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ABSTRACT

In this thesis, we investigate the blow-up and global existence of solutions to the following time fractional nonlinear diffusion equations

$$\begin{cases} {}_0^C D_t^\alpha u - \Delta u = |u|^{p-1} u, x \in R^N, t > 0, \\ u(0,x) = u_0(x), x \in R^N, \end{cases}$$

where $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0(R^N)$ and ${}_0^C D_t^\alpha u = (\partial/\partial t) {}_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, ${}_0 I_t^{1-\alpha}$ denotes left Riemann-Liouville fractional integrals of order $1-\alpha$. We prove that if $1 < p < 1 + 2/N$, then every nontrivial nonnegative solution blow-up in finite time, and if $p \geq 1 + 2/N$, and $\|u_0\|_{L^{q_c}(R^N)}$, $q_c = N(p-1)/2$ is sufficiently small, then the problem has global solution.

Introduction

This thesis is concerned with the blow-up and global existence of solutions to the following Cauchy problems for time fractional diffusion equation

$$ {}_0^C D_t^\alpha u - \Delta u = |u|^{p-1} u, x \in R^N, t > 0, \tag{1} $$

$$ u(0,x) = u_0(x), x \in R^N. \tag{2} $$

where $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0(R^N) = \left\{ u \in C(R^N) : \lim_{|x|\to\infty} u(x) = 0 \right\}$ and ${}_0^C D_t^\alpha u = (\partial/\partial t) {}_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, ${}_0 I_t^{1-\alpha}$ denotes left Riemann-Liouville fractional integrals of order $1-\alpha$ and is defined by

$$ {}_0 I_t^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s)\, ds. $$

When $\alpha = 1$, the problem (1)-(2) reduces to the semilinear heat equation

$$ u_t - \Delta u = |u|^{p-1} u, x \in R^N, t > 0, \tag{3} $$

with (2). Fujita showed that if $1 < p < 1 + 2/N$ and $u_0 \neq 0$, then every solution of (3)-(2) blows up in a finite time. If $p > 1 + 2/N$, then for initial values bounded by a sufficiently small Gaussian, that is for $\tau > 0$, there is $\varepsilon = \varepsilon(\tau) > 0$ such that if $0 \leq u_0(x) \leq \varepsilon G_\tau(x)$, then the solution of (3)-(2) is global. The critical case $p = 1 + 2/N$ was later proved to be in the blow-up category. Weissler proved that if the initial value u_0 is small enough in $L^{q_c}(R^N)$, $q_c = N(p-1)/2 > 1$, then the solution of (3)-(2) exists globally.
Kirane, Laskyi an Tatar studied the following evolution problem

$$ {}_0^C D_t^\alpha u + (-\Delta)^{\beta/2} u = h(x,t) |u|^{1+\sim p}, x \in R^N, t > 0, \tag{4} $$

with (2), where $0 < \alpha < 1$, $0 < \beta \leq 2$, ${}_0^C D_t^\alpha u = (\partial/\partial t) {}_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, $\sim p > 0$, h satisfies $h(x,t) \geq C_h |x|^\sigma t^\rho$ for $x \in R^N$, $t > 0$, $C_h > 0$, and σ, ρ satisfy some conditions. $(-\Delta)^{\beta/2} u =^{-1} (|\zeta|^\beta (u))$, where denotes Fourier transform and $^{-1}$ denotes its inverse. They obtained that if $0 < \sim p \leq (\alpha(\beta + \sigma) + \beta\rho)/(\alpha N + \beta(1-\alpha))$, then the problem (4)-(2) admits no global weak nonnegative solution other than the trivial one.
Cazenave, Dickstein and Weissler considered the followin heat equation with nonlinear memory,

$$ u_t - \Delta u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u\, ds, x \in R^N, t > 0, \tag{5} $$

with (2), where $p > 1$, $0 \leq \gamma < 1$, and $u_0 \in C_0(R^N)$.
Let $p_\gamma = 1 + 2(2-\gamma)/(N - 2 + 2\gamma)_+$, $(N-2+2\gamma)_+ = \max\{0, N-2+2\gamma\}$ and $p_* = \max\{1/\gamma, p_\gamma\} \in (0,\infty]$. They obtained that if $p \leq p_*$, $u_0 \geq 0$, $u_0 \neq 0$, then the solution u of (5)-(2) blows up in finite time and if $p > p_*$ and $u_0 \in L^{q_{sc}}(R^N)$, $q_{sc} = N(p-1)/(4-2\gamma)$ with $\|u_0\|_{L^{q_{sc}}(R^N)}$ sufficiently small, then the solution

exists golbally.
Fino and Kirane discussed the following equation

$$u_t + (-\Delta)^{\beta/2}u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} uds, x \in R^N, t > 0, \tag{6}$$

with (2), where $0 < \beta \leq 2$, $0 \leq \gamma < 1$, they got the blow-up and global existence results by using the test function method. The method based on rescalings of a compactly support test function to prove the blow-up results which is used by Mitidieri and Pohozaev to show the blow-up results.

Chapter 1

Preliminaries

In this chapter, we present some preliminaries that will be used in the next chapters.

1.1 Functional analysis

1.1.1 L^p spaces

Definition 1 *Let $X = [a,b]$ provided with the Borel tribe and a measure on (X, B_X). For $1 \leq p < \infty$, We denote by $L^p(X,x)$ the set of measurable functions $f : X \to R$ as*

$$\|f\|_p = \left(\int_X |f|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

It is clear that $L^1(X,x)$ is a vector space. To obtain a similar result in the case $p > 1$, We need the following theorem.

Theorem 2 *Let $p, q \in]1, \infty[$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. So for any measurable functions $f, g : X \to R$ we have*

$$\left|\int_X f\, g\, dx\right| \leq \|f\|_p \|g\|_q \ (Hölder).$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \ (Minkowski).$$

Proof. We first demonstrate the inequality of Hölder. Without loss of generality, we can suppose that $\|f\|_p = \|g\|_q = 1$. For everyone $x, y \geq 0$, we have

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Then

$$\left|\int_X f\,g\,dx\right| \le \int_X |fg|\ dx \le \int_X \left(\frac{|f|^p}{p} + \frac{|g|^q}{q}\right) dx = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = 1.$$

Let us now show Minkowski's inequality. We obtain

$$\begin{aligned}
&\|f+g\|_p^p \\
&\qquad = \int_X |f+g|^p\ dx \le \int_X |f+g|^{p-1}\left(|f|+|g|\right)\ dx \\
&\qquad \le \left(\int_X |f+g|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_X |f|^p\ dx\right)^{\frac{1}{p}} + \left(\int_X |g|^p\ dx\right)^{\frac{1}{p}}.
\end{aligned}$$

This inequality immediately implies the desired result. ■

1.1.2 Banach space

A Banach space is a vector space X over the field R of real numbers, or over the field C of complex numbers, which is equipped with a norm and which is complete with respect to that norm, that is to say, for every Cauchy sequence $\{x_n\}$ in X, there exists an element x in X such that

$$lim_{n\to\infty} x_n = x,$$

or equivalently

$$lim_{n\to\infty} \|x_n - x\|_X = 0.$$

The vector space structure allows one to relate the behavior of Cauchy sequences to that of converging series of vectors. A normed space X is a Banach space if and only if each absolutely convergent series in X converges,

$$\textstyle\sum_{n=1}^{\infty} \|v_n\|_X < \infty \text{ implies that } \sum_{n=1}^{\infty} v_n \text{converges in } X.$$

Completeness of a normed space is preserved if the given norm is replaced by an equivalent one. All norms on a finite-dimensional vector space are equivalent. Every finite-dimensional normed space over R or C is a Banach space.

1.1.3 Complete Metric space

Definition 3 *Let (X, d) be a metric space. A sequence (x_n) in X is called a Cauchy sequence if for any $\varepsilon > 0$, there is an $n_\varepsilon \in N$ such that $d(x_m, x_n) < \varepsilon$ for any $m \ge n_\varepsilon$, $n \ge n_\varepsilon$.*

Theorem 4 *Any convergent sequence in a metric space is a Cauchy sequence.*

Proof. Assume that (x_n) is a sequence which converges to x. Let $\varepsilon > 0$ be given. Then there is an $N \in N$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Let $m, n \in N$ be such that $m \geq N$, $n \geq N$. Then

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (x_n) is a Cauchy sequence.
Then converse of this theorem is not true. For example, let $X = (0, 1]$. Then $(\frac{1}{n})$ is a Cauchy sequence which is not convergent in X. ■

Definition 5 *A metric space (X, d) is said to be complete if every Cauchy sequence in X converges (to a point in X).*

1.1.4 Theorem of Riesz-Thorin

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, let T:$L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu) \to L^{q_0}(Y,\upsilon) \cap L^{q_1}(Y,\upsilon)$, be a linear operator, and suppose that there exist positive real numbers c_0, c_1 such that, for all $u_0 \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu)$,

$$\|T(t)u_0\|_{L^{q_0}} \leq c_0 \|u_0\|_{L^{p_0}}, \ \|T(t)u_0\|_{L^{q_1}} \leq c_1 \|u_0\|_{L^{p_1}}.$$

Fixe a real number $0 < \lambda < 1$ and define the numbers p_λ, q_λ, c_λ by

$$\frac{1}{p_\lambda} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \frac{1}{q_\lambda} = \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}, c_\lambda = c_0^{1-\lambda} c_1^{\lambda}.$$

If $q_\lambda = \infty$ assume that (Y, B, υ) is semi-finite. Then

$$\|T(t)u_0\|_{L^{q_\lambda}} \leq c_\lambda \|u_0\|_{L^{p_\lambda}},$$

for all $u_0 \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu) \subset L^{p_\lambda}(X,\mu)$.

1.1.5 Hölder's inequality

Let

$$\frac{1}{p} + \frac{1}{q} = 1,$$

with $p, q > 1$. Then Hölder's inequality for integrals states that

$$\int_a^b |f(x)g(x)| \, dx \leq \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}} \left[\int_a^b (x)|^q \, dx\right]^{\frac{1}{q}},$$

with equality when

$$|g(x)| = c\,|f(x)|^{p-1}.$$

If $p = q = 2$, this inequality becomes Schwarz's inequality. Similarly, Hölder's inequality for sums states that

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}},$$

with equality when

$$|b_k| = c\,|a_k|^{p-1}.$$

If $p = q = 2$, this becomes Cauchy's inequality.

1.1.6 Young's inequality

Let f be a real-valued, continuous, and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, a in $[0, c]$, and b in $[0, f(c)]$, then

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab,$$

where f^{-1} is the inverse function of f. Equality holds iff $b = f(a)$. Taking the particular function $f(x) = \widehat{x}(p-1)$ gives the special case

$$\frac{a^p}{p} + \left(\frac{p-1}{p} \right) b^{\frac{p}{p-1}} \geq ab,$$

which is often written in the symmetric form

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab,$$

where $a, b \geq 0$, $p > 1$, and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

1.1.7 Gronwall's inequality

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality

$$u'(t) \leq \beta(t)\,u(t),\ t \in I^\circ,$$

then u is bounded by the solution of the corresponding differential equation $y(t) = \beta(t)y(t)$

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)\,\mathrm{d}s \right),$$

for all $t \in I$.

1.1.8 Fubini's theorem

Fubini's theorem, sometimes called Tonelli's theorem, establishes a connection between a multiple integral and a repeated one. If $f(x,y)$ is continuous on the rectangular region $R = \{(x,y) \in R^2 : a \leq x \leq b, c \leq y \leq d\}$, then the equality

$$\int\int_R f(x,y)\, d(x,y) = \int_a^b \int_c^d f(x,y)\, dy dx.$$

1.1.9 Uniform convergence

Definition 6 *A sequence of functions $\{f_n\}, n = 1, 2, 3, ...$ is said to be uniformly convergent to f for a set E of values of x, if for each $\varepsilon > 0$, an integer n_0 can be found such that*

$$|f_n(x) - f(x)| < \varepsilon,$$

for $n \geq n_0$ and all $x \in E$.
A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$S_n(x) = \sum_{k=0}^{n} f_k(x),$$

converges uniformly on E.
To test for uniform convergence, use Abel's uniform convergence test or the Weierstrass M-test. If individual terms $u_n(x)$ of a uniformly converging series are continuous, then the following conditions are satisfied.

Proposition 7 *1. The series sum*

$$f(x) = \sum_{n=1}^{\infty} u_n(x),$$

is continuous.
2. The series may be integrated term by term

$$\int_a^b f(x)\, dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx.$$

For example, a power series $\sum_{n=0}^{\infty}(x - x_0)^n$ is uniformly convergent on any closed and bounded subset inside its circle of convergence.
3. The situation is more complicated for differentiation since uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ does not tell anything about convergence of $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$. Suppose that $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in [a,b]$, that each $u_n(x)$ is differentiable on $[a,b]$, and that $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$ converges uniformly on [a,b]. Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$ to a function f, and for each $x \in [a,b]$,

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

1.1.10 Dominated convergence theorem

Theorem 8 *(Lebesgue dominated convergence theorem)*
Suppose $f_n : R \to [-\infty, +\infty]$ are (Lebesgue) mesurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrale $g : R \to [0, \infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in R$. Then f is integrable as is f_n for each n, and

$$\lim_{n\to\infty} \int_R f_n \, dx = \int_R \lim_{n\to\infty} f_n \, dx = \int_R f \, dx.$$

1.1.11 Semi group

Definition 9 *A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators, on X is said to be a semi graoupe on X, if it satisfies*
(a) $T(0) = I$.
(b) $T(t + s) = T(t)T(s)$, $t, s \geq 0$.

The Semigroup Property

By transitionally breaking down the process of evolution, it is evident that we can reach the state of the system at time $t+s$ by either going directly from the initial condition to the state at time $t+s$ or by allowing the state to evolve over s time units (taking a snapshot), and then allowing it to evolve t more time units. Here the $T(\cdot)$ is acting like a transition operator. The uniqueness of the solution gives reveals the semigroup property which is given by

$$T(t + s) = T(t)T(s) \, (t > 0, \, s > 0).$$

The semigroup property of the family of functions, $\{T(t); t \geq 0\}$, is a composition (not a multiplication). Notice that $T(0)$ is the identity operator I (i.e. there is no transition at time zero and the initial data exists).

More Properties

Now that we have seen the fundamental semigroup property, we want to understand how A (which governs the evolution of the system) and T relate to one another. We will first examine the scalar case. Two observations which may be preliminary indicators of the relationship are given as follows

$$T(t)(f) = T(t)(u(0)) = u(t) = e^{At} f,$$

and

$$\frac{d}{dt} T(t)(f) = A(T(t)(f)),$$

where A is the derivative of $T(t)$. In addition, each $T(t) : f \to e^{At} f$ is a continuous operator on R, (or in an infinite dimensional setting, a Banach space X), which indicates the continuous dependence of $u(t)$ on f. The initial data f

should belong to the domain of A. We have the following results:
a) $T(t)$ is a continuous function.
b) $T(0)f = f$.
c) $T(t) : R \to R$ is linear provided A is linear.
Again, since we are interested in linear semigroups, we will assume that A is linear. These observations bring for the notion of C_0 semigroups.

1.1.12 Semi-group strongly continuous on a Banach space

Definition 10 *A family of operators $(T(t))_{t\geq 0}$ of $L(X)$ is a strongly continuous semigroup on X when the following conditions are realized*
(a) $T(0) = I$,
(b) $T(t+s) = T(t)T(s)$ *for every* $t \geq 0$ *and all* $s \geq 0$,
(c) $\lim_{t\to 0} \|T(t)x - x\| = 0$, *for every* $x \in X$.

Theorem 11 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semi-group on X. Then there exist constants $w \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{wt},$$

for every $t \geq 0$.

Theorem 12 *If $(T(t))_{t\geq 0}$ is a strongly continuous semi-group on X then, for all $x \in X$, the application*

$$t \longmapsto S(t)x,$$

is continuous from $[0, \infty)$ in X.

