P(s) + Y(s), \tag{2.35}$$

the inverse Laplace transform is applying to (2.35):

$$z(t) = {}_0\mathscr{D}_t^{-q}p(t) + y(t), \tag{2.36}$$

the uniform formula of fractional integral of order $q \in (0,1)$ in the sense of caputo and the fact that $p(t) \geq 0$ that:

$$z(t) \geq y(t). \tag{2.37}$$

■

Theorem 2.4.1. *Let $z = 0$ be an equilibrium point for the nonautonomous FOS given by (2.38) as:*

$$^{C}_{0}D^{\alpha}_{t}z(t) = \phi(t,z), \; q \in (0,1). \tag{2.38}$$

Assume that there exists a Lyapunov function $V(t,z(t))$ and class- $\mathcal{K}$ functions $\beta_i (i = 1,2,3)$ satisfying

$$\beta_1(\|z\|) \leq V(t,z) \leq \beta_2(\|z\|) \tag{2.39}$$

and

$$^{C}_{0}D^{q}_{t}V(t,z(t)) \leq -\beta_3(\|z\|) \tag{2.40}$$

where $q \in (0,1)$. Then the system (2.38) is asymptotically stable.

Proof. From (2.39) and (2.40) we get:

$$^{C}_{0}D^{q}_{\mathrm{r}}V \leq -\beta_3\left(\beta_2^{-1}(V)\right), \tag{2.41}$$

as shown in lemma (2.4.1) that $V(t,z(t))$ is bounded by the unique nonnegative solution of the scalar differential equation:

$$ {}_0^C D_r^q f(t) = -\beta_3\left(\beta_2^{-1}(f(t))\right), \quad f(0) = V(0, z(0)), \tag{2.42}$$

it follows from z_0 is an equilibrium point of $({}_0^C D_t^q z(t) = \phi(t,z),\ q \in (0,1))$ then $\phi(t, z_0) = 0$, then $f(t) = 0$ for $t \geq 0$ if $f(0) = 0$, because ${}_3\beta_2^{-1}$ is a class- $\mathcal{K}$ function. Otherwise, $f(t) \geq 0$ on $t \in [0, +\infty)$, it result from (2.42) that ${}_0^C D_r^q f(t) \leq 0$. The same idea in the proof of Lemma (2.4.1) is applied and gives:

$$ f(t) \leq f(0), \tag{2.43}$$

for $t \in (0, +\infty)$. Then the asymptotic stability of (2.42) is proved by contradiction.

Case 1: Assume that there exists a constant $t_1 \geq 0$ satisfying:

$$ {}_0^C D_{r_1}^3 f(t) = -\beta_3\left(\beta_2^{-1}\left(f\left(t_1\right)\right)\right) = 0, \tag{2.44}$$

which implies that

$$ {}_0^C D_r^q f(t) = {}_{r_1}^C D_r^q f(t) = -\beta_3\left(\beta_2^{-1}(f(t))\right), \tag{2.45}$$

for any $t \geq t_1$. $z = 0$ is the equilibrium point of ${}_{t_1}^C D_t^q f(t) = -\beta_3\left(\beta_2^{-1}(f(t))\right)$. Then $f(t) = 0$ for $t \geq t_1$ if $f\left(t_1\right) = 0$.

Case 2: Suppose that there exists a positive constant ε such that $f(t) \geq \varepsilon$ for $t \geq 0$. Then from (2.43) it result:

$$ 0 < \varepsilon \leq f(t) \leq f(0), \quad t \geq 0. \tag{2.46}$$

After substitute (2.46) into (2.42) we get:

$$ \begin{aligned} -\beta_3\left(\beta_2^{-1}(f(t))\right) &\leq -\beta_3\left(\beta_2^{-1}(\varepsilon)\right) \\ &= -\frac{\beta_3\left(\beta_2^{-1}(\varepsilon)\right)}{f(0)} f(0) \leq -\lg(t), \end{aligned} \tag{2.47}$$

here $0 < l = \frac{\beta_3\left(a_2^{-1}(a)\right)}{f(0)}$,

it then follows that

$$ {}_0^c D_r^q f(t) = -\beta_3\left(\beta_2^{-1}(f(t))\right) \leq -\lg(t). \tag{2.48}$$

By the same idea of the proof in theorem (2.3.1) we obtain

$$f(t) \leq f(0)E_q\left(-lt^q\right), \tag{2.49}$$

which contradicts the assumption that $f(t) \geq \varepsilon$. From the both cases one and two, we have $f(t)$ tends to zero as $t \to \infty$. Because $V(t, x(t))$ is bounded by $f(t)$, it follows from (2.39) that $\lim_{t\to\infty} z(t) = 0$. ∎

Theorem 2.4.2. *If the assumptions in theorem (2.4.1) are satisfied except replacing* ${}_0D_t^q$ *by* ${}_0D_{\mathrm{r}}^q$*, then we have* $\lim_{\mathrm{t}\to\infty} z(t) = 0$.

Proof. From lemma (2.3.1) and $V(t, z) \geq 0$ it result that:

$${}_0^cD_{\mathrm{r}}^q V(t, z(t)) \leq {}_0D_{\mathrm{r}}^\beta V(t, z(t)), \tag{2.50}$$

which implies

$${}_0^cD_r^q V(t, z(t)) \leq {}_0D_1^z V(t, z(t)) \leq -\beta_q(\|z\|). \tag{2.51}$$

The same idea of the proof in theorem (2.4.1) result $\lim_{t\to\infty} z(t) = 0$. ∎

2.5 Lyapounov candidate functions for stability of fractional order system

In 2014, Norelys Aguila-Camacho et al [3] publish a paper presents a novel property for fractional derivatives of Caputo when $0 < q < 1$, which makes it possible to identify a candidate Lyapunov function for numerous FOS, utilizing the Lyapunov direct method's of FO extension.

Theorem 2.5.1. *(FO extension of Lyapunov direct method).*

Let $\underline{z}^{eq} = 0$ *be an equilibrium point for the FOS (2.25). Suppose that there exists class-K functions* $\delta_i (i = \overline{1,3})$ *and Lyapunov candidate function* $V(t, \underline{z}(t))$ *satisfying*

$$\begin{aligned} &\delta_1(\|\underline{z}\|) \leqslant V(t, \underline{z}(t)) \leqslant \delta_2(\|\underline{z}\|) \\ &{}^{C}_{t_0}D^{\beta}_t V(t, \underline{z}(t)) \leqslant -\delta_3(\|\underline{z}\|) \end{aligned} \tag{2.52}$$

where $q \in (0, 1)$*. So (2.25) is asymptotically stable.*

Lemma 2.5.1. *Let* $z(t) \in \mathbb{R}$ *be a derivable and continuous function, then:*

$$\forall \alpha \in (0,1),\ \forall t \geqslant t_0 :\ \frac{1}{2}\ {}^{C}D^q_t z^2(t) \leqslant z(t)\ {}^{C}_{t_0}D^q_t z(t), \tag{2.53}$$

Proof. The expression (2.53) is equivalent to:

$$z(t)\ {}^{c}_{t_0}D^q_t z(t) - \frac{1}{2}{}^{C}D^q_t z^2(t)\ \geq\ 0; \forall \alpha \in (0,1), \tag{2.54}$$

using the definition of Caputo of fractional derivative:

$${}^{c}_{t_0}D^q_t z(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{\dot{z}(\tau)}{(t-\tau)^q} d\tau, \tag{2.55}$$

also

$$\frac{1}{2}{}_{t_0}D^q_t z^2(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{\chi(\tau)\dot{z}(\tau)}{(t-\tau)^q} d\tau, \tag{2.56}$$

so, formula (2.53) can be written as

$$\frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{[z(t) - z(\tau)]\dot{z}(\tau)}{(t-\tau)^q} d\tau \geqslant 0, \tag{2.57}$$

define now the auxiliar variable $\zeta(\tau) = (t) - z(\tau)$, which suggests that $\zeta'(\tau) = \frac{d\zeta(\tau)}{d\tau} = -\frac{dz(\tau)}{d\tau}$.

Then, expression (2.57) can be expressed as

$$\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}\frac{\zeta(\tau)\zeta'(\tau)}{(t-\tau)^q}d\tau \leqslant 0, \tag{2.58}$$

an integration by parts of Expression (2.58) give us:

$$\begin{aligned} &du = \zeta(\tau)\zeta'(\tau)d\tau \quad u = \frac{1}{2}\zeta^2 \\ &v = \frac{1}{\Gamma(1-q)}(t-\tau)^{-q} \quad dv = \frac{q}{\Gamma(1-q)}(t-\tau)^{-q-1} \end{aligned} \tag{2.59}$$

Expression (2.58) can be expressed as

$$-\left[\frac{\zeta^2(\tau)}{2\Gamma(1-q)(t-\tau)^q}\right]\Bigg|_{\tau=t} + \left[\frac{\zeta_0^2}{2\Gamma(1-q)\,(t-t_0)^q}\right] + \frac{q}{2\Gamma(1-q)}\int_{t_0}^{t}\frac{\zeta^2(\tau)}{(t-\tau)^{q+1}}d\tau \geqslant 0, \tag{2.60}$$

the first term of (2.60) contain an indetermination at $\tau = t$, so we analyze the corresponding limit.

$$\lim_{\tau\to t}\frac{\zeta^2(\tau)}{2\Gamma(1-q)(t-\tau)^q} = \frac{1}{2\Gamma(1-q)}\lim_{\tau\to t}\frac{[z(t)-z(\tau)]^2}{(t-\tau)^q} = \frac{1}{2\Gamma(1-q)}\lim_{\tau\to t}\frac{\left[z^2(t)-2z(t)z(\tau)+z^2(\tau)\right]}{(t-\tau)^q} = \frac{0}{0}, \tag{2.61}$$

since the function is derivable, applying the L'Hopital rule as follow:

$$\begin{aligned} \frac{1}{2\Gamma(1-q)}\lim_{\tau\to t}\frac{\left[z^2(t)-2z(t)z(\tau)+z^2(\tau)\right]}{(t-\tau)^q} &= \frac{1}{2\Gamma(1-q)}\lim_{\tau\to t}\frac{[-2z(t)\dot{z}(\tau)+2z(\tau)\dot{z}(\tau)]}{-q(t-\tau)^{q-1}} \\ &= \frac{1}{2\Gamma(1-q)}\lim_{\tau\to t}\frac{[2z(t)\dot{z}(\tau)-2z(\tau)\dot{z}(\tau)](t-\tau)^{1-q}}{q} = 0, \end{aligned} \tag{2.62}$$

then we can reduce (2.60) to:

$$\frac{\zeta_0^2}{2\Gamma(1-q)\,(t-t_0)^q} + \frac{q}{2\Gamma(1-q)}\int_{t_0}^{t}\frac{\zeta^2(\tau)}{(t-\tau)^{q+1}}d\tau \geqslant 0, \tag{2.63}$$

so the formula (2.60) is clearly true, and the proof ends here. ■

Remark 2.5.1. *Lemma (2.5.1) is still valid when $z(t) \in \mathbb{R}^m$, So:*

$$\forall q \in (0,1)\ , \forall t \geqslant t_0 : \ \frac{1}{2}\ {}_{t_0}D_t^q z^T(t)z(t) \leqslant z^T(t)\ {}^{c}_{t_0}D_t^q z(t) \tag{2.64}$$

The idea of the proof is the application of lemma (2.5.1) after decomposing the expression (2.64) into a sum of scalar products.

Corollary 2.5.1. *Consider the FOS:*

$$ {}^{C}_{t_0}D_t^q z(t) = \phi(z(t)),\ q \in (0,1),\ z(t) \in \mathbb{R}, \tag{2.65}$$

where $z^{eq} = 0$ is the equilibrium point, if the next condition is satisfied

$$z(t)\phi(z(t)) \leqslant 0, \quad \forall z, \tag{2.66}$$

so the origin of (2.23) is stable.

And if:

$$z(t)\phi(z(t)) < 0, \quad \forall z \neq 0, \tag{2.67}$$

so the origin of (2.23) is asymptotically stable.

Proof. Let the following positive definite Lyapunov candidate function:

$$V(z(t)) = \frac{1}{2}z^2(t), \tag{2.68}$$

The lemma (2.5.1) give us:

$$ {}^{C}_{t_0}D_t^q V(z(t)) \leqslant z(t) {}^{C}_{t_0}D_t^q z(t), \tag{2.69}$$

if $z(t)\phi(z(t)) \leqslant 0$, so $z(t){}^{C}_{t_0}D_t^q z(t) \leqslant 0$, and the fractional derivative (2.69) of the Lyapunov candidate function outcomes negative semidefinite. Using the comparison principle [45], this implies that $V(z(t)) \leqslant V(z(0)), \forall z$.

The definition of $V(z(t))$ implies:

$$\frac{1}{2}z^2(t) \leqslant \frac{1}{2}z^2(0), \quad \forall z, \tag{2.70}$$

expression (2.70) give us the conclusion that the origin of (2.65) is stable in the sense of Lyapunov, in accordance with the definition of stability in the sense of Lyapunov [38].
If $z(t)\phi(z(t)) < 0$ for all $z \neq 0$, then $z(t){}_{t_0}^{C}D_t^q z(t) < 0$, and the fractional derivative (2.69) of Lyapunov function results negative definite.
Given the relation between the class-K functions in [44] and positive definite functions, it can be concluded through theorem (2.5.1), that the origin of (2.65) is asymptotically stable. ■

Remark 2.5.2. *When the system (2.65) is vectorial, i.e.* $z(t) \in \mathbb{R}^m$*, the corollary (2.5.1) is still valid.*
The idea of the proof is to Apply the lemma (2.5.1) and choosing a Lyapunov candidate function defined by $V(z(t)) = \frac{1}{2}z^T(t)z(t)$.

Example 2.5.1. *We aim in this example to study the stability of the following* $3-D$ *system using a fractional Lyapunov function:*

$$\begin{cases} {}^{C}D_t^q x_1(t) = -x_1^3 - 2x_2 + x_2x_3, \\ {}^{C}D_t^q x_1(t) = x_1 - x_2^3 - x_1x_3, \\ {}^{C}D_t^q x_1(t) = x_1x_2 - x_3^3 - x_3. \end{cases} \tag{2.71}$$

Let's define the following Lyapunov function:

$$V(x_1, x_2, x_3) = \frac{1}{2}\left(x_1^2 + x_2^2 + \frac{1}{2}x_3^2\right), \tag{2.72}$$

which is a positive definite function on $\mathbb{R}^3$*,*
applying the fractional derivative of Caputo on (2.73) and the lemma (2.5.1) , we get:

$$V(x_1, x_2, x_3) = \left(\frac{1}{2}x_1^2 + x_2^2 + \frac{1}{2}x_3^2\right), \tag{2.73}$$

$$\begin{aligned}
{}^{C}D_t^{q}V(x_1,x_2,x_3)(t) &= {}^{C}D_t^{q}(\frac{1}{2}x_1^2+x_2^2+\frac{1}{2}x_3^2),\\
&\le \frac{1}{2}{}^{C}D_t^{q}x_1^2+{}^{C}D_t^{q}x_2^2+\frac{1}{2}{}^{C}D_t^{q}x_3^2,\\
&\le x_1{}^{C}D_t^{q}x_1+2x_2{}^{C}D_t^{q}x_2+x_3{}^{C}D_t^{q}x_3,\\
&\le x_1(-x_1^3-2x_2+x_2x_3)+2x_2(x_1-x_2^3-x_1x_3)+x_3(x_1x_2-x_3^3-x_3),\\
&\le -x_1^4-2x_2^4-x_3^4-x_3^2,\\
&< 0, \forall(x_1,x_2,x_3)\neq(0,0,0),
\end{aligned} \tag{2.74}$$

from (2.74), ${}^{C}D_t^{q}V(x_1,x_2,x_3)(t)$ is strictly negative definite on $\mathbb{R}^3$, then $(0,0,0)$ is asstmptotically stable equilibrium point of the system (2.71).

The figure (2.2) depict the evolution in time of the system (2.71) with initial conditions $x_1(0),x_2(0),x_3(0)=(3,1,-1)$ and fractional order $q=0.98$.

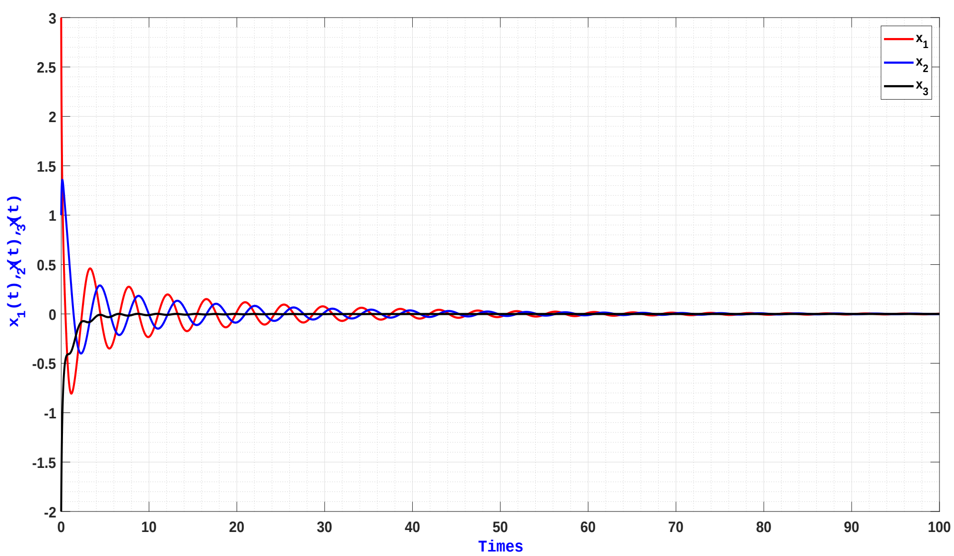

Figure 2.2: Evolution in time of the states x_1,x_2 and x_3 of the system (2.71).

2.6 Detection Of Chaos

2.6.1 History and development of chaos

The roots of chaos theory occured in the late 19th century with the work of the mathematician and physicist Henri Poincaré who studied the three body problem in celestial mechanics and he was trying to understand the motion of three interacting celestial bodies under the influence of gravity. Henri Poincare discovered that even seemingly simple systems like this could exhibit complex and unpredictable behavior over time. Also, he introduced the "butterfly effect" or "sensitive dependence on initial conditions", which suggests that small variations in initial conditions can lead to vastly different outcomes on the behavior of nonlinear dynamical systems. The study of chaos theory began in the 1960s and 1970s with the work of mathematicians and physicists: Edward Lorenz, Mitchell Feigenbaum and Robert May. In 1963, the meteorologist Edward Norton Lorenz made a significant contribution to chaos theory with his work on weather prediction models. Edward Lorenz discovered that small changes in initial conditions could lead to drastically different weather forecasts, which he famously referred to as the "butterfly effect ". in the late 1970s, Mitchell Feigenbaum made important discoveries related to chaos theory where he found that in certain nonlinear dynamical systems, there exist universal constants (called Feigenbaum constants) that govern the transition to chaos. These constants played a crucial role in understanding the onset of chaos in various systems. From 1980 to 1990, chaos theory was gained widespread recognition across various fields, including mathematics, physics, biology, economics, and even the social sciences. As a consequence, the researchers began applying chaos theory to a wide range of phenomena, from fluid dynamics to population dynamics to the behavior of financial markets. Since the late 20th century, research in chaos theory has continued to advance, with new discoveries and applications emerging in fields such as network theory, complex systems science and cryptosytems. It is necessary to detect chaos in nonlinear dynamical systems, we reminder in our work some methods such Lyapunov exponents and the binary $0-1$ test. The diagram of bifurcation or the spectral entropy analysis are used to show also the chaos in the system. We present here a briefly description of those methods. We use these methode to detect chaos in deterministic systems in the last chapter.

2.6.2 Definitions, properties and chaos transition scenarios

There is no universally accepted definition of chaos that has was associated with an iterative application formally introduced by Li and Yorke in 1975, where they established a simple criterion for chaos in one dimentional differences equations. So, the chaos in the sense of Li-Yorke is defined as follows:

Definition 2.6.1. *[51]*
Let $S = [0, 1]$ is the unit interval, a continuous map $\phi : S \to S$ is a chaos in the sense of Li-Yorke if there is an uncountable set $\Omega \subset S$ such that trajectories of any two points distinct y, x in Ω are proximal and not asymptotic, that is

$$\liminf_{n\to\infty} d\left(\phi^m(y), \phi^m(x)\right) = 0 \textit{and} \limsup d\left(\phi^m(y), \phi^m(x)\right) > 0. \tag{2.75}$$

Remark 2.6.1. *The need for uncountability of Ω in this definition (that is not in a general compact metric space, but for continuous maps of the interval) is equivalent with the condition that Ω contains two points, or that S is a perfect set (that is nonempty, compact and without isolated points).*

In the sense of devaney, the chaos is defined by:

Definition 2.6.2. *[51] A continuous map $\phi : U \to U$ (where U be a set) is said to be chaotic on U if:*

- *ϕ is topologically transitive: for any pair of open non-empty sets $V, W \subset U$ there exists a $\xi > 0$ such that $\phi^{\xi}(u) \cap W \neq \emptyset$.*
- *The periodic points of ϕ are dense in U.*
- *ϕ has sensitive dependence on the initial conditions: $\exists \rho > 0$ such that, $\forall y \in U$ and any neighborhood M of y, there exists a $x \in M$ and an $m \geq 0$ such that $\left|\phi^m(y) - \phi^m(x)\right| > \rho$.*

Remark 2.6.2. *Chaotic phenomena are not random but we can said that they obey to deterministic laws.*

The chaotic phenomena have characterized by basics properties, which are: the nonlinearity, the fractal structure and strange attractors.
Among possible scenarios of transition to chaos we have: the intermittence to the chaos, the doubling of the period and the scenario via the quasi-periodicity.

Quantification of chaos

In order to quantify chaos, we can use one of the following methods, reconstruction of the phase space, bifurcation diagrams, Lyapunov Exponents, $0-1$ test, Spectral entropy and C_0−complexity. As a point of view, the spectre of Lyapunov is the first robust method from them since its rich information about system behavior, the bifurcation diagrams is the second robust method, the other methods are used to confirm the results obtained from the spectre of lyapunov and of bifurcation diagrams.

Bifurcation diagrams

Henri poincare in his work is the first one who report the notion of bifurcation of a dynamical system. The bifurcation means a structural change in the orbit of a dynamical system. The study of bifurcation is concerned with how the structural change occurs when the parameter of the system are changing. The structural change and the transition behavior of a system are the central part of dynamical evolution. The point at which bifurcation occurs is known as the bifurcation point. The behavior of fixed point and the nature of trajectories may change dramatically at bifurcation points. The characters of attractor and repellor are altered, in general when bifurcation occurs. The diagram of the parameter values versus the fixed points of the system is knownas the bifurcation diagram. There are many type of bifurcation such that: saddle-node, pitchfork, transcritical, supercritical hopf, subcritical hopf and homoclinic and heteroclinic bifurcations, see in details the previous types with examples in [34]. In our work in the application chapters, we use the numerical simulation to plot the bifurcation diagrams, we found in the most case a succession of bifurcation fourche (doubling period or its inverse). This method contain some negatives under the plot of diagrams likes time of execution, it take a long time to plot the diagram, especially in the fractional cases when the bifurcation diagram is influenced by the memory effect of the system. So we opt in the first position to calculate the Lyapunov exponents.

2.6.3 Lyapunov Exponents

Alexandre Lyapunov has developed a quantity to measure the divergence of trajectories that are close at the start, this quantity is called "Lyapunov exponent" which is often used to determine whether a system is chaotic or not. For a continus DS we should discritize our system after start calculating the Lyapunov exponents.

