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The stability study of a Gierer-Meinhardt system

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Note: .... **Mention:** ....

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Dedication

- We devote this modest work to all, who from near and far have given us their moral and physical support for the realization of this work.
- To our parents for their support during all our studies and who never cease to lavish us with their love.
- To our brothers and sisters hwo will find here the expression of our respect and love.
- To all of our colleagues within the Mathematics & Informatics department.
- To anyone we know and with whom we exchange feelings of friendship, love and respect.

Acknowledgment

In the name of Allah, Most Gracious, Most Merciful, to Whom all praise is due.

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Departement of Mathematics & Informatics.

First of all, we would like to thank all of our teachers for having given us the necessary knowledge in recent years.

We would also like to thank Dr. Salem ABDELMALEK, our assets framed and supported during these five months to accomplish this project.

We also wish to thank our colleagues and friends from the University and all of those who are in our hearts, especially our dear parents, for their sacrifices, kindness and never–ending support.

Abstract

In this work, we study the Turing patterns appearing in a Gierer-Meinhardt model of the activator-inhibitor type with different sources. First, we investigate the corresponding kinetic equations and derive the conditions for the stability of the equilibrium and then, we turn our attention to the Hopf bifurcation of the system. In certain parameter range, the equilibrium experiences a Hopf bifurcation; the bifurcation is supercritical and the bifurcated periodic solution is stable. With added diffusions, we show that both the equilibrium and the stable Hopf periodic solution experience Turing instability, if the diffusion coefficients of the two species are sufficiently different. And we prove the global existence in time of the solutions of this system.

ملخص بالعربية

إن الهدف من هذا العمل هو دراسة استقرار نقاط التوازن في نمـوذج Gierer-Meinhardt مـن نـوع المنشط و المثبط.

ولقد اعتمدنا في ذلك علىٰ دراســة المعادلات الحركيــة واستخلاص ظــروف الاسـتقرار والتـــوازن كما وجهنا اهتمامنا للتشعبات (Hopf Bifurcation)، وبرهان الوجود الكلي للحل بالنسبة للزمن باستعمال نظرية ليابانوف (Liapunov).

Introduction

The formation of patterns by Turing instability has been investigated in different models to explain how these can emerge from a merely uniform environment. The interaction of two biochemical substances with different diffusion rates having the capacity to generate biological patterns was introduced by Turing ([15]). Some twenty years later, Gierer and Meinhardt found that the two substances, in fact, opposed the action of each other giving rise to the activator-inhibitor model ([5]). Which can be used to explain the formation of polar, symmetric and periodic structures (spots on animals).

Much of this work is devoted to study the stability of a Gierer-Meinhardt system, To address this work, we must address the theories of stability (system stability) as well as the definition of reaction systems.

This work is divided into three chapters

- **Chapter 1**: **(Stability theory)**. This chapter deals with theories of stability of the linear and nonlinear (via linearization) systems (systems with dimension 2×2), then we talk about Lyapunov direct method to study the global stability. Then we took as un application the Lotka-Voltera system where we studied its local and global stability, in this study we relied mainly on the eigenvalues of the Jacobian matrix attached to the system and on Lyapunov's theorem.

- **Chapter 2: (Introduction to Reaction diffusion Systems)**. In this chapter we present an introduction to Reaction-Diffusion system, activator-inhibitor system. Then we talked about the Hopf bifurcation and the global existence of solutions using the Lyapunov's theorem, in the later we gave some examples of these systems and we devoted the system of Gierer-Meinhardt.

- **Chapter 3: (The Gierer-Meinhardt Activator-Inhibitor Model)**. This is the important chapter, we have studied the local stability, the Hopf bifurcation and the global existence of solutions of the Gierer-Meinhardt system, first we started with the ODE system and then we studied PDE system.

Preliminaries

This chapter recalls some useful preliminaries that are necessary for the dissertation at hand. We introduce some basic notation and notions.

0.1 Notation

the following are some notations the are used in the memorie.

- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- A denotes the matrix
- $\det(A)$ denotes the determinant of a matrix A.
- $\operatorname{tr}(A)$ denotes the trace of real and complex matrice A.
- A^{-1} denotes the inverse of matrix A.
- $A^{\top}$ denotes the transpose of matrix A.
- $\operatorname{sgn}(a)$ denotes the signal of a.
- $\frac{d^k f}{dt^k} = f^{(k)}$ denotes the k^{th} derivative of $f(t)$.
- $\frac{\partial f}{\partial x_i} = \partial_{x_i} f = f_{x_i}$ denotes the partial derivative of $f(x_1, ..., x_n)$ with respect to x_i.

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0.2 General Notions

- ∇u denotes the gradient of u, where

$$\text{Grad}\, u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n} \right).$$

Δu denotes the laplacian of u, Δu is defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

0.2.1 Important Spaces

The set of $\mathbb{L}^p$ functions (where $p \geq 1$) generalizes $\mathbb{L}^2$ space. Instead of square integrable, the measurable function f must be p integrable for f to be in $\mathbb{L}^p$.

On a measure space Ω, the $\mathbb{L}^p$ norm of a function f is

$$|f|_{L^p} = \left(\int_X |f|^p \right)^{\frac{1}{p}}.$$

The $\mathbb{L}^p$ functions are the functions for which this integral converges. For $p \neq 2$, the space of L^p functions is a Banach space which is not a Hilbert space.

The $\mathbb{L}^p$ space on R^n, and in most other cases, is the completion of the continuous functions with compact support using the $\mathbb{L}^p$ norm. As in the case of an $\mathbb{L}^2$ space, an $\mathbb{L}^p$ function is really an equivalence class of functions which agree almost everywhere. It is possible for a sequence of functions f_n to converge in $\mathbb{L}^p$ but not in $\mathbb{L}^{p'}$ for some other p', e.g., $f_n = (1 + x^2)^{-\frac{1}{2} - \frac{1}{n}}$ converges in $\mathbb{L}^2(\Omega)$ but not $\mathbb{L}^1(\Omega)$. However, if a sequence converges in $\mathbb{L}^p$ and in $\mathbb{L}^{p'}$, then its limit must be the same in both spaces.

For $p > 1$, the dual vector space to $\mathbb{L}^p$ is given by integrating against functions in $\mathbb{L}^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. This makes sense because of Höder's inequality for integrals. In particular, the only $\mathbb{L}^p$ space which is self-dual is $\mathbb{L}^2$.

While the use of $\mathbb{L}^p$ functions is not as common as $\mathbb{L}^2$, they are very important in analysis and partial differential equations. For instance, some operators are only bounded in $\mathbb{L}^p$ for some $p > 2$.

For $d \geq 1$, Ω an open subset of $\mathbb{R}^d$, $p \in [1; +\infty]$ and $s \in N$, the Sobolev space $W^{s,p}(\mathbb{R}^d)$ is defined by

$$W^{s,p}(\Omega) = \left\{ f \in \mathbb{L}^P(\Omega) : \forall \, |\alpha| \leq s, \; \partial_x^\alpha f \in \mathbb{L}^P(\Omega) \right\}, \tag{1}$$

where $\alpha = (\alpha_1, ..., \alpha_d)\,, |\alpha| = \alpha_1 + ... + \alpha_d$, and the derivatives partial derivative $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d} f$ are taken in a weak sense.When endowed with the norm.

In the special case $p=2, W^{s,2}(\Omega)$ is denoted by $\mathbb{H}^s(\Omega)$. This space is a Hilbert space for the inner product

$$\langle f,g\rangle_{s,\Omega}=\sum_{|\alpha|\leq s}\langle\partial_x^\alpha f,\partial_x^\alpha g\rangle_{\mathbb{L}^2(\Omega)}=\sum_{|\alpha|\leq s}\int_\Omega \partial_x^\alpha f\partial_x^\alpha g d\mu.$$

Sobolev spaces play an important role in the theory of **partial differential equations**.

0.2.2 Important Formulas

Taylor's Formula

Recall Taylors for $f:\mathbb{R}\to\mathbb{R}$:

$$f(x)=f(a)+\frac{\partial f}{\partial x}(a)(x-a)+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(a)(x-a)^2+...+\frac{1}{k!}\frac{\partial^k f(a)}{\partial x^k}(x-a)^k+O(x,a).$$

Recall Taylors for $f:\mathbb{R}^2\to\mathbb{R}$:

$$\begin{aligned} f(x,y) &= f(a,b)+\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2\\ &+\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2+\frac{\partial^2 f}{\partial x\partial y}(a,b)(x-a)(y-b)\\ &+O\left((x-a)^2+(y-b)^2\right).\end{aligned}$$

Green's Formula([1])

We recall now some green's formulas whitch generalize the multidimensional case the formula of integration by parts of dimension one. They write as following:

Theorem 1 *we suppose that Ω is an open domain of boundry $\partial\Omega$ continue with part. Then, if u and v are function of $\mathbb{H}^1(\Omega)$, we have*

$$\int_\Omega \frac{\partial u}{\partial x_i}v dx=-\int_\Omega u\frac{\partial v}{\partial x_i}dx+\int_{\partial\Omega}uv\eta_i d\sigma,\qquad 1\leqslant i\leqslant n, \tag{2}$$

we design by η_i the i^{th} consinus director of normal η in $\partial\Omega$ directed towards the outside of Ω and we write $\eta_i=(\vec{\eta}.\vec{e}_i)$. $d\sigma$ the superficial measure on $\partial\Omega$.

Proof. If u (resp v) belongs to $\mathbb{H}^1(\Omega)$, there exists a suite (u_m) (resp (v_p)) from $D(\bar{\Omega})$ witch converge to u on $\mathbb{H}^1(\Omega)$(resp to v on $\mathbb{H}^1(\Omega)$) $\left[D(\bar{\Omega})\text{ dense on }\mathbb{H}^1(\Omega)\right]$.

We have for the functions u_m and v_p of $D(\bar{\Omega})$

$$\int_{\Omega} \frac{\partial u_m}{\partial x_i} v_p dx = -\int_{\Omega} u_m \frac{\partial v_p}{\partial x_i} dx + \int_{\partial\Omega} u_m v_p \eta_i d\sigma, \qquad 1 \leqslant i \leqslant n, \tag{3}$$

We obtain the expretion (2) by switching to the limite in Green formula precedent. ■

Corollary 2 *For all function u of $\mathbb{H}^1(\Omega)$ and all function v of $\mathbb{H}^1(\Omega)$, we have the Green formula*

$$\int_{\Omega} (\Delta u) v = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma - \int_{\Omega} \nabla u \nabla v. \tag{4}$$

Proof. Let given a consequence of Theorem precedent.
On suppose $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, the laplacian of a distribution u. Then, if u is a function of $\mathbb{H}^1(\Omega)$, we have from (2) for all function v of $\mathbb{H}^1(\Omega)$

$$\begin{aligned} -\int_{\Omega} (\Delta u) v &= -\sum_{i=1}^{n} \int_{\Omega} \frac{\partial^2 u}{\partial x_i^2} v dx \\ &= \sum_{i=1}^{n} \left\{ \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_{\Omega} \frac{\partial u}{\partial x_i} v \eta_i d\sigma \right\} \\ &= \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_{\Omega} \frac{\partial u}{\partial \eta} v d\sigma \\ &= \int_{\Omega} \nabla u \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma. \end{aligned}$$

■

Remark 1 *The Green formula rest remains valid if $u, v \in C^1$ and the formula (4) remains valid if $u \in C^2$, $v \in C^1$.*

The Jordan Normal Form([4])

If A is a nonsingular matrix, there exist two nonsingular matrices J and B such that $A = B^{-1}JB$, or equivalently $BA = JB$. J is called the Jordan normal form (or simply Jordan matrix) of A. The Jordan matrix J is triangular (but not necessarily diagonal).

Let us show what happens if $n = 2$, which is the case we will deal with in the sequel. Let A be a 2×2 matrix with eigenvalues λ_1, λ_2. Then the Jordan matrix is as follows.

1. If λ_1, λ_2 are real and distinct, then their algebraic and geometric multiplicity is 1 and hence

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{5}$$

2. If $\lambda_1 = \lambda_2$ is real, then its algebraic multiplicity is 2. Either its geometric multiplicity is also 2, a case where

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \tag{6}$$

or its geometric multiplicity is 1, a case where

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \tag{7}$$

Furthermore, if the eigenvalues are complex conjugate, $\lambda_{1,2} = \alpha \pm \iota\beta$, then one can show that

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{8}$$

Quadratic Formula([1])

A quadratic formula is a homogeneous polynomial of the second degree with respect to n variables $u_1, u_2, ..., u_n$ a quadratic form always has the representation

$$\sum_{i,j=1}^{n} a_{ij} u_i u_j, \tag{9}$$

where

$$A = (a_{ij})_{1 \leq i,j \leq n},$$

is a symetrica matrix.

If we denote the matrix-column $(u_1, u_2, ..., u_n)$ with u and the quadratic formula with

$$A(u,u) = \sum_{i,j=1}^{n} a_{ij} u_i u_j, \tag{10}$$

we can write

$$A(u,u) = u^{\top} A u = Au \cdot u. \tag{11}$$

If

$$A = (a_{ij})_{1 \leq i,j \leq n},$$

is a real symmetric matrix, the form (10) is called the real quadratic foormula. In this work, we are interested in real quadratic formula.

Definition 1 *a quadratic formula (10) is called defined non-negative if, for arbitrary real values of the variables*

$$A(u,u) \geq 0. \tag{12}$$

Definition 2 *a quadratic formula (10) is called defined positive if, for arbitrary values of non-zero variables* $(u \neq 0)$*, we have*

$$A(u,u) > 0. \tag{13}$$

Theorem 3 *a quadratic formula (10) is called defined positive if, and only if, all the principls determinants succesifs of her matrix a coefficients, are positives*

$$\det 1 > 0, \det 2 > 0, ..., \det n > 0. \tag{14}$$

Corollary 4 *In a positive quadratic formula (10) all the determinent principls of the matrix of coefficients, are positifs, when the principals determinants successivesof a real symmetric matrix are positive, all the remaining principals determinants are posxitives.*

Remark 2 *if the principals successives determinants are non-negatives*

$$\det 1 \geq 0, \det 2 \geq 0, ..., \det n \geq 0, \tag{15}$$

it does not follow that $A(u,u)$ *is defined as non-negative. Thus the forme*

$$a_{11}u_1^2 + 2a_{12}u_1u_2 + a_{22}u_2^2, \tag{16}$$

in which $a_{11} = a_{12} = 0$*,* $a_{22} < 0$*, satisfied (15) but is not defined non-negative.*

We have, however, the following theorem:

Theorem 5 *a quadratic formula (10) is said defined as non-negative if, and only if, all the principal determinants of its matrix of coefficients are non-negative*

$$\det A[i_1, i_2, ..., i_n | i_1, i_2, ..., i_p] \geq 0,$$

where

$$1 \leq i_1 < i_2 < ... < i_p \leq n, \qquad and \qquad p \leq n.$$

0.2.3 Important Inequalities

Young's Inequality

Let f be a continuous and increasing function on $[0, c]$ ou $c > 0$.

$f(0) = 0$, $a \in [c, 0]$ and $b \in [0,\ f(c)]$, then

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx, \tag{17}$$

or f^{-1} is the inverse function of f.

Proof. we begin with the expression

$$g(a) = ab - \int_0^a f(x)\,dx, \tag{18}$$

we take $b > 0$ as a parameter. since $g'(a) = b - f(a)$ and since the function f is inceasing, we have

$$\begin{aligned} g'(a) &> 0 \text{ for } 0 < a < f^{-1}(b), \\ g'(a) &= 0 \text{ for } a = f^{-1}(b), \\ g'(a) &< 0 \text{ for } a > f^{-1}(b). \end{aligned}$$

From this, $g(a)$ is a maximum value of the function g reached $a = f^{-1}(b)$.
so

$$g(a) \leq \max g(x) = g\left(f^{-1}(b)\right) \tag{19}$$

applying an integration by parts

$$\begin{aligned} g\left(f^{-1}(b)\right) &= bf^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,dx \\ &= \int_0^{f^{-1}(b)} xf'(x)\,dx \end{aligned}$$

if we take $y = f(x)$, the above equation becomes:

$$g\left(f^{-1}(b)\right) = \int_0^b f^{-1}(y)\,dy \tag{20}$$

by comparing (18),(19), and (20), one obtains (17)

the function $f(x) = x^{p-1}$ with $p > 1$ in each interval $[0;c]$ satisfies the condition apply(1.2.27) utilisant $\frac{1}{p} + \frac{1}{q} = 1$ on obtient

$$\forall a, b \in \mathbb{R}^+ : ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{21}$$

if we replace the function $f(x)$ by ϵx^{p-1} on (17) then the inequality from Yong with ϵ :

$$\forall a, b \in \mathbb{R}^+ : ab \leq \epsilon a^p + \frac{(\epsilon p)^{\frac{-q}{p}}}{q} b^q \tag{22}$$

ce qui donne

$$\forall a, b \in \mathbb{R}^+ : ab \leq \epsilon a^p + \frac{(\epsilon p)^{\frac{-1}{p-1}}}{q} b^q$$

■

Hölder's Inequality

Let $p > 1$ and q be real numbers connected by the relation $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\forall (f, g) \in L^p(\Omega) \times L^q(\Omega) : \int_\Omega |f(x)\,g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{23}$$

Proof. Using inequality (23), we obtain

$$|f(x)\,g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

it results that $fg \in \mathbb{L}^1(\Omega)$ and

$$\int_\Omega |f(x)\,g(x)|\,dx \leq \frac{1}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q}\int_\Omega |g(x)|^q\,dx. \tag{24}$$

Replacing in (24) f by λf $(\lambda > 0)$ he comes

$$\int_{\Omega} |f(x)\, g(x)|\, dx \leq \frac{\lambda^{p-1}}{p} \int_{\Omega} |f(x)|^p\, dx + \frac{1}{\lambda q} \int_{\Omega} |g(x)|^q\, dx,$$

we choose $\lambda = \left(\int_{\Omega} |f(x)|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q \right)^{\frac{1}{q}}$, we obtain then (23). ■

Definition 3 *Let* $(V, \langle \cdot, \cdot \rangle)$ *be a* n *dimensional euclidean vector space and* $T : V \to V$ *a linear operator. We will call the adjoint of* T*, the linear operator* $T^* : V \to V$ *such that:*

$$\langle Tu, v \rangle = \langle u, T^* v \rangle, \qquad \textit{for all } u, v \in V.$$

CHAPTER 1

Stability theory

In this chapter we present an introduction to the theory of stability. Since this is a very broad area which includes not only many topics but also various notions of stability, we mainly focus on Liapunov stability of equilibrium points. Some of the proofs are omitted or carried out in special simple cases.