Theorem 13 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semi-group over X and $(A, D(A))$ its infinitesimal generator. The following properties are verified*
(a) For all $x \in X$, we have

$$\lim_{t\to 0} \frac{1}{h} \int_t^{t+h} T(s)\, x\, ds = T(t)x.$$

(b) For every $x \in X$ and every $t > 0$, $\int_0^t T(s)x\, ds$ belongs to $D(A)$ and

$$A\left(\int_0^t \int_t^{t+h} T(s)x\, ds\right) = T(t)x - x.$$

(c) If $x \in D(A)$, then $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)x\, A.$$

(d) If $x \in D(A)$, then

$$T(t)x - T(s)x = \int_s^t T(\tau)A\, x\, d\tau = \int_s^t AT(\tau)x\, d\tau.$$

Definition 14 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semi-group on X. The generator infinitesimal of the semigroup $(T(t))_{t\geq 0}$, the unbounded operator $(A, D(A))$ defined by*

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \, exists \, in \, X \right\},$$

and

$$Ax = \lim_{t\to 0} \tfrac{T(t)x-x}{t} \text{ for all } x \in X.$$

1.1.13 Contractive semigroup

Definition 15 *A strongly continuous semi-group $(T(t))_{t\geq 0}$ over X is a semi-group of contractions if*

$$\|T(t)\| \leq 1, \text{ for all } t \geq 0.$$

Theorem 16 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semi-group over X and $(A, D(A))$ its infinitesimal generator. The following properties are verified*
(a) For all $x \in X$, we have

$$\lim_{t\to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x.$$

(b) For every $x \in X$ and every $t > 0$, $\displaystyle\int_0^t T(s)x \, ds$ belongs to $D(A)$ and

$$A\left(\int_0^t \int_t^{t+h} T(s)x \, ds \right) = T(t)x - x.$$

(c) If $x \in D(A)$ then $T(t)x \in D(A)$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)xA.$$

(d) If $x \in D(A)$ then

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax \, d\,\tau = \int_s^t AT(\tau)x \, d\tau.$$

Theorem 17 *(Hille-Yosida theorem 1) An unbounded linear operator $(A, D(A))$ in X is the infinitesimal generator of a semi-group of contractions on X if and only if the following conditions are satisfied*
(a) A is closed,
(b) $D(A)$ is dense in X,
(c) for everything $\lambda > 0$, $(\lambda I - A)$ is a bijective mapping of $D(A)$ into X, and $(\lambda I - A)^{-1}$ is A bounded operator on X satisfying

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{1}{\lambda}.$$

Theorem 18 *(Hille-Yosida theorem 2) An unbounded linear operator $(A, D(A))$ in X is the infinitesimal generator of a semi-group of contractions on X if and only if A is m-dissipative and dense domain in X.*

1.1.14 Laplace transform

The Laplace transform intervenes in the resolution of equations and differential systems.

Definition 19 *The Laplace transform of a function f of the real variable $t \in R^+$ is defined by*

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt, \lambda \in R.$$

$F(t)$ is called the original of $f(\lambda)$.
The Laplace transform of a function exists if the previous integral is convergent, for which the original must be exponential order a, i.e: there exists $M > 0$ such that

$$|f(t)| \leq Me^{at} f(t)\, dt, for\, t > T.$$

In this case, the Laplace transform exists for $Re(\lambda) > a$. The original $f(t)$ is called the inverse Laplace transform

$$\mathcal{L}^{-1}(\mathcal{L}f)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} f(\lambda)\, d\lambda,\ c > a.$$

Proposition 20 *The Laplace transform is linear i.e*

$$\mathcal{L}(f(t))(\lambda) = \mathcal{L}\left(\sum_{i=0}^{n} c_i f_i(t)\right)(\lambda) = \sum_{i=0}^{n} c_i \mathcal{L} f_i(\lambda).$$

Definition 21 *When the product $f(x - t)g(t)$ is integrable over any interval $[0, x]$ of R^+, the convolution product of f and g is defined by*

$$(f * g)(x) = \int_0^x f(x - t) g(t)\, dt.$$

Proposition 22 *If the Laplace transforms of f and g exist, then the Laplace transform of the convolution product satisfies*

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)\mathcal{L}(g).$$

Proposition 23 *The Laplace transform of the derivative of order $n \in N$ of the function f is given by*

$$\mathcal{L}(f^{(n)})(\lambda) = \lambda^n \mathcal{L}(f)(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1} f^{(k)}(0) = \lambda^k \mathcal{L}(f)(\lambda) - \sum_{k=0}^{n-1} \lambda^k f^{(n-k-1)}(0).$$

1.2 Calculates fractional

1.2.1 Gamma function

Definition 24

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,\ Re(z) > 0,\ z \in C.$$

From this definition it is clear that $\Gamma(z)$ *is analytic for* $Re(z) > 0$*. By using integration by parts we find that*

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t} t^z\, dt = -\int_0^\infty t^z\, de^{-t} = -e^{-t} t^z \mid_0^\infty + \int_0^\infty e^{-t}\, dt^z \\ &= z \int_0^\infty e^{-t} t^{z-1}\, dt = z\Gamma(z),\ Re(z) > 0.\end{aligned}$$

Hence we have.

Theorem 25

$$\Gamma(z+1) = z\Gamma(z),\ Re(z) > 0,\ z \in C.$$

Further we have

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\, dt = -e^{-t} \mid_0^\infty = 1. \\ \Gamma(n+1) &= n!,\ n = 0, 1, 2, 3, \ldots .\end{aligned}$$

Now we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},\ -1 < Re(z) \leq 0, z \neq 0.$$

Proposition 26 *(of the Gamma function)*

$$\begin{aligned}\frac{\Gamma(z+n)}{\Gamma(z)} &= z(z+1)(z+2)....(z+n-1), n \in N. \\ \frac{\Gamma(z)}{\Gamma(z-n)} &= z(z-1)(z-2)....(z-n), n \in N. \\ \frac{\Gamma(-z)}{\Gamma(-z-n)} &= (-1)^n \frac{\Gamma(1+z+n)}{\Gamma(1+z)},\ n\, non\, negative\, integer. \\ \lim_{n\to\infty} \frac{\Gamma(z+n)}{\Gamma(n) n^z} &= 1.\end{aligned}$$

1.2.2 Beta function

Definition 27

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt,\ Re(u) > 0,\ Re(v) > 0.$$

This integral is often called the beta integral. From the de nition we easily obtain the symmetry

$$B(u,v) = B(v,u),$$

since we have by using the substitution $t = 1 - s$

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt = -\int_1^0 (1-s)^{u-1}s^{v-1}\,ds \\ &= \int_0^1 s^{v-1}(1-s)^{u-1}\,ds = B(v,u). \end{aligned}$$

The connection between the beta function and the gamma function is given by the following theorem:

Theorem 28

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\ Re(u) > 0,\ Re(v) > 0.$$

In order to prove this theorem we use the definition of gamma function to obtain

$$\begin{aligned} \Gamma(u)\Gamma(v) &= \int_0^\infty e^{-t}t^{u-1}\,dt \int_0^\infty e^{-s}s^{v-1}\,ds \\ &= \int_0^\infty e^{-(t+s)}t^{u-1}s^{v-1}\,dtds. \end{aligned}$$

Now we apply the change of variables $t = xy$ *and* $s = x(1-y)$ *to this double integral. Note that* $t + s = x$ *and that* $0 < t < \infty$ *and* $0 < s < \infty$ *imply that* $0 < x < 1$ *and* $0 < y < 1$*. There exist many useful forms of the beta integral which can be obtained by an appropriate change of variables. For instance, if we set* $t = s = (s+1)$ *into* $B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$*, we obtain*

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt \\ &= \int_0^\infty s^{u-1}(s+1)^{-u+1}(s+1)^{-v+1}(s+1)^{-2}\,ds \\ &= \int_0^\infty \frac{s^{u-1}}{(s+1)^{u+v}}\,ds,\ Re(u) > 0,\ Re(v) > 0. \end{aligned}$$

1.2.3 Mittag-Leffler function

Definition 29 *The Mittag-Leffler function is defined for complex $z \in C$*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)},\ \alpha \in C,\ Re(\alpha) > 0,\ z \in C,$$

and its general form

$$E_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)},\ \alpha,\ \beta \in C,\ Re(\alpha) > 0,\ Re(\beta) > 0,\ z \in C.$$

Originally Mittag-Leffler assumed only the parameter and assumed it as positive, but soon later the generalization with two complex parameters was considered by Wiman. In both cases the Mittag-Leffler functions are entire of order $\frac{1}{Re(\alpha)}$. Generally $E_{\alpha,1}(z) = E_\alpha(z)$.

If $Re(\alpha) > 0$, $Re(\beta) > 0$ and $z \in C$. The following representations are obtained

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\,ds,$$

and

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\,ds,$$

where the path of integration sparates all the poles of $\Gamma(s)$ at the points $s = -v, v = 0, 1, \cdots$ from those of $\Gamma(1-s)$ at the points $s = 1+v, v = 0, 1, \cdots$. On evaluating the residues at the poles of the gamma function $\Gamma(1-s)$ we obtain the following analytic continuation formulas for the Mittag-Leffler functions:

$$\begin{aligned} E_\alpha(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\,ds \\ &= -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}. \end{aligned}$$

$$\begin{aligned} E_{\alpha,\beta}(z) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\,ds \\ &= -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(\beta-\alpha k)}. \end{aligned}$$

1.2.4 Some special cases

We begin our study by giving the special cases of the Mittag-Leffler function $E_\alpha(z)$

$$\begin{aligned}
E_0(z) &= \frac{1}{1-z},\ |z|<1.\\
E_1(z) &= E_{1,1}(z) = e^z.\\
E_2(z) &= \cosh(\sqrt{z}),\ z\in C.\\
E_2(-z^2) &= \cos(z),\ z\in C.\\
E_{1,2}(z) &= \frac{e^z-1}{z},\ z\in C.
\end{aligned}$$

We obtain

$$\begin{aligned}
E_{1,2}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}\\
&= \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z-1).\\
E_{2,2}(z) &= \frac{\sinh(\sqrt{z})}{\sqrt{z}},\ z\in C.\\
E_3(z) &= \frac{1}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right], z\in C.\\
E_{1,3}(z) &= \frac{e^z-z-1}{z^2},\ z\in C.
\end{aligned}$$

We obtain

$$\begin{aligned}
E_{1,3}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!} = \frac{1}{z^2}\left(e^z-z-1\right).\\
E_4(z) &= \frac{1}{2}\left[\cos\left(z^{\frac{1}{4}}\right) + \cosh\left(z^{\frac{1}{4}}\right)\right],\ z\in C.\\
E_{1/2}\left(\pm z^{\frac{1}{2}}\right) &= e^z\left[1+erf\left(\pm z^{\frac{1}{2}}\right)\right] = e^z erfc\left(\pm z^{\frac{1}{2}}\right), z\in C.
\end{aligned}$$

Where $erfc$ denotes the complimentary error function and the error function is defines as

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp(-t^2)\,dt, erfc(z) = 1-erf(z),\ z\in C.$$

Theorem 30 *The Mittag-Leffler has the following properties*
1) For $|z| < 1$, the generalized Mittag-Leffler function satisfies

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\,\beta}(zt^\alpha)\,dt = \frac{1}{z-1}.$$

2) The Laplace transform of this function is given by

$$\mathcal{L}[z^{\alpha k+\beta-1} E^k_{\alpha,\beta}(\alpha z^\alpha)](\lambda) = \frac{k!\lambda^{\alpha-\beta}}{(\lambda^\alpha-\alpha)^{k+1}},\ Re(\lambda) > |a|^{\frac{1}{\alpha}}\,.$$

Or $E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y)$.
3) Integration of Mittag-Leffler function

$$\int_0^z E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1}\,dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha).$$

4) The derivative of order $n \in N$ of the function of Mittag-Leffler is given by

$$\frac{d^n}{dz^n}\left(z^{\beta-1} E_{\alpha,\beta}\left(\lambda z^\alpha\right)\right) = z^{\beta-n-1} E_{\alpha,\beta-n}(z^\alpha).$$

1.2.5 Integral representation of Mittag-Leffler function

In this section several integrals associated with Mittag-Leffler functions are presented, which can be easily established by the application by means of beta and gamma function formulas and other techniques

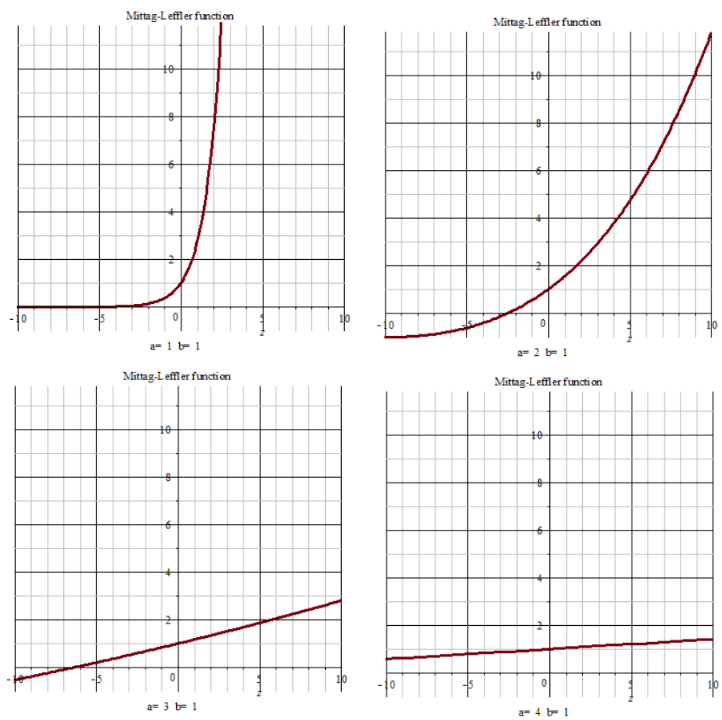
$$\begin{aligned}
\int_0^\infty e^{-\zeta} E_\alpha(\zeta^\alpha z)\,d\zeta &= \frac{1}{1-z'},\ |z| < 1,\ \alpha \in C, Re(\alpha) > 0,\\
\int_0^\infty e^{-x} x^{\beta-1} E_{\alpha,\beta}(x^\alpha z)\,dx &= \frac{1}{1-z'},\ |z| < 1,\ \alpha,\ \beta \in C,\ Re(\alpha) > 0,\ Re(\beta) > 0,\\
\int_0^x (x-\zeta)^{\beta-1} E_\alpha(\zeta^\alpha)\,d\zeta &= \Gamma(\beta) x^\beta E_{\alpha,\beta+1}(x^\alpha),\ Re(\alpha) > 0,\ Re(\beta) > 0,\\
\int_0^\infty e^{-\lambda\zeta} E_\alpha(-\zeta^\alpha)\,d\zeta &= \frac{\lambda^{\alpha-1}}{1+\lambda^\alpha},\ Re(\lambda) > 0,
\end{aligned}$$

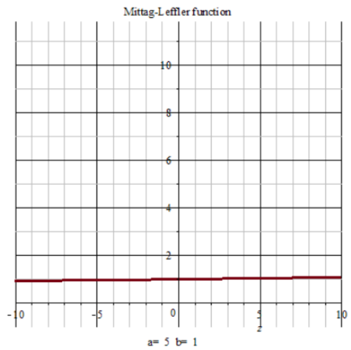
$$\begin{aligned}
&\int_0^\infty e^{-\lambda\zeta} \zeta^{m\alpha+\beta-1} E^{(m)}_{\alpha,\beta}(\pm\alpha\zeta^\alpha)\,d\zeta\\
=\ &\frac{m!\lambda^{\alpha-\beta}}{(\lambda^\alpha \pm \alpha)^{m+1}},\ Re(\alpha) > 0,\ Re(\lambda) > 0,\ Re(\beta) > 0,
\end{aligned}$$

where $\alpha,\ \beta \in C$ and

$$
\begin{aligned}
E_{\alpha,\beta}^{(m)}(z) &= \frac{d^m}{dz^m} E_{\alpha,\beta}(z), \\
E_\alpha(-x^\alpha) &= \frac{2}{\pi}\sin(\frac{\alpha\pi}{2}) \int_0^\infty \frac{\zeta^{\alpha-1}\cos(x\zeta)}{1+2\zeta^\alpha\cos(\frac{\alpha\pi}{2})+\zeta^{2\alpha}}\, d\zeta,\ \alpha\in C,\ Re(\alpha)>0, \\
E_\alpha(-x) &= \frac{1}{\pi}\sin(\alpha\pi) \int_0^\infty \frac{\zeta^{\alpha-1}}{1+2\zeta^\alpha\cos(\alpha\pi)+\zeta^{2\alpha}} e^{\zeta x^{\frac{1}{\alpha}}}\zeta\, d\zeta,\ \alpha\in C,\ Re(\alpha)>0, \\
E_\alpha(-x) &= 1-\frac{1}{2\alpha}+\frac{x^{\frac{1}{\alpha}}}{\pi}\int_0^\infty \arctan\left[\frac{\zeta^\alpha\cos(\alpha\pi)}{\sin(\alpha\pi)}\right] e^{-\zeta x^{\frac{1}{\alpha}}}\zeta\, d\zeta,\ \alpha\in C,\ Re(\alpha)>0.
\end{aligned}
$$

