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A posteriori error estimates for the generalized overlapping domain decomposition method for a class of Advection-Diffusion equation

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Abstract

Abstract: In this thesis, a posteriori error estimates for the generalized Schwarz method with Dirichlet boundary conditions and mixed boundary condition on the interfaces for advection-diffusion equation with second order boundary value problems are proved by using the Euler time scheme combined with Galerkin spatial method. Furthermore, an asymptotic behavior in Sobolev norm is deduced by using Benssoussan-Lions algorithm. Key words: A posteriori error estimates, GODDM, advection-diffusion, Galerkin method, an asymptotic behavior, Benssoussen-Lions algorithm. Résumé: Dans cette thèse, les erreurs a posteriori sont estimées pour la méthode de Schwarz généralisée avec des conditions aux limites de Dirichlet et des conditions aux limites mixtes sur les interfaces pour l'équation advection-diffusion avec des problèmes de valeurs aux limites de second ordre, en utilisant le schéma de temps d'Euler combiné à la méthode spatiale de Galerkin . De plus, un comportement asymptotique dans la norme de Sobolev est déduit en utilisant l'algorithme de Benssoussan-Lions. Mots clés: Estimations de l'erreur a posteriori, GODDM, advection-diffusion, méthode de Galerkin, comportement asymptotique, algorithme de Benssoussen-Lions.

Notation and conventions

In the remainder of this thesis, we will use the following notations:

Ω: given domain in space.
$\partial\Omega = \Gamma$: topological boundary of Ω.
$meas\Omega$: measure of Ω.
Q_T : given domain for time-depending problems.
$x = (x_1, x_2)$: generic point of $\mathbb{R}^2$.
$dx = dx_1 dx_2$: Lebesgue measuring on Ω.
η : outer unit normal vector with respect to $\partial\Omega$.
$\frac{\theta}{\partial\eta}$: directional derivative with respect to η.
∇u : gradient of u.
Δu: Laplacien of u.
$D(\Omega)$: space of differentiable functions with compact support in Ω.
$D^{'}(\Omega)$: distribution space.
$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.
$a(.,.)$: bilinear form.
∂^α : derivative of order $|\alpha|$ with respect to the multi-index α.
I : identity.
π : projection operator.
π_h : interpolation or projection operator which maps onto the finite element space.
τ_j, τ : discretization parameters with respect to time.
P_k : the set of all polynomials of degree k.
V, V^* : Banach space and its dual.
V_h : finite-dimensional finite element space.
$\dim V$: dimension of V.
$\langle f, v\rangle$: value of the functional $f \in V*$ applied to $v \in V$.
$\mathcal{L}(V)$: space of continuous linear mappings of V in$V$.
$L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.
$\|f\|_p = \left(\int\limits_\Omega \left(|f|^P\right)\right)^{\dfrac{1}{p}}$ .norm of the linear functional f.
H : Hilbert space.
$(.,.)$: scalar product in$V$, if V is Hilbert space.
$W^{1,p}(\Omega) = \{u \in L^p(\Omega),\ \nabla u \in L^p(\Omega)\}$ Sobolev space.
$H_0^1(\Omega) = W_0^{1,2}$.

$L^p(0,\ T;\ X) = \left\{f : (0,T) \longrightarrow X \text{ is measurable; } \int\limits_0^T \|f(t)\|_X^p\ dt < \infty\right\}$ p-th integrable functions with values in the Banach space X.

$$L^{\infty}(0,\ T;\ X)=\left\{f:(0,T)\longrightarrow X \text{ is measurable; } ess-\sup_{t\ \in[0,\ T]}\|f(t)\|_{X}^{p}\ <\infty\right\}.$$

$C^{k}([0,\ T];\ X)$:Space of functions $k-$times continuously differentiable for $[0,\ T]\longrightarrow X$.

$D([0,\ T];\ X)$: space of functions continuously differentiable with compact support in $[0,\ T]$.

Published papers

1-Salah Boulaaras, Mohammed Said Touati Brahim and Smail Bouzenada, An asymptotic behavior and a posteriori error estimates for the generalized Schwarz method of advection-diffusion equation, Acta Mathematatica Scientia, 2018, 38B(4): 1227â€“1244.

2-Salah Boulaaras, Mohammed Said Touati Brahim and Smail Bouzenada, A posteriori error estimates for the generalized Schwarz method of a new class of advection-diffusion equation with mixed boundary condition, Math. Meth. Appl. Sci. 2018; 41, 5493â€“5505.

3-Mohammed Said Touati Brahim, Tarek Abdulkafi Alloush, Bahri Belgacem Cherif and Ahmed Himadan Ahmed, The Study of Asymptotic Behavior of Positive Solutions and its Stability for a New Class of Hyperbolic Differential System, Appl. Math. Inf. Sci. 13, No. 3, 341-349 (2019).

4-Mohammed Said Touati Brahim, Salah Boulaaras, and Rafik Guefaifia and Tarek Alloush, Existence of positive weak solutions for sublinear Kirchhoff parabolic systems with multiple parameters, Applied Sciences, Vol. 22, 2020, pp. 52-65.

General Introduction

Originally, the domain decomposition methods were used to solve large problems, because of the large size of the domain calculation, or the precision required for the resolution of the linear system associated with the discretisation of a PDE. Domain decomposition methods then allow the initial linear system to be decomposed into smaller linear subsystems that can be solved by a single processor, according to an iterative algorithm, the different processors responsible for solving the problems exchange, at the interfaces between the subproblems.

Schwarz domain decomposition methods are the oldest domain decomposition methods. They were invented by Hermann Amandus Schwarz in 1869 as an analytical tool to rigorously prove results obtained by Riemann through a minimization principle, [51], known today under the name of Schwarz multiplication method (or Schwarz alternating method), to prove the existence of harmonic functions in irregular domains (it takes the example of a domain composed of a rectangle and a circle that intersect). The idea was not repeated until more than a century later, at the time of development in the 1980s, parallel computer architectures and supercomputers multiprocessors. Domain Decomposition Methods, well adapted massively parallel calculations, then became a new field of study of numerical analysis for the resolution of linear partial equations And nonlinear (see [46], [40], [52] and the references therein). In particular, P. L. Lions [33], [34] and [35] proposes at the end of the 1980s a version parallel of the Schwarz algorithm, with overlapping subdomains where the connection of the solution is ensured by the transmission of a Dirichlet data. This algorithm nevertheless has two drawbacks: on the one hand, the hypothesis of recovery of subdomains is needed for convergence and secondly convergence is slower as the area of coverage is small. For to remedy this problem, he proposes in 1990 a variant for which do not overlap, based on Robin transmission conditions. These conditions are the key point of many works and we will use this type of conditions in this thesis. P.L. Lions noted that it is possible to replace constants intervening in Robin's condition by functions on the interface or local and even non-local operators. The question was therefore to determine the best operators that would make the Schwarz algorithm optimal. The operators non-local, difficult to numerical calculation, approximations using Taylor's developments were first proposed ([22] for a small diffusion in the case of a convection-diffusion problem dominated by convection, then [42] for a low frequency approximation). The first the use of optimized conditions has been introduced in [28] and [29]. These are such as the parameter (s) of the Robin transmission conditions (or more generally of type Ventcell) are chosen in order to optimize the rate of convergence of the algorithm. An analysis and a synthesis of these conditions, these so-called optimized conditions give the methods their name, so-called Schwarz optimizations. These methods converge necessarily faster than classical Schwarz methods, for the same cost per iteration.

For parabolic problems several approaches are possible is:

- One is to discretize the equation in time using a schema implicit, then to use at each time step the domain decomposition in space. The major disadvantage of this step is that the time steps must be the same in the different subdomains. It is more expensive since it does not need to change very often a small amount of information.

- A second approach is to discretize the equation in space, then to solve the system of ordinary differential (time) equations obtained by a wave relaxation type method.

- A third approach, called space-time domain decomposition, consists of to solve independently the subproblems in space and time in the subdomains, then iterated on the values defined on the interfaces space temps between the subdomains to connect the solution between the subdomains adjacent. The strength of this global approach in time is that different discretizations in space and time can be chosen in the different subdomains, the data being transferred in this case to the space-time interface, with the help of projections between the different space-time grids. This algorithm converging slowly, an optimized version, based on the ideas of Schwarz's methods Optimized [33], [34] and [35]. This method is called the Optimized Wave Relaxation Method or Optimized Schwarz wave form relaxation method (OSWR). It is well adapted for the resolution problems in porous media described above.

Allows to link in a natural way the conditions of Robin in multidomains on any meshes. A iterative algorithm is proposed and analyzed to solve this multidomain schema discreet. Another difficulty is to find the right parameters of Robin allowing a rapid convergence of this multidomain algorithm: we have compared continuous and discrete approaches to optimize this parameter. We proposed an approach taking into account the numerical diffusion of the schema, which allows to significantly improve convergence in the case of convection very strongly dominant.

The outline of the thesis is as follows: we will recall in In the first chapter, the fundamental theorems, definitions and properties of the principal spaces, This is a preparatory chapter. And is devoted to the study the formulation of finite element method by variational approach.

In chapter 2 some numerical analysis of evaluation boundary value problems are given, We will explain in this chapter the main numerical methods that will be used later.

In third chapter we expose an introduction to decomposition methods of domains in space with recovery (Schwarz method) as well as so-called non-recovery methods (Schur complement method).

In the chapter 4 an a posteriori error estimate is proposed for the convergence of the discrete solution using Euler time scheme combined with a finite element method on subdomains. Than, we associate with the introduced discrete problem a fixed point mapping and use that in proving the existence of a unique discrete solution.

Finally, in the fifth chapter an-asymptotic behavior estimate for each subdomain is derived with mixed boundary of advevtion-diffusion equation.

We were able to publish the following articles:

1-An asymptotic behavior and a posteriori error estimates for the generalized Schwarz method of advection-diffusion equation, Salah Boulaaras, Mohammed Said Touati Brahim and Smail Bouzenada, Acta Mathematica Scientia 2018,38B(4):1227â€“1244.

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2-A posteriori error estimates for the generalized Schwarz method of a new class of advection-diffusion equation with mixed boundary condition, Salah Boulaaras, Mohammed Said Touati Brahim and Smail Bouzenada, Math Meth Appl Sci. 2018;41 5493â€“5505.

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3-The Study of Asymptotic Behavior of Positive Solutions and its Stability for a New Class of Hyperbolic Differential System, Mohammed Said Touati Brahim, Tarek Abdulkafi Alloush, Bahri Belgacem Cherif and Ahmed Himadan Ahmed, Appl. Math. Inf. Sci. 13, No. 3, 341-349 (2019).

link: http://www.naturalspublishing.com/Article.asp?ArtcID=19692

Chapter 1

Preliminaries

We introduce in this chapter some of the basic concepts of functional spaces, and we present a brief description of those aspects of the Hilbert space, the L^p space and Sobolev spaces, which lie at the heart of the modern theory of Partial Differential Equations.

1.1 Banach fixed-point theorem-the contraction mapping principal

Definition 1. *[19] Let (X, d) be a metric space. A function (map) $T : X \to X$ is called a contraction mapping on X if there exists $k \in [0, 1)$ such that*

$$d(T(x), T(y)) \leqslant kd(x, y), \forall x, y \in X, \tag{1.1}$$

a contraction map "contracts" or " shrinks" the distance between points by the factor k.

Theorem 1. *[19] Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T has a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows:*
start with an arbitrary element x^0in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$.
Then $x_n \longrightarrow x^*$.

Theorem 2. *[49] Consider X, Y to be two Banach spaces, and let $f; X \to Y$ be a mapping such that $f(tx)$ is continuos in t for each fixed x. Assume that there exists $\theta \geq 0$, and $p \in [0, 1)$ such that $\frac{\|f(x,y)-f(x)-f(y)\|}{\|x\|^p+\|y\|^p} \leq \theta$, for any $x, y \in X$. Then there exists a unique linear mapping $T : X \to Y$ such that*

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p}; \forall x \in X. \tag{1.2}$$

1.2 The $L^p(\Omega)$ spaces

Definition 2. *[19] Let Ω be a domain in $\mathbb{R}^n$ and $p \in \mathbb{R}$ with $1 \leq p < \infty$, we denote*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \textit{ is measurable and } \int_\Omega |u(x)|^p \, dx < \infty \right\}. \tag{1.3}$$

We shall presently that the functional $\|.\|_p$ defined by

$$\|u\|_{L^p} = \|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \tag{1.4}$$

is a norm on $L^p(\Omega)$ provided $1 \leq p < \infty$, (it is not norm if $0 < p < 1$).

Definition 3. *We set*

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}; u \text{ is measurable and } |u(x)| \leq C \text{ a.e on } \Omega\},$$

with

$$\|u\|_{L^\infty} = \|u\|_\infty = \inf \{C; |u(x)| \leq C \text{ a.e on } \Omega\}.$$

Remark 1. *[2] If $u \in L^\infty$ then we have $|u(x)| \leq \|u\|_\infty$ a.e on Ω, that implies $\|.\|_\infty$ is a norm.*

Lemma 1. *(**Minkowski's inequality**) If $1 \leq p < \infty$ and $u, v \in L^p$, then*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

Notation 1.3. *Let $1 \leq p \leq \infty$; we denote by q the number $\frac{p}{p-1}$ so that $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = q$ is called the exponent conjugate to p.*

Theorem 3. *[2] (**Holder's inequality**) Assume that $u \in L^p$ and $v \in L^q$ with $1 \leq p \leq \infty$. Then $uv \in L^1$ and*

$$\int_\Omega |uv| \leq \|u\|_p \|v\|_q.$$

Theorem 4. *[19] Assume that $u \in L^p$ and $v \in L^q$ with $0 < p < 1$. Then $uv \in L^1$ and*

$$\int_\Omega |uv| \geq \|u\|_p \|v\|_q.$$

Proposition 1. *[19] If $1 < p < \infty$, L^p is reflexive, separable, and the dual of L^p is L^q for any p.*
If $p = 1$, L^1 is no reflexive, separable, and the dual of L^1 is L^∞.
If $p = \infty$, L^∞ is no reflexive, no separable, and the L^1 dual of L^∞.

1.4 Hilbert spaces

Many interesting questions in theory of variational equalities or inequalities my be formulated in terms of bilinear forms on Hilbert spaces, which has numerous applications in Mechanics and in Physics, in free boundary value problems and in optimal and stochastic control. this theory is a generalization of the variational theory.

Definition 4. *A Hilbert space H is a vectorial space equipped with a scalar product (u, v) such that H is complete for the norm $\|u\| = (u,\ u)^{\frac{1}{2}}$.*

Lemma 2. ***(The Cauchy-Schwarz inequality)*** *Let the inner product* (u,v) *, then*

$$|(u,v)| \leq \|u\| \, \|v\| \, .$$

The equality sign holds if and only if u *and* v *are dependent.*

Proposition 2. *[2]* H *is uniformly convex, and thus it is reflexive.*

Theorem 5. ***(Riez-Fréchet)*** *Given any* $\varphi \in H'$ *there exists a unique* $f \in H$ *such that*

$$\langle \varphi, u \rangle = (f,u) \, ; \forall u \in H.$$

Moreover

$$|f| = \|\varphi\|_{H'} \, .$$

Corollary 1. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence which converges to* u*, in the weak topology and* $(v_n)_{n\in\mathbb{N}}$ *is an other sequence which converge weakly to* v*, then*

$$\lim_{n\longrightarrow\infty} (v_n, u_n) = (v,u) \, .$$

Definition 5. *[6] Let* H *be a real Hilbert space, a bilinear form* $a : H \times H \to \mathbb{R}$ *is said to be:*
i) Continuous if there exists a constant $C > 0$ *such that:* $\forall u, v \in H : |a(u,v)| \leq C \|u\| \, \|v\| \, .$
ii) Coercive if there exists a constant $\alpha > 0$ *such that:* $\forall v \in H : a(v,v) \geq \alpha \|v\|^2 \, .$

Theorem 6. ***(Stampacchia)*** *Assume that* $a(u,v)$ *is a continuous coercive bilinear form on* H*.*
Let $K \subset H$ *be a nonempty closed and convex subset. Then, given any* $\varphi \in H'$*; there exists a unique elements* $u \in K$ *such that:*

$$a(u, v-u) \geq \langle \varphi, v-u \rangle , \forall v \in K.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in K, \quad \textit{and} \ \frac{1}{2} a(u,u) - \langle \varphi, v \rangle = \min_{v\in K} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \right\}.$$

Corollary 2. ***(Lax-Milgram)*** *Assume that* $a(u,v)$ *is a continuous coercive bilinear form on* H*.*

Then, given any $\varphi \in H'$; there exists a unique elements $u \in H$ such that:

$$a(u, v-u) \geq \langle \varphi, v-u \rangle , \forall v \in H.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \ \text{ and } \frac{1}{2} a(u,u) - \langle \varphi, v \rangle = \min_{v\in H} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \right\}. \tag{1.5}$$

Remark 2. *The Lax-Milgram theorem is a very simple and efficient tool for solving linear elliptic partial differential equations.*

Theorem 7. ***(Lax-Milgram) [50]*** *Let* V *be a real Hilbert space,* $L(.)$ *a continuous linear form on* V*,* $a(\cdot,\cdot)$ *a continuous coercive bilinear form on* V*. Then the problem*

$$\begin{cases} \textit{find } u \in V \textit{ such that} \\ a(u,v) = L(v) \textit{ for every } v \in V, \end{cases}$$

has a unique solution. Further, this solution depends continuously on the linear form L*.*

1.5 The Sobolev space

In this section, we introduce Sobolev spaces and establish some of their elementary properties. Now the use of Sobolev spaces is essential for our study, we do not intend to develop in detail the properties of these spaces; some aspects of them basic are described in later section. As applications we give weak formulations of some PDE problems to obtain their solutions.

1.5.1 The space $H^1(\Omega)$

Definition 6. *We call Sobolev space of order 1 on Ω the space*

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) ; \frac{\partial v}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n \right\}, \tag{1.6}$$

we provide $H^1(\Omega)$ with the inner product

$$(u, v)_{1,\Omega} = \int_{\Omega} (uv + \nabla u \nabla v)\, dx, \tag{1.7}$$

and denote $\|v\|_{1,\Omega} = (v, v)_{1,\Omega}^{\frac{1}{2}}$ the corresponding norm.

Definition 7. *Let $C_c^{\infty}(\Omega)$ be the functions space of class C^{∞} with compact support in Ω. Sobolev space $H_0^1(\Omega)$ is defined as the adhesion of $C_c^{\infty}(\Omega)$ in $H^1(\Omega)$.*

Proposition 3. *(**Poincaré's inequality**) Let Ω be an open set of $\mathbb{R}^n$ bounded in at least one direction of space. There exist a constant $C > 0$ such that*

$$\forall v \in H_0^1(\Omega) ; \|v\|_{0,\Omega} \leq C \int |\nabla v(x)|^2 \, dx.$$

Proposition 4. *[19] Let Ω be an open domain in $\mathbb{R}^n$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$(T, \varphi) = \int_{\Omega} f(x)\, g(x) \, dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p < \infty$, and it is well-known that f is unique.

1.5.2 The space $H^m(\Omega)$

Definition 8. *For an integer $m \geq 0$, the space $H^m(\Omega)$ is defined by*

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) ; \forall \alpha \text{ with } |\alpha| \leq m, \partial^{\alpha} v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \in L^2(\Omega) ; |\alpha| = \sum_{i=1}^{N} \alpha_i \right\}. \tag{1.8}$$

Proposition 5. *[19] $H^m(\Omega)$ is a Hilbert space with their usual norm*

$$\|u\|_{H^m(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha| \leq m} \partial^{2\alpha} u(x)\, dx \right)^{\frac{1}{2}}.$$

Theorem 8. *[6] Let Ω is a regular open bounded class C^2. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_{\Omega} \Delta u(x)\, v(x)\, dx = -\int_{\Omega} \nabla u(x)\, \nabla v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta}(x)\, v(x)\, dx. \tag{1.9}$$

Definition 9. *$H_0^m(\Omega)$ is given by the completion of $D(\Omega)$ with respect to the norm $\|.\|_{H^m(\Omega)}$.*

Remark 3. *Clearly $H_0^m(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^m(\Omega)}$.*

The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega) := [H_0^m(\Omega)]^*$.

Lemma 3. *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$D(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow D'(\Omega).$$

1.5.3 The spaces $W^{m,p}(\Omega)$

More generally, we can define three spaces for any integer $m \geq 0$ and for a reel $p; 1 \leq p < \infty$.

Definition 10. *We denote by $H^{m,p}(\Omega)$ the completion of $C^m(\overline{\Omega})$ with respect the norm*

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}} \quad \|u\|_{m,p} = \max_{|\alpha| \leq m} \|\partial^\alpha u\|_\infty.$$

The $W^{m,\,p}(\Omega)$ is the space of all $u \in L^p(\Omega)$, defined as

$$W^{m,\,p}(\Omega) = \left\{ \begin{array}{l} u \in L^p(\Omega), \text{ such that } \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^m \text{ such that,} \\ \\ |\alpha| = \sum_{j=1}^{n} \alpha_j \leq m, \text{ where, } \partial^\alpha = \partial_1^{\alpha_1}\ \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}. \end{array} \right\}. \tag{1.10}$$

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Theorem 9. *$W^{m,\,p}(\Omega)$ is a Banach space with their usual norm $\|u\|_{m,p}$.*

1.5.4 Duality

Recall that the dual H' of a Hilbert space H is the set of continuous linear forms on H.

Definition 11. *We denote the dual of $H_0^{m,p}(\Omega)$ by $H^{-m,p'}(\Omega); \frac{1}{p} + \frac{1}{p'} = 1$, or simply $H^{-m}(\Omega)$ when $p = 2$.*

Proposition 6. *The space $H^{-1}(\Omega)$ is characterized by*

$$H^{-1}(\Omega) = \left\{ f = v_0 + \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}; v_0, v_1, ..., v_n \in L^2(\Omega) \right\}.$$

We denote

$$\langle L, \phi \rangle_{H^{-1}, H_0^1} = L(\phi) = \int_{\Omega} \left(v_0 \phi - \sum_{i=1}^{n} v_i \frac{\partial \phi}{\partial x_i} \right) dx, \tag{1.11}$$

where $L \in H^{-1}(\Omega)$ is continuous linear form and $\phi \in H_0^1(\Omega)$.

Lemma 4. *[6] Let $v \in L^2(\Omega)$. For $1 \le i \le n$, we can define a continuous linear form $\frac{\partial v}{\partial x_i}$ in $H^{-1}(\Omega)$ by the formula*

$$\left\langle \frac{\partial v}{\partial x_i}, \phi \right\rangle_{H^{-1},H_0^1} = -\int v \frac{\partial \phi}{\partial x_i} dx; \forall \phi \in H_0^1(\Omega), \tag{1.12}$$

which verifies

$$\left\| \frac{\partial v}{\partial x_i} \right\|_{H^{-1}(\Omega)} \le \|v\|_{L^2(\Omega)}.$$

Now the smoothness of the boundary $\partial\Omega := \overline{\Omega} - \Omega$ can be described:

Definition 12. *Let Ω be an open subset of $\mathbb{R}^d$, $0 \leqslant \lambda \leqslant 1$, $m \in \mathbb{N}$. We say that its boundary that its boundary $\partial\Omega$ is of class $C^{m;\lambda}$ if the following conditions are satisfied:*

For every $x \in \partial\Omega$ there exist a neighborhood V of x in $\mathbb{R}^d$, and new orthogonal coordinates $\{y_1, ..., y_d\}$ such that V is a hypercube in the new coordinates:

$$V = \{(y_1, ..., y_d) : -a_i < y_i < a_i, i = 1, ...d\},$$

and there exists a function $\varphi \in C^{m;\lambda}(V')$, with

$$V' = \{(y_1, ..., y_{d-1}) : -a_i < y_i < a_i, i = 1, ...d-1\},$$

and such that

$$|\varphi(y')| \leqslant \frac{1}{2} a_d, \forall y' := (y_1, ..., y_{d-1}) \in V'$$
$$\Omega \cap V = \{(y', y_d) \in V : y_d < \varphi(y')\},$$

and

$$\partial\Omega \cap V = \{(y', y_d) \in V : y_d = \varphi(y')\}.$$

A boundary of class $C^{0;1}$ is called Lipschitz boundary.