The fundamental idea of Lyapunov is that adjacent trajectories of a fixed point, or any other point, are attracted or repelled at an exponentially fast rate. This gives a mean value at the rate of exponential growth for neighboring orbits of a map $\phi : \mathrm{E} \rightarrow \mathrm{E}$ move apart. If the map exhibits sensitive dependence on initial conditions then the distance between the neighboring orbits should increase exponentially. This would necessitate an average exponential rate that is positive and diverging from nearby orbits; as a result, a positive Lyapunov exponent is indicative of chaos. Let's consider a one-dimensional map $y_{m+1} = \phi(y_m)$ with y_0 and $y_0 + \delta$ are two neighboring initial points. The M th step of iteration have the form $y_0 \rightarrow \phi^M(y_0)$ and $(y_0 + \delta) \rightarrow \phi^M(y_0 + \delta)$, δ present a small quantity. λ is a number depending on the initial point y_0 and it is defined in the limit $M \rightarrow \infty$ as

$$\lim_{M\rightarrow\infty} e^{M\lambda} = \lim_{M\rightarrow\infty} \frac{\left|\phi^M(y_0+\delta) - \phi^M(y_0)\right|}{\delta}. \tag{2.76}$$

evidently, $\delta \rightarrow 0$ as $M \rightarrow \infty$. λ is a number which called the Lyapunov exponent. Evidently, the dynamics of a system is said chaotic if the number $\lambda > 0$ and it is said non chaotic or regular if $\lambda \leq 0$. Let's entering the logarithm in Eq. (2.76), we obtain the following formula for λ :

$$\begin{aligned}\lambda &= \lim_{M\to\infty} \frac{1}{M} \ln \frac{\left|\phi^M(y_0+\delta)-\phi^M(y_0)\right|}{\varepsilon} \\ \delta &\to 0 \\ &= \lim_{M\to\infty} \frac{1}{M} \ln \left|\frac{d\phi^M}{\mathrm{d}y}(y_0)\right| = \lim_{M\to\infty} \frac{1}{M} \ln \left|\left(\left(\phi^M\right)'(y_0)\right)\right|,\end{aligned} \tag{2.77}$$

we use the chain rule of differentiation, we get the formula:

$$\begin{aligned}\left(\phi^M\right)'(y_0) &= \left(\phi\left(\phi^{M-1}\right)\right)'(y_0) \\ &= \left(\phi'\left(\phi^{M-1}\right)\right)(y_0)\cdot\left(\phi^{M-1}\right)'(y_0) \\ &= \phi'\left(\phi^{M-1}(y_0)\right)\cdot\left(\phi'\left(\phi^{M-2}\right)\right)(y_0)\cdot\left(\phi^{M-2}\right)'(y_0) \\ &= \phi'\left(\phi^{M-1}(y_0)\right)\phi'\left(\phi^{M-2}(y_0)\right)\cdot\phi'\left(\phi^{M-3}(y_0)\right)\ldots\phi'\left(\phi(y_0)\right)\cdot\phi'(y_0),\end{aligned} \tag{2.78}$$

considering $\phi^k(y_0) = y_k \forall k \in \mathbb{Z}$, i.e , $y_1 = \phi(y_0), y_2 = \phi^2(y_0) = \phi(y_1), y_3 = \phi^3(y_0) = \phi(y_2), \ldots$, we write (2.78) as:

$$\begin{aligned}\left(\phi^M\right)'(y_0) &= \phi'(y_{M-1})\cdot\phi'(y_{M-2})\cdot\phi'(y_{M-3})\ldots\phi'(y_1)\cdot\phi'(y_0) \\ &= \prod_{i=0}^{M-1}\phi'(y_i),\end{aligned} \tag{2.79}$$

then,

$$\lambda = \lim_{m\to\infty} \frac{1}{M} \ln \left|\prod_{i=0}^{N-1} f'(x_i)\right| = \lim_{M\to\infty} \frac{1}{M} \sum_{i=0}^{M-1} \ln \left|f'(x_i)\right|, \tag{2.80}$$

it is necessary to highlight that λ have a dependence on the initial condition y_0. As a consequence, for a given initial condition y_0 and provided the limit exists, the Lyapunov exponent λ or $\lambda(y_0)$ of ϕ is obtained by:

$$\lambda = \lim_{M\to\infty} \frac{1}{M} \sum_{i=0}^{M-1} \ln \left|\phi'(y_i)\right|. \tag{2.81}$$

Our aim now is to present the method of Lyapunov exponents in the systems of higher dimension.

Let's a system of dimension m defined by:

$$y_{m+1} = \phi(y_m) \ , y_m \in \mathbb{R}^m, \tag{2.82}$$

select a point y in the system's phase space. Let $M(y)$, a small neighborhood of y_0, choose also another point $y_0+\Delta y_0$ in $M(y_0)$, Δy_0 is the separation of the neighboring y_0 and $y_0+\Delta y_0$. Note that (y_0 and $y_0 + \Delta y_0$) correspond to two orbits of the system during iterations. Let $y_0, y_1, y_2, \ldots, y_M, \ldots$ and $y_0 + \Delta y_0, y_1 + \Delta y_1, y_2 + \Delta y_2, \ldots, y_M + \Delta y_M, \ldots$, are respectively the successive points on the orbits , where $y_i = \phi(y_{i-1})$ and $y_i+\Delta y_i = \phi(y_{i-1}+\Delta_{y_{i-1}}), i = 1, 2, \ldots, M, \ldots$. Consequently, the successive neighboring points (y_0 and $y_0+\Delta y_0$) have a the separations between them on the orbits which given by:

$$\begin{aligned}
\Delta y_1 &= \phi(y_0 + \Delta y_0) - \phi(y_0) \simeq D\phi(y_0)\Delta y_0, \\
\Delta y_2 &= \phi(y_1 + \Delta y_1) - \phi(y_1) \simeq D\phi(y_1)\Delta y_1, \\
\Delta y_3 &= \phi(y_2 + \Delta y_2) - \phi(y_2) \simeq D\phi(y_2)\Delta y_2, \\
&\vdots \\
\Delta y_{M-1} &= \phi(y_{M-2} + \Delta y_{M-2}) - \phi(y_{M-2}) \simeq D\phi(y_{M-2})\Delta y_{M-2} , \\
\Delta y_M &= \phi(y_{M-1} + \Delta y_{M-1}) - \phi(y_{M-1}) \simeq D\phi(y_{M-1})\Delta y_{M-1} , \\
&\vdots
\end{aligned} \tag{2.83}$$

here $D\phi(y_i)_{m\times m}$ is the jacobian matrix of the map ϕ at the point y_i. Consequently, the separation Δy_M at the M th iteration is given by:

$$\Delta y_M = D\phi(y_{M-1})\, D\phi(y_{M-2}) \ldots D\phi(y_1)\, D\phi(y_0)\Delta y_0 = D_M \, \Delta \, y_0, \tag{2.84}$$

here $D_M = D\phi(y_{M-1})\, D\phi(y_{M-2}) \ldots D\phi(y_1)\, D\phi(y_0)$, is an $m \times m$ matrix depends on M and y_0. The separation between two neighboring orbits at the M th iteration is determined by the initial separation Δy_0 and the matrix D_M. Due to the multidimensionality of the phase space, the eigenvectors of the matrix D_M must be introduced as a natural basis for the vectors' decomposition in the phase space . We can get the eigenvectors $e(M)$ of D_M from the folowing formulla:

$$D_M e_i(M) = v_i(M) e_i(M), i = 1, 2, \ldots, M, \tag{2.85}$$

$e_i(M)$ and $v_i(M)$ are depends on M. We choose the eigenvectors generally orthonormal and if they are not orthonornal, we use the technique of GramSchmidt orthonormalization in order to make them orthonormal. The quantities $v_i(M)$ represents the eigenvalues of the matrix D_M which we use the caracteristic equation to determine them.

$$\det\left(D_M - v(M)I\right) = 0, \tag{2.86}$$

$I_{M\times M}$ represent the identity matrix. We obtain m eigenvalues from equation (2.86) . The initial separation Δy is obtained by the following formula by assuming in terms of the basis vectors $e_i(N)$.

$$\Delta y_0 = \sum_{i=1}^{n} c_i e_i(N), \tag{2.87}$$

c_i 's represents the coordinates of Δy in the basis $e(M)$. By substitute (2.87) into (2.84), we obtain:

$$\Delta y_M \;=\; D_M \sum_{i=1}^{M} c_i e_i(M) = \sum_{i=1}^{m} c_i\, D_M\, e_i(M) = \sum_{i=1}^{m} c_i\, v_i(M)\, e_i(M), \tag{2.88}$$

it is evident from (2.87) and (2.88) that $v_i(N)$ of the coefficient matrix D_M characterizes the separation of the orbits along the i th direction. We can find the Lyapunov exponents by utilizing the exponential rate of orbital separation. Let λ_i denote the Lyapunov exponent along the i th direction. So

$$|v_i(M)| \approx e^{\lambda_i M} \text{ for large } M, \tag{2.89}$$

this suggests

$$\lambda_i = \lim_{M\to\infty} \frac{1}{M} \ln\left(|v_i(M)|\right), i = 1, 2, \ldots, m. \tag{2.90}$$

The equation (2.90) gives us an estimation of the Lyapunov exponents for the multidimensional systems. The Lyapunov exponents are negative, zero as well as positive. A positive Lyapunov exponent signifies that we have a chaotic behavior. It should be noted that the number of Lyapunov exponents is equal to the dimension of the phase space.

Kaplan York dimension

Among the measures of complexity in chaotic systems we have the dimension of Kaplan-Yorke, it isemphasized that the systems that have the largest dimension of Kaplan-Yorke

(KYD) are the systems that have the more complex behavior than the others [15].

Definition 2.6.3. *(Kaplan York dimensione [56])*

Let i_0 a positive integer such that:

$$\sum_{j=1}^{i_0} \lambda_j \geq 0 \ \ and \sum_{j=1}^{i_0+1} \lambda_j < 0, \tag{2.91}$$

the dimension of Kaplan and Yorke is then defined by the following relation :

$$D_{KY} = i_0 + \frac{\sum_{j=1}^{i_0} \lambda_j}{|\lambda_{i_0+1}|}. \tag{2.92}$$

Definition 2.6.4. *(Dimension of Mori)*

Let m_0 be the number of Lyapunov exponents that are zero, m_+ be the number of positive exponents, $\bar{\lambda}_+$ the average of the positive exponents, and $\bar{\lambda}_-$ the average of the negative exponents. The Mori dimension is given by the following relation:"

$$D_m = m_0 + m_+\left(1 + \frac{\bar{\lambda}_+}{|\bar{\lambda}_-|}\right). \tag{2.93}$$

In the following table the different possible cases are given after the calculation of Lyapunov exponents.

State	Attractors	Dimension	Lyapunov Exponent
Equilibrium point	Point	0	$\lambda_n \leq \cdots \leq \lambda_1 < 0$
Periodic	Circl	1	$\lambda_1 = 0$ and $\lambda_n \leq \cdots \leq \lambda_2 < 0$
Period of ordre 2	Torus	2	$\lambda_1 = \lambda_2 = 0$ and $\lambda_n \leq \cdots \leq \lambda_3 < 0$
Period of ordre 2	k-Torus	k	$\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_n \leq \cdots \leq \lambda_{2k+1} < 0$
Chaotic		Not integer	$\lambda_1 > 0$ and $\sum_{i=1}^{n} \lambda_i < 0$
Hyperchaotic		Not integer	$\lambda_1 > 0, \lambda_2 > 0$ and $\sum_{i=1}^{n} \lambda_i < 0$

Table 2.1: Lyapunov Exponenents and dimension

2.6.4 Spectral Entropy Analysis

By using the correlation algorithm, meaning by the complexity of chaotic dynamical systems the degree of a chaotic sequence close to random sequence [46]. It is necessary to say that the more a sequence is close to the random sequence is the greater the complexity. The complexity is one of the tools that is used to characterize the dynamics of chaotic systems, its usage is similar with the bifurcation diagram and the LE. There are various ways to define the SE, we choose the following algorithm:

Spectral Entropy Complexity Algorithm

A step by step description of the spectral entropy complexity algorithm is given below. Using the Shannon entropy, we can use the Fourier transform to create an energy distribution and find the associated spectrum entropy [46].

Step 1: Let $A^M(m), m = 0, 1, \ldots, M-1$ define a chaotic pseudo-randomness sequence where its length is M. In order for the spectrum to more effectively reflect the energy information contained in the signal, we must first eliminate the DC (Direct Current) component of A^M using the formula below:

$$a(m) = a(m) - \bar{\mu}\ ; \text{ where } \bar{\mu} = \frac{1}{M}\sum_{m=0}^{M-1} a(m) \tag{2.94}$$

Step 2: Applying now to the sequence A(n) a discrete Fourier transform.

$$A(k) = \sum_{m=0}^{M-1} a(n)e^{-j\frac{2\pi}{M}mk} = \sum_{m=0}^{M-1} a(m)W_M^{mk}, \quad \text{where } k = 0, 1, 2, \ldots, M-1. \tag{2.95}$$

Step 3: We calculate in this step the relative power spectrum. Paserval's theorem is used to take and calculate the first half of the sequence for converted A(k). The power spectrum's value at one of its frequency points is given by:

$$p(k) = \frac{1}{M}|A(K)|^2, \quad \text{where } k = 0, 1, 2, \ldots, M/2-1, \tag{2.96}$$

so, the relative power spectrum of the sequence is:

$$P_k = \frac{p(k)}{p_{\text{tot}}} = \frac{\frac{1}{M}|A(k)|^2}{\frac{1}{M}\sum_{k=0}^{M/2-1}|A(k)|^2} = \frac{|A(k)|^2}{\sum_{k=0}^{M/2-1}|A(k)|^2}. \text{ Such that } \sum_{k=0}^{M/2-1} P_k = 1. \tag{2.97}$$

Step 4: Calculate now the spectral entropy. The Shannon entropy and the relative power spectral density P_k are used to calculate the spectral entropy se.

$$se = -\sum_{k=0}^{M/2-1} P_k \ln P_k, \tag{2.98}$$

take caution if that is $P_k = 0$ then we define $P_k \ln P_k = 0$. It can be shown that the spectral entropy value converges to $\ln(M/2)$. In order to facilitate comparison, the spectral entropy SE is normalized by:

$$SE(M) = \frac{se}{\ln(M/2)}. \tag{2.99}$$

It is evident that a non-uniform distribution of the sequence power spectrum leads to a signal with a distinct oscillation pattern and a simpler sequence spectrum structure. Here in this case, the SE measure is smaller, which implies that the complexity is smaller. If not, the complexity is larger.

2.6.5 The $0-1$ test

In this subsection we will present another method to quantify chaos in a deterministic dynamical systems, this method was introduced in 2003 by *Gottwald* and *Melbourne* [17]. The authors have illustrated in their publication that unlike the usual method of calculating the maximum exponent of Lyapunov, the method is applied directly to the data of the time series (temporal) and does not require the reconstruction of the phase space, also the dimension and complexity of the dynamic system do not matter to the method. Changes to the $0-1$ test algorithm are proposed in [17, 32, 19].

Starting with the description of the $0-1$ test :

Let the following system:

$$\begin{cases} \theta_c(m+1) = \theta_c(m) + c, \\ \\ \alpha_c(m+1) = \alpha_c(m) + \psi(m)\cos(\theta_c(m)), \\ \\ \beta_c(n+1) = \beta_c(m) + \psi(m)\sin(\theta_c(m)), \end{cases} \tag{2.100}$$

where $m = 1, 2, \cdots$. After the resolution of the system (2.100) using the theory of the difference equations and we choose a constant $c > 0$ and we get the following system:

$$\begin{cases} \theta_c(m) = cm \\ \\ \alpha_c(m) = \sum_{j=1}^{m} \psi(j)\cos(\theta_c(j)) \\ \\ \beta_c(m) = \sum_{j=1}^{m} \psi(j)\sin(\theta_c(j)) \end{cases} \tag{2.101}$$

Where: c is a strictly positive arbitrary constant, $\psi(m)$ is the observable of the deterministic dynamical system. The nature of the trajectory behavior in the $(\alpha - \beta)$ plane gives us the nature of the dynamical system tested.

- If the trajectories in the plane $(\alpha - \beta)$ are bounded then the dynamic system tested is regular (stationary).
- If trajectories in the $(\alpha - \beta)$ plane behave asymptotically as a Brownian motion then the dynamical system tested is chaotic.

Remark 2.6.3. *The method is independent of the shoice of ψ, for example if $y = (y_1, y_2, \dots, y_n)$, so the choice $\psi(y) = y_1$ is possible and simple, we can choose also $\psi(y) = y_2$, or $\psi(y) = y_1 + y_3$. indeed, there is a lot of possibility of choice. just we shoul choose an observable which depend on the other states in order to garantee that are all forme the Bronian motion.*

Calculates the mean square deplacement

For a time series of observations $\psi(j)$ wher $j = 1, \cdots, M$, calculate the mean square displacement $\mathcal{M}_c(m)$ of the two variables $\alpha_c(m)$ and $\beta_c(m)$ defined in (2.101):

$$\mathcal{M}_c(m) = \lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \Big((\alpha_c(j+n) - \alpha_c(j))^2 + (\beta_c(j+n) - \beta_c(j))^2 \Big), \tag{2.102}$$

if the trajectory behavior in $(\alpha - \beta)$ is Brownian, that is to say that the dynamical system tested is chaotic then $\mathcal{M}_c$ increases linearly compared to m. If the trajectory behavior in the $(\alpha - \beta)$ plane is bounded, that is to say that the dynamical system tested is non chaotic then $\mathcal{M}_c(m)$ is nonlinear. For each $c \in [0, \Pi]$, then:

$$\mathcal{M}_c(m) = V(c)m - V_{osc}(c, m) + \epsilon(c, m), \tag{2.103}$$

where the error $\epsilon(c, m)/nm \to 0$ as $m \to \infty$ uniformly in $c \in [0, \pi]$ and

$$V_{osc}(c, m) = E(\psi)^2 \frac{1 - \cos(mc)}{1 - \cos(c)}. \tag{2.104}$$

such that

$$E(\psi(y)) = \lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \psi(j),$$

the form (2.103) defines the modified mean square displacement $D(m)$:

$$D_c(m) = \mathcal{M}_c(m) - V_{osc}(c, m). \tag{2.105}$$

Remark 2.6.4. *The autocorrelation function for the observation $\psi(j)$ is given by:*

$$\rho(k) = E(\psi(1)\psi(k+1)) - (E(\psi))^2, k = 0, 1, 2, \cdots, \tag{2.106}$$

then $V(c)$ is given by

$$V(c) = \sum_{k=-\infty}^{+\infty} e^{ikc} \rho(|k|) = \lim_{m\to\infty} \frac{1}{m} E \left| \sum_{j=0}^{m-1} e^{ijc} \psi(j) \right|^2. \tag{2.107}$$

To calculate the asymptotic growth rate K we can use the correlation method or the linear regression method.

Linear regression method

One can determine K numerically by means of the linear regression of $log\mathcal{M}(m)$ compared to $log(m)$ So the asymptotic growth rate K is given by the formula [17, 19]:

$$K = \lim_{m\to\infty} \frac{\log \mathcal{M}(m)}{\log(m)}, \tag{2.108}$$

to avoid the negative logarithm we can calculate:

$$K = \lim_{m\to\infty} \frac{\log(\mathcal{M}(m) + 1)}{\log(m)}. \tag{2.109}$$

Correlation method

The correlation method for calculating the asymptotic growth rate is defined by the median value of the correlation coefficient K [20] :

$$K = \text{median}(K_c), \tag{2.110}$$

where

$$K_c = \frac{cov(\xi, \Delta)}{\sqrt{var(\xi)var(\Delta)}} \in [-1, 1], \tag{2.111}$$

such that

$$\xi = (1, 2, \cdots, m_{cut}) \tag{2.112}$$

and

$$\Delta = (D_c(1), D_c(2), \cdots, D_c(m_{cut})), \quad m_{cut} = \text{round}(M/10). \tag{2.113}$$

The variance and covariance are defined for the y and x length vectors β by:

$$cov(y, x) = \frac{1}{\beta} \sum_{j=1}^{\beta} (y(j) - \bar{y})(x(j) - \bar{x}), \tag{2.114}$$

where $$\bar{y} = \frac{1}{\beta} \sum_{j=1}^{\beta} y(j), \text{ and } var(y) = cov(y, y). \tag{2.115}$$

The rate **K** takes a binary value **0** or **1** which distinguishes whether the dynamical system tested is chaotic or not chaotic. If $\mathbf{K} \simeq 0$ means that the dynamic is not chaotic. If $\mathbf{K} \simeq 1$ means that the dynamic are chaotic.

Remark 2.6.5. *When comparing the presented test to the exhibitor method Lyapunov, there are a few advantages. These advantages include:*

1. *Without requiring the reconstruction of the phase spaces, the test is applied directly to a temporal series. By doing this, all potential difficulties with choosing the delay or diving dimension are avoided.*
2. *Because this test is binary (with outputs of either 0 or 1), conclusions can be concluded definitive on the behaviour of the system. For example $K = 0.01$ indicates that the dynamic is not chaotic, while this value for the largest exponent of Lyapounov leaves us in doubt.*
3. *Inspection of trajectories in the $(\alpha - \beta)$ plane provides a simple visual test to conclude whether the dynamics is chaotic or not.*

Conclusion 2.1.

The fundamental ideas of fractional dynamical systems and their use in chaos detection were examined in this chapter. In the framework of fractional-order systems, we first defined dynamical systems and investigated the stability of equilibrium points. In order to describe the long-term behavior of fractional systems, the idea of generalized Mittag-Leffler stability was presented as an essential extension of stability theory. Additionally, we talked about the Fractional Lyapunov Direct Method, which uses class K functions to look at system stability. This method provides a framework for examining the stability of fractional-order systems, which includes identifying potential Lyapunov functions that can be used as instruments for stability analysis. Lastly, the topic of detecting chaos in fractional dynamical systems was covered, including information on how these systems may behave chaotically and how to spot it with the right instruments and techniques.

CHAPTER 3

METHODS OF SYNCHRONIZATION OF FRACTIONAL DYNAMICAL SYSTEMS

3.1 Definition and different methods of synchronization of FOS

3.1.1 Intuitive definition of synchronization

We consider car drivers at a racetrack so that the first rider is going at 100 km/h and the second rider is going at 120 km/h. We want the drivers to go with the same speed in the rest of the race. For this, the driver of the first car can increase its speed from 100 to 120 km/h, the driver of the second car can also reduce its speed from 120 km/h to 100 km/h and the first could increase its speed by about 110 km/h, while the second rider would reduce its speed to 110 km/h. The process in which driver change their speeds to the same speed of the other driver in order to going with the same speed is called synchronization of the speed of the cars, and the process in which drivers decreeze or increase their speeds is called speed control. In the other hands, the synchronization in the oxford dictionary mean the fact of happening at the same time or moving at the same speed as something else. Generally, synchronization is a way of maintaining a periodic or chaotic movement. Synchronizing two dynamic systems means that each system evolves according to the behavior of the other system in the following subsection we give the mathematical definition of synchronization.

3.1.2 Mathematical definition of synchronization

A first attempt to define a synchronized movement was presented in [8]. In [9], Brown and Kocarev give a generalisation of the definition given in [8]. To construct the definition, they assume that a dynamical system, global, finite-dimensional and deterministic is divisible into two subsystems:

$$\begin{cases} \frac{dx(t)}{dt} = \phi_1(x, y, t), \\ \frac{dy(t)}{dt} = \phi_2(y, x, t), \end{cases} \tag{3.1}$$

where, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are vectors that can have different dimensions. Let $\psi(X_0)$ a global system trajectory given by (3.1) with the initial condition $X_0 = [x_0, y_0] \in \mathbb{R}^m \times \mathbb{R}^n$. For each subsystem a trajectory is formed $\psi_x(X_0)$ et $\psi_y(X_0)$ (X_0 being a given initial condition).

Note by ζ the space of all trajectories of the first subsystem, and by η the space of all trajectories of the second subsystem, and consider two functions (properties)
$f_x : \zeta \times \mathbb{R} \to \mathbb{R}^\delta$ and $f_y : \eta \times \mathbb{R} \to \mathbb{R}^\delta$, which are not identically null, the first $\mathbb{R}$ represents time, we say that the functions, f_x and f_y, are properties of the subsystems defined by (3.1) respectively. Finally, to define a synchronized state, Brown and Kocarev [9] require a function $\chi\left(f_x, f_y\right) : \mathbb{R}^\delta \times \mathbb{R}^\delta \to \mathbb{R}\delta$ such as $|\chi| = 0$ or $|\chi| \to 0$, (where $|\cdot|$ is any norms).
We say that the χ, which is time independent, compares the measured properties on the

two subsystems. Both measures are appropriate over time if and only if $\chi\left(g_x, g_y\right)$ =0. With these preliminaries, [9] proposes the following definition for synchronization:

Definition 3.1.1. *Subsystems in equations (3.1) are synchronized on the trajectory of* $\psi\,(\mathrm{X}_0)$*, compared to the properties* f_x *and* f_y*, if there is an instant independent of the* χ *application such as* $\left|\chi\left(f_x, f_y\right)\right| = 0$*.*

Definition 3.1.2. *Subsystems in equations (3.1) are synchronized with* f_x *and* f_y*, if there is an instant independent of the* χ *application such as* $\left|\chi\left(f_x, f_y\right)\right| = 0$ *on all trajectories. With the choice of* f_x, f_y *and* χ *we can determine the type of synchronization. This approach leads to the idea that there are different types of synchronization that could be engaged in the same formalism.*

Theorem 3.1.1. *The master system and the slave system are synchronized if and only if all Lyapunov exponents of the slave system, called conditional Lyapunov exponents, are negative.*

3.2 Some type of synchronizaion

In this section, we introduce some types of synchronization namely Full (complete) synchronization (CS), anti-synchronization, delayed synchronization, projective synchronization (PS), generalized synchronization (GS) , Q-S synchronization, adaptive synchronization, FSHPS synchronization and IFSHPS.

3.2.1 Complete synchronization or Full synchronization

We consider a master (drive) chaotic system represented by:

$$D^q X(t) = \phi(X(t)), \tag{3.2}$$

where $X(t)$ is the state vector of the master system of dimension $m, \phi : \mathbb{R}^m \to \mathbb{R}^m, D^q$ is the fractional derivative in the sense of Caputo, and the slave (response) system represented by the following formula:

$$D^q Y(t) = \psi(Y(t)) + U(t), \tag{3.3}$$

where $Y(t)$ is the state vector of the slave system of dimension $n, \psi : \mathbb{R}^n \to \mathbb{R}^n, U = (\eta_i)_{i=1}\, m \in \mathbb{R}^m$ determine the control vector and D^q is the fractional derivative in the sense of Caputo.