Plot of the Mettag-Meffler function





1.2.6 Wright function $W_{\lambda,\mu}(z)$

Definition 31 *The Wright function, that we denote by $W_{\lambda,\mu}(z)$ is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions.The function is de ned by the series representation, convergent in the whole z-complex plane,*

$$W_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{k!\Gamma(\lambda k + \mu)}, \lambda > -1, \mu \in C.$$

so $W_{\lambda,\mu}(z)$ is an entire function. Originally Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$.
We also need the following Wright type function which was considered by Mainardi

$$\begin{aligned}
\phi_\alpha(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} \\
&= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(k+1) \sin(\pi(k+1)\alpha)}{k!}, 0 < \alpha < 1. \\
M(z;\beta) &= \frac{1}{2\pi i} \int_{H_a} e^{\delta - z\delta^\beta} \frac{d\delta}{\delta^{1-\beta}}, 0 < \beta < 1,
\end{aligned}$$

the Hankel representation for the reciprocal of the gamma function. Writing

$$\begin{aligned}
2\pi i M(z;\beta) &= \int_{H_a} e^{\delta} \left[\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \delta^{\beta k} \right] \frac{d\delta}{\delta^{1-\beta}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \left[\int_{H^a} e^{\delta} \delta^{\beta k + \beta - 1} d\delta \right],
\end{aligned}$$

and using the Hankel representation of the reciprocal of the gamma function, we obtain the following series representation

$$M(z;\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\Gamma(-+(1-\beta))}, 0 < \beta < 1.$$

1.2.7 Integral representation of Wright function

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{H_a} e^{\sigma+z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}}, \lambda > -1, \mu \in C,$$

where H_a denotes the Hankel path. The equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function

$$\frac{1}{\Gamma(\zeta)}\int_{H_a} e^u u^{-\zeta}\, du,\ \zeta \in C,$$

and performing a term-by-term integration. In fact,

$$\begin{aligned} W_{\lambda,\mu}(z) &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma+z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}} = \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\left[\sum_{k=0}^{\infty}\frac{z^k}{k!}\sigma^{-\lambda k}\right]\frac{d\sigma}{\sigma^{\mu}} \\ &= \sum_{k=0}^{\infty}\frac{z^k}{k!}\left[\frac{1}{2\pi i}\int_{H_a} e^{\sigma}\sigma^{-\lambda k-\mu} d\,\sigma\right] = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\lambda k+\mu)}. \end{aligned}$$

It is possible to prove that the Wright function is entire of order $1/(1+\lambda)$; hence of exponential type if $\lambda \geq 0$. The case $\lambda = 0$ is trivial since

$$W_{0,\mu}(z) = \frac{e^z}{\Gamma(\mu)}.$$

Proposition 32 *ϕ_α is an entire function and has the following properties*
(a) $\phi_\alpha(\theta) \geq 0,\ for\, \theta \geq 0\, and \int_0^\infty \phi_\alpha(\theta)\, d\theta = 1$.
(b) $\int_0^\infty \phi_\alpha(\theta)\theta^r\, d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)},\ for\, r > -1$.
(c) $\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}\, d\theta = E_{\alpha,1}(-z),\ z \in C$.
(d) $\alpha\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}\, d\theta = E_{\alpha,\alpha}(-z),\ z \in C$.

Proof.
(a)

$$\begin{aligned} \phi_\alpha(\theta) &= \frac{1}{2\pi i}\int_{H_a} e^{\delta-\delta^{\alpha}\theta}\frac{d\delta}{\delta^{1-\alpha}}. \\ \int_0^\infty \phi_\alpha(\theta)\, d\theta &= \frac{1}{2\pi i}\int_{H_a} e^{\delta}\left[\int_0^\infty e^{-\delta^{\alpha}\theta}\, d\theta\right]\frac{d\delta}{\delta^{1-\alpha}} \\ &= \frac{1}{2\pi i}\int_{H_a}\frac{e^{\delta}}{\delta}d\delta = 1. \end{aligned}$$

(b) For the M-Wright functions, the following rule for absolute moments in R^+ holds. M-Wright functions

$$\begin{aligned}\phi_\alpha(\theta) &= \sum_{k=0}^{\infty}\frac{(-\theta)^k}{k!\Gamma(-\alpha k+1-\alpha)},\ \theta\in C.\\ \int_0^\infty \phi_\alpha(\theta)\theta^r d\theta &= \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)},\ for\, r>-1,\ 0\le\alpha<1.\end{aligned}$$

In order to derive this fundamental result, we proceed as follows on the basis of the integral representation

$$\begin{aligned}\phi_\alpha(\theta) &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma-r\sigma^\alpha\theta}\frac{d\sigma}{\sigma^{1-\alpha}}.\\ \int_0^\infty \theta^r\phi_\alpha(\theta)d\theta &= \int_0^\infty \theta^r\left[\frac{1}{2\pi i}\int_{H_a} e^{\sigma-r\sigma^\alpha\theta}\frac{d\sigma}{\sigma^{1-\alpha}}\right]d\theta\\ &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\left[\int_0^\infty e^{-\theta\sigma^r}\theta^r d\theta\right]\frac{d\sigma}{\sigma^{1-\alpha}}\\ &= \frac{\Gamma(1+r)}{2\pi i}\int_{H_a}\frac{e^\sigma}{\sigma^{\alpha r+1}}d\sigma = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}.\end{aligned}$$

(c)

$$\begin{aligned}\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}d\theta &= \frac{1}{2\pi i}\int_0^\infty e^{-z\theta}\left[\int_{H_a} e^{\sigma-\theta\sigma^\alpha}\frac{d\sigma}{\sigma^{1-\alpha}}\right]d\theta\\ &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\sigma^{\alpha-1}\left[\int_0^\infty e^{-\theta(z+\sigma^\alpha)}\,d\theta\right]d\sigma\\ &= \frac{1}{2\pi i}\int_{H_a}\frac{e^\sigma e^{\alpha-1}}{\sigma^\alpha+z}\,d\sigma = E_\alpha(-z) = E_{\alpha,1}(-z),\ z\in C.\end{aligned}$$

In the second approach we develop in series the exponential kernel of the Laplace transform and we use the expression (b) for the absolute moments of the M-Wright function arriving to the following series representation of the Mittag-Leffler function

$$\begin{aligned}\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}\,d\theta &= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\int_0^\infty \theta^k\phi_\alpha(\theta)\,d\theta\\ &= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\frac{\Gamma(k+r)}{\Gamma(1+\alpha k)}\\ &= \sum_{k=0}^{\infty}\frac{(-z)^k}{\Gamma(1+\alpha k)} = E_{\alpha,1}(-z),\ z\in C.\end{aligned}$$

(d)

$$\alpha\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}\,d\theta = \alpha\frac{1}{2\pi i}\int_0^\infty e^{-z\theta}\left[\int_{H_a} e^{\sigma-\theta\sigma^\alpha}\frac{d\sigma}{\sigma^{1-\alpha}}\right]d\theta$$

$$= \alpha \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\sigma^{\alpha-1}\left[\int_0^{\infty} e^{-\theta(z+\sigma^{\alpha})}\,d\theta\right] d\sigma$$
$$= \frac{1}{2\pi i}\int_{H_a} \alpha\frac{e^{\sigma}e^{\alpha-1}}{\sigma^{\alpha}+z}\,d\sigma = E_{\alpha,\alpha}(-z),\ z \in C.$$

■

1.2.8 Riemann-Liouville Fractional integral

Definition 33 *Integration of order $n \in N$ is described by the operation*

$$(\mathcal{L}_a^n)[u](x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}u(t)\,dt.$$

The natural extension of such a definition to real order $s > 0$ is

$$(\mathcal{L}_a^s)[u](x) = \frac{1}{\Gamma(s)}\int_a^x (x-t)^{s-1}u(t)\,dt.$$

This is called the Left Riemann-Liouville Fractional Integral of order s (because we integrate to x from the left). We will discuss the Right Riemann-Liouville Fractional Integral later.

1.2.9 Riemann-Liouville Fractional derivative

Definition 34 *To define a fractional derivative we **cannot** just formally replace s by $-s$ in the Riemann-Liouville integral. For a given u, we do not have a nite integral for all $x \in [a; b]$ (except if u is identically zero)*

$$(D_a^s)[u](x) = \frac{1}{\Gamma(-s)}\int_a^x (x-t)^{-s-1}u(t)\,dt.$$

There is, however a nice trick we can use to get around this.
To define a fractional derivative of order $s \in (0; 1]$ we integrate to order $1-s$ then differentiate to order 1

$$(D_a^s)[u](x) = \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_a^x (x-t)^{-s}u(t)\,dt.$$

More generally, to define a fractional derivative of order $s \in (k-1; k]$ for $k \in N$ we integrate to order $k-s$ then differentiate to order k

$$(D_a^s)[u](x) = \frac{1}{\Gamma(k-s)}\frac{d^k}{dx^k}\int_a^x (x-t)^{k-1-s}u(t)\,dt.$$

This is the Left Riemann-Liouville Fractional Derivative.

1.2.10 Relation with Reimann-Liouville Fractional Calculus Operators

In this section, we present the relations of Mittag-Leffler functions with the left and rightsided operators of Riemann-Liouville fractional calculus, which are defined

$$\begin{aligned}
(I_{0+}^{\alpha}\phi)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty}(x-t)^{\alpha-1}\phi(t)\,dt,\ Re(\alpha)>0.\\
(I_{-}^{\alpha}\phi)(x) &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}\phi(t)\,dt, Re(\alpha)>0.\\
\left(D_{0+}^{\alpha}\phi\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1}\left[I_{0^+}^{1-\{\alpha\}}\phi\right](x)\\
&= \frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dt}\right)^{[\alpha]+1}\int_0^x(x-t)^{\alpha-1}\phi(t)\,dt,\ Re(\alpha)>0.\\
\left(D_{-}^{\alpha}\phi\right)(x) &= -\frac{d}{dx}^{[\alpha]+1}\left[I_{-}^{1-\{\alpha\}}\phi\right](x)\\
&= \frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dt}\right)^{[\alpha]+1}\int_x^{\infty}(t-x)^{-\{\alpha\}}\phi(t)\,dt,\ Re(\alpha)>0.
\end{aligned}$$

Where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

1.2.11 Caputo fractional derivative

Definition 35 *The fractional derivative of Caputo of order $\alpha \in R^+$ of a function f is Given by*

$$^{c}D_a^{\alpha}f(x) = I^{n-\alpha}f^n(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}f^n(t)\,dt,\ x>a,$$

with: $n-1<\alpha\leq n$, $n\in N^*$.

Proposition 36 *(Caputo)*
1/ Properties: Let $n-1<\alpha<n$, $n\in N$, $\alpha\in R$ and $f(t)$ be such that ${}^cD_t^{\alpha}f(t)$ exists. Then the following properties for the Caputo operator hold

$$\begin{aligned}
\lim_{\alpha\to n}{}^cD_t^{\alpha}f(t) &= f^{(n)}(t).\\
\lim_{\alpha\to n-1}{}^cD_t^{\alpha}f(t) &= f^{(n-1)}(t)-f^{(n-1)}(0).
\end{aligned}$$

2/ Linearity: Let $n-1<\alpha<n$, $n\in N$, $\alpha,\lambda\in C$ and the functions $f(t)$ and $g(t)$ be such that ${}^{\alpha}D_t^{c}f(t)$ and ${}^cD_t^{\alpha}g(t)$ exists. The Caputo fractional derivatives

is a linear operator i.e

$$^cD_t^\alpha(\lambda f(t)+g(t)) \quad = \quad \lambda\, {^cD_t^\alpha} f(t) +^c D_t^\alpha g(t).$$

3/

$$^cD_t^\alpha\, {^cD_t^\beta} f(t) \quad = \quad {^cD_t^{\alpha+\beta}} f(t) \neq {^cD_t^\beta}\, {^cD_t^\alpha} f(t).$$

Example 37 *Let* $a=0$, $\alpha=\frac{1}{2}$, $(n=1)$, $f(t)=1$, *then a pluing formula*

$$\frac{1}{\Gamma(n-1)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-1-a}}\, d\tau, n-1<\alpha<n, n\in N,$$

we get

$$^cD_t^{\frac{1}{2}}(t)=\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\tau)^{\frac{-1}{2}}}\, d\tau.$$

Taking into account the properties of the Gamma function $\Gamma(\frac{1}{2})=\sqrt{\pi}$ *and using the substitution* $n:=(t-\tau)^{\frac{1}{2}}$ *the final result for the Caputo fractional derivative of the function* $f(t)=t$ *is obtain else*

$$^cD_t^{\frac{1}{2}}(t)=\frac{-1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\, d(t-\tau)=\frac{-1}{\sqrt{\pi}}\int_{\sqrt{t}}^0 \frac{1}{u}\, du^2=\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}}\frac{2u}{u}\, du=\frac{2}{\sqrt{\pi}}(\sqrt{t}-0),$$

thus, it holds

$$^cD_t^{\frac{1}{2}}(t)=\frac{2\sqrt{t}}{\sqrt{\pi}}.$$

1.3 Abstract equation problem

Lemma 38 *If* ${}_0^cD_t^\alpha f\in L^1(0,T)$, $g\in C^1([0,T])$ *and* $g(T)=0$, *then we have the following formula of integration by parts*

$$\int_0^T g\, {}_0^cD_t^\alpha f\, dt \quad = \quad \int_0^T (f(t)-f(0))\, {}_t^cD_T^\alpha g\, dt, \tag{1.1}$$

where

$$\begin{aligned} {}_0^cD_t^\alpha g &= -\frac{d}{dt}\, {}_tI_T^{1-\alpha} g. \\ {}_tI_T^{1-\alpha} g &= \frac{1}{\Gamma(1-\alpha)}\int_t^T (s-t)^{-\alpha} g(s)\, ds. \end{aligned}$$

We need know the Caputo fractional derivative of the following function, which will be used in the next sections. For given $T > 0$ and $n > 0$, if we get

$$\varphi(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^n, t \leq T, \\ 0, t > T, \end{cases}$$

then

$${}_t^cD_T^\alpha \varphi(t) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{n-\alpha}, \ t \leq T.$$

Proof.

$$\begin{aligned}
{}_t^cD_T^\alpha \varphi(t) &= -\frac{d}{dt} {}_tI_T^{1-\alpha} \varphi(t) \\
&= -\frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \varphi(s)\, ds\right] \\
&= -\frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \left(1 - \frac{s}{T}\right)^n ds\right], t \leq T \\
&= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\int_t^T (s-t)^{-\alpha} (1 - \frac{s}{T})^n\, ds\right], t \leq T = uv \mid - \int u'v. \\
I_1(t) &= \frac{n}{T(1-\alpha)} \frac{(n-1)}{T(2-\alpha)} \int_t^T \left(1 - \frac{s}{T}\right)^{n-1} (s-t)^{-\alpha+2}\ ds.
\end{aligned}$$

.

.

.

$$\mathrm{I}_k(t) = \frac{n(n-1)(n-k+1)}{T^k(1-\alpha)(2-\alpha)......(k-\alpha)} \int_t^T \left(1 - \frac{s}{T}\right)^{n-k} (s-t)^{-\alpha+k}\, ds.$$

$$\frac{n!}{(k-3)!} = n(n-1)(n-2); \frac{(-\alpha)!}{(k-\alpha)!} = \frac{\Gamma(1-\alpha)}{\Gamma(n-\alpha+1)}.$$

$$\frac{n(n-1)(n-k+1)}{T^k(1-\alpha)(2-\alpha)......(k-\alpha)} \frac{(n)!}{(n-\alpha)!} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} = I(t).$$

$$\begin{aligned}
{}_t^cD_T^\alpha \left(1 - \frac{t}{T}\right)^n &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \left(1 - \frac{s}{T}\right)^n (s-t)^{-\alpha}\, ds, \ t \leq T \\
&= \frac{\Gamma(n+1)}{\Gamma(1+n-\alpha)} \int_t^T \left(1 - \frac{s}{T}\right)^{n-\alpha} ds, t \leq T \\
&= \frac{\Gamma(n+1)}{\Gamma(1+n-\alpha)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{n-\alpha}, \ t \leq T.
\end{aligned}$$

■

Theorem 39 *We denote* $A = \Delta$ *and it generates a semigroup* $\{T(t)\}_{t\geq 0}$ *on* $C_0(R^N)$ *with domain*

$$D(A) = \{u \in C_0(R^N) : \Delta u \in C_0(R^N)\}.$$

Then $T(t)$ *is an analytic and contractive semigroup on* $C_0(R^N)$ *and, for* $t > 0$, $x \in R^N$.

$$T(t)u_0 = \int_{R^N} G(t, x-y)u_0(y)\,dy,\; G(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{(4t)}}, \tag{1.2}$$

and $T(t)$ *is a contractive semigroup on* $L^q(R^N)$ *for* $q \geq 1$, *and*

$$\|T(t)u_0\|_{L^p(R^N)} \leq (4\pi t)^{\frac{-N}{2}(\frac{1}{q}-\frac{1}{p})} \|u_0\|_{L^q(R^N)}, \tag{1.3}$$

for $u_0 \in L^q(R^N)$, $q \leq p \leq +\infty$.