1.6 The $L^p(0,\ T;\ X)$ spaces

Definition 13. *Let X be a Banach space, denote by $L^p(0,\ T; X)$ the space of measurable functions*

$$\begin{aligned} f :]0,\ T[&\longrightarrow X \\ t &\longrightarrow f(t), \end{aligned}$$

such that

$$\int_0^T \left(\|f(t)\|_X^p\right)^{\frac{1}{p}} dt = \|f\|_{L^p(0,\ T,\ X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,\ T,\ X)} = \sup_{t\ \in]0,\ T[} ess \|f(t)\|_X.$$

Theorem 10. *The space $L^p(0,\ T,\ X)$ is a Banach space.*

Lemma 5. *Let $f \in L^p(0,\ T,\ X)$ and $\dfrac{\partial f}{\partial t} \in L^p(0,\ T,\ X)$, $(1 \le p \le \infty)$, then, the function f is continuous from $[0,\ T]$ to X. i. e. $f \in C^1(0, T, X)$.*

1.7 Sobolev spaces of fractional order and trace theorems

In this section let $\Omega \subset \mathbb{R}^d$ is a measurable set with Lipschitz boundary $\partial\Omega$. The boundary $\partial\Omega$ of Ω will be denoted by $\Gamma := \partial\Omega$.

On the $(d-1)$ -dimensional set it is also possible to define Sobolev spaces:

Definition 14. *$H^{\frac{1}{2}}(\Gamma)$ is defined by*

$$H^{\frac{1}{2}}(\Gamma) := \left\{ u \in L^2(\Gamma) : |u|_{\frac{1}{2},\Gamma} < \infty \right\}, \tag{1.13}$$

where the semi norm $|.|_{\frac{1}{2},\Gamma}$ is given by

$$|u|_{\frac{1}{2},\Gamma} := \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|}{|x-y|^d} ds(x)ds(y), \quad u \in H^{\frac{1}{2}}(\Gamma). \tag{1.14}$$

Theorem 11. *[19] $H^{\frac{1}{2}}(\Gamma)$ with the scalar product*

$$(u,v)_{\frac{1}{2},\Gamma} := \int_\Gamma uvds + \int_\Gamma \int_\Gamma \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^d} ds(x)ds(y), \tag{1.15}$$

is a Hilbert space.

Definition 15. *Let $\Gamma_1 \subset \Gamma$ be a proper, connected $(d-1)$-dimensional relative open subset. Then we define*

$$H^{\frac{1}{2}}(\Gamma_1) := \left\{ u \in L^2(\Gamma_1) : \exists \widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \text{ with } u = \widetilde{u}\,|_{\Gamma 1} \right\}, \tag{1.16}$$

with norm

$$\|u\|_{\frac{1}{2},\Gamma_1} := \inf_{\substack{\widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \\ \widetilde{u}\,|_{\Gamma_1} = u}} \|\widetilde{u}\|_{\frac{1}{2},\Gamma}, \qquad u \in H^{\frac{1}{2}}(\Gamma_1). \tag{1.17}$$

Now we construct a particular subspace of $H^{\frac{1}{2}}(\Gamma_1)$. For $v \in H^{\frac{1}{2}}(\Gamma_1)$ the zero extension of v into $\Gamma - \Gamma_1$ will be denoted by $\widetilde{v}$.

Definition 16. *$H^{\frac{1}{2}}_{00}(\Gamma_1)$ is defined by*

$$H^{\frac{1}{2}}_{00}(\Gamma_1) := \left\{ v \in L^2(\Gamma_1) : \widetilde{v} \in H^{\frac{1}{2}}(\Gamma) \right\}. \tag{1.18}$$

Notice that

$$(u,v)_{H^{\frac{1}{2}}_{00}(\Gamma_1)} := (u,v)_{\frac{1}{2},\Gamma_1} + \int_{\Gamma_1} \frac{uv}{\rho(x,\partial\Gamma_1)} ds(x), \tag{1.19}$$

where $\rho(x, \partial\Gamma_1)$ is a positive function which behaves like the distance between x and $\partial\Gamma_1$, defines a scalar product in $H^{\frac{1}{2}}_{00}(\Gamma_1)$.

Remark 4. *[2] By a direct calculation, for all $v \in L^2(\Gamma_1)$ we obtain tow positive constants c_1, c_2 such that:*

$$c_1 \|v\|_{\frac{1}{2},\Gamma_1} \leqslant \|v\|_{H^{\frac{1}{2}}_{00}(\Gamma_1)} \leqslant c_2 \|v\|_{\frac{1}{2},\Gamma_1}. \tag{1.20}$$

Therefore $H^{\frac{1}{2}}_{00}(\Gamma_1)$ is a Hilbert space.

The dual of these spaces are denoted by

$$H^{-\frac{1}{2}}(\Gamma_1) := \left[H_{00}^{\frac{1}{2}}(\Gamma_1)\right]^*, \quad H_{00}^{-\frac{1}{2}}(\Gamma_1) := \left[H^{\frac{1}{2}}(\Gamma_1)\right]^*. \tag{1.21}$$

Next we present some trace theorems.
Let be $u \in C(\overline{\Omega})$. Then we can define the trace of u on $\partial\Omega$:

$$\gamma_0(u) := u\,|_{\partial\Omega}\,.$$

This trace operator can be extended:

Theorem 12. *[50] Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with boundary $\partial\Omega \in C^{0;1}$. Then the trace mapping γ_0 defined on $C^{0;1}(\overline{\Omega})$ extends uniquely to a bounded, surjective linear map:*

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega).$$

Moreover the right inverse of the trace operator exists.

Theorem 13. *[31] Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, with Lipschitz boundary $\partial\Omega$. Then there exists a linear bounded operator*

$$E : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega), \quad \textit{such hat} \gamma_0(E(\varphi)) = \varphi, \quad \forall\varphi \in H^{\frac{1}{2}}(\partial\Omega).$$

Note that the preceding theorems allow the definition of the following equivalent norm on $H^{\frac{1}{2}}(\partial\Omega)$:

$$\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)} := \inf_{\substack{u \in H^1(\Omega)\\ \gamma_0(u) = u}} \|u\|_{H^1(\Omega)}\,, \quad \forall\varphi \in H^{\frac{1}{2}}(\partial\Omega).$$

Sometimes the simpler notation $u\,|_{\partial\Omega} = \gamma_0(u)$ is used for functions $u \in H^1(\Omega)$.
With the trace operator γ_0 we can characterize the space $H_0^1(\Omega)$.

Theorem 14. *[43] Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, with boundary $\partial\Omega \in C^{0;1}$. Then $H_0^1(\Omega)$ is the kernel of trace operator γ_0, i.e,*

$$H_0^1(\Omega) = N(\gamma_0) = \left\{u \in H^1(\Omega) : \gamma_0(u) = 0\right\} = \left\{u \in H^1(\Omega) : u\,|_{\partial\Omega} = 0\right\}.$$

Definition 17. *Let Ω is an open smooth domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Gamma_D \subsetneq \partial\Omega$ such hat* $\mathrm{mes}(\Gamma_D) > 0$. *We set*

$$H_{\Gamma_D}^1(\Omega) = \left\{u \in H^1(\Omega) : \gamma_0(u) = 0 \textit{ on } \Gamma_D\right\} \left\{u \in H^1(\Omega) : u\,|_{\Gamma_D} = 0\right\}. \tag{1.22}$$

Lemma 6. *$H_{\Gamma_D}^1(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^1(\Omega)}$.*

Theorem 15. *[19] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Furthermore let $\Gamma_D \subset \Omega$ be a connected part of the boundary of Ω with $mes_{d-1}(\Gamma_D) > 0$. Then the inequality*

$$\|u\|_{0,\Omega} \leqslant C(\Omega, \Gamma_D)\ \ |u|_{1,\Omega}$$

is true for all $u \in H^1(\Omega)$ with $\gamma_0(u)\,|_{\Gamma_D} = 0$. The constant $C(\Omega, \Gamma_D)$ depend only on Ω and Γ_D and is bounded by the diameter of Ω.

Remark 5. *By he Inequality of Poincaré we deduce that the semi-norm $|.|_{1,\Omega}$ is an equivalent norm to $\|.\|_{1,\Omega}$ in $H_{\Gamma_D}^1(\Omega)$.*

1.7.1 Green's formula

Proposition 7. *[6] Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary. Then for all $u, v \in H^1(\Omega)$, we have*

$$\int_\Omega \left(\frac{\partial u}{\partial x_i}v + \frac{\partial v}{\partial x_i}u\right)\, dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \quad i = 1, ..., d, \tag{1.23}$$

where η_i is the i-th component of the outward normal vector η.

1.8 Abstract variational problems

The principal of the variational approach for solving the Partial Differential Equations is to replace the equation with an equivalent formulation called variational, obtained by integrating the equation multiplied by a function test. We describe the general abstract framework for all variational problems. Let V be a real Hilbert space with scalar product $(.,.)_V$ and norm $\|.\|_V$.

Theorem 16. *Let S be non empty, convex, closed subset of H.*

$$\forall x \in V, \exists! p(x) \in S : \|x - p(x)\| = \inf_{y \in S} \|x - y\|.$$

The vector $p(x)$ is called the orthogonal projection of x on S. It is also characterized by the inequality

$$\forall y \in S : (x - p(x), y - p(x)) \leq 0$$

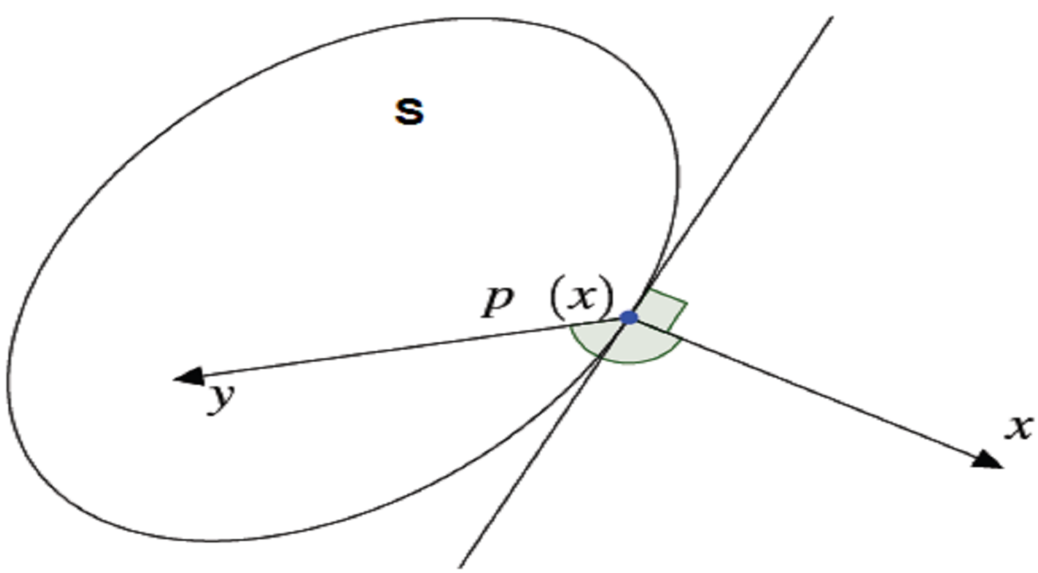


Figure 1.1: Orthogonal projection of x on S

Theorem 17. ***(Riesz)*** *Let V be a Hilbert space and L an element of its dual V', there exist a unique $u \in V$ such that*

$$\forall v \in V : L(v) = (u, v).$$

Moreover,

$$\|L\|_{V'} = \|u\|_V .$$

and the linear mapping $T : V' \to V, L \mapsto u$ is isometry

Theorem 18. *(Lax-Millgram) Let V be a Hilbert space, a be a bilinear form continuous coercive, and L be a linear form is continuous.*

There exists a unique $u \in V$ that solves the abstract variational problem:

$$\text{Find } u \in V \text{ such that; } \forall v \in V : a(u, v) = L(v).$$

Proposition 8. *The mapping $V' \to V, L \mapsto u$ defined by theorem Lax-Millgram is linear and continuous.*

Proposition 9. *Let the hypotheses of the Lax-Millgram theorem be satisfied. Assume in addition that the bilinear form a is symmetric.*

Then the solution u of the variational problem (18) is also the unique solution of the minimization problem

$$J(u) = \inf_{v \in V} J(v) \text{ with } J(v) = \frac{1}{2} a(v, v) - L(v).$$

Proof. Let u be the Lax-Millgram solution. For all $v \in V$, we let $w = v - u$, then

$$\begin{aligned} J(v) = J(w+u) &= \tfrac{1}{2} a(u, u) + \tfrac{1}{2} a(u, w) + \tfrac{1}{2} a(w, u) + \tfrac{1}{2} a(w, w) - L(u) - L(w) \\ &= J(u) + a(u, w) - L(w) + \tfrac{1}{2} a(w, w) \geq J(u). \end{aligned}$$

Assume the u minimizes J on V, for all $\lambda > 0$ and all $v \in V$, we have

$$J(u + \lambda v) \geq J(u) \Longrightarrow \frac{1}{2} a(u, u) + \lambda a(u, v) + \frac{\lambda^2}{2} a(v, v) - L(u) - \lambda L(v) \geq J(u)$$

$$\Longrightarrow J(u) + \lambda a(u, v) + \frac{\lambda^2}{2} a(v, v) - \lambda L(v) \geq J(u) \quad \Longrightarrow a(u, v) + \frac{\lambda}{2} a(v, v) - L(v) \geq 0,$$

we then let $\lambda \to 0$, hence $a(u, v) \geq L(v)$, change v in $-v$ obtain $a(u, v) \leq L(v)$,

then $a(u, v) = L(v)$. □

1.8.1 The inf-sup conditions

A more general result on the existence and uniqueness of weak solutions is provided by the inf-sup conditions.

Theorem 19. *[47] Let U, V be a reflexive Banach spaces with the norm $\|.\|_U$, and $\|.\|_V$. Furthermore, let the bilinear form a be bounded in U, V. Then the following statement are equivalent: i) For all $f \in V'$ the linear variational problem $a(u, v) = L(v)$ has a unique solution $u_f \in U$ that satisfies:*

$$\|u_f\|_U \leq \frac{1}{\gamma} \|f\|_{V'} \quad \text{with } \gamma > 0 \text{ independent of } f.$$

ii) The bilinear form a satisfies the inf-sup conditions

$$\exists \gamma > 0 : \inf_{w \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{\|a(w, v)\|}{\|w\|_U \|v\|_V}.$$

$$\forall v \in V \setminus \{0\} \sup_{w \in U \setminus \{0\}} |a(w, v)| > 0.$$

1.9 The Galerkin method

Now, we study the Galerkin method, and verifying of its fundamental properties.

Boris Grigoryevich Galerkin, born on February 20, 1871 in Polotsk (Belarus) and died on July 12, 1945, is a mathematician and an engineer Russian Federation for its contributions to the study of beam trusses and plates elastics His name is linked to a hut of soles approximation of the elastic structures, which is one of the bases of the finite element method.

1.9.1 Analysis of the Galerkin method

The Galerkin finite element method as a general tool for numerical solution of differential equations. Iterations procedures and interpolation techniques are necessary to drive basic a priori and posteriori error estimates.

We place under the following assumptions

$$\begin{cases} H \text{ is a Hilbet space} \\ a\left(.,.\right) \text{ bilinear form continuous and coercive} \\ f \in H'. \end{cases}$$

We consider the problem

$$u \in Ha\left(u,v\right) = f\left(v\right); v \in H. \tag{1.24}$$

By the Lax-Miligram theorem, there is existence and uniqueness of $u \in H$ solution of (1.24). We give $V_h \subset H$ such that $dimH < \infty$ and we try to solve the problem approached:

$$u_h \in H_h; a\left(u_h, v\right) = f\left(v\right); v \in H_h. \tag{1.25}$$

By the Lax-Miligram theorem, we have immediately:

1.9.1.1 Existence and uniqueness

Theorem 20. *[48] Under the hypotheses, if $V_h \subset H$ and $\dim V_h = N$, there exists a unique $u_h \in V_h$ solution of (1.25).*

Proof. As $\dim V_h = N$, there exists a base $(\phi_1, ..., \phi_N)$ of V_h. Let $v \in V_h$, we can thus develop v on this basis:

$$v = \sum_{i=1}^{N} v_i \phi_i, (\phi_1, ..., \phi_N) \in \mathbb{R}^N.$$

by developing u on the basis $(\phi_i)_{i=1;...,N}$, we obtain : $\sum_{j=1}^{N} a\left(\phi_j, \phi_i\right) u_j = f\left(\phi_i\right), \forall i = 1, ..., N$.

We can write this last equality in the form of a linear system: $KU = G$. So, Let $w \in \mathbb{R}^N$ such that: $Kw = 0$, we have:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a\left(\phi_j, \phi_i\right) w_j w_i = 0.$$

for coercivity of a: $a\left(w,w\right) = 0 \Rightarrow w = 0$, then $w_i = 0, \forall i = 1, ..., N$. □

1.9.1.2 Stability

Corollary 3. *[48] The Galarkin method is stable, uniformly with respect to h, and*

$$\|u_h\|_V \leq \frac{1}{\alpha}\|f\|_{V'}. \tag{1.26}$$

1.9.1.3 Convergence

Lemma 7. *(Cé a lemma) Under the hypothesis of the theorem precedent, if u the solution of (1.24), and u_h the solution of (1.25), then:*

$$\|u - u_h\| \leq \frac{M}{\alpha}\|u - v_h\|\,;\forall v_h \in V_h, \tag{1.27}$$

where M and α are satisfies: $\alpha\|u\|^2 \leq a(u,v) \leq M\|u\|^2, \forall u \in V$, and therefore we find:

$$\|u - u_h\|_V \leq \frac{M}{\alpha}\inf_{w_h \in V_h}\|u - w_h\|_V.$$

Proof. i) As the bilinear form a is coercive of constant $\alpha : a(u - u_h, u - u_h) \geq \alpha\|u - u_h\|^2$.
We have:

$$\begin{gathered}
a(u - u_h, u - v) + a(u - u_h, v - u_h) \geq \alpha\|u - u_h\|_H^2 \\
a(u - u_h, u - v) + a(u, v - u_h) - a(u_h, v - u_h) \geq \alpha\|u - u_h\|_H^2 \\
a(u - u_h, u - v) + f(v - u_h) - f(v - u_h) \geq \alpha\|u - u_h\|_H^2 \\
a(u - u_h, u - v) \geq \alpha\|u - u_h\|_H^2.
\end{gathered}$$

ii) As the continuity of the bilinear form a:

$$\alpha\|u - u_h\|_V^2 \leq M\|u - u_h\|_H\|u - v\|_V.$$

we obtain:

$$\|u - u_h\| \leq \frac{M}{\alpha}\|u - v\|_H\,;\forall v \in V_h.$$

□

1.9.2 Application on parabolic problem[48]

We consider parabolic equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f \; in \; Q_T = \Omega \times (0,T) \\ \\ u(x,0) = u_0(x) \;\; in \; \Omega, \end{cases} \tag{1.28}$$

with $u(x,t) = 0$ and $\dfrac{\partial u(x,t)}{\partial \eta} = 0$ on $\partial\Omega$,
where Ω is a domain of $\mathbb{R}^d, d = 1,2,3. L = L(x)$ is a generic elliptic operator.
The weak formulation of the problem (1.28) is:

$$\int_\Omega \frac{\partial u(t)}{\partial t} v dx + a(u(t), v) = \int_\Omega f(t) v dx; \quad \forall v \in V u(0) = u_0. \tag{1.29}$$

The bilinear form $a(.,.)$ is continuous and weakly coercive, that is

$$\exists \lambda \geq 0, \exists \alpha > 0 : a(v,v) + \lambda \|v\|_{L^2}^2 \geq \alpha \|v\|_V^2 .$$

Moreover, we require $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$. Then, problem (1.29) admits a unique solution $u \in L^2(\mathbb{R}_+, V) \cap C^0(\mathbb{R}_+, L^2(\Omega))$ with $V = H^1_\Gamma(\Omega)$. We now consider the Galerkin approximation of problem (1.28) :

For each $t > 0$ find $u_h(t) \in\in V_h$ such that:

$$\int_\Omega \frac{\partial u_h(t)}{\partial t} v_h dx + a(u_h(t), v_h) = \int_\Omega f(t) v_h dx; \quad \forall v_h \in V_h \subset V u_h(0) = u_{0h}. \tag{1.30}$$

We will have $u_h(x,t) = \sum_{j=1}^{N_h} u_j(t) \varphi_j(x)$, then:

$$\sum_{j=1}^{N_h} u_j'(t) \int_\Omega \varphi_j \varphi_i dx + \sum_{j=1}^{N_h} u_j(t) a(\varphi_j, \varphi_i) = \int_\Omega f(t) \varphi_i dx. \tag{1.31}$$

The system (1.31) can be written in matrix form as $Mu'(t) + Au(t) = f(t)$. For the numerical solution of this ODE system, many finite difference methods are available.

By using the Poincaré©, Young and Cauchy-Schwarz inequality, and with coercivity constant α, we obtain the following a priori estimate

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^T \|f(s)\|_{L^2} ds. \tag{1.32}$$

We therefore conclude with the additional a priori estimate

$$\left(\|u(t)\|_{L^2}^2 + 2\alpha \int_0^T \|\nabla u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \leq \|u_0\|_{L^2} + \int_0^T \|f(s)\|_{L^2} ds. \tag{1.33}$$

Similarly, to what we did for problem (1.29) we can prove the following a priori (stability) estimates for the solution to problem (1.30):

$$\|u_h(t)\|_{L^2}^2 + 2\alpha \int_0^T \|\nabla u_h(s)\|_{L^2}^2 ds \leq \|u_{0h}\|_{L^2}^2 + \frac{C^2}{\alpha} \int_0^T \|f(s)\|_{L^2} ds. \tag{1.34}$$

Let us now suppose thatV_h is the space of finite elements of degree r, the convergence of u_h to u in suitable norms.

$$\left(\|(u - u_h)((t))\|_{L^2}^2 + 2\alpha \int_0^t \|(u - u_h)(s)\|_{H^1}^2 ds \right)^{\frac{1}{2}} \leq C' h^r \left(\sqrt{N(u)} + \int_0^t \left| \frac{\partial u(s)}{\partial t} \right|_{H^1} ds \right), \tag{1.35}$$

where $N(u)$ is a suitable function depending on u and on $\dfrac{\partial u}{\partial t}$, and C' is a suitable positive constant.

1.10 Discrete variational formulation

The attribute discrete means that the solution can be characterized by a finite number of real (or complex) numbers.

1.10.1 The Finite Element Method

The finite element method is a way of choosing the bases of the approximation spaces for the methods from Galerkin.

1.10.1.1 Principle of the method

Let $\Omega \subset \mathbb{R}^d\,(\mathbb{R}^2 or \mathbb{R}^3)$, let V be the functional space which we search for the solution (for example $H_0^1(\Omega)$). We search $V_h \subset V$, and the basic functions $\phi_1, ..., \phi_N$.

We will determine these basic function from a split of Ω in a finite number cells, called "element". The procedure is as follows:

i) We build a "mesh" $\mathcal{T}$ of Ω (in triangles or rectangles) that we call elements $\mathcal{K}$.

ii) In each element, we give the points so called "nodes".

iii) We define V_h by

$$V_h = \{u_h : \Omega \to \mathbb{R}/u_h\,|_K \in P_{\mathcal{K}}, \forall K \in \mathcal{T}\} \cap V,$$

where $P_{\mathcal{K}}$ denotes the set of polynomials of degree lower than or equal to k. The values at the nodes are also the "degree of freedom"(which are the components of the approximate solution u_h in a baseV_h, then we define this approximate solution as the solution to the following problem:

$$\begin{cases} \text{find } u_h \in V_h \text{ such hat :} \\ a(u_h, v_h) = L(v_h), \qquad \forall v_h \in V_h, \end{cases}$$

which thus reduces to the resolution of a linear system whose matrix is called the stiffness matrix.

iv) We are building a base $\{\phi_1, ..., \phi_N\}$ of V_h such that the support of ϕ_i is a small as possible. The functions ϕ_i are called "function of form".