The complete synchronization error is defined by:

$$\epsilon(t) = Y(t) - X(t) \text{ such that } \lim_{t \to +\infty} |\epsilon(t)| = 0, \tag{3.4}$$

where $\|.\|$ the Euclidean norm. We destinguish two cases:

1. If $\phi = \psi$, the relationship between (3.2) and (3.3) is called an identical full synchronization.

2. If $\phi \neq \psi$, the relationship between (3.2) and (3.3) is called a not identical full synchronization.

Therefore, the synchronization (CS) corresponds to a complete coincidence between the state variables of the two synchronized systems.

3.2.2 Anti synchronization

Theoretically, we said that two systems are anti-synchronized if the master system and the slave system have identical state vectors in absolute value but with opposite signs and on the other hand, the sum of the state vectors of the two systems tends towards zero when time tends towards infinity [1]. The anti synchronization error can be defined by:

$$\epsilon(t) = Y(t) + X(t). \tag{3.5}$$

3.2.3 Delayed synchronization

The researchers found that two different chaotic dynamic systems can expose a synchronization phenomenon in which the dynamic variables of the two systems become synchronized, but with a time lag [10]. It is said that one has a delayed (or anticipated) synchronization if the $Y(t)$ state variables of the slave chaotic system converge to the $X(t)$ state variables lagged in time of the master chaotic system as indicated by the relationship below:

$$\lim_{t\to+\infty} |Y(t) - X(t-\tau)| = 0, \forall X(0), \tag{3.6}$$

with τ is a very small positive number.

3.2.4 Projective synchronization

We have a projective synchronization if the state variables $y_i(t)$ of the slave chaotic system $Y(t) = (y_i(t))_{1\leq i\leq m}$ in (3.3) synchronize with a multiple constant of the state $x_i(t)$ of the master chaotic system $X(t) = (x_i(t))_{1\leq i\leq m}$ in 3.4, such as [**?**]:

$$\exists \xi_i \neq 0, \lim_{t\to+\infty} \left\| y_i(t) - \xi_i x_i(t) \right\| = 0, \forall (x(0), y(0)), i = 1, 2, \ldots, m, \tag{3.7}$$

in (3.7) we destinguish according to the value of ξ the following cases:

1. If all ξ_i are equal to 1, the case represents the complete synchronization.

2. If all ξ_i are equal to -1, the case represents the complete anti-synchronization.

3.2.5 Generalized synchronization

Generalized synchronization is considered a generalization of complete synchronization, anti-synchronization and projective synchronization in the case of chaotic systems of different dimensions and models [59]. So, we consider a couple of master-slave systems represented by:

Definition 3.2.1.

$$\begin{cases} D^q X(t) = \phi(X(t)), \\ D^q Y(t) = \psi(Y(t)) + U, \end{cases} \tag{3.8}$$

where $X(t) \in \mathbb{R}^m, Y(t) \in \mathbb{R}^n$ are the states of the master system and the slave system, respectivelyt, $\phi : \mathbb{R}^m \to \mathbb{R}^m, \psi : \mathbb{R}^n \to \mathbb{R}^n, D^q$ is the fractional derivative of Caputo and

$U = (\eta_i)_{i=1}^{m} \in \mathbb{R}^m$ is a controller to be determined.

Definition 3.2.2. *If there is a function $\Theta : \mathbb{R}^m \to \mathbb{R}^n$, such that all trajectories of the master and slave systems, with the initial conditions $x(0)$ and $y(0)$ verify:*

$$\lim_{t\to+\infty} \|Y(t) - \Theta(X(t))\| = 0, \forall x(0), y(0), \tag{3.9}$$

then, the master-slave systems (3.9) synchronize in the generalized sense with respect to the function Θ.

3.2.6 QS synchronization

Q-S synchronization is considered a generalization of all previous synchronizations [28]. We will say that a master system $X(t)$ with m-dimensional, and a slave system $Y(t)$ with n-dimensional, are in Q-S synchronization in the dimension δ, if there is a controller $U = (_i)_{1\leq i\leq n}$ et deux fonctions $Q : \mathbb{R}^m \to \mathbb{R}^\delta, S : \mathbb{R}^n \to \mathbb{R}^\delta$ such as the synchronization error

$$e(t) = Q(X(t)) - S(Y(t)), \tag{3.10}$$

verifies

$$\lim_{t\to+\infty} \|\epsilon(t)\| = 0. \tag{3.11}$$

3.2.7 Adaptive synchronization

Consider the master and slave systems described by [47].

$$\begin{cases} \dot{x} = g(x) + G(x)Q, \\ \dot{y} = f(y) + F(y)P + U, \end{cases} \tag{3.12}$$

where $y \in R^m, x \in R^m$ are the state vectors in slave system and master system, respectively; $g(x)$ is an $m \times 1$ matrix and $G(x)$ is an $m \times p$ matrix in master system. In the same way, in slave system, $f(y)$ is an $m \times 1$ matrix and $F(y)$ is an $m \times q$ matrix. Note that $Q \in R^q$ and $P \in R^p$ are uncertain parameter vectors.

We have to give firstly the following lemma which will be utilized throughout our designing process for the stability analysis.

Lemma 3.2.1. *[29]*

If $g, \dot{g} \in L_\infty$, and $fg \in L_p$ for some $p \in [1, \infty)$, then $g(t) \to 0$ as $t \to \infty$.

The synchronization error is defined as $\epsilon(t) = y(t) - x(t)$, describe the parameter estimation error in more detail as $Q(t) = Q - \hat{Q}$ and $P(t) = P - \hat{P}$. Then we get the error as:

$$\begin{aligned}\dot{\epsilon} =& \dot{y} - \dot{x} = f(y) - g(x) + F(y)Q - G(x)P + U \\ =& F(y)Q(t) - G(x)P(t) + (U + f(y) - g(x) \\ &+ F(y)\hat{Q} - G(x)\hat{P}) \\ =& F(y)Q(t) - G(x)P(t) + \bar{U},\end{aligned} \tag{3.13}$$

where $\bar{U} = (U + f(y) - g(x) + F(y)\hat{Q} - G(x)\hat{P})$. Achieving synchronization and identifying the unknown parameters are our two main objectives. Using the common quadratic form, we create the next positive definite Lyapunov function:

$$\begin{aligned}&V_1(\epsilon) = \epsilon^T \epsilon / 2, \\ &V_2(P, Q) = \left(P(t)^T (P(t) + (Q(t)^T (Q(t))\right) / 2,\end{aligned} \tag{3.14}$$

and $V = V_1(\epsilon) + V_2(P, Q)$. Now, differentiating V along the error trajectory (3.12) gives:

$$\begin{aligned}\dot{V} &= \epsilon^T (F(y)Q(t) - G(x)P(t) + \bar{U}) + Q(t)^T \dot{Q(t)} + P(t)^T \dot{P(t)} \\ &= \epsilon^T \bar{U} + \epsilon^T F(y)Q(t) + Q(t)^T \dot{Q(t)} - \epsilon^T G(x)P(t) + P(t)^T \dot{P(t)},\end{aligned} \tag{3.15}$$

Additionally, if we choose

$$\begin{cases} \bar{U} = -K\epsilon, \\ \dot{Q(t)} = -\dot{\hat{Q}} = -F^T(y)\epsilon, \\ \dot{P(t)} = -\dot{\hat{P}} = G^T(x)\epsilon, \end{cases} \tag{3.16}$$

After that, we can get $\dot{V} = -K\epsilon^2 \leq 0$, where K are predefined positive controlling gains. Since $V(0)$ and $V(t)$ are bounded. As a consequence, from (3.12), the parameter estimates and the state trajectories are also bounded, i.e., $y, x, \epsilon, \hat{P}, \hat{Q} \in L_\infty$.

From (3.15) we know that for $t \in [0, +\infty)$, $\dot{\epsilon}(t)$ exists and is bounded . From $\dot{V} = -K\epsilon^2 \leq 0$, we get

$$\int_0^\infty \epsilon^2(\tau)d\tau = -\frac{1}{K}\int_0^\infty \dot{V}d\tau = \frac{1}{K}(V_0 - V_\infty), \tag{3.17}$$

where $V_0 = V(\epsilon(0), P(0), Q(0))$. The equation above shows that $\epsilon \in L_2$. Based on the outcome of Lemma (3.2.1), we can conclude that $\epsilon(t) \to 0$ as $t \to \infty$, which implies that the synchronization would eventually be achieved. In addition to that, this in turn also sugests that $\dot{Q(t)}, \dot{P(t)} \to 0$ as $t \to \infty$.

3.2.8 FSHPS synchronization

Consider the master system (3.18) and the slave system (3.19) defined by:

$${}^C D_t^{q_i} x_i(t) = f_i(X(t)), \tag{3.18}$$

$${}^C D_t^{p_i} y_i(t) = \sum_{j=1}^{m} b_{ij} y_j(t) + g_i(Y(t)) + \eta_i, \tag{3.19}$$

where:

$$\begin{cases} \bullet b_{ij} \in M_{m\times m}(\mathbb{R}), \\ \bullet X(t) = (x_1, x_2, \ldots, x_m)^T \text{ and } Y(t) = (y_1, y_2, \ldots, y_m)^T \text{ are the states vecors of systems (3.18) and (3.19) ,} \\ \bullet f_i, g_i : \mathbb{R}^m \longrightarrow \mathbb{R}^m \text{ are nonlinear function,} \\ \bullet 0 < p_i, q_i < 1, \\ \bullet D^{p_i} \text{ and } D^{q_i} \text{ are the caputo fractional derivative,} \\ \bullet \eta_i (i = 1, 2, \ldots, m) \text{ are controllers to be designed.} \end{cases} \tag{3.20}$$

Now we can give the definition of (FSHPS) as follow:

Definition 3.2.3. *Full State Hybrid Projective Synchronization (FSHPS) occur in the synchronization between the master system (3.18) and the slave system (3.19) if there exist controllers* $\eta_i, i = \overline{1,m}$ *and given real numbers* $\left(b_{ij}\right)_{1\le i,j\le m}$ *such that the synchronization error are given by:*

$$\epsilon_i(t) = y_i(t) - \sum_{j=1}^{n} b_{ij}x_j(t), i = 1, \ldots, m, \tag{3.21}$$

satisfy $\lim_{t\to+\infty} \epsilon_i(t) = 0$*, for* $i = 1, 2, \ldots, m$*.*

3.2.9 IFSHPS synchronization

Consider the master system (3.22) and the slave system (3.23) defined by:

$$^{C}D_t^{p_i} x_i(t) = \sum_{j=1}^{m} a_{ij}x_j(t) + f_i(X(t)), \tag{3.22}$$

$$^{C}D_t^{q_i} y_i(t) = g_i(Y(t)) + \eta_i, \tag{3.23}$$

where:

$$\begin{cases} \bullet a_{ij} \in M_{m\times m}(\mathbb{R}), \\ \bullet X(t) = (x_1, x_2, \ldots, x_m)^T \text{ and } Y(t) = (y_1, y_2, \ldots, y_m)^T \text{ are the states vecors of systems (3.22) and (3.23) ,} \\ \bullet f_i, g_i : \mathbb{R}^m \longrightarrow \mathbb{R} \text{ are nonlinear function,} \\ \bullet 0 < p_i, q_i < 1, \\ \bullet D^{p_i} \text{ and } D^{q_i} \text{ are the caputo fractional derivative,} \\ \bullet \eta_i (i = 1, 2, \ldots, m) \text{ are controllers to be designed,} \end{cases} \tag{3.24}$$

Now we can give the definition of (IFSHPS) as follow:

Definition 3.2.4. *Inverse Full State Hybrid Projective Synchronization (IFSHPS) occur in the synchronization between the master system (3.22) and the slave system (3.23) if there exist controllers $\eta_i, i = \overline{1,m}$ and given real numbers $\left(a_{ij}\right)_{1\leq i,j\leq m}$ such that the synchronization error are given by:*

$$\epsilon_i(t) = x_i(t) - \sum_{j=1}^{m} a_{ij} y_j(t), i = 1, \ldots, m \tag{3.25}$$

satisfy $\lim_{t\to+\infty} \epsilon_i(t) = 0$*, for* $i = 1, 2, \ldots, m$*.*

Example 3.2.1. *We consider the slave system given by:*

$$\begin{cases} {}_0^C D_t^{q_1} x(t) = \sigma(y - x) + sv + v_1, \\ {}_0^C D_t^{q_2} y(t) = rx - y - xz + v_1, \\ {}_0^C D_t^{q_3} z(t) = xy - bz + v_1, \\ {}_0^C D_t^{q_3} z(t) = -x - \sigma v + v_1, \end{cases} \tag{3.26}$$

and the master system given by:

$$\begin{cases} {}_0^C D_t^{q_1} x(t) = -ax - ey^2, \\ {}_0^C D_t^{q_2} y(t) = by - kxz, \\ {}_0^C D_t^{q_3} z(t) = -cz + mxy, \end{cases} \tag{3.27}$$

the state error between (3.27) and (3.26) of FSHPS is defined by:

$$\epsilon_i(t) = y_i - \left(\sum_{j=1}^{3} \xi_{ij} x_j\right), i = \overline{1,4}, \tag{3.28}$$

this yield:

$$\mathcal{D}_t^{q_i} \epsilon_i(t) = \mathcal{D}_t^{q_i} y_i - \mathcal{D}_t^{q_i} \left(\sum_{j=1}^{3} \xi_{ij} x_j\right), i = \overline{1,4}, \tag{3.29}$$

with respect to (3.29), then the error system between the slave system (3.27) and the master system (3.26) result as:

$$\begin{cases} \mathcal{D}_t^{q_1}\epsilon_1(t) = \sigma(y-x) + sv + v_1 - \mathcal{D}_t^{q_1}\Big(\xi_{11}x_1 + \xi_{12}x_2 + \xi_{13}x_3\Big), \\ \mathcal{D}_t^{q_1}\epsilon_2(t) = rx - y - xz + v_2 - \mathcal{D}_t^{q_2}\Big(\xi_{21}x_1 + \xi_{22}x_2 + \xi_{23}x_3\Big), \\ \mathcal{D}_t^{q_3}\epsilon_3(t) = xy - bz + v_3 - \mathcal{D}_t^{q_3}\Big(\xi_{31}x_1 + \xi_{32}x_2 + \xi_{33}x_3\Big), \\ \mathcal{D}_t^{q_4}\epsilon_4(t) = -x - \sigma v + v_4 - \mathcal{D}_t^{q_3}\Big(\xi_{31}x_1 + \xi_{32}x_2 + \xi_{33}x_3\Big), \end{cases} \tag{3.30}$$

we can write the system (3.30) as:

$$\begin{cases} \mathcal{D}_t^{q_1}\epsilon_1(t) = \Big(\sum_{j=1}^{3} b_{1j}\epsilon_j\Big) + R_1 + V_1, \\ \mathcal{D}_t^{q_1}\epsilon_2(t) = \Big(\sum_{j=1}^{3} b_{2j}\epsilon_j\Big) + R_2 + V_2, \\ \mathcal{D}_t^{q_3}\epsilon_3(t) = \Big(\sum_{j=1}^{3} b_{3j}\epsilon_j\Big) + R_3 + V_3, \\ \mathcal{D}_t^{q_4}\epsilon_4(t) = \Big(\sum_{j=1}^{3} b_{4j}\epsilon_j\Big) + R_4 + V_4, \end{cases} \tag{3.31}$$

with :

$$\begin{cases} R_1 = -ax - ey^2 - \Big(\sum_{j=1}^{4} b_{1j}\epsilon_j\Big) - \mathcal{D}_t^{q_1}\Big(\xi_{11}x_1 + \xi_{12}x_2 + \xi_{13}x_3\Big), \\ R_2 = by - kxz - \Big(\sum_{j=1}^{4} b_{2j}\epsilon_j\Big) - \mathcal{D}_t^{q_2}\Big(\xi_{21}x_1 + \xi_{22}x_2 + \xi_{23}x_3\Big), \\ R_3 = -cz + mxy - \Big(\sum_{j=1}^{4} b_{3j}\epsilon_j\Big) - \mathcal{D}_t^{q_3}\Big(\xi_{31}x_1 + \xi_{32}x_2 + \xi_{33}x_3\Big), \\ R_4 = -cz + mxy - \Big(\sum_{j=1}^{4} b_{4j}\epsilon_j\Big) - \mathcal{D}_t^{q_4}\Big(\xi_{31}x_1 + \xi_{32}x_2 + \xi_{33}x_3\Big), \end{cases} \tag{3.32}$$

the system (3.31) can be written as follows:

$$\mathcal{D}_t^{q_i}\epsilon_i(t) = \Big(\sum_{j=1}^{4} b_{ij}e_j\Big) + R_i + V_i, \tag{3.33}$$

in the compact form the error system (3.33) become:

$$ {}^{C}_{t}D^{q}_{t}\epsilon = Be + R + V, \tag{3.34}$$

where $B = (b_{ij})_{3\times 5}$, $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)^T$, $R = (R_i)_{i=\overline{1,4}}$ *and* $V = (V_i)_{i=\overline{1,4}}$.

In order to achieve the required synchronization, let define the control law by:

$$\begin{aligned}
V_1 &= -\Big(- ax - ey^2 - \Big(\sum_{j=1}^{4} b_{1j}\epsilon_j\Big) - {}^{C}D_t^{q_1}\Big(\xi_{11}x_1 + \xi_{12}x_2 + \xi_{13}x_3\Big)\Big) \\
&\quad - \Big(\sum_{j=1}^{4} c_{1j}\epsilon_j\Big), \\
V_2 &= -\Big(by - kxz - \Big(\sum_{j=1}^{3} b_{2j}\epsilon_j\Big) - {}^{C}D_t^{q_2}\Big(\xi_{21}x_1 + \xi_{22}x_2 + \xi_{23}x_3\Big)\Big) \\
&\quad - \Big(\sum_{j=1}^{4} c_{2j}\epsilon_j\Big), \\
V_3 &= -\Big(- cz + mxy - \Big(\sum_{j=1}^{4} b_{3j}\epsilon_j\Big) - {}^{C}D_t^{q_3}\Big(\xi_{31}x_1 + \xi_{32}x_2 + \xi_{33}x_3\Big)\Big) - \Big(\sum_{j=1}^{3} c_{3j}\epsilon_j\Big). \\
V_4 &= -\Big(- \Big(\sum_{j=1}^{4} b_{4j}\epsilon_j\Big) - {}^{C}D_t^{q_4}\Big(\xi_{41}x_1 + \xi_{42}x_2 + \xi_{43}x_3\Big)\Big) - \Big(\sum_{j=1}^{4} c_{4j}\epsilon_j\Big).
\end{aligned} \tag{3.35}$$

Theorem 3.2.1. *The FSHPS occurs between the master system (3.27) and the slave system (3.26) under the control law defined by:*

$$V = -R - C\epsilon, \tag{3.36}$$

where $C = (c_{ij}) \in M_3(\mathbb{R})$ *is feedback gain matrix selected in such way* $B - C$ *is a negative definite matrix.*

Using the FSHPS we get : $B = \begin{pmatrix} -1 & 1 & 0 & 1.5 \\ 26 & -1 & 0 & 0 \\ 0 & 0 & -0.7 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$ *We choose the matrix* C *and* $(\xi_{ij})_{3\times 4}$ *as follows:*

$$C = \begin{pmatrix} 10 & 15 & 0 \\ -5 & 7.5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } (\xi_{ij})_{3\times 3} = \begin{pmatrix} -1 & 5 & 1 & -1 \\ 4 & 2 & -3 & 4 \\ -2 & 1 & 6 & 7 \end{pmatrix},$$

with the previous data , finally we obtain the error system as follows:

$$\begin{cases} {}_0^C D_t^{q_1} \epsilon_1(t) = -2\epsilon_1, \\ {}_0^C D_t^{q_1} \epsilon_2(t) = -5\epsilon_2, \\ {}_0^C D_t^{q_3} \epsilon_3(t) = -8\epsilon_3, \\ {}_0^C D_t^{q_3} \epsilon_3(t) = -11\epsilon_3, \end{cases} \tag{3.37}$$

all eigenvalues of system (3.37) are negative ($\lambda_1 = -2, \lambda_1 = -5, \lambda_1 = -8, \lambda_1 = -11$*), so the system (3.37) is asymptotically stable. Hence, the synchronization between the master system (3.27) and the slave system (3.26) is achieved.*

Conclusion 3.1.

The many approaches for synchronizing fractional-order dynamical systems (FOS) have been reviewed in this chapter. We started by defining synchronization in terms of fractional-order systems. After that, we looked at a number of important synchronization techniques, including the CS, AS, DS, PS, GS, QS, FSHPS, IFSHPS and the adaptive synchronization strategy.

CHAPTER 4

STABILIZATION AND SYNCHRONIZATION VIA ADAPTIVE CONTROL WITH CIRCUIT DESIGN OF SOME FRACTIONAL CHAOTIC SYSTEMS

The aim of this chapter is to studies some new fractional chaotic systems [4, 5]. Firstly, we introduce a $3-D$ *chaotic system, its dynamical analysis, adaptive synchronization and its circuit desing. Furthermore, an extention to fractional order case with stabilization and fractional circuits design are implemented. Secondly, an hyperchaotic fractional system is given with stabilization and fractional circuit design also implemented. finally, the system of Ma is investigated throught the adaptive synchronization and circuit design.*

All electronic circuits are implemented to show the reliability of the proposed systems in secure communication.

4.1 Study of the new 3D chaotic system

4.1.1 Description of the new 3-D chaotic system

Let α, β, γ three positive reals parameters and x_1, x_2, x_3 are the states variables. With a simple modification in the famous Lorenz system, we can get a new $3-D$ chaotic system as follows:

$$\begin{cases} \frac{dx_1}{dt} = \alpha(x_2 - x_1), \\ \frac{dx_2}{dt} = \gamma x_1 - x_2 - x_1 x_3, \\ \frac{dx_3}{dt} = \beta x_1 x_2 x_3 - x_2 - \beta x_3, \end{cases} \tag{4.1}$$

for $\gamma = 35, \alpha = 10$ and $\beta = 3$, the system (4.1) display chaotic behavior. From the caracteristic of strange attractors is that it volume in the space is null (we will confirm this idea in the next subsections using the Kaplan York dimension), that is what we see when we plot the phase portrait of the system (4.1). Indeed, we obtain a strange attractor with no volume in the space and the projections in different planes $(x_1 - x_2)$, $(x_1 - x_3)$ and $(x_2 - x_3)$ in figure (4.1) and figure (4.2).

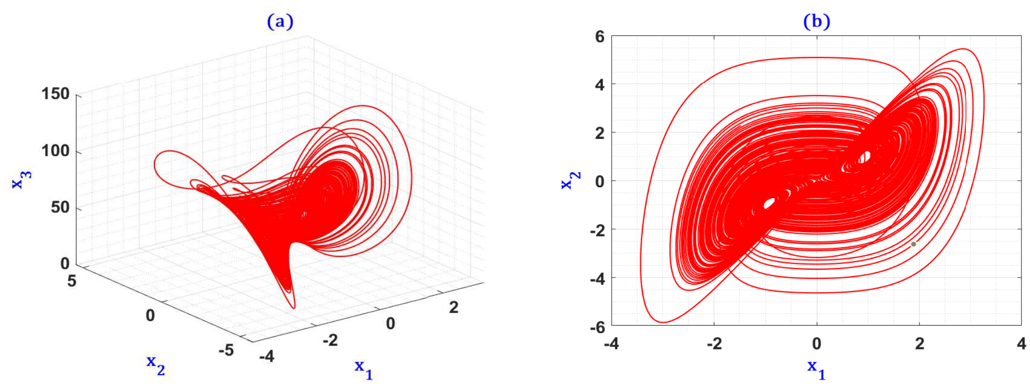

Figure 4.1: Strange attractor and projection in $(x_1 - x_2)$ plane of (4.1).

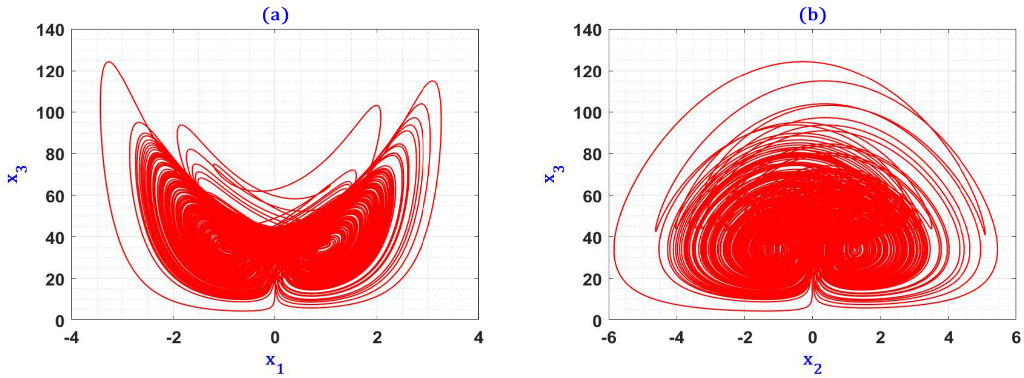

Figure 4.2: Projection in $(x_1 - x_3), (x_2 - x_3)$ planes of (4.1).

4.1.2 Elementary properties of the new chaotic system

The study of any dynamical system need to analyze their comportment. We need firstly to study the nature of our system, it is dissipative or conservative, this notion is so important and he give us a first vision regarding their structure. Also, we requisite to calculate the equilibrium points of our system, study their stability in order to discover their comportment. The next subsections are aims to clarivate the previous points about the system (4.1).

Dissipative or Conservative ?