1.1 Introduction

The term "stable" informaly means resistant to change. For technical use the term has to be defined more precisely in term of the mathematical model, but the same connotation.

In mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. The heat equation, for example, is a stable partial differential equation because small perturbations of initial data lead to small variations in temperature at a later time as a result of the maximum principle. In partial differential equations one may measure the distances between functions using $\mathbb{L}^p$ norms or the sup norm.

Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories what happens with the system after a long period of time. The simplest kind of behavior is exhibited by equilibrium points, or fixed points, and by periodic orbits. If a particular orbit is well understood, it is natural to ask next whether a small change in the initial condition will lead to similar behavior. Stability theory addresses the following questions: Will a nearby orbit indefinitely stay close to a given orbit? Will it converge to the given orbit? (The latter is a stronger property). In the former case, the orbit is called stable; in the latter case, it is called asymptotically stable and the given orbit is said to be attracting.

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. In particular, at each equilibrium of a smooth dynamical system with an n-dimensional phase space, there is a certain $n \times n$ matrix A whose eigenvalues characterize the behavior of the nearby points.

More precisely, if all eigenvalues are negative real numbers or complex numbers with negative real parts then the point is a stable attracting fixed point, and the nearby points converge to it at an exponential rate, Liapunov stability and exponential stability. If none of the eigenvalues are purely imaginary (or zero) then the attracting and repelling directions are related to the eigenspaces of the matrix A with eigenvalues whose real part is negative and, respectively, positive. Analogous statements are known for perturbations of more complicated orbits.

1.2 Stability of Fixed Points

The simplest kind of an orbit is a fixed point, or an equilibrium. If a mechanical system is in a stable equilibrium state then a small push will result in a localized motion, for example, small oscillations as in the case of a pendulum. In a system with damping, a stable equilibrium state is moreover asymptotically stable. On the other hand, for an unstable equilibrium, such as a ball resting on a top of a hill, certain small pushes will result in a motion with a large amplitude that may or may not converge to the original state.

There are useful tests of stability for the case of a linear system. Stability of a nonlinear system can often be inferred from the stability of its linearization.

In this work we will study the stability of the (2×2) system

1.2.1 The Stability of a Linear System

The stability of fixed points of a system of constant coefficient linear differential equations of first order can be analyzed using the eigenvalues of the corresponding matrix.

Let be the system

$$\frac{\partial X}{\partial t} = f(X(t)), \qquad X = (x(t), y(t)), \tag{1.1}$$

we write the system (1.1) in the form

$$\begin{cases} \frac{\partial x(t)}{\partial t} = a_{11}x + a_{12}y = f(x,y) \\ \frac{\partial y(t)}{\partial t} = a_{21}x + a_{22}y = g(x,y) \end{cases}, \tag{1.2}$$

where the coefficients a_{ij} are real numbers. Letting

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we put

$$\begin{cases} f(x,y) = a_{11}x + a_{12}y \\ g(x,y) = a_{2I1}x + a_{22}y \end{cases},$$

(x^*, y^*) is a equilibrium point of (1.2) :

$$\begin{cases} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{cases}, \tag{1.3}$$

then

$$x^* = 0, \; y^* = 0,$$

hence, the unique equilibriun point of (1.2) is $(0,0)$.

The Jacobian matrix of (1.2) is

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1.4}$$

$$J(0,0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1.5}$$

Typology of the Solutions of the Linear Systems in the $(\mathrm{tr}, \det)$ **−Plane**

The typology of the solutions of the planar linear systems which we established with to leave the nature of the eigenvalues of the matrix of the system (1.2)can be also summarized in a plan, $(\mathrm{tr}, \det)$.(see Figure 1.1)Eigenvalues of J are solutions of the characteristic

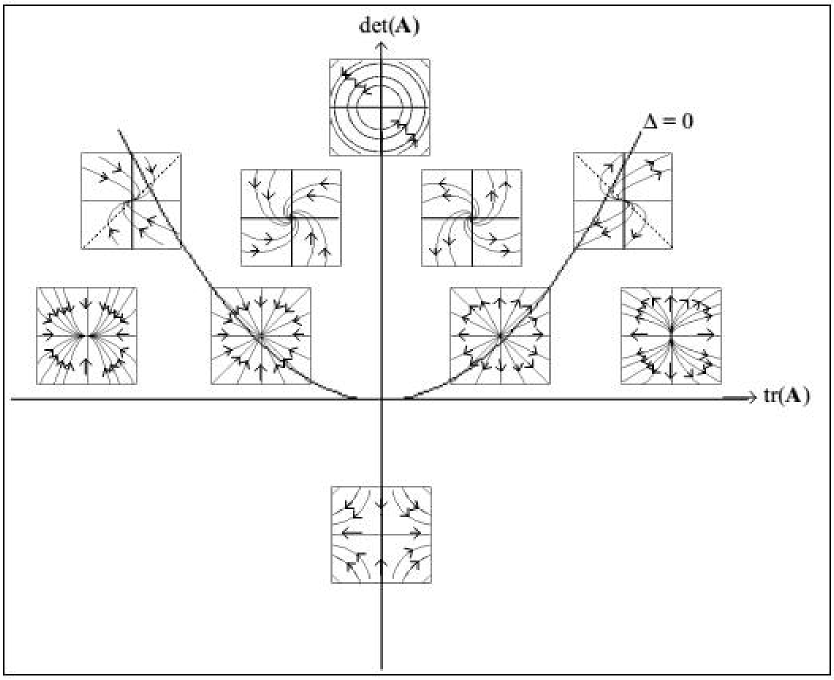


Figure 1.1: Summary of the different possible phase portraits of the system $\dot{X} = AX$ as a function of the sign of the trace and the determinant of the matrix A.

equation:

$$\lambda^2 - \mathrm{tr}\,(J) + \det\,(J) = 0 \text{ with } \begin{cases} \mathrm{tr}\,(A) = \lambda_1 + \lambda_2 \\ \det\,(A) = \lambda_1 \lambda_2 \end{cases}.$$

The nature of the eigenvalues depends on the sign of the discriminant $\Delta = (\mathrm{tr}\,(J))^2 - 4\det\,(J)$.

In the plan (tr, det), the equation $\Delta = 0$ is that of a parabola passing by the origin:

$$\det(J) = \frac{1}{4}(\operatorname{tr}(J))^2.$$

This parabola divides the plan into two great areas: above the parabola ($\Delta < 0$), one finds the portraits of phase of the hearths and the centers; below ($\Delta > 0$), one finds them nodes and the points saddles.

- Case $\Delta = 0$

1. There is then $\lambda_1 = \lambda_2 = \lambda_0$, i.e. $\det(J) = \lambda_0^2 > 0$ and $\operatorname{tr}(J) = 2\lambda_0$. Consequently, if trace is positive ($\lambda_0 > 0$), we have a star or an unstable degenerated node; if the trace is negative ($\lambda_0 < 0$), we have a star or a stable degenerated node.(see Figure 1.2).

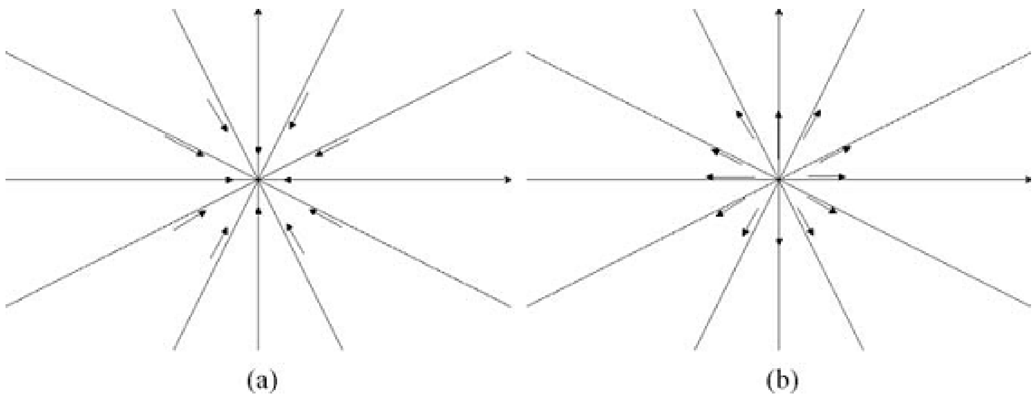


Figure 1.2: (a) Stable node, with $\lambda_1 = \lambda_2 < 0$; (b) Unstable node, with $\lambda_1 = \lambda_2 > 0$.

- Case $\Delta > 0$

We have two distinct real eigenvalues then. is in the area under the parabola who still can is shared in three zones:

$\det(J) < 0 : \lambda_1$ and λ_2 are of opposed sign, the origin is a point saddles (see Figure 1.3);

$\det(J) > 0$ and $\operatorname{tr}(J) > 0 : \lambda_1, \lambda_2 > 0$, the origin are an unstable node (see Figure 1.4);$\det(J) > 0$ and $\operatorname{tr}(J) < 0 : \lambda_1, \lambda_2 < 0$, the origin are a stable node (see Figure 1.5).

- Case $\Delta < 0$

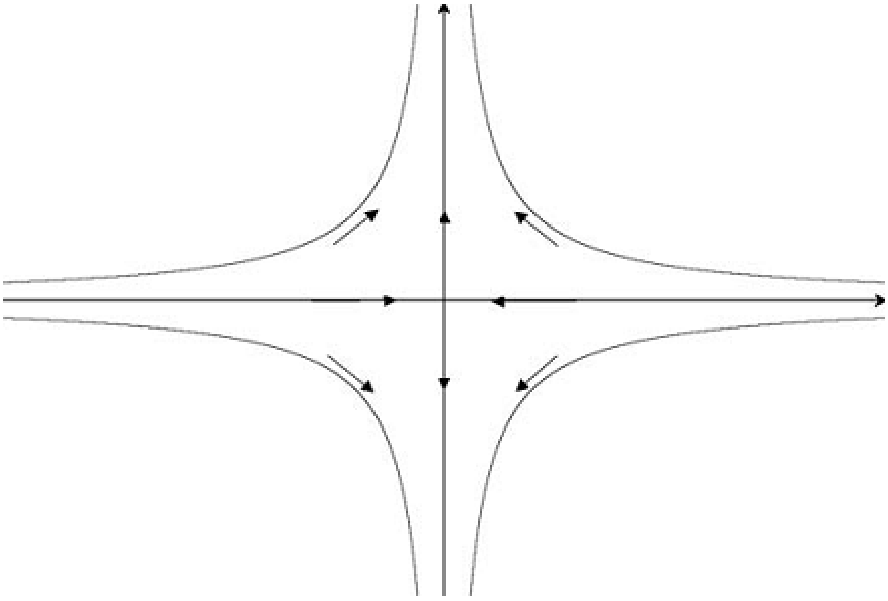


Figure 1.3: Saddle, wich $\lambda_1 < 0 < \lambda_2$

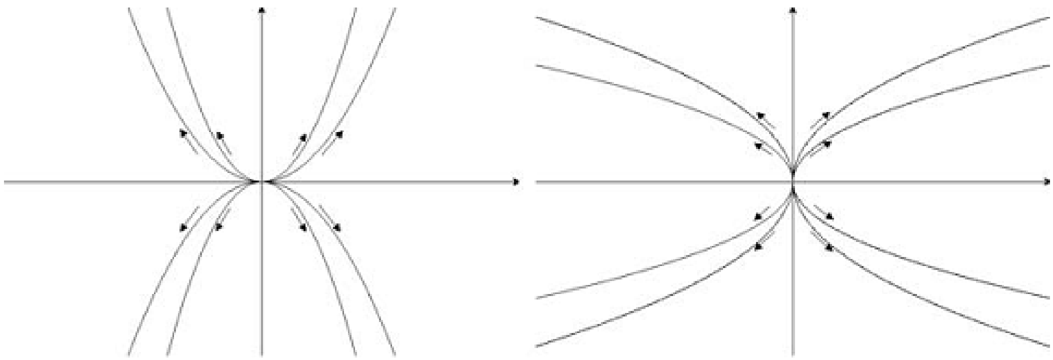


Figure 1.4: Unstable node. (a) $0 < \lambda_1 < \lambda_2$; (b) $0 < \lambda_2 < \lambda_1$

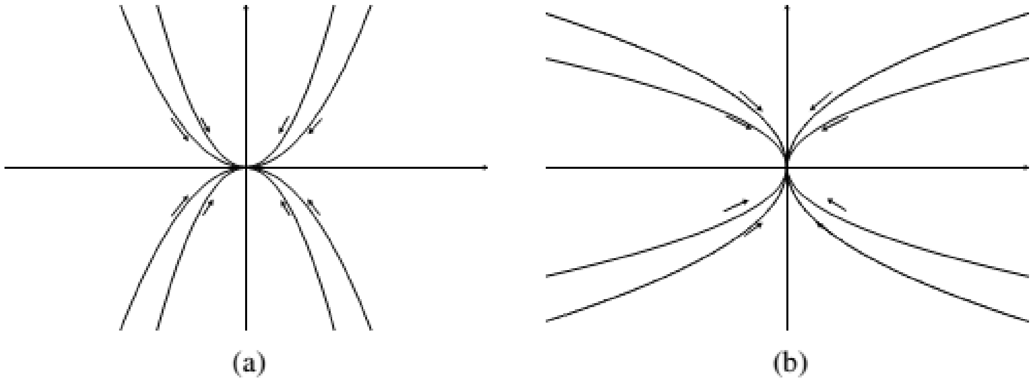


Figure 1.5: Stable node. (a) $\lambda_1 < \lambda_2 < 0$; (b) $\lambda_2 < \lambda_1 < 0$

we have two combined complex eigenvalues then, $\lambda_{1,2} = \alpha \pm i\beta$, i.e det $(J) = \alpha^2 + \beta^2 > 0$ and tr $(J) = 2\alpha$. One is in the area above the parabola, which division there still in three distinct zones:

tr $(J) < 0$: The real part of the eigenvalues is negative, the origin is a asymptotically stable focus (see Figure 1.6);

tr $(J) > 0$: The real part of the eigenvalues is positive, the origin is a unstable focus (see Figure 1.6);

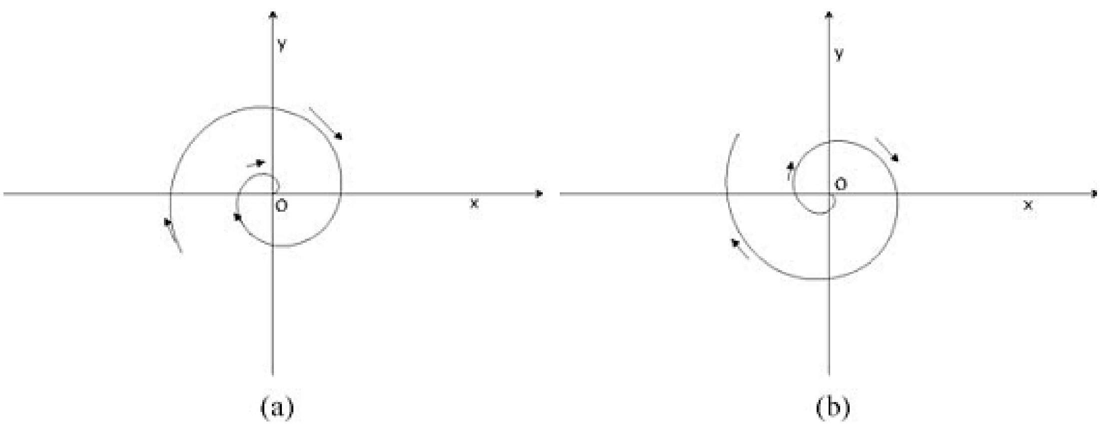


Figure 1.6: (a) $\alpha < 0$: Stable focus; (b) $\alpha > 0$: unstable focus

tr $(J) = 0$: The real part of the eigenvalues is worthless, the origin is a stable center (see Figure 1.7).