In practice the familiesV_h must represent an approximation of the space$V$, in the sense that the number of degrees of freedom can be as large as possible, so as to approach the exact solution of precisely as possible. In other word :

$$\lim_{h\to 0}\left[\inf_{v_h\in V_h} \|v - v_h\|\right] = 0.$$

Remark 6. *(Non conforming finite elements) Note that we introduced here a method of finite elements compliant, that is, the $V_h \subset V$. In non-conforming method, we will not have $V_h \subset V$. we will also have to build an approaching bilinear form a_T.*

1.10.1.2 Finite element of Lagrange

Definition 18. *We say that the set $\Sigma = \{a_j\}_{j=1}^N$ and P the vectorial space of finite dimension and composed of functions defined on a compact part K, connected with real values involving if and only if there exists a function p of the space P and is a single such as*

$$p\,(a_j) = \alpha_j; 1 \leq j \leq N,$$

where α_j is real scalars given.

Definition 19. *When the set Σ and $P-$ insolvent, the triplet (K, P, Σ) is called finite element of Lagrange.*

1.10.2 Triangulation of the domain

Let us consider a bounded, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$.

We consider a finite decomposition of the domain: $\overline{\Omega} = \underset{K \in \mathcal{T}_h}{\cup} K$ as:

i- each element K of $\mathcal{T}_h$ is a polyhedron of $\mathbb{R}^d$, of non-empty interior;

ii-the interior of two distinct polyhedrons de $\mathcal{T}_h$ are disjoint;

iii-any face of a polyhedral $K_1 \in \mathcal{T}_h$ is of another polyhedral $K_2 \in \mathcal{T}_h$, are said to be adjacent, or part of the boundary Γ in Ω.

Definition 20. *Any decomposition of $\overline{\Omega}$ satisfying the properties $(i), (ii), (iii)$ is called the triangulation of $\overline{\Omega}$.*

$\mathcal{T}_h$ will denote a triangulation of $\overline{\Omega}$ such that $h = \underset{K \in \mathcal{T}_h}{max}\, h_K$,

where h_K is the diameter of the polyhedron K.

We assume that for each polyhedron K of $\mathcal{T}_h$ is associated a finite element of Lagrange (K, P_K, Σ_K)such that: $P_K \subset H^1(K)$

and we define

$$X_h = \left\{ v \in C^0\left(\overline{\Omega}\right) : v_{|K} \in P_K; \forall K \in \mathcal{T}_h \right\}; X_{0h} = \left\{ v \in X_h : v \mid \Gamma = 0 \right\}.$$

The variational approximation theory leads to looking u_h for a solution in V_h of the problem

$$\forall v_h \in V_h, a(u_h, v_h) = L(v_h),$$

with $V_h = X_h$ if $V = H^1(\Omega)$; $V_h = X_{0h}$ if $V = H_0^1(\Omega)$.

Example 1. *The finite element method in the one-dimensional case.*

Let us suppose that Ω is an interval (a, b), we introduce a partition $\mathcal{T}_h$ of (a, b) in $N+1$ subintervals $Kj = (xj_{-1}, x_j)$, also called elements, having width $h_j = x_j - x_{j_{-1}}$ with

$$a = x_0 < x_1 < ... < x_N < x_{N+1} = b,$$

and set $h = max\ h_j$. Since the functions of $H^1(a, b)$ are continuous functions on $[a, b]$, we can construct the following family of space

$$X_h^r = \left\{ v_h \in C^0\left(\overline{\Omega}\right) : v_{h|K_j} \in P_r; \forall K_j \in \mathcal{T}_h \right\}, r = 1.$$

Consequently, having assigned $N+2$ basis functions $\phi_i, i = 0, ..., N+1$, the whole space X_h^1 will be completely defined. The characteristic Lagrangian basis functions are characterized by the following property $\phi_i \in X_h^1$ such that $\varphi_i(x_j) = \delta_{i_j}, i, j = 0, 1, ..., N+1$, The function ϕ_i is therefore piecewise linear and equal to one at xi and zero at the remaining nodes of the partition. Its expression is given by

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} : x_{i-1} \leq x \leq x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} : x_i \leq x \leq x_{i+1} \\ \quad 0 : \ otherwise \end{cases}$$

Let $\Omega =]0,1[\subset \mathbb{R}$ *and let* $V = H_0^1([0,1[), h = \dfrac{1}{N+1}$; *we posed* $x_i = i, i = 0, N+1$ *and* $K_i =]x_i, x_{i+1}[; P_1 = \{ax+b; a,b \in \mathbb{R}\}$,

$V_h = \{u : \Omega \to \mathbb{R}/u|_K \in P_1, u \in C^0([0,1])$ *and* $u(0) = u(1) = 0\}$, $supp(\phi_i) = [x_{i-1}, x_{i+1}]$ *and* $\phi_i(x_i) = 1, \phi_i(x_{i-1}) = \phi_i(x_{i+1}) = 0$.

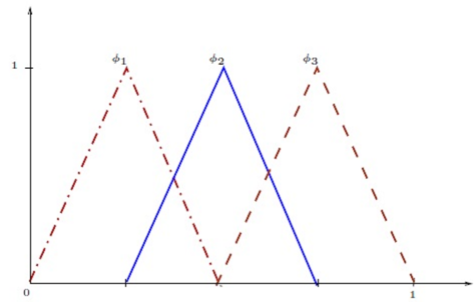


Figure 1.2: The basis function

1.10.2.1 Galerkin discretization

We replace V , e. g, $H_1(\Omega)$ or $H_0^1(\Omega)$, in (LVP) by finite dimensional subspaces equipped with the same norm.

The most general approach relies on two subspaces of V, i. e,

$W_n \subset V$ " trial space", $dimW_n = N \in \mathbb{N}$, and $V_h \subset V$ " test space", $dimV_h = N \in \mathbb{N}$.

If $W_n = V_h$ we speak of a classical Galerkin discretization.

1.10.2.2 Galerkin orthogonality of the discretization error

$e_h := u - u_h$ to the test space V_h, The error is the smallest possible when measured in the a-optimality. The error can be estimated by Cé a's lemma.

1.10.2.3 Boundary value problem

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, ightthe solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

Types of Boundary conditions

1. Dirichlet boundary condition

The Dirichlet (or first-type) boundary condition is a specifies the values that a solution needs to take on along the boundary of the domain. A Dirichlet boundary condition may also be referred to as a fixed boundary condition.

2. Neumann boundary condition

The Neumann (or second-type) boundary condition is a specifies the values that the derivative of a solution is to take on the boundary of the domain.

3. Cauchy boundary condition

Cauchy boundary condition augments an ordinary differential equation or a partial differential equation with conditions that the solution must satisfy on the boundary; ideally so to ensure that a unique solution exists. A Cauchy boundary condition specifies both the function value and normal derivative on the boundary of the domain. This corresponds to imposing both a Dirichlet and a Neumann boundary condition.

4. Robin boundary condition

Robin boundary conditions are a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions. This contrasts to mixed boundary conditions, which are boundary conditions of different types specified on different subsets of the boundary. Robin boundary conditions are also called impedance boundary conditions.

1.10.2.4 The finite element method in the multi-dimensional case

We will consider domains $\Omega \subset \mathbb{R}^2$ with polygonal shape and meshes (or grids). Th which represent their cover with non-overlapping triangles. The discretized domain $\Omega_h = int\left(\underset{K\in\mathcal{T}_h}{\cup} K\right)$

represented by the internal part of the union of the triangles of $\mathcal{T}_h$ perfectly coincides with Ω. Also in the multidimensional case, the parameter h is related to the spacing of the grid. Having set $h_K = diam(K)$, for each $K \in \mathcal{T}_h$, where $diam(K) = \underset{x,y\in K}{max}|x-y|$ is the diameter of element K, we define $h = \underset{K\in\mathcal{T}_h}{max} h_K$

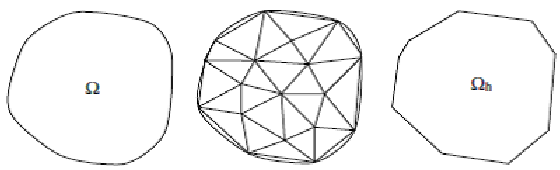


Figure 1.3: polygonal shape

Let ρ_K be the diameter of the circle inscribed in the triangle K (also called sphericity of K); a family of girds $\{\mathcal{T}_h, h>0\}$ is squid to be regular if for suitable $\delta > 0$, the condition

$$\frac{h_K}{\rho_K} \leq \delta; \forall K \in \mathcal{T}_h,$$

is verified. We find

$$P_r = \left\{ p(x_1,x_2) = \sum_{i,j\geq 0; i+j\leq r} a_{ij} x_1^j x_2^j; a_{ij} \in \mathbb{R} \right\}.$$

Corollary 4. *Let $(\mathcal{T}_h)$ be a regular family of triangulations of $\overline{\Omega}$ associated with a n- (simplex, parallel, parallelogram) with k type.*

Then the finite element method is convergent. Moreover, there exists a constant C independent of h such that if the solution $u \in H^{l+1}(\Omega)$, $1 \leq l \leq k; n \leq 3$, we have

$$\|u-u_h\|_{1,\Omega} \leq C h^l |u|_{l+1,\Omega}.$$

Proof. We posed $u \in H^{l+1}(\Omega)$, it's continuous on $\overline{\Omega}$ and we can build the function $\Pi_h u$ belongs to the subspace V_h of V, then

$$\|u-u_h\|_{1,\Omega} \leq C_1 \|u-\Pi_h u\|_{1,\Omega},$$

with $C_1 = \dfrac{M}{\alpha}$, since the restriction of$\Pi_h u$ to any $K \in \mathcal{T}_h$.

$$\|u - \Pi_h u\|_{1,\Omega} = \left(\sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{1,K}^2 \right)^{\frac{1}{2}},$$

but there exist C_2 and C_3 such that

$$|u - \Pi_K u|_{1,K} \leq C_2 \frac{h_K^{l+1}}{\rho_K} |u|_{k+1,K} .$$

and

$$\|u - \Pi_K u\|_{0,K} \leq C_3 h_K^{l+1} |u|_{k+1,K} ,$$

it result $\|u - \Pi_K u\|_{1,K} \leq C_4 h_K^{l+1} |u|_{k+1,K}$; with $C_4 = \sigma \left(C_2^2 + \left(C_3 \; diam \left(\overline{\Omega} \right) \right)^2 \right)^{\frac{1}{2}}$.

So an increase of the interpolation error in $\overline{\Omega}$ is

$$\|u - \Pi_h u\|_{1,\Omega} \leq C_4 h^l \left(\sum_{K \in \mathcal{T}_h} |u|_{k+1,K}^2 \right)^{\frac{1}{2}} = C_4 h^l |u|_{k+1,\Omega} .$$

The increase of the corollary obtained with $C = C_1 C_4$. □

For parabolic problems several approaches are possible. A third approach, called space-time domain decomposition, consists of to solve independently the sub-problems in space and time in the subdomains then iterated on the values defined on the interfaces space temps between the subdomains to connect the solution between the subdomains adjacent. The strength of this global approach in time is that different discretizations in space and time can be chosen in the different subdomains.

To ensure the continuity of the discrete spaces, defined with the help of the partitions, we need the following additional condition:

Definition 21. *A partition of* Ω *is called* admissible, *if two elements* T_i, T_j *are either disjoint or share a complete k-face,* $0 \leqslant k \leqslant d-1$.

Remark 7. *The condition of admissibility means, that there are no hanging nodes in* Ω.

Denoting h_T as the diameter of a simplex $T \in \tau_h$ and ρ_T as the diameter of the largest ball inscribed into T, and

$$h = \max_{T \in \tau_h} h_T.$$

We can formulate another important property of the partition τ_h.

Definition 22. *A partition* τ_h *is called shape regular if there exists a positive constant* C *independent of* h, *such that*

$$\sigma_T = \frac{h_T}{\rho_T} \leqslant C, \quad \forall T \in \tau_h .$$

Definition 23. *A partition τ_h is called is called quasi-uniform, if there is a constant $\tau > 0$, such that*

$$\max_{T\in\tau_h} h_T \geqslant \tau h.$$

Remark 8. *The first condition ensures that asymptotically the simplifies do not degenerate. The meaning of quasi-uniformity is, that the size of all simplifies of one partition is asymptotically equal up to a constant not depending on the parameter h,*

(for more details see [6], [48] and [50]).

Chapter 2

Some Numerical Analysis of Evolution Problems

In this chapter we shall briefly discuss the generalization of our previous error analysis to initial-boundary value problems for more general evolution problems.

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions.

The problem under examination is:

$$\begin{cases} u_t + Lu = f \quad \text{in } Q_T = \Omega \times [0, T], \\ u = 0 \quad \text{on } \Gamma \times [0, T], \\ u = g \quad \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{2.1}$$

where Ω is an open domain in $\mathbb{R}^n$, and $\Gamma = \partial\Omega$ is the boundary of Ω, and L linear elliptic differential operators

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t) u$$

for given coefficients $a_{i,j}$, b_i,$c \in L^\infty(Q_T)\,(i,j = 1, 2, .., n)$.

It is assumed that we have:

Definition 24. *We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is uniformly parabolic if there exists a constant $\lambda > 0$ such that:*

$$\sum_{i,j=1}^{n} a_{ij}(x,t) \zeta_i \zeta_j \geq \lambda |\zeta|^2 ; \ \ \forall (x,t) \in Q_T, \zeta \in \mathbb{R}^n.$$

Remark 9. *For each fixed time $0 \leq t \leq T$, The operator L is a uniformly elliptic operator in the spatial variable x.*

Theorem 21. ***(boundary maximum principle)*** *Let $u \in W^{1,2} \cap L^\infty\left(0, T; L^2\left(\overline{\Omega}\right)\right)$ satisfied:*

$$u_t - Lu \leq 0 \ \textit{in } Q_T,$$

then

$$\max_{(x,t)\in\overline{Q}_T} u(x,t) = \max_{(x,t)\in\partial Q_P} u(x,t),$$

where

$$\partial Q_P = \left\{(x,t) \in \overline{Q}_T : x \in \partial\Omega \ \text{or} \ t = 0\right\}.$$

Lemma 8. ***(comparison principle)*** *Let $u, v \in W^{1,2} \cap L^{\infty}\left(0,T; L^2\left(\overline{\Omega}\right)\right)$ then*

$$\begin{cases} u_t - Lu \leq v_t - Lv \quad in \ \ Q_T \\ \\ u \leq v \quad on \ \partial\Omega \times (0,T) \\ \\ u \leq v \quad for \ \ t = 0 \end{cases} \Longrightarrow u \leq v \quad in \ \ \overline{Q}_T$$

This lemma immediately implies the uniqueness of classical solutions of (2.1).

A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. It's called the strong solution of the problem, and (2.1) is called the strong formulation of the problem.

Aside from the boundary condition, boundary value problems are also classified according to the type of differential operator involved. For an *elliptic operator*, one discusses *elliptic boundary value problems* and for an parabolic operator, one discusses parabolic boundary value problems.

In most cases it is not possible to find analytical solutions of these problems i.e. that the explicit computation of the exact solution of such equations is often out to be achieved. Therefore, in general, the exact problem is approached by a discrete problem that can be solved by numerical methods.

We will explain in this chapter the main numerical methods that will be used later.

2.1 The variational formulation of some boundary value problems

We may write the parabolic problem in variational form as

$$\begin{cases} (u_t, v) + a(u,v;t) = (f,v), \quad \forall v \in H_0^1(\Omega), t \in [0,T] \\ \\ u(0) = g \end{cases} \tag{2.2}$$

where $a(u,v;t)$ the time dependent bilinear form

$$a(u,v;t) = \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} v + c(x,t)\, uv \right) dx.$$

Theorem 22. ***(Garding's Inequality)*** *([47]) Let Ω be a regular open set of class C^1. Let v be a function of $H_0^1(\Omega)$, both with bounded support in the closed set Ω. Then*

$$a(v,v;t) \geq M\|v\|_1^2 - \lambda\|v\|^2 \quad \forall v \in H_0^1, \ \text{with} \ M > 0, \lambda \in \mathbb{R}.$$

Proof.

$$a(v,v;t)+\lambda\|v\|^2=\int_\Omega\left(\sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^n b_i\frac{\partial v}{\partial x_i}v+(c+\lambda)v^2\right)dx$$

$$=\int_\Omega\left(\sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}+\left(c+\lambda-\frac{1}{2}\sum_{i=1}^n b_i\frac{\partial v}{\partial x_i}\right)v^2\right)dx\geq M\|v\|_1^2.$$

with

$$M>0, if \lambda>\sup_{\Omega\times[0,T]}\left(\frac{1}{2}\sum_{j=1}^n b_j\frac{\partial v}{\partial xj}-c\right).$$

We shall consider λ to be fixed in this manner in the sequel (weakly coercive). ☐

We assume that

$$a_{i,j},b_i,c\in L^\infty(Q_T)\,(i,j=1,2,..,n)\,,f\in L^2(Q_T)\,,g\in L^2(\Omega)\,.$$

Remark 10. *For $\lambda=0$ the standard definition of coercivity.*

Remark 11. *Observe that: $u_t=g_0+\sum_{j=1}^n\frac{\partial g_j}{\partial xj}$ in Q_T. for*

$$g_0:=f-\sum_{j=1}^n b_j\frac{\partial u}{\partial xj}-cu,$$

and

$$g_j:=\sum_{j=1}^n a_{ij}\frac{\partial v}{\partial xj}\,(j=1,...,n)\,.$$

We imply

$$\|u_t\|_{H^{-1}}\leq\left(\sum_{j=0}^n\|g_j\|_{L^2}^2\right)^{\frac{1}{2}}\leq C\left(\|u\|_{H_0^1}+\|f\|_{L^2}\right).$$

Definition 25. *[26] We say a function $u\in L^2(0,T,H_0^1(\Omega))$ with $u_t\in L^2(0,T,H^{-1}(\Omega))$, is a weak solution of the parabolic boundary value problem (2.1) provided.*
i) $(u_t,v)+a(u,v;t)=(f,v),\quad\forall v\in H_0^1(\Omega)\,,t\in[0,T]$.
ii) $u(0)=g$.

2.1.1 Galarkin approximation

We assume the functions $\varphi_k=\varphi_k(x)\,,(k=1,...)$ are smooth: $\{\varphi_k\}_{k\in\mathbb{N}}^*$ is an orthogonal basis of $H_0^1(\Omega)$ and $\{\varphi_k\}_{k\in\mathbb{N}}^*$ is an orthonormal basis of $L^2(\Omega)$, we take $u_m:[0,T]\longrightarrow H_0^1(\Omega)$, of the form

$$u_m(t)=\sum_{k=1}^m d_m^k(t)\,\varphi_k, \tag{2.3}$$

where

$$d_m^k(0)=(g,\varphi_k)\,;(k=1,...,m) \tag{2.4}$$

and

$$\left(u_m', \varphi_k\right) + a\left(u_m, \varphi_k; t\right) = \left(f, \varphi_k\right), \quad \forall k = 1, ..., m, t \in [0, T]. \tag{2.5}$$

$(.,.)$ denotes the inner product in $L^2(\Omega)$.

Theorem 23. *[47] For each integer $m = 1, 2, ...$ there exists a unique function u_m of the form (2.3) satisfying (2.4, 2.5).*

Theorem 24. *[47] (Energy estimates) There exists a constant C, depending only on Ω, T and the coefficients of L, such that*

$$\max \|u_m(t)\|_{L^2(\Omega)} + \|u_m\|_{L^2\left(0,T,H_0^1(\Omega)\right)} + \|u\prime_m\|_{L^2(0,T,H^1(\Omega))} \leq C\left(\|f\|_{L^2(0,T,L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right); m = 1, 2, ...$$

Theorem 25. *(Existence and uniqueness) There exists a unique weak solution of (2.1).*

Proof. 1- According to the energy estimates, the sequence $\{u_m\}_{m\in\mathbb{N}^*}$ is bounded in $L^2\left(0, T, H_0^1(\Omega)\right)$ and $\{u_m'\}_{m\in\mathbb{N}^*}$ in $L^2\left(0, T, H^{-1}(\Omega)\right)$, then there exists a subsequence $\{u_{m_l}\}_{l\in\mathbb{N}^*} \subset \{u_m\}_{m\in\mathbb{N}^*}$.
and $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ with $u_t \in L^2\left(0, T, H^{-1}(\Omega)\right)$ such that:

$$\begin{cases} u_{m_l} \longrightarrow u \text{ weakly in } L^2\left(0, T, H_0^1(\Omega)\right), \\ \\ u_{m_l}' \longrightarrow u' \text{ weakly in } L^2\left(0, T, H^{-1}(\Omega)\right). \end{cases} \tag{2.6}$$

We choose a function $\upsilon \in C^1\left([0, T], H_0^1(\Omega)\right)$ having the form

$$\upsilon = \sum_{k=1}^{N} d^k(t)\varphi_k,$$

then

$$\int_0^T \left(\left(u\prime_m, \upsilon\right) + a\left(u_m, \upsilon; t\right)\right) dt = \int_0^T (f, \upsilon)\, dt.$$

we set $m = m_l$ and recall (2.6), to find upon passing to weak limits that:

$$\int_0^T \left(\left(u', \upsilon\right) + a\left(u, \upsilon; t\right)\right) dt = \int_0^T (f, \upsilon)\, dt.$$

because $\upsilon \in L^2\left(0, T, H_0^1(\Omega)\right)$ are dense in this space, then:

$$\left(u\prime, v\right) + a\left(u, v; t\right) = (f, v), \forall v \in H_0^1(\Omega), 0 \leq t \leq T.$$

since $u_{m_l} \longrightarrow g$ in $L^2((\Omega))$, we conclude $u(0) = g$.
2- If $f \equiv g \equiv 0$, the only weak solution of 2.1 is $u \equiv 0$ (by setting $u = v$ and Gronwell's inequality and

$$\left(u', u\right) + a\left(u, u; t\right) = 0).$$

□

Theorem 26. *[47] (regularity) Assume that $g \in H_0^1(\Omega), f \in L^2\left(0, T, L^2(\Omega)\right)$.*
Suppose also $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ with $u_t \in L^2\left(0, T, H^{-1}(\Omega)\right)$, is the weak solution of (2.1).
Then in fact $u \in L^\infty\left(0, T, H_0^1(\Omega)\right) \cap L^2\left(0, T, H^2(\Omega)\right)$with $u_t \in L^2\left(0, T, L^2(\Omega)\right)$, and we have the estimate

$$ess \sup_{0\leq t\leq T} \|u(t)\|_{H_0^1} + \|u\|_{L^2(0,T,H^2)} + \left\|u'\right\|_{L^2(0,T,L^2)} \leq C\left(\|f\|_{L^2(0,T,L^2)} + \|g\|_{H_0^1}\right).$$

Theorem 27. *[26] (Infinite differentiability) Assume* $g \in C^{\infty}\left(\overline{\Omega}\right), f \in C^{\infty}\left(\overline{Q}_T\right)$, *and the* m^{th}*-order compatibility conditions hold for m=1,2,...,then the problem (2.1) has a unique solution*

$$u \in C^{\infty}\left(\overline{Q}_T\right).$$

2.1.1.1 The approximation

For each $t > 0$ the solution to the Galerkin problem belongs to the subspace as well, we will have

$$u_h\left(x,t\right) = \sum_{k=1}^{N_h} u_j\left(t\right)\varphi_j\left(x\right).$$

than

$$\sum_{j=1}^{N_h} u_j'\left(t\right)\int_{\Omega}\varphi_j\varphi_i d\Omega + \sum_{j=1}^{N_h} u_j\left(t\right)a\left(\varphi_j,\varphi_i\right) = \int_{\Omega} f\left(t\right)\phi_i d\Omega, i = 1,2,...,N_h. \tag{2.7}$$

The system (2.7) can be rewritten in matrix forms as

$$Mu'\left(t\right) + Au\left(t\right) = f\left(t\right).$$

For numerical solution of this ODE system, the θ-method is:

$$M\frac{u^{k+1}-u^k}{\Delta t} + A\left[\theta u^{k+1} + \left(1-\theta\right)u^k\right] = \theta f^{k+1} + \left(1-\theta\right)f^k,$$

the matrix M is invertible, being positive definite.