A dynamical system is dissipative if the divergence of its vector field is negative. We put system (4.1) under vector notation as:

$$\dot{X} = g(X) = \begin{pmatrix} g_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) \\ g_3(x_1, x_2, x_3) \end{pmatrix}, \tag{4.2}$$

let κ a region in R^3 with a smooth boundary. Applying ϕ_t the flow of g on κ we get $\kappa(t) = \phi_t(\kappa)$. Moreover, $V(t)$ is the volume of $\kappa(t)$. Using Liouville's theorem, we obtain:

$$\dot{V}(t) = \int_{\kappa(t)} (\nabla \cdot g) dx_1 dx_2 dx_3, \tag{4.3}$$

with

$$\nabla \cdot g = \frac{\partial \dot{g}_1}{\partial x_1} + \frac{\partial \dot{g}_2}{\partial x_2} + \frac{\partial \dot{g}_3}{\partial x_3} = -(\alpha + \beta + 1) + \beta x_1 x_2, \tag{4.4}$$

it is clear that for $\beta = 3, \gamma = 35, \beta = 3, \alpha = 10$ *and* $x_1 x_2 < \frac{14}{3}$*, we obtain:*

$\nabla.V < 0$ *which mean that the volume comprising of the trajectories as* $t \longrightarrow \infty$ *with an exponential decay eventually shrink to zero. Finally, the asymptotic motion settles on the attractor. As a result the novel system (4.1) is dissipative.*

Lyapunov exponents and 0-1 test for detecting chaos

In this subsection we will prove the chaoticity of the system (4.1) by calculating their lEs (figure 4.4). Using also the binary $0-1$ *test we calculate the rapport* k *and the nature of trajectories are inspected in the plane* $(p-q)$ *[19].*
Numerical simulation can be implemented in Matlab and it gives us:

$$LE_1 = 1.940505, LE_2 = 0.000099, LE_3 = -12.939005, \tag{4.5}$$

the novel system (4.1) exhibit chaotic behavior because LE_1 *is a positive Lyapunov exponent and* $\sum_{i=1}^{3} LE_i < 0$*. Also, using the* $0-1$ *test we choose the random constant* $(C \in [0;\pi])$*, as a consequence we find* $k_C = 0.9968 \cong 1$ *(figure 4.4). Moreover, a brownian motion is obtained in* $(p-q)$ *plane, which confirm the behavior obtained by the Lyapunov exponents (figure 4.5).*

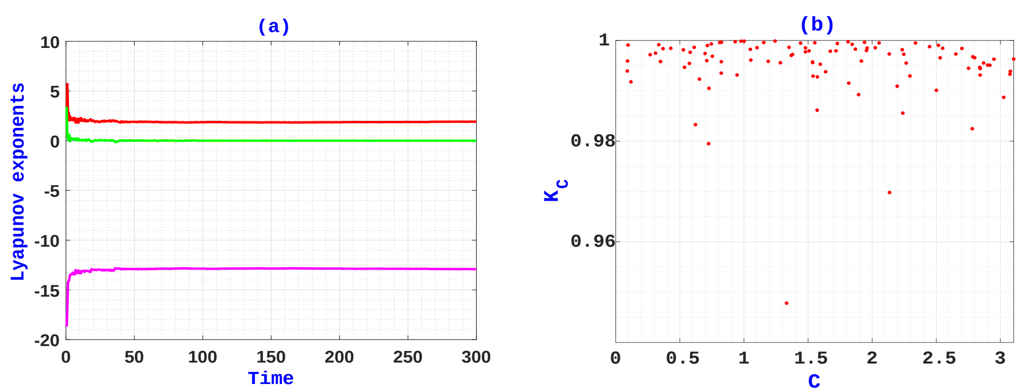

Figure 4.3: Evolution of $(LE_i, i = 1,2,3)$ in times; plot of the rate K_C vs C through the correlation method.

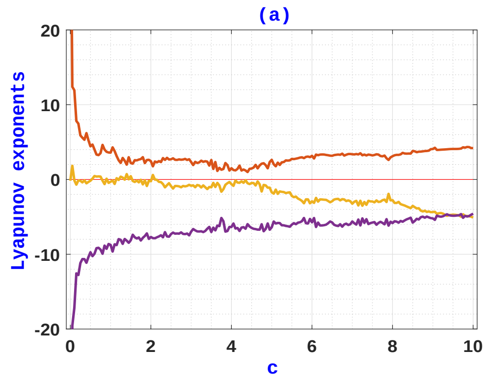

Figure 4.4: Evolution of ($LE_i, i = 1,2,3$) in times; Plot of the rate K_C vs C through the correlation method.

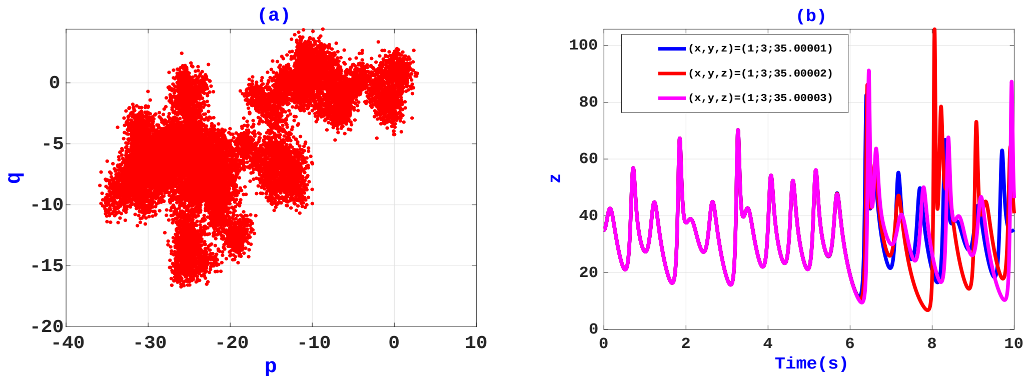

Figure 4.5: Brownian-like trajectories are displayed in the plane ($p-q$); the novel chaotic system (4.1) depict a higher sensitivity to initial conditions.

Chaotic systems are caracterized by the fact that the dimension of the attractor is fractal, so we calculate the dimension of Kaplan-Yorke for the chaotic system (4.1) we get:

$$D_{KY} = \frac{L_1 + L_2}{|L_3|} + 2 = 2.14998093 \tag{4.6}$$

The equation (4.6) suggests that the new attractor has fractal structure. Furthermore, the systems that exhibit chaotic behavior are characterized by their sensitivity to initial conditions, to be sure we show in the figure (4.5) a higher sensitivity to the initial conditions.

Stability of equilibrium points

Putting the three equations of system (4.1) equal to zero, we get the following resault:

For $\gamma \neq 1$ we have three fixed points:

$$p_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } p_{2,3} = \begin{bmatrix} \dfrac{\pm\sqrt{4\beta^2\gamma^2 - 8\beta^2\gamma + 4\beta^2 + 1} \mp 1}{2(\beta - \beta\gamma)} \\ \\ \dfrac{\pm\sqrt{4\beta^2\gamma^2 - 8\beta^2\gamma + 4\beta^2 + 1} \mp 1}{2(\beta - \beta\gamma)} \\ \\ \gamma - 1 \end{bmatrix} \tag{4.7}$$

calculating the jacobian matrix of (4.1) we obtain:

$$Df(x_1, x_2, x_3) = \begin{pmatrix} -\alpha & \alpha & 0 \\ \gamma - z & -1 & -x \\ \beta x_2 x_3 & -1 + \beta x_1 x_3 & -\beta + \beta x_1 x_2 \end{pmatrix}, \tag{4.8}$$

throught the theory of stability of Lyapunov we obtain the results:

For P_0 we have :

$$Df(0,0,0) = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & -1 & -b \end{pmatrix}, \tag{4.9}$$

so, the characteristic equation:

$$(\lambda + \beta)(\lambda^2 + (\alpha + 1)\lambda + \alpha(1 - \gamma)) = 0, \tag{4.10}$$

this equation give us the following eigenvalues:

$$\lambda_2 = \frac{-(\alpha + 1) - \sqrt{(\alpha - 1)^2 + 4\alpha\gamma}}{2}, \quad \lambda_1 = -\beta, \quad \lambda_3 = \frac{-(\alpha + 1) + \sqrt{(\alpha - 1)^2 + 4\alpha\gamma}}{2}, \tag{4.11}$$

we note that for all $\alpha, \beta, \gamma \in \mathbb{R}_+^*$ and $\gamma \neq 1$ positive integers, $\lambda_1, \lambda_2 < 0$. So we inspect the sign of λ_3. Then after simple calculus, we obtain:

1. *P_0 is a stable node for $0 < \gamma < 1$ because $\lambda_{1,2,3} < 0$.*

2. P_0 is an unstable saddle point for $\gamma > 1$ because $\lambda_{1,2} < 0$ and $\lambda_3 > 0$.

For $P_{2,3}$ and with $d = \sqrt{1 + \beta^2(\gamma - 1)^2}$, the jacobian matrix are given by:

$$\begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & -\frac{1+\sqrt{d}}{2\beta(\gamma-1)} \\ \frac{1+\sqrt{d}}{2} & -1 + \frac{1+\sqrt{d}}{2} & -1 + \frac{1+\sqrt{d}}{\beta(\gamma-1)^2} \end{pmatrix}, \tag{4.12}$$

characteristic polynomial is given by:

$$\begin{aligned} &\lambda^3 + \left(\alpha - \frac{1}{\beta(\gamma-1)^2}(\sqrt{d}+1) + 2\right)\lambda^2 \\ &+ \left(\frac{1}{2\beta(\gamma-1)}(\sqrt{d}+1)\left(\frac{1}{2}\sqrt{d} - \frac{1}{2}\right) - \left(\frac{1}{\beta(\gamma-1)^2}(\sqrt{d}+1) - 1\right)(\alpha+1)\right)\lambda \\ &+ \left(\frac{1}{2}\frac{\alpha}{\beta(\gamma-1)}(\sqrt{d}+1)\left(\frac{1}{2}\sqrt{d} + \frac{1}{2}\right) - \frac{1}{2\beta(\gamma-1)}(\sqrt{d}+1)\left(\frac{1}{2}\sqrt{d} - \frac{1}{2}\right) + \frac{1}{2\beta}\frac{\alpha+1}{\gamma-1}(\sqrt{d}+1)\left(\frac{1}{2}\sqrt{d} - \frac{1}{2}\right)\right), \end{aligned} \tag{4.13}$$

after substituting $\alpha = 10, \beta = 3$ and $\gamma = 35$ we get:

$$P_{2,3} = \left(\frac{1 \mp \sqrt{41617}}{204}; \frac{1 \mp \sqrt{41617}}{204}; 34\right), \tag{4.14}$$

then we get the following eigenvalues:

$$\lambda_{2,3} = 1.5574 \pm 11.963i \text{ and } \lambda_1 = -14.085, \tag{4.15}$$

the equilibrium points $P_{2,3}$ according to Hartman Gropman theorem are unstable saddle-focus points.

4.1.3 Comparison of the new proposed system with thirty other chaotic systems

*The dimension of Kaplan-Yorke (KYD) is one tool used to quantify complexity in chaotic systems [**?**]; it is important to note that systems with higher KYD dimensions also tend to exhibit more complicated behavior than other systems [**?**]. Our novel system (4.1) is compared to other 30 chaotic systems in the table (4.6). The system that we have* *presented has the largest Kaplan-Yorke dimension, indicating that it is at least more complicated than the other* 30 *systems in the table (4.6).*

System of	Reference	KYD	System of	Reference	KYD
1. Lorenz	[20]	2.062	16. Yang el al	[34]	2.112
2. Rossler	[20]	2.013	17. Lassoued et al	[35]	2.124
3. Chen et al	[21]	2.010	18. Li el al	[36]	2.105
4. Li	[22]	2.112	19 . Kim el al	[37]	2.033
5. Zhou et al	[23]	2.097	20. Abooee	[38]	2.037
6. Lü	[24]	2.066	21. Qiao et al	[39]	2.038
7. Sambas et al	[25]	2.072	22. Deng et al	[40]	2.117
8. Kapitaniak et al	[26]	2.107	23. Gholizadeh	[41]	2.105
9. Tigan et al	[27]	2.121	24. Wu et al	[42]	2.015
10. Modified Lü	[28]	2.034	25. Lai et al	[43]	2.091
11. Liu el al	[29]	2.116	26. Su	[44]	2.044
12. Li et al	[30]	2.101	27. Akgul	[45]	2.024
13. Pan el al	[31]	1.9921	28. Zhang et al	[46]	2.025
14. Wei	[32]	2.0528	29. Gholamin	[47]	2.055
15. Xu et al	[33]	2.047	30. Vaidyanathan et al	[48]	2.136
31	**the system (1)**	**2.14998093**			

Figure 4.6: Comparison of the proposed system and thirty other systems

Remark 4.1.1. *All reference in the previous table are from [4] and not from this thesis.*

4.1.4 Dynamic analysis of the proposed system

The bifurcations diagrams and the evolution of the LLE versus the system parameters will be used to evaluate the dynamic of our system in this section. The system (4.1) passes from its regular case, quasiperiodic case, periodic case until chaotic case. So that this process is repeated several times in the form of windows, we take as an example the parameter b. Indeed, we have the following dynamics:

1. *For $b = 0.5 \in [0; 0.72]$ the dynamic of (4.1) is periodic.*

2. *For $b \in [0.72; 0.9]$ the dynamic of (4.1) is quasi-periodic.*

3. *For* $b \in [0.9;9]$ *the dynamic of (4.1) is chaotic.*

4. *For* $b \in [9;30]$ *the dynamic of (4.1) is quasi-periodic.*

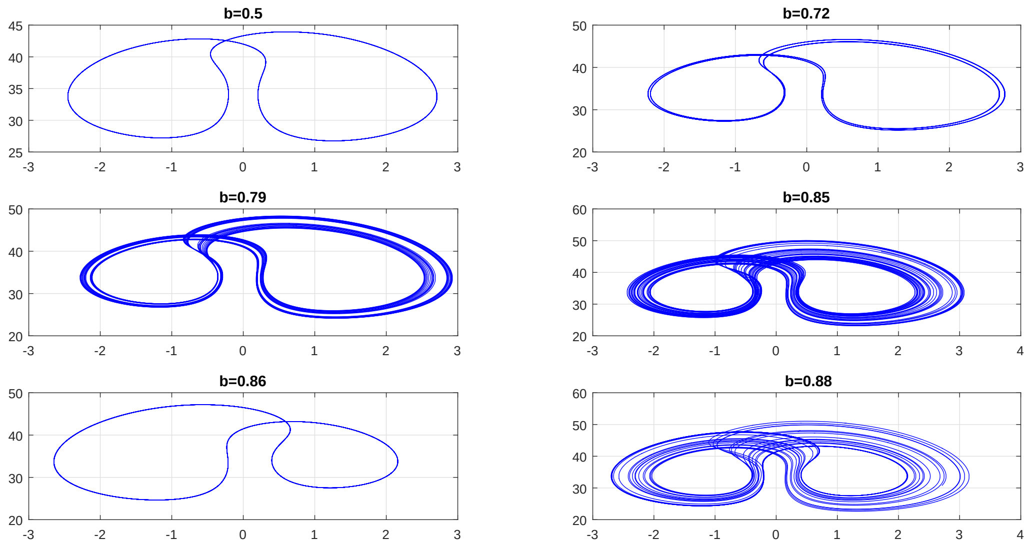

Figure 4.7: The transition to chaos by period doubling where $c = 35$, $a = 10$, and varying b.

Bifurcation diagrams versus Largest Lyapunov exponent

In the following figure we plot the bifurcation diagrams and the LLEs in order to view that the resault that are obtained by them are identical and give us the same informations about the dynamics of system (4.1), the resault of numerical simulation are shown respectively in figure 4.9, figure 4.10 and figure 4.11.

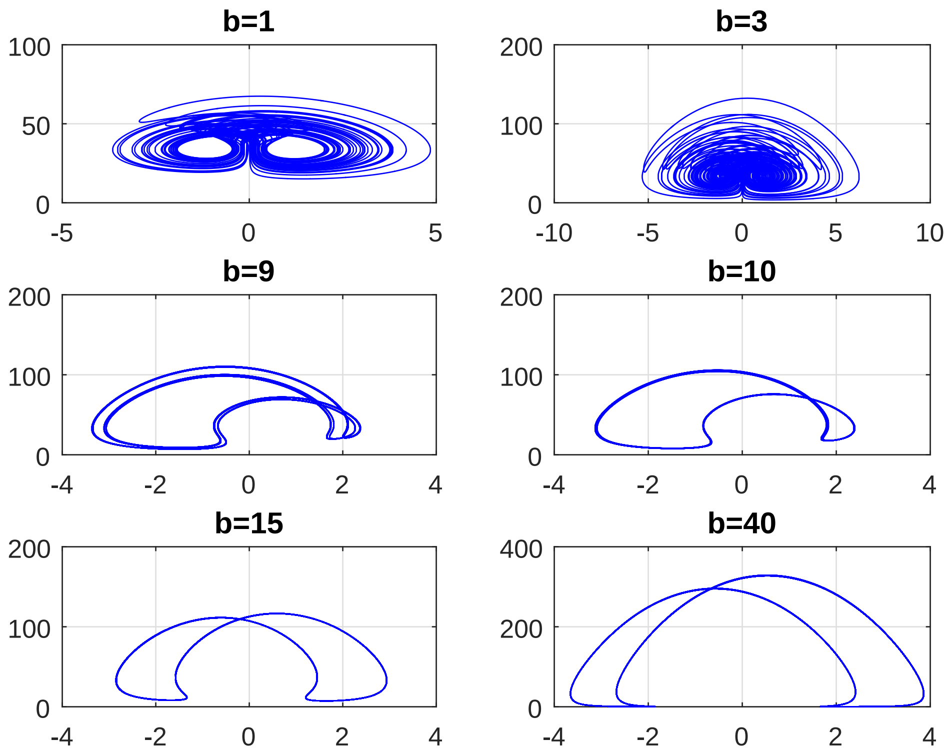

Figure 4.8: The transition to chaos by period doubling where $c = 35$, $a = 10$, and varying b.

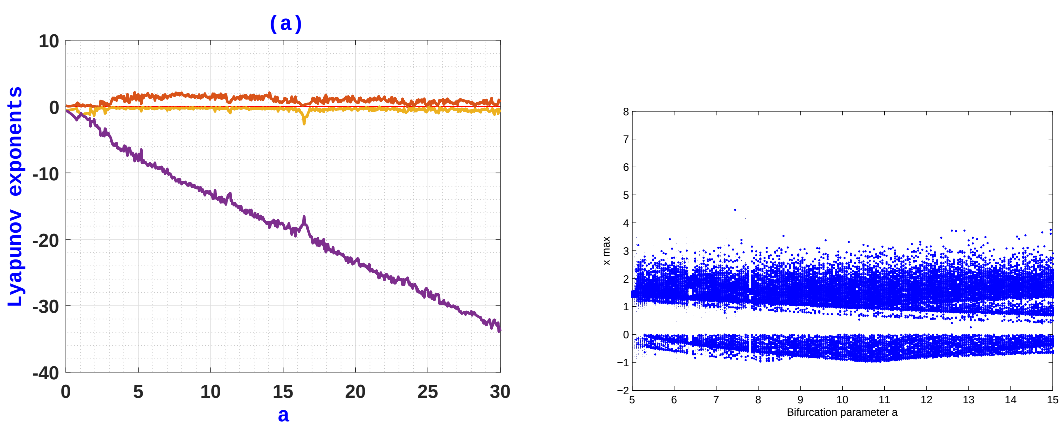

Figure 4.9: LLEs and bifurcation diagrams of (4.1) at initial values $(v_1(0); v_2(0); v_3(0)) = (1, 2, 40)$ with respect to a

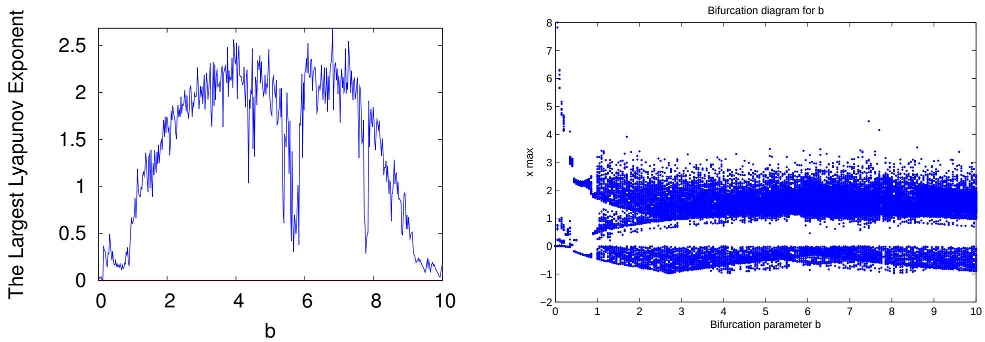

Figure 4.10: LLEs and bifurcation diagrams of (4.1) at initial values $(v_1(0); v_2(0); v_3(0)) = (1, 2, 40)$ with respect to b

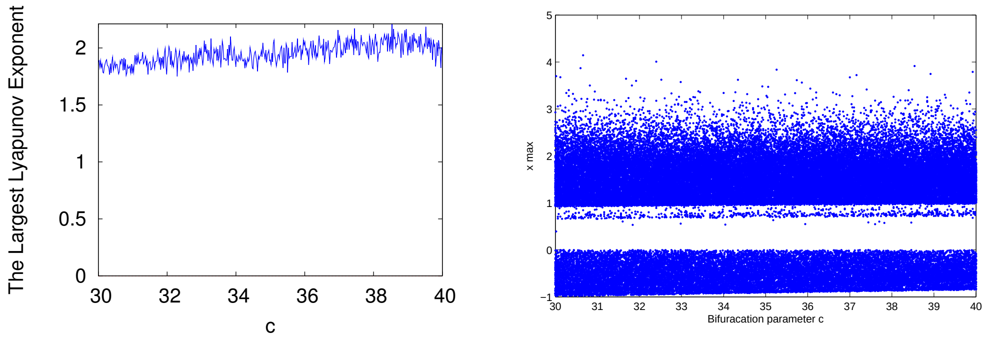

Figure 4.11: LLEs and bifurcation diagrams of (4.1) at initial values $(v_1(0); v_2(0); v_3(0)) = (1, 2, 40)$ with respect to c

4.1.5 Identical Adaptive synchronization of the proposed chaotic systems

In order to synchronize the system (4.1) we apply an adaptive control law with unknown system parameters. So, the master system is given as follow:

$$\begin{cases} \frac{dv_1}{dt} = \alpha(v_2 - v_1), \\ \frac{dv_2}{dt} = \gamma v_1 - v_1 - v_1 v_3, \\ \frac{dv_3}{dt} = \beta v_1 v_2 v_3 - v_2 - \beta v_3, \end{cases} \tag{4.16}$$

after that we define the slave system by:

$$\begin{cases} \frac{dw_1}{dt} = \alpha(w_2 - w_1) + \eta_1, \\ \frac{dw_2}{dt} = \gamma w_1 - w_1 - w_1 w_3 + \eta_2, \\ \frac{dw_3}{dt} = \beta w_1 w_2 w_3 - w_2 - \beta w_3 + \eta_3, \end{cases} \tag{4.17}$$

the slave system (4.16) and the master system (4.17) contains unknown parameters α, β and γ. Now our goal is to find the controllers η_1, η_2 and η_3.

Calculating the error of synchronization between the systems (4.16) and (4.17) is defined as:

$$\epsilon_1 = w_1 - v_1,\ \epsilon_2 = w_2 - v_2 \text{ and } \epsilon_3 = w_3 - v_3 \text{ implies } \dot{\epsilon}_1 = \dot{w}_1 - \dot{v}_1,\ \dot{\epsilon}_2 = \dot{w}_2 - \dot{v}_2 \text{ and } \dot{\epsilon}_3 = \dot{w}_3 - \dot{v}_3. \quad (4.18)$$

Thus, the error of synchronization between (4.16) and (4.17) result as follows:

$$\begin{cases} \dot{\epsilon}_1 = \alpha(\epsilon_2 - \epsilon_1) + \eta_1, \\ \dot{\epsilon}_2 = (\gamma - 1)\epsilon_1 - w_1 w_3 + v_1 v_3 + \eta_2, \\ \dot{\epsilon}_3 = -\epsilon_2 - \beta\epsilon_3 + \beta(w_1 w_2 w_3 - v_1 v_2 v_3) + \eta_3, \end{cases} \quad (4.19)$$

let define the adaptive control law by:

$$\begin{cases} \eta_1 = -\alpha_1(\epsilon_2 - \epsilon_1) - \xi_1\epsilon_1, \\ \eta_2 = -(\gamma_1 - 1)\epsilon_1 + w_1 w_3 - v_1 v_3 - \xi_2\epsilon_2, \\ \eta_3 = \epsilon_2 + \beta_1\epsilon_3 - \beta_1(w_1 w_2 w_3 - v_1 v_2 v_3) - \xi_3\epsilon_3, \end{cases} \quad (4.20)$$

where ξ_1, ξ_2 and ξ_3 are positive constants.