Lemma 40 *Les deux estimations*

Proof. fghjklm ■
Proof. theorem
We use the theorem of Riesz-Thorin, we search (λ): $\frac{1}{q} = \frac{1-\lambda}{1} + \frac{\lambda}{2} = 1-\lambda + \frac{1}{2}\frac{2}{p} = 1 - \lambda + \frac{1}{p}$, so $\lambda = \frac{2}{p}$.
In order to verify the result
$\frac{1}{p} = \frac{1-\lambda}{\infty} + \frac{\lambda}{2} = \frac{\lambda}{2}$,
so $\lambda = \frac{2}{p}$.
We get

$$\|T(t)u_0\|_{L^p(R^N)} \leq (4\pi t)^{(-N/2)(1-\lambda)} \|u_0\|_{L^q(R^N)} \leq (4\pi t)^{(-N/2)(\frac{1}{p}+\frac{1}{q}-\frac{1}{p}-\frac{1}{p})} \|u_0\|_{L^q(R^N)},$$

finally

$$\|T(t)u_0\|_{L^p(R^N)} \leq (4\pi t)^{(-N/2)(\frac{1}{q}-\frac{1}{p})} \|u_0\|_{L^q(R^N)}.$$

■

Theorem 41 *Define the operators* $P_\alpha(t)$ *and* $S_\alpha(t)$ *as*

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta, t \geq 0. \tag{1.4}$$

$$S_\alpha(t)u_0 = \alpha\int_0^\infty \theta\phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta,\; t \geq 0. \tag{1.5}$$

Consider the following linear time fractional equation

$$\begin{cases} {}_0^cD_t^\alpha u - \Delta u = |u|^{p-1}u,\; x \in R^N,\; t > 0, \\ u(0,x) = u_0(x),\; x \in R^N, \end{cases} \tag{1.6}$$

where $u_0 \in C_0(R^N)$ *and* $f \in L^1((0,T), C_0(R^N))$. *If* u *is a solution of (1.6), we get*

$$u(t,x) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds.$$

Proof. we denote $A = -\Delta$, $|u|^{p-1}u(x) = f(t,x)$.

$$\left\{\begin{array}{c} {}_0^cD_t^\alpha u + Au = f(t,x), x \in R^N, \\ u(0,x) = u_0(x), x \in R^N, \end{array}\right. \quad (P)$$

we discus the existence and uniqueness of mild solution of the inhomogeneous linear time fractional (Cauchy problem) where ${}^cD_t^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative of order α, and u_0 is given belong to a subset of R^N

Assumption.

Assume that $u(.,.) : [0,T] \to R^N$, $u(t,x) \in D(A)$ for $t \in [0,T]$, $Au \in L^1((0,T); R^N)$ and u satisfies (P). We can rewrite (P) as

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x)\, ds \; for\, t \in [0,T],$$

if $u : [0,T] \to R^N$, is a functions satisffying Assumption (H^*), then $u(t)$ satisfies the following integral equation

$$u(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,x)\, ds, \; for\, t \in [0,T].$$

Not thet the Laplace transform of an abstract function $f \in L^1(R^+, R^N)$ is defines by

$$F(\lambda) = \int_0^\infty e^{-\lambda t}\, dt, \lambda \in C, (\lambda > 0).$$

Appliying the Laplace transform to

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x)\, ds,$$

for the Laplace transform of the convolution

$$f(t) * g(t) = \int_0^t f(t-\lambda) g(s) d\, s = \int_0^t f(t) g(t-s)\, ds,$$

we have

$$\mathcal{L}\{f(t) * g(t); \lambda\} = F(\lambda)G(\lambda),$$

we start with

$${}^cD^\alpha u(t) = {}^{RL} D^\alpha (u(t) - u_0).$$

$$\begin{aligned} {}_0^{RL}I_t^\alpha \left({}^{RL}D^\alpha (u(t) - u_0) \right) &= {}_0^{RL}I_t^\alpha(-Au) + {}_0^{RL} I_t^\alpha(f(t,x)) \\ u(t) - u_0 &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \; for\, t \in [0,T] \end{aligned}$$

$$\mathcal{L}\left\{u(t),\lambda\right\} = \mathcal{L}\left\{u_0,\lambda\right\} - \frac{1}{\Gamma(\alpha)}\mathcal{L}\left\{\int_0^t (t-s)^{\alpha-1}Au(s)\,ds,\lambda\right\} + \frac{1}{\Gamma(\alpha)}\mathcal{L}\left\{\int_0^t (t-s)^{\alpha-1}f(s)\,ds,\lambda\right\},$$

where $\mathcal{L}\left\{x^{\alpha-1},\lambda\right\} = \Gamma(\alpha)\lambda^{-\alpha}$. We get

$$\begin{aligned}
\widehat{u}(\lambda) &= \frac{u_0}{\lambda} - \frac{1}{\Gamma(\alpha)}\Gamma(\alpha)\lambda^{-\alpha}A\widehat{u}(\lambda) + \frac{1}{\Gamma(\alpha)}\Gamma(\alpha)\lambda^{-\alpha}\widehat{f}(\lambda)\\
\widehat{u}(\lambda) &= u_0\lambda^{-1} - \lambda^{-\alpha}A\widehat{u}(\lambda) + \lambda^{-\alpha}\widehat{f}(\lambda)\\
\widehat{u}(\lambda) + \lambda^{-\alpha}A\widehat{u}(\lambda) &= u_0\lambda^{-1} + \lambda^{-\alpha}\widehat{f}(\lambda)\\
\left(1+\lambda^{-\alpha}A\right)\widehat{u}(\lambda) &= u_0\lambda^{-1} + \lambda^{-\alpha}\widehat{f}(\lambda)\\
\left(\lambda^{\alpha}+\lambda^{\alpha}\lambda^{-\alpha}A\right)\widehat{u}(\lambda) &= u_0\lambda^{\alpha-1} + \lambda^{-\alpha+\alpha}\widehat{f}(\lambda)\\
\left(\lambda^{\alpha}+A\right)\widehat{u} &= u_0\lambda^{\alpha-1} + \widehat{f}(\lambda),
\end{aligned}$$

by composition $(\lambda^{\alpha}+A)^{-1}$ is obtained

$$\left(\lambda^{\alpha}+A\right)^{-1}\left(\lambda^{\alpha}+A\right)\widehat{u} = u_0\lambda^{\alpha-1}\left(\lambda^{\alpha}+A\right)^{-1} + \left(\lambda^{\alpha}+A\right)^{-1}\widehat{f}(\lambda),$$

that is

$$\widehat{u}(\lambda) = u_0\lambda^{\alpha-1}(\lambda^{\alpha}+A)^{-1} + (\lambda^{\alpha}+A)^{-1}\widehat{f}(\lambda),$$

on the other hand, using for every $\lambda \in C$ with $Re(\lambda) > 0$, one has

$$R(\lambda,-A) = \int_0^\infty e^{-\lambda t}T(t)\,dt = (\lambda^{\alpha}+A)^{-1},$$

we deduce that

$$\begin{aligned}
\widehat{u}(\lambda) &= \lambda^{\alpha-1}(\lambda^{\alpha}+A)^{-1}u_0 + (\lambda^{\alpha}+A)^{-1}\widehat{f}(\lambda)\\
&= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^{\alpha}t}T(t)u_0dt + \int_0^\infty e^{-\lambda^{\alpha}t}T(t)\widehat{f}(\lambda)\,dt,
\end{aligned}$$

we use the change of variable in the first and the second term $t = t^{\alpha}$

$$= -u_0\int_0^\infty \frac{d}{d\lambda}e^{-(\lambda t)^{\alpha}}T(t^{\alpha})\,dt + \int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^{\alpha}}T(t^{\alpha})f(s)e^{-s\lambda}\,dsdt,$$

where

$$e^{-(\lambda)^{\alpha}} = \int_0^\infty \frac{\alpha}{t^{\alpha+1}}\phi_{\alpha}\left(\frac{1}{t^{\alpha}}\right)e^{-\lambda t}\,dt,$$

we use the change of variable in the second term $t = \dfrac{t}{\tau}$

$$\begin{aligned}
= {} & u_0\int_0^\infty\int_0^\infty \frac{\alpha t}{\tau^{\alpha}}\phi_{\alpha}\left(\frac{1}{\tau^{\alpha}}\right)e^{-\lambda t\tau}T(t^{\alpha})\,d\tau dt\\
&+ \int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha}{\tau^{2\alpha}}t^{\alpha-1}\phi_{\alpha}\left(\frac{1}{\tau^{\alpha}}\right)e^{-\lambda t}T\left(\frac{t^{\alpha}}{\tau^{\alpha}}\right)f(s)e^{-s\lambda}\,d\tau dsdt,
\end{aligned}$$

we use the change of variable in the first term $t=\dfrac{t}{\tau}$, and in the second term $\dfrac{1}{\tau^{\alpha}}=\tau$, so

$$
\begin{aligned}
= \quad & u_0\int_0^{\infty}\int_0^{\infty}\frac{\alpha}{\tau^{\alpha+1}}\phi_\alpha\left(\frac{1}{\tau^\alpha}\right)e^{-\lambda t}T\left(\frac{t^\alpha}{\tau^\alpha}\right)d\tau dt \\
& +\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\alpha\tau t^{\alpha-1}\phi_\alpha(\tau)T\left(t^\alpha\tau\right)f(s)e^{-(s+t)\lambda}\,d\tau ds dt,
\end{aligned}
$$

we use the change of variable in the first term $\dfrac{1}{\tau}^{\alpha}=\tau$, and in the second term $t=t-s$, we get

$$
\begin{aligned}
= \quad & u_0\int_0^{\infty}e^{-\lambda t}\int_0^{\infty}\phi_\alpha(\tau)T\left(t^\alpha\tau\right)\,d\tau dt \\
& +\int_0^{\infty}e^{-\lambda t}\int_0^{t}(t-s)^{\alpha-1}f(s)\left(\int_0^{\infty}\alpha\tau\phi_\alpha\left(\tau\right)T\left((t-s)^\alpha\tau\right)\,d\tau\right)dsdt, \\
= \quad & \int_0^{\infty}e^{-\lambda t}P_\alpha(t)u_0\,dt+\int_0^{\infty}e^{-\lambda t}\int_0^{t}(t-s)^{\alpha}S_\alpha(t-s)f(s)\,dsdt \\
= \quad & \int_0^{\infty}e^{-\lambda t}\left(P_\alpha(t)u_0+\int_0^{t}(t-s)^{\alpha-1}S_\alpha(t-s)f(s)\,ds\right)dt,
\end{aligned}
$$

this implies that

$$
\widehat{u}(\lambda)=\int_0^{\infty}e^{-\lambda t}\left(P_\alpha(t)u_0+\int_0^{t}(t-s)^{\alpha-1}S_\alpha(t-s)f(s)\,ds\right)dt,
$$

so

$$
u(\lambda)=P_\alpha(t)u_0+\int_0^{t}(t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds.
$$

■

Lemma 42 *We denote*

$$
K(t,x)=\int_0^{\infty}\phi_\alpha(\theta)G(t^\alpha\theta,x)\,d\theta,\ x\in R^N\backslash\{0\},\ t>0.
$$

Note that for given $t>0$ and $x\in R^N\backslash\{0\}$, $G(t^\alpha\theta,x)\to 0$ as $\theta\to 0$, so K is well defined. Since $\int_0^\infty\phi_\alpha(\theta)\,d\theta=1, \int_{R^N}G(t,x)dx=1$, we know that

$$
\|K(t,.)\|_{L^1(R^N)}=1\ \text{for}\ t>0.
$$

Proof.

$$
K(t,x)=\int_0^{\infty}\phi_\alpha(\theta)G(t^\alpha\theta,x)\,d\theta, x\in R^N\backslash\{0\}, t>0.
$$

$$
\int_0^{\infty}\phi_\alpha(\theta)\,d\theta=1,\int_{R^N}G(t,x)\,dx=1.
$$

$$
\begin{aligned}
\|K(t,x)\|_{L^1(R^N)} &= \int_{R^N}\left[\int_0^\infty |\phi_\alpha(\theta)G(t^\alpha\theta,x)|\,d\theta\right]dx,\ for\,t>0,\\
&= \int_0^\infty |\phi_\alpha(\theta)|\,d\theta\int_{R^N}|G(t^\alpha\theta,x)|\,dx=1,\ for\,t>0.
\end{aligned}
$$

■

Lemma 43 *The operator* $\{P_\alpha(t)\}_{t>0}$ *has the following properties*
(a) If $u_0\geq 0, u_0\not\equiv 0,$ *then* $P_\alpha(t)u_0>0$ *and* $\|P_\alpha(t)u_0\|_{L^1(R^N)}=\|u_0\|_{L^1(R^N)}$.
(b) If $1\leq p\leq q\leq+\infty$ *and* $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}<\frac{2}{N}$, *then*

$$
\|P_\alpha(t)u_0\|_{L^q(R^N)}\leq(4\pi t^\alpha)^{\frac{-N}{2r}}\frac{\Gamma\left(1-\frac{N}{2r}\right)}{\Gamma\left(1-\alpha\frac{N}{2r}\right)}\|u_0\|_{L^p(R^N)}. \qquad (1.7)
$$

Proof. (a) Follows from $T(t)u_0\geq 0,\phi_\alpha\geq 0$. The operator $\{P_\alpha(t)\}_{t>0}$ define as

$$
\begin{aligned}
P_\alpha(t)u_0 &= \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta,\ t\geq 0.\\
\|P_\alpha(t)u_0\|_{L^1(R^N)} &= \int_{(R^N)}\left[\int_0^\infty|\phi_\alpha(\theta)T(t^\alpha\theta)u_0|\,d\theta\right]dx\\
&= \int_{(R^N)}\left[\int_0^\infty\left|\phi_\alpha(\theta)\int_{(R^N)}G(t^\alpha\theta,x-y)u_0(y)\,dy\right|d\theta\right]dx\\
&= \int_0^\infty\phi_\alpha(\theta)\,d\theta\int_{(R^N)}\int_{(R^N)}G(t^\alpha\theta,x-y)u_0(y)\,dy\,dx\\
&= \int_0^\infty\phi_\alpha(\theta)\,d\theta\int_{(R^N)}u_0(y)\int_{(R^N)}G(t^\alpha\theta,x-y)\,dx\,dy\ (byFubini)\\
&= \int_0^\infty\phi_\alpha(\theta)\,d\theta\,\|G(t^\alpha\theta,y)\|_{L^1(R^N)}\int_{(R^N)}|u_0|\,dy,\\
\|P_\alpha(t)u_0\|_{L^1(R^N)} &= \int_{(R^N)}|u_0|\,d\,\theta=\|u_0\|_{L^1(R^N)}.
\end{aligned}
$$

(b) By (1.3) and the properties of $\phi_\alpha(\theta)$, we have

$$
\begin{aligned}
\|P_\alpha(t)u_0\|_{L^q(R^N)} &= \left\|\int_0^\infty\phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta\right\|_{L^q(R^N)}\\
&\leq \left(\int_{(R^N)}\left[\int_0^\infty|\phi_\alpha(\theta)T(t^\alpha\theta)u_0|^q\,d\theta\right]dx\right)^{\frac{1}{q}},
\end{aligned}
$$

according to previous result

$$
\|T(t)u_0\|_{L^p(R^N)}\leq(4\pi t)^{\frac{-N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q(R^N)},
$$

so

$$\begin{aligned}
\|P_\alpha(t)u_0\|_{L^q(R^N)} &\leq \int_0^\infty \phi_\alpha(\theta)(4\pi t^\alpha\theta)^{\frac{-N}{2r}}\,d\theta\,\|u_0\|_{L^p(R^N)} \\
&\leq \int_0^\infty \phi_\alpha(\theta)(4\pi t^\alpha)^{\frac{-N}{2r}}\theta^{\frac{-N}{2r}}\,d\theta\,\|u_0\|_{L^p(R^N)} \\
&\leq (4\pi t^\alpha)^{\frac{-N}{2r}}\int_0^\infty \theta^{\frac{-N}{2r}}\phi_\alpha(\theta)\,d\theta\,\|u_0\|_{L^p(R^N)},
\end{aligned}$$

we have

$$\begin{aligned}
\int_0^\infty \theta^{\frac{-N}{2r}}\phi_\alpha(\theta)\,d\theta &= \int_0^\infty \theta^{\frac{-N}{2r}}\left[\frac{1}{2\pi i}\int_{H_a} e^{\sigma-(\frac{-N}{2r})\sigma^\alpha\theta}\frac{d\sigma}{\sigma^{1-\alpha}}\right]d\theta \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\left[\int_0^\infty e^{-\theta\sigma^{\frac{-N}{2r}}}\theta^{\frac{-N}{2r}}\,d\theta\right]\frac{d\sigma}{\sigma^{1-\alpha}} \\
&= \frac{\Gamma(1+(\frac{-N}{2r}))}{2\pi i}\int_{H_a}\frac{e^\sigma}{\sigma^{\alpha(\frac{-N}{2r})+1}}d\sigma \\
&= \frac{\Gamma(1-\frac{N}{2r})}{\Gamma(1-\alpha\frac{N}{2r})},
\end{aligned}$$

for $\frac{-N}{2r} > -1 \Longrightarrow \frac{N}{2r} < 1$,
so

$$\|P_\alpha(t)u_0\|_{L^q(R^N)} \leq (4\pi t^\alpha)^{\frac{-N}{2r}}\frac{\Gamma(1-\frac{N}{2r})}{\Gamma(1-\alpha\frac{N}{2r})}\|u_0\|_{L^p(R^N)}.$$