In short, the zone where the point of balance is asymptotically stable is that corres-

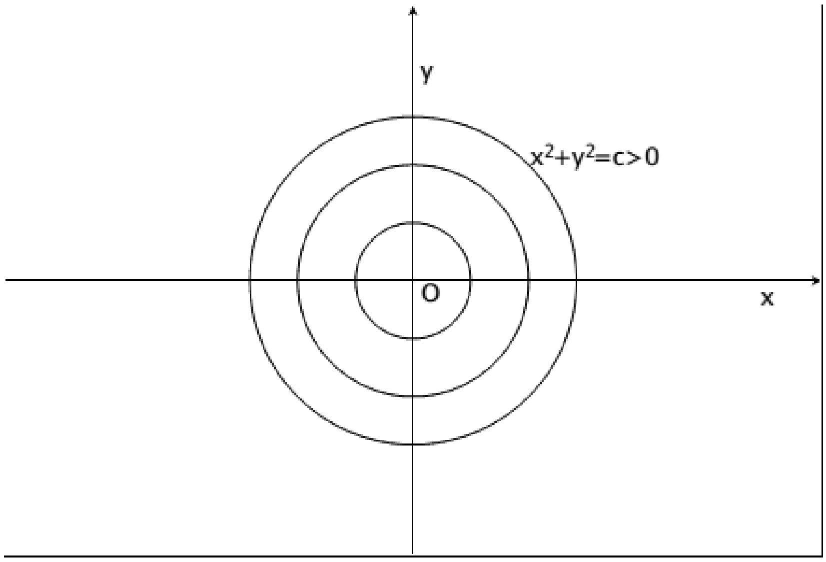


Figure 1.7: Stable center: with $\alpha = 0$

pondent with:

$$\begin{cases} \det(J) > 0 \\ \operatorname{tr}(J) < 0 \end{cases}.$$

In the typical case where

$$\begin{cases} \det(J) > 0 \\ \operatorname{tr}(J) = 0 \end{cases},$$

the origin is a center.

The following table sumarizes the natur of the equilibrium $(0,0)$:

Eigenvalues	Equilibrium
$\lambda_{1,2} \in \mathbb{R}, \lambda_1, \lambda_2 < 0$	asymptotically stable node
$\lambda_{1,2} \in \mathbb{R}, \lambda_1, \lambda_2 > 0$	unstable node
$\lambda_{1,2} \in \mathbb{R}, \lambda_1 * \lambda_2 < 0$	unstable saddle
$\lambda_{1,2} = \alpha \pm i\beta, \alpha < 0$	asymptotically stable focus
$\lambda_{1,2} = \alpha \pm i\beta, \alpha > 0$	unstable focus
$\lambda_{1,2} = \pm i\beta$	stable center

1.2.2 The Stability of a Nonlinear System

Definition 4 *Given a system $x'(t) = f(x(t))$ with equilibrium $x^* = 0$, its linearization at $x^* = 0$ is the linear system $x'(t) = Ax$, where $A = \nabla f(0)$. Developing f in Taylor's expansion we find $f(x) = Ax + O(|x|)$.*

Then the linearized system is $x'(t) = Ax$. We have seen that a sufficient condition for the asymptotic stability of $x = 0$ for $x'(t) = Ax$ is that all the real parts of the eigenvalues of A be negative, whilst if at least one eigenvalue is positive, or has positive real part, then $x^* = 0$ is unstable. This result is extended to the nonlinear case in the next theorem, whose proof is omitted.

Theorem 6 ([4]) *.Suppose that all the eigenvalues of $\nabla f(0)$ have negative real parts.Then the equilibrium $x^* = 0$ is asymptotically stable with respe:ct to the system $x'(t) = \nabla f(0)x + O(|x|)$. If at least one eigenvalue of $\nabla f(0)$ has positive real part, then the equilibrium $x^* = 0$ is unstable.*

Example 1 *.Consider the Van der Pol system*

$$\begin{cases} x' = -y \\ y' = x - 2\mu(x^2 - 1)y \end{cases},$$

with $|\mu| < 0$. Here the eigenvalues of

$$A = \nabla f(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 2\mu \end{pmatrix},$$

are $\lambda_1 = \mu + \sqrt{\mu^2 - 1}$, $\lambda_2 = \mu - \sqrt{\mu^2 - 1}$.if $0 < \mu < 1$, both the eigenvalues have positive real part and the equilibrium $(0,0)$ is unstable. On the other hand, if $-1 < \mu < 0$, both the eigenvalues have negative real part and the equilibrium $(0,0)$ is asymptotically stable.

1.3 Liapunov Direct Method

At the beginning of the 1900's, the Russian mathematician Aleksandr Liapunov developed what is called the Liapunov Direct Method for determining the stability of an equilibrium point. We will describe this method and illustrate its applications.

Definition 5 *Let $x^* \in \mathbb{R}^n$ be an equilibrium point of*

$$x'(t) = f(x(t)). \quad x(0) = p. \tag{1.6}$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set containing x^*. A real valued function $V \in C^1(\Omega, \mathbb{R})$ is called a Liapunov function for (1.6) if
$(V1): V(x) > V(x^*)$ for all $x \in \Omega, x \neq x^*$.

$(V2) : \dot{V}(x) = (\nabla V(x) | f(x)) \leq 0$, *for all* $x \in \Omega$.
Recall that $(x|y)$ *denotes the euclidean scalar product of the vectors* x, y, *see Notations. Moreover,* $\nabla V = (V_{x_1}, ..., V_{x_n})$ *denotes the gradient of* V *and the subscripts denote partial derivatives.*

Remark 3 *Note that, since* $x'(t) = f(x(t))$ *we have that*

$$\begin{aligned}
\dot{V}(x(t)) &= V_{x_1}(x(t)) f_1(x(t)) + V_{x_2}(x(t)) f_2(x(t)) ... + V_{x_n}(x(t)) f_n(x(t)) \\
&= V_{x_1}(x(t)) x_1'(t) + V_{x_2}(x(t)) x_2'(t) + ... + V_{x_n}(x(t)) x_n'(t) \\
&= (\nabla V(x(1)) | x'(t)) \\
&= \frac{d}{dt}[V(x(t))].
\end{aligned}$$

In other word, $\dot{V}(x(t)) = \frac{dV(x(t))}{dt}$ *is nothing but the derivative of* V *along the trajectories* $x(t)$*.Therefore* $(V2)$ *implies that* $V(x(t))$*. is non-increasing along the trajectories* $x(t)$*.*

Theorem 7 ([4]) *(Liapunov stability theorem).*
(i) If (1.2) has a Liapunov function, then x^* *is stable.*
(ii) If in $(V2)$ *one has that* $\dot{V}(x) < 0$, *for all* $x \neq 0$, *then* x^* *is asymptotically stable.*

Proof. We will prove only the statement (i).By the change of variable $y = x - x^*$, the autonomous system $x'(t) = f(x(t))$ becomes $y'(t) = f(y + x^*)$ which has $y = 0$ as equilibrium. Thus, without loss of generality, we can assume that $x^* = 0$. Moreover, still up to a translation, we can assume without loss of generality that $V(x^*) = 0$.

Finally, for simplicity, we will assume that $\Omega = \mathbb{R}^n$. The general case requires only minor changes. Set

$$\varphi_p(t) = V(x(t, p)).$$

The function $\varphi_p(t)$ is defined for all $t \geq 0$ and all $p \in \mathbb{R}^n$. Moreover $\varphi_p(t)$ is differentiable and one has

$$\varphi_p'(t) = V_{x_1}x_1' + ... + V_{x_n}x_n' = \left(\nabla V x(t, p) \left| x'(t, p)'\right.\right) = \dot{V}(x(t, p)).$$

By $(V2)$ it follows that $\varphi_p'(t) \leq 0$ for all $t \geq 0$. Hence $\varphi_p(t)$ is non-increasing and thus

$$0 \leq V(x(t, p)) \leq V(x(0, p)) = V(p), \qquad \forall t \geq 0.$$

Given any ball T_r centered at $x = 0$ with radius $r > 0$, let S_r denote its boundary.

From $(V1)$ it follows that

$$m = m(r) = \min\{V(x) : y \in S_r\} > 0.$$

Let $U = \{p \in T_r, V(p) < m\}$. From $(V1)$one has that U is a neighborhood of $x = 0$. Moreover, by (12.2) it follows that $V(x(t,p)) < m$ for all $t \geq 0$ and all $p \in U$.Since m is the minimum of V in S_r, the solution $x(t,p)$ has to remain in T_r,provided $p \in U$, namely $p \in U \Longrightarrow x(t,p) \in T_r$ and this proves that $x = 0$ is stable.

Roughly, the Liapunov function V is a kind of potential well with the property that the solution with initial value p in the well remain confined therein for all $t \geq 0$. ■

Remark 4 *If $\dot{V} = 0$ for all $t \geq 0$, then $V(x(t,p))$ is constant, namely $V(x(t,p)) = V(x(0,p)) = V(p)$ all $t \geq 0$. Then $x(t,p)$ cannot tend to x^* as $t \to +\infty$. As a consequence, x^* is stable but not asymptotically stable.*

1.3.1 Example: Lotka Voltera Equations

In the 1920"s Vito Voltera was asked whether it would be possible to explain the fluctuations that had been observed in the fish population of the Adriatic sea fluctuations that were of great concern ti fishermen in times of low fish populations.Volterra 1926 constructed the model that has become known as the Lotka Voltera model (because A.J.Lotka (1925) constructed a similar model in a different context about the same time), based on the assumptions that fish and sharks were in a predator-prey relationship.Here is a description of the model suggested by Volterra. Let $x(t)$ be the number of fish and $y(t)$ the number of sharks at time t. We assume that the plankton; which is the food supply for the fish, is unlimited, and thus that the per capita growth rate of the fish population in the absence of sharks would be constant. Thus, if there were no sharks the fish population would satisfy a differential equation of the form $\partial x/\partial t = ax$. The sharks, on the other hand, depend on fish as their food supply,and we assume that if there were no fish, the sharks would have a constant per capita death rate; thus, in the absence of fish, the shark population would satisfy a differential equation of the form $\partial y/\partial t = -cy$. We assume that the presence of fish increases the shark growth rate, changing the per capita growth rate from $-c$ to $-c + dx$. The presence of sharks reduces the fish population, changing the per capita fish growth rate from a to $a - by$. This gives the Lotka-Volterra equation.

$$\begin{cases} \frac{\partial x}{\partial t} = ax - bxy \\ \frac{\partial y}{\partial t} = -cx + dxy \end{cases} . \qquad (1.7)$$

The predator population lags behind that of the prey in achieving its maximum values. This lag is shown in Figure 1.8, witch graphs both $x\left(t\right)$ and $y\left(t\right)$.

Now we want to stady the equilibruim point the system (1.7):

We put

$$\begin{cases} ax - bxy = f(x,y) \\ -cx + dxy = g(x,y) \end{cases}, \tag{1.8}$$

the point (x^*, y^*) is an equilibruim:

$$\begin{cases} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{cases},$$

$$\begin{cases} ax^* - bx^*y^* = 0 \\ -cx^* + dx^*y^* = 0 \end{cases} \Longrightarrow \begin{cases} x^* = 0,\ y^* = 0 \\ x^* = \frac{c}{d},\ y^* = \frac{a}{b} \end{cases},$$

so $(x^*, y^*) = \left\{(0,0), (\frac{c}{d}, \frac{a}{b})\right\}$.

The Jacobian matrix of (1.7) is

$$J = \begin{pmatrix} a - by & -by \\ dy & -c + dx \end{pmatrix}, \tag{1.9}$$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = -a$, $\lambda_2 = c$. It follows that $(0,0)$ is unstable saddle.

On the contrary, let us show that the equilibrium $(\frac{c}{d}, \frac{a}{b})$ is stable. Here

$$J(\frac{c}{d}, \frac{a}{b}) = \begin{pmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix},$$

$$\det(J - \lambda I) = \lambda^2 + ac = 0,$$

then eigenvalues are $\lambda_{1,2} = \pm i\sqrt{ac}$. It follows that $(\frac{c}{d}, \frac{a}{b})$ is stable center.

Global stability:

We want to study the global stability of the nontrivial equilibrium $(\frac{c}{d}, \frac{a}{b})$ of (1.7) :

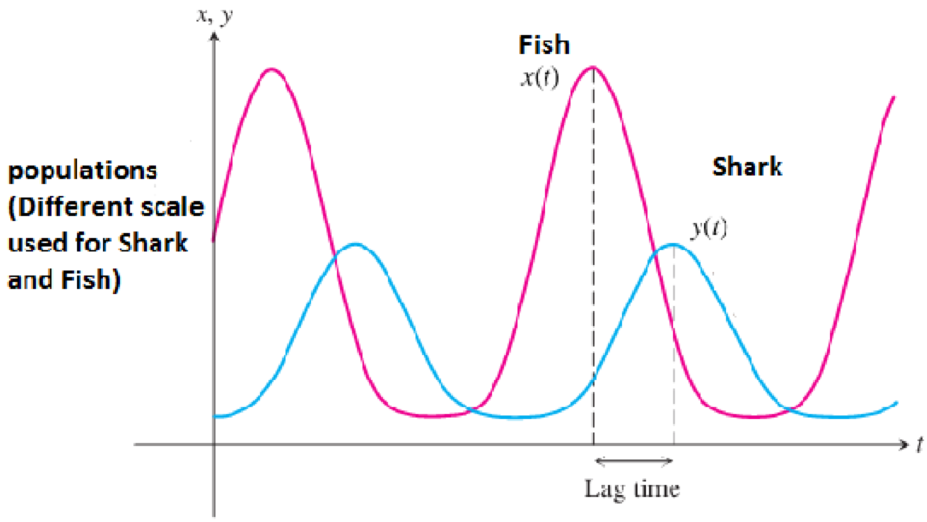


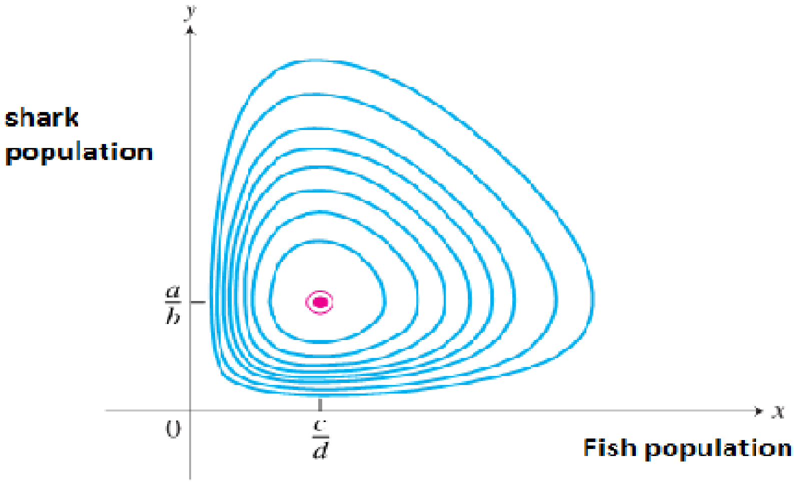
Figure 1.8: The shark and fish populations oscillante periodically, with the mlaximum shark population lagging the maximum fish population.

Let the liapunov function be defined as

$$\begin{aligned}
H(x,y) &= dx + by - c\ln x - a\ln y, x>0, y>0 \\
\frac{\partial H}{\partial t}(x,y) &= H_x\frac{\partial x}{\partial t} + H_y\frac{\partial y}{\partial t} = \left(d - \frac{c}{x}\right)\frac{\partial x}{\partial t} + \left(b - \frac{a}{y}\right)\frac{\partial y}{\partial t} \\
&= \left(d - \frac{c}{x}\right)(ax - bxy) + \left(b - \frac{a}{y}\right)(-cx + dxy), \\
\frac{\partial H}{\partial t}(\frac{c}{d},\frac{a}{b}) &= 0,
\end{aligned} \tag{1.10}$$

then $(\frac{c}{d},\frac{a}{b})$ is stable (but not asymptotically stable).

The shark and fish populations oscillate through repeated cycles along a fixed trajectory. Figure 1.3.1 shows several trajectories for the predator-prey system.



Some trajectories along wich H is conserved

CHAPTER 2

Reaction–Diffusion System

In this chapter we present an introduction to R-D systems, Activator-Inhibitor systems. Then we talked about the Hopf bifurcation and the global existence of solutions using the Liapunov's theorem, in the later we gave some examples of these systems and we devoted the system of Gierer-Meinhardt.