Remark 12. - *For* $\theta = 0$, *we obtain the explicit Euler method.*
- *For* $\theta = 1$, *we obtain the implicit Euler method.*

Let V_h is a sub subspace of finite elements. We have the following stability condition:

$$\exists C > 0 : \triangle t \leq Ch^2; \forall h > 0.$$

2.1.2 Stability analysis of the $\theta-$method

Applying the θ-method to Galerkin problem, we obtain

$$\left(\frac{u_h^{k+1}-u_h^k}{\Delta t}, v_h\right) + a\left(\theta u_h^{k+1} + \left(1-\theta\right)u^k, v_h\right) = \theta f^{k+1}\left(v_h\right) + \left(1-\theta\right)f^k\left(v_h\right),$$

for each $k \geq 0$ with $u_h^0 = u_{0h}$, f^k indicates the functional is evaluated at time t^k.
Using the discrete Gronwall lemma, it can be proven that:

$$\left\|u_h^n\right\|_{L^2(\Omega)}^2 + 2\alpha\triangle t\sum_{k=1}^{n}\left\|u_h^k\right\|_V^2 \leq C\left(t\right)\left(\left\|u_{0h}\right\|_{L^2(\Omega)}^2 + \sum_{k=1}^{n}\triangle t\left\|f^k\right\|_{L^2(\Omega)}^2\right).$$

For each given $\triangle t > 0$,

$$\lim_{k\longrightarrow\infty}\left\|u_h^k\right\|_{L^2((\Omega))} = 0.$$

2.1.3 Convergence analysis of the $\theta-$method

Theorem 28. *[26] Under the hypothesis that u_0, f and the exact solution are sufficiently regular, the following a priori error estimate holds: $\forall n \geq 1$,*

$$\|u(t^n) - u_h^n\|_{L^2(\Omega)}^2 + 2\alpha\triangle t\sum_{k=1}^{n} \left\|u\left(t^k\right) - u_h^k\right\|_V^2 \leq C\left(u_o, f, u\right)\left(\triangle t^{P(\theta)} + h^{2r}\right),$$

where $p(\theta) = 2$ if $\theta \neq 1/2, p(1/2) = 4$ and C depends on its arguments but not on h and Δt.

2.1.3.1 A priori estimate

By integrating in time, we obtain the following energy a priori estimate, for all $t > 0$

$$\|u(t)\|_{L^2(\Omega)}^2 + \alpha\int_0^t \|\nabla u(s)\|_{L^2(\Omega)}^2 ds \leq \|u_0\|_{L^2(\Omega)}^2 + \frac{C'^2}{\alpha}\int_0^t \|f(s)\|_{L^2(\Omega)}^2 ds,$$

and the further a priori estimate

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} ds.$$

Finally, we obtain the a priori estimate for all $t > 0$

$$\|u(t) - u_h(t)\|_{L^2(\Omega)}^2 + 2\alpha\int_0^t \|u(s) - u_h(s)\|_{H^1(\Omega)}^2 ds \leq Ch^{2r}N(u)e^t.$$

2.1.4 Nonvariational approach

In this subsection, we gather various techniques and properties of solutions for nonlinear PDE that are not of variational form.

2.1.4.1 Fixed point methods

These are:
i) Fixed point theorems for strict contraction.
ii) Fixed point theorems for compact mapping.
iii) Fixed point theorems for order-prescring operator.

Definition 26. *Assume $A : X \longrightarrow X$ the nonlinear mapping A is a strict contraction if:*

$$\|A(u) - A(\widetilde{u})\| \leq C\|u - \widetilde{u}\|; \forall u, \widetilde{u} \in X, C < 1.$$

Theorem 29. *[48] Assume $A : X \longrightarrow X$ is a nonlinear mapping and a strict contraction, then A has a unique fixed points.*

Consider the problem $Au = b$ where the $N \times N$ matrix A is invertible and $b \in \mathbb{R}^n$ is given. The problem is equivalent to $Bu = (B - A)\, u + b$, where B the matrix $n \times n$ invertible.

The fixed-point iteration

$$Bu^{k+1} = (B - A)\, u^k + b; k = 0, 1, ... \tag{2.8}$$

The convergence behavior of (2.8) follows form Banach's fixed point theorem for the simpler iteration

$$u^{k+1} = Tu^k + b; k = 0, 1, ...,$$

with

$$T := B^{-1}\,(B - A) \;\; and \; t := B^{-1} b.$$

Lemma 9. *Assume that $T \in L(\mathbb{R}^n)$ with $\|T\| < 1$, then the fixed point problem: $u = Tu + b$ has a unique solution $u \in \mathbb{R}^n$ for any $u^0 \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$ and*

$$\left\|u^{k+1} - u\right\| \leq \|T\| \left\|u^k - u\right\|; k = 0, 1, ...,$$

and

$$\left\|u^k - u\right\| \leq \frac{\|T\|^k}{1 - \|T\|} \left\|u^1 - u^0\right\|; k = 1, 2, ...$$

2.1.4.2 Method of Subsolutions and Supersolutions

The idea is to exploit ordering properties for solutions. If we can find a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of a boundary value problems and if furthermore $\underline{u} \leq \overline{u}$, then there exists a solution u satisfying

$$\underline{u} \leq u \leq \overline{u}.$$

2.2 Advective Diffusion Problem

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection. Depending on context, the same equation can be called the advection-diffusion equation.

We describe this convection-diffusion equation in porous media

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \quad \text{in } \Omega \times (0, T)\,, \tag{2.9}$$

where x is the distance measured down the column, t the time, u is the flow velocity down the column and D> 0 (constant) the diffusion coefficient.

We presume that the layer of diffusion of length l>0, then $\Omega := (0, l)$ .We assume the conditions

$$\begin{cases} C\,(x, 0) = h\,(x)\,; 0 \leq x \leq l, \\ \\ C\,(l, t) = f_l\,(t)\,; t > 0, \\ \\ D\frac{\partial C}{\partial x}\,(l, t) = g_l\,(t)\,; t > 0. \end{cases}$$

The equation (2.9) can be rewritten as

$$L_t(C) + uL_x(C) = DL_{xx}(C). \tag{2.10}$$

The solution of equation (2.10) as given by the decomposition series, $C = \sum_{n\geq 0} c_n$, where the sequence $\{c_n\}_{n\geq 0}$ satisfies

$$\begin{cases} C_0(x,t) = f_l(t) + \frac{x-l}{D} g_l(t), \\ C_{n+1}(x,t) = \frac{1}{D} L_{xx}^{-1} \left[L_t(C_n(x,t)) + u(x,t) L_x(C_n(x,t)) \right]; n \geq 0. \end{cases}$$

for the noncharacteristic cauchy-problem, and

$$\begin{cases} C_0(x,t) = h(x), \\ C_{n+1}(x,t) = L_t^{-1} \left[DL_{xx}(C_n(x,t)) - u(x,t) L_x(C_n(x,t)) \right]; n \geq 0, \end{cases}$$

for characteristic cauchy-problem.

2.2.1 Initial spatial concentration distribution

We will consider the equation

$$\begin{cases} \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}, \\ C(x, t_0) = C_0, x \leq 0. \end{cases}$$

The solution can be written as

$$C(x,t) = \frac{C_0}{2} \left(1 - erf\left(\frac{x}{\sqrt{4Dt}} \right) \right).$$

Example 2. *Dissolving sugar in coffee*

Add 2 g of sugar in cup of coffee. The diameter of the cup is 5 cm, its height is 7 cm. The concentration of sugar is fixed at the saturation concentration at the bottom of the cup and is initially zero everywhere else. These are the same conditions as for the fixed concentration solution, thus; the sugar distribution at height z above the bottom of the cup is

$$C(z,t) = \frac{C_0}{2} \left(1 - erf\left(\frac{z}{\sqrt{4Dt}} \right) \right).$$

The characteristic height of the concentration boundary layer is proportional to $\sigma = \sqrt{2Dt}$. *Assume the concentration boundary layer first reaches the top of the cup when* $2\sigma = h = 7cm$. *Solving for time gives* $t_{mixt,bl} = \frac{h^2}{8D} \approx 6.10^5 s$ *with* $D \sim 10^{-9} m^2.s^{-1}$. *Assuming* $C_{sat} = 0.58 g/cm^3$, *the time needed to dissolve all the sugar is* $t_d = 5.10^4 s$.

Chapter 3

Domain Decomposition Methods

3.1 Introduction

The domain decomposition (DD) method has been considered as a parallel algorithm for solving elliptic and parabolic partial differential equations. The DD method involves overlapping and non-overlapping decompositions. Domain decomposition methods are a family of methods to solve problems of linear algebra on parallel machines in the context of simulation.

3.2 The Dirichlet principle

The Dirichlet principle states that an harmonic function, which is a function satisfying:

$$\mathcal{L}u = f \text{ in } \Omega, u = g \text{ on } \partial\Omega.$$

The Dirichlet principle could be rigorously proved for simple domains, where Fourier analysis was applicable.

3.3 Classical iterative DDM

We consider the following problem

$$\mathcal{L}u = f \text{ in } \Omega, \tag{3.1}$$

where $\mathcal{L}$ is a partial differential operator, f is a given function, and u is the unknown solution. Should Ω be partitioned into two disjoint subdomains Ω_1 and Ω_2. Denote for $i = 1, 2$ by u_i the restriction of u to Ω_i, it follows form (3.1) that

$$\mathcal{L}u_1 = f \text{ in } \Omega_1; \ \mathcal{L}u_2 = f \text{ in } \Omega_2.$$

We need the transmission conditions between u_1 and u_2 across Γ, such conditions expressed by

$$\boldsymbol{\Phi}(u_1) = \Phi(u_2) \text{ on } \Gamma; \ \boldsymbol{\Psi}(u_1) = \Psi(u_2) \text{ on } \Gamma,$$

where the functions Φ and Ψ will depend upon the nature of the problem. These interface conditions are most often determined noting that:

i) The solution u belongs to a space of functions defined over the whole Ω. This requires that $u_{\backslash\Omega_1}$ in Ω_1 and $u_{\backslash\Omega_2}$ in Ω_2 enjoy a certain regularity therein, and in addition that they satisfy a suitable matching on Γ.

ii) The restrictions $u_{\backslash\Omega_1}$ and $u_{\backslash\Omega_2}$ are distributional solutions to the given equation in Ω_1 and Ω_2 respectively. Another interface condition between them comes from the fact that u in fact satisfies the equation in the sense of distributions in the whole Ω; namely, through the interface Γ and not only separately in Ω_1 and Ω_2.

Example 3. *Consider the Poisson problem which consists in finding $u : \Omega \to \mathbb{R}$ such that:*

$$-\Delta u = f \ \textit{in}\ \Omega,\ \ u = 0\ \textit{on}\ \partial\Omega, \tag{3.2}$$

then the problem (3.2) can be reformulated in equivalent multi-domain form:

$$\begin{cases} -\Delta u_1 = f \quad \textit{in}\ \Omega_1, \\ u_1 = 0\ \textit{on}\ \partial\Omega_1 \cap \partial\Omega, \\ u_1 = u_2 \quad \textit{on}\ \Gamma. \end{cases}$$

$$\begin{cases} \dfrac{\partial u_1}{\partial \eta} = \dfrac{\partial u_2}{\partial \eta}\ \textit{on}\ \Gamma, \\ u_2 = 0\ \textit{on}\ \partial\Omega_2 \cap \partial\Omega,\ -\Delta u_2 = f\ \textit{in}\ \Omega_2, \end{cases}$$

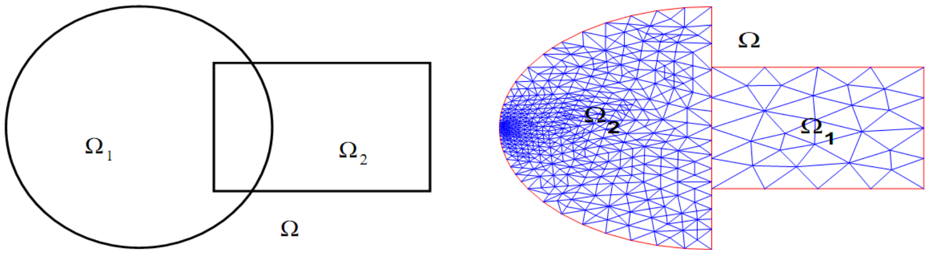
then $\Phi(v) = v$ and $\Psi(v) = \dfrac{\partial v}{\partial \eta}$.

There are two large families of methods for this subdivision by subdomains:
1. Methods with overlapping.
2. The methods without overlap- nonoverlapping-.
We will illustrate these two types of methods applied to problems in space for linear operators.

3.4 The methods with overlapping

Methods with overlap or methods of Schwarz. For the case of a decomposition into two subdomains. The global domain is divided into regions overlapping and Dirichlet local problems are solved on each subdomain. The link between the solutions of the different subdomains is ensured by the common region called overlap. These methods were originally proposed by Schwarz in 1870 to demonstrate the existence and uniqueness of solutions to elliptic problems on complex domains. At his time, there were no Sobolev spaces no Lax-Milgram theorem. The only available tool was the Fourier transform, limited by its very nature to simple geometries.

H.A. SCHWARZ in 1870, in order to consider more general situations, devised an iterative algorithm for solving Poisson problem set on a union of simple geometries: this is the alternating Schwarz method. (See figure 3.1)



(a) The original example of H.A. SCHWARZ (b) Decomposition into simple domains

Figure 3.1: The figure shows two simple decompositions

3.4.1 Schwarz methods at the continuous level

3.4.1.1 Original Schwarz algorithm

Let the domain Ω be the union of a disk and a rectangle (see the figure).

Consider the Poisson problem which consists in finding $u : \Omega \longrightarrow \mathbb{R}$ such that:

$$\begin{cases} -\Delta u = f, \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.3}$$

Definition 27. *[18] The Schwarz algorithm is an iterative method based on solving alternatively sub-problems in domains Ω_1 and Ω_2.*

It updates $(u_1^n, u_2^n) \longrightarrow (u_1^{n+1}, u_2^{n+1})$ by

$$\begin{cases} -\Delta u_1^{n+1} = f \quad in\ \Omega_1, \\ u_1^{n+1} = u_2^n \quad on\ \partial\Omega_1 \cap \overline{\Omega_2}, \\ u_1 = 0 \quad on\ \partial\Omega_1 \cap \partial\Omega. \end{cases}$$

Then,

$$\begin{cases} -\Delta u_2^{n+1} = f \ in\ \Omega_2, \\ u_2^{n+1} = u_1^{n+1} \ on\ \partial\Omega_2 \cap \overline{\Omega_1}, \\ u_2 = 0 \ on\ \partial\Omega_2 \cap \partial\Omega. \end{cases}$$

H. Schwarz proved the convergence of the algorithm and thus the well posedness of the Poisson problem in complex geometries.

With the advent of digital computers, this method also acquired a practical interest as an iterative linear solver.

Subsequently, parallel computers became available and a small modification of the algorithm (cf. [9]) makes it suited to these architectures. Its convergence can be proved using the maximum principle.

We present this method in a general case:

Let given a model problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Lu = f \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.4}$$

where L being a generic second order elliptic operator. By integrating by parts in Ω, it is easily seen that the weak formulation of reads

$$\text{find } u \in V = H_0^1(\Omega) \text{ such that } a(u,v) = (f,v), \quad \forall v \in V,$$

being $a(\cdot)$ the bilinear form associated with L.
Consider a decomposition of the domain Ω in two subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset, \quad \partial\Omega_i \cap \Omega_j = \Gamma_i, \; i \neq j \;\text{ and } i,j = 1,2.$$

Consider the following **iterative method**. Given u_2^0 on Γ_1, solve the following problems for $n \in \mathbb{N}^*$

$$\begin{cases} Lu_1^n = f \text{ in } \Omega_1, \\ \\ u_1^n = u_2^{n-1} \text{ on } \Gamma_1, \\ \\ u_1^n = 0 \text{ on } \partial\Omega_1 - \Gamma_1 \end{cases}$$

and

$$\begin{cases} Lu_2^n = f \text{ in } \Omega_2, \\ \\ u_2^n = \begin{cases} u_1^{n-1} \\ \\ u_1^n \end{cases} \text{ on } \Gamma_2, \\ \\ u_2^n = 0 \quad \text{on } \partial\Omega_2 - \Gamma_2. \end{cases} \tag{3.5}$$

In the case in which one chooses u_1^n on Γ_2 in (3.5) the method is named **multiplicative Schwarz** (MSM), it's algorithm is sequential. Whereas that in which we choose u_1^{n-1}, is named **additive Schwarz** (ASM), problems in domains Ω_1 and Ω_2 may be solved concurrently. The reason of this appointment is clarified in ([34]).

Denoting the solution of iteration step i in subdomain Ω_j by u_j^i for the two-domain case the multiplicative variant can be described as follows: Starting with an initial guess, first a new solution in Ω_1 is computed. Then, already using this solution, the solution in Ω_2 is solved, and so on.

In contrast the additive algorithm uses the solution of the previous step instead of the current solution. The second method has got the advantage that the solution of all subdomain problems can be completely done in parallel.

In the multi-domain case the multiplicative variant requires a coloring of the subdomains.

We have thus two elliptic boundary-value problems with Dirichlet conditions for the two subdomains Ω_1 and Ω_2, and we would like the two sequences $(u_1^n)_{n\in\mathbb{N}^*}$ and $(u_2^n)_{n\in\mathbb{N}^*}$ to converge to the restrictions of the solution u of

problem (3.4), that is

$$\lim_{m\to+\infty} u_1^n = u^n\,|_{\Omega_1} \quad \text{and} \lim_{m\to+\infty} u_2^n = u^n\,|_{\Omega_2} \quad .$$

It can be proven that the Schwarz method applied to problem (3.4) always converges, with a rate that increases as the measure Ω_{12} of the overlapping region Ω_{12} increases.

It is easy to see that if the algorithm converges, the solutions $u_i^{\infty}, i = 1, 2$, in the intersection of the subdomains take the same values.

The original algorithms ASM and MSM are very slow. Another weakness is the need of overlapping subdomains. Indeed, only the continuity of the solution is imposed and nothing is imposed on the matching of the fluxes. When there is no overlap convergence is thus impossible.

The alternating Schwarz method converges to the solution u of (3.1), provided some mild assumptions on the subdomains Ω_1 and Ω_2 are satisfied. Precisely, there exist C_1, $C_2 \in (0, 1)$ such that:

for all $k > 0$

$$\left\| u_{|\Omega_1} - \widetilde{u}_1^{n+1} \right\|_{L^\infty(\Omega_1)} \leq C_1^k C_2^k \left\| u - \widetilde{u}^0 \right\|_{L^\infty(\Gamma_1)},$$

$$\left\| u_{|\Omega_2} - \widetilde{u}_2^{n+1} \right\|_{L^\infty(\Omega_2)} \leq C_1^{k+1} C_2^k \left\| u - \widetilde{u}^0 \right\|_{L^\infty(\Gamma_2)}.$$

3.4.2 Additive and multiplicative Schwarz algorithm for two subdomains

```
Additive Schwarz algorithm
1- Initial guess u_{1}^{0},u_{2}^{0}
2- i=0
3- Until convergence
4- i=i+1
5- Compute u_{j}^{i} using u_{j}^{i-1}; j=1,2
6- end
Multiplicative Schwarz algorithm
1- Initial guess u_{2}^{0}
2- i=0
3- Until convergence
4- i=i+1
5- Compute u_{1}^{i} using u_{2}^{i-1}
6- Compute u_{2}^{i} using u_{1}^{i}
7- end.
```

In order to remedy the drawbacks of the original Schwarz method, Modify the original Schwarz method by replacing the Dirichlet interface conditions on $\partial\Omega_i \cap \partial\Omega$, $i = 1, 2$, by Robin interface conditions ($\partial\eta_i + \alpha$, where η_i is the outward normal to subdomain Ω_i, see [47]).

3.4.2.1 Parallel Schwarz algorithm

Definition 28. *Iterative method which solves concurrently in all subdomains, $i = 1, 2$*

$$\begin{cases} Lu_i^{n+1} = f \quad in\ \Omega_i, \\ u_i^{n+1} = u_{3-i}^n \ \ on\ \partial\Omega_i \cap \overline{\Omega}_{3-i}, \\ u_i^{n+1} = 0 \quad on\ \partial\Omega_i \cap \partial\Omega. \end{cases} \tag{3.6}$$

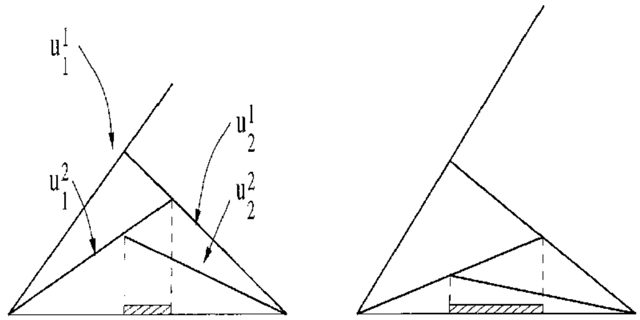


Figure 3.2: Error behavior for Schwarz method

It is easy to see that if the algorithm converges, the solutions u_i^∞,$i = 1; 2$ in the intersection of the subdomains take the same values. Indeed, in the overlap $\Omega_{12} := \Omega_1 \cap \Omega_2$, let $e^\infty := u_1^\infty - u_2^\infty$. By the last line of (3.6), we know that $e^\infty = 0$ on $\partial\Omega_{12}$. By linearity of the Poisson equation, we also have that e^∞ is harmonic.

The discretization of this algorithm yields a parallel algebraic method for solving the linear system $AU = F \in \mathbb{R}^{\#N}$ (N is the set of degrees of freedom) arising from the discretization of the original Poisson problem set on domain.

Definition 29. *(First global Schwarz iteration) Let u^n be an approximation to the solution to the Poisson problem (3.1), u^{n+1} is computed by solving first local sub-problems:*

$$\begin{cases} -\Delta w_i^{n+1} = f \quad in\ \Omega_i, \\ w_i^{n+1} = u^n \ \ on\ \partial\Omega_i \cap \overline{\Omega}_{3-i}, \\ w_i^{n+1} = 0 \quad on\ \partial\Omega_i \cap \partial\Omega. \end{cases}$$

3.4.2.2 Algebraic formulation of the discrete problem

The unknowns of the finite dimensional problem (3.1) are given by the point values of u_h at the finite element nodes a_j. For each element $u_h \in V_h$ can be represented through

$$u_h(x) = \sum_{j=1}^{N_h} u_h(a_j)\, \varphi_j(x).$$

Problem (3.1) can be rewritten as $AU = F$ where

$$U := \{u_h(a_j)\}_{j=1,\ldots,N_h}; F := \{(f, \varphi_j)\}_{g=1,\ldots,N_h}.$$

The matrix A is called the finite element stiffness matrix and is given by

$$A_{ij} = a(\varphi_i, \varphi_j); i, j = 1, \ldots, N_h.$$

The stiffness matrix A is symmetric and positive definite. In particular, any eigenvalue of A has a positive real part. Then we have the simplified relation

$$\kappa(A) := \|A\|_2 \left\|A^{-1}\right\|_2 = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} = O\left(h^{-2}\right).$$

Definition 30. *(**ASM algorithm**) The iterative ASM algorithm is the preconditioned fixed point iteration defined by*

$$U^{n+1} = U^n + M^{-1}\left(F - AU^n\right),$$

where the matrix $M^{-1} := \sum_{i=1}^{N} R_i^T \left(R_i A R_i^T\right)^{-1} R_i$ *is called the ASM preconditioner.*

In other words: $I_d = \sum_{i=1}^{N} R_i^T D_i R_i$; *where* I_d *is the identity matrix, and* D_i *is a local diagonal matrix.*

3.4.3 Non-Overlapping decomposition

We partition now the domain Ω in two disjoint subdomains Ω_1 and Ω_2:

The following equivalence result holds.