Hence, we derive (1.7) holds. ■

Lemma 44 *For the operator $\{S_\alpha(t)\}_{t>0}$, we have the following results.*
(a) If $u_0 \geq 0$ and $u_0 \not\equiv 0$, then $S_\alpha(t)u_0 > 0$ and

$$\|S_\alpha(t)u_0\|_{L^1(R^N)} = \frac{1}{\Gamma(\alpha)}\|u_0\|_{L^1(R^N)}.$$

(b) For $1 \leq p \leq q \leq +\infty$, let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, if $\frac{1}{r} < \frac{4}{N}$, then

$$\|S_\alpha(t)u_0\|_{L^q(R^N)} \leq \alpha(4\pi t^\alpha)^{\frac{-N}{2r}}\frac{\Gamma(2-\frac{N}{2r})}{\Gamma(1+\alpha-\alpha\frac{N}{2r})}\|u_0\|_{L^p(R^N)}. \tag{1.8}$$

Proof. (a) Follows from $T(t)u_0 \geq 0, \phi_\alpha \geq 0$. The operator $\{S_\alpha(t)\}_{t>0}$ define as

$$\begin{aligned}
S_\alpha(t)u_0 &= \alpha\int_0^\infty \theta\phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta,\ t \geq 0. \\
\|S_\alpha(t)u_0\|_{L^1(R^N)} &= \int_{(R^N)}\left[\alpha\int_0^\infty |\theta\phi_\alpha(\theta)T(t^\alpha\theta)u_0|\,d\theta\right]dx
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \int_{(R^N)} \left[\int_0^\infty \left|\theta\phi_\alpha(\theta) \int_{(R^N)} G(t^\alpha\theta, x-y)u_0(y)\, dy\right| \, d\theta\right] \, dx \\
&= \alpha \int_0^\infty \theta\phi_\alpha(\theta) \int_{(R^N)} \int_{(R^N)} G(t^\alpha\theta, x-y)u_0(y)\, dy\, dx\, d\theta \\
&= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\, d\theta \int_{(R^N)} u_0(y) \int_{(R^N)} G(t^\alpha\theta, x-y)\, dx\, dy\, (byFubini) \\
&= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\, d\theta \left\|G(t^\alpha\theta, y)\right\|_{L^1(R^N)} \int_{(R^N)} |u_0| \, dy \\
&= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\, d\theta \left\|u_0\right\|_{L^1(R^N)},
\end{aligned}
$$

we have

$$
\begin{aligned}
\alpha \int_0^\infty \theta\phi_\alpha(\theta)\, d\theta &= \alpha \int_0^\infty \theta \left[\frac{1}{2\pi i} \int_{H_a} e^{\sigma - \sigma^\alpha\theta} \frac{d\sigma}{\sigma^{1-\alpha}}\right] d\theta \\
&= \frac{\alpha}{2\pi i} \int_{H_a} e^\sigma \left[\int_0^\infty e^{-\theta\sigma}\theta\, d\theta\right] \frac{d\sigma}{\sigma^{1-\alpha}} \\
&= \frac{\alpha\Gamma(1)}{2\pi i} \int_{H_a} \frac{e^\sigma}{\sigma^{\alpha+1}} d\sigma = \frac{\alpha\Gamma(1)}{\alpha\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)},
\end{aligned}
$$

so

$$
\left\|S_\alpha(t)u_0\right\|_{L^1(R^N)} = \frac{1}{\Gamma(\alpha)} \left\|u_0\right\|_{L^1(R^N)}.
$$

(b) by (1.3) and the properties of $\phi_\alpha(\theta)$, we have

$$
\begin{aligned}
\left\|S_\alpha(t)u_0\right\|_{L^q(R^N)} &= \left\|\alpha \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\, d\theta\right\|_{L^q(R^N)} \\
&\leq \left(\int_{(R^N)} \left[\alpha \int_0^\infty \left|\phi_\alpha(\theta)T\left(t^\alpha\theta\right)u_0\right|^q \, d\theta\right] \, dx\right)^{\frac{1}{q}},
\end{aligned}
$$

according to previous result

$$
\left\|T(t)u_0\right\|_{L^p(R^N)} \leq (4\pi t)^{\left(\frac{-N}{2}\right)\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|u_0\right\|_{L^q(R^N)},
$$

so

$$
\begin{aligned}
\left\|S_\alpha(t)u_0\right\|_{L^q(R^N)} &\leq \alpha \int_0^\infty \theta\phi_\alpha(\theta)(4\pi t^\alpha\theta)^{\frac{-N}{2r}} \, d\theta \left\|u_0\right\|_{L^p(R^N)} \\
&\leq \alpha \int_0^\infty \theta\phi_\alpha(\theta)(4\pi t^\alpha)^{\frac{-N}{2r}}\theta^{\frac{-N}{2r}} \, d\theta \left\|u_0\right\|_{L^p(R^N)} \\
&\leq \alpha(4\pi t^\alpha)^{\frac{-N}{2r}} \int_0^\infty \phi_\alpha(\theta)\theta^{1-\frac{N}{2r}} \, d\theta \left\|u_0\right\|_{L^p(R^N)},
\end{aligned}
$$

we have

$$\begin{aligned}
\int_0^\infty \theta^{1-\frac{N}{2r}}\phi_\alpha(\theta)\,d\theta &= \int_0^\infty \theta^{1-\frac{N}{2r}}\left[\frac{1}{2\pi i}\int_{H_a} e^{\sigma-(1-\frac{N}{2r})\sigma^\alpha\theta}\frac{d\sigma}{\sigma^{1-\alpha}}\right]d\theta \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\left[\int_0^\infty e^{-\theta\sigma^{(1-\frac{N}{2r})}}\theta^{(1-\frac{N}{2r})}\,d\theta\right]\frac{d\sigma}{\sigma^{1-\alpha}} \\
&= \frac{\Gamma(1+1-\frac{N}{2r})}{2\pi i}\int_{H_a}\frac{e^\sigma}{\sigma^{\alpha(1-\frac{N}{2r})+1}}d\sigma \\
&= \frac{\Gamma(2-\frac{N}{2r})}{\Gamma(1+\alpha(1-\frac{N}{2r}))} = \frac{\Gamma(2-\frac{N}{2r})}{\Gamma(1+\alpha-\alpha\frac{N}{2r})},
\end{aligned}$$

for $\dfrac{-N}{2r} > -1 \Longrightarrow \frac{N}{2r} < 1$, so

$$\|S_\alpha(t)u_0\|_{L^q(R^N)} \qquad \le \alpha(4\pi t^\alpha)^{\frac{-N}{2r}}\frac{\Gamma(2-\frac{N}{2r})}{\Gamma(1+\alpha-\alpha\frac{N}{2r})}\|u_0\|_{L^p(R^N)}.$$

Hence, we derive (1.8) holds. ■

Lemma 45 *For $u_0 \in C_0(R^N)$, we have $P_\alpha(t)u_0 \in D(A)$ for $t > 0$, and*

$${}_0^cC_t^\alpha P_\alpha(t)u_0 = AP_\alpha(t)u_0,\ t>0.$$

$$\|AP_\alpha(t)u_0\|_{L^\infty(R^N)} \le \frac{C}{t^\alpha}\|u_0\|_{L^\infty(R^N)},\ t>0,$$

for some constant $C > 0$.

Proof. Let $X = C_0(R^N)$. First, we prove if $u_0 \in X$, then $P_\alpha(t)u_0 \in D(A)$.

$$\begin{aligned}
P_\alpha(t)u_0 &= \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta \\
&= \int_0^\infty [\phi_\alpha(\theta)T(t^\alpha\theta)u_0 + T(t^\alpha\theta)u_0\phi_\alpha(0) - T(t^\alpha\theta)u_0\phi_\alpha(0)]\,d\theta \\
&= \int_0^\infty [\phi_\alpha(\theta)-\phi_\alpha(0)]T(t^\alpha\theta)u_0 + \phi_\alpha(0)T(t^\alpha\theta)u_0\,d\theta \\
&= \int_0^1 [\phi_\alpha(\theta)-\phi_\alpha(0)]T(t^\alpha\theta)u_0 d\theta + \phi_\alpha(0)\int_0^1 T(t^\alpha\theta)u_0 d\theta + \int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta.
\end{aligned}$$

Clearly, $\displaystyle\int_0^1 T(t^\alpha\theta)u_0 d\,\theta \in D(A)$. Note that there exists positive constant C such that

$$\|AT(t^\alpha\theta)u_0\|_X \le C\frac{\|u_0\|_X}{t^\alpha\theta},\ t>0,\ \theta>0,$$

we get that $\displaystyle\int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta \in D(A)$. Next, we show that

$$\int_0^1 (\phi_\alpha(\theta)-\phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta \in D(A).$$

$$
\begin{aligned}
A\int_0^t T(t^\alpha\theta)u_0 d\theta &= T(t)u_0 - u_0, \forall t\geq 0. \\
\frac{T(h)-I}{h}\int_0^t T(t^\alpha\theta)u_0\, d\theta &= \frac{1}{h}\int_0^t (T(h)-I)T(t^\alpha\theta)u_0\, d\theta \\
&= \frac{1}{h}\int_0^t (T(h)T(t^\alpha\theta)u_0 - T(t^\alpha\theta)u_0)\, d\theta \\
&= \frac{1}{h}\int_0^t T(h)T(t^\alpha\theta)u_0 d\theta - \frac{1}{h}\int_0^t T(t^\alpha\theta)u_0\, d\theta \\
&= \frac{1}{h}\int_0^t T(t^\alpha\theta+h)u_0\, d\theta - \frac{1}{h}\int_0^t T(t^\alpha\theta u_0)\, d\theta,
\end{aligned}
$$

we use change variable $t^\alpha\theta + h = \tau$

$$
\begin{aligned}
&= \frac{1}{h}\int_h^{t+h} T(\tau)u_0 d\tau - \frac{1}{h}\int_0^t T(t^\alpha\theta)u_0\, d\theta \\
&= \frac{1}{h}\int_0^{t+h} T(\tau)u_0\, d\tau - \frac{1}{h}\int_0^h T(\tau)u_0\, d\theta - \frac{1}{h}\int_0^t T(\tau)u_0\, d\tau \\
&= \frac{1}{h}\int_t^{t+h} T(\tau)u_0\, d\tau - \frac{1}{h}\int_0^h T(\tau)u_0\, d\theta.
\end{aligned}
$$

By passing the limit for $h\to 0$ and considering the lemma (Let $T(t)_{t\geq 0}$ be a $C_0 0-$semi-group. So: $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\, ds = T(t)$, $\forall t\geq 0$) we get

$$
A\int_0^t T(t^\alpha\theta)u_0\, d\theta = T(t)u_0 - u_0, \forall t\geq 0.
$$

In fact, for every $h>0$,

$$
\frac{T(h)-I}{h}\int_0^1 (\phi_\alpha(\theta)-\phi_\alpha(0))T(t^\alpha\theta)u_0\, d\theta
$$

$$
\begin{aligned}
&= \frac{1}{h}\left[T(h)\int_0^1 (\phi_\alpha(\theta)-\phi_\alpha(0))\, T(t^\alpha\theta)u_0\, d\theta - \int_0^1 (\phi_\alpha(\theta)-\phi_\alpha(0))T(t^\alpha\theta)\, u_0\, d\theta\right] \\
&= \frac{1}{h}\int_0^1 [(\phi_\alpha(\theta)-\phi_\alpha(0))T(t^\alpha\theta)T(h) - T(t^\alpha\theta)]u_0\, d\theta \\
&= \frac{1}{h}\int_0^1 (\phi_\alpha(\theta)-\phi_\alpha(0))(T(t^\alpha\theta+h) - T(t^\alpha\theta))u_0\, d\theta.
\end{aligned}
$$

Since

$$
\left\|\frac{(T(t^\alpha\theta+h)-T(t^\alpha\theta))u_0}{h}\right\|_X \leq \frac{C}{t^\alpha\theta}\left\|u_0\right\|_X, \quad \left|\frac{\phi_\alpha(\theta)-\phi_\alpha(0)}{\theta}\right| \leq C,
$$

for some constant $C > 0$ independent of θ and h, so, by dominated convergence theorem, we know

$$\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)\, u_0\, d\theta \in D(A).$$

Note that

$$\begin{aligned} AP_\alpha(t)u_0 &= A\left[\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\, d\theta + \phi_\alpha(0)\int_0^1 T(t^\alpha\theta)u_0\, d\theta + \int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\, d\theta\right] \\ &= A\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))\, T(t^\alpha\theta)u_0\, d\theta + \phi_\alpha(0)A\int_0^1 T(t^\alpha\theta)u_0\, d\theta + A\int_1^\infty \phi_\alpha(\theta)T\,(t^\alpha\theta)\, u_0\, d\theta \\ &= \int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))\, AT(t^\alpha\theta)u_0\, d\theta + \frac{\phi_\alpha(0)\,(T(t^\alpha)\, u_0 - u_0)}{t^\alpha} + \int_1^\infty \phi_\alpha(\theta)AT\,(t^\alpha\theta)\, u_0\, d\theta. \end{aligned}$$

Therefore

$$\|AP_\alpha(t)u_0\|_X \leq \frac{C}{t^\alpha\theta}\,\|u_0\|_X\,. \tag{1.9}$$

For some positive constant C. By dominated convergence theorem, we obtain that for $u_0 \in X$,

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}AS_\alpha(t)u_0,\ t > 0.$$

Furthermore, if $u_0 \in D(A)$, then

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}S_\alpha(t)Au_0,\ t > 0.$$

Since

$$\begin{aligned} {}_0I_t^{1-\alpha}\left(t^{\alpha-1}S_\alpha(t)Au_0\right) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{s^{\alpha-1}}{(t-s)^\alpha}\int_0^\infty \alpha\theta\phi_\alpha(\theta)T(s^\alpha\theta)Au_0\, d\theta ds. \\ \int_0^\infty \alpha\theta\phi_\alpha(\theta)T(s^\alpha\theta)Au_0\, d\theta &= \frac{1}{2\pi i}\int_{\Gamma'} E_{\alpha,\alpha}(\lambda s^\alpha)(\lambda - A)^{-1}Au_0\, d\lambda, \end{aligned}$$

where $\Gamma^{'}$ is a path composed from two rays $\rho e^{i\pi}$ $\rho \geq 1$, $\pi/2 < \tau < \pi$ and $\rho e^{-i\pi}$and a curve $e^{i\pi}$, $-\tau \leq \beta \leq \tau$,

$${}_0I_t^{1-\alpha}(t^{\alpha-1}S_\alpha(t)Au_0) = P_\alpha(t)Au_0 = AP_\alpha(t)u_0,$$

by similar argument with ${}^c_0D_t^\alpha P_\alpha(t)u_0 = AP_\alpha(t)u_0$, one can prove that $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ belongs to $\mathcal{F}_0^\gamma(s_\mu^0)$ for $t > 0$ and hence, such as (Let $-1 < \gamma < 0$ and let s_μ^0 with $0 < \mu < \pi$ be the open sector $\{z \in C\setminus\ \{0\} : |\arg(z)| < \mu\}$ and s_μ bits closure, that is $s_\mu = \{z \in C\setminus\ \{0\} : |\arg(z)| \leq \mu\} \cup \{0\}$. Set

$$\mathcal{F}_0^\gamma(s_\mu^0) = \cup_{s<0}\Psi_s^\gamma(s_\mu^0) \cup \Psi_0(s_\mu^0).$$

$\mathcal{F}(s_\mu^0) = \{f \in H(s_\mu^0)$; there $k, n \in N$ such that $f\Psi_n^k \in \mathcal{F}_0(s_\mu^0)\}$; where

$$H(s_\mu^0) = \{f : s_\mu^0 \to c; f\, is\, holomorphic\},$$

$$H^{\infty}(s_{\mu}^{0}) = \{f \in H(s_{\mu}^{0}); f\, is\, bounded\},$$

$$\varphi_0(z) = \frac{1}{1+z}, \Psi_n(z) = \frac{1}{(1+z)^n}, z \in C\backslash\{-1\}, n \in N \cup \{0\},$$

$$\Psi_0(s_{\mu}^{0}) = \{f \in H(s_{\mu}^{0}) : s_{\mu}^{0} \sup_{z\in C}\left|\frac{f(z)}{\varphi_0(z)}\right| < \infty\}.$$

$$\begin{aligned} {}_0I_t^{1-\alpha}(t^{\alpha-1}S_{\alpha}(t)Au_0) &= {}_0I_t^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}))Au_0 \\ &= (E_{\alpha}(\lambda t^{\alpha}))Au_0 \\ &= P_{\alpha}(t)Au_0, \end{aligned}$$

in view of ${}_0I_t^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})) = E_{\alpha}(\lambda t^{\alpha})$ this completes the proof.
Befor proceeding with our theory further, we present the following result.