2.1 Introduction to Reaction-Diffusion Systems

In recent years, reactions-diffsion systems have received a great deal of attention motivated by their widespread incident in models of biological and chemical phenomena, and by the richness of the structure of their sets of solutions. Considering the numerous and varied applications of these systems; The Approaches to modeling certain chemical problems such as reactions oscillating chemicals (Brussellateur). Individuals diverge from one problem to another:

In chemistry, for example, they are chemical substances. In biochemistry, they May represent molecules. In metallurgy, atoms. In dynamics of Populations, they are humans. In population genetics, they represent characters. In biophysics, electrical charges or potential differences. In the environment, they can represent the animals or plants of a forest, a sea or an ocean ...

For most of these problems, we show that results in reaction-diffusion systems. The conditions at the edges will be chosen according to the origin and the nature of the problem Studied: if there is no immigration of individuals across the boundary of the domain Ω on which the problem is posed, the conditions at the homogeneous edges of Neumann. If there are no individuals on the border, we take the conditions at the edges Homogeneous of Dirichlet. The unknown (the solution one seeks) is a vector of which The components are generally positive functions: in chemistry, for example, it is a vector of chemical concentrations. In biochemistry or metallurgy, a vector of concentrations in numbers of molecules or atoms respectively. In population dynamics and in the environment, it is a vector of densities of Human, animal or plant populations ...

Initial conditions are generally positive; Since they are concentrations, densities, electrical charges, etc. All these problems are in the form of:

$$\frac{\partial u}{\partial t} - D\Delta u = f(u),$$

Where $u(x,t) = (u_1(x,t), ..., u_m(x,t))$ is a vector of dependent variables, and $f(x,t,u(x,t)) = (f_1(x,t,u(x,t)), ..., f_m(x,t,u(x,t)))$ is the reaction (usually nonlinear) and D is a square matrix $m \times m$ Positive and diagonalizable called dissemination matrix. The terms of reaction are the result of any interaction between the components of u:

For example, in chemistry u is a vector of chemical concentrations and f represents the chemical reactions on these concentrations. The term $D\Delta u$ represents the Molecular diagnoses across the reaction boundary.

2.1.1 Turing Instabilities

In 1952, Turing published a paper titled The chemical basis of morphogenesis([15]), where he proposed a reaction-diffusion model for pattern formation, in which diffusion was the source of the instability that caused patterns to form. However, we will see that the key is that this instability comes from the interaction of the reactive and diffusive terms that govern interacting chemical species that are diffusing within some spatial domain.

We introduce a definition:

Definition 6 *A diffusion-driven instability, or Turing instability, occurs when a steady state, stable in the absence of diffusion, becomes unstable when diffusion is present.*

2.2 Activator Inhibitor Systems

2.2.1 Introduction

Pattern formation is a very important process in the development of all organisms. For example the colony formation of small marine animals is triggered by an activator-inhibitor system. Furthermore, the regular spacing of leaves or the the ordering of stomata on a leaf can be explained with the help of such interacting system. There exist different mathematical models that are able to simulate such processes. These models consist of at least two substances that influence each other. The system has to be globally stable and locally unstable to form patterns. In order to achieve theses characteristics the diffusion plays a very important role as it is shown in the following.

2.2.2 Behaviour of Activator-Inhibitor Systems

An activator-inhibitor system consists of two substances that act on each other. The activator stimulates its own production via autocatalysis as well as the production of the inhibitor. The inhibitor in turn represses the production of the activator. In addition, the inhibitor diffuses more rapidly than the activator such that patterns of activator and inhibitor concentrations can arise. Two cases are considered:

1) An equal activator increase at all positions of a linear array of cells.

2) A random perturbation in just a few cells of the array. Both situations will lead to different behaviours of the system.

Theorem 8 ([2]) *In the system (1.2)*
If

$$f_y(x^*, y^*) < 0, g_y(x^*, y^*) < 0$$

and

$$f_x(x^*, y^*) > 0, g_x(x^*, y^*) > 0$$

is satisfied, then we call x an activator, y an inhibitor, and the system (1.2) is an activator–inhibitor system.

2.3 Hopf Bifurcation

The term Hopf bifurcation (also sometimes called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes called dynamic (as opposed to static) bifurcation theory. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.

That a periodic solution should be generated in this event is intuitively clear from Figure 2.1. When the real parts of the eigenvalues are negative the fixed point is a stable focus (Figure 2.1.a); when they cross zero and become positive the fixed point becomes an unstable focus, with orbits spiralling out. But this change of stability is a local change and the phase portrait sufficiently far from the fixed point will be qualitatively unaffected: if the nonlinearity makes the far flow contracting then orbits will still be coming in and we expect a periodic orbit to appear where the near and far flow find a balance (as in Figure 2.1.b).

The Hopf bifurcation theorem makes the above precise. Consider the planar system where μ is a parameter. Suppose it has a fixed point $(x, y) = (x_0, y_0)$, which may depend on μ. Letthe eigenvalues of the linearised system about this fixed point be given by $\lambda(\mu)$, $\bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$.

Suppose further that for a certain value of μ, say $\mu = \mu_0$, the following conditions are satisfied (As mentioned in [9] and [17]):

1.$\alpha(\mu_0) = 0$, $\beta(\mu_0) = w \neq 0$, where $sgn(w) = sgn[(\partial g_\mu/\partial x)|_{\mu=\mu_0}(x_0, y_0)]$ (non-hyperbolicity condition: conjugate pair of imaginary eigenvalues).

2.$\frac{d\alpha(\mu)}{d\mu}\big|_{\mu=\mu_0} = d \neq 0$ (transversality condition: the eigenvalues cross the imaginary axis with non-zero speed).

3. $a \neq 0$, where

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16w}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}),$$

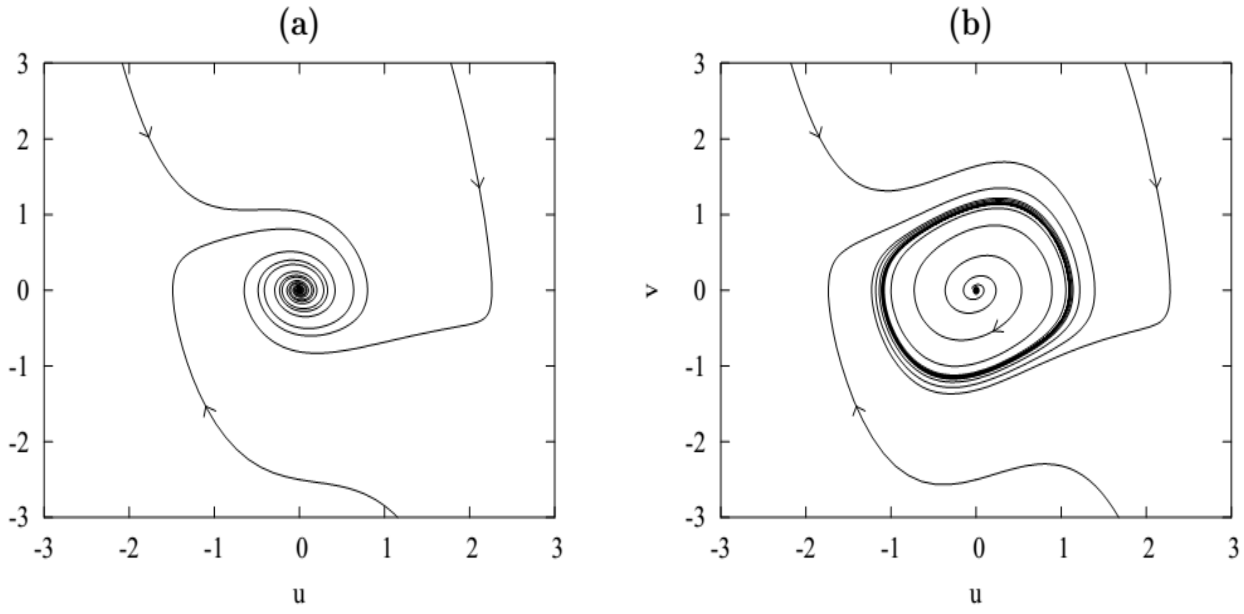


Figure 2.1: Phase portraits of (2) for (a) $\mu = -0.2$, (b) $\mu = 0.3$. There is a supercritical Hopf bifurcation at $\mu = 0$.

with $f_{xy} = (\partial^2 f_\mu / \partial x \partial y)\big|_{\mu=\mu_0}(x_0, y_0)$, etc. (genericity condition).

Then a unique curve of periodic solutions bifurcates from the fixed point into the region $\mu > \mu_0$ if $ad < 0$ or $\mu < \mu_0$ if $ad > 0$. The fixed point is stable for $\mu > \mu_0$ (resp. $\mu < \mu_0$) and unstable for $\mu < \mu_0$ (resp. $\mu > \mu_0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable) if the fixed point is unstable (resp. stable) on the side of $\mu = \mu_0$ where the periodic solutions exist.

The amplitude of the periodic orbits grows like $\sqrt{|\mu - \mu_0|}$ whilst their periods tend to $2\pi / |w|$ as μ tends to μ_0. The bifurcation is called supercritical if the bifurcating periodic solutions are stable, and subcritical if they are unstable.

This $2D$ version of the Hopf bifurcation theorem was known to Andronov and his co-workers from around 1930, and had been suggested by Poincaré in the early 1890s. Hopf, in 1942, proved the result for arbitrary (finite) dimensions. Through centre manifold reduction the higher-dimensional version essentially reduces to the planar one provided that apart from the two purely imaginary eigenvalues no other eigenvalues have zero real part. In his proof (which predates the centre manifold theorem), Hopf assumes the functions f_μ and g_μ to be analytic, but C^5differentiability is sufficient. Extensions exist to infinite-dimensional problems such as differential delay equations and certain classes of partial differential equations (including the Navier-Stokes equations).

2.4 Examples

Now we will provide some **examples** of these systems.

2.4.1 System of Lengyel-Epstein

The Lengyel–Epstein System is the following reaction-diffusion equations

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2} \\ \frac{\partial v}{\partial t} = (\sigma b)\Delta v + (\sigma b)\left(u - \frac{uv}{1+u^2}\right) \end{cases}, \qquad x \in \Omega,\ t > 0$$

which was derived from the chlorite iodide malonic acid (CIMA) chemical reaction introduced by Lengyel and Epstein and can be used to design chemical systems capable of displaying stationary, symmetry breaking reaction diffusion patterns (Turing structures). Here u and v are the concentrations of the active iodide I^- and inhibitor (ClO_2^-) at time t, respectively, a and b are positive parameters related to the feed concentrations, $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch.

A closely related system for this chemical reaction mechanism is the chlorine dioxide-iodine-malonic acid (CDIMA) reaction shown below.

$$\begin{cases} MA + I_2 \rightarrow IMA + I + H^+ \\ ClO_2 + I^- \rightarrow ClO_2 + \frac{1}{2}I_2 \\ CIO_2^- + 4I^- + 4H^+ \rightarrow Cl + 2I_2 + 2H_2O \\ S + I_2 + I_- \rightarrow SI^{-3} \end{cases}.$$

The first reaction serves as a source of the activator I^-, the second produces the inhibitor chlorite ion, the third shows regeneration of iodine, and the last reaction shows the complex formation between the activator iodide (I^-) and the indicator starch.

In the CDIMA system of reaction, the concentration of Malonic acid (MA), Chloride Dioxide (ClO_2) and Iodine (I_2) displays very little variation and essentially they can be considered constant. Since only the activator iodine ion (I^-) and the inhibitor chlorite ion (ClO_2^-) show wide concentration variation, the system can be approximated by two variables model.

In the presence of starch which is used as indictor, the diffusion rate of the activator (I^-) is slower than that of the inhibitor (ClO_2^-). The starch which is much bigger molecule forms a chemical complex with I^- effectively reducing the diffusion rate of I^-. This allows the inhibitor to diffuse faster creating a condition that leads to oscillatory phenomenon.(For more information, see [11])

2.4.2 The Gray-Scott Model: Pearson's Parametrization

The reaction-diffusion system described here involves two generic chemical species U and V, whose concentration at a given point in space is referred to by variables u and v. As the term implies, they react with each other, and they diffuse through the medium. Therefore the concentration of U and V at any given location changes with time and can differ from that at other locations.

The system is described by the following formula

$$\begin{cases} \frac{\partial u}{\partial t} = D_u \Delta u - uv^2 + F(1-u) \\ \frac{\partial v}{\partial t} = D_v \Delta v + uv^2 + (F+k)v \end{cases},$$

$u = [U]$, the concentration of U, and $v = [V]$, the concentration of V. For the sake of simplicity we can consider Du, Dv, F and k to be constants. (For more information, see [12])

2.4.3 The Gierer-Meinhardt System

The G-M Model is a reaction-diffusion system of the activator-inhibitor type that appears to account for many important types of pattern formation and morphogenesis observed in development, in their seminal paper, Gierer and Meinhardt ([5]) proposed the model

$$\begin{cases} \frac{\partial a}{\partial t} = -\mu a + c\rho \frac{a^r}{h^s} + \rho\rho_0 + D_a \frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = c'\rho' \frac{a^T}{h^u} - \gamma h + D_h \frac{\partial^2 h}{\partial x^2}. \end{cases} \tag{2.1}$$

Where $a(x,t)$ represents the population density of the activator and $h(x,t)$ the inhibitor, and $\rho_0, \rho, \rho', c', c, \mu, \gamma, r, s, T, u, D_a, D_h$ are all positive constants. The activator a and the inhibitor h act on the sources with density $\rho(x)$ and $\rho'(x)$, respectively.For simplicity, we assume the sources are evenly distributed in space, i.e. $\rho(x) = \rho$, $\rho'(x) = \rho'$ and the basal production of the activator is proportional to ρ. The terms $-\mu a$ and $-\gamma h$ represent the rates that a and h are removed by either enzyme degradation, or leakage, or reuptake by the source, or by any combination of these mechanisms. D_a and D_h are the diffusion coefficients of the activator and inhibitor, respectively. We will also assume that D_a, ρ_0 small and D_h large, so that the inhibitor will have nearly equal distribution over the entire area ([5]). A further approximation leads (2.1) to the following two simplest models:

The first model is the activator and inhibitor system with common sources

$$\begin{cases} \frac{\partial a}{\partial t} = -\mu a + c\rho \frac{a^2}{h^4} + \rho\rho_0 + D_a \frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = c'\rho' \frac{a^2}{h^4} - \gamma h + D_h \frac{\partial^2 h}{\partial x^2} \end{cases}, \tag{2.2}$$

where correspondingly $u = s = 4, r = T = 2$ and $\rho = \rho'$ in (2.1), and the second model is the activator and inhibitor system with different sources

$$\begin{cases} \frac{\partial a}{\partial t} = -\mu a + c\rho \frac{a^2}{h^s} + \rho\rho_0 + D_a \frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = c'\rho' a - \gamma h + D_h \frac{\partial^2 h}{\partial x^2} \end{cases}, \tag{2.3}$$

where correspondingly $u = 0, r = 2, T = 1$ in (2.1).

In [[14]], the stability of the equilibrium and the Hopf bifurcation of (2.3) with $s = 1$ were analyzed, and the author used spectral analysis and Floquet exponent to show that under certain conditions, the equilibrium (Hopf periodic solution) is asymptotically stable for the ODE system while, with added diffusions under Neumann boundary conditions, the equilibrium (Hopf periodic solution) may lose its stability and a spatial pattern may occur.

In contrast, there have been a paucity of studies on system (2.3) with $s = 2$. In particular, we would like to understand the pattern generating mechanism in this case, namely, the parametric range for the pattern to form and the dynamics that are responsible for the patterns such as stripes and spots. As far as we know, the results for the Turing patterns for the Geirer-Meinhardt model in this case are new, and they are important in theory and implication.

2.5 Global Existence

To demonstrate the existence of the solutions of the reaction systems, there are several methods such as the invariant region method, the smoothing effect method, functional methods based on a priori estimates or on Liapunov functional. Here we do not expose the first two methods since they do not always give the global existence in view of the diffculty and the complexity of the reaction terms of certain reaction-diffusion systems, but we devote ourselves to the last method which gives satisfactory results.

Definition 7 *(Functional of Lyapunov) Functional Lyapunov is associated with a system of reaction-diffusion formed by m-equations, any function*

$$L : \mathbb{R}^+ \to \mathbb{R}^+,$$

such that

$$\frac{d}{dt}\left(L\left(u_1(t,\cdot), ..., u_m(t,\cdot)\right)\right) \leq 0,$$

for all $t > 0$ *and all solution* $\left(u_1(t,\cdot), ..., u_m(t,\cdot)\right)$ *of the system.*

Remark 5 *We can use only a bounded functional to demonstrate the global existence of solutions.Here are some of the theories we have used in the following chapters.*

Theorem 9 ([13]) *Assume Ω bounded.*
(i) Assume that

$$\frac{r-1}{T} < \min\left(\frac{s}{u+1}, 1\right). \tag{2.4}$$

Then, for all $a_0, h_0 \in C\left(\bar{\Omega}\right)$, with $a_0, h_0 > 0$, the solution (a, h) of problem (2.1) is global. If in addition $\mu, \gamma, \rho, \rho_0 > 0$, then a, h, are uniformly bounded in $\bar{\Omega} \times [0, \infty)$.
(ii) Assume that

$$\frac{r-1}{T} > \min\left(\frac{s}{u+1}, 1\right), \qquad \frac{r-1}{T} \neq 1.$$

Then there exist space-independent initial data $a_0, h_0 > 0$ such that the solution $(a, h) = (a(t), h(t))$ of problem satisfies $T_{\max} < \infty$.