Theorem 30. *The solution* u *of the model problem is such that* $u|_{\Omega_i} = u_i$ *for* $i = 1, 2$, *where* u_i *is the solution to the problem*

$$\begin{cases} Lu_i = f \quad \text{in } \Omega_i, \\ \\ u_i = 0 \ \text{ on } \partial\Omega_i \setminus \Gamma, \end{cases}$$

with interface conditions $u_1 = u_2$ *and* $\dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n}$ *on* Γ

3.4.3.1 Dirichlet-Neumann method

Given u_2^0 on Γ, for $k \geq 1$ solve the problems:

$$\begin{cases} Lu_1^{(k)} = f \quad \text{in } \Omega_1, \\ \\ u_1^{(k)} = u_2^{(k-1)} \text{ on } \Gamma, \\ \\ u_1^{(k)} = 0 \text{ on } \partial\Omega_1 \setminus \Gamma. \end{cases}$$

and

$$\begin{cases} Lu_2^{(k)} = f \quad \text{in } \Omega_2, \\ \\ \dfrac{\partial u_1^{(k)}}{\partial n} = \dfrac{\partial u_2^{(k)}}{\partial n} \text{ on } \Gamma, \\ \\ u_2^{(k)} = 0 \text{ on } \partial\Omega_2 \setminus \Gamma. \end{cases}$$

The DN algorithm is therefore consistent. Its convergence however is not always guaranteed.

3.5 Optimized Schwarz methods

During the last decades, a new class of non-overlapping and overlapping Schwarz methods was developed for partial differential equations, namely the optimized Schwarz methods. These methods are based on a classical domain decomposition, but they use more effective transmission conditions than the classical Dirichlet conditions at the interfaces between subdomains.

Optimized Schwarz methods (OSM) are very popular methods which were introduced by P.L. Lions [34] for elliptic problems and by B. Despré s for propagative wave phenomena.

For elliptic problems, Schwarz method is defined only for overlapping subdomains. The domain decomposition method introduced by P.L. Lions is a third type of methods. It can be applied to both overlapping and nonoverlapping subdomains. It is based on improving Schwarz methods by replacing the Dirichlet interface conditions by Robin interface conditions.

Let α be a positive number, the modified algorithm reads

$$\begin{cases} -\triangle u_1^m = f \quad \text{in } \Omega_1, \\ \dfrac{\partial u_1^{m+1}}{\partial \eta_1} + \alpha_1 u_1^{m+1} = \dfrac{\partial u_2^m}{\partial \eta_1} + \alpha_1 u_2^m, \ \text{ on } \Gamma_1, \\ u_1^m = 0 \text{ on } \partial\Omega_1 - \Gamma_1, \end{cases} \tag{3.7}$$

and

$$\begin{cases} -\triangle u_2^m = f \quad \text{in } \Omega_2, \\ \dfrac{\partial u_2^{m+1}}{\partial \eta_2} + \alpha_2 u_1^{m+1} = \dfrac{\partial u_1^m}{\partial \eta_2} + \alpha_2 u_1^m \text{ on } \Gamma_2, \\ u_2^m = 0 \text{ on } \partial\Omega_2 - \Gamma_2, \end{cases} \tag{3.8}$$

where η_1 and η_2 are the outward normals on the boundary of the subdomains.

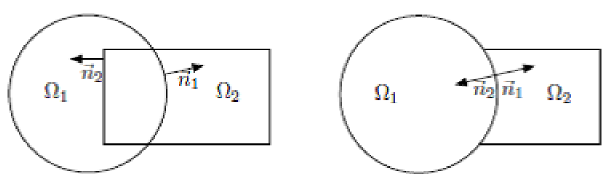


Figure 3.3: outward normals for overlapping and non overlapping subdomain for P.L.Lions algorithm

We use Fourier transform to analyze the problem of the Robin interface conditions in a simple case.

It is also possible to consider other interface conditions than Robin conditions and optimize their choice with respect to the convergence factor.

The algebraic formulation of the P.L. Lions algorithm in the case of overlapping subdomains. It is based on the introduction of the ORAS (Optimized Restricted Additive Schwarz) preconditioned:

$$M_{ORAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i,$$

where$(B_i)_{1\leq n\leq N}$ is the discretization matrix of the Robin problem in subdomain Ω_i.

The following fixed point method

$$U^{n+1} = U^n + M_{ORAS}^{-1}\left(F - AU^n\right),$$

yields, iterates that are equivalent to that of the discretization of P.L.Lions' Algorithm.

Chapter 4

A posteriori error estimates for the generalized Schwarz method of a new class of advection-diffusion equation

4.1 Introduction

In this chapter, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains for a class of advection-diffusion equations with linear source terms using Euler time scheme combined with a finite element spatial approximation, similar to that in [12], which investigated Laplace equation and parabolic free boundary problems which are mentioned above. Moreover, an asymptotic behavior in Sobolev norm is deduced using Benssoussan–Lions' algorithms ([13]).

We consider the following advection diffusion equation:
find $u\left(x,t\right)$ such that $u\in L^{2}\left(0,T,H_{0}^{1}\left(\Omega\right)\right),u_{t}\in L^{2}\left(0,T,L^{2}\left(\Omega\right)\right)$

$$\begin{cases} \dfrac{\partial u}{\partial t}-(D\Delta u+\overrightarrow{v}\nabla u)=f \quad \text{ on } \Sigma, \\ u=0 \text{ in } \Gamma\times\left[0,T\right], \\ u(.,0)=u_{0} \text{ in } \Omega, \end{cases} \tag{4.1}$$

where D is a diffusion coefficient satisfies

$$D\geq\beta>0.$$

$\overrightarrow{v}$ is the average velocity satisfies

$$\bar{v}\in L^{2}\left(0,T,L^{\infty}\left(\Omega\right)\right)\cap C^{\ 0}\left(0,T,H^{-1}\left(\Omega\right)\right).$$

Σ is a set in $\mathbb{R}^{N}\times\mathbb{R}$ defined as $\Sigma=\Omega\times\left[0,T\right]$ with $T<+\infty$, and Ω is a smooth bounded domain of $\mathbb{R}^{N}$ with boundary Γ and the right hand side f is a regular function satisfies

$$f\in L^{2}\left(0,T,L^{\infty}\left(\Omega\right)\right)\cap C^{1}\left(0,T,H^{-1}\left(\Omega\right)\right). \tag{4.2}$$

For equation (4.1) one has the following weak formulation:
find $u \in L^2(0,T,H^1(\Omega)), u_t \in L^2(0,T,L^2(\Omega))$

$$\begin{cases} (u_t, v) + a(u,v)_\Omega = (f,v)_\Omega, \\ u(.,0) = u_0, \ v \in H^1(\Omega), \\ u = 0 \text{ in } \Gamma \times [0,T], \end{cases} \tag{4.3}$$

where

$$a(u,v) = D(\nabla u, \nabla v)_\Omega + \frac{1}{2}\left((\overrightarrow{v}\nabla u, v)_\Omega - (\overrightarrow{v}\nabla v, u)_\Omega\right). \tag{4.4}$$

The symbol $(.,.)_\Omega$ signifies the inner product in $L^2(\Omega)$ and $(.,.)_\Gamma$ indicates the inner product of $L^2(\Gamma)$.

4.2 The generalized overlapping domain decomposition of advection-diffusion equations.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise $C^{1,1}$ boundary $\partial\Omega$. We consider a simple decomposition of Ω into two overlapping subdomaine Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12}, \ \partial\Omega_s \cap \Omega_t = \Gamma_s, \ s \neq t \text{ and } s,t = 1,2. \tag{4.5}$$

We need the spaces

$$V_i = H^1(\Omega) \cap H^1(\Omega_i) = \left\{ v \in H^1(\Omega_i): \ v_{\partial\Omega_i \cap \partial\Omega} = 0 \right\}, \tag{4.6}$$

and

$$W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \{ v_{\Gamma_s}, \ v \in V_s \ \text{ and } v = 0 \text{ on } \ \partial\Omega_s \backslash \Gamma_s \}, \tag{4.7}$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_s) = \left\{ \psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \ \text{ for } \varphi \in V_s, \ s = 1,2 \right\}, \tag{4.8}$$

equipped with the norm

$$\|\varphi\|_{W_s} = \inf_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}, \tag{4.9}$$

and set

$$\Gamma = \partial\Omega_1 \cap \partial\Omega_2; \Gamma_i = \partial\Omega_i \cap \partial\Omega, \Gamma_{ij},$$

as the part of $\partial\Omega_i$ inside denote by $\overrightarrow{n}_{ij}$ the outward normal vector on Γ_{ij} .

We discretize the problem (4.3) with respect to time using the Euler time scheme, then we have

$$\begin{cases} \left(\dfrac{u^k - u^{k-1}}{\Delta t}, v \right)_\Omega + a\left(u^k, v\right)_\Omega = \left(f^k, v\right)_\Omega \ \text{ in } \Omega, \\ u^0(x) = u_0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \end{cases} . \tag{4.10}$$

implies

$$\begin{cases} \left(\dfrac{u^k}{\Delta t}, v\right)_\Omega + a\left(u^k, v\right)_\Omega = \left(f^k + \dfrac{u^{k-1}}{\Delta t}, v\right)_\Omega \text{ in } \Omega, \\ u^0(x) = u_0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.11}$$

Problem (4.11) can be reformulated as the following coercive system of elliptic variational equation

$$\begin{cases} b\left(u^k, v\right) = \left(f^k + \lambda u^{k-1}, v\right) = \left(F\left(u^{k-1}\right), v\right), \\ u^0(x) = u_0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.12}$$

such that

$$\begin{cases} b\left(u^k, v\right) = \lambda\left(u^k, v\right) + a\left(u^k, v\right), \ u^k \in \ H_0^1(\Omega), \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n}, \ k = 1, ..., n. \end{cases} \tag{4.13}$$

We define the continuous counterparts of Schwarz sequences for problem (4.3), respectively by $u_1^{k,m+1} \in H_0^1(\Omega), \ m = 0, 1, 2, ...$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - L_D u_i^{m+1} = f \text{ in } \Omega_i, \\ u_i^{m+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega, \\ u_i^0 = u_0 \text{ in } \Omega, \\ D\dfrac{\partial u_i^{m+1}}{\partial n_i} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_i + z_i\right)u_i^{m+1} = D\dfrac{\partial u_j^m}{\partial n_i} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_i + z_i\right)u_j^m \text{ on } \Gamma_{ij}, i \neq j, \end{cases} . \tag{4.14}$$

where $\overrightarrow{n}_i$ is the exterior normal to Ω_i and $L_D u_i^{m+1} = \left(D\Delta u_i^{m+1} + \overrightarrow{v}\nabla u_i^{m+1}\right)$.
The weak formulation of problems (5.15) is: find $u_i^{m+1} \in V_i$

$$\begin{aligned} &\left(\frac{u^{m+1,k} - u^{m+1,k-1}}{\Delta t}, v_i\right)_\Omega + a\left(u_i^{m+1,k}, v_i\right)_\Omega + \sum_i \left\langle z_i u_i^{m+1,k}, v_i\right\rangle_{\Gamma_{ij}} \\ &= \left(f^k, v_i\right)_\Omega + \sum_{i,j}\left(D\frac{\partial u_j^{m,k}}{\partial n_i} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n}_i + z_i\right)u_j^{m,k}, v_i\right)_{\Gamma_{ij}}. \end{aligned} . \tag{4.15}$$

4.2.1 The space-continuous for generalized overlapping domain decomposition

According to (4.12), (4.14) and (4.15), we can write the following problem, respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$, for $m = 0, 1, 2, ...$such that

$$\begin{cases} b\left(u_1^{k,m+1}, v\right) = \left(F\left(u_1^{k-1,m+1}\right), v\right)_{\Omega_1}, \\ u_1^{k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ D\dfrac{\partial u_1^{k,m+1}}{\partial n_1} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_1 + z_1\right) u_i^{k,m+1}, \\ = D\dfrac{\partial u_2^{k,m}}{\partial n_1} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_1 + z_1\right) u_2^{k,m} \text{ on } \Gamma_1, \end{cases} \tag{4.16}$$

and $u_2^{k,m+1} \in H_0^1(\Omega)$ is a solution of

$$\begin{cases} b\left(u_2^{k,m+1}, v\right) = \left(F\left(u_2^{k-1,m+1}\right), v\right)_{\Omega_2}, \quad m = 0, 1, 2, ..., \\ u_2^{k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega = \partial\Omega_2 - \Gamma_2, \\ D\dfrac{\partial u_2^{k,m+1}}{\partial n_2} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_2 + z_2\right) u_2^{k,m+1}, \\ = D\dfrac{\partial u_1^{k,m}}{\partial n_2} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_2 + z_2\right) u_1^{k,m} \text{ on } \Gamma_2, \end{cases} \tag{4.17}$$

where n_i is the exterior normal to Ω_i and $z_i \in L^\infty(\partial\Omega_i \setminus \partial\Omega)$, $z_i > 0$ is a real parameter, $i = 1, 2$ to accelerate the convergence, this is accomplished by

$$\lim_{D \to 0^+} z_i = \frac{1}{2} |\overrightarrow{v_i}\overrightarrow{n_i}| \text{ on } \Gamma_i. \tag{4.18}$$

4.3 A Posteriori Error Estimate in the Continuous Case

We need to introduce two auxiliary problems defined on nonoverlapping subdomains of Ω nonoverlapping subdomains of Ω. This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [44]. We get these auxiliary problems by coupling each one of the problems (4.17) and (4.18) with a different problem in a nonoverlapping way over Ω.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint sub-domains: (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \quad \Omega = \Omega_2 \cup \Omega_4, \text{ with } \Omega_2 \cap \Omega_4 = \varnothing. \tag{4.19}$$

Let $(u_1^{k,m}, u_2^{n+1,m})$ be the solution of problems (4.16) and (4.17), we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems

$$
\begin{cases}
\dfrac{u_1^{k,m+1}-u_1^{k-1,m+1}}{\Delta t}-\left(D\Delta u_1^{k,m+1}+\overrightarrow{v}\,u_1^{k,m+1}\right)=f^k \text{ in } \Omega_1,\\
u_1^{k,m+1}=0,\ \text{ on } \partial\Omega_1\cap\partial\Omega,\ \ k=1,...,n,\\
D\dfrac{\partial u_1^{k,m+1}}{\partial n_1}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_1}+z_1\right)u_1^{k,m+1}\\
=D\dfrac{\partial u_3^{k,m}}{\partial n_1}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_1}+z_1\right)u_3^{k,m},\ \text{ on } \Gamma_1
\end{cases}
\tag{4.20}
$$

and

$$
\begin{cases}
\dfrac{u_3^{k,m+1}-u_3^{k-1,m+1}}{\Delta t}-\left(D\Delta u_3^{k,m+1}+\overrightarrow{v}\,u_3^{k,m+1}\right)=f^{\ k}\text{in } \Omega_3,\\
u_3^{k,m+1}=0,\ \ \text{ on } \partial\Omega_3\cap\partial\Omega,\\
D\dfrac{\partial u_3^{k,m+1}}{\partial n_3}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_3}+z_3\right)u_3^{n+1,m+1}\\
=D\dfrac{\partial u_1^{k,m}}{\partial n_3}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_3}+z_3\right)u_1^{k,m},\ \text{ on } \Gamma_1.
\end{cases}
\tag{4.21}
$$

Lube and Otto [36] proved there exist a constant $C>0$ such that for the error

$$
E_i^{k,n+1}=U_i^{k,n+1}-U, n\in\mathbb{N}\ and\ i=1,2
$$

holds

$$
\left\|E_i^{k,n+1}\right\|=C\left(K_j\left\|z_i-\frac{1}{2}\overrightarrow{v_i}\overrightarrow{n_i}\right\|_{0,\infty,\Gamma}\left\|E_3^{k,n+1}-E_1^{k,n}\right\|+M_j\left\|E_3^{k,n}-E_1^{k,n+1}\right\|\right),
\tag{4.22}
$$

with

$$
C:=\frac{1}{C_0\left(\theta\right)},
$$

and constants

$$
K_j=\frac{C_{Tr_j}}{\sqrt{D}}, M_j;\ =C_{Tr_j^{-1}}\left(\sqrt{D}+\sqrt{C_\infty}C_F+2\min\left(\frac{C}{\sqrt{w_j}},\frac{C_F}{\sqrt{D}}\right)\right).
\tag{4.23}
$$

Applying Green formula with the new boundary conditions of generalized Schwarz alternating method defined in (5.17), we obtain

$$\left(-D_1\Delta u_1^{k,m+1}, v_1\right)_{\Omega_1} = \left(D_1\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(D_1\frac{\partial u_1^{k,m+1}}{\partial n_1}, v_1\right)_{\partial\Omega_1-\Gamma_1} + \left(D_1\frac{\partial u_1^{k,m+1}}{\partial n_1}, v_1\right)_{\Gamma_1}$$

$$= \left(D_1\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(D_1\frac{\partial u_1^{k,m+1}}{\partial n_1}, v_1.\right)_{\Gamma_1}$$

$$= \left(D_1\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1}$$

$$- \left(D_1\frac{\partial u_2^{k,m}}{\partial n_2} + \left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_2^{k,m} - \left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_1^{k,m+1}, v_1\right)_{\Gamma_1}$$

$$= \left(D_1\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1}$$

$$+ \left(\left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_1^{k,m+1}, v_1\right)_{\Gamma_1} - \left(D_1\frac{\partial u_2^{k,m}}{\partial n_2} + \left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_2^{k,m}, v_1\right)_{\Gamma_1},$$

thus the problem (1.16) is equivalent to:
find $u_1^{k,m+1} \in V_1$ such that:

$$\begin{aligned} b(u_1^{k,m+1}, v_1) + \left(\left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_1^{k,m+1}, v_1\right)_{\Gamma_1} &= \left(F(u^{k-1}), v_1\right)_{\Omega_1} \\ + \left(D_1\frac{\partial u_2^{k,m}}{\partial \eta_1} + \left(-\tfrac{1}{2}\overrightarrow{v_1}\overrightarrow{n_1} + z_1\right)u_2^{k,m}, v_1\right)_{\Gamma_1} &\quad \text{for all } v_1 \in V_1 \end{aligned} \tag{4.24}$$

and for (4.17) $u_2^{k,m+1} \in V_2$, we have

$$\begin{aligned} b(u_2^{k,m+1}, v_2) + \left(\left(-\tfrac{1}{2}\overrightarrow{v_2}\overrightarrow{n_2} + z_2\right)u_2^{k,m+1}, v_2\right)_{\Gamma_2} &= \left(F(u^{k-1}), v_2\right)_{\Omega_2} \\ + \left(D_2\frac{\partial u_1^{k,m}}{\partial n_2} + \left(-\tfrac{1}{2}\overrightarrow{v_2}\overrightarrow{n_2} + z_2\right)u_1^{k,m}, v_2\right)_{\Gamma_2} &\quad \text{for all } v_2 \in V_2. \end{aligned} \tag{4.25}$$

We can set $E_1^{n+1,m} + u_3^{n+1,m} = u_2^{n+1,m}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{n+1,m}$ and $u_3^{n+1,m}$ in problems (4.16), (4.17) and resp., (4.20) and (4.21) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [42] that is, $u_2^{k,m}$ and $u_3^{k,m}$ tend to u_2 (resp. u_3), $E_1^{k,m}$ should tend to naught and m tends to infinity in V_2 .

Multiply the first equation by $v_1 \in V_1$ and integration by part and by putting

$$\Lambda_1^{k,m} = D\frac{\partial u_1^{k,m}}{\partial n_3} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) u_1^{k,m}.$$

$$\Lambda_2^{k,m} = D\frac{\partial u_2^{n+1,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_2^{n+1,m}, \tag{4.26}$$

$$\Lambda_3^{k,m} = D\frac{\partial u_3^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m} + D\frac{\partial E_1^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m}.$$

Then, (4.16) can be reformulated as the following system of elliptic variational equations, using the Green formula

$$\begin{aligned} b(u_1^{k,m+1}, v_1) + \Big(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_1^{k,m+1}, v_1\Big)_{\Gamma_1} &= \left(F(u^{k-1,m+1}), v_1\right)_{\Omega_1} + \\ + \Big(\Lambda_3^{k,m+1}, v_1\Big)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{4.27}$$

and (4.21)

$$\begin{aligned} b(u_3^{k,m+1}, v_3) + \Big(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) u_3^{k,m+1}, v_3\Big)_{\Gamma_1} &= \left(F(u^{k-1,m+1}), v_3\right)_{\Omega 3} + \\ + \Big(\Lambda_1^{k,m+1}, v_3\Big)_{\Gamma_1}, \forall v_3 \in V_3. \end{aligned} \tag{4.28}$$

On the other hand by setting

$$\theta_1^{k,m} = D\frac{\partial \epsilon_1^{k,m}}{\partial n_1} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m}, \tag{4.29}$$

we get

$$\begin{aligned} \Lambda_3^{k,m} &= D\frac{\partial u_3^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m} + D\frac{\partial (u_2^{k,m} - u_3^{k,m})}{\partial n_1} \\ &\quad + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) (u_2^{k,m} - u_3^{k,m}) \\ &= D\frac{\partial u_3^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m} + D\frac{\partial E_1^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m} \\ &= D\frac{\partial u_3^{k,m}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m} + \theta_1^{k,m}. \end{aligned} \tag{4.30}$$

Using (4.29) we have

$$\begin{aligned} \Lambda_3^{k,m+1} &= D\frac{\partial u_3^{k,m+1}}{\partial n_1} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m+1} + \theta_1^{k,m+1} \\ &= -D\frac{\partial u_3^{k,m+1}}{\partial n_3} + \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) u_3^{k,m+1} + \theta_1^{k,m+1} \\ &= \left(-\tfrac{1}{2}\overrightarrow{v}\left(\overrightarrow{n_1} + \overrightarrow{n_3}\right) + (z_1 + z_3)\right) u_3^{k,m+1} - \Lambda_1^{k,m} + \theta_1^{k,m+1}. \end{aligned} \tag{4.31}$$

We can write the following algorithm which is equivalent to the auxiliary nonoverlapping problem (4.27) and (4.28). We need this algorithm to get an a posteriori error estimate for the presented problem.

4.3.1 Algorithm

The sequences $(u_1^{k,m}, u_3^{k,m})_{m\in\mathcal{U}2115}$ solutions of (4.27) and (4.28) satisfy the following domain decomposition algorithm:

Step 1: $k = 0$.