Substituting (4.20) in (4.19), we get:

$$\begin{cases} \dot{\epsilon}_1 = (\alpha - \alpha_1)(\epsilon_2 - \epsilon_1) - \xi_1\epsilon_1, \\ \dot{\epsilon}_2 = (\gamma - \gamma_1)\epsilon_1 - \xi_2\epsilon_2, \\ \dot{\epsilon}_3 = (\beta_1 - \beta)\epsilon_3 + (\beta - \beta_1)(w_1 w_2 w_3 - v_1 v_2 v_3) - \xi_3\epsilon_3, \end{cases} \quad (4.21)$$

the estimation errors of the parameters are defined by:

$$\begin{cases} \epsilon_\alpha(t) = \alpha - \alpha_1(t), \\ \epsilon_\beta(t) = \beta - \beta_1(t), \\ \epsilon_\gamma(t) = \gamma - \gamma_1(t), \end{cases} \quad (4.22)$$

with respect to t, differentiating (4.22) , we get:

$$\begin{cases} \frac{d\epsilon_\alpha(t)}{dt} = -\frac{d\alpha_1(t)}{dt}, \\ \frac{d\epsilon_\beta(t)}{dt} = -\frac{d\beta_1(t)}{dt}, \\ \frac{d\epsilon_\gamma(t)}{dt} = -\frac{d\gamma_1(t)}{dt}, \end{cases} \tag{4.23}$$

after substitution, equation (4.21) give us:

$$\begin{cases} \dot{\epsilon}_1 = \epsilon_\alpha(\epsilon_2 - \epsilon_1) - \xi_1\epsilon_1, \\ \dot{\epsilon}_2 = \epsilon_\gamma\epsilon_1 - \xi_2\epsilon_2, \\ \dot{\epsilon}_3 = -\epsilon_\beta\epsilon_3 + \epsilon_\beta(w_1w_2w_3 - v_1v_2v_3) - \xi_3\epsilon_3, \end{cases} \tag{4.24}$$

to achieve the required synchronization between (4.16) and (4.17), we can use a function V, positive definite on $\mathbb{R}^6$ to push synchronization error to zero. Hence:

$$V\left(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma\right) = \frac{1}{2}\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_\alpha^2 + \epsilon_\beta^2 + \epsilon_\gamma^2\right), \tag{4.25}$$

differentiating V with respect to t, we obtain:

$$\dot{V} = -\sum_{i=1}^{3} \xi_i\epsilon_i^2 + \epsilon_a\left(\epsilon_1\epsilon_2 - \epsilon_1^2 - \frac{d\alpha_1(t)}{dt}\right) + \epsilon_\beta\left(-\epsilon_3^2 + \epsilon_3(w_1w_2w_3 - v_1v_2v_3) - \frac{d\beta_1(t)}{dt}\right) + \epsilon_\gamma\left(\epsilon_1\epsilon_2 - \frac{d\gamma_1(t)}{dt},\right) \tag{4.26}$$

as a consequence, the parameter update law are taken by:

$$\begin{cases} \frac{d\alpha_1(t)}{dt} = \epsilon_1\epsilon_2 - \epsilon_1^2, \\ \frac{d\beta_1(t)}{dt} = -\epsilon_3^2 + \epsilon_3(w_1w_2w_3 - v_1v_2v_3), \\ \frac{d\gamma_1(t)}{dt} = \epsilon_1\epsilon_2, \end{cases} \tag{4.27}$$

putting (4.27) into (4.26), the function $\dot{V}$ become:

$$\dot{V} = -\sum_{i=1}^{3} \xi_i\epsilon_i^2, \tag{4.28}$$

then in (4.28), $\dot{V}$ is a negative definite function on $\mathbb{R}^3$. Through Lyapunov stability theory [21], it result that $\epsilon_i(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $i = 1, 2, 3$. Hence, we have demonstrated the *following theorem.*

Theorem 4.1.1. *The 3-D novel chaotic systems (4.16) and (4.17) with unknown parameters are exponentially and globally synchronized for all initial conditions through the the parameter update law (4.27) and the adaptive feedback control law (4.20), where ξ_1, ξ_2 and ξ_3 are positive constants.*

Numerical results and simulation

Passing now to the visualization of theoretical results using Matlab software. In order to solve the required systems ((4.16), (4.17) and (4.27)) we use the Runge Kutta method of order four with step size $h = 10^{-8}$. We choose the initial conditions as follows: $\big(v_1(0), v_2(0), v_3(0)\big) = (-5, -10, 15)$, $\big(w_1(0), w_2(0), w_3(0)\big) = (15, -20, 20)$ and $\xi_1 = 10$, $\xi_2 = 20$, $\xi_3 = 30$. Furthermore, the initail conditions of the parameters estemate are taken as: $\big(\alpha_1(0), \beta_1(0), \gamma_1(0)\big) = (15, 5, 40)$.

The figure (4.12(a)) depict that in less than 0.4 seconds, the synchronization errors achieve the origine.

When we apply the adaptive control law (4.20) and the parameter update law (4.27), the synchronization of the master system (4.16) and the slave system (4.17) is shown in figures (4.13) and (4.12 (b).

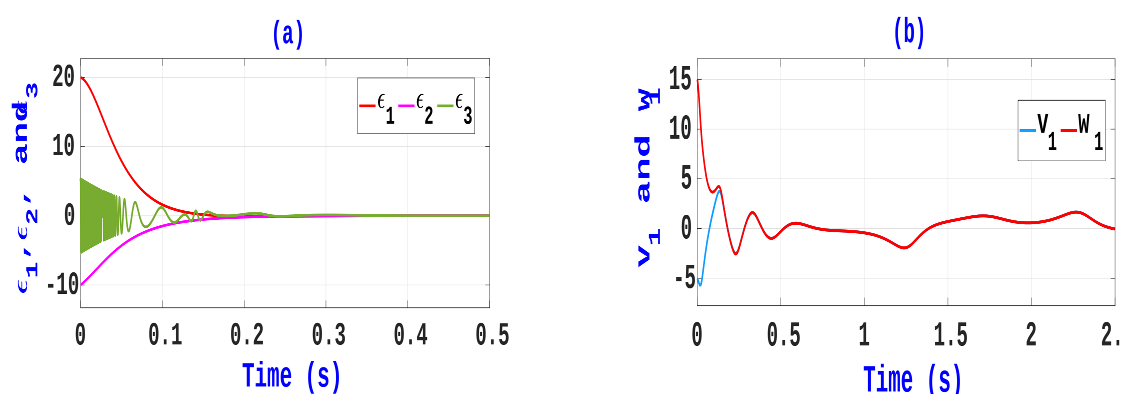

Figure 4.12: (a) Evolution in time of the synchronization errors states between (4.16) and (4.17); (b) synchronization between $v_1(t)$ and $w_1(t)$.

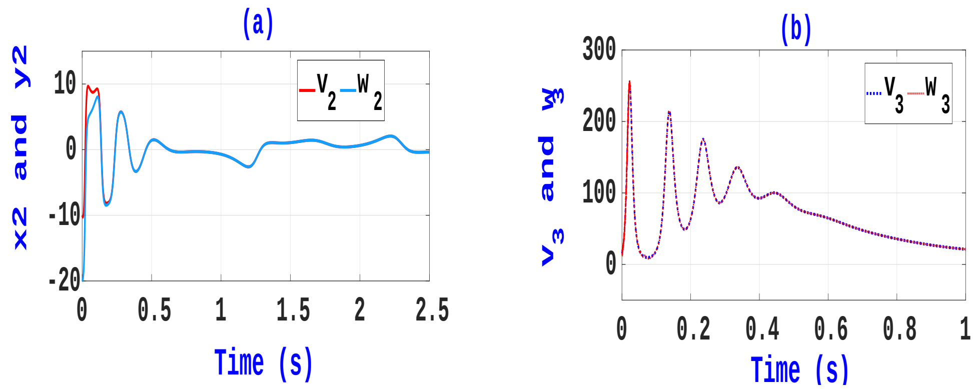

Figure 4.13: (a) Synchronization between $v_2(t)$ and $w_2(t)$; (b) synchronization between $v_3(t)$ and $w_3(t)$.

4.1.6 Circuit design of the proposed integer chaotic system

This subsection presents an electronic circuit equivalent to our proposed chaotic system (4.1). The electronic circuit is shown in figure (4.14) and he was designed using the Multisim software. For q = 1, the analog circuit of the proposed 3-D chaotic system (4.1) is implemented by adopting capacitors, resistors, analog multipliers AD633, operational amplifiers TL084ACN. We give here the steps to design the analog circuit that is equivalent to system (4.1).

From the first equation of to system (4.1), we have:

$$\frac{dx_1}{dt} = -10x_1 + 10x_2, \tag{4.29}$$

by applying the Kirchhoff law to equation (4.29) we get:

$$\begin{aligned} x_1 &= \int \Big[-10x_1 + 10x_2 \Big] dt \\ &= \int \Big[10(-x_1) + 10(x_2) \Big] dt \\ &= \frac{-1}{c_1} \int \Big[\frac{(-x_1)}{R_1} + \frac{x_2}{R_2} \Big] dt \\ &= \frac{1}{c_1} \int \Big[\frac{x_1}{R_1} + \frac{R_4}{R_3} \frac{(-x_2)}{R_2} \Big] dt, \end{aligned} \tag{4.30}$$

from the second equation of (4.1), we have:

$$\frac{dx_2}{dt} = 35x_1 - x_2 - x_1x_3, \tag{4.31}$$

By applying the Kirchhoff law to equation (4.31) we get:

$$\begin{aligned}
x_2 &= \int \Big[35x_1 - x_2 - x_1x_3\Big]dt \\
&= \int \Big[35x_1 + (-x_2) + (-x_1)x_3\Big]dt \\
&= \frac{-1}{c_2}\int \Big[\frac{x_1}{R_5} + \frac{-x_2}{R_6} + \frac{-x_1x_3}{10R_7}\Big]dt \\
&= \frac{1}{c_2}\int \Big[\frac{(-x_1)}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7}\Big]dt \\
&= \frac{1}{c_2}\int \Big[\frac{R_9}{R_8}\frac{(-x_1)}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7}\Big]dt,
\end{aligned} \tag{4.32}$$

from the third equation of (4.1), we have:

$$\frac{dx_3}{dt} = 3x_1x_2x_3 - x_2 - 3x_3 \tag{4.33}$$

by applying the Kirchhoff law to equation (4.33) we get:

$$\begin{aligned}
x_3 &= \int \Big[3x_1x_2x_3 - x_2 - 3x_3\Big]dt \\
&= \int \Big[3x_1x2x_3 + (-x_2) + 3(-x_3)\Big]dt \\
&= \frac{-1}{c_3}\int \Big[x_1x_2x_3 + (-x_2) + (-x_3)\Big]dt \\
&= \frac{1}{c_3}\int \Big[x_1x_2(-x_3) + x_2 + x_3\Big]dt \\
&= \frac{1}{c_3}\int \Big[\frac{R_{14}}{R_{13}}\frac{x_1x_2(-x_3)}{10R_{10}} + \frac{x_2}{R_{11}} + \frac{x_3}{R_{12}}\Big]dt,
\end{aligned} \tag{4.34}$$

finally, we obtain the follwing electrical system:

$$\begin{aligned}
x_1 &= \frac{1}{c_1}\int \Big[\frac{x_1}{R_1} + \frac{R_4}{R_3}\frac{(-x_2)}{R_2}\Big]dt, \\
x_2 &= \frac{1}{c_2}\int \Big[\frac{R_9}{R_8}\frac{(-x_1)}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7}\Big]dt, \\
x_3 &= \frac{1}{c_3}\int \Big[\frac{R_{14}}{R_{13}}\frac{x_1x_2(-x_3)}{10R_{10}} + \frac{x_2}{R_{11}} + \frac{x_3}{R_{12}}\Big]dt,
\end{aligned} \tag{4.35}$$

the circuital components values of (4.60) are selected after identities between (4.1) and (4.60) as:

$$R_1 = R_2 = R_7 = R_{10} = 1\text{k}\Omega, R_5 = 0.285714\text{k}\Omega, R_{12} = 3.333\text{k}\Omega, R_3 = R_4 = R_8 = R_9 = R_{13} = R_14 = 100\text{k}\Omega,$$

$$R_6 = R_{11} = 10\text{k}\Omega, R_{10} = 0.333\text{k}\Omega, c_1 = c_2 = c_3 = 100\text{nf}. \tag{4.36}$$

As can be seen from Figures 4.14, 4.15 and 4.16, the circuit simulation results and numerical simulation results are consistent and show that chaotic systems have existense physique.

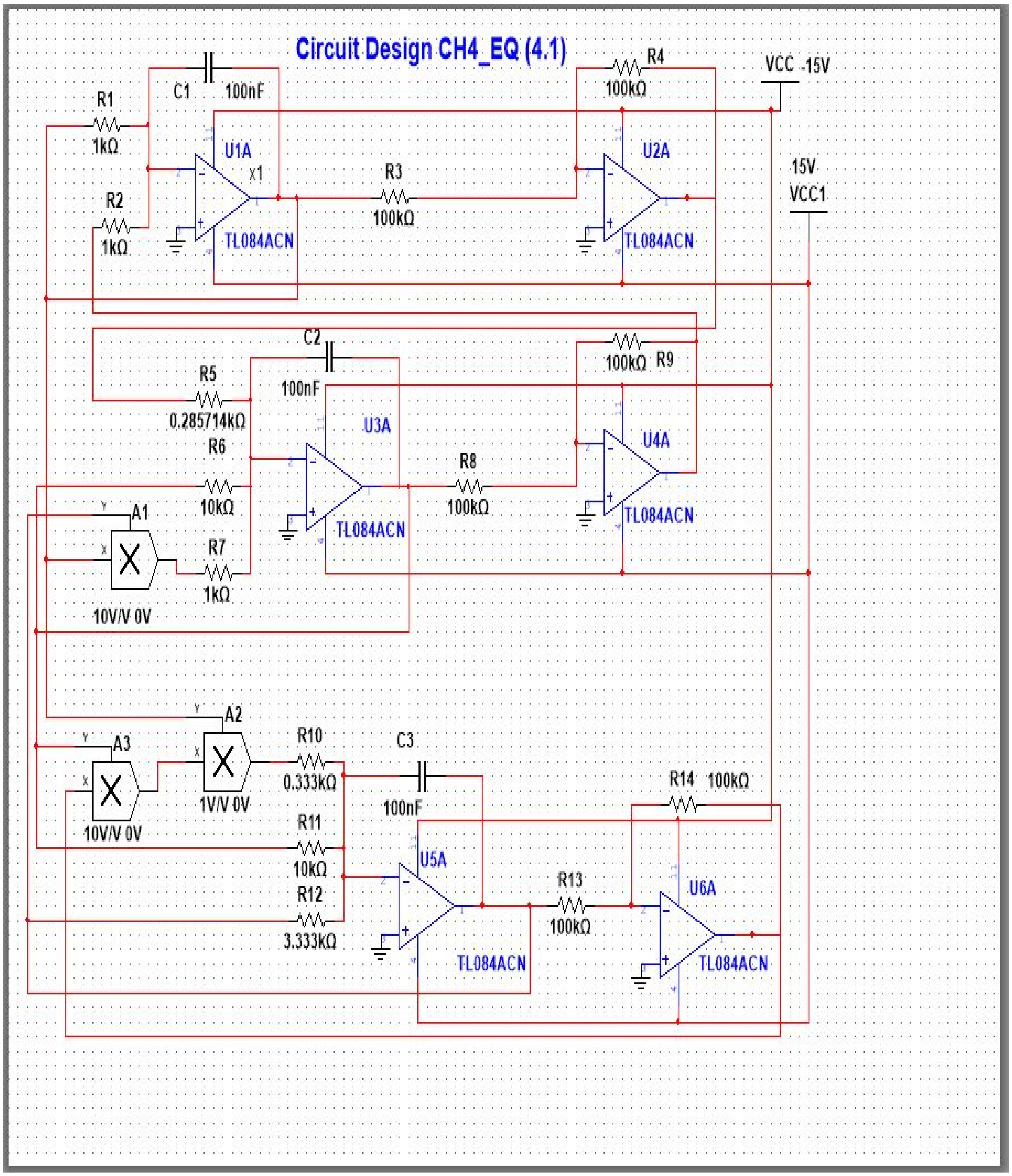

Figure 4.14: Cicuit design in multisim of the proposed chaotic system (4.1)

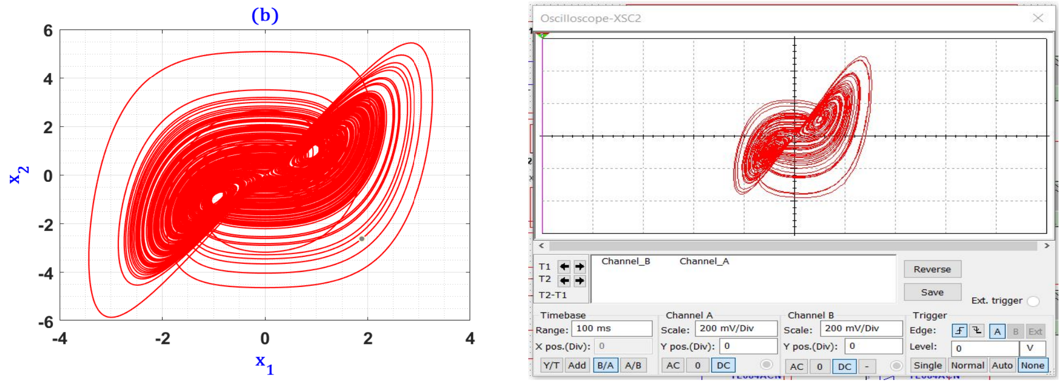

Figure 4.15: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_2 plane

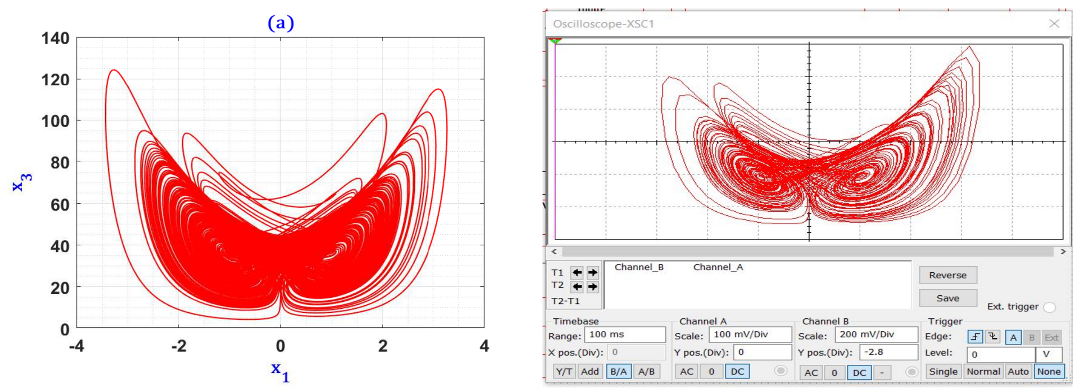

Figure 4.16: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_3 plane

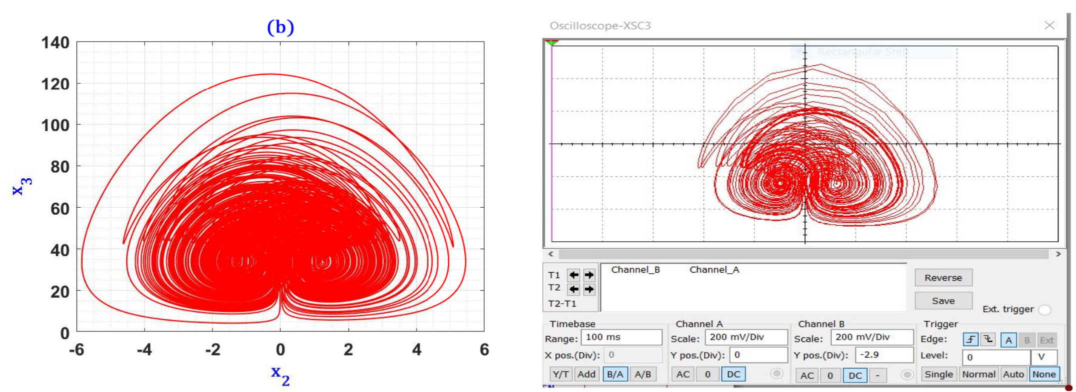

Figure 4.17: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_2x_3 plane

4.2 Extension to fractional case with stabilization via adaptive control and circuit design

In this section we extend the integer proposed system (4.1) to its factional form in the sense of Caputo then we investigate their stabilizeation throught the adaptive control. So, we consider the fractional version of (4.1) which given by:

$$\begin{cases} \mathcal{D}_t^{q_1} x(t) = a(y - x), \\ \mathcal{D}_t^{q_2} y(t) = cx - y - xz, \\ \mathcal{D}_t^{q_3} z(t) = bxyz - y - bz, \end{cases} \tag{4.37}$$

where D^q is the Caputo derivative operator, a, b, c are positive reals parameters and x, y, z are the state variables. With $a = 10, b = 3, c = 35$; $(q_1; q_2; q_3) = (0.98; 0.9; 0.98)$;$(x(0); y(0); z(0)) = (1; 2; 20)$. The evolution in times of the Lyapunov exponents are depicted in (4.37) , so LEs are given by:

$$L_1 = 2.0573; L_2 = -0.0012 \simeq 0; L_3 = -14.0377, \tag{4.38}$$

so we have $L_1 > 0$, then the system (4.37) is chaotic. Indeed the Kaplan-york dimension for this chaotic system is calculated as:

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1466, \tag{4.39}$$

it is so fractional, that is confirms the characteristic of the fractional dimension of strange attractor in our system.

4.2.1 Commensurate and incommensurate necessary conditions for existing chaos

According to [48] a necessary conditions for the existence of chaos in a fractional chaotic system (4.37) can be given as:

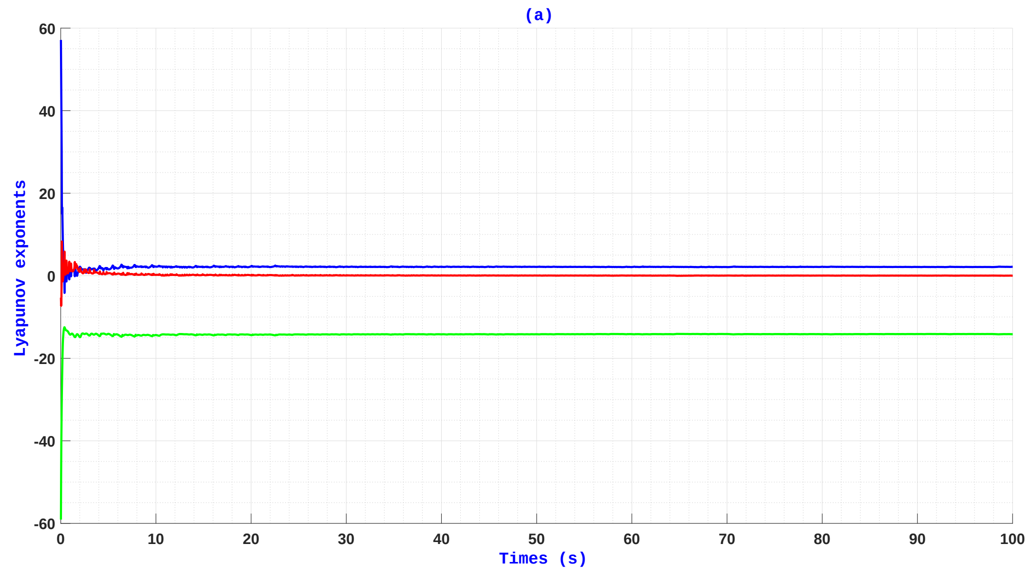

Figure 4.18: Evolution in time of the fractional chaotic system (4.37)

If $q_1 = q_2 = q_3 = q$*, a necessary condition for the system (4.37) to be chaotic is:*

$$q > \frac{2}{\pi} \tan \frac{\left(|Im(\lambda)|\right)}{Re(\lambda)} = 0.7547, \tag{4.40}$$

also in the incommensurate case the necessary condition for the system (4.37) to be chaotic is :

$$\frac{\pi}{2M} - min_{i=1}|arg\lambda_i| \geq 0, \tag{4.41}$$

if we put $q_i = \frac{n_i}{d_i}$*, we mean by* M *the lowest common multiple of the denominators* d_i *and* $\lambda_i's$ *are the roots of:*

$$det(diag(\lambda^{Mq_1}, \lambda^{Mq_2}, \lambda^{Mq_3}) - J(E_i)) = 0, \tag{4.42}$$

for the parameter $a = 3, b = 1$ *and* $c = 1$*, the characteristic equation of the equilibrium points* E_2 *and* E_3 *is given by:*

$$\lambda^{290} + \lambda^{194} + 3\lambda^{193} + 3\lambda^{97} + 599.96 = 0, \tag{4.43}$$

in Matlab we can obtain the roots of the characteristic equation, we found

$\frac{\pi}{2M} - 0.01226336539 = 0.00344 > 0$,

Then the necessary incommensurate condition is satisfied in the new proposed fractional system (4.37).