Lemma 10 ([13]) *Assume that r, s, T, u satisfy For all $\eta, \alpha, \beta > 0$, there exist $C = C(\eta, \alpha, \beta) > 0$ and $\theta = \theta(\alpha) \in (0.1)$ such that*

$$\alpha \frac{x^{r-1+\alpha}}{y^{u+1+\beta}} \leq \beta \frac{x^{T+\alpha}}{y^{u+1+\beta}} + C\left(\frac{x^{\alpha}}{y^{\beta}}\right)^{\theta}, \qquad x \geq 0, y \geq \eta. \tag{2.5}$$

Proof. Let $x > 0$ and $y \geq \eta$. Inequality is equivalent to

$$\alpha \frac{x^{r-1}}{y^{s}} \leq \beta \frac{x^{T}}{y^{u+1}} + C\left(\frac{y^{\beta}}{x^{\alpha}}\right)^{1-\theta},$$

write

$$\alpha \frac{x^{r-1}}{y^{s}} = \left(\frac{x^{T}}{y^{u+1}}\right)^{(r-1)/T} y^{(r-1)(u+1)/r-s} = C\left(\beta \frac{x^{r}}{y^{u+1}}\right)^{\kappa} y^{-m},$$

where $\kappa = (p-1)/r < 1$ and $m = s - (r-1)(u+1)/T > 0$. For each $0 < \varepsilon < \min(m/(u+1), 1-\kappa)$, using $y \geq \eta$ and Young's inequality, we obtain

$$\begin{aligned} \alpha \frac{x^{r-1}}{y^{s}} &= C\left(\beta \frac{x^{T}}{y^{u+1}}\right)^{\kappa+\epsilon} y^{-m+(u+1)\epsilon} x^{-T\epsilon} \leq C\left(\beta \frac{x^{T}}{y^{u+1}}\right)^{\kappa+\epsilon} \left(\frac{y^{\beta}}{x^{\alpha}}\right)^{T\epsilon/\alpha} \\ &\leq \beta \frac{x^{T}}{y^{u+1}} + C\left(\frac{y^{\beta}}{x^{\alpha}}\right)^{T\epsilon/(1-\kappa-\epsilon)\alpha}, \end{aligned}$$

and follows by taking ϵ sufficiently small. ■

Lemma 11 ([?]) *Let $T > 0$ and $f = f(t)$ be a non-negative integrable function on $[0, T)$. Let $0 < \theta < 1$ and $W = W(t)$ be a positive function on $[0, T)$ satisfying the differential* *inequality*

$$\frac{dW}{dt} \leq -W\left(t\right) + f\left(t\right) W^{\theta}\left(t\right), \ \ 0 \leq t < T.$$

Then $W\left(t\right) \leq k$*, where* k *is the positive root of the algebraic equation*

$$x - \left(\sup_{0<t<T} \int e^{-(t-\xi)} f\left(\xi\right) d\xi\right) x^{\theta} = W\left(0\right).$$

CHAPTER 3

The Gierer-Meinhardt Activator-Inhibitor Model

As we see before about Gierer-Meinhardt now we will investigate this system by studying the local stability in all case (**EDO** and **PDE** models), the global existence, How changes occur in stability, Bifurcations.

3.1 The ODE Model

3.1.1 Local stability

The Stability of the Equilibrium of the Kinetic System (2.3) with $s = 1$

In this subsubsection we will study the stability of the equilibrium of (2.3) with $s = 1$, for the ODE system.

Following the Gierer-Meinhardt equations for the activator concentration a and the inhibitor concentration h we can write:

$$\begin{cases} a_t = -\mu a + c\rho \frac{a^2}{h} + \rho\rho_0 \\ h_t = c'\rho' a^2 - vh \end{cases}. \tag{3.1}$$

System (3.1) can be interpreted in this way: two molecules of activator are necessary to activate and one to inhibit the source.

For convenience, we denote

$$\begin{cases} -\mu a + c\rho \frac{a^2}{h} + \rho\rho_0 = f(a, h) \\ c'\rho' a^2 - vh = g(a, h) \end{cases}.$$

The equilibrium (a^*, h^*) of (3.1) satisfies the equations

$$\begin{cases} -\mu a + c\rho \frac{a^2}{h} + \rho\rho_0 = 0 \\ c'\rho' a^2 - vh = 0 \end{cases},$$

so

$$\begin{cases} -\mu u + c\rho \frac{{a^*}^2}{h^*} + \rho\rho_0 = 0 \\ -vh^* + c'\rho' a^{*2} = 0 \end{cases},$$

then

$$\begin{cases} a^* = \left(\frac{c\rho v + c'\rho'\rho\rho_0}{c'\rho'\mu}\right) \\ h^* = \frac{c'\rho' a^{*2}}{v} \end{cases},$$

the nontrivial equilibrium is $(a^*, h^*) = \left(\left(\frac{c\rho v + c'\rho'\rho\rho_0}{c'\rho'\mu}\right), \frac{c'\rho'}{v}\left(\frac{c\rho v + c'\rho'\rho\rho_0}{c'\rho'\mu}\right)^2\right)$.

The jacobian matrix of the system (3.1) is:

$$J = \begin{pmatrix} \frac{2c\mu v}{cv + c'\rho'\rho_0} - \mu & -\frac{c}{\rho}\left(\frac{\mu v}{cv + c'\rho'\rho_0}\right)^2 \\ \frac{2\rho(cv + c'\rho'\rho_0)}{\mu} & -v \end{pmatrix},$$

$$\begin{aligned}\det(J) &= \left(\frac{2c\mu v}{cv+c'\rho'\rho_0}-\mu\right)(-v)+\left(\frac{2\rho\left(cv+c'\rho'\rho_0\right)}{\mu}\right)\left(-\frac{c}{\rho}\left(\frac{\mu v}{cv+c'\rho'\rho_0}\right)^2\right)\\ &= \mu v.\end{aligned}$$

$$\begin{cases} tr(J)=\frac{2c\mu v}{cv+c'\rho'\rho_0}-\mu-v \\ \det(J)=\mu v>0\end{cases},$$

$$tr(J)=\frac{2c\mu v}{cv+c'\rho'\rho_0}-\mu-v<0,$$

then

$$\frac{-\left(c'\rho'\rho_0-cv\right)}{c'\rho'\rho_0+cv}<\frac{v}{\mu}$$

$$\begin{aligned}\det(J-\lambda I) &= \begin{vmatrix}\frac{2c\mu v}{cv+c'\rho'\rho_0}-\mu-\lambda & -\frac{c}{\rho}\left(\frac{\mu\nu}{c\nu+c'\rho'\rho_0}\right)^2\\ \frac{2\rho(cv+c'\rho'\rho_0)}{\mu} & -v-\lambda\end{vmatrix}\\ &= \lambda^2-\left(\frac{2c\mu\nu}{cv+c'\rho'\rho_0}-\mu-v\right)\lambda+\mu v\\ &= 0,\end{aligned}$$

the characteristic equation is

$$\lambda^2-\lambda t\,\mathrm{tr}\,J+\det J=0$$

$$\begin{aligned}\Delta &= \left(-\,\mathrm{tr}\,J\right)^2-4\det J\\ &= i^2\left(4\det J-(\mathrm{tr}\,J)^2\right),\end{aligned}$$

then

$$\lambda_{1,2}=\frac{1}{2}\,\mathrm{tr}\,J\pm i\frac{1}{2}\sqrt{4\det J-(\mathrm{tr}\,J)^2}$$

the characteristic roots can be expressed as

$$\lambda_{1,2}=\alpha\left(\mu\right)\pm iw(\mu),$$

where

$$\alpha\left(\mu\right)=\frac{1}{2}\,\mathrm{tr}\,J,\ w(\mu)=\frac{1}{2}\sqrt{4\det J-\mathrm{tr}\,J^2}.$$

It is easy to see that $\lambda_{1,2}$ have negative real parts if

$$\mu>0\Longleftrightarrow v>\frac{c'\rho'\rho_0}{c}, \tag{3.2}$$

the point (a^*, h^*) is asymptatically stable if

$$\mu < \frac{v\,(cv + c'\rho'\rho_0)}{cv - c'\rho'\rho_0} = \mu_0 \tag{3.3}$$

and

$$v > \frac{c'\rho'\rho_0}{c}.$$

The Stability of the Equilibrium of the Kinetic System (2.3) with $s = 2$:

From now on, we fix $s = 2$ in (2.3).To make the later exposition easier we nondimensionalize (2.3) with $s = 2$. Let $a = \frac{c\rho D_a}{(c'\rho')^2}\bar{a}$, $h = \frac{c\rho}{c'\rho'}\bar{h}$, $c_0 = \frac{\rho_0}{c(D_a)^2}\left(c'\rho'\right)^2$, $t = \frac{\bar{t}}{D_a}$, $\mu = D_a\bar{\mu}$, $\gamma = D_a\bar{\gamma}, D_h = rD_a$, we have

$$\begin{cases} \frac{\partial \bar{a}}{\partial t} = c_0 + \frac{\bar{a}^2}{\bar{h}^2} - \mu\bar{a} + \frac{\partial^2 \bar{a}}{\partial x^2} \\ \frac{\partial \bar{h}}{\partial t} = \bar{a} - \gamma\bar{h} + r\frac{\partial^2 \bar{h}}{\partial x^2} \end{cases}. \tag{3.4}$$

For notational convenience, we will still use $a, h, t,\ \mu,\ \gamma$ instead of $\bar{a},\ \bar{h},\ \bar{t},\ \bar{\mu},\ \bar{\gamma}$. (3.4) now reads

$$\begin{cases} \frac{\partial a}{\partial t} = c_0 + \frac{a^2}{h^2} - \mu a + \frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = a - \gamma h + r\frac{\partial^2 h}{\partial x^2} \end{cases}. \tag{3.5}$$

We will study the stability of the equilibrium and the Hopf bifurcation of the system (3.5)

Without diffusion, we can write the system (3.5) as

$$\begin{cases} \frac{\partial a}{\partial t} = c_0 + \frac{a^2}{h^2} - \mu a \\ \frac{\partial h}{\partial t} = a - \gamma h \end{cases}, \tag{3.6}$$

we denote

$$\begin{cases} c_0 + \frac{a^2}{h^2} - \mu a = f(a,h) \\ a - \gamma h = g(a,h) \end{cases}$$

(a^*, h^*) is an equilibruim point of (3.6) if :

$$\begin{cases} f(a^*, h^*) = 0 \\ g(a^*, h^*) = 0 \end{cases},$$

then

$$\begin{cases} c_0 + \frac{a^2}{h^2} - \mu a = 0 \\ a - \gamma h = 0 \end{cases},$$

then

$$\begin{array}{l} h^* = \frac{c_0+\gamma^2}{\mu v} \\ a^* = \frac{c_0+\gamma^2}{\mu} \end{array}.$$

The unique équilibruim point of this system is :

$$(a^*, h^*) = (\frac{c_0 + \gamma^2}{\mu}, \frac{c_0 + \gamma^2}{\mu\gamma}). \tag{3.7}$$

The Jacobian matrice of (3.6) is:

$$J = \begin{pmatrix} \frac{2a}{h^2} - \mu & \frac{-2a^2}{h^3} \\ 1 & -\gamma \end{pmatrix}, \tag{3.8}$$

$$J(a^*, h^*) = \begin{pmatrix} \frac{v^2-c_0}{c_0+\gamma^2}\mu & \frac{-2v^3}{c_0+\gamma^2}\mu \\ 1 & -\gamma \end{pmatrix}, \tag{3.9}$$

$$\begin{aligned} \det(J - \lambda I) &= (\frac{\gamma^2 - c_0}{c_0 + \gamma^2}\mu - \lambda)(-\gamma - \lambda) + \frac{2\gamma^3}{c_0 + \gamma^2}\mu \\ &= 0, \end{aligned}$$

$$\lambda^2 - (\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\mu - \gamma)\lambda + \mu\gamma = 0 \tag{3.10}$$

$$\begin{aligned} \Delta &= (\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\mu - \gamma)^2 - 4\mu\gamma \qquad (3.11) \\ &= \left(\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\right)^2 \mu^2 - 2\frac{-c_0 + v^2}{c_0 + v^2}\mu\gamma + \gamma^2 - 4\mu v \\ &= \left(\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\right)^2 \mu^2 - \left(2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma + 4\gamma\right)\mu + \gamma^2. \end{aligned}$$

$$\begin{aligned} \Delta' &= \left(2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma + 4\gamma\right)^2 - 4\left(\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\right)^2\gamma^2 \\ &= \left(2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma + 4\gamma - 2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma\right)\left(2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma + 4\gamma + 2\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma\right) \\ &= (4\gamma)\left(4\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma + 4\gamma\right) \\ &= 16\frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\gamma^2 + 16\gamma. \end{aligned}$$

The eigenvalues of (3.11) are

$$\begin{aligned}\mu_1 &= \frac{\left(\sqrt{c_0+\gamma^2}-\sqrt{2}\gamma\right)^2}{-c_0+\gamma^2}\mu_0\\ \mu_2 &= \frac{\left(\sqrt{c_0+\gamma^2}+\sqrt{2}\gamma\right)^2}{-c_0+\gamma^2}\mu_0,\end{aligned}$$

where

$$\mu_0 = \frac{-c_0+\gamma^2}{c_0+\gamma^2}\gamma. \tag{3.12}$$

Theorem 12 *System (3.6) has a unique equilibrium* $(a^*, h^*) = (\frac{c_0+\gamma^2}{\mu}, \frac{c_0+\gamma^2}{\mu\gamma})$, *whish is asymptotically stable if*

$$(H1) \qquad \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu < \gamma,$$

and is unstable if

$$(H2) \qquad \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu > \gamma.$$

Furthermore,
(i) (a^*, h^*)*is a stable node if one of the following conditions is satisfied:*

$$(ia)\ \ \gamma^2 \neq c_0, 0<\mu\leq\mu_1;\ \ (ib)\ \ \gamma^2 = c_0,\ \ 0<\mu<\frac{\gamma}{4};\ \ (ic)\ \ \gamma^2 < c_0,\ \ \mu\geq\mu_2.$$

(ii) (a^*, h^*)*is a stable focus if one of the following conditions is satisfied:*

$$(iia)\, \gamma^2 < c_0, \mu_1<\mu\leq\mu_2; (iib)\, \gamma^2 = c_0, \mu > \frac{\gamma}{4}; (iic)\, \gamma^2 > c_0, \mu_1<\mu\leq\mu_0.$$

(iii) (a^*, h^*) *is an unstable focus if*

$$\gamma^2 > c_0, \mu_0 < \mu < \mu_2,$$

(iv) (a^*, h^*)*is an unstable node if*

$$\gamma^2 > c_0, \mu \geq \mu_2.$$

Proof. the characteristic roots of (3.10) are

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma \pm \sqrt{\left(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma\right)^2 - 4\mu\gamma}\right].$$

We have two cases:
The first case $\gamma^2 = c_0$
(A_1)

$$\begin{aligned}\Delta &= \gamma^2 - 4\mu v > 0\\ \text{so } \gamma - 4\mu &> 0 \text{ then } 0 < \mu < \frac{\gamma}{4},\end{aligned}$$

the eigenvalues become

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\mu\gamma}\right),\\ \lambda_1 &= \frac{1}{2}\left(-\gamma - \sqrt{\gamma^2 - 4\mu\gamma}\right) < 0,\\ \lambda_2 &= \frac{1}{2}\left(-\gamma + \sqrt{\gamma^2 - 4\mu\gamma}\right) < 0.\end{aligned}$$

Then if $\gamma^2 = c_0$ and $0 < \mu < \frac{\gamma}{4}$; $\lambda_1, \lambda_2 < 0$, the equilibrium (a^*, h^*) is asymptotically stable node.
(A_2)

$$\Delta = \gamma^2 - 4\mu\gamma < 0 \text{ so } \gamma < 4\mu \text{ then } \mu > \frac{\gamma}{4}$$

the eigenvalues become

$$\lambda_{1,2} = \frac{1}{2}\left[-\gamma \pm i\sqrt{\gamma^2 - 4\mu\gamma}\right],$$

then if $\gamma^2 = c$ and $\mu > \frac{\gamma}{4} > 0$; the equilibrium (a^*, h^*) is asymptotically stabe focus.
The second case $\gamma^2 \neq c_0$
(B_1) $\quad (\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma)^2 - \mu\gamma \geq 0$ if $(\mu \leq \mu_1$ or $\mu \geq \mu_2)$

$$\begin{cases}\mu_1 = \frac{\left(\sqrt{c_0+\gamma^2} - \sqrt{2}\gamma\right)^2}{-c_0+\gamma^2}\mu_0\\ \mu_2 = \frac{\left(\sqrt{c_0+\gamma^2} + \sqrt{2}\gamma\right)^2}{-c_0+\gamma^2}\mu_0\end{cases},$$