Step 2: Let $\Lambda_i^{k,0} \in W_1^*$ be an initial value, $i = 1, 3$ (W_1^* is the dual of W_1

Step 3; Given $\Lambda_j^{k,m} \in W^*$ solve for $i, j = 1, 3, i \neq j$: Find $u_i^{k,m+1} \in V_i$ solution of

$$b_i(u_i^{k,m+1}, v_i) + \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i} + z_i\right) u_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(F(u^{k-1,m+1}), v_i\right)_{\Omega_i} + + \left(\Lambda_j^{k,m+1}, v_i\right)_{\Gamma_i}, \forall v_i \in V_i. \tag{4.32}$$

Step 4: Compute

$$\theta_1^{k,m+1} = D\frac{\partial E_1^{k,m+1}}{\partial n_1} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m+1}. \tag{4.33}$$

Step 5: Compute new data $\Lambda_j^{k,m+1} \in W^*$ solve for $i, j = 1, 3$, from

$$\left(\Lambda_i^{k,m+1}, \varphi\right)_{\Gamma_i} = \left(\left(-\tfrac{1}{2}\overrightarrow{v}\left(\overrightarrow{n_1} + \overrightarrow{n_3}\right) + (z_1 + z_3)\right) u_i^{k,m+1}, v_i\right)_{\Gamma_i} - \left(\Lambda_j^{k,m+1}, \varphi\right)_{\Gamma_i} + \left(\theta_j^{k,m+1}, \varphi\right)_{\Gamma i}, \forall \varphi \in W_i, i \neq j\ . \tag{4.34}$$

Step 6: Set $m = m + 1$ go to **Step 3.**

Step 7: Set $k = k + 1$ go to **Step 2.**

Lemma 10. *Let $u_i^k = u^k$ in Ω_i solution of (4.32), $E_i^{k,m+1} = u_i^{k,m+1} - u_i^k$ and $\eta_i^{k,m+1} = \Lambda_i^{k,m+1} - \Lambda_i^k$. Then for $i, j = 1, 3, i \neq j$, the following relations hold*

$$b_i(E_i^{k,m+1}, v_i) + \left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_i} + z_i\right) E_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(\eta_j^{k,m}, v_i\right)_{\Gamma_i}, \forall v_i \in V_i, \tag{4.35}$$

where

$$\left(\eta_i^{k,m+1}, \varphi\right)_{\Gamma i} = \left(\left(-\tfrac{1}{2}\overrightarrow{v}\left(\overrightarrow{n_i} + \overrightarrow{n_j}\right) + (z_i + z_j)\right) E_i^{k,m+1}, v_1\right)_{\Gamma_i} - \left(\eta_j^{k,m}, \varphi\right)_{\Gamma_i} + \left(\theta_j^{k,m+1}, \varphi\right)_{\Gamma i}, \forall \varphi \in W_1. \tag{4.36}$$

Proof. **First**, We have

$$b_i(u_i^{k,m+1}, v_i) + \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i} + z_i\right) u_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(F(u^{k-1,m+1}), v_i\right)_{\Omega_i}$$

$$+ \left\langle \Lambda_j^{k,m}, v_i \right\rangle_{\Gamma_i}, \forall v_i \in V_i.$$

Since $b(.,.)$ is a coercive bilinear form, then

$$b_i(u_i^{k,m+1} - u_i^{n+1}, v_i) + \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i} + z_i\right)\left(u_i^{k,m+1} - u_i^{n+1}\right), v_i\right)_{\Gamma_i}$$

$$= \left(\Lambda_j^{k,m} - \Lambda_i^k, v_i\right)_{\Gamma_i}, \forall v_i \in V_i,$$

and so

$$b_i(E_i^{k,m+1}, v_i) + \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i} + z_i\right) E_i^{k,m+1}, v_i\right)_{\Gamma_i}$$

$$= \left(\eta_j^{k,m}, v_1\right)_{\Gamma_i}, \forall v_i \in V_i.$$

Second, we have

$$\lim_{m \to +\infty} E_1^{n+1,m} = \lim_{m \to +\infty} \theta_1^{n+1,m} = 0.$$

Than

$$\Lambda_i^k = (\left(-\frac{1}{2}\overrightarrow{v}\,(\overrightarrow{n_i} + \overrightarrow{n_j}) + (z_i + z_j)\right))u_i^k - \Lambda_j^k.$$

Therefore,

$$\eta_i^{k,m+1} = \Lambda_i^{k,m+1} - \Lambda_i^{n+1}$$

$$= \left(-\tfrac{1}{2}\overrightarrow{v}\,(\overrightarrow{n_i} + \overrightarrow{n_j}) + (z_i + z_j)\right) u_i^{k,m+1} - \Lambda_j^{k,m} + \theta_j^{k,m+1}$$

$$- \left(-\tfrac{1}{2}\overrightarrow{v}\,(\overrightarrow{n_i} + \overrightarrow{n_j}) + (z_i + z_j)\right) u_i^k + \Lambda_j^k$$

$$= \left(-\tfrac{1}{2}\overrightarrow{v}\,(\overrightarrow{n_i} + \overrightarrow{n_j}) + (z_i + z_j)\right)(u_1^{k,m+1} - u_i^k) - (\Lambda_j^{k,m} - \Lambda_j^k) + \theta_j^{k,m+1}.$$

□

Lemma 11. *By letting C be a generic constant which has different values at different places, we get for $i, j = 1, 3, i \neq j$*

$$\left(\eta_i^{k,m-1} - \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right) E_i^{k,m}, w\right)_{\Gamma_1} \leqslant C \left\|E_i^{k,m}\right\|_{1,\Omega_i} \|w\|_{W_1}, \tag{4.37}$$

and

$$\left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right) w_i + \theta_1^{k,m+1}, E_i^{k,m+1}\right)_{\Gamma_1} \leqslant C \left\|E_i^{k,m+1}\right\|_{1,\Omega_i} \|w\|_{W_1}. \tag{4.38}$$

Proof. By using Lemma 1 and the fact that the trace mapping $Tr_i : V_i \longrightarrow W_i$, and its is inverse are continuous, we obtain $i, j = 1, 3, i \neq j$

$$\left(\eta_i^{k,m-1} - (-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i})\, E_i^{k,m}, w\right)_{\Gamma_i} = b_i(E_i^{k,m}, Tr^{-1}w) = \left(\nabla E_i^{k,m}, \nabla Tr^{-1}w\right)_{\Omega_i}$$

$$+\left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n}\right) E_i^{k,m}, Tr^{-1}w\right)_{\Omega_i} + \lambda\left(E_i^{k,m}, Tr^{-1}w\right)_{\Omega_i}$$

$$\leqslant \left|E_i^{k,m}\right|_{1,\Omega_i} |Tr^{-1}w|_{1,\Omega_i} + \left\|-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n}\right\|_{\infty} \left\|E_i^{k,m}\right\|_{0,\Omega_i} \|Tr^{-1}w\|_{0,\Omega_i}$$

$$+|\lambda| \left\|E_i^{k,m}\right\|_{0,\Omega_i} \|Tr^{-1}w\|_{0,\Omega_i}$$

$$\leqslant C\left\|E_i^{k,m}\right\|_{1,\Omega_i} \|w\|_{W_1}.$$

For the second estimate, we have

$$\left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right) w_i + \theta_1^{k,m+1}, E_i^{k,m+1}\right)_{\Gamma_i} = \int_{\Gamma_i} \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right) w_i + \theta_1^{k,m+1}\right) E_i^{k,m+1} ds$$

$$\leqslant \left\|\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right) w_i + \theta_1^{k,m+1}\right\|_{0,\Gamma_1} \left\|E_i^{k,m+1}\right\|_{0,\Gamma_1}$$

$$\leqslant \left(\left|-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right| \|w_i\|_{0,\Gamma_1} + \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}\right) \left\|E_i^{n+1,m+1}\right\|_{0,\Gamma_1}$$

$$\leqslant \max(\left|-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_i}\right|, \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}) \|w_i\|_{0,\Gamma_1} \left\|E_i^{k,m+1}\right\|_{0,\Gamma_1}$$

$$\leqslant C\left\|E_i^{k,m+1}\right\|_{0,\Gamma_1} \|w_i\|_{0,\Gamma_1} \leqslant C\left\|E_i^{k,m+1}\right\|_{0,\Gamma_1} \|w_i\|_{W_1}.$$

Proposition 10. *For the sequences $(u_1^{k,m}, u_3^{k,m})_{m\in\mathcal{N}}$ solutions of (4.27) and (4.28) we have the following a posteriori error estimation*

$$\left\|E_1^{k,m+1}\right\|_{1,\Omega_1} + \left\|E_3^{k,m}\right\|_{1,\Omega_3} \leqslant C\left\|u_1^{k,m+1} - u_3^{k,m}\right\|_{W_1}. \tag{4.39}$$

Proof. From (4.31) and (4.33), we have

$$b_1(E_1^{k,m+1}, v_1) + b_3(E_3^{k,m}, v_3)$$

$$= \left(\eta_3^{k,m} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m+1}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_3\right)_{\Gamma_1}$$

$$= \left(\eta_3^{k,m} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m+1}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_3\right)_{\Gamma_1}$$

$$+ \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_1\right)_{\Gamma_1} - \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_1\right)_{\Gamma_1}$$

$$= \left(\eta_3^{k,m} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right) E_1^{k,m+1} + \eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_1\right)_{\Gamma_1}$$

$$+ \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_3 - v_1\right)_{\Gamma_1}$$

$$= \left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right)(E_3^{k,m} - E_1^{k,m+1}) + \theta_1^{k,m}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, v_3 - v_1\right)_{\Gamma_1}.$$

Taking $v_1 = E_1^{k,m+1}$ and $v_3 = E_3^{k,m}$. Then using lemma 2, we get

$$\frac{1}{2}\left(\left\|E_1^{k,m+1}\right\|_{1,\Omega_1} + \left\|E_3^{k,m}\right\|_{1,\Omega_3}\right)^2 \leq \left\|E_1^{k,m+1}\right\|_{1,\Omega_1}^2 + \left\|E_3^{k,m}\right\|_{1,\Omega_3}^2$$

$$= \left(\nabla E_1^{k,m+1}, \nabla E_1^{k,m+1}\right)_{\Omega_1} + \left(\nabla E_3^{k,m}, \nabla E_3^{k,m}\right)_{\Omega_3}$$

$$\leq \tfrac{1}{D} b_1\left(E_1^{k,m+1}, E_1^{k,m+1}\right) + \tfrac{1}{D} b_3\left(E_3^{k,m}, E_3^{k,m}\right)$$

$$\leqslant \tfrac{1}{D}\left(\left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_1} + z_1\right)(E_3^{k,m} - E_1^{k,m+1}) + \theta_1^{k,m}, E_1^{k,m+1}\right)_{\Gamma_1}$$

$$+ \left(\eta_1^{k,m-1} - \left(-\tfrac{1}{2}\overrightarrow{v}\overrightarrow{n_3} + z_3\right) E_3^{k,m}, E_3^{k,m} - E_1^{k,m}\right)_{\Gamma_1}$$

$$\leq C_1 \left\|E_1^{k,m+1}\right\|_{1,\Omega_1} \left\|E_3^{k,m} - E_1^{k,m+1}\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} + C_2 \left\|E_3^{k,m}\right\|_{1,\Omega_3} \left\|E_3^{k,m} - E_1^{k,m+1}\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)},$$

then

$$\frac{1}{2}\left(\left\|E_1^{k,m+1}\right\|_{1,\Omega_1} + \left\|E_3^{k,m}\right\|_{1,\Omega_3}\right)^2 \leqslant \max(C_1, C_2)\left[\left\|E_1^{k,m+1}\right\|_{1,\Omega_1} + \left\|E_3^{k,m}\right\|_{1,\Omega_3}\right] \left\|E_3^{k,m} - E_1^{k,m+1}\right\|_{W_1}$$

or

$$\left\|E_1^{k,m+1}\right\|_{1,\Omega_1} + \left\|E_3^{k,m}\right\|_{1,\Omega_3} \leqslant \left\|E_1^{k,m+1} - E_3^{k,m}\right\|_{W_1}.$$

Therefore,

$$\left\|u_1^{n+1,m+1} - u_1^{n+1}\right\|_{1,\Omega_1} + \left\|u_3^{n+1,m} - u_3^{n+1}\right\|_{3,\Omega_3} \leqslant 2\max(C_1, C_2)\left\|u_1^{n+1,m+1} - u_3^{n+1,m}\right\|_{W_1}.$$

$\square$

Similarly, we define another nonoverlapping auxiliary problems over (Ω_2, Ω_4), we get the same result.

Proposition 11. *For the sequences $(u_2^{k,m}, u_4^{k,m})_{m\in\mathcal{N}}$ We get the the similar following a posteriori error estimation*

$$\left\|u_2^{k,m+1} - u_2^k\right\|_{2,\Omega_2} + \left\|u_4^{k,m} - u_4^k\right\|_{4,\Omega_4} \leqslant C\left\|u_2^{k,m+1} - u_4^{k,m}\right\|_{W_2}. \tag{4.40}$$

Proof. The proof is very similar to the proof of Proposition 1. □

Theorem 31. *Let $u_i^k = u_{\Omega_i}^k$ for the sequences $(u_1^{k,m}, u_2^{k,m})_{m\in\mathcal{N}}$ be solutions of problems (4.20 and (4.21), we have the following a posteriori error estimate result*

$$\left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_2^{k,m} - u_2^k\right\|_{2,\Omega_2} \leqslant C(\left\|u_1^{k,m+1} - u_2^{k,m}\right\|_{W_1} + \left\|u_2^{k,m} - u_1^{k,m-1}\right\|_{W_2} + \left\|E_1^{k,m}\right\|_{W_1} + \left\|E_2^{k,m-1}\right\|_{W_2}).$$

Proof. By using two nonoverlapping auxiliary problems over (Ω_1, Ω_3) and (Ω_2, Ω_4) resp. From the previous two proposition we have

$$\begin{aligned}
&\left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_2^{k,m} - u_2^k\right\|_{2,\Omega_2} \\
&\leqslant \left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_3^{k,m} - u_3^k\right\|_{3,\Omega_3} \\
&+ \left\|u_2^{k,m} - u_2^k\right\|_{2,\Omega_2} + \left\|u_4^{k,m-1} - u_4^k\right\|_{4,\Omega_4} \\
&\leqslant C\left\|u_1^{k,m+1} - u_3^{n+1,m}\right\|_{W_1} + C\left\|u_2^{k,m} - u_4^{k,m-1}\right\|_{W_2} \\
&\leqslant C\left\|u_1^{k,m+1} - u_2^{k,m} + E_1^{k,m}\right\|_{W_1} + C\left\|u_2^{k,m} - u_1^{k,m-1} + E_2^{k,m-1}\right\|_{W_2}.
\end{aligned}$$

Thus, it can be deduced

$$\begin{aligned}
&\left\|u_1^{k,m+1} - u_1^k\right\|_{\Omega_1} + \left\|u_2^{k,m} - u_2^k\right\|_{\Omega_2} \\
&\leqslant C\left(\left\|u_1^{k,m+1} - u_2^{k,m} + E_1^{k,m}\right\|_{W_1} + \left\|u_2^{k,m} - u_1^{k,m-1} + E_2^{k,m-1}\right\|_{W_2} + \left\|E_1^{k,m}\right\|_{W_1} + \left\|E_2^{k,m-1}\right\|_{W_2}\right).
\end{aligned}$$

□

4.4 A Posteriori Error Estimate in the Discrete Case

In this section, we consider the discretization of the problem (4.13). Let τ_h be a decomposition of Ω into open triangles,compatible with the discretization. A triangle is denote by K which its diameter by h_K, an edge by E, and the length of the edge by h_E and $V_h \subset H_0^1$ is the subspace of continuous functions which vanish over $\partial\Omega$. We have

$$V_{i,h} = V_{h|\Omega_i}, \qquad W_{i,h} = V_{h|\Gamma_i}, \quad i = 1, 2., \tag{4.41}$$

where $W_{i,h}$ is a subspace of $H_{00}^{\frac{1}{2}}(\Gamma_i)$ which consists of continuous piecewise polynomial functions on Γ_i which vanish at the end points of $\Gamma_i\,(i = 1, 2)$.

4.4.0.1 The space discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h>0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i=\{1,...,m(h)\}$ defined by $\varphi_i(M_j)=\delta_{ij}$ where M_j is a summit of the considered triangulation.

We discretize in space, i.e, that we approach the space H_0^1 by a space discretization of finite dimensional $V^h\subset H_0^1$. In a second step, we discretize the problem with respect to time using the Euler scheme. Therefore, we search a sequence of elements $u_h^n\in V^h$ which approaches $u^n(t_n)$, $t_n = n\Delta t$, with initial data $u_h^0=u_{0h}$. Now, we apply Euler scheme on the following to the semi-discrete approximation for $v_h\in V^h$.

Let $u_h^{m+1}\in V_h$ be the solution of discrete problem associated with (4.32), $u_{i,h}^{m+1}=u_h^{m+1}|_{\Omega_i}$.

We construct the sequences $(u_{i,h}^{n+1,m+1})_{m\in\mathbb{N}}, u_{i,h}^{n+1,m+1}\in V_{i,h}, (i=1,2)$ as solutions of discrete problems associated with (4.20) and (4.21).

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for (Ω_1,Ω_3):

$$\begin{cases} b_1(u_{1,h}^{k,m+1},v_h)+\left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_1}+z_1\right)u_{1,h}^{k,m+1},v_h\right)_{\Gamma_1}=\left(F(u_{1,h}^{k,m+1}),v_h\right)_{\Omega_1}+\\ +\left(\Lambda_3^{k,m},v_h\right)_{\Gamma_1}\text{ for all } v_h\in V^h,\\ u_{1,h}^{k,m+1}=0\text{ on }\partial\Omega_1\cap\partial\Omega,\\ D\dfrac{\partial u_{1,h}^{k,m+1}}{\partial n_1}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_1+z_1\right)u_{1,h}^{k,m+1}=D\dfrac{\partial u_{2,h}^{k,m}}{\partial n_1}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_1+z_1\right)u_{2,h}^{k,m}\text{ on }\Gamma_1,\end{cases}\tag{4.42}$$

and

$$\begin{cases} b_3(u_{3,h}^{k,m+1},v_h)+\left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_3}+z_3\right)u_{3,h}^{k,m+1},v_h\right)_{\Gamma_1}\\ =\left(F(u_{3,h}^{k,m+1}),v_h\right)_{\Omega 3}+\left(\Lambda_1^{k,m},v_h\right)_{\Gamma_1}\text{ for all } v_h\in V^h,\\ u_{3,h}^{k,m+1}=0\text{ on }\partial\Omega_3\cap\partial\Omega,\\ D\dfrac{\partial u_{3,h}^{k,m+1}}{\partial n_3}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_3+z_3\right)u_{3,h}^{k,m+1}=D\dfrac{\partial u_{1,h}^{k,m}}{\partial n_3}+\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_3+z_3\right)u_{1,h}^{k,m}\text{ on }\Gamma_1,\end{cases}\tag{4.43}$$

and for (Ω_2, Ω_4)

$$\begin{cases} b_2(u_{2,h}^{k,m+1}, v_h) + \left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_2} + z_2\right) u_{2,h}^{k,m+1}, v_h\right)_{\Gamma_1} \\ = \left(F(u_{2,h}^{k,m+1}), v_h\right)_{\Omega 3} + \left(\Lambda_4^{k,m}, v_h\right)_{\Gamma_1} \text{ for all } v_h \in V^h, \\ u_{2,h}^{k,m+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega, \\ D\dfrac{\partial u_{2,h}^{k,m+1}}{\partial n_2} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_2 + z_2\right) u_{2,h}^{k,m+1}, \\ = D\dfrac{\partial u_{1,h}^{k,m}}{\partial n_2} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_2 + z_2\right) u_{1,h}^{k,m} \text{ on } \Gamma_2, \end{cases} \tag{4.44}$$

and

$$\begin{cases} b_4(u_{4,h}^{k,m+1}, v_4) + \left(\left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n_4} + z_4\right) u_{4,h}^{k,m+1}, v_h\right)_{\Gamma_1} \\ = \left(F(u_{4,h}^{k,m+1}), v_h\right)_{\Omega 3} + \left(\Lambda_2^{k,m}, v_h\right)_{\Gamma_1}, \text{for all } v_h \in V^h. \\ u_{4,h}^{k,m+1} = 0 \text{ on } \partial\Omega_4 \cap \partial\Omega, \\ D\frac{\partial u_{4,h}^{k,m+1}}{\partial n_4} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_4 + z_4\right) u_{3,h}^{k,m+1} \\ = D\frac{\partial u_{2,h}^{k,m}}{\partial n_4} + \left(-\frac{1}{2}\overrightarrow{v}\overrightarrow{n}_4 + z_4\right) u_{2,h}^{k,m} \text{ on } \Gamma_2. \end{cases} \tag{4.45}$$

It can be written

$$u_{2,h}^{k,m} = u_{3,h}^{k,m} - E_{1,h}^{k,m} \textit{ on } \Gamma_1 \textit{ and } u_{1,h}^{k,m} = u_{4,h}^{k,m} - E \textit{ on } \Gamma_2,$$

that is $E_{1,h}^{k,m}, E_{2,h}^{k,m}$ is the difference between the discrete overlapping and nonoverlapping solution $u_{2,h}^{k,m}, u_{3,h}^{k,m}$ in Ω_3, $\left(u_{1,h}^{k,m}, u_{4,h}^{k,m} \textit{ in } \Omega_4\right)$.

Because both $u_{2,h}^{k,m}$ and $u_{3,h}^{k,m}$ converge to u_2, $\left(u_{1,h}^{k,m}\text{and } u_{4,h}^{k,m}\text{converge to } u_1\right)$, $E_{1,h}^{k,m}$, $E_{2,h}^{k,m}$ should tend to naught as m tend to infinity.

Proposition 12. *We can obtain the discrete counterparts of propositions 1 and 2 by doing almost the same analysis as in section above (i.e, passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,*

$$\left\|u_{h1}^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_{h3}^{k,m} - u_3^k\right\|_{1,\Omega_3} \leqslant C \left\|u_{h1}^{k,m+1} - u_{h3}^{k,m}\right\|_{W_1}, \tag{4.46}$$

and

$$\left\|u_{h2}^{k,m+1} - u_{h2}^{n+1}\right\|_{1,\Omega_2} + \left\|u_{h4}^{k,m} - u_{h4}^{n+1}\right\|_{1,\Omega_4} \leqslant C \left\|u_{h2}^{k,m+1} - u_{h4}^{k,m}\right\|_{W_2}, \tag{4.47}$$

and so we get the discrete case of the theorems 2, (4.46) and (4.47)

$$\begin{aligned}\left\|u_{1,h}^{k,m+1}-u_{1,h}^{k}\right\|_{1,\Omega_1}+\left\|u_{2,h}^{k,m}-u_{2,h}^{k}\right\|_{1,\Omega_2}\leqslant C(\left\|u_{1,h}^{k,m+1}-u_{2,h}^{k,m}\right\|_{W_1}+\left\|u_{2,h}^{k,m}-u_{1,h}^{k,m-1}\right\|_{W_2}\\ +\left\|E_{1,h}^{k,m}\right\|_{W_1}+\left\|E_{2,h}^{k,m-1}\right\|_{W_2}).\end{aligned} \tag{4.48}$$

4.5 An asymptotic behavior for the problem

4.5.1 A fixed point mapping associated with discrete problem

We define for $i=1,2,3,4$ the following mapping:

$$\begin{aligned}T_h : V_{i,h} \longrightarrow H_0^1\left(\Omega_i\right)\\ W_i \longrightarrow TW_i=\xi_{h,i}^{k,m+1}=\partial_h\left(F\left(w_i\right)\right),\end{aligned} \tag{4.49}$$

where $\xi_{h,i}^{k}$ is the solution of the following problem:

$$\begin{cases} b_i(\xi_{i,h}^{k,m+1},v_i)+\left(\left(-\frac{1}{2}\overrightarrow{v_i}\overrightarrow{n}_i+z_i\right)\xi_{i,h}^{k,m+1},v_{i,h}\right)_{\Gamma_i}=\left(F(w_i),v_{i,h}\right)_{\Omega_i}, \\ \xi_{i,h}^{k,m+1}=0, \quad \text{on } \partial\Omega_i\cap\partial\Omega, \\ D_i\dfrac{\partial\xi_{i,h}^{k,m+1}}{\partial\eta_i}+\left(-\frac{1}{2}\overrightarrow{v_i}\overrightarrow{n}_i+z_i\right)\xi_{i,h}^{k,m+1} \\ =D_i\dfrac{\partial\xi_{j,h}^{k,m}}{\partial\eta_i}+\left(-\frac{1}{2}\overrightarrow{v_i}\overrightarrow{n}_i+z_i\right)\xi_{j,h}^{k,m}, \ \text{on } \Gamma_i, \ i=1,...,4, \ j=1,2. \end{cases} . \tag{4.50}$$

4.5.2 An iterative discrete algorithm

We choose the initial data $u_h^{i,0}=u_{h0}^{i}$ as a solution of the following discrete equation

$$b^i\left(u_{h,i}^0,v_h\right)=\left(g_i^0,v_h\right), \ v_h\in V^h, \tag{4.51}$$

with $g^{i,0}$ is a linear and a regular function.
Now, we give the following discrete algorithm

$$u_{i,h}^{k,m+1}=T_h u_{i,h}^{k-1,m+1}, k=1,...,n, \ i=1,...,4, \tag{4.52}$$

where $u_{i,h}^{k}$ is the solution of the problem (4.50).