4.2.2 Stabilisation of the novel fractional system via adaptive control

This section aim to design an adaptive control law for globally controlling the identical novel fractional chaotic system with unknown system parameters. The controlled system is given by:

$$\begin{cases} {}^{C}D_t^{q_1} x_1(t) = a(x_2 - x_1) + \eta_1, \\ \\ {}^{C}D_t^{q_2} x_2(t) = cx_1 - x_1 - x_1x_3 + \eta_2, \\ \\ {}^{C}D_t^{q_3} x_3(t) = bx_1x_2x_3 - x_2 - bx_3 + \eta_3, \end{cases} \tag{4.44}$$

where the parameters a, b, d *are unknown and their estimates* $a_1(t), b_1(t), d_1(t)$*, respectively. We will later search an adaptive controllers* η_1, η_2, η_3.
We take the adaptive control law as follow:

$$\begin{cases} \eta_1 = -\hat{a}(x_2 - x_1) - \xi_1 xi_1, \\ \eta_2 = -\hat{c}x_1 - +1 + x_1x_3 - \xi_2 x_2, \\ \eta_3 = -\hat{b}x_1x_2x_3 + \hat{b}x_3 + x_2 - \xi_3 x_3, \end{cases} \tag{4.45}$$

where ξ_1, ξ_2 *and* ξ_3 *are positive constants and* $\hat{a}, \hat{b}$*, and* $\hat{c}$ *are estimates of the system parameters* a, b*, and* c*, respectively. In order to get the closed loop system we substitute (4.45) in (4.44), so we get:*

$$\begin{cases} {}^{C}D_t^{q_1} x_1(t) = (a - \hat{a})(x_2 - x_1) - \xi_1 x_1, \\ {}^{C}D_t^{q_1} x_1(t) = (c - \hat{c})x_1 - \xi_2 x_2, \\ {}^{C}D_t^{q_1} x_1(t) = (b - \hat{b})x_1x_2x_3 - (b - \hat{b})x_3 - \xi_3 x_3, \end{cases} \tag{4.46}$$

let define the parameter estimation error as:

$$\epsilon_a = a - \hat{a},\ \epsilon_b = b - \hat{b},\ \epsilon_c = c - \hat{c}, \tag{4.47}$$

applying the Caputo derivative of order q in (4.47), we obtain:

$$\begin{cases} {}^C\mathcal{D}_t^q \epsilon_a(t) = -{}^C\mathcal{D}_t^q \hat{a}, \\ {}^C\mathcal{D}_t^q \epsilon_b(t) = -{}^C\mathcal{D}_t^q \hat{b}, \\ {}^C\mathcal{D}_t^q \epsilon_c(t) = -{}^C\mathcal{D}_t^q \hat{c}, \end{cases} \tag{4.48}$$

after substituting (4.47) in (4.46) we obtain:

$$\begin{cases} {}^CD_t^{q_1} x_1(t) = \epsilon_a(x_2 - x_1) - \xi_{11}, \\ {}^CD_t^{q_1} x_2(t) = \epsilon_c x_1 - \xi_2 x_2, \\ {}^CD_t^{q_1} x_3(t) = \epsilon_b x_1 x_2 x_3 - \epsilon_b x_3 - \xi_3 x_3, \end{cases} \tag{4.49}$$

we aim now to adjust the parametre estimates, let's define the following Lyapunov function:

$$V(x_1, x_2, x_3, e_a, e_d) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2\right), \tag{4.50}$$

which is a positive definite function on $\mathbb{R}^6$. Utilizing (4.48) and the diferentiation of the function V along the trajectories of equation (4.49), we get:

$$\begin{cases} {}^C\mathcal{D}_t^q V(x_1, x_2, x_3, e_a, e_b, e_c) = \frac{1}{2}D^q x_1^2 + \frac{1}{2}D^q x_2^2 + \frac{1}{2}D^q x_3^2 + \frac{1}{2}D^q e_a^2 + \frac{1}{2}D^q e_b^2 + \frac{1}{2}D^q e_c^2 \\ \leq x_1 D^q x_1 + x_2 D^q x_2 + x_3 D^q x_3 + e_a D^q e_a + e_b D^q e_b + e_c D^q e_c \\ \leq x_1(\epsilon_a(x_2 - x_1) - \xi_1 x_1) + x_2(\epsilon_c x_1 - \xi_2 x_2) + x_3(\epsilon_b x_1 x_2 x_3 - \epsilon_b x_3 - \xi_3 x_3) \\ +\epsilon_a(-{}^C\mathcal{D}_t^q \hat{a}) + \epsilon_b(-{}^C\mathcal{D}_t^q \hat{b}) + \epsilon_c(-{}^C\mathcal{D}_t^q \hat{c}) \\ \leq -\xi_1 x_1^2 - \xi_2 x_2^2 - \xi_3 x_3^2 + \epsilon_a(x_1 x_2 - x_1^2 -{}^C D_t^q \hat{a}) + +\epsilon_b(x_1 x_2 x_3^2 - x_3^2 -{}^C D_t^q \hat{b}) + \epsilon_c(x_1 x_2 -{}^C D_t^q \hat{c}), \end{cases} \tag{4.51}$$

in view of (4.51), we take the parameter update law as follows:

$$\begin{cases} {}^C\mathcal{D}_t^q \hat{a}(t) = x_1 x_2 - x_1^2 + \xi_4 \epsilon_a \\ {}^C\mathcal{D}_t^q \hat{b}(t) = x_1 x_2 x_3^2 - x_3^2 + \xi_5 \epsilon_b \\ {}^C\mathcal{D}_t^q \hat{c}(t) = x_1 x_2 + \xi_4 \epsilon_c, \end{cases} \tag{4.52}$$

substituting (4.52) into (4.51), we obtain:

$$^{C}\mathcal{D}_t^q V(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_a,\epsilon_b,\epsilon_c) \le -\xi_1^2\epsilon_1^2 - \xi_2^2\epsilon_2^2 - \xi_3^2\epsilon_3^2 - \xi_4^2\epsilon_a^2 - \xi_5^2\epsilon_b^2 - \xi_6^2\epsilon_6^2 \le 0, \tag{4.53}$$

which shows that $\dot{V}$ is a negative definite function on $\mathbb{R}^6$, it follows that

$$x_1(t) \to 0, \quad x_2(t) \to 0, \quad x_3 \to 0, \quad e_a \to 0, \quad e_b \to 0 \quad \text{and} \quad e_c \to 0, \tag{4.54}$$

exponentially as $t \to \infty$. Hence, we have proved the following theorem.

Theorem 4.2.1. *The novel fractional chaotic systems (4.44) with unknown parameters is globally and exponentially stabiliized for all initial conditions by the adaptive feedback control law (4.45) and the parameter update law (4.52), were $\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6$ are positive constants. The errors for parameter estimates $\epsilon_a,\epsilon_b,\epsilon_c$ decay to zero exponentially as $t \longrightarrow +\infty$.*

4.2.3 Circuit design of the proposed fractional chaotic system

The fractional frequency domain approximation

Throught the standard integer order operators, we can devolope an approximation to the fractional aperators. In view of circuit theory, we have the following approximation for $q = 0.98$ [22]:

$$\frac{1}{s^\alpha} = \frac{1}{s^{0.98}} = \frac{1.2974(s+1125)}{s+0.01125}, \tag{4.55}$$

where $s = jw$ represent the complex frequency. The representation of chain ship circuit unit is describerd in the figure (4.19).

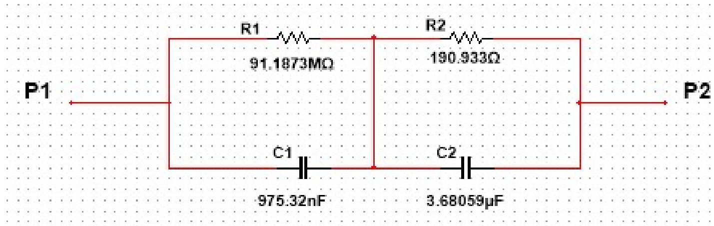

Figure 4.19: Chain ship unit of $q = 0.98$

Also, the transfer function between p_1 and p_2 can be obtained by the next formulla:

$$\begin{aligned} H_{0.98}(s) &= \frac{R_1}{sR_1C_1+1} + \frac{R_2}{sR_2C_2+1} \\ &= \frac{1}{C_0}\frac{\left(\frac{C_0}{C_1}+\frac{C_0}{C_2}\right)\left[s+\frac{\frac{1}{R_1}+\frac{1}{R_2}}{C_1+C_2}\right]}{(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}. \end{aligned} \tag{4.56}$$

In equation (4.56) we get $R_1 = 91.1873, R_2 = 190.933w, C_1 = 975.32nF$, and $C_2 = 3.6806\mu F$, by putting $C_0 = 1\mu F$ and $H(s)C_0 = \frac{1}{s^{0.98}}$. Using kirshof law we get the necessary value of components of the circuit design depicted in figure (4.20). In the following figures we chow a simple comparisons (see figures 4.21, 4.22 and 4.23) which proves that analog circuit for system (4.37) is well coincident with numerical simulations. A conclusion can be made that the chaotic behaviors exist in the fractional order system (4.37), which verifies its existence and validity.

4.3 Hyperchaotic integer system and its circuit design

Let $\alpha, \beta, \gamma, \delta$ four positive reals parameters and x_1, x_2, x_3, x_4 are the states variables. We can get a new $4-D$ hyperchaotic system as follows:

$$\begin{cases} \frac{dx_1}{dt} = \alpha(x_2 - x_1), \\ \frac{dx_2}{dt} = \gamma x_1 - x_2 - x_1x_3, \\ \frac{dx_3}{dt} = \beta x_1x_2x_3 - x_2 - \beta x_3, \\ \frac{dx_4}{dt} = -\delta(x_1^3 + x_2^3 + x_4), \end{cases} \tag{4.57}$$

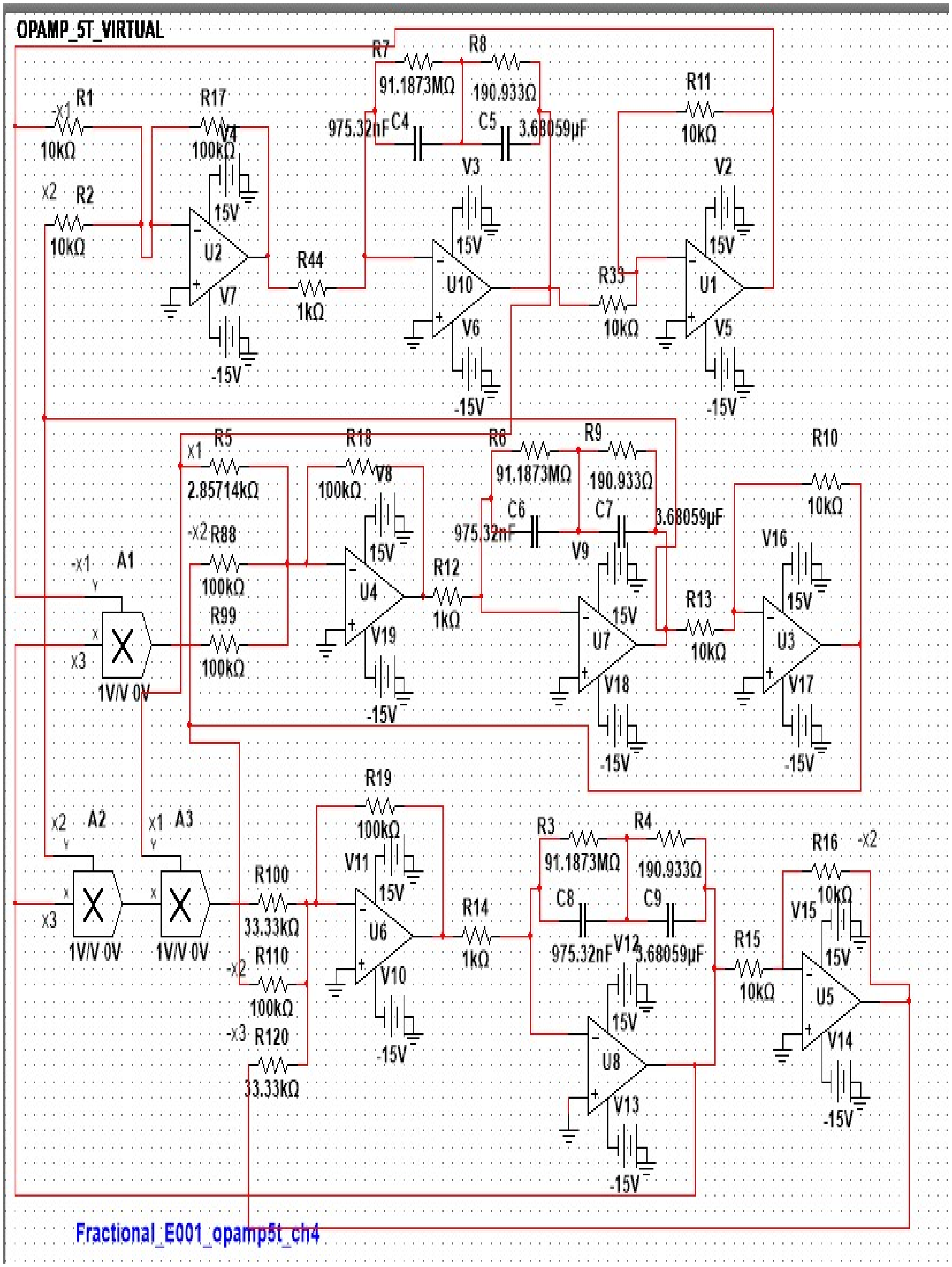

Figure 4.20: Cicuit design in multisim of the proposed fractional chaotic system for $q = 0.98$

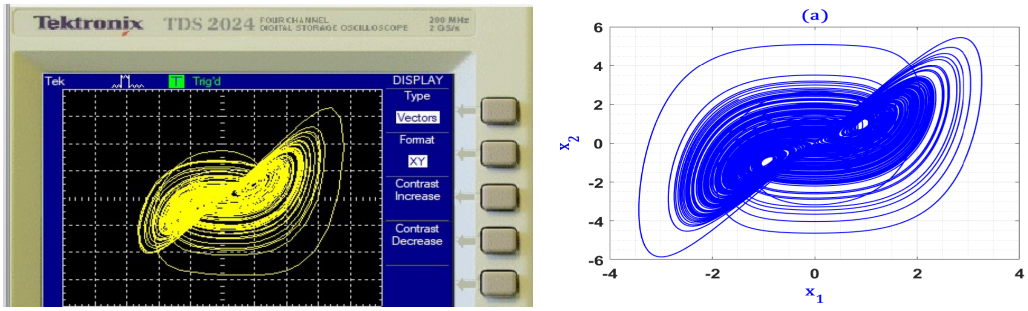

Figure 4.21: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed fractional chaotic system in x_1x_2 plane for $q = 0.98$

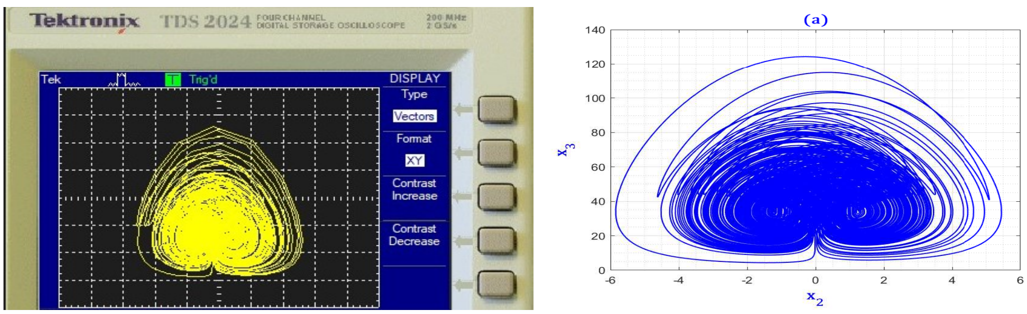

Figure 4.22: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed fractional chaotic system in x_2x_3 plane for $q = 0.98$

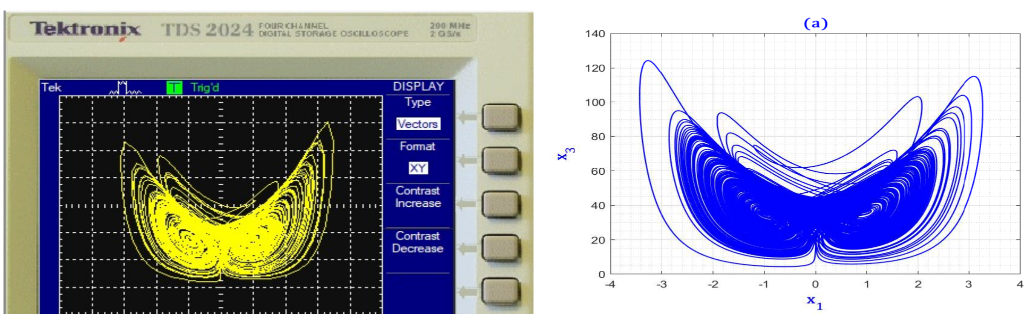

Figure 4.23: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed fractional chaotic system in x_1x_3 plane for $q = 0.98$

for $\gamma = 35, \alpha = 10, \beta = 3, \delta = 10,$ *and* $\rho = 10$ *with initiales conditions:* $(x_1, x_2, x_3, x_4) = (1; 2; 3; 1)$*, the system (4.57) display chaotic behavior since we have the following Lyapunov exponents:*

$$L_1 = 1.8963; L_2 = 0.0456; L_3 = -9.9992; L_4 = -12.9372, \tag{4.58}$$

We have the Kaplan York dimention as:

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1943, \tag{4.59}$$

which suggest that the volume of the attractors in the space is null. Indeed, we obtain a strange attractor with no volume in the space and the projections in different planes $(x_1 - x_4)$, $(x_{12} - x_4)$ *and* $(x_3 - x_4)$ *in figure (4.24). We use to calculate the Lyapunov exponent* 10000 *iterations and* $(x_1, x_2, x_3, x_4) = (1; 20; 70; 10)$ *as we can see in figure (4.25).*

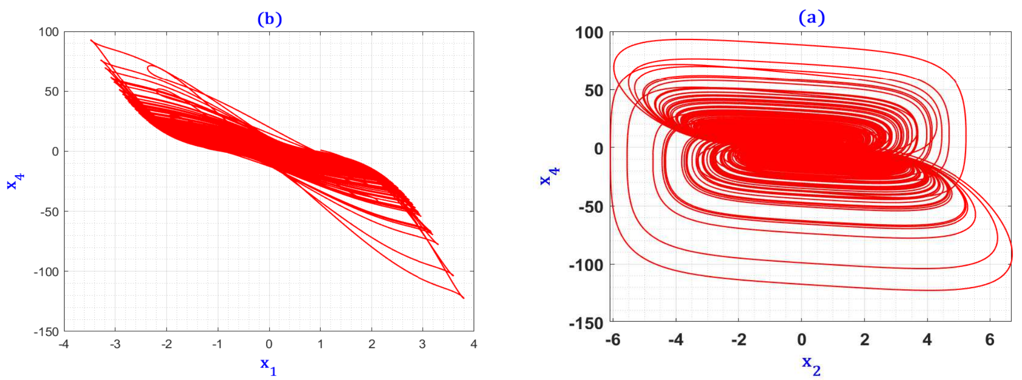

Figure 4.24: Projection in $(x_1 - x_4)$ and $(x_{12} - x_4)$ plane of (4.57).

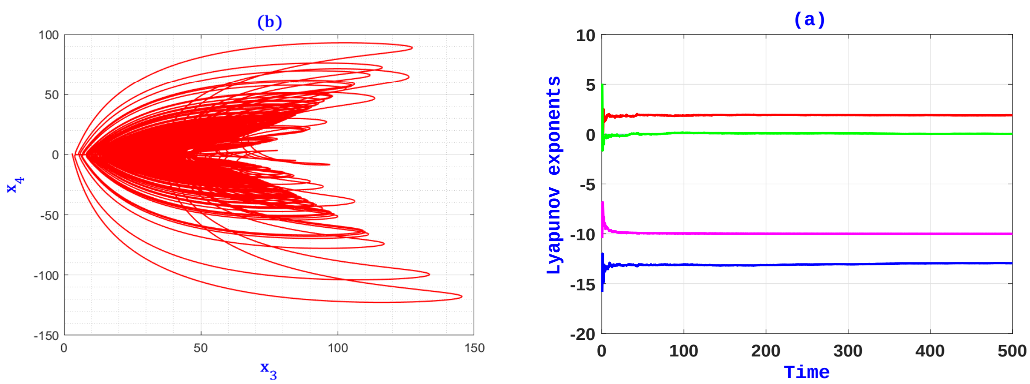

Figure 4.25: Projection in $(x_1 - x_4)$ plane and Lyapunov Exponents of (4.57).

We passed now to the presentation of the proposed hyperchaotic system as an electronic circuit. The electronic circuit is shown in figure (4.26) and he was designed using the Multisim software. For $q = 1$*, the analog circuit of the proposed 4-D hyperchaotic system is implemented by adopting capacitors, resistors, analog multipliers AD633, operational amplifiers TL084ACN. By the same previous method we obtain the follwing electrical* *system which equivalent to system (4.57):*

$$\begin{aligned} x_1 &= \frac{1}{c_1}\int\Big[\frac{x_1}{R_1}+\frac{R_4}{R_3}\frac{(-x_2)}{R_2}\Big]dt,\\ x_2 &= \frac{1}{c_2}\int\Big[\frac{R_9}{R_8}\frac{(-x_1)}{R_5}+\frac{x_2}{R_6}+\frac{x_1x_3}{10R_7}\Big]dt,\\ x_3 &= \frac{1}{c_3}\int\Big[\frac{R_{14}}{R_{13}}\frac{x_1x_2(-x_3)}{10R_{10}}+\frac{x_2}{R_{11}}+\frac{x_3}{R_{12}}\Big]dt,\\ x_4 &= \frac{1}{c_4}\int\Big[\frac{x_1x_1(x_1)}{10R_{18}}+\frac{x_2x_2x_2}{R_{15}}+\frac{x_4}{R_{16}}\Big]dt, \end{aligned} \tag{4.60}$$

the circuital components values of (4.60) are selected after identities between (4.1) and 4.60 as:

$$\begin{aligned} &R_1 = R_2 = R_7 = R_{10} = R_{15} = R_{16} = R_{18} = 1\text{k}\Omega,\\ &R_5 = 0.285714\text{k}, R_12 = 3.333\text{k}\Omega,\\ &R_3 = R_4 = R_8 = R_9 = R_13 = R_{14} = R_{17} = 100\text{k}\Omega, R_6 = R_{11} = 10\text{k}\Omega, R_{10} = 0.333\text{k}\Omega,\\ &c_1 = c_2 = c_3 = c_4 = 100\text{nf}, \end{aligned} \tag{4.61}$$

In the following figures we chow a simple comparisons (see figures 4.27, 4.28 and 4.29) which proves that analog circuit for system (4.57) is well coincident with numerical simulations. A conclusion can be made that the chaotic behavior exist in the fractional order system (4.57), which verifies its existence and validity.