(B_{11}) $\quad\gamma^2 > c_0$ and $\mu_1 < \mu_0 < \mu_2$
if $0 < \mu \leq \mu_1$ stable node,
if $\mu \geq \mu_2$ unstable node.
(B_{12}) $\quad\gamma^2 < c_0$ and $\mu_0 < 0 < \mu_1 < \mu_2$

$$\begin{cases} 0 < \mu < \mu_1 \text{ then } (\lambda_{1,2} \text{ réal négative}) \\ \mu \geq \mu_2 \end{cases},$$

(a^*, h^*) stable node
(B_2) $\quad(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma)^2 - 4\mu\gamma < 0$ if $\mu_1 < \mu < \mu_2$
(B_{21}) $\quad\gamma^2 < c_0$

$$\alpha(\mu) < 0,$$

(a^*, h^*) is asyptotically stable focus.
(B_{22}) $\quad\gamma^2 > c_0$
we have $\mu_0 = \frac{c_0+\gamma^2}{-c_0+\gamma^2}\gamma > 0$
(B_{221}) $0 < \mu_1 < \mu < \mu_0$: (a^*, h^*) stable focus $[\lambda_{1,2}]$ $(\alpha < 0)$,
(B_{222}) $\mu_0 < \mu < \mu_2$ (a^*, h^*) unstable focus $(\alpha > 0)$. ■

3.1.2 Global Existence of Solutions

Global Existence of Solutions of the Kinetic System (2.3) with $s = 1$

We claim that, for all large $\alpha, \beta > 0$, the function

$$\phi = \phi_{\alpha,\beta}(t) = \frac{a^\alpha}{h^\beta}, \tag{3.13}$$

satisfies

$$\sup_{t\in(0,T)} \phi(t) < \infty. \tag{3.14}$$

We have:

$$\begin{aligned}
\phi'(t) &= \frac{\alpha a^{\alpha-1}a_t h^{\beta} - \beta h^{\beta-1}h_t a^{\alpha}}{h^{2\beta}} \\
&= \frac{\alpha a^{\alpha-1}\left[\rho\rho_0 + c\rho\frac{a^2}{h} - \mu a\right]}{h^{\beta}} - \frac{\beta a\alpha\left[c'\rho' a^2 - vh\right]}{h^{\alpha+1}} \\
&= (-\alpha\mu + \beta v)\frac{a^{\alpha}}{h^{\beta}} + \left(\alpha c\rho\frac{a^{\alpha+1}}{h^{\beta+1}} - \beta c'\rho'\frac{a^{\alpha+2}}{h^{\beta+1}} + \alpha\rho\rho_0\frac{a^{\alpha-1}}{h^{\beta}}\right) \\
&= (-\alpha\mu + \beta v)\phi + \left(\alpha c\rho\frac{a^{\alpha+1}}{h^{\beta+1}} - \beta c'\rho'\frac{a^{\alpha+2}}{h^{\beta+1}} + \alpha\rho\rho_0\frac{a^{\alpha-1}}{h^{\beta}}\right).
\end{aligned}$$

We also have

$$\frac{a^{\alpha-1}}{h^{\beta}} = \left(\frac{a^{\alpha}}{h\beta}\right)^{(\alpha-1)/\alpha} h^{-\beta/\alpha} \leq C\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{(\alpha-1)/\alpha}. \tag{3.15}$$

Using Lemma (2) , (3.15) and Holder's inequality, we obtain

$$\begin{aligned}
\phi'(t) &\leq (-\alpha\mu + \beta v)\phi + C\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{\theta} + \alpha\rho\rho_0\frac{a^{\alpha-1}}{h^{\beta}} \\
&\leq (-\alpha\mu + \beta v)\phi + C\left(\phi^{\theta} + \phi^{(\alpha-1)/\alpha}\right),
\end{aligned} \tag{3.16}$$

for some $\theta \in (0,1)$.

Since $-\alpha\mu + \beta v < 0$, the function

$$f(Y) = (-\alpha\mu + \beta v)Y + C\left(Y^{\theta} + Y^{(\alpha-1)/\alpha}\right),$$

has a largest positive zero, say $Y = K$. Since, by 3.16, $\phi'(t) < 0$ whenever $\phi(t) > K$, we deduce easily that $\sup_{t\in(0.T)}\phi(t) \leq \max(\phi(0), K)$, hence 3.14. Since h is bounded below, it is clear that 3.14 remains true if we enlarge β.The claim is proved.

Global Existence of Solutions of the Equilibrium of the Kinetic System (2.3) with $s = 2$:

We claim that, for all large $\alpha, \beta > 0$, the function

$$\phi = \phi_{\alpha,\beta}(t) = \frac{a^{\alpha}}{h^{\beta}}, \tag{3.17}$$

satisfies

$$\sup_{t\in(0,T)}\phi(t) < \infty. \tag{3.18}$$

By (3.6)we have:

$$\begin{aligned}
\phi'(t) &= \frac{\alpha a^{\alpha-1}a_t h^{\beta}-\beta h^{\beta-1}h_t a^{\alpha}}{h^{2\beta}} \\
&= \frac{\alpha a^{\alpha-1}\left[c_0+\frac{a^2}{h^2}-\mu a\right]}{h^{\beta}}-\frac{\beta a^{\alpha}\left[a-\gamma h\right]}{h^{\beta+1}} \\
&= \frac{\alpha c_0 a^{\alpha-1}}{h^{\beta}}+\frac{\alpha a^{\alpha+1}}{h^{\beta+2}}-\frac{\alpha\mu a^{\alpha}}{h^{\beta}}-\frac{\beta a^{\alpha+1}}{h^{\beta+1}}+\frac{\beta\gamma a^{\alpha}}{h^{\beta}} \\
&= \left(-\alpha\mu+\beta\gamma\right)\frac{a^{\alpha}}{h^{\beta}}+\left(\alpha\frac{a^{\alpha+1}}{h^{\beta+2}}-\beta\frac{a^{\alpha+1}}{h^{\beta+1}}+\alpha c_0\frac{a^{\alpha-1}}{h^{\beta}}\right) \\
&= \left(-\alpha\mu+\beta\gamma\right)\phi+\left(\alpha\frac{a^{\alpha+1}}{h^{\beta+2}}-\beta\frac{a^{\alpha+1}}{h^{\beta+1}}+\alpha c_0\frac{a^{\alpha-1}}{h^{\beta}}\right).
\end{aligned}$$

Owing to , we also have

$$\frac{a^{\alpha-1}}{h^{\beta}}=\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{(\alpha-1)/\alpha}h^{-\beta/\alpha}\leq C\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{(\alpha-1)/\alpha}. \tag{3.19}$$

Using Lemma (2) ,(3.19) and Holder's inequality, we obtain

$$\begin{aligned}
\phi'(t) &\leq \left(-\alpha\mu+\beta\gamma\right)\phi+C\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{\theta}+\alpha\rho\rho_0\frac{a^{\alpha-1}}{h^{\beta}} \\
&\leq \left(-\alpha\mu+\beta\gamma\right)\phi+C\left(\phi^{\theta}+\phi^{(\alpha-1)/\alpha}\right),
\end{aligned} \tag{3.20}$$

for some $\theta\in(0,1)$.

Since $-\alpha\mu+\beta v<0$, the function

$$f(Y)=\left(-\alpha\mu+\beta\gamma\right)Y+C\left(Y^{\theta}+Y^{(\alpha-1)/\alpha}\right),$$

has a largest positive zero, say $Y=K$. Since, by (3.20), $\phi'(t)<0$ whenever $\phi(t)>K$, we deduce easily that $\sup_{t\in(0.T)}\phi(t)\leq\max(\phi(0),K)$, hence (3.18). Since h is bounded below, it is clear that (3.18) remains true if we enlarge β.The claim is proved.

3.1.3 The Hopf bifurcation

The Hopf bifurcation of the Kinetic System (2.3) with $s=2$:

When $\mu=\mu_0$ the Jacoobian matrix of the system (3.6) evaluated at (a^*,h^*) has a pair of conjugate pure imaginary eigenvalues. This indicates that system may undergo a Hopf bifurcation at $\mu=\mu_0$.

From now on,we will set μ to be the bifurcation parameter, and study the direction and stability of **the Hopf bifurcation**. We first transforme into normal form and translate the equilibrium (a^*, h^*) to the origin by the translation $x = a - a^*, y = h - h^*$. The following computation is tedious but straightforward:

We will develope f and g in Taylor's expansion.

We can calculate that

$$f_{ah}(a^*, h^*) = -4(\frac{c_0 + \gamma^2}{\mu})(\frac{\mu\gamma}{c_0 + \gamma^2})^3 = \frac{-4\mu^2\gamma^3}{(c_0 + \gamma^2)^2},$$

$$f_a(a^*, h^*) = 2(\frac{c_0 + \gamma^2}{\mu})(\frac{\mu\gamma}{c_0 + \gamma^2})^2 - \mu = \frac{-c_0 + \gamma^2}{c_0 + \gamma^2}\mu,$$

$$f_{aa}(a^*, h^*) = 2(\frac{\mu\gamma}{c_0 + \gamma^2})^2,$$

$$\begin{aligned} f_h(a^*, h^*) &= -2(\frac{c_0 + \gamma^2}{\mu})^2(\frac{\mu\gamma}{c_0 + \gamma^2})^3 \\ &= -2\frac{\gamma^3\mu}{c_0 + \gamma^2}, \end{aligned}$$

$$\begin{aligned} f_{hh}(a^*, h^*) &= 6(\frac{c_0 + \gamma^2}{\mu})^2(\frac{\mu\gamma}{c_0 + \gamma^2})^4 \\ &= 6\frac{\gamma^4\mu^2}{(c_0 + \gamma^2)^2}, \end{aligned}$$

$$f_{aaa}(a^*, h^*) = 0$$

$$f_{aah}(a^*, h^*) = -4\left(\frac{c_0 + \gamma^2}{\mu\gamma}\right)^3,$$

$$f_{ahh}(a^*, h^*) = 12\left(\frac{c_0 + \gamma^2}{\mu}\right)\left(\frac{\mu\gamma}{c_0 + \gamma^2}\right)^4,$$

$$\begin{aligned} f_{hhh}(a^*, h^*) &= -24\left(\frac{c_0 + \gamma^2}{\mu}\right)^2\left(\frac{\mu\gamma}{c_0 + \gamma^2}\right)^5 \\ &= 24\frac{\gamma^5\mu^3}{(c_0 + \gamma^2)^3}. \end{aligned}$$

Thus

$$
\begin{aligned}
f(a,h) &= f(a^*,h^*)+(a-a^*)(\frac{-c_0+\gamma^2}{c_0+\gamma^2})\mu-(h-h^*)2\frac{\gamma^3\mu}{c_0+\gamma^2} \\
&+\frac{1}{2!}\left[(a-a^*)^2*2\left(\frac{\mu\gamma}{c_0+\gamma^2}\right)^2-6(a-a^*)(h-h^*)\frac{\mu^2\gamma^3}{(c_0+\gamma^2)^2}+6(h-h^*)^2\frac{\gamma^4\mu^2}{(c_0+\gamma^2)^2}\right] \\
&+\frac{1}{3!}\left[(a-a^*)^3*0+3(a-a^*)^2(h-h^*)f_{aah}+3(a-a^*)(h-h^*)^2f_{ahh}+(h-h^*)^3f_{hhh}\right] \\
&+O(4),
\end{aligned}
$$

then

$$
\begin{aligned}
f(x,y) &= \left(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right)x-2\frac{\gamma^3\mu}{c_0+\gamma^2}y+\frac{(\mu\gamma)^2}{(c_0+\gamma^2)^2}x^2-4xy\gamma\frac{(\mu\gamma)^2}{(c_0+\gamma^2)^2}+y^2\gamma^2\frac{(\mu\gamma)^2}{(c_0+\gamma^2)^2} \\
&+\frac{1}{6}\left[3x^2y(-4\left(\frac{\mu\gamma}{c_0+\gamma^2}\right)^3)+3xy^2(12\frac{\mu^3\gamma^4}{(c_0+\gamma^2)^4})+y^3(24\frac{\mu^3\gamma^5}{(c_0+\gamma^2)^3})\right]+O(4).
\end{aligned}
$$

Now we get Taylor'sexpansion of f and g

$$
\begin{aligned}
f(x,y) &= \left(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right)x-2\frac{\gamma^3\mu}{c_0+\gamma^2}y+\frac{(\mu\gamma)^2}{(c_0+\gamma^2)^2}\left[x^2-4xy\gamma+3y^2\gamma^2\right] \\
&-2\frac{\mu^3\gamma^3}{(c_0+\gamma^2)^3}\left[x^2y-3xy^2\gamma-2y^3\gamma^2\right]+O(4),
\end{aligned}
$$

$$g(x,y)=x-\gamma y.$$

Now the system (3.6) becomes

$$
\frac{\partial}{\partial t}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu & -2\frac{\gamma^3\mu}{c_0+\gamma^2} \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} f_1(x,y,\mu) \\ g_1(x,y,\mu) \end{pmatrix}. \tag{3.21}
$$

Where

$$
\begin{aligned}
f_1(x,y,\mu) &= \frac{(\mu\gamma)^2}{(c_0+\gamma^2)^2}\left[x^2-4xy\gamma+3y^{2\gamma^2}\right]-2\frac{\mu^3\gamma^3}{(c_0+\gamma^2)^3}\left[x^2y-3xy^2\gamma-2y^3\gamma^2\right] && (3.22) \\
&+o(4) && (3.23) \\
g_1(x,y,\mu) &= 0,
\end{aligned}
$$

and $o(4)$represents the remaining terms with order greater than or equal to 4.

For $\mu=\mu_0$, we verify $\lambda_{1,2}=\pm iw_0$ and $w_0=w(\mu_0)=\gamma\sqrt{\frac{c_0+\gamma^2}{-c_0+\gamma^2}}>0$ and $\frac{d}{d\mu}\operatorname{Re}(\lambda_{1,2})\Big|_{\mu=\mu_0}=\frac{-c_0+\gamma^2}{2(c_0+\gamma^2)}>0$ and the eigenvector of $J(\mu_0)$ correspending to iw_0 is ξ, where $(J - iw_0 I)\,\xi = 0$

$$\begin{pmatrix} \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - iw_0 & \frac{-2\gamma^3}{c_0+\gamma^2}\mu \\ 1 & -\gamma - iw_0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{cases} \left(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - iw_0\right)x - \frac{2\gamma^3}{c_0+\gamma^2}\mu y = 0 \\ x + (-\gamma - iw_0)\,y = 0 \Rightarrow x = (\gamma + iw_0) \end{cases},$$

$$\Rightarrow \xi = (\gamma + iw_0, 1)^\top .$$

The eigenvector of $J(\mu_0)$ corresponding to iw_0 is $\xi = (\gamma + iw_0, 1)^\top$.

Setting

$$P = \begin{pmatrix} w_0 & \gamma \\ 0 & 1 \end{pmatrix},$$

to calculate P^{-1}we assume that

$$P^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\begin{pmatrix} w_0 & \gamma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$P^{-1} = \begin{pmatrix} \frac{1}{w_0} & \frac{-\gamma}{w_0} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P\begin{pmatrix} 0 & -w_0 \\ w_0 & 0 \end{pmatrix}P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1 \\ g_1 \end{pmatrix},$$

with transformation:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= P\begin{pmatrix} u \\ v \end{pmatrix} \\ P\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= P\begin{pmatrix} 0 & -w_0 \\ -w_0 & 0 \end{pmatrix}P^{-1}P\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \\ \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= P^{-1}P\begin{pmatrix} 0 & -w_0 \\ w_0 & 0 \end{pmatrix}P^{-1}P\begin{pmatrix} u \\ v \end{pmatrix} + P^{-1}\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \end{aligned}$$

(3.21) turns into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -w_0 \\ w_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}.$$

Where

$$\begin{aligned} f_2(u,v,\mu_0) &= \frac{1}{w_0}\left[\begin{array}{c} \frac{\gamma^4}{(-c_0+\gamma^2)^2}\left[(w_0u+\gamma v)^2 - 4\gamma(w_0u+8v)v + 3\gamma^2v^2\right] \\ -\frac{2\gamma}{(-c_0+\gamma^2)^3}\left[(w_0u+\gamma v)^2 v + 3\gamma(w_0u+\gamma v)v^2 + 2\gamma^2v^3\right] \end{array}\right] + O(4) \\ &= -\frac{\gamma^5}{(c_0-\gamma^2)^4}u\left[\left(c_0^2-\gamma^4\right)u + 2\left(c_0-\gamma^2\right)^2 v + 2\gamma^2\left(c_0+\gamma^2\right)uv + 2\gamma^2\left(c_0-\gamma^2\right)v^2\right] \\ &\quad +O(4), \\ g_2(u,v,\mu_0) &= 0. \end{aligned}$$

The stability of Hopf bifurcation of (3.6) at (a^*, h^*) is determined by the sign of the following

$$\begin{aligned} \rho &= \frac{1}{16}\left(f_{2uuu} + g_{2uuv} + f_{2uvv} + g_{2vvv}\right) \\ &\quad + \frac{1}{16w_0}\left[f_{2uv}\left(f_{2uu}+f_{2vv}\right) - g_{2uv}\left(g_{2uu}+g_{2vv}\right) - f_{2uu}g_{2vv} + f_{2vv}g_{2vv}\right], \end{aligned}$$

where all the partial derivatives are evaluated at the bifurcation point $(u,v,\mu) = (0,0,\mu_0)$.