Proposition 13. *Let $\xi_h^{i,k}$ be a solution of the problem (4.50) with the right hand side $F^i\left(w_i\right)$ and the boundary condition*

$$D_i\frac{\partial\xi_{i,h}^{k,m+1}}{\partial\eta_i}+\left(-\frac{1}{2}\overrightarrow{v_i}\overrightarrow{n}_i+z_i\right)\xi_{i,h}^{k,m+1},$$

$\tilde{\xi}_h^{i,k}$, *the solution for* $\tilde{F}^i$ *and*

$$D_i \frac{\partial \tilde{\xi}_{i,h}^{k,m+1}}{\partial \eta_i} + \left(-\frac{1}{2} \overrightarrow{v_i} \overrightarrow{n}_i + z_i \right) \tilde{\xi}_{i,h}^{k,m+1}.$$

The mapping T_h *is a contraction in* $V_{i,h}$ *with the rate of contraction* $\dfrac{\lambda}{\frac{(\Delta t)\,\beta}{D} + \lambda}$. *Therefore,* T_h *admits a unique fixed point which coincides with the solution of the problem (4.50).*

Proof. We note that

$$\|W\|_{H_0^1(\Omega_i)} = \|W\|_1 .$$

Setting

$$\phi = \frac{1}{\frac{(\Delta t)\,\beta}{D} + \lambda} \left\| F(w_i) - F(\tilde{w}_i) \right\|_1 .$$

Then, we have $\xi_{i,h}^{k,m+1} + \phi$ is a solution of

$$\begin{cases} b\left(\xi_{i,h}^{k,m+1} + \phi, (v_{i,h} + \phi) \right) = \left(F(w_i) + \alpha_i \phi, (v_{i,h} + \phi) \right), \\ \xi_{i,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \alpha_i \xi_{i,h}^{k,m+1} = \dfrac{\partial \xi_{j,h}^{k,m}}{\partial \eta_i} + \alpha_i \xi_{j,h}^{k,m}, \text{ on } \Gamma_i, \ i = 1, ..., 4, \ j = 1, 2. \end{cases}$$

On the other hand, we have

$$F(w_i) \leq F(\tilde{w}_i) + \|F(w_i) - F(\tilde{w}_i)\|_1 \leq F(\tilde{w}_i) + \frac{\alpha}{\beta + \lambda} \|F(w_i) - F(\tilde{w}_i)\|_1 \leq F(\tilde{w}_i) + a\phi.$$

It is very clear that if $F^i(w_i) \geqq F^i(\tilde{w}_i)$ then $\xi_{i,h}^{k,m+1} \geqq \tilde{\xi}_{i,h}^{k,m+1}$. Thus,

$$\xi_{i,h}^{k,m+1} \leq \tilde{\xi}_{i,h}^{k,m+1} + \phi.$$

But the role of w_i and $\tilde{w}_i$ are symmetrical, thus we have a similar prof

$$\tilde{\xi}_{i,h}^{k,m+1} \leq \xi_{i,h}^{k,m+1} + \phi,$$

yields

$$\begin{aligned} \|T(w) - T(\tilde{w})\|_\infty &\leq \frac{1}{\frac{(\Delta t)\,\beta}{D} + \lambda} \|F(w_i) - F(\tilde{w}_i)\|_1 \\ &= \frac{1}{\frac{(\Delta t)\,\beta}{D} + \lambda} \left\| f^i + \lambda w_i - f^i - \lambda \tilde{w}_i \right\|_1 \\ &\leq \frac{\lambda}{\frac{(\Delta t)\,\beta}{D} + \lambda} \|w_i - \tilde{w}_i\|_1 . \end{aligned}$$

□

Proposition 14. *Under the previous hypotheses and notations, we have the following estimate of convergent*

$$\left\|u_{i,h}^{n,m=1}-u_{i,h}^{\infty,m=1}\right\|_1 \leq \left(\frac{1}{1+\frac{(\Delta t)\,\beta}{D}}\right)^n \left\|u_{i,h}^{\infty,m=1}-u_{i,h_0}\right\|_1, \;\; k=0,...,n, \tag{4.53}$$

where $u^{\infty,m+1}$ is an asymptotic continuous solution and u_{i,h_0} is a solution of (4.51).

Proof. We have

$$u_h^{i,\infty}=T_h u_h^{i,\infty},$$

$$\left\|u_{i,h}^{1,m+1}-u_{i,h}^{\infty,m+1}\right\|_1=\left\|T_h u_{i,h}^{0,m+1}-T_h u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\frac{(\Delta t)\,\beta}{D}}\right)\left\|u_{i,h}^{i,0}-u_{i,h}^{\infty,m+1}\right\|_1$$

and also we have

$$\left\|u_h^{n+1,m+1}-u_h^{i,\infty}\right\|_1=\left\|T_h u_{i,h}^{n,m+1}-T_h u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\frac{(\Delta t)\,\beta}{D}}\right)\left\|u_{i,h}^{n,m+1}-u_{i,h}^{i,\infty}\right\|_1.$$

Then

$$\left\|u_{i,h}^{n,m+1}-u_{i,h}^{\infty}\right\|_1 \leq \left(\frac{1}{1+\frac{(\Delta t)\,\beta}{D}}\right)^n \left\|u_{i,h}^{\infty,m+1}-u_{i,h_0}\right\|_1.$$

□

Theorem 32. *Under the previous hypotheses, notations, results, we have for $i=1,...,4,\ k=1,...,n,\ m=1,2,...$*

$$\left\|u_{i,h}^{n,m+1}-u^{\infty}\right\|_1 \leq C\left[\begin{array}{c}\left\|u_{1,h}^{k,m+1}-u_{2,h}^{k,m}\right\|_{W_1}+\left\|u_{2,h}^{k,m}-u_{1,h}^{k,m-1}\right\|_{W_2}+\left\|E_{1,h}^{n+1,m}\right\|_{W_1}\\ +\left\|E_{2,h}^{n+1,m-1}\right\|_{W_2}+\left(\frac{1}{1+\frac{(\Delta t)\,\beta}{D}}\right)^n.\end{array}\right] \tag{4.54}$$

Proof. Using Proposition 3 and 5, it can be easily deduced (4.54) using the triangulation inequality.
□

4.6 Numerical example

In this section, we give a simple numerical example. Consider the following advection diffusion equation

$$\begin{cases} \dfrac{\partial^i u}{\partial t}+\max\limits_{1\leq i\leq 2}\left(A^i u^i-f^i\right)=0, \text{ in } \Omega\times[0,T] \\ u\,(0,t) \qquad \text{in } \Omega=0 \end{cases} \tag{4.55}$$

where $\Omega =]0.1[\,,\ u(0,x) = 0,\ T = 1$ and

$$A^1 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x},\ A^2 u = \frac{\partial^2 u}{\partial x^2} + u \text{ and } \ f^1 = f^2 = x + t$$

The exact solution of the problem is

$$u(x,t) = x^5 \sin(10x) \cos(20\pi t).$$

For the finite element approximation, we take uniform partition and linear conforming element. For the domain decomposition, we use the following decompositions $\Omega_1 =]0,\ 0.55[\,,\ \Omega_2 =]0.45,\ 1[\,.$

We compute the bilinear semi-implicit scheme combined with Galerkin solution in Ω and we apply the generalized overlapping domain decomposition method to compute the bilinear sequences $u_{h,s}^{i,k,m+1}$, $(s = 1, 2)$ to be able to look at the behavior of the constant C, where the space steps $h = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ and the time steps of discetization $\Delta t = \frac{1}{10}, \frac{1}{50}, \frac{1}{100}$.

We denote by

$$E_s = \left\| u_s^{i,k} - u_{h,s}^{i,k,m} \right\|_{1,\Omega_s},\ T_1 = \left\| u_{h,1}^{i,k,m+1} - u_{h,2}^{i,k,m} \right\|_{W_h^1} \text{and } \ T_2 = \left\| u_{h,2}^{i,k,m} - u_{h,1}^{i,k,m-1} \right\|_{W_h^2}.$$

The generalized overlapping domain decomposition method, with $\alpha_1 = \alpha_2 = 0.55$, converges. The iterations have been stopped when the relative error between two subsequent iterates is less than 10^{-6}, we get the following results

	h	1/10	1/100	1/1000
$\Delta t = \frac{1}{10}$	E_s	0.487588 (−4)	0.198746 (−6)	0.4147712 (−6)
	E_s	0.517474 (−4)	03284541 (−6)	0.362897 (−6)
	T_1	0.800825 (−4)	0.5874121 (−6)	0.851795 (−6)
	T_2	0.918475 (−4)	0.6123898 (−6)	0.923192 (−6)
	Iterations	9	18	20
	h	1/10	1/100	1/1000
$\Delta t = \frac{1}{20}$	E_s	0.460399 (−3)	0.8496273 (−4)	0.901941 (−4)
	E_s	0.498788 (−3)	0.7892758 (−4)	0.817449 (−4)
	T_1	0.7148525 (−3)	0.280914 (−4)	0.795864 (−4)
	T_2	0.81744568 (−3)	0.109839 (−4)	0.810876 (−4)
	Iterations	9	18	24
	h	1/10	1/100	1/1000
$\Delta t = 1/40$	E_s	0.9276183 (−2)	0.2937842 (−3)	0.8297682 (−4)
	E_s	0.8524725 (−2)	0.2572064 (−3)	0.87085497 (−4)
	T_1	0.9793482 (−2)	0.6079027 (−3)	0.5433127 (−4)
	T_2	0.7582921 (−2)	0.51975802 (−3)	0.517528 (−4)
	Iterations	8	14	24

Finally, we can deduce the asymptotic behavior

$$As = \sum_{s=1}^{2} \left\| u_{h,s}^{i,n,m+1} - u^{i,\infty} \right\|_1 \text{ for } \Delta t = 1/1000 \text{ ie., } n = 1000$$

as the following result

h	$1/10$	$1/100$	$1/1000$
As	$0.284756\,(-3)$	$0.157846\,(-4)$	$0.127845\,(-4)$
Iterations	9	18	24

In the tables above, we also see that the iteration number is roughly related to h and Δt, and the order of convergence is in a good agreement with our estimates (4.54).

Chapter 5

A posteriori error estimates for the generalized Schwarz method of a new class of advection-diffusion equation with mixed boundary condition

In this chapter, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary conditions on the boundaries for the discrete solutions on subdomains for a class of advection-diffusion equations with linear source terms using theta time scheme combined with a finite element spatial approximation, similar to that in [12], which investigated Laplace equation and parabolic free boundary problems which are mentioned above. Moreover, an asymptotic behavior in Sobolev norm is deduced using Benssoussan–Lions' algorithms ([13]).

We consider the following advection diffusion equation:
find $u(x,t)$ such that $u \in L^2\left(0,T,H_0^1(\Omega)\right), u_t \in L^2\left(0,T,L^2(\Omega)\right)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - (D\Delta u + \overrightarrow{v}\nabla u) = f \quad \text{ on } \Sigma, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0 \times [0,T], \\ \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{5.1}$$

where D is a diffusion coefficient satisfies

$$D \geq \beta > 0.$$

$\overrightarrow{v}$ is the average velocity satisfies

$$\bar{v} \in L^2\left(0,T,L^\infty(\Omega)\right) \cap C^{\,0}\left(0,T,H^{-1}(\Omega)\right).$$

Σ is a set in $\mathbb{R}^N \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, and Γ is a smooth bounded domain of $\mathbb{R}^N$ with boundary Γ and the right hand side, Γ_0 is the part of the boundary given by [45] and defined as:

$$\Gamma_0 = \left\{ x \in \partial\Omega = \Gamma \text{ such that } \forall \xi \;\; > 0,\; x + \xi \notin \bar{\Omega} \right\}.$$

f is a regular function satisfies

$$f \in L^2\left(0, T, L^\infty\left(\Omega\right)\right) \cap C^1\left(0, T, H^{-1}\left(\Omega\right)\right). \tag{5.2}$$

For equation (5.1) one has the following weak formulation:
find $u \in L^2\left(0, T, H^1\left(\Omega\right)\right), u_t \in L^2\left(0, T, L^2\left(\Omega\right)\right)$ solution of:

$$\begin{cases} \left(u_t, v\right) + a\left(u, v\right)_\Omega = \left(f, v\right)_\Omega + \left(\varphi, v\right)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0 \times [0, T], \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u(., 0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{5.3}$$

where

$$a(u, v) = D\left(\nabla u, \nabla v\right)_\Omega + \left(\overrightarrow{v} \nabla u, v\right)_\Omega - \left(a_0 u, v\right)_\Omega. \tag{5.4}$$

5.1 The generalized overlapping domain decomposition of advection-diffusion equations

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise $C^{1,1}$ boundary $\partial\Omega$. We consider a simple decomposition of Ω into two overlapping subdomaine Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12},\; \partial\Omega_s \cap \Omega_t = \Gamma_s,\; s \neq t \text{ and } s, t = 1, 2. \tag{5.5}$$

We need the spaces

$$V_i = H^1(\Omega) \cap H^1(\Omega_i) = \left\{ v \in H^1(\Omega_i) : \; v_{\partial\Omega_i \cap \partial\Omega} = 0 \right\}, \tag{5.6}$$

and

$$W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \left\{ v_{\Gamma_s},\; v \in V_s \;\text{ and } v = 0 \text{ on } \; \partial\Omega_s \backslash \Gamma_s \right\}, \tag{5.7}$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_s) = \left\{ \psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \;\text{ for } \varphi \in V_s,\; s = 1, 2 \right\}, \tag{5.8}$$

equipped with the norm

$$\|\varphi\|_{W_s} = \inf_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}, \tag{5.9}$$

and set

$$\Gamma = \partial\Omega_1 \cap \partial\Omega_2; \Gamma_i = \partial\Omega_i \cap \partial\Omega, \Gamma_{ij}$$

as the part of $\partial\Omega_i$ inside denote by $\overrightarrow{n}_{ij}$ the outward normal vector on Γ_{ij}.

5.1.1 The space discretization

Let Ω open bounded be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform,associated with a finite element of Lagrange (K, P_K, Σ_K). We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$, where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$V_h = \begin{cases} v \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right), \\ \text{such that } v_h|_K \in P_K, k \in \tau_h, \\ v_h(., 0) = v_{h0} \text{ (initial data) in } \Omega, \\ \dfrac{\partial v_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \mathrm{v}_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{cases} \tag{5.10}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1.
We consider r_h be the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(M_i)\, \varphi_i(x).$$

The discrete maximum principle assumption (dmp) [36]. We assume the matrices whose coefficients $a(\varphi_i, \varphi_j)$ are $M-$matrix. For convenience in all the sequels, C will be a generic constant independent on h.

It can be approximated the problem 5.1 by a weakly coupled system of the following parabolic equation $v \in H^1(\Omega)$

$$\left(\frac{\partial u}{\partial t}, v\right)_\Omega + a(u, v) = (f, v)_\Omega + (\varphi, v)_{\Gamma_0}. \tag{5.11}$$

We discretize in space, i.e., we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right)\right)$, we get the following semi-discrete system of parabolic equation

$$\left(\frac{\partial u_h}{\partial t}, v_h\right)_\Omega + a(u_h, v_h) = (f, v_h)_\Omega + (\varphi, v_h)_{\Gamma_0}. \tag{5.12}$$

5.1.2 The time discretization

We apply the θ-scheme in the semi-discrete approximation (5.10). Thus we have, for any $\theta \in [0, 1]$ and $k = 1, ..., p$

$$\begin{gathered} \left(u_h^k - u_h^{k-1}, v_h\right)_\Omega + (\Delta t)\, a\left(u_h^{\theta,k}, v_h\right) = \\ (\Delta t)\left[\left(f^{i,\ \theta,k}, v_h\right)_\Omega + \left(\varphi^{i,\theta,k}, v_h\right)_{\Gamma_0}\right], \end{gathered} \tag{5.13}$$

where

$$\begin{aligned} u_h^{\theta,k} &= \theta u_h^k + (1-\theta)\, u_h^{k-1}, \\ f^{\ \theta,k} &= \theta f^k + (1-\theta)\, f^{k-1}, \end{aligned} \tag{5.14}$$

and

$$\varphi^{\ \theta,k} = \theta \varphi^k + (1-\theta)\, \varphi^{k-1}. \tag{5.15}$$

By multiplying and dividing by θ and by adding $\left(\dfrac{u_h^{k-1}}{\theta \Delta t}, v_h\right)$ to both parties of the inequalities (5.13), we get

$$\begin{aligned} &\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right) = \left(f^{\ \theta,k} + \frac{u_h^{\theta,k-1}}{\theta\Delta t}, v_h\right)_\Omega + \\ &+ \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0}, \ \ v_{ih} \in V_h. \end{aligned} \tag{5.16}$$

Then, the problem (5.16) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$b\left(u_h^{\theta,k}, v_h\right) = \left(f^{i,\ \theta,k} + \mu u_h^{k-1}, v_h\right)_\Omega + \left(\varphi^{\theta,k}, v_h\right)_{\Gamma_0}, \ \ v_h, \ u_h^{\theta,k} \in V_h, \tag{5.17}$$

where

$$\begin{cases} b\left(u_h^{\theta,k}, v_h\right) = \mu\left(u_h^{\theta,k}, v_h\right)_\Omega + a\left(u_h^{\theta,k}, v_h\right), \ \ v_h \in V_h^i, \\[2ex] \mu = \dfrac{1}{\theta\Delta t} = \dfrac{p}{\theta T}. \end{cases} . \tag{5.18}$$

5.1.3 The space continuous for the generalized Schwarz method

We define the continuous counterparts of the continuous Schwarz sequences defined in (5.18), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)\,,\ m = 0, 1, 2, ..., i = 1, ..., M$ solution of

$$\begin{cases} b\left(u_1^{\theta,k,m+1}, v\right) = \\[1ex] \left(F^\theta\left(u_1^{\theta,k-1,m+1}\right), v\right)_{\Omega_1} + \left(\varphi^{\theta,k-1,m+1}, v\right)_{\Gamma_0}, \\[1ex] u_1^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\[1ex] D\dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_1^{\theta,k,m+1} = D\dfrac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 u_2^{\theta,k,m} \text{ on } \Gamma_1, \end{cases} \tag{5.19}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1, 2$.

In the next section, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions of generalized Schwarz alternating method, we get

$$\left(-\Delta u_1^{\theta,k,m+1}, v_1\right)_{\Omega_1} = \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1}$$

$$-\left(D\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\partial\Omega_1-\Gamma_1} + \left(D\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1}$$

$$= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} -$$

$$\left(D\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_2} + \overrightarrow{v}_1\ u_2^{\theta,k,m} - \overrightarrow{v}_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1}$$

$$= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\overrightarrow{v}_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1}$$

$$= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\overrightarrow{v}_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1}$$

$$-\left(D\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1},$$

thus the problem (5.19) equivalent to; find $u_1^{\theta,k,m+1} \in V_1$ such that

$$b(u_1^{\theta,k,m+1}, v_1) + \left(\overrightarrow{v}_1 u_1^{\theta,k,m}, v_1\right)_{\Gamma_1}$$
$$= \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} \tag{5.20}$$
$$+\left(D\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1$$

and we have $u_2^{\theta,k,m+1} \in V_2$

$$b(u_2^{\theta,k,m+1}, v_2) + \left(\overrightarrow{v}_2 u_2^{\theta,k,m+1}, v_2\right)_{\Gamma_2}$$
$$= \left(F(u_2^{\theta,k-1,m+1}), v_2\right)_{\Omega_2} + (\varphi, v)_{\Gamma_0} + \tag{5.21}$$
$$\left(D\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_2} + \overrightarrow{v}_2 u_1^{\theta,k,m}, v_2\right)_{\Gamma_2}.$$

5.2 A posteriori error estimate in continuous case

We define these auxiliary problems by of (5.19) with another problem in a nonoverlapping way over Ω. These auxiliary problems are needed for analysis and not for the computation section.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing,$$

$$\Omega = \Omega_2 \cup \Omega_4, \text{and } \Omega_2 \cap \Omega_4 = \varnothing.$$

Let $(u_1^{k,m}, u_2^{k,m})$ be the solution of problems (5.19), we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems

$$\begin{cases} \dfrac{u_1^{k,m+1} - u_1^{k-1,m+1}}{\Delta t} - D\Delta\ u_1^{\theta,k,m+1} - \overrightarrow{v_1}\nabla u_1^{\theta,k,m+1} + a_0^k u_1^{\theta,k,m+1} = \\ F^{\ \theta}\ \left(u_1^{\theta,k-1,m+1}\right) + \varphi^{\theta,k-1,m+1} \text{ in } \Omega_1, \\ u_1^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\ k = 1, ..., n, \\ D\dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_1^{\theta,k,m} = D\dfrac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_2^{\theta,k,m}, \text{ on } \Gamma_1 \end{cases} \tag{5.22}$$

and

$$\begin{cases} \dfrac{u_3^{k,m+1} - u_3^{k-1,m+1}}{\Delta t} - D\Delta\ u_3^{\theta,k,m+1} - \overrightarrow{v_1}\nabla u_3^{\theta,k,m+1} + a_0^k u_3^{\theta,k,m+1} \\ = F^{\ \theta}\ \left(u_3^{\theta,k-1,m+1}\right) + \varphi^{\theta,k-1,m+1} \text{in } \Omega_3, \\ u_3^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, k = 1, ..., n, \\ D\dfrac{\partial u_3^{\theta,k,m+1}}{\partial \eta_3} + \overrightarrow{v}_3 u_3^{\theta,k,m} = D\dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_3} + \overrightarrow{v}_3 u_1^{\theta,k,m}, \text{ on } \Gamma_2. \end{cases} \tag{5.23}$$

It can be taken $\epsilon_1^{\theta,k,m} = u_2^{\theta,k,m+1} - u_3^{\theta,k,m+1}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ of the problem (5.19) and (resp.(5.22) and (5.23) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [52] that is, $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ tend to $u_3^{\theta,k}$ (resp. $u_3^{\theta,k}$), then $\epsilon_1^{\theta,k,m}$ should tend to naught when m tends to infinity in V_2 .

By taking

$$\begin{aligned} \Lambda_3^{k,m} &= D\frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 u_2^{\theta,k,m} \\ &= D\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 u_3^{\theta,k,m} + D\frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 \epsilon_1^{\theta,k,m}, \end{aligned} \tag{5.24}$$

$$\Lambda_1^{k,m} = D\frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \overrightarrow{v}_3 u_1^{\theta,k,m}.$$

Using Green formula, (5.22) and (5.23) can be reformulated to the following system of elliptic variational equations

$$
\begin{aligned}
& b(u_1^{\theta,k,m+1}, v_1) + \left(\overrightarrow{\vartheta}_1 u_1^{\theta,k,m}, v_1\right)_{\Gamma_1} \\
& = \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} \\
& + \left(\Lambda_3^{k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1
\end{aligned}
\tag{5.25}
$$

and

$$
\begin{aligned}
& b(u_3^{\theta,k,m+1}, v_3) + \left(\overrightarrow{\vartheta}_3 u_3^{\theta,k,m+1}, v_3\right)_{\Gamma_1} \\
& = \left(F^\theta(u_3^{\theta,k-1,m+1}), v_3\right)_{\Omega_3} + (\varphi, v)_{\Gamma_0} \\
& + \left(\Lambda_1^{k,m}, v_3 - u_3^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_3 \in V_3.
\end{aligned}
\tag{5.26}
$$

On the other hand by taking

$$
\theta_1^{k,m} = D\frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{\theta,k,m},
\tag{5.27}
$$

we get

$$
\Lambda_3^{\theta,k,m} = D\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m}.
\tag{5.28}
$$

Using (5.27) we have

$$
\begin{aligned}
\Lambda_3^{k,m+1} & = D\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i\theta,k,m} + \theta_1^{k,m+1} \\
& = -D\frac{\partial u_3^{\theta,k,m}}{\partial \eta_3} + \overrightarrow{\vartheta}_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\
& = \overrightarrow{\vartheta}_3 u_3^{\theta,k,m} - D\frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} - \overrightarrow{\vartheta}_3 u_1^{\theta,k,m} \\
& + \overrightarrow{\vartheta}_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\
& = (\overrightarrow{\vartheta}_1 + \overrightarrow{\vartheta}_3) u_3^{\theta,k,m} - \Lambda_1^{k,m} + \theta_1^{k,m+1}
\end{aligned}
\tag{5.29}
$$

and the last equation in (5.29), we have

$$
\begin{aligned}
& \Lambda_1^{k,m+1} = -\frac{\partial u_1^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{\vartheta}_3 u_1^{\theta,k,m} = \overrightarrow{\vartheta}_1 u_1^{\theta,k,m} - \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} - \overrightarrow{\vartheta}_1 u_2^{\theta,k,m} + \\
& \overrightarrow{\vartheta}_3 u_1^{\theta,k,m} + \overrightarrow{\vartheta}_3 u_1^{\theta,k,m} = (\overrightarrow{\vartheta}_1 + \overrightarrow{\vartheta}_3) u_1^{\theta,k,m} - \Lambda_3^{k,m} + \theta_3^{k,m+1}.
\end{aligned}
\tag{5.30}
$$

Lemma 12. *Let* $u_s^k = u_{\Omega s}^k$, $e_s^{\theta,k,m+1} = u_s^{\theta,k,m+1} - u_s^k$ *and* $\eta_s^{k,m+1} = \Lambda_s^{k,m+1} - \Lambda_s^k$.