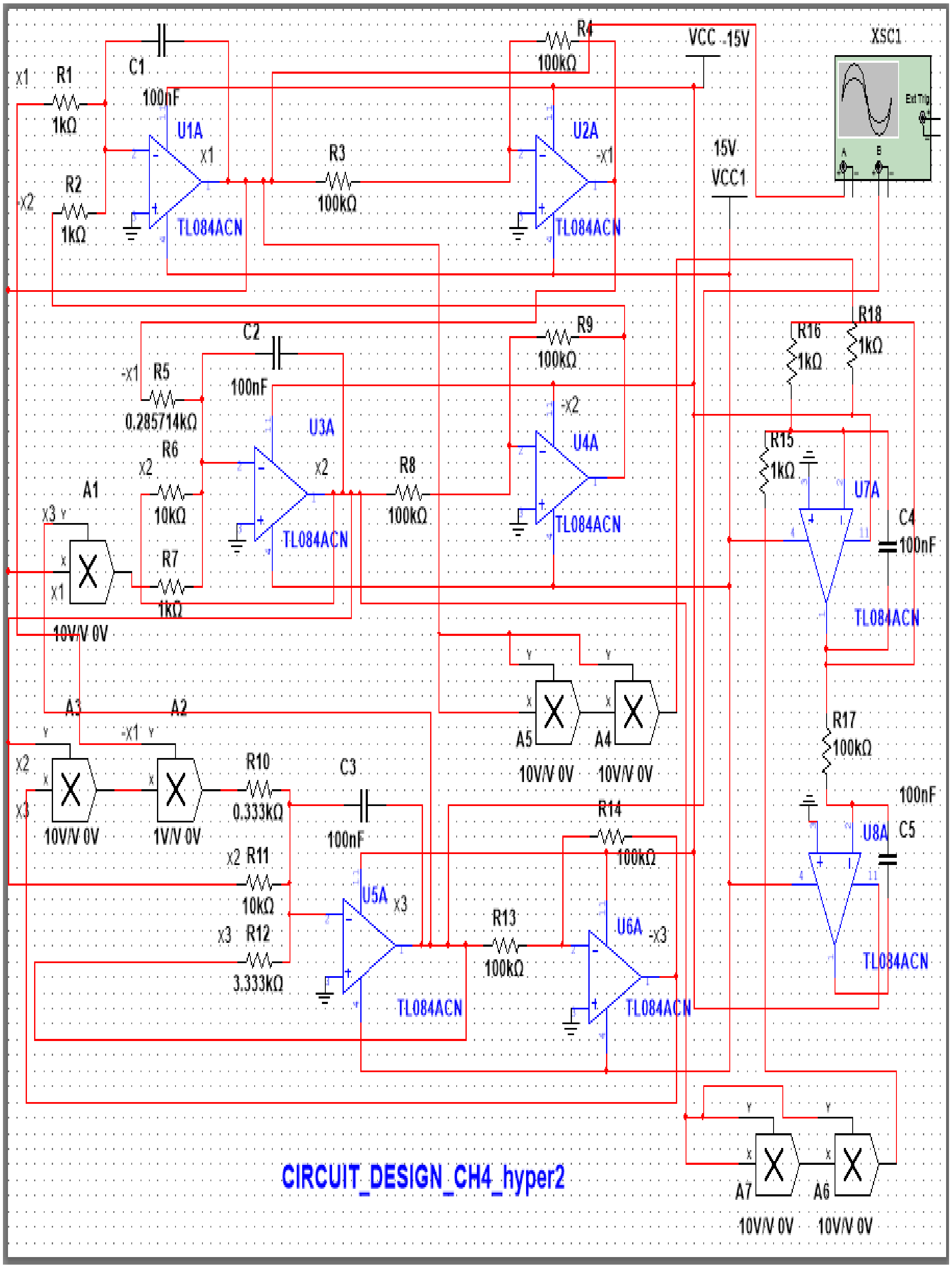

Figure 4.26: Cicuit design in multisim of the proposed hyperchaotic system

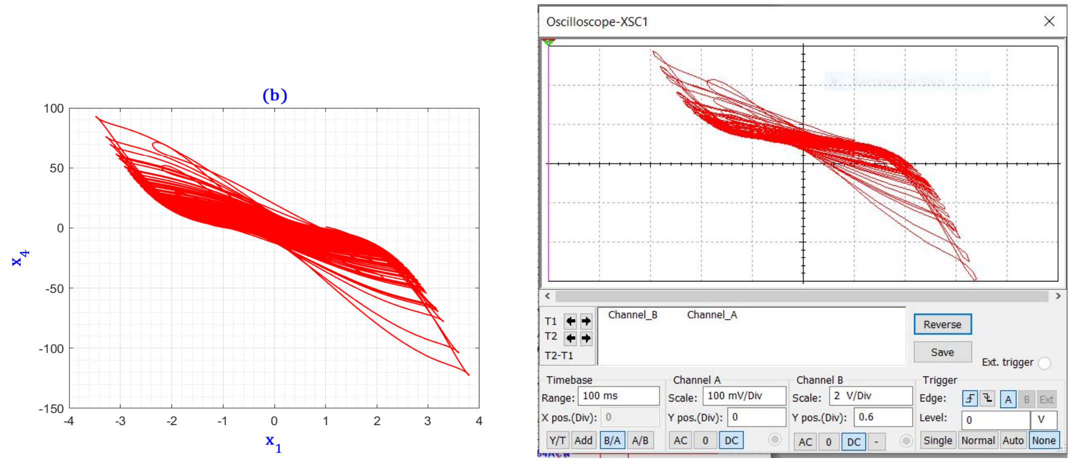

Figure 4.27: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_4 plane

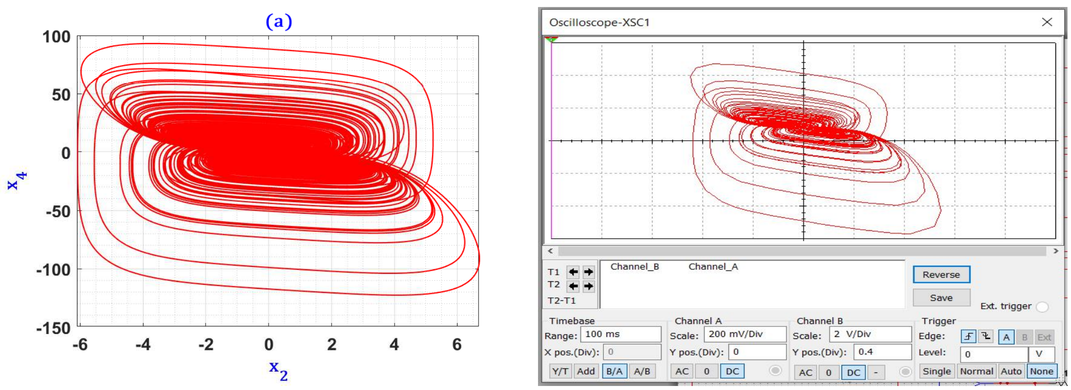

Figure 4.28: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_2x_4 plane

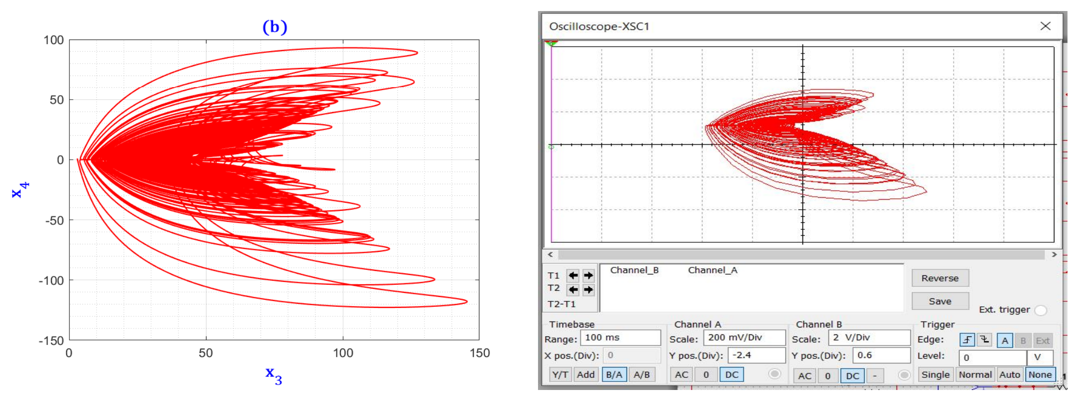

Figure 4.29: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_3x_4 plane

4.4 Extension of the hyperchaotic system to fractional case with stabilization via adaptive control

In this section we extend the integer proposed hyperchaotic system (4.57) to its factional form in the sense of Caputo then we investigate their stabilizeation throught the adaptive control. So, we consider the fractional version of (4.57) which given by:

$$\begin{cases} {}^{C}\mathcal{D}_t^{q_1} x_1(t) = \alpha(x_2 - x_1), \\ {}^{C}\mathcal{D}_t^{q_1} x_1(t) = \gamma x_1 - x_2 - x_1 x_3, \\ {}^{C}\mathcal{D}_t^{q_1} x_1(t) = \beta x_1 x_2 x_3 - x_2 - \beta x_3, \\ {}^{C}\mathcal{D}_t^{q_1} x_1(t) = -\delta(x_1^3 + x_2^3 + x_4), \end{cases} \tag{4.62}$$

where D^q *is the Caputo derivative operator,* $\alpha, \beta, \gamma, \delta$ *are positive reals parameters and* x, y, z, w *are the state variables. With* $\alpha = 10, \beta = 3, \gamma = 35, \delta = 10$*;* $(q_1; q_2; q_3; q_4) = (0.98; 0.9; 0.98; 0.98)$*;* $(x_1(0); x_2(0); x_3(0); x_4(0) =) = (1; 2; 3; 1)$. *The evolution in times of the Lyapunov exponents are depicted in (4.30) , so LEs are given by:*

$$L_1 = 2.0442; L_2 = 0.0593; L_3 = -10.8789; L_4 = -14.0681, \tag{4.63}$$

so we have $L_1 > 0$*, then the system (4.62) is chaotic. Indeed the Kaplan-york dimension for this chaotic system is calculated as:*

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1934, \tag{4.64}$$

it is so fractional, that it confirms the characteristic of the fractional dimension of strange attractor in our system.

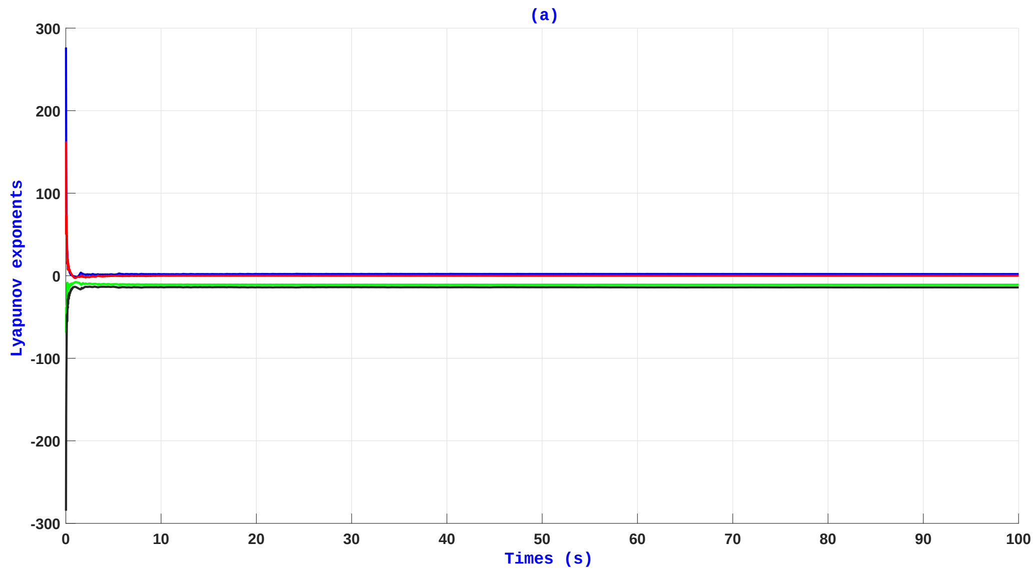

Figure 4.30: Evolution in time of the LEs of the fractional chaotic system (4.57)

4.4.1 Stabilisation of the novel fractional hyperchaotic system via adaptive control

This section aim to design an adaptive control law for globally controlling the identical novel fractional chaotic system with unknown system parameters. The controlled system is given by:

$$\begin{cases} {}^{C}D_t^{q_1} x_1(t) = a(x_2 - x_1) + \eta_1, \\ {}^{C}D_t^{q_2} x_2(t) = cx_1 - x_1 - x_1x_3 + \eta_2, \\ {}^{C}D_t^{q_3} x_3(t) = bx_1x_2x_3 - x_2 - bx_3 + \eta_3, \\ {}^{C}D_t^{q_4} x_4(t) = -d(x_1^3 + x_2^3 + x_4) + \eta_4, \end{cases} \tag{4.65}$$

where the parameters a, b, c, d *are unknown and their estimates* $a_1(t), b_1(t), c_1(t), d_1(t)$*, respectively. We will later search an adaptive controllers* $\eta_1, \eta_2, \eta_3, \eta_4$.

We take the adaptive control law as follow:

$$\begin{cases} \eta_1 = -\hat{a}(x_2 - x_1) - \xi_1 x_1, \\ \eta_2 = -\hat{c}x_1 + x_1 + x_1 x_3 - \xi_2 x_2, \\ \eta_3 = -\hat{b}x_1 x_2 x_3 + \hat{b}x_3 + x_2 - \xi_3 x_3, \\ \eta_4 = \hat{d}(x_1^3 + x_2^3 + x_4) - \xi_4 x_4, \end{cases} \tag{4.66}$$

where ξ_1, ξ_2, ξ_3 and ξ_4 are positive constants and $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ are estimates of the system parameters a, b, and c, respectively. In order to get the closed loop system we substitute (4.66) in (4.65), so we get:

$$\begin{cases} \mathcal{D}_t^{q_1} x_1(t) = (a - \hat{a})(x_2 - x_1) - \xi_1 x_1, \\ \mathcal{D}_t^{q_1} x_2(t) = (c - \hat{c})x_1 - \xi_2 x_2, \\ \mathcal{D}_t^{q_1} x_3(t) = (b - \hat{b})x_1 x_2 x_3 - (b - \hat{b})x_3 - \xi_3 x_3, \\ \mathcal{D}_t^{q_1} x_4(t) = (d - \hat{d})(x_1^3 + x_2^3 + x_4) - \xi_4 x_4, \end{cases} \tag{4.67}$$

let define the parameter estimation error as:

$$\epsilon_a = a - \hat{a},\ \epsilon_b = b - \hat{b},\ \epsilon_c = c - \hat{c},\ \epsilon_d = d - \hat{d}, \tag{4.68}$$

applying the Caputo derivative of order q in (4.68), we obtain:

$$\begin{cases} \mathcal{D}_t^{q} \epsilon_a(t) = -\mathcal{D}_t^{q} \hat{a}, \\ \mathcal{D}_t^{q} \epsilon_b(t) = -\mathcal{D}_t^{q} \hat{b}, \\ \mathcal{D}_t^{q} \epsilon_c(t) = -\mathcal{D}_t^{q} \hat{c}, \\ \mathcal{D}_t^{q} \epsilon_d(t) = -\mathcal{D}_t^{q} \hat{d}, \end{cases} \tag{4.69}$$

after substituting (4.68) in (4.67) we obtain:

$$\begin{cases} \mathcal{D}_t^{q_1} x_1(t) = \epsilon_a(x_2 - x_1) - \xi_{11}, \\ \mathcal{D}_t^{q_1} x_2(t) = \epsilon_c x_1 - \xi_2 x_2, \\ \mathcal{D}_t^{q_1} x_3(t) = \epsilon_b x_1 x_2 x_3 - \epsilon_b x_3 - \xi_3 x_3, \\ \mathcal{D}_t^{q_1} x_4(t) = \epsilon_d(x_1^3 + x_2^3 + x_4) - \xi_4 x_4, \end{cases} \tag{4.70}$$

we aim now to adjust the parametre estimates, let's define the following Lyapunov function:

$$V(x_1, x_2, x_3, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right) \tag{4.71}$$

which is a positive definite function on $\mathbb{R}^8$. *Utilizing (4.69) and the diferentiation of the function* V *along the trajectories of equation (4.70), we get:*

$$\begin{cases} {}^{C}\!D_t^q V(x_1, x_2, x_3, x_4, e_a, e_b, e_c, e_d) = \frac{1}{2}D^q x_1^2 + \frac{1}{2}D^q x_2^2 + \frac{1}{2}D^q x_3^2 + \frac{1}{2}D^q x_4^2 + \frac{1}{2}D^q e_a^2 + \frac{1}{2}D^q e_b^2 + \frac{1}{2}D^q e_c^2 + \frac{1}{2}D^q e_d^2 \\ \le x_1 D^q x_1 + x_2 D^q x_2 + x_3 D^q x_3 + x_4 D^q x_4 + e_a D^q e_a + e_b D^q e_b + e_c D^q e_c + e_d D^q e_d \\ \le x_1(\epsilon_a(x_2 - x_1) - \xi_1 x_1) + x_2(\epsilon_c x_1 - \xi_2 x_2) + x_3(\epsilon_b x_1 x_2 x_3 - \epsilon_b x_3 - \xi_3 x_3) + x_4(\epsilon_d(x_1^3 + x_2^3 + x_4) - \xi_4 x_4) \\ +\epsilon_a(-{}^{C}\!D_t^q \hat{a}) + \epsilon_b(-{}^{C}\!D_t^q \hat{b}) + \epsilon_c(-{}^{C}\!D_t^q \hat{c}) + \epsilon_d(-{}^{C}\!D_t^q \hat{d}) \\ \le -\xi_1 x_1^2 - \xi_2 x_2^2 - \xi_3 x_3^2 - \xi_4 x_3^4 + \epsilon_a(x_1 x_2 - x_1^2 -^{C} D_t^q \hat{a}) + \epsilon_b(x_1 x_2 x_3^2 - x_3^2 -^{C} D_t^q \hat{b}) + \epsilon_c(x_1 x_2 -^{C} D_t^q \hat{c}) \\ +\epsilon_d(x_1^3 x_4 + x_2^3 x_4 + x_4^2 -^{C} D_t^q \hat{d}), \end{cases} \tag{4.72}$$

in view of (4.72), we take the parameter update law as follows:

$$\begin{cases} {}^{C}\!D_t^q \hat{a}(t) = x_1 x_2 - x_1^2 + \xi_4 \epsilon_a, \\ {}^{C}\!D_t^q \hat{b}(t) = x_1 x_2 x_3^2 - x_3^2 + \xi_5 \epsilon_b, \\ {}^{C}\!D_t^q \hat{c}(t) = x_1 x_2 + \xi_4 \epsilon_c, \\ D_t^q \hat{d}(t) = x_1^3 x_4 + x_2^3 x_4 + x_4^2 + \xi_4 \epsilon_d, \end{cases} \tag{4.73}$$

substituting (4.73) into (4.72), we obtain:

$${}^{C}\!D_t^q V(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_a, \epsilon_b, \epsilon_c, \epsilon_d) \le -\xi_1^2\epsilon_1^2 - \xi_2^2\epsilon_2^2 - \xi_3^2\epsilon_3^2 - \xi_4^2\epsilon_4^2 - \xi_5^2\epsilon_a^2 - \xi_6^2\epsilon_b^2 - \xi_7^2\epsilon_c^2 - \xi_8^2\epsilon_d^2 \le 0, \tag{4.74}$$

which shows that ${}^{C}\!D_t^q V$ *is a negative definite function on* $\mathbb{R}^8$, *it follows that:*

$$x_1(t) \to 0, \quad x_2(t) \to 0, \quad x_3 \to 0, \quad x_4 \to 0, \quad e_a \to 0, \quad e_b \to 0, \quad e_c \to 0 \quad \textit{and} \quad e_d \to 0, \tag{4.75}$$

exponentially as $t \to \infty$. *Hence, we have proved the following theorem.*

Theorem 4.4.1. *The novel fractional chaotic systems (4.65) with unknown parameters is globally and exponentially stabiliized for all initial conditions by the adaptive feedback control law (4.66) and the parameter update law (4.73), were* $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8$ *are positive constants. The errors for parameter estimates* $\epsilon_a, \epsilon_b, \epsilon_c, \epsilon_d$ *decay to zero exponentially as* $t \longrightarrow +\infty$.

4.5 The novel Fractional Order Ma System and its circuit design

A 3-D dynamic model of finance, known as the Ma system, has been reported in many works [29, 21], and provided its systematized equations. The fractional version of this streamlined financial system [21] is examined in this work. Let $(\alpha, \beta, \gamma) = (4; 20; 4)$ *three positive reals parameters and* v_1, v_2, v_3 *are the states variables. We can get a new fractional* $3-D$ *chaotic system as follows [5]:*

$$\begin{cases} \mathcal{D}_t^{q_1} v_1(t) = -\alpha(v_1 + v_2), \\ \mathcal{D}_t^{q_2} v_2 = -v_2 - \gamma v_1 v_3, \\ \mathcal{D}_t^{q_3} v_3 = \beta + \alpha v_1 v_2, \end{cases} \tag{4.76}$$

where D^q *represent the Caputo derivative operator. In the finance system (4.76)* v_1 *represent the interest rate,* v_2 *represent the the investment demand and* v_3 *is the price index. Putting* α, β *and* γ*;* $(q_1; q_2; q_3) = (0.98; 0.98; 0.98)$*;* $(v_1(0); v_2(0); v_3(0)) = (0.1; 0.1; 0.1)$.
Using Benettin–Wolf algorithm [11], the results of the Lyapunov exponents and the $0-1$ *test are given by numerical simulations in Matlab with t=10000 iteration (figure (4.31)), and are given by:*

$$L_1 = 1.2726; L_2 = -0.00025; L_3 = -6.7457, \tag{4.77}$$

so we have $L_1 > 0$ *and* $\sum_{i=1}^{3} LE_i < 0$*, then the new financial fractional system (4.76) is chaotic.*

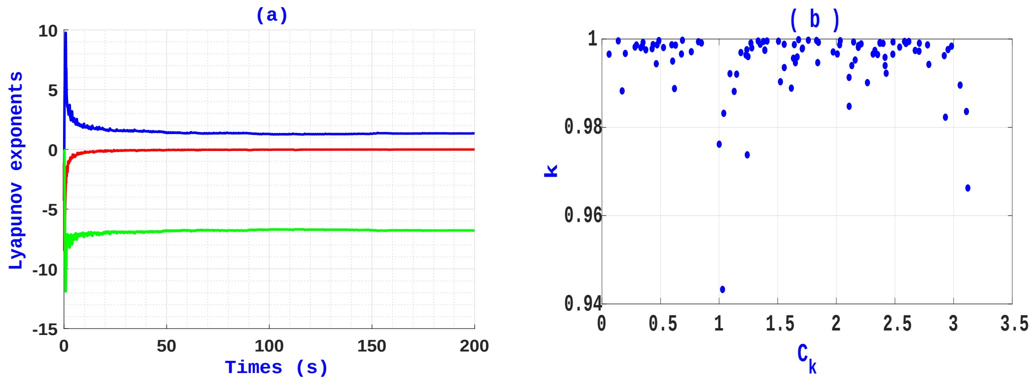

Figure 4.31: Lyapunov exponents and 0-1 test for Ma system

Indeed the Kaplan-york dimension for this chaotic system is calculated as:

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1892, \tag{4.78}$$

the $0-1$ *test give us* $k = 0.9976 \cong 1$ *which confirm that the system (4.76) exhibit chaotic behavior.*

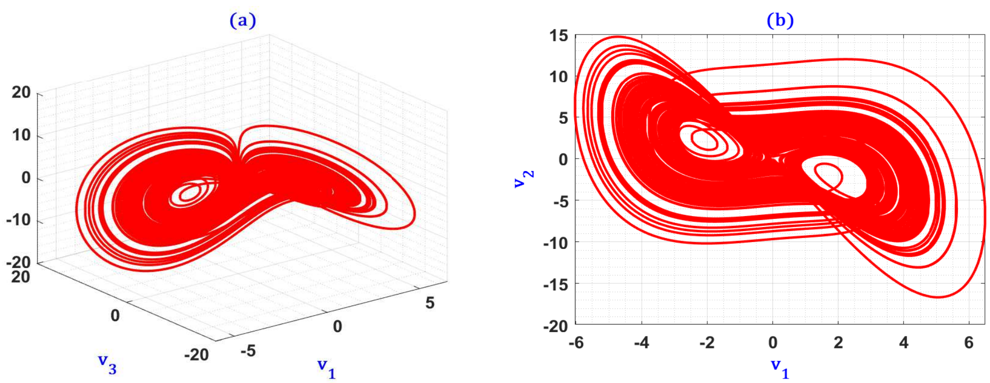

Figure 4.32: Attractor of Ma system and projection in v_1v_2 plane

The series of figures (4.32, 4.33) depict the strange attarctor of the fractional financial system with its projections in the planes v_1-v_2, v_1-v_3 *and* v_2-v_3*. Also The series of figures (4.34, 4.35, 4.31) depicts the variation of such system parameters and the fractional order of the system with the three lyapunov exponents. These figures give us a deep view how the system evolve to the chaotic behavior.*

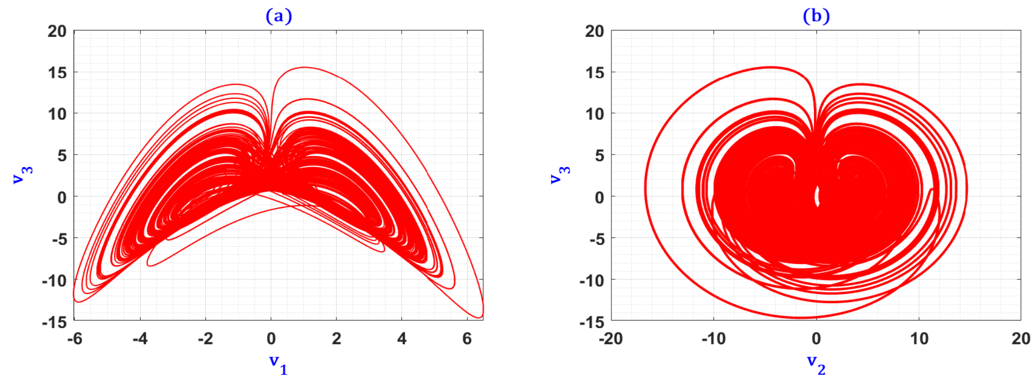

Figure 4.33: Projection in v_1v_3 and v_2v_3 plane of Ma system

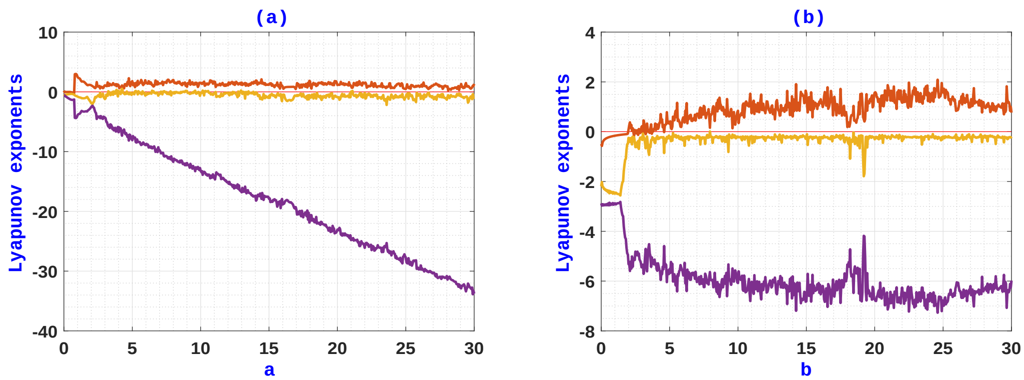

Figure 4.34: Evolution in time of parametre a and b versus the largest Lyapunov Exponent

4.5.1 Adaptive synchronization of the novel system

The goal of this section is to create an adaptive control law that will allow the identical novel fractional chaotic system with unknown system parameters to be globally synchronized. We announce the next theorem.