Since $g_2(u,v,\mu_0) = 0$ we have that

$$\rho = \frac{1}{16}\left(f_{2uuu} + f_{2uvv}\right) + \frac{1}{16w_0} f_{2uv}\left(f_{2uu}+f_{2vv}\right).$$

We can calculate that

$$\begin{aligned} f_{2uuu} &= 0, \\ f_{2uvv} &= \frac{4\gamma^7}{(-c_0+\gamma^2)^3}, \\ f_{2uv} &= \frac{2\gamma^5\left(c_0+\gamma^2\right)}{(-c_0+\gamma^2)^3}, \\ f_{2uu} &= -\frac{2\gamma^5}{(-c_0+\gamma^2)^2}, \\ f_{2vv} &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \rho &= \frac{1}{16} f_{2uvv} + \frac{1}{16 w_0} f_{2uv} f_{2uu} \\ &= -\frac{c_0 \gamma^7}{4\left(-c_0 + \gamma^2\right)^4} < 0. \end{aligned}$$

Now from Poincaré-Andronov-Hopf Bifurcation Theorem, $\frac{d}{d\mu} \operatorname{Re}\left(\lambda_{1,2}\right)\Big|_{\mu=\mu_0} > 0$ and the above calculation of ρ, we summarize our results as the following theorem.

Theorem 13 *Suppose $c_0 < \gamma^2$, then system (3.6) experiences a Hopf bifurcation at $(a^*, h^*)$for $\mu = \mu_0$. The Hopf bifurcation is supercritical and the bifurcated limit cycle is stable.*

3.2 The PDE Model

3.2.1 Diffusion Driven Instability of the Equilibrium

In this subsection, we consider the influence of diffusion of the stability of equilibrium (a^*, h^*) of the system

$$\begin{cases} \frac{\partial a}{\partial t} = c_0 + \frac{a^2}{h^2} - \mu a + \frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = a - vh + r\frac{\partial^2 h}{\partial x^2} \end{cases}. \tag{3.24}$$

We first assume condition $(H1)$ so that (a^*, h^*) is stable equilibrium for system (3.6) and we consider the corresponding reaction diffusion system (3.24) with the following Neumann boundary conditions

$$\begin{cases} \frac{\partial a}{\partial x}(0,t) = \frac{\partial a}{\partial x}(\pi,t) = 0 \\ \frac{\partial h}{\partial x}(0,t) = \frac{\partial h}{\partial x}(\pi,t) = 0 \end{cases}, \tag{3.25}$$

in the Banach space $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$, where

$$\mathbb{H}^2([0,\pi]) = \left\{ w(.,t) \mid \frac{\partial^i w}{\partial x_i} \in \mathbb{L}^2([0,\pi]), i = 0,1,2 \right\}.$$

It is well known that the operator $a \to a_{xx}$ with the above boundary condition has eigenvalues and eigenfunctions as follows:

$$\lambda_0 = 0,\ \varphi_0 = \sqrt{\frac{1}{\pi}},\ \lambda_k = k^2,\ \varphi_k = \sqrt{\frac{2}{\pi}} \cos(kx),\ k = 1,2,3,...$$

From the standard linear operator theory, it is known that if all the eigenvalues of the operator L have negative real parts, then (a^*, h^*) is asymptotically stable, and if some eigenvalues have positive real parts, the (a^*, h^*) is unstable.

It is easy to see that (a^*, h^*) is a steady state solution of (3.24)-(3.25) and if (a^*, h^*) is lineary unstable in $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$ then, it is nonlineary unstable for (3.24)-(3.25).

Let $u_1 = a - a^*, u_2 = h - h^*$, we can write the linearized system of Eq (3.24) at (a^*, h^*) as

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := D\begin{pmatrix} u_{1xx} \\ u_{2xx} \end{pmatrix} + J\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.26}$$

where

$$D := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}, \quad J := \begin{pmatrix} \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu & -\frac{2\gamma^3}{c_0+\gamma^2}\mu \\ 1 & -\gamma \end{pmatrix}.$$

Let $(u_1, u_2) \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$.

We consider the following characteristic equation of the operator

$$L\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \delta \begin{matrix} \varphi \\ \psi \end{matrix}.$$

Let $(\varphi(x), \psi(x))$ be an eigenfunction of L corresponding to the eigenvalue δ, and let

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx,$$

where a_k and b_k are coefficients, we obtain that

$$-k^2 D \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx + J \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx = \delta \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx.$$

Hence

$$\left(J - k^2 D\right) \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \delta \begin{pmatrix} a_k \\ b_k \end{pmatrix} (k = 0, 1, 2, ...).$$

Denote

$$J_k = J - k^2 D = \begin{pmatrix} \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - k^2 & -\frac{2\gamma^3}{c_0+\gamma^2}\mu \\ 1 & -\gamma - rk^2 \end{pmatrix} (k = 0, 1, 2, ...), \tag{3.27}$$

It follows from this, that the eigenvalues of L are given by the eigenvalues of J_k for $k = 0, 1, 2, ...$ The characteristic equation of J_k is

$$\det(J_k - \delta I) = \begin{vmatrix} \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - k^2 - \delta & -\frac{2\gamma^3}{c_0+\gamma^2}\mu \\ 1 & -\gamma - rk^2 - \delta \end{vmatrix} \quad (3.28)$$

$$= \left(\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - k^2 - \delta\right)\left(-\gamma - k^2 - \delta\right) + \frac{2\gamma^3}{c_0+\gamma^2}\mu \quad (3.29)$$

$$= \delta^2 - \delta\left(-k^2(1+r) + \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma\right)$$
$$+ \left(rk^2\left(k^2 - \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right) + \gamma\left(k^2+\mu\right)\right) \quad (3.30)$$

$$= 0, \quad (3.31)$$

we are searching conditions on the solutions of(3.28) such that $\operatorname{Re}(\delta(k)) < 0$ and $\delta(k)$ satisfies the equation

$$\delta^2 - \delta \operatorname{tr}(k) + \det(k) = 0, \ k = 0, 1, 2, ... \quad (3.32)$$

where

$$\operatorname{tr}(k) = -k^2(1+r) + \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu - \gamma < 0,$$

and

$$\det(k) = rk^2\left(k^2 - \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right) + \gamma\left(k^2+\mu\right).$$

Under condition $(H1)$ we have $tr(k) < 0$ for all $k = 0, 1, 2, ...$ and $\det(0) > 0$.

$$\det(k) > 0 \text{ so } rk^2\left(k^2 - \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right) > -\gamma\left(k^2+\mu\right),$$

$$r > \frac{-\gamma\left(k^2+\mu\right)}{k^2\left(k^2 - \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu\right)}, \qquad k \neq 0.$$

Then $\det(k) \geq \det(0) > 0$ for $k = 1, 2, ...$ if $\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu \leq 1$ or $m < \sqrt{\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu} \leq m+1$ and $r < r_m$, where $m \in \mathbb{N}^+$ and

$$r_m = \min_{1 \leq k \leq m} \frac{\gamma\left(k^2+\mu\right)\left(c_0+\gamma^2\right)}{k^2\left[\left(-c_0+\gamma^2\right)\mu - k^2\left(c_0+\gamma^2\right)\right]}.$$

It follows that (3.32) can only have solutions with $\operatorname{Re}(\delta(k)) < 0$ for all $k = 0, 1, 2, ...$, i.e. (a^*, h^*) is linearly asymptotically stable for (3.24), the linear stability implies the non linear stability of the equilibrium. It follows that (a^*, h^*) is nonlinearly stable for (3.24). Hence, there is no Turning pattern foe system (3.24) in this case. If $m < \sqrt{\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu} \leq$

$m+1$ and $r > r_m$, then there exists at least one negative in $\det(1), ..., \det(m)$, and so (a^*, h^*) is unstable for (3.24). Hence, Turing instability takes place in this case.

Theorem 14 *The following result is a summary of the obove discussion:*
Assume condition $(H1)$ and let $r_m = \min\limits_{1 \le k \le m} \frac{\gamma(k^2+\mu)(c_0+\gamma^2)}{k^2[(-c_0+\gamma^2)\mu - k^2(c_0+\gamma^2)]}$, *then (a^*, h^*) is unstable equilibrium for (3.24) if*
$(H3)$ $\qquad m < \sqrt{\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu} \le m+1$ $m \in \mathbb{N}^+$ *and* $r > r_m$;
and is a stable equilibrium for (3.24) if either
$(H4)$ $\qquad \frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu \le 1$,
or
$(H5)$ $\qquad m < \sqrt{\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu} \le m+1$ *and* $r < r_m$.

3.2.2 Diffusion-Driven Instability of the Limit Cycle

In this subsection, we investigate the stability of limit cycle as derived in the last theorem under spatially inhomogeneous perturbations. Recall that the limit cycle is a small amplitude periodic solution bifurcated from (a^*, h^*) for μ sufficiently close to μ_0. For the rest of the section we assume condition $(H2)$ so that the limit cycle is stable under spatially homogeneous perturbation.Let $u^1 = a - a^*, u^2 = h - h^*.$, $\mu = \mu_0$ and $U = (u_1, u_2)^\top$, then system (3.24)becomes

$$\begin{cases} U_t = \left[J(\mu_0) + D \begin{pmatrix} \delta_{xx} & 0 \\ 0 & \delta_{xx} \end{pmatrix} \right] U + F(U, \mu_0) \\ U_x(0,t) = U_x(\pi, t) = (0,0)^\top \end{cases}, \tag{3.33}$$

where

$$J(\mu_0) = \begin{pmatrix} \gamma & -\frac{2\gamma^4}{-c_0+\gamma^2} \\ 1 & -\gamma \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix},$$
$$F(U, \mu_0) = (f_1(u_1, u_2, \mu_0), g_1(u_1, u_2, \mu_0))^\top,$$

and f_1 and g_1 are defined in (3.22) .

We write $F(U, \; \mu_0)$ in the form

$$F(U, \; \mu_0) = \frac{1}{2} Q(U,U) + \frac{1}{6}(U,U,U) + O\left(|U|^4\right),$$

where Q and C are in the following form:

$$\begin{aligned} Q(U,U) &= (Q_1(U,U), Q_2(U,U,))^\top, \\ C(U, \; U, \; U) &= (C_1(U,U,U), C_2(U,U,U))^\top, \end{aligned}$$

with

$$
\begin{aligned}
Q_1(U,V) &= f_{1uu}u_1v_1 + f_{1uv}u_1v_2 + f_{1vu}u_2v_1 + f_{1vv}u_2v_2 \\
&= \frac{\gamma^4}{(-c_0+\gamma^2)^2}\left(2u_1v_1 - 4\gamma u_1v_2 - 4\gamma u_2v_1 + 6\gamma^2 u_2v_2\right), \\
Q_2(U,V) &= g_{uu}u_1v_1 + g_{2uv}u_1v_2 + g_{1vu}u_2v_1 + g_{1vv}u_2v_2 \\
&= 0, \\
C_1(U,V,W) &= f_{1uuu}u_1v_1w_1 + f_{1uuv}u_1v_1w_2 + f_{1uvu}u_1v_2w_1 + f_{1uvv}u_1v_2w_2 \\
&\quad + f_{1vuu}u_2v_1w_1 + f_{1vuv}u_2v_1w_2 + f_{1vvu}u_2v_2w_1 + f_{1vvv}u_2v_2w_2 \\
&= -\frac{2\gamma^6}{(-c_0+\gamma^2)^3}[2u_1v_1w_2 + 2u_1v_2w_1 - 6\gamma u_1v_2w_2 \\
&\quad + 2u_2v_1w_1 - 6\gamma u_2v_1w_2 - 6\gamma u_2v_2w_1 + 12\gamma^2 u_2v_2w_2], \\
C_2(U,V,W) &= g_{1uuu}u_1v_1w_1 + g_{1uuv}u_1v_1w_2 + g_{1uvu}u_1v_2w_1 + g_{1uvv}u_1v_2w_2 \\
&\quad + g_{1vuu}u_2v_1w_1 + g_{1vuv}u_2v_1w_2 + g_{1vvu}u_2v_2w_1 + g_{1vvv}u_2v_2w_2 \\
&= 0,
\end{aligned}
$$

for any $U = (u_1, v_2)^\top$, $V = (v_1, v_2)^\top$, $W = (w, w)^\top$ and $U,\ V,\ W \in \mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$.

The linear operator L defined in (3.26) for $\mu = \mu_0$ is

$$
LU = \left[J(\mu_0) + D \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix} \right] U,
$$

for $U \in \mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$.

Let L^* be the adjoint operator of L defined in $\mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$, then

$$
L^*U = \left[J^*(\mu_0) + D \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix} \right],
$$

with

$$
J^*(\mu_0) = \begin{pmatrix} \gamma & 1 \\ -\frac{2\gamma^4}{-c_0+\gamma^2} & -\gamma \end{pmatrix}.
$$

Obviously, $\langle L^*U,\ V\rangle = \langle U,\ LV\rangle$ for any $U,\ V \in \mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$ and the inner product in $\mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$ is defined as $\langle U,\ V\rangle \frac{1}{\pi} \times \int_0^\pi \bar{U}^\top V dx$ for any U, $V \in \mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$.

The linearized system of(3.33) at the equilibrium $(0,0)$ is

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = L \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.34}$$

which are subject to the boundary conditions:

$$U_x(0,\ t) = U_x(\pi,\ t) = (0,0)^\top. \tag{3.35}$$

Let $U = (u_1, u_2)^\top \in \mathbb{H}^2([0,\ \pi]) \times \mathbb{H}^2([0,\ \pi])$ be a solution of (**??**). Since (3.34) is linear, we can formally write the solution as

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t + ikx}, \tag{3.36}$$

where $\lambda \in \mathbb{C}$ is the temporal spectrum, k is wave number (spatial spectrum) and a_k, h_k are real number for $k = 0,\ 1,\ 2....$ Plugging (u_1, u_2) into (3.34), we have

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda \begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t + ikx} &= L \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t + ikx} \\ &= \sum_{k=0}^{\infty} L_k \begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t + ikx}. \end{aligned}$$

Collecting the like terms about k we have

$$(\lambda I - L_k) \begin{pmatrix} a_k \\ h_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ k = 0,\ 1,\ 2, ..., \tag{3.37}$$

where

$$L_k = \begin{pmatrix} -k^2 + \gamma & -\frac{2\gamma^4}{-c_0 + \gamma^2} \\ 1 & -rk^2 - \gamma \end{pmatrix}.$$

For some k, Eq.(3.37) has a nonzero solution $(a_k, h_k)^\top$ if and only if the following dispersion relation is satisfied:

$$\det(\lambda I - L_k) = 0.$$

We are searching conditions such that $\operatorname{Re}(\lambda(k)) > 0$ and $\lambda(k)$ satisfies equation

$$\lambda^2 - T_k \lambda + D_k = 0,\ k = 0,\ 1,\ 2, ..., \tag{3.38}$$

where $T_k = -(1+r)k^2$ and $D_k = rk^2(k^2 - \gamma) + k^2\gamma + w_0^2$.

Under condition $c_0 < \gamma^2$ and $\mu = \mu_0$ we have $T_0 = 0$, $D_0 = w_0^2 > 0$, $T_k < 0$ for $k = 0$, 1, 2, ..., it follows that for $k = 0$, L has eigenvalues with zero real parts. We then proceed to the center manifold reduction.

First of all, if $0 < \gamma \leq 1$, then $D_k \geq \gamma + w_0^2 > 0$ for $k = 1, 2, ...$ Ferthermore, for $m < \sqrt{\gamma} \leq m+1$ and $r < \bar{r}$, where $m \in \mathbb{N}^+$ and $\bar{r} = \min_{1 \leq k \leq m} \frac{k^2\gamma + w_0^2}{k^2(\gamma - k^2)}$, we have $D_k > 0$ for $k = 1, 2, ...$; and for $m < \sqrt{\gamma} \leq m+1$ and $r > \bar{r}$, there exists at least one negative in $D_1, ..., D_m$.

Letting $L_q = iw_0 q$ and $L^* q^* = -iw_0 q^*$, we have $q = (iw_0 + \gamma, 1)^\top$ and $q^* = \frac{1}{2w_0}(i, w_0 - i\gamma)^\top$, respectively. It is easy to see that $\langle q^*, q \rangle = 1$ and $\langle q^*, q \rangle = 0$.