$${}_0I_t^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})) = E_{\alpha,1}(\lambda t^{\alpha}),$$

so,

$${}_0I_t^{1-\alpha}(t^{\alpha-1}S_{\alpha}(t)Au_0) = P_{\alpha}(t)Au_0 = AP_{\alpha}(t)u_0. \tag{1.10}$$

Therefore, we get

$${}_0^C D_t^{\alpha} P_{\alpha}(t)u_0 = AP_{\alpha}(t)u_0,$$

if $u_0 \in D(A), t > 0$.
To prove ${}_0^c D_t^{\alpha} P_{\alpha}(t)u_0 = AP_{\alpha}(t)u_0$ first it is easy to see that $\frac{1}{\varphi_0} \in \mathcal{F}(s_{\mu}^{0})$ and the operator $\varphi_0(A)$ is infective. Taking $u_0 \in D(A)$, by $f(A)g(A) = (fg)(A)$ provided that $g(A)$ is bounded or $D((fg))(A) \subset D(g(A))$ one has

$$P_{\alpha}(t)u_0 = E_{\alpha}(\lambda t^{\alpha})(A)u_0 = (E_{\alpha}(\lambda t^{\alpha})\varphi_0)(A)(\frac{1}{\varphi_0})(A)x,$$

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}} + \varepsilon_{\alpha,\beta}(z), |\arg(z)| \le \frac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), |\arg(-z)| < (1-\frac{1}{2}\alpha)\pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{k=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)} + 0(|z|^{-N}),\, as\, z \to \infty,$$

we have

$$\sup_{z\to\infty}|zt^{\alpha}E_{\alpha}(\lambda t^{\alpha})| < \infty,$$

which implies that

$$\left|zE_{\alpha}(\lambda t^{\alpha})(1+\lambda)^{-1}\right| \le c\,|z|^{-1}\,t^{-\alpha},\, as\, z \to \infty,$$

where c is a constant which is independent of t. Consequently

$$zE_\alpha(\lambda t^\alpha)(1+\lambda)^{-1} \in \mathcal{F}_0^\gamma(s_\mu^0).$$

Notice also that

$${}_0^c D_t^\alpha E_\alpha(\lambda t^\alpha)(1+\lambda)^{-1} R(\lambda; A) = (z) E_\alpha(\lambda t^\alpha)(1+\lambda)^{-1} R(\lambda; A),$$

combining $(f(A)g(A) \subset (fg)(A)$ for all $f, g \in \mathcal{F}(s_\mu^0))$ and $zE_\alpha(\lambda t^\alpha)(1+\lambda)^{-1}$ we get

$$\begin{aligned} {}_0^c D_t^\alpha [E_\alpha(\lambda t^\alpha)(1+\lambda^\beta)^{-1}(A)] &= \frac{1}{2\pi i} \int_{\mu_0} zE_\alpha(\lambda t^\alpha)(1+\lambda)^{-1} R(\lambda; A)\, dz \\ &= zA[E_\alpha(\lambda t^\alpha)(1+\lambda)^{-1}](A) \\ &= A[E_\alpha(\lambda t^\alpha)(1+\lambda)^{-1}](A). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} {}_0^c D_t^\alpha P_\alpha(t) u_0 &= A[E_\alpha(\lambda t^\alpha)(1+\lambda)^{-1}](A)(1+\lambda)(A) u_0 \\ &= A[E_\alpha(\lambda t^\alpha)](A) u_0 = AP_\alpha(t) u_0. \end{aligned}$$

Next, we prove that the conclusion also holds if $u_0 \in X$. In fact, if $u_0 \in X$, then we can find $\{u_{0,n}\} \subset D(A)$ such that $u_{0,n} \to 0$ in X. By (1.10) and Lemma 1.2, we have ${}_0^c D_t^\alpha P_\alpha(t) u_0 = AP_\alpha(t) u_0$ and $\|P_\alpha(t) u_0\|_{L^1(R^N)}$ we know ${}_0^C D_t^\alpha P_\alpha(t) u_{0,n} = AP_\alpha(t) u_{0,n}$, and $\|P_\alpha(t) u_{0,n}\|_X \leq \|u_{0,n}\|_X$.
We denote $u_n = P_\alpha(t) u_{0,n}$. Then, there exists $u \in X$ such that for every $T > 0$, $u_n \to u$ uniformly in X for $t \in [0, T]$ as $n \to \infty$. Since

$$\left\| {}_0 I_t^{1-\alpha} u_n \right\|_X \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \|u_n\|_{L^\infty((0,T),X)},\ t \in [0, T].$$

So we know ${}_0 I_t^{1-\alpha} u_n \to_0 I_t^{1-\alpha} u$ in X. By (1.9),

$$\left\| {}_0^C D_t^\alpha u_n \right\|_X = \|{}_0^c D_t^\alpha P_\alpha(t) u_{0,n}\|_X = \|AP_\alpha(t) u_{0,n}\|_X \leq \frac{C}{t^\alpha \theta} \|u_{0,n}\|_X,$$

for some constant $C > 0, t > 0$.
Hence, for every $\delta > 0$, there exists $w \in C([\delta, \infty), X)$ such that ${}_0^C D_t^\alpha u_n \to w$ uniformly in X on $t \in [\delta, \infty)$. Note that for $t \in [\delta, \infty)$,

$$\begin{aligned} {}_0^C D_t^\alpha u_n &=_0^c D_t^\alpha P_\alpha(t) u_{0,n} = AP_\alpha(t) u_{0,n} =_0 I_t^{1-\alpha}(t^{\alpha-1} S_\alpha(t) A u_{0,n}) =_0 I_t^{1-\alpha}(\frac{d}{dt} P_\alpha(t) u_{0,n}) \\ &= \frac{d}{dt}\left({}_0 I_t^{1-\alpha}(P_\alpha(t) u_{0,n} - u_{0,n})\right) = Au_n. \end{aligned}$$

We have $u_n = P_\alpha(t) u_{0,n} \to u$, $u_{0,n} \to u_0$, ${}_0^c D_t^\alpha u_n \to_0^c D_t^\alpha u$ and ${}_0^c D_t^\alpha u_n \to w$, so

$$ {}_0^cD_t^\alpha u_n = \frac{d}{dt}({}_0I_t^{1-\alpha}(P_\alpha(t)u_{0,n} - u_{0,n})) \to \frac{d}{dt}({}_0I_t^{1-\alpha}(u - u_0)) ={}_0^c D_t^\alpha u, $$

and we have

$$ {}_0^cD_t^\alpha u_n \to w ={}_0^c D_t^\alpha u. $$

So

$$ w = \frac{d}{dt}({}_0I_t^{1-\alpha}(u - u_0)) ={}_0^C D_t^\alpha u,\ t \in [\delta, \infty). $$

We have ${}_0^cD_t^\alpha u_n = Au_n$ and ${}_0^cD_t^\alpha u_n \to w$, so $Au_n \to Au$. Since A is closed, we have $w = Au$, that is ${}_0^C D_t^\alpha u = Au = P_\alpha(t)u_0, t \in [\delta, \infty)$. By arbitrariness of δ, we have ${}_0^C D_t^\alpha u = P_\alpha(t)u_0,\ t > 0$. ■

Lemma 46 *Assume that* $f \in L^q((0,T),\ C_0(R^N)), q > 1$. *Let*

$$ z(t) = \int_0^t (t-s)^{\alpha-1} S_{\alpha-1}(t-s) f(s)\, ds, $$

then

$$ {}_0I_t^{1-\alpha} z = \int_0^t P_\alpha(t-s) f(s)\, ds. $$

Furthermore, if $q\alpha > 1$, *then* $z \in C((0,T), C_0(R^N))$.

Proof. Let $X = C_0(R^N)$. By Fubini theorem and (1.10), we have

$$ \begin{aligned} {}_0I_t^{1-\alpha} z &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau)\, d\tau ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_\tau^s (t-s)^{-\alpha} (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau)\, ds d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^{t-\tau} (t-s-\tau)^{-\alpha} s^{\alpha-1} S_\alpha(s) f(\tau)\, ds d\tau \\ &= \int_0^t P_\alpha(t-\tau) f(\tau)\, d\tau. \end{aligned} $$

For every $h > 0$ and $t + h \le T$, we have $z(t+h) - z(t) = I_1 + I_2$, where

$$ I_1 = \alpha \int_t^{t+h} \int_0^\infty \theta\phi_\alpha(\theta)(t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha\theta) f(\tau)\, d\theta d\tau. $$

$$ I_2 = \alpha \int_0^t \int_0^\infty \theta\phi_\alpha(\theta)[(t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha\theta) - (t-\tau)^{\alpha-1} T((t-\tau)^\alpha\theta)] f(\tau)\, d\theta d\tau. $$

By Hölder inequality, we have

$$ \|I_1\|_X \ \le\ \alpha \int_t^{t+h} \int_0^\infty \theta\phi_\alpha(\theta)(t+h-\tau)^{\alpha-1} \|f(\tau)\|_X\, d\theta d\tau $$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-\tau)^{\alpha-1} \|f(\tau)\|_X \, d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \|f\|_{L^q((0,T),X)} \left(\int_t^{t+h} (t+h-\tau)^{\frac{(q\alpha-1)}{q}} \, d\tau \right)^{\frac{q-1}{q}} \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{q-1}{q\alpha-1} \right)^{\frac{q-1}{q}} \|f\|_{L^q((0,T),X)} \, h^{\frac{(q\alpha-1)}{q}},
\end{aligned}
$$

so

$$
\|I_1\|_X = \frac{1}{\Gamma(\alpha)} \left(\frac{q-1}{q\alpha-1} \right)^{\frac{q-1}{q}} \|f\|_{L^q((0,T),X)} \, h^{\frac{(q\alpha-1)}{q}}. \tag{1.11}
$$

Note that, for $0 < \tau < t$,

$$
\left\| (t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha \theta) f(\tau) - (t-\tau)^{\alpha-1} T((t-\tau)^\alpha \theta) f(\tau) \right\|_X \leq 2(t-\tau)^{\alpha-1} \|f(\tau)\|_X ,
$$

and there exists constant $C > 0$ such that

$$
\begin{aligned}
&\left\| [(t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha \theta) - (t-\tau)^{\alpha-1} T((t-\tau)^\alpha \theta)] f(\tau) \right\|_X \\
&\leq \left| (t+h-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1} \right| \|T((t+h-\tau)^\alpha \theta) f(\tau)\|_X \\
&\quad + (t-\tau)^{\alpha-1} \|T((t+h-\tau)^\alpha \theta) - T((t-\tau)^\alpha \theta)) f(\tau)\|_X \\
&\leq C (t-\tau)^{\alpha-2} h \|f(\tau)\|_X .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|I_2\|_X &\leq C \int_0^t \int_0^\infty \alpha\theta\phi_\alpha(\theta) \min\left\{ \frac{1}{(t-\tau)^{1-\alpha}}, \frac{h}{(t-\tau)^{2-\alpha}} \right\} d\theta \, \|f(\tau)\|_X \, d\tau \\
&\leq \frac{C}{\Gamma(\alpha)} \left(\int_0^t \left(\min\left\{ \frac{1}{(t-\tau)^{1-\alpha}}, \frac{h}{(t-\tau)^{2-\alpha}} \right\} \right)^{\frac{q}{(q-1)}} d\tau \right)^{\frac{(q-1)}{q}} \|f\|_{L^q((0,T),X)} .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_0^t \left(\min\left\{ \frac{1}{(t-\tau)^{1-\alpha}}, \frac{h}{(t-\tau)^{2-\alpha}} \right\} \right)^{\frac{q}{q-1}} d\tau = \int_0^t \left(\min\left\{ \frac{1}{\tau^{1-\alpha}}, \frac{h}{\tau^{2-\alpha}} \right\} \right)^{\frac{q}{q-1}} d\tau \\
\leq \; &\int_0^\infty \left(\min\left\{ \frac{1}{\tau^{1-\alpha}}, \frac{h}{\tau^{2-\alpha}} \right\} \right)^{\frac{q}{q-1}} d\tau = \int_0^h \tau^{\frac{q(\alpha-1)}{(q-1)}} \, d\tau + \int_h^\infty h^{\frac{q}{q-1}} \tau^{\frac{q(\alpha-2)}{q-1}} \, d\tau \\
= \; &\frac{q(q-1)}{(q\alpha-1)(q+1-q\alpha)} h^{\frac{q\alpha-1}{q-1}},
\end{aligned}
$$

so,

$$
\|I_2\|_X < C \|f\|_{L^q((0,T),X)} \, h^{\frac{q\alpha-1}{q}}. \tag{1.12}
$$

Hence, (1.11)-(1.12) imply that the conclusion of Lemma 2.4 also holds. ■

Chapter 2

Local existence

In this chapter, we give the local existence and uniqueness of mild solution of the problem (1)-(2). First, we give the definition of the mild solution of (1)-(2).

Definition 47 *Let $u_0 \in C_0(R^N)$, $T > 0$. We call that $u \in C([0,T], C_0(R^N))$ is a mild solution of the problem (1)-(2) if u satisfies the following integral equation*

$$u(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \left|u\right|^{p-1} u(s)\, ds, t \in [0,T].$$

For the problem (1)-(2), we have the following local existence result.

Theorem 48 *Given $u_0 \in C_0(R^N)$, then there exists a maxial time $T_{\max} = T(u_0) > 0$ such that the problem (1)-(2) has a unique mild solution u in $C([0,T], C_0(R^N))$ and either $T_{\max} = +\infty$ or $T_{\max} < +\infty$ and $\|u\|_{L^\infty((0,t),C_0(R^N))} \to +\infty$ as $t \to T_{\max}$. If, in addition, $u_0 \geq 0$, $u_0 \neq 0$, then $u(t) > 0$ and $u(t) \geq P_\alpha(t)u_0$ for $t \subset (0, T_{\max})$. Moreover, if $u_0 \subset L^r(R^N)$ for some $r \subset [1,\infty)$, then $u \in C([0,T_{\max}), L^r(R^N))$.*

Proof. For given $T > 0$ and $u_0 \in C_0(R^N)$, let

$$E_T = \left\{ u : u \in C([0,T], C_0(R^N)),\ \|u\|_{L^\infty((0,T),L^\infty(R^N))} \leq 2\left\|u_0\right\|_{L^\infty(R^N)} \right\},$$

$$d(u,v) = \max_{t\in[0,T]} \left\|u(t) - v(t)\right\|_{L^\infty(R^N)} \; for\, u, v \in E_T.$$

Since $C([0,T], C_0(R^N))$ is a Banach space, (E_T, d) is a complete metric space. We define the operator G on E_T as

$$G(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \left|u(s)\right|^{p-1} u(s)\, ds,\ u \in E_T,$$

then $G(u) \in C([0,T], C_0(R^N))$ in view of lemma(2.4). If $u \in E_T$, then by lemma (2.1)(b) and lemma (2.2)(b), for $t \subset [0,T]$,

$$\left\|G(u)(t)\right\|_{L^\infty(R^N)} \quad \leq \quad \left\|u_0\right\|_{L^\infty(R^N)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\|u(s)\right\|^p_{L^\infty(R^N)}\, ds$$

$$\leq \quad \|u_0\|_{L^\infty(R^N)} + \frac{2^p T^\alpha}{\alpha\Gamma(\alpha)} \|u_0\|^p_{L^\infty(R^N)}.$$

Hence, we have choose T small enough such that

$$\frac{2^p T^\alpha}{\alpha\Gamma(\alpha)} \|u_0\|^{p-1}_{L^\infty(R^N)} \leq 1,$$

so we get $\|G(u)\|_{L^\infty((0,T),L^\infty(R^N))} \leq 2\|u_0\|_{L^\infty(R^N)}$. Furthermore, for $u, v \in E_T$, we have for $t \in [0,T]$

$$\begin{aligned} \|G(u)(t) - G(v)(t)\|_{L^\infty(R^N)} &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| |u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s) \right\|_{L^\infty(R^N)} ds \\ &\leq \frac{4^{(p-1)} p T^\alpha \|u_0\|^{p-1}_{L^\infty(R^N)}}{\alpha\Gamma(\alpha)} \|u - v\|_{L^\infty((0,T),L^\infty(R^N))}. \end{aligned}$$