Theorem 4.5.1. *The 3-D novel fractional chaotic systems (4.79) and (4.80) with unknown parameters are exponentially and globally synchronized for all initial conditions through the parameter update law (4.91) and the adaptive feedback control law (4.84), where ξ_1, ξ_2 and ξ_3 are positive constants.*

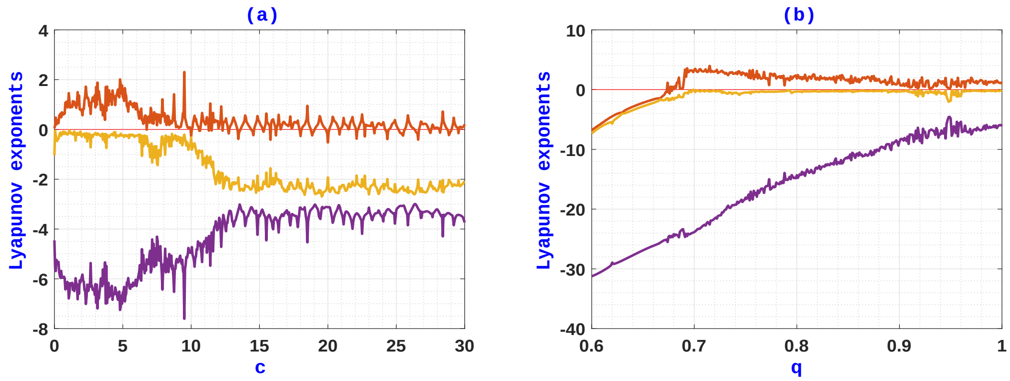

Figure 4.35: Evolution in time of parametres a and the order q versus the largest Lyapunov Exponent

Proof. The master system is provided by:

$$\begin{cases} {}^{C}\!D_t^{q_1} v_1(t) = -\alpha(v_1 + v_2), \\ {}^{C}\!D_t^{q_2} v_2(t) = -v_2 - \gamma v_1 v_3, \\ {}^{C}\!D_t^{q_3} v_3(t) = \beta + \alpha v_1 v_2, \end{cases} \tag{4.79}$$

the following is how to obtain the slave system:

$$\begin{cases} {}^{C}\!D_t^{q_1} w_1(t) = -\alpha(w_1 + w_2) + \eta_1, \\ {}^{C}\!D_t^{q_2} w_2(t) = -w_2 - \gamma w_1 w_3 + \eta_2, \\ {}^{C}\!D_t^{q_3} w_3(t) = \beta + \alpha w_1 w2 + \eta_3, \end{cases} \tag{4.80}$$

the slave system (4.79) and the master system (4.80) contains unknown parameters α, β and γ.

Now our goal is to find the controllers η_1, η_2 and η_3.

Calculating the error of synchronization between the systems (4.79) and (4.80) is defined as:

$$\epsilon = w_i - v_i, i = \overline{1,3}, \tag{4.81}$$

the previous Equation implies:

$$ {}^{C}D_t^q \epsilon(t) = {}^{C}D_t^q w_i(t) - {}^{C}D_t^q v_i(t), i = \overline{1,3} \tag{4.82}$$

Thus, the error of synchronization between (4.79) and (4.80) result as follows:

$$\begin{cases} {}^{C}D_t^q \epsilon_1(t) = \alpha(\epsilon_1 + \epsilon_2) + \eta_1, \\ {}^{C}D_t^q \epsilon_2(t) = -\epsilon_2 + \gamma v_1 v_3 - \gamma w_1 w_3 + \eta_2, \\ {}^{C}D_t^q \epsilon_3(t) = \alpha(w_1 w_2 - v_1 v_2) + \eta_3, \end{cases} \tag{4.83}$$

define now the law of the adaptive control by:

$$\begin{cases} \eta_1 = -\alpha_1(\epsilon_2 + \epsilon_2) - \xi_1 \epsilon_1, \\ \eta_2 = \epsilon_2 - \gamma_1 v_1 v_3 + \gamma_1 w_1 w_3 - \xi_2 \epsilon_2, \\ \eta_3 = -\alpha_1(w_1 w_2 - v_1 v_2) - \xi_3 \epsilon_3, \end{cases} \tag{4.84}$$

where ξ_1, ξ_2 and ξ_3 are positive constants.

Substituting (4.84) in (4.83), we get:

$$\begin{cases} {}^{C}D_t^q \epsilon_1(t) = (\alpha - \alpha_1)(\epsilon_2 + \epsilon_1) - \xi_1 e_1, \\ {}^{C}D_t^q \epsilon_2(t) = (\gamma - \gamma_1)(v_1 v_3 - w_1 w_3) - \xi_2 \epsilon_2, \\ {}^{C}D_t^q \epsilon_3(t) = (\alpha - \alpha_1)(w_1 w_2 - v_1 v_2) - \xi_3 \epsilon_3, \end{cases} \tag{4.85}$$

the estimation errors of the parameters are defined by:

$$\begin{cases} \epsilon_\alpha(t) = \alpha - \alpha_1(t), \\ \epsilon_\beta(t) = \beta - \beta_1(t), \\ \epsilon_\gamma(t) = \gamma - \gamma_1(t), \end{cases} \tag{4.86}$$

applying the operator of order q of Caputo in (4.86), we get:

$$\begin{cases} \mathcal{D}_t^q \epsilon_a(t) = -\mathcal{D}_t^q \alpha_1(t), \\ \mathcal{D}_t^q \epsilon_b(t) = -\mathcal{D}_t^q \beta_1(t), \\ \mathcal{D}_t^q \epsilon_d(t) = -\mathcal{D}_t^q \gamma_1(t), \end{cases} \tag{4.87}$$

by using (4.87), rewriting the closed-loop system (4.85) as:

$$\begin{cases} \mathcal{D}_t^q \epsilon_1(t) = \epsilon_\alpha(\epsilon_2 + \epsilon_1) - \xi_1 \epsilon_1, \\ \mathcal{D}_t^q \epsilon_2(t) = \epsilon_\gamma(v_1 v_3 - w_1 w_3) - \xi_2 \epsilon_2, \\ \mathcal{D}_t^q \epsilon_3(t) = \epsilon_\alpha(w_1 w_2 - v_1 v_2) - \xi_3 \epsilon_3, \end{cases} \tag{4.88}$$

our aim now is to lead (4.88) to zero, we choose the Lyapunov function as:

$$V\left(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_\alpha, \epsilon_\gamma\right) = \frac{1}{2}\left(\xi_1 \epsilon_1^2 + \xi_2 \epsilon_2^2 + \xi_3 \epsilon_3^2 + \epsilon_\alpha^2 + \epsilon_\gamma^2\right), \tag{4.89}$$

where V a positive function which definite on $\mathbb{R}^5$. Along the trajectories of the systems (4.87) and (4.87), we apply the Caputo operator of differentiation in V to get:

$$\begin{cases} \mathcal{D}_t^q V\left(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_\alpha, \epsilon_\gamma\right) = \frac{1}{2} D^q k_1 \epsilon \epsilon_1^2 + \frac{1}{2}\xi_2 D^q \epsilon_2^2 + \frac{1}{2}\xi_3 D^q \epsilon_3^2 + \frac{1}{2} D^q \epsilon_\alpha^2 + \frac{1}{2} D^q \epsilon_\gamma^2 \\ \qquad \leq \xi_1 \epsilon_1 D^q \epsilon_1 + \xi_2 \epsilon_2 D^q \epsilon_2 + \xi_3 \epsilon_3 D^q \epsilon_3 + \epsilon_\alpha D^q \epsilon_\alpha + \epsilon_\gamma D^q \epsilon_\gamma \\ \leq -\xi_1^2 \epsilon_1^2 - \xi_2^2 \epsilon_2^2 - \xi_3^2 \epsilon_3^2 + \epsilon_\alpha \left(\xi_1 \epsilon_1^2 + \xi_1 \epsilon_1 \epsilon_2 + \xi_3 \epsilon_3 w_1 w_2 - \xi_3 \epsilon_3 v_1 v_2 - D^q \alpha_1(t)\right) \\ \qquad + \epsilon_\gamma \left(\xi_2 \epsilon_2 v_1 v_3 - \xi_2 \epsilon_2 w_1 w_3 - D^q \gamma_1(t)\right), \end{cases} \tag{4.90}$$

in view of (4.90), we can put the parameter update law as:

$$\begin{cases} \mathcal{D}_t^q \alpha_1(t) = \xi_1 \epsilon_1^2 + \xi_1 \epsilon_1 \epsilon_2 + \xi_3 \epsilon_3 w_1 w_2 - \xi_3 \epsilon_3 v_1 v_2, \\ \mathcal{D}_t^q \gamma_1(t) = \xi_2 \epsilon_2 v_1 v_3 - \xi_2 \epsilon_2 w_1 w_3, \end{cases} \tag{4.91}$$

substituting (4.91) into (4.90), we get the closed-loop error dynamics as follows:

$$\mathcal{D}_t^q V\left(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_\alpha, \epsilon_\gamma\right) \leq -\xi_1^2 \epsilon_1^2 - \xi_2^2 \epsilon_2^2 - \xi_3^2 \epsilon_3^2 \leq 0,$$

Hence, the zero solution of the error system (4.90) is assymptoticly stable and the previous theorem is proved. ■

4.5.2 Numerical simulation

We solve the system of differential equations (4.79, 4.80, 4.88, 4.91) using the algorithm of Adams-Bashforth-Moulton [42] for the fractional-order system in order to validate our findings. The initial conditions of the drive system (4.79) and the response system (4.80) are chosen as: $(v_1(0), v_2(0), v_3(0)) = (0.4, 0.5, 0.2)$, $(w_1(0), w_2(0), w_3(0)) = (5.4, -4.5, 2.2)$, *respectively.*

$(\epsilon_1(0), \epsilon_2(0), \epsilon_3(0)) = (5, -5, 2)$, $(\alpha_1(0), \beta_1(0)) = (5, 4.5)$. *In figure (4.36, 4.38, 4.38), the time-history of the synchronization of states* $v_1(t)w_1(t); v_2(t)w_2(t); v_3(t)w_3(t)$*, the errors of the synchronization* $e_1(t); e_2(t); e_3(t)$ *and the evolution of the parameters estimations are depicted.*

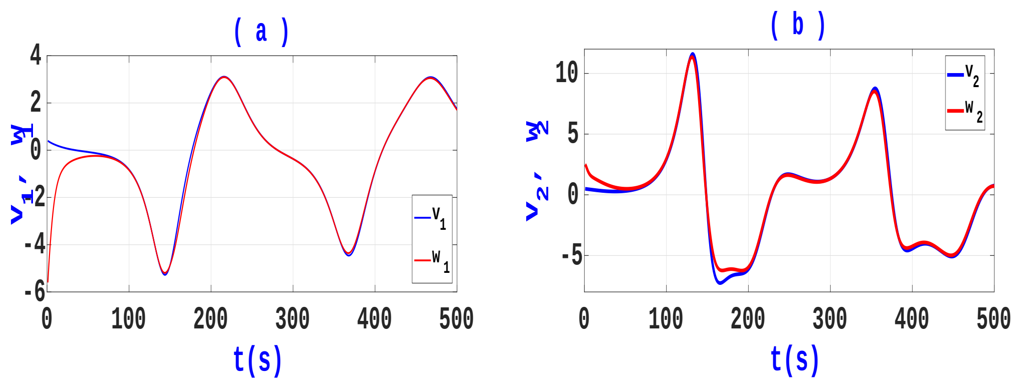

Figure 4.36: Synchronization between $v_i, w_i, i = 1, 2$..

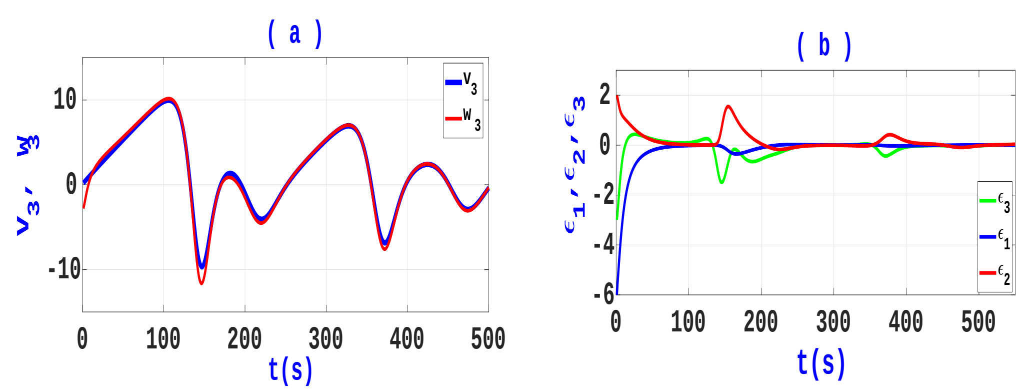

Figure 4.37: (a) Evolution in time of the synchronization between v_3 and y_3 (b) Evolution in time of the synchronization errors $\epsilon_1(t); \epsilon_2(t); \epsilon_3(t)$ and

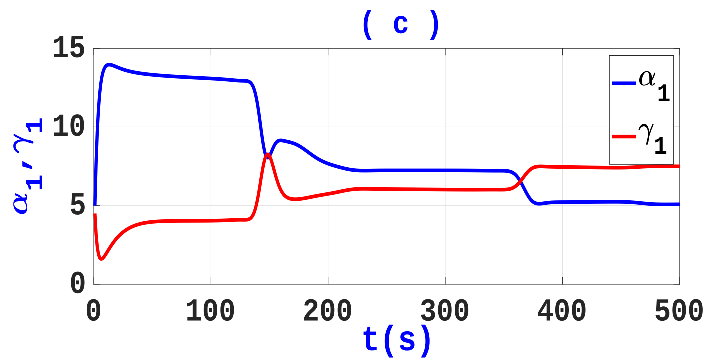

Figure 4.38: (a) Evolution in time of parameters estimation.

4.5.3 Circuit design of the proposed chaotic system

This subsection presents an electronic circuit equivalent to our proposed chaotic system (4.76). The electronic circuit is shown in figure (4.39) and he was designed using the Multisim software. For q = 1, the analog circuit of the proposed 3-D chaotic system (4.76) is implemented by adopting capacitors, resistors, analog multipliers AD633, operational amplifiers TL084ACN. The analoge circuit can produce nonlinear equations that are represented as the following by applying the Kirchhoff law to it where we get the value given in (4.93).

$$\begin{aligned} x_1 &= \frac{1}{c_1}\int\left[\frac{x_1}{R_1} + \frac{(x_2)}{R_2}\right]dt, \\ x_2 &= \frac{1}{c_2}\int\left[\frac{x_2}{R_{55}} + \frac{x_1x_3}{10R_{66}}\right]dt, \\ x_3 &= \frac{1}{c_3}\int\left[\frac{1}{R_5} + \frac{x_1x_2}{10R_{10}}\right]dt, \end{aligned} \tag{4.92}$$

the circuital components values of (2.78) are selected after identities between (4.1) and (4.92) as:

$$\begin{aligned} &R_1 = R_2 = 2.5\text{k}\Omega, \\ &R_{55} = 10\text{k}\Omega, R_{66} = 0.25\text{k}\Omega, \\ &R_3 = R_4 = R_6 = R_7 = 100\text{k}\Omega, \\ &R_{10} = 0.25\text{k}\Omega, R_5 = 0.5\text{k}\Omega, \\ &c_1 = c_2 = c_3 = 100\text{nf}. \end{aligned} \tag{4.93}$$

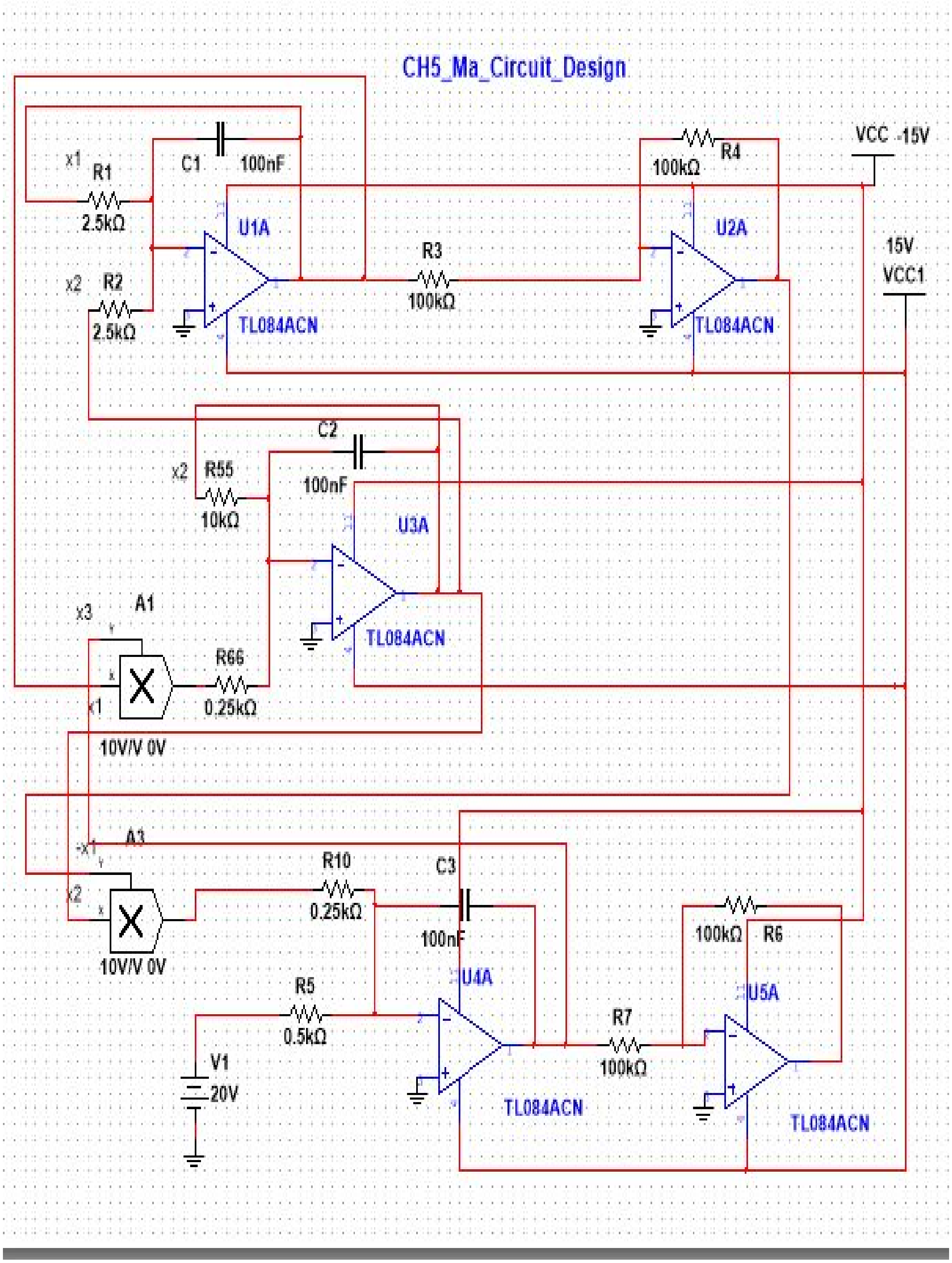

Figure 4.39: Cicuit design in multisim of the proposed chaotic Ma system

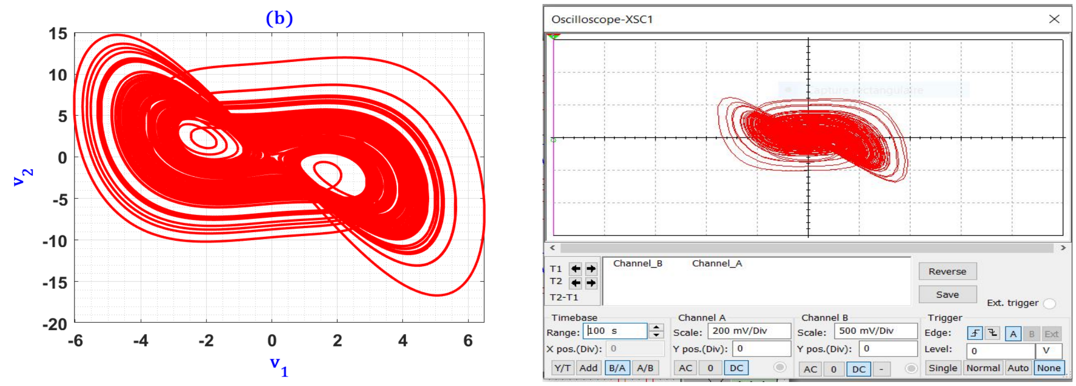

Figure 4.40: (a) Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in v_1v_2 plane

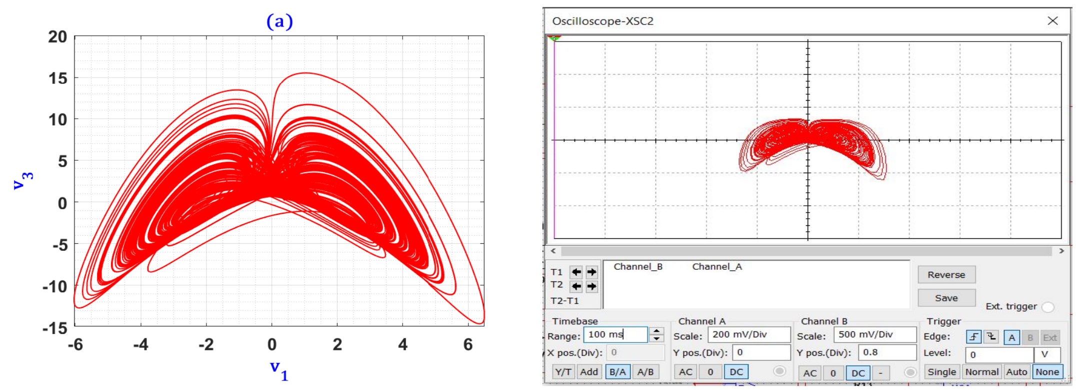

Figure 4.41: (a) Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in v_1v_3 plane

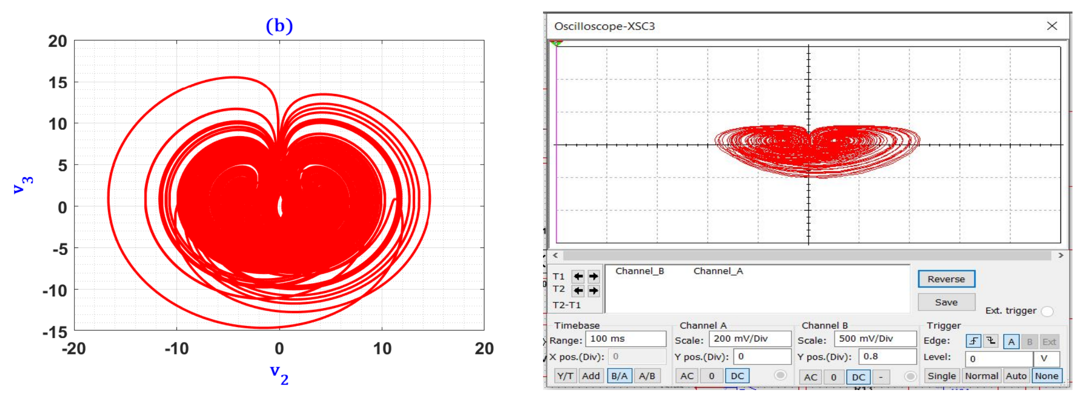

Figure 4.42: (a) Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in v_2v_3 plane

GENERAL CONCLUSION AND PERSPECTIVES

Stabilization and synchronization of chaotic systems are very interesting topic because they exists many problems contain these phenomena. In this thesis, we studied the stabilization and the synchronization in chaotic systems. We used the Caputo sense derivative because of its fractional initial conditions were compatible with ordinary initial conditions and fractional derivative equals to zero, where we introduced two chaotic systems and one hyperchaotic. The process of finding or creating chaotic systems depends on proof that they have chaotic or hyperchaotic behavior using Liapunov exponents as the basic method to give us the stability zone, the periodicity zone and finally the chaotic zone, bifurcation diagrams and 0-1 test are also used for chaos inspection in the systems studied in this thesis. One of the problems we encountered is that the three methods of chaos detection are numerical methods in which we relied on the computer to find numerical results and make simulations to read and see the results, but working in fractional systems made the time to implement simulations in the computer quite long. We used the Adams Bashfort Moulton method to solve the fractional differential equations, after calculating the Lyapunov exponents, we calculated the Kaplan York dimension and compared it with other dimensions of other chaotic systems, because this dimension *every time it is large, the chaotic system is stronger in encryption. Secondly, we have demonstrated that the chaotic systems proposed are convertible into real electronic circuits, when we used a Multisim software we designate a circuit diagram representing the proposed chaotic systems and check the validity of the results we had previously found in numerical simulations, Comparing the results allowed us to ensure that the proposed chaotic systems have a physical meaning. We use the adaptive stabilization and adaptive synchronization as one of the most powerful methods, this is due to the fact that it contains parameters that change at every moment which makes data hacking difficult. In the next studies, we will move on to realistically implement these electronic circuits by installing them in a laboratory, we will also use this proposed chaotic and hyperchaotic fractional systems in the encryption and trasmition of the information.*

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