We write

$$U = zq + \bar{z}\bar{q} + w, \; z = \langle q^*, U \rangle, \; w = (w_1, w_2)^\top,$$

and

$$\begin{cases} u_1 = (iw_0 + \gamma) z + (-iw_0 + \gamma) \bar{z} + w_1 \\ u_2 = z + \bar{z} + w_2 \end{cases}.$$

The system (3.33) in $(z, \; w)$ coordinates becomes

$$\begin{cases} \dot{z} = iw_0 z + \langle q^*, \; \bar{f} \rangle \\ \dot{w} = Lw + H(z, \; \bar{z}, \; w) \end{cases}, \tag{3.39}$$

with

$$\begin{aligned} \tilde{f} &= F(zq + \bar{z}\bar{q} + w, \mu_0) \\ H(z, \; \bar{z}, \; w) &= \bar{f} - \langle q^*, \bar{f} \rangle q - \langle \bar{q}^*, \bar{f} \rangle \bar{q}, \end{aligned}$$

and

$$\begin{aligned} \tilde{f} &= \frac{1}{2}Q(zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w) + \frac{1}{6}C(zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w) \\ &\quad + O\left(|zq + \bar{z}\bar{q} + w|^4\right) \\ &= \frac{1}{2}[Q(zq, zz) + Q(\bar{z}\bar{q}, \bar{z}\bar{q}) + Q(w, w) + 2Q(zq, \bar{z}\bar{q}) + 2Q(zq, w) + 2Q(\bar{z}\bar{q}, w)] \\ &\quad + \frac{1}{6}[C(zq, zq, zq) + C(\bar{z}\bar{q}, \bar{z}\bar{q}, \bar{z}\bar{q}) + C(w, w, w) + 3C(zq, zq, \bar{z}\bar{q}) + 3C(zq, zq, w) \\ &\quad + 3C(\bar{z}\bar{q}, \bar{z}\bar{q}, zq) + 3C(\bar{z}\bar{q}, \bar{z}\bar{q}, w) + 3C(w, w, zq) + 3C(w, w, \bar{z}\bar{q}) + 6C(zq, \bar{z}\bar{q}, w)] \\ &\quad + O\left(|zq + \bar{z}\bar{q} + w|^4\right) \\ &= \frac{1}{2}Q(q, q) z^2 + Q(q, \bar{q}) z\bar{z} + \frac{1}{2}Q(\bar{q}, \bar{q}) \bar{z}^2 + O\left(|z|^3, |z| \cdot |w|, |w|^2\right), \end{aligned}$$

$$\begin{aligned}
\left\langle q^*, \tilde{f} \right\rangle &= \frac{1}{2} \left\langle q^*, Q\left(q,q\right)\right\rangle z^2 + \left\langle q^*, Q\left(q,\bar{q}\right)\right\rangle z\bar{z} \\
&\quad + \frac{1}{2} \left\langle q^*, Q\left(\bar{q},\bar{q}\right)\right\rangle \bar{z}^2 + O\left(\left|z\right|^3, \left|z\right| \cdot \left|w\right|, \left|w\right|^2\right) \\
\left\langle q^*, \tilde{f} \right\rangle &= \frac{1}{2} \left\langle \bar{q}^*, Q\left(q,q\right)\right\rangle z^2 + \left\langle \bar{q}^*, Q\left(q,\bar{q}\right)\right\rangle z\bar{z} + \frac{1}{2} \left\langle \bar{q}^*, Q\left(\bar{q},\bar{q}\right)\right\rangle \bar{z}^2 \\
&\quad + O\left(\left|z\right|^3, \left|z\right| \cdot \left|w\right|, \left|w\right|^2\right).
\end{aligned}$$

so

$$H\left(z,\bar{z},w\right) = \frac{1}{2} z^2 H_{20} + z\bar{z} H_{11} + \frac{1}{2} \bar{z}^2 H_{02} + O\left(\left|z\right|^3, \left|z\right| \cdot \left|w\right|, \left|w\right|^2\right),$$

where

$$\begin{cases}
H_{20} = Q\left(q,q\right) - \left\langle q^*, Q\left(q,q\right)\right\rangle q - \left\langle \bar{q}^*, Q\left(q,q\right)\right\rangle \bar{q}, \\
H_{11} = Q\left(q,\bar{q}\right) - \left\langle q^*, Q\left(q,\bar{q}\right)\right\rangle q - \left\langle \bar{q}^*, Q\left(q,\bar{q}\right)\right\rangle \bar{q}, \\
H_{02} = Q\left(\bar{q},\bar{q}\right) - \left\langle q^*, Q\left(\bar{q},\bar{q}\right)\right\rangle q - \left\langle \bar{q}^*, Q\left(\bar{q},\bar{q}\right)\right\rangle \bar{q}.
\end{cases}$$

Furthermore,

$$\begin{aligned}
H_{20} &= \left(Q_1\left(q,q\right), Q_2\left(q,q\right)\right)^\top - \frac{1}{2w_0}[-iQ_1\left(q,q\right) \\
&\quad + \left(w_0 + i\gamma\right) Q_2\left(q,q\right)] \left(iw_0 + \gamma w_0 + \gamma, 1\right)^\top \\
&\quad - \frac{1}{2w_0}[iQ_1\left(q,q\right) + \left(w_0 - i\gamma\right) Q_2\left(q,q\right)] \times \left(-iw_0 + \gamma, 1\right)^\top \\
&= \left(0,0\right)^\top.
\end{aligned}$$

Similarly, we have

$$H_{11} = H_{02} = \left(0,0\right)^\top.$$

Therefore,

$$H\left(z,\bar{z},w\right) = O\left(\left|z\right|^3, \left|z\right| \cdot \left|w\right|, \left|w\right|^2\right).$$

It follows that system (3.39) pessesses a center manifold, which we write

$$w = \frac{1}{2} z^2 w_{20} + z\bar{z} w_{11} + \frac{1}{2} \bar{z}^2 w_{02} + O\left(\left|z\right|^3\right),$$

then from $Lw + H\left(z,\bar{z},w\right) = \dot{w} = \frac{\partial w}{\partial \bar{z}} \dot{z} + \frac{\partial w}{\partial z} \dot{z}$, we have

$$\begin{cases}
w_{20} = \left[2iw_0 - L\right]^{-1} H_{20} = \left[2iw_0 - J\left(\mu_0\right)\right]^{-1} H_{20} = \left(0,0\right)^\top, \\
w_{11} = -L^{-1} H_{11} = -J\left(\mu_0\right) H_{11} = \left(0,0\right)^\top, \\
w_{02} = \left[-2iw_0 - L\right]^{-1} H_{02} \\
= \left[-2iw_0 - J\left(\mu_0\right)\right]^{-1} H_{02} = \left(0,0\right)^\top,
\end{cases}$$

then $w = O\left(|z|^3\right)$.

Now the reaction-diffusion system restricted to the center manifold is

$$\dot{z} = iw_0 z + \left\langle q^*, \tilde{f} \right\rangle = iw_0 z + \sum_{2 \leq i+j \leq 3} \frac{g_{ij}}{i!j!} z^i \bar{z}^j + O\left(\left|z^4\right|\right) \tag{3.40}$$

where

$$\begin{aligned} g_{20} &= \langle q^*, Q(q,q) \rangle, \\ g_{11} &= \langle q^*, Q(q,\bar{q}) \rangle, \\ g_{02} &= \langle q^*, Q(\bar{q},\bar{q}) \rangle, \\ g_{21} &= 2\langle q^*, Q(w_{11},q) \rangle + \langle q^*, Q(w_{20},\bar{q}) \rangle + \langle q^*, C(q,q,\bar{q}) \rangle \\ &= \langle q^*, C(q,q,\bar{q}) \rangle, \end{aligned}$$

the dynamics of (3.39) is determined by that of (3.40), we write the Poincaré normal form of (3.33) in the following form

$$\dot{z} = (\alpha(\mu) + iw(\mu)) z + z \sum_{j=1}^{M} c_j(\mu) (z\bar{z})^i, \tag{3.41}$$

where z is a complex variable, $M \geq 1$,and $c_j(\mu)$ are complex-valued coefficients. Then we have

$$\begin{aligned} c_1(\mu) &= \frac{g_{20} g_{11} [3\alpha(\mu) + iw(\mu)]}{2[\alpha^2(\mu) + w^2(\mu)]} + \frac{|g_{11}|^2}{\alpha(\mu) + iw(\mu)} \\ &\quad + \frac{|g_{02}|^2}{2[\alpha(\mu) + 3iw(\mu)]} + \frac{g_{21}}{2}, \end{aligned}$$

and $\operatorname{Re}(c_1(\mu_0)) = \operatorname{Re}\left[\frac{g_{20}g_{11}}{2w_0} i + \frac{g_{21}}{2}\right]$ since $\alpha(\mu_0) = 0$ and $w(\mu_0) = w_0 > 0$.

Since

$$\begin{aligned} g_{20} &= \langle q^*, Q(q,q) \rangle = \frac{w_0^4 (-2\gamma + iw_0)}{(c_0 + \gamma^2)^2}, \\ g_{11} &= \langle q^*, Q(q,\bar{q}) \rangle = \frac{w_0^5 i}{(c_0 + \gamma^2)^2} \\ g_{21} &= \langle q^*, C(q,q,\bar{q}) \rangle = \frac{w_0^6 (\gamma + iw_0)}{(c_0 + \gamma^2)^3}, \end{aligned}$$

we then have $\operatorname{Re}[c_1(\mu_0)] = -\frac{c_0 \gamma^7}{(c_0+\gamma^2)^4} < 0$, Therefore, the supercritical Hopf bifurcation occurs at $\mu = \mu_0$. By setting $\bar{r} = \min_{1\leq k\leq m} \frac{k^2\gamma + w_0^2}{k^2(\gamma - k^2)}$, we summarize the above conclusions as in the following theorem:

Theorem 15 *Assume condition* $(H2)$*so the spatially homogeneous periodic of* (γ) *bifurcated from the equilibrium is stable. Then the spatially homogeneous periodic solution for (3.24) is unstable if*
(H6) $m < \sqrt{\gamma} \leq m+1, m \in \mathbb{N}^+$ *and* $r > \bar{r}$; *and is stable if either*
(H7) $0 < \gamma \leq 1$,
or
(H8) $m < \sqrt{\gamma} \leq m+1, m \in \mathbb{N}^+$ *and* $0 < r < \bar{r}$.

Remark 6 *The previous theorem states that system (3.24) experiences a Turing instability for* $c_0 < \gamma^2, \mu > \mu_0, \gamma > 1$ *and* $r > \bar{r}$. *which accounts for the spot and stripe patterns in system.*

3.2.3 Global Existence of Solutions

We claim that, for all large $\alpha, \beta > 0$, the function

$$\phi = \phi_{\alpha,\beta}(t) = \int_{\Omega} \frac{a^\alpha}{h^\beta} dx, \tag{3.42}$$

satisfies

$$\sup_{t\in(0,T)} \phi(t) < \infty.$$

By (3.24)we have:

$$\begin{aligned}
\phi' &= \int_\Omega \left(\frac{\alpha a^{\alpha-1} a_t h^\beta - \beta h^{\beta-1} h_t a^\alpha}{h^{2\beta}} \right) dx \\
&= \int_\Omega \left(\alpha \frac{a^{\alpha-1} a_t}{h^\beta} - \beta \frac{a^\alpha h_t}{h^{\beta+1}} \right) dx \\
&= \alpha \int_\Omega \frac{a^{\alpha-1}}{h^\beta} \left(c_0 + \frac{a^2}{h^2} - \mu a + \Delta a \right) dx - \beta \int_\Omega \frac{a^\alpha}{h^{\beta+1}} \left(a - \gamma h + r \Delta h \right) dx \\
&= \alpha \int_\Omega \left(\frac{a^{\alpha-1}}{h^\beta} \Delta a + \frac{a^{\alpha+1}}{h^{\beta+2}} - \mu \frac{a^\alpha}{h^\beta} + c_0 \frac{a^{\alpha-1}}{h^\beta} \right) dx - \beta \int_\Omega \left(\frac{a^\alpha}{h^{\beta+1}} r \Delta h + \frac{a^{\alpha+1}}{h^{\beta+1}} - \gamma \frac{a^\alpha}{h^\beta} \right) dx \\
&= \left(-\alpha\mu + \beta\gamma \right) \phi + \int_\Omega \left(\alpha c_0 \frac{a^{\alpha-1}}{h^\beta} + \alpha c_0 \frac{a^{\alpha-1}}{h^\beta} - \beta \frac{a^{\alpha+1}}{h^{\beta+1}} \right) dx \\
&\quad + \int_\Omega \left(\alpha \frac{a^{\alpha-1}}{h^\beta} \Delta a - \beta r \frac{a^\alpha}{h^{\beta+1}} \Delta h \right) dx,
\end{aligned}$$

using Green's formula, we deduce that

$$\begin{aligned}
\Phi'(t) &= \left(-\alpha\mu + \beta\gamma \right) \phi + \int_\Omega \left(\alpha c_0 \frac{a^{\alpha-1}}{h^\beta} + \alpha c_0 \frac{a^{\alpha-1}}{h^\beta} - \beta \frac{a^{\alpha+1}}{h^{\beta+1}} \right) dx \\
&\quad + \int \left(-\alpha(\alpha-1) \frac{a^{\alpha-2}}{h^\beta} |\nabla a|^2 - \beta(\beta+1) r \frac{a^\alpha}{h^{b+2}} |\nabla h|^2 + \alpha\beta(1+r) \frac{a^{\alpha-1}}{h^{\beta+1}} \nabla a \cdot \nabla h \right) dx.
\end{aligned}$$

the last integrand can be rewritten as

$$Q = \left[\left(-\alpha(\alpha-1) h^2 |\nabla a|^2 - \beta(\beta+1) r a^2 |\nabla h|^2 + \alpha\beta(1+r)(h \nabla a \cdot a \nabla h) \right) \right] \frac{a^{\alpha-2}}{h^{\beta+2}},$$

$$\begin{vmatrix} -\alpha(\alpha-1) & \frac{\alpha\beta(1+r)}{2} \\ \frac{\alpha\beta(1+r)}{2} & -\beta(\beta+1) r \end{vmatrix} = \alpha\beta(\alpha-1)(\beta+1) - \frac{(\alpha\beta)^2 (1+r)^2}{4} \leq 0,$$

which guarantees that the discriminant$(1+r)^2 \alpha^2 \beta^2 - 4\alpha\beta r (\alpha-1)(\beta+1)$ of the quadratic form Q is nonpositive.

Consequently we have $Q \leq 0$, provided we assume

$$\frac{\alpha\beta}{(\alpha-1)(\beta+1)} \leq \frac{4r}{(1+r)^2}, \tag{3.43}$$

we also have

$$\frac{a^{\alpha-1}}{h^{\beta}} = \left(\frac{a^{\alpha}}{h^{\beta}}\right)^{(\alpha-1)/\alpha} h^{-\beta/\alpha} \leq C\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{(\alpha-1)/\alpha}. \tag{3.44}$$

Using Lemma (9) , (3.44) and Holder's inequality, we obtain

$$\begin{aligned} \phi'(t) &\leq (-\alpha\mu+\beta\gamma)\phi + C\int_{\Omega}\left(\frac{a^{\alpha}}{h^{\beta}}\right)^{\theta} dx + \alpha c_0 \int_{\Omega} \frac{a^{\alpha-1}}{h^{\beta}} dx \\ &\leq (-\alpha\mu+\beta\gamma)\phi + C\left(\phi + \phi^{(\alpha-1)/\alpha}\right), \end{aligned} \tag{3.45}$$

for some $\theta \in (0,1)$.

Now assume $\alpha \geq 2\max(1, \mu/\gamma)$ and $\beta \leq 2r/(1+r)^2 \leq 1$. Then (3.43) is satisfied and,since $-\alpha\mu + \beta v < 0$, the function

$$f(Y) = (-\alpha\mu+\beta\gamma)Y + C\left(Y^{\theta} + Y^{(\alpha-1)/\alpha}\right),$$

has a largest positive zero, say $Y = K$. Since, by (3.45), $\phi'(t) < 0$ whenever $\phi(t) > K$, we deduce easily that $\sup_{t\in(0.T)}\phi(t) \leq \max(\phi(0), K)$, hence (3.18). Since h is bounded below, it is clear that (3.18) remains true if we enlarge β.The claim is proved.

For the Gierer-Meinhardt model(3.1), it is known that under the condition (3.2), if $\mu < \mu_0$, where μ_0 is a (critical) value defined in (3.3), then the positive equilibrium E^* is asymptotically stable. when μ passes through the critical value μ_0, a Hopf bifurcation occurs.

For the Gierer-Meinhardt model(3.6), under the condition $\mu < \mu_0$, where μ_0 is a (critical) value defined in (3.12), then the positive equilibrium E^* is asymptotically stable. When $\gamma^2 > c_0