We can choose T small enough such that

$$\frac{p4^{(p-1)} T^\alpha \|u_0\|^{p-1}_{L^\infty(R^N)}}{\alpha\Gamma(\alpha)} \leq \frac{1}{2},$$

then

$$\|G(u)(t) - G(v)(t)\|_{L^\infty(R^N)} \leq \frac{1}{2} \|u - v\|_{L^\infty((0,T),C_0(R^N))}.$$

Therefore, G is contractive on E_T. So, G has a fixed point $u \in E_T$ by the contraction mapping principle.
Now, we prove the uniqueness. Let $u, v \in C([0,T], C_0(R^N))$ be the mild solutions of (1)-(2) for some $T > 0$, then there exists positive constant $C > 0$ such that

$$\begin{aligned} \|u(t) - v(t)\|_{L^\infty(R^N)} &= \|G(u)(t) - G(v)(t)\|_{L^\infty(R^N)} \\ &\leq C \int_0^t (t-s)^{\alpha-1} \|u(s) - v(s)\|_{L^\infty(R^N)} ds. \end{aligned}$$

Hence, by Gronwall's inequality, we know $u = v$.
Next, using the uniqueness of solution, we conclude that the existence of solution on a maximal interval $[0, T_{\max})$, where

$$T_{\max} = \sup\{T > 0 \, : \, there\, exists\, a\, mild\, solution\, \mathrm{u}\, of (1) - (2)\, in\, C([0,T], C_0(R^N))\}.$$

Assume that $T_{\max} < +\infty$ and there exists $M > 0$ such that for $t \in [0, T_{\max})$,

$$\|u(t)\|_{L^\infty(R^N)} \leq M.$$

Next, we will verify that $\lim_{t \to T_{\max}} u(t)$ exists in $C_0(R^N)$. In fact, for $0 < t < \tau < T_{\max}$, by the proof of lemma (2.4), there exists constant $C > 0$ such that

$$\|u(t) - u(\tau)\|_{L^\infty(R^N)} \quad \leq \quad \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^\infty(R^N)}$$

$$+\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u(s)|^{p-1}\,u(s) - (\tau-s)^{\alpha-1} S_\alpha(\tau-s)\,|u(s)|^{p-1}\,u(s)\,ds\right\|_{L^\infty(R^N)}$$

$$+\left\|\int_t^\tau (\tau-s)^{\alpha-1} S_\alpha(\tau-s)\,|u(s)|^{p-1}\,u(s)\,ds\right\|_{L^\infty(R^N)}$$

$$\le \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^\infty(R^N)} + \frac{M^p}{\Gamma(\alpha)}\int_t^\tau (\tau-s)^{\alpha-1} d\,s + CM^p \int_0^t \min\{(t-s)^{\alpha-1}, (t-s)^{\alpha-2}(\tau-t)\}\,ds$$

$$\le \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^\infty(R^N)} + \frac{M^p}{\alpha\Gamma(\alpha)}(\tau-t)^\alpha + CM^p\frac{1}{\alpha(1-\alpha)}(\tau-t)^\alpha.$$

Since $P_\alpha(t)u_0$ is uniformly continuous in $[0, T_{\max}]$, so $\lim_{t\to T_{\max}} u(t)$ exists.
We denote $u_{T_{\max}} = \lim_{t\to T_{\max}} u(t)$ and define $u(T_{\max}) = u_{T_{\max}}$. Hence,
$u \in C([0, T_{\max}], C_0(R^N))$ and then, by Lemma (2.4),

$$\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u(s)|^{p-1}\,u(s)\,ds \in C([0, T_{\max}], C_0(R^N)).$$

For $h > 0$, $\delta > 0$, let

$$E_{h,\delta} = \{u \in C([T_{\max}, T_{\max}+h], C_0(R^N)) : u(T_{\max}) = u_{T_{\max}}, d(u, u_{T_{\max}}) \le \delta\},$$

where

$$d(u,v) = \max_{t\in[T_{\max}, T_{\max}+h]} \|u(t) - v(t)\|_{L^\infty(R^N)}\ for\, u, v \in E_{h,\delta}.$$

Via $C([T_{\max}, T_{\max}+h], C_0(R^N))$ is a Banach space, we know $(E_{h,\delta}, d)$ is a complete metric space.
We define the operator G on $E_{h,\delta}$ as

$$\begin{aligned} G(v)(t) &= P_\alpha(t)u_0 + \int_0^{T_{\max}} (t-\tau)^{\alpha-1} S_\alpha(t-\tau)\,|u|^{p-1}\,u(\tau)d\tau \\ &+ \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)\,|v|^{p-1}\,v(\tau)d\tau\, v \in E_{h,\delta}. \end{aligned}$$

Clearly, $G(v) \in C([T_{\max}, T_{\max}+h], C_0(R^N))$ and $G(v)(T_{\max}) = u_{T_{\max}}$.
If $v \in E_{h,\delta}$, then for $t \in [T_{\max}, T_{\max}+h]$,

$$\|G(v)(t) - u_{T_{\max}}\|_{L^\infty(R^N)} \le \|P_\alpha(t)u_0 - P_\alpha(T_{\max})u_0\|_{L^\infty(R^N)} + \|I_3\|_{L^\infty(R^N)} + \|I_4\|_{L^\infty(R^N)},$$

where

$$\begin{aligned} I_3 &= \int_0^{T_{\max}} (t-\tau)^{\alpha-1} S_\alpha(t-\tau)\,|u(\tau)|^{p-1}\,u(\tau)d\tau - (T_{\max}-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau)\,|u(\tau)|^{p-1}\,u(\tau)d\tau, \\ I_4 &= \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)\,|v|^{p-1}\,v(\tau)d\tau. \end{aligned}$$

Taking h small enough such that

$$\begin{aligned}
\|P_\alpha(t)u_0 - P_\alpha(T_{\max})u_0\|_{L^\infty(R^N)} &< \frac{\delta}{3}\, for\, t \in [T_{\max}, T_{\max}+h], \\
\|I_3\|_{L^\infty(R^N)} &\leq \frac{\delta}{3},
\end{aligned}$$

$$\begin{aligned}
\|I_4\|_{L^\infty(R^N)} &\leq \left\| \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau)(|v|^{p-1} v(\tau) - |u_{T_{\max}}|^{p-1} u_{T_{\max}}) d\tau \right\|_{L^\infty(R^N)} \\
&+ \left\| \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau) |u_{T_{\max}}|^{p-1} u_{T_{\max}} d\tau \right\|_{L^\infty(R^N)} \\
&\leq C\delta \int_{T_{\max}}^t (t-\tau)^{\alpha-1} d\tau + \|u_{T_{\max}}\|^p_{L^\infty(R^N)} \frac{1}{\Gamma(\alpha)} \int_{T_{\max}}^t (t-\tau)^{\alpha-1} d\tau \\
&= \frac{C\delta}{\alpha}(t-T_{\max})^\alpha + \frac{\|u_{T_{\max}}\|^p_{L^\infty(R^N)}}{\Gamma(\alpha+1)}(t-T_{\max})^\alpha \leq \frac{\delta}{3},
\end{aligned}$$

for $t \in [T_{\max}, T_{\max}+h]$. Then, we have $\|G(v)(t) - u_{T_{\max}}\|_{L^\infty(R^N)} \leq \delta$, $t \in [T_{\max}, T_{\max}+h]$. Next, we will prove that G is contractive on $E_{h,\delta}$ for h small enough. In fact, for $w, v \in E_{h,\delta},\ t \in [T_{\max}, T_{\max}+h]$,

$$\begin{aligned}
&\|w(t) - v(t)\|_{L^\infty(R^N)} \\
&\leq \int_{T_{\max}}^t (t-\tau)^{\alpha-1} \left\| S_\alpha(t-\tau)(|w|^{p-1} w(\tau) - |v|^{p-1} v(\tau) \right\|_{L^\infty(R^N)} d\tau \\
&\leq \|w-v\|_{L^\infty((T_{\max}, T_{\max}+h), L^\infty(R^N))} (\|w\|_{L^\infty((T_{\max}, T_{\max}+h), L^\infty(R^N))} \\
&\quad + \|v\|_{L^\infty((T_{\max}, T_{\max}+h), L^\infty(R^N))})^{p-1} \frac{p}{\Gamma(\alpha)} \int_{T_{\max}}^t (t-\tau)^{\alpha-1} d\tau \\
&\leq \frac{2^{p-1}p}{\Gamma(\alpha+1)} (\delta + \|u_{T_{\max}}\|_{L^\infty(R^N)})^{p-1} (t-T_{\max})^\alpha d(w,v).
\end{aligned}$$

Choosing h small cnough such that

$$\frac{2^{p-1}p}{\Gamma(\alpha+1)} (\delta + \|u_T\|_{L^\infty(R^N)})^{p-1} h^\alpha \leq \frac{1}{2}.$$

Then, G is contractive on $E_{h,\delta}$. So, we know G has a fixed point $v \in E_{h,\delta}$. Since

$$v(T_{\max}) = G(v(T_{\max})) = u(T_{\max}),$$

if we let

$$\sim u(t) = \begin{cases} u(t), t \in [0, T_{\max}), \\ v(t), t \in [T_{\max}, T_{\max}+h], \end{cases}$$

then $\sim u \in C([0, T_{\max}+h], C_0(R^N))$ and

$$\sim u(t) = P_\alpha(t)u_0 + \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) |\sim u|^{p-1} \sim u(\tau) d\tau.$$

Therefore, $\sim u(t)$ is a mild solution of (1)-(2), which contradicts with the definition of $T_{\max}$. If $u_0 \in L^r(R^N)$ for some $1 \leq r < \infty$, then repeating the above argument, we get the conclusion. Moreover, if $u_0 \geq 0$, then we can obtain the nonnegative solution of (1) applying the above argument in the set

$$E_T^+ = \{u \in E_T : u \geq 0\}.$$

Then, we know $u(t) \geq P_\alpha(t)u_0 > 0$ on $t \in (0, T_{\max})$. ■

Chapter 3

Blow-up and global existence

In this chapter, we prove the blow-up results and global existence of solutions of (1.1)-(1.2). First, we give the definition of weak solution of (1)-(2).

Definition 49 *We call $u \in L_p((0,T), L^{\infty}_{loc}(R^N))$, for $u_0 \in L^{\infty}_{loc}(R^N)$ and $T > 0$, is a weak solution of (1) if*

$$\int_{R^N}\int_0^T (|u|^{p-1}\, u\varphi + u_0 {}_t^c D_T^{\alpha}\varphi)\, dtdx = \int_{R^N}\int_0^T u(-\Delta\varphi)\, dtdx + \int_{R^N}\int_0^T {}_t^c D_T^{\alpha}\varphi\, dtdx,$$

for every $\varphi \in C^{2,1}_{x,t}(R^N \times [0,T])$ with $supp_x\varphi \subset\subset R^N$ and $\varphi(.,T) = 0$.

Lemma 50 *Assume $u_0 \in C_0(R^N)$, let $u \in C([0,T], C_0(R^N))$ be a mild solution of (1)-(2), then u is also a weak solution of (1)-(2).*

Proof. Assuming that $u \in C([0,T], C_0(R^N))$ is a mild solution of (1)-(2), we have

$$u - u_0 = P_{\alpha}(t)u_0 - u_0 + \int_0^t (t-\tau)^{\alpha-1} S_{\alpha}(t-\tau)\, |u|^{p-1}\, u\, d\tau.$$

Note that by Lemma 2.4,

$${}_0I_t^{1-\alpha}\left(\int_0^t (t-\tau)^{\alpha-1} S_{\alpha}(t-\tau)\, |u|^{p-1}\, u(\tau) d\tau\right) = \int_0^t P_{\alpha}(t-s)\, |u|^{p-1}\, u(s)\, ds,$$

so, we know

$${}_0I_t^{1-\alpha}(u-u_0) ={}_0 I_t^{1-\alpha}(P_{\alpha}(t)u_0 - u_0) + \int_0^t P_{\alpha}(t-\tau)\, |u|^{p-1}\, u(\tau)\, d\tau.$$

Then, for every $\varphi \in C^{2,1}_{x,t}(R^N \times [0,T])$ with supp $supp_x\varphi \subset\subset R^N$ and $\varphi(x,T) = 0$, we get

$$\int_{R^N} {}_0I_t^{1-\alpha}(u-u_0)\varphi\, dx = I_5(t) + I_6(t), \tag{3.1}$$

where

$$I_5(t)=\int_{R^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0-u_0)\varphi\,dx,\ I_6(t)=\int_{R^N}\int_0^t P_\alpha(t-s)\,|u|^{p-1}\,u(s)\varphi\,dx.$$

By Lemma 2.3,

$$\frac{dI_5}{dt}=\int_{R^N} A(P_\alpha(t)u_0)\varphi\,dx+\int_{R^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0-u_0)\varphi_t\,dx. \tag{3.2}$$

For every $h>0$, $t\in[0,T)$ and $t+h\to T$, we have

$$\frac{1}{h}\left(I_6(t+h)-I_6(t)\right)=\frac{1}{h}\int_0^{t+h}\int_{R^N} P_\alpha(t+h-s)\,|u|^{p-1}\,u\,ds\varphi(t+h,x)\,dx-$$

$$\frac{1}{h}\int_0^t\int_{R^N} P_\alpha(t-s)\,|u|^{p-1}\,u\,ds\varphi(t,x)\,dx=I_7+I_8+I_9,$$

where

$$\begin{aligned}
I_7 &= \frac{1}{h}\int_{R^N}\int_t^{t+h}\int_0^\infty \phi_\alpha(\theta)T((t+h-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\varphi(t+h,x)\,dx,\\
I_8 &= \frac{1}{h}\int_{R^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)T((t+h-s)^\alpha\theta)-T((t-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\theta ds\varphi(t,x)\,dx,\\
I_9 &= \frac{1}{h}\int_{R^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)T((t+h-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\theta ds(\varphi(t+h,x)-\varphi(t,x))\,dx.
\end{aligned}$$

By dominated convergence theorem, we conclude that

$$\begin{aligned}
I_7 &\to \int_{R^N}|u|^{p-1}\,u\varphi\,dx\,as\,h\to 0,\\
I_9 &\to \int_{R^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)T((t-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\theta ds\varphi_t\,dx\\
&= \int_{R^N}\int_0^t P_\alpha(t-s)\,|u|^{p-1}\,u(s)\,ds\varphi_t\,dx\,as\,h\to 0.
\end{aligned}$$

Since

$$\begin{aligned}
I_8 &= \int_{R^N}\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}A(T((t+\tau h-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\tau d\theta ds\varphi dx\\
&\quad -\int_{R^N} A\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}T((t+\tau h-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\tau d\theta ds\varphi dx\\
&= \int_{R^N}\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}T((t+\tau h-s)^\alpha\theta)\,|u|^{p-1}\,u(s)\,d\tau d\theta dsA\varphi dx,
\end{aligned}$$

by dominated convergence theorem, we know

$$I_8\to\int_{R^N}\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\,|u|^{p-1}\,u(s)\,dsA\varphi dx\,as\,h\to 0.$$

Hence, the right derivative of I_6 on $[0,T)$ is

$$\int_{R^N} |u|^{p-1} u\varphi dx+\int_{R^N}\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u|^{p-1} u(s)\,ds A\varphi dx+\int_{R^N}\int_{0\alpha}^t P_\alpha(t-s)\,|u|^{p-1} u(s)\,ds\varphi_t dx,$$

and it is continuous in $[0,T)$. Therefore

$$\begin{aligned}\frac{dI_6}{dt} &= \int_{R^N} |u|^{p-1} u\varphi\,dx+\int_{R^N}\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u|^{p-1} u(s)\,ds A\varphi\,dx \\ &\quad +\int_{R^N}\int_0^t P_\alpha(t-s)\,|u|^{p-1} u(s)\,ds\varphi_t\,dx \\ &= \int_{R^N} |u|^{p-1} u\varphi\,dx+\int_{R^N}\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u|^{p-1} u(s)\,ds A\varphi\,dx \\ &\quad +\int_{R^N} I_{0/t}^{1-\alpha}(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,|u|^{p-1} u(s)\,ds)\varphi_t\,dx, \qquad (3.3)\end{aligned}$$

for $t\in[0,T)$. It follows from (3.1)-(3.3) that

$$\begin{aligned}0 &= \int_0^T \frac{d}{dt}\int_{R^N} I_{0/t}^{1-\alpha}(u-u_0)\varphi\,dx = \int_0^T \frac{dI_5}{dt}+\frac{dI_6}{dt}\,dt \\ &= \int_0^T\int_{R^N} P_\alpha(t)u_0\Delta\varphi\,dxdt - \int_0^T\int_{R^N}(u-u_0)\,{}_0^C D_T^\alpha\varphi\,dxdt + \int_0^T\int_{R^N}|u|^{p-1}u\varphi\,dxdt \\ &\quad +\int_0^T\int_{R^N}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\,|u|^{p-1}\;ds\Delta\varphi\,dxdt \\ &= \int_0^T\int_{R^N} u\Delta\varphi dxdt - \int_0^T\int_{R^N}(u-u_0)\,{}_0^C D_T^\alpha\varphi\,dxdt + \int_0^T\int_{R^N}|u|^{p-1}u\varphi\,dxdt.\end{aligned}$$

Hence, we get the conclusion. We say the solution u of the problem (1)-(2) blows up in a finite time T if $\lim_{t\to T} = \|u(t,.)\|_{L^\infty(R^N)} = +\infty$. Now, we give a blow-up result of the problem (1)-(2). ■

Theorem 51 *Let $u_0\in C_0(R^N)$ and $u_0\geq 0$, if*

$$\int_{R^N} u_0(x)\chi(x)\,dx > 1,$$

where

$$\chi(x) = \left(\int_{R^N} e^{-\sqrt{N^2+|x|^2}}\,dx\right)^{-1} e^{-\sqrt{N^2+|x|^2}},$$

then the mild solutions of (1)-(2) blow up in a finite time.