Then for $s,t=1,3, s\neq t$, we have

$$\begin{aligned} & b_s(e_s^{\theta,k,m+1}, v_s - e_s^{\theta,k,m+1}) + \left(\overrightarrow{v}_s e_s^{\theta,k,m+1}, v_s - e_s^{\theta,k,m+1}\right)_{\Gamma s} \\ & = \left(\eta_t^{k,m}, v_s - e_s^{\theta,k,m+1}\right)_{\Gamma_s}, \forall v_s \in V_s; \ \text{with } D\overrightarrow{v}^{-1} = \overrightarrow{e} \end{aligned} \tag{5.31}$$

and

$$\left(\eta_s^{k,m+1}, \psi^i\right)_{\Gamma_s} = \left((\overrightarrow{v}_s + \overrightarrow{v}_t)e_s^{k,m+1}, v_s\right)_{\Gamma_s} - \left(\eta_t^{k,m}, \psi\right)_{\Gamma_s} + \left(\theta_t^{k,m+1}, \psi\right)_{\Gamma s}, \forall \psi \in V_s. \tag{5.32}$$

Proof. The proof is very similar to that in [14]. □

Lemma 13. *By letting C be a generic constant which has different values at different places, we get for $s,t=1,3, s\neq t$*

$$\left(\eta_s^{k,m-1} - \overrightarrow{v}_s e_s^{k,m}, w\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m}\right\|_{1,\Omega s} \|w\|_{W_1}, \tag{5.33}$$

and

$$\left(\overrightarrow{v}_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1}\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m+1}\right\|_{1,\Omega_s} \|w\|_{W_1}, \tag{5.34}$$

where C is a constant independent of h and k.

Proof. The proof is very similar to that in [14]. □

Proposition 15. *[14] For the sequences $(u_1^{k,m}, u_3^{k,m})_{m\in\mathcal{N}}$ solutions of (5.23) and (5.24) we have the following a posteriori error estimation*

$$\left\|e_1^{\theta,k,m+1}\right\|_{1,\Omega_1} + \left\|e_3^{\theta,k,m}\right\|_{3,\Omega_3} \leqslant C \left\|u_1^{\theta,k,m+1} - u_3^{\theta,k,m}\right\|_{W_1}, \tag{5.35}$$

where C is a constant independent ofh and k.

Proposition 16. *For the sequences $(u_2^{\theta,k,m+1}, u_4^{\theta,k,m+1})_{m\in\mathcal{N}^*}$. We get the the similar following a posteriori error estimation*

$$\left\|e^{\theta,k,m+1}\right\|_{2,\Omega_2} + \left\|e^{\theta,k,m}\right\|_{4,\Omega_4} \leqslant C \left\|u_2^{\theta,k,m+1} - u_4^{\theta,k,m+1}\right\|_{W_2}. \tag{5.36}$$

where C is a constant independent ofh and k.

Proof. The proof is very similar to proof of Proposition 2 which proved in our published paper on [14]. □

Theorem 33. *Let $u_s^{\theta,k} = u_{\Omega_s}^{\theta,k}, s=1,2$. For the sequences $(u_1^{\theta,k,m+1}, u_2^{\theta,k,m+1})_{m\in U^*(F)2115}$ solutions of problems (5.20) and (5.21), one have the following result*

$$\begin{aligned} & \left\|u_1^{\theta,k,m+1} - u_1^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_2^{\theta,k,m} - u_2^{\theta,k}\right\|_{2,\Omega_2} \leqslant \\ & C \ (\left\|u_1^{\theta,k,m+1} - u_2^{\theta,k,m}\right\|_{W_1} + \left\|u_1^{\theta,k,m} - ui,_1^{\theta,k,m+1}\right\|_{W_2} + \\ & + \left\|e_1^{k,m}\right\|_{W_1} + \left\|e_2^{k,m+1}\right\|_{W_2}), \end{aligned}$$

where C is a constant independent of h and k.

5.3 A Posteriori Error Estimate: discrete Case

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_s $s = \{1, ..., m(h)\}$ defined by $\varphi_l(M_j) = \delta_{lj}$, where M_j is a vertex of the considered triangulation.

In the first step, we approach the space H_0^1 by a suitable discretization space of finite dimensional $V^h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements $u_h^{\theta,n} \in V^h$ which approaches $u_h(t_n, .)$, $t_n = n\Delta t$, $k = 1, ..., n$, with initial data $u_h^0 = u_{0h}$.

Let $u_h^{\theta,k,m+1} \in V^h$ be the solution of the discrete problem associated with (5.19)

$$u_{s,h}^{\theta,k,m+1} = u_{h,\Omega_s}^{\theta,k,m+1}.$$

We construct the sequences $(u_{s,h}^{\theta,k,m+1})_{m\in mathcalN}, u_{s,h}^{\theta,k,m+1} \in V_s^h, (s = 1, 2)$ solutions of discrete problems associated with (5.25).

We define the discrete space K_h is a suitable set given by

$$K_h = \begin{cases} u_h \in \left(L^2(0, T, H_0^1(\Omega)) \cap C\left(0, T, H_0^1\left(\overline{\Omega}\right)\right)\right), \\ \\ u_h = 0 \text{ in } \Gamma, \quad \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \, u_h = 0 \text{ in } \Gamma \backslash \Gamma_0, \end{cases} \tag{5.37}$$

where r_h is the usual interpolation operator defined by $r_h v = \sum\limits_{i=1}^{m(h)} v(M_j)\varphi_i(x)$.

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for (Ω_1, Ω_3) the following full-discrete problems: find $u_{1,h}^{\theta,k,m+1} \in K_h$ solution of

$$\begin{cases} b(u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h}) + \left(\overrightarrow{v}_{1,h} u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h}\right)_{\Gamma_1}, \\ \\ = \left(F^\theta(u_{1,h}^{\theta,k-1,m+1}), \tilde{v}_{1,h}\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0}, \\ \\ u_{1,h}^{\theta,k,m+1} = 0, \text{ on} \partial\Omega_1 \cap \partial\Omega, \ tildev_{1,h} \in K_h \\ \\ D\dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_{1,h}^{\theta,k,m+1} = D\dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0, \end{cases} \tag{5.38}$$

by taking the trial function $\tilde{v}_{1,h} = v_{1,h} - u_{1,h}^{\theta,k,m+1}$ in (5.38), we get

$$\begin{cases} b(u_{1,h}^{\theta,k,m+1}, v_{1,h}) + \left(\overrightarrow{v}_{1,h} u_{1,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} = \left(F(u_{1,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_1} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{1,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \ v_{1,h} \in K_h, \\ D\dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \overrightarrow{v}_1 u_{1,h}^{\theta,k,m+1} = D\dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \overrightarrow{v}_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0. \end{cases} \tag{5.39}$$

Similarly, we get

$$\begin{cases} b(u_{3,h}^{\theta,k,m+1}, v_{1,h}) + \left(\overrightarrow{v}_{3,h} u_{3,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} = \left(F^{\theta}(u_{3,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_3} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{3,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ D\dfrac{\partial u_{3,h}^{\theta,k,m+1}}{\partial \eta_3} + \overrightarrow{v}_3 u_{3,h}^{\theta,k,m+1} = D\dfrac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \overrightarrow{v}_3 u_1^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0. \end{cases} \tag{5.40}$$

For (Ω_2, Ω_4), are similar in (5.39) and (5.40).

We can obtain the discrete counterparts of propositions 1 and 2 by doing almost the same analysis as in section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{3,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k} \right\|_{1,\Omega_3} \leqslant C \left\| u_{1,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k,m} \right\|_{W_1} \tag{5.41}$$

and

$$\left\| u_{2,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} + \left\| u_{4,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k} \right\|_{1,\Omega_4} \leqslant C \left\| u_{2,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k,m} \right\|_{W_2}. \tag{5.42}$$

Similar to that in the proof of Theorem 2 we get the following discrete estimates

$$\begin{aligned} &\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{\theta,k,m} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} \leqslant \\ &C \; (\left\| u_{1,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k,m} \right\|_{W_1} + \left\| u_{2,h}^{\theta,k,m} - u_{1,h}^{\theta,k,m} \right\|_{W_2} \\ &+ \left\| e_{1,h}^{k+1,m} \right\|_{W_1} + \left\| e_{2,h}^{k+1,m} \right\|_{W_2}). \end{aligned} \tag{5.43}$$

Next we will obtain an error estimate between the approximated solution $u_{s,h}^{\theta,k,m+1}$ and the semi discrete solution in time $u^{i,\theta,k}$. We introduce some necessary notations. We denote by

$$\varepsilon_h = \{E \in T : T \in \tau_h \text{ and } E \notin \partial\Omega\},$$

and for every $T \in \tau_h$ and $E \in \varepsilon_h$, we define as

$$\omega_T = \{T' \in \tau_h : T' \cap T \neq \varnothing\}, \quad \textit{and } \omega_E = \{T' \in \tau_h : T' \cap E \neq \varnothing\}.$$

The right hand side f is not necessarily continuous function across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E.

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

5.4 An asymptotic behavior for the problem

Theorem 34. *Let* $u_s^{\theta,k} = u^{\theta,k} \mid_{\Omega_s}$ *where* u *is the solution of problem (5.1), the sequences* $\left(u_{1,h}^{\theta,k,m+1}, u_{2,h}^{\theta,k,m}\right)_{m\in U^*(F)}$ *are solutions of the discrete problems (5.25) and (5.26). Then there exists a constant* C *independent of* h *such that*

$$\left\|u_{1,h}^{\theta,k,m+1} - u_1^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{\theta,k,m} - u_2^{\theta,k}\right\|_{1,\Omega_2} \leqslant C\left\{\sum_{i=1}^{2}\sum_{T\in\tau_h}\left(\eta_i^T\right) + \eta_{\Gamma_s}\right\}, \tag{5.44}$$

where

$$\eta_{\Gamma_s} = \left\|u_{h,s}^{\theta,k,*} - u_{h,t}^{i,\theta,k,*-1}\right\|_{W_{h,s}} + \left\|\epsilon_{i,h}^{\theta,k,*}\right\|_{W_{h,s}},$$

and

$$\eta_s^T = h_T\left\|\begin{array}{c} F\left(u_{h,s}^{\theta,k-1,*}\right) + u_{h,s}^{\theta,k-1} + \\ \Delta\ u_{h,s}^{\theta,k,*} - \left(1+\lambda a_{h0}^k\right) u_{h,s}^{\theta,k} \end{array}\right\|_{0,T} + \sum_{E\in\varepsilon_h} h_E^{U^*(F)bd}\left\|\left[\frac{\partial u_{h,s}^{\theta,k,*}}{\partial \eta_E}\right]\right\|_{0,E},$$

where C *is a constant independent of* h *and* k *and the symbol* $*$ *is corresponds to* $m+1$ *when* $s=1$ *and to* m *when* $s=2$.

Proof. The proof is based on the technique of the residual a posteriori estimation see [52] and Theorem 3. We give the main steps by the triangle inequality we have

$$\sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k,*}\right\|_{1,\Omega_s} \leqslant \sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{1,\Omega_s} + \sum_{s=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s}. \tag{5.45}$$

The second term on the right hand side of (5.45) is bounded by

$$\sum_{s=1}^{2}\sum_{i=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s} \leqslant C\sum_{s=1}^{2}\eta_{\Gamma_s}.$$

To bound the first term on the right hand side of (5.45) we use the residual equation and apply the technique of the residual a posteriori error estimation [52] , to get for $v_h \in V^h$

$$b(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) = b(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s - v_{h,s})$$

$$\leq \sum_{T\subset\Omega_s} \int_T \left(\begin{array}{c} F^\theta\left(u_{h,s}^{\theta,k-1}\right) + u_{h,s}^{\theta,k-1} + \mu D\Delta\ u_{h,s}^{\theta,k} - \\ \left(1+\mu a_{h0}^k \overrightarrow{\vartheta}\right) u_{h,s}^{\theta k} \end{array} \right) (v_s - v_{h,s})\, ds$$

$$-\sum_{E\subset\Omega_s} \int_E \left[D\frac{\partial u_{h,s}^{\theta k}}{\partial \eta_E} \right] (v_s - v_{h,s})\, ds$$

$$-\sum_{E\subset\Gamma_s} \int_E D\frac{\partial u_{h,s}^{\theta k}}{\partial \eta_E} (v_s - v_{h,s})\, ds'$$

$$+\sum_{E\subset\Omega_s} \int_T \left(F^\theta\left(u_s^{\theta,k}\right) - F^\theta\left(u_{h,s}^{\theta k}\right)\right)(v_s - v_{h,s}) d\sigma$$

$$+\left(D\frac{\partial u_{h,s}^{\theta k}}{\partial \eta_s}, v_s - v_{h,s} \right)_{\Gamma_s},$$

where $F^\theta\left(u_{h,s}^{\theta,k}\right)$ is any approximation of $F^\theta\left(u_s^{\theta,k}\right)$. Therefore,

$$\begin{aligned}
&\sum_{s=1}^{2} c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) \\
&\leq \sum_{s=1}^{2}\sum_{T\subset\Omega_s} \left\| \begin{array}{c} F^\theta\left(u_{h,s}^{\theta,k}\right) + u_{h,s}^{\theta,k-1} + \mu D\Delta\ u_{h,s}^{\theta,k} \\ -\left(1+\mu a_{h0}^k \overrightarrow{\vartheta}\right) u_{h,s}^{\theta,k} \end{array} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} \\
&+\sum_{s=1}^{2}\sum_{E\subset\Omega_s} \left\| \left[D\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} \right] \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} + \sum_{s=1}^{2}\sum_{E\subset\Gamma_s} \left\| D\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} \\
&+\sum_{s=1}^{2}\sum_{T\subset\Omega_s} c \left\| u_s^{\theta,k} - u_{h,s}^{\theta,k} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} + \sum_{s=1}^{2}\sum_{T\subset\Omega_s} \left\| D\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_s} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T}.
\end{aligned} \tag{5.46}$$

Using the following fact

$$\left\| u_s^{\theta,k} - u_{h,s}^{\theta,k} \right\|_{1,\Omega_s} \leqslant \sup_{v_s^i \in K} \frac{c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s + ch_s^T)}{\|v_s + ch_s^T\|_{1,\Omega_i}},$$

we get

$$\sum_{s=1}^{2} c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s + ch_s^{i,T}) \leq \sum_{s=1}^{2} \left(\sum_{T\subset\Omega_s} \eta_s^{i,T} \right) \sum_{s=1}^{2} \|v_s\|_{1,\Omega_s}. \tag{5.47}$$

Finally, by combining (5.44), (5.45) and (5.46) the required result follows. □

5.4.1 A fixed point mapping associated with discrete problem

We define for $i = 1, 2, 3, 4$ the following mapping:

$$\begin{aligned} T_h : V_{i,h} &\longrightarrow H_0^1(\Omega_i) \\ W_i &\longrightarrow TW_i = \xi_{h,i}^{k,m+1} = \partial_h\left(F\left(w_i\right)\right), \end{aligned} \tag{5.48}$$

where $\xi_{h,i}^{k}$ is the solution of the following problem:

$$\begin{cases} b_i(\xi_{i,h}^{k,m+1}, v_i) + \left(\overrightarrow{v_i}\,\xi_{i,h}^{k,m+1}, v_{i,h}\right)_{\Gamma_i} = (F(w_i), v_{i,h})_{\Omega_i}, \\ \xi_{i,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ D_i \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} = \varphi^{k,m+1} \text{ in } \Gamma_0 \\ D_i \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \overrightarrow{v_i}\,\xi_{i,h}^{k,m+1} = D_i \dfrac{\partial \xi_{j,h}^{k,m}}{\partial \eta_i} + \overrightarrow{v_i}\,\xi_{j,h}^{k,m}, \ \text{on } \Gamma_i, \ i = 1, ..., 4, \ j = 1, 2. \end{cases} . \tag{5.49}$$

5.4.2 An iterative discrete algorithm

We choose the initial data $u_h^{i,0} = u_{h0}^{i}$ as a solution of the following discrete equation

$$b^i\left(u_{h,i}^0, v_h\right) = \left(g_i^0, v_h\right), \ v_h \in V^h, \tag{5.50}$$

with $g^{i,0}$ is a linear and a regular function. Now, we give the following discrete algorithm

$$u_{i,h}^{k,m+1} = T_h u_{i,h}^{k-1,m+1}, k = 1, ..., n, \ i = 1, ..., 4, \tag{5.51}$$

where $u_{i,h}^{k}$ is the solution of the problem (5.49).

Proposition 17. *Let $\xi_h^{i,k}$ be a solution of the problem (5.49) with the right hand side $F^i\left(w_i\right)$ and the boundary condition $D_i \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \overrightarrow{v_i}\,\xi_{i,h}^{k,m+1}$, $\tilde{\xi}_h^{i,k}$ the solution for $\tilde{F}^i$ and $D_i \dfrac{\partial \tilde{\xi}_{i,h}^{k,m+1}}{\partial \eta_i} + \overrightarrow{v_i}\,\tilde{\xi}_{i,h}^{k,m+1}$. The mapping T_h is a contraction in $V_{i,h}$ with the rate of contraction $\dfrac{\lambda}{\frac{(\Delta t)\,\beta}{D} + \lambda}$. Therefore, T_h admits a unique fixed point which coincides with the solution of the problem (5.49).*

Proof. We note that

$$\|W\|_{H_0^1(\Omega_i)} = \|W\|_1 .$$

Setting

$$\phi = \frac{1}{\frac{(\Delta t)\,\beta}{D} + \lambda} \left\|F\left(w_i\right) - F\left(\tilde{w}_i\right)\right\|_1 .$$

Then, we have $\xi_{i,h}^{k,m+1}+\phi$ is a solution of

$$\begin{cases} b\left(\xi_{i,h}^{k,m+1}+\phi,(v_{i,h}+\phi)\right)=\left(F(w_i)+\alpha_i\phi,(v_{i,h}+\phi)\right), \\ \xi_{i,h}^{k,m+1}=0, \quad \text{on } \partial\Omega_i\cap\partial\Omega, \\ D\dfrac{\partial\xi_{i,h}^{k,m+1}}{\partial\eta_i}+\alpha_i\xi_{i,h}^{k,m+1}=D\dfrac{\partial\xi_{j,h}^{k,m}}{\partial\eta_i}+\alpha_i\xi_{j,h}^{k,m}, \text{ on } \Gamma_i, \ i=1,...,4, \ j=1,2. \end{cases}$$

On the other hand, we have

$$F(w_i)\le F(\tilde{w}_i)+\|F(w_i)-F(\tilde{w}_i)\|_1 \le F(\tilde{w}_i)+\frac{\alpha}{\beta+\lambda}\|F(w_i)-F(\tilde{w}_i)\|_1 \le F(\tilde{w}_i)+a\phi.$$

It is very clear that if $F^i(w_i)\geqq F^i(\tilde{w}_i)$ then $\xi_{i,h}^{k,m+1}\geqq\tilde{\xi}_{i,h}^{k,m+1}$. Thus

$$\xi_{i,h}^{k,m+1}\le\tilde{\xi}_{i,h}^{k,m+1}+\phi.$$

But the role of w_i and $\tilde{w}_i$ are symmetrical, thus we have a similar prof

$$\tilde{\xi}_{i,h}^{k,m+1}\le\xi_{i,h}^{k,m+1}+\phi,$$

yields

$$\begin{aligned} \|T(w)-T(\tilde{w})\|_\infty &\le \frac{1}{\frac{(\Delta t)\beta}{D}+\lambda}\|F(w_i)-F(\tilde{w}_i)\|_1 \\ &= \frac{1}{\frac{(\Delta t)\beta}{D}+\lambda}\|f^i+\lambda w_i-f^i-\lambda\tilde{w}_i\|_1 \\ &\le \frac{\lambda}{\frac{(\Delta t)\beta}{D}+\lambda}\|w_i-\tilde{w}_i\|_1 . \end{aligned}$$

□

Proposition 18. *Under the previous hypotheses and notations, we have the following estimate of convergent*

$$\left\|u_{i,h}^{n,m=1}-u_{i,h}^{\infty,m=1}\right\|_1\le\left(\frac{1}{1+\dfrac{(\Delta t)\beta}{D}}\right)^n\left\|u_{i,h}^{\infty,m=1}-u_{i,h_0}\right\|_1, \ k=0,...,n \tag{5.52}$$

where $u^{\infty,m+1}$ is an asymptotic continuous solution and u_{i,h_0} is a solution of (5.50).

Proof. We have

$$u_h^{i,\infty}=T_hu_h^{i,\infty},$$

$$\left\|u_{i,h}^{1,m+1}-u_{i,h}^{\infty,m+1}\right\|_1=\left\|T_hu_{i,h}^{0,m+1}-T_hu_{i,h}^{\infty,m+1}\right\|_1\le\left(\frac{1}{1+\dfrac{(\Delta t)\beta}{D}}\right)\left\|u_{i,h}^{i,0}-u_{i,h}^{\infty,m+1}\right\|_1$$

and also we have

$$\left\|u_h^{n+1,m+1} - u_h^{i,\infty}\right\|_1 = \left\|T_h u_{i,h}^{n,m+1} - T_h u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\dfrac{(\Delta t)\,\beta}{D}}\right) \left\|u_{i,h}^{n,m+1} - u_{i,h}^{i,\infty}\right\|_1 .$$

Then

$$\left\|u_{i,h}^{n,m+1} - u_{i,h}^{\infty}\right\|_1 \leq \left(\frac{1}{1+\dfrac{(\Delta t)\,\beta}{D}}\right)^n \left\|u_{i,h}^{\infty,m+1} - u_{i,h_0}\right\|_1 .$$

□

Theorem 35. *Under the previous hypotheses, notations and results, we have for* $i = 1, ..., 4$, $k = 1, ..., n$, $m = 1, 2, ...$

$$\left\|u_{i,h}^{n,m+1} - u^{\infty}\right\|_1 \leq C \left[\begin{array}{c} \left\|u_{1,h}^{k,m+1} - u_{2,h}^{k,m}\right\|_{W_1} + \left\|u_{2,h}^{k,m} - u_{1,h}^{k,m-1}\right\|_{W_2} + \left\|e_{1,h}^{n+1,m}\right\|_{W_1} \\ \\ + \left\|e_{2,h}^{n+1,m-1}\right\|_{W_2} + \left(\dfrac{1}{1+\dfrac{(\Delta t)\,\beta}{D}}\right)^n . \end{array}\right] \tag{5.53}$$

Proof. Theorem can be easily proved by using the results of Theorem 1 and Proposition 4. □

General conclusion

In this thesis, we have presented a general approach to error estimates. A posteriori error estimates for the generalized Shwarz method with Dirichlet boundary conditions on the interfaces for advection-diffusion equation with second order boundary value problems are derived using Euler time scheme combined with Galerkin spatial method. Furthermore, a result of asymptotic behavior in uniform norm is deduced by using Benssoussan- Lion's algorithm. Then, a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary conditions on the interfaces for parabolic equation with second order boundary value problems are studied using theta time scheme combined with a Galerkin approximation. Furthermore, a result of an asymptotic behavior using H_0^1 -norm is presented using Benssoussan-Lion's Algorithm. These error bounds can be evaluated by numerically solving versions of the problems. In future, . The geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate for linear and a new class of non linear elliptic PDEs will be established and the results of some numerical experiments will be presented to support the theory. Moreover, we will try to connect concept the proposed area with a mechanic fluid such as a compressible single and two-phase flows which extensively studied in ([56]-[57]). The results we obtained encouraging us to extend our study to a wide class of advection-diffusion equation. Finally, we will expand all results to problems and re-work with spectral methods.

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