Proof. We take $\Psi\in C_0^\infty(R)$ such that

$$\Psi(x) = \begin{cases} 1, |x|\leq 1, \\ 0, |x|\geq 2,\end{cases}$$

and $0 \le \Psi(x) \le 1$, $x \in R$. Let $\Psi_n(x) = \Psi(x/n)$, $n = 1, 2, \cdots$. By Lemma 4.2, a mild solution of (1)-(2) is also a weak solution of it. So, using the definition of weak solution of (1)-(2), taking $\varphi_n(x,t) = \chi(x)\Psi_n(x)\varphi_1(t)$ for $\varphi_1 \in C^1([0,T])$ with $\varphi_1(T) = 0$ and $\varphi_1 \ge 0$, we have

$$\int_{R^N}\int_0^T u^p\varphi_n dxdt + \int_{R^N}\int_0^T u_0{}_t^C D_T^\alpha \varphi_n dxdt = \int_{R^N}\int_0^T (-u\Delta\varphi_n + u_t^C D_T^\alpha\varphi_n)\, dxdt. \quad (3.4)$$

Since $\Delta(\chi\Psi_n) = (\Delta\chi)\Psi_n + 2\nabla\chi.\nabla\Psi_n + (\Delta\Psi_n)\chi$ and $\Delta\chi \ge -\chi$, by (3.4) and the dominated convergence theorem, let $n \to \infty$, we have

$$\int_{R^N}\int_0^T u^p\chi\varphi_1 dxdt + \int_{R^N}\int_0^T u_0\chi_t^C D_T^\alpha\varphi_1 dxdt \le \int_{R^N}\int_0^T (u\chi\varphi_1 + u\chi_t^C D_T^\alpha\varphi_1)\, dxdt. \quad (3.5)$$

Hence, by Jensen's inqequality and (3.5), we have

$$\int_0^T \left(\int_{R^N} u\chi\, dx\right)^p \varphi_1 dt + \int_{R^N}\int_0^T u_0\chi_t^C D_T^\alpha\varphi_1 dxdt \le \int_{R^N}\int_0^T (u\chi\varphi_1 + u_0\chi_t^C D_T^\alpha\varphi_1)\, dxdt.$$

So, if we denote $f(t) = \int_{R^N} u\chi\, dx$, then

$$\int_0^T (f^p - f)\varphi_1 dt \le \int_0^T (f - f(0))_t^C D_T^\alpha\varphi_1\, dt. \quad (3.6)$$

We take $\varphi_1 =_t I_T^\alpha \sim\Psi(t)$ where $\sim\Psi \in C_0^1((0,T))$ and $\sim\Psi \ge 0$, then (3.6) implies

$$\int_0^T {}_t I_T^\alpha (f^p - f)\sim\Psi dt = \int_0^T (f^p - f)\sim{}_t I_T^\alpha\Psi(t) dt \le \int_0^T (f - f(0))\sim\Psi\, dt.$$

Hence,

$${}_t I_T^\alpha(f^p - f) + f(0) \le f. \quad (3.7)$$

In view of $f(0) = \int_{R^N} u_0(x)\chi(x)dx > 1$ and the continuity of f, we obtain $f(t) > 1$ when t is small enough. Then (3.7) implies $f(t) \ge f(0) > 1$ for $t \in [0,T]$. Taking $\varphi_1(t) = (1 - \frac{t}{T})^m$, $t \in [0,T]$ $m \ge \max\{1, p\alpha/(p-1)\}$, we know there exists constant $C > 0$ such that

$$\int_0^T (f^p - f)\varphi_1 dt + Cf(0)T^{1-\alpha} \le \varepsilon\int_0^T f^p\varphi_1 dt + C(\varepsilon)T^{1-p\alpha/(p-1)}.$$

Choosing ε small enough such that $f(0) > (1-\varepsilon)^{-1/(p-1)}$, we then have $f(0) \le CT^{\alpha - p\alpha/(p-1)}$ for some constant $C > 0$. If the solution of (1.1)-(1.2) exists globally, we get $f(0) = 0$ by taking $T \to \infty$, which contradicts with $f(0) > 1$. Hence, we give the main result of this paper. ■

Theorem 52 *Let $u_0 \in C_0(R^N)$ and $u_0 \ge 0$, $u_0 \not\equiv 0$, then*
(a) If $1 < p < 1 + 2/N$, then the mild solition of (1)-(2) blows up in a finite time.
(b) If $p \ge 1 + 2/N$ and $\|u_0\|_{L^{q_c}(R^N)}$ is sufficienlly small, where $q_c = N(p-1)/2$, then the solutions of (1)-(2) exist globally.

Proof. (a) Let $\Phi \in C_0^\infty(R)$ such that $\Phi(s) = 1$ for $|s| \leq 1$, $\Phi(s) = 0$ for $|s| > 2$ and $0 \leq \Phi(s) \leq 1$. For $T > 0$, we define

$$\varphi_1(x) = \left(\Phi\left(T^{-\alpha/2}|x|\right)\right)^{2p/(p-1)}, \ \varphi_2(x) = \left(1 - \frac{t}{T}\right)^m, \ m \geq \max\left\{1, \frac{p\alpha}{p-1}\right\},$$

for $t \in [0, T]$. Assuming that u is a mild solution of (1)-(2), then by Lemma (3.2) we have

$$\int_{R^N}\int_0^T (u^p\varphi_1\varphi_2 + u_0\varphi_1{}_t^C D_T^\alpha \varphi_2)dtdx = \int_{R^N}\int_0^T (u(-\Delta\varphi_1)\varphi_2 + u\varphi_1{}_t^C D_T^\alpha\varphi_2)\, dtdx. \quad (3.8)$$

Note that

$$\left|(-\Delta\varphi_1)\varphi_2 + \varphi_1{}_t^C D_T^\alpha\varphi_2\right| \leq CT^{-\alpha}\varphi_1^{1/p}\varphi_2^{1/p}, \quad (3.9)$$

for some positive constant C independent of T. Then by (3.8), (3.9) and Hölder inequality, we have

$$\int_{R^N}\int_0^T (u^p\varphi_1\varphi_2 + u_0\varphi_1{}_t^C D_T^\alpha\varphi_2)\, dtdx \leq CT^{-\alpha}\int_{R^N}\int_0^T u\varphi_1^{1/p}\varphi_2^{1/p}\, dtdx$$

$$\leq CT^{-\alpha+(1+\alpha N/2)(p-1)/p}(\int_{R^N}\int_0^T u^p\varphi_1\varphi_2\, dtdx)^{1/p}.$$

Hence

$$T^{1-\alpha}\int_{R^N} u_0\varphi_1\, dx \leq CT^{1+\alpha N/2 - p\alpha/(p-1)}.$$

It follows from $p < 1 + 2/N$ that $(N/2 + 1)\alpha - p\alpha/(p-1) < 0$. Therefore, if solution of (1)-(2) exists globally, then taking $T \to \infty$, we obtain

$$\int_{R^N} u_0\varphi_1\, dx = 0,$$

and then $u_0 \equiv 0$. Hence, by Theorem(4.3), we know u blows up in a finite time.
(b) We construct the global solution of (1)-(2) by the contraction mapping principale.
Since $p \geq 1 + 2/N > 1 + 2\alpha/(\alpha N + 2 - 2\alpha)$, we know

$$\frac{\alpha N(p-1)}{2(p\alpha - p + 1)_+} > 1, \quad (3.10)$$

where $(p\alpha - p + 1)_+ = \max\{0, p\alpha - p + 1\}$.
In view of $p \geq 1 + 2/N > (4 - N + \sqrt{N^2 + 16})/4$, we have

$$\frac{N(p-1)}{2p(2-p)_+} > 1. \quad (3.11)$$

Hence, by Lemma (3.10), (3.11) and
$(p-1)N/(2p) < (\alpha N(p-1))/(2(p\alpha - p+1)_+)$, we can choose $q > p \geq 1+2/N$ such tha

$$\frac{\alpha}{p-1} - \frac{1}{p} < \frac{\alpha N}{2q} < \frac{\alpha}{p-1}, \tag{3.12}$$

and

$$\frac{\alpha}{p-1} - \alpha < \frac{\alpha N}{2q}. \tag{3.13}$$

Let

$$\beta = \frac{\alpha N}{2}(\frac{1}{q_c} - \frac{1}{q}) = \frac{\alpha}{p-1} - \frac{\alpha N}{2q}. \tag{3.14}$$

Using (3.12) and (3.14), one verifies that

$$0 < p\beta < 1, \alpha = \frac{\alpha N(p-1)}{2q} + (p-1)\beta. \tag{3.15}$$

Assume that the initial value u_0 satisfies

$$\sup_{t>0} t^{\beta} \|P_\alpha(t)u_0\|_{L^q(R^N)} = \eta < +\infty. \tag{3.16}$$

Note that (3.13) implies $1/q_c - 1/q < 2/N$. If $u_0 \in L^{q_c}(R^N)$, (1.7) implies (3.16) holds. If $u_0(x) \leq C|x|^{-2/(p-1)}$ for some constant $C > 0$, then $\|T(t)u_0\|_{L^q(R^N)} \leq Ct^{N/(2q)-1/(p-1)}$. Hence

$$\|P_\alpha(t)u_0\|_{L^q(R^N)} \leq Ct^{\alpha(N/(2q)-1/(p-1))} \int_0^\infty \phi_\alpha(\theta)\theta^{N/(2q)-1/(p-1)} d\theta.$$

Since $N/(2q) - 1/(p-1) > -1$,

$$\int_0^\infty \phi_\alpha(\theta)\theta^{N/(2q)-1/(p-1)} d\theta < \infty.$$

Therefore, we also obtain that (3.16) is staisfied in this case.
Let $Y = \{u \in L^\infty((0,\infty), L^q(R^N)) : \|u\|_Y < \infty\}$, where

$$\|u\|_Y = \sup_{t>0} t^{\beta} \|u(t)\|_{L^q(R^N)}.$$

For $u \in Y$, we define

$$\Phi(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) |u|^{p-1} u(s)\, ds.$$

Denote $B_M = \{u \in Y : \|u\|_Y \leq M\}$. For any u, $v \in B_M$, $t \geq 0$,

$$t^{\beta} \|\Phi(u)(t) - \Phi(v)(t)\|_{L^q(R^N)} \leq t^{\beta} \int_0^t (t-s)^{\alpha-1} \|S_\alpha(t-s)(u^p(s) - v^p(s))\|_{L^q(R^N)} ds \tag{3.17}$$

Since $q > p > N(p-1)/4$, so $p/q-1/q < 4/N$. Hence, Hölder inequality, Lemma 2.2, (3.15) and (3.17) imply that there exists constant $C > 0$ such that

$$\begin{aligned}
t^{\beta}\left\|\Phi(u)-\Phi(v)\right\|_{L^{q}(R^{N})} &\leq Ct^{\beta}\int_{0}^{t}(t-s)^{\alpha-1-\alpha N(p/q-1/q)/2}\left\|u^{p}-v^{p}\right\|_{L^{\frac{q}{p}}(R^{N})}\,ds \\
&\leq Ct^{\beta}\int_{0}^{t}(t-s)^{\alpha-1-\alpha N(p-1)/(2q)}\left(\left\|u^{p-1}_{L^{q}(R^{N})}\right\|+\left\|v^{p-1}_{L^{q}(R^{N})}\right\|\right)\left\|u-v\right\|_{L^{q}(R^{N})}\,ds \\
&\leq Ct^{\beta}M^{p-1}\int_{0}^{t}(t-s)^{\alpha-1-\alpha N(p-1)/(2q)}s^{-p\beta}ds\left\|u-v\right\|_{Y} \\
&= CM^{p-1}t^{\beta-p\beta-\alpha N(p-1)/(2q)+\alpha}\int_{0}^{1}(1-\tau)^{-\alpha N(p-1)/(2q)+\alpha-1}\tau^{-p\beta}d\tau\left\|u-v\right\|_{Y} \\
&= CM^{p-1}\int_{0}^{1}(1-\tau)^{-\alpha N(p-1)/(2q)+\alpha-1}\tau^{-p\beta}d\tau\left\|u-v\right\|_{Y} \\
&= CM^{p-1}\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)}\left\|u-v\right\|_{Y}.
\end{aligned}$$

If we choose M small enough such that

$$CM^{p-1}\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)} < \frac{1}{2},$$

then $\left\|\Phi(u)-\Phi(v)\right\|_{Y} \leq \frac{1}{2}\left\|u-v\right\|_{Y}$. Since

$$\begin{aligned}
t^{\beta}\left\|\Phi(u)(t)\right\|_{L^{q}(R^{N})} &\leq \eta+CM^{p}t^{\beta}\int_{0}^{t}(t-s)^{-\alpha N(p/q-1/q)/2-1+\alpha}s^{-p\beta}\,ds \\
&\leq \eta+CM^{p}\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)}, t\in[0,+\infty),
\end{aligned}$$

we can choose η and M small enough such that

$$\eta+CM^{p}\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)} \leq M.$$

Therefore, by contraction mapping principale we know Φ has a fixed point $u \in B_M$. Next, we will prove $u \in C([0,\infty), C_0(R^N))$.
First, we prove that for $T > 0$ small enough, $u \in C([0,T], C_0(R^N))$. In fact, the above proof shows that u is the unique solution in

$$B_{M,T} = \left\{u \in L^{\infty}((0,T), L^{q}(R^{N})) : \sup_{0<t<T} t^{\beta}\left\|u(t)\right\|_{L^{q}(R^{N})} \leq M\right\}.$$

By Theorem 3.2 and $u_0 \in C_0(R^N) \cap L^q(R^N)$, we know that for T small enough, (1.1) has a unique solution $\sim u \in C([0,T], C_0(R^N) \cap L^q(R^N))$.
Hence, we can take T small enough such that $\sup_{0<t<T} t^{\beta}\left\|\sim u(t)\right\|_{L^{q}(R^{N})} \leq M$.
Then, by uniqueness, we know $u \equiv \sim u$ for $t \in [0,T]$ and then $u \in C([0,T], C_0(R^N)) \cap C([0,T], L^q(R^N))$.

Next, we show that $u \in C([0,T], C_0(R^N))$ by a bootstrap argument. For $t > T$, we have

$$\begin{aligned} u - P_\alpha(t)u_0 &= \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) u^p \, ds \\ &= \int_0^T (t-s)^{\alpha-1} S_\alpha(t-s) u^p ds + \int_T^t (t-s)^{\alpha-1} S_\alpha(t-s) u^p \, ds \\ &= I_{10} + I_{11}. \end{aligned}$$

It follows from $u \in C([0,T], C_0(R^N))$ that

$$I_{10} \in C([T,\infty), C_0(R^N)) \cap C([T,\infty), L^q(R^N)).$$

For $T_1 > T$, we know $u^p \in L^\infty((T,T_1), L^{q/p}(R^N))$. Note that $q > N(p-1)/2$, we can choose $r > q$ such that $N(p/q - 1/r)/2 < 1$. Then analogous to the proof of Lemma 2.4, we can show that $I_{11} \in C([T,T_1], L^r(R^N))$. By the arbitrariness of T_1, we know $I_{11} \in C([T,\infty), L^r(R^N))$ and so $u \in C([T,\infty), L^r(R^N))$.
We take $r = q\chi^i$, $\chi > 1$ such that

$$\frac{N}{2}\left(\frac{p}{q} - \frac{1}{q\chi^i}\right) < 1, i = 1, 2, \cdots,$$

then $u \in C([T,\infty), L^{q\chi^i}(R^N))$. By finite steps, we have $p/(q\chi^i) < 2/N$, so $u \in C([0,\infty), C_0(R^N))$. ■

Conclusion

xcfvgbjhnk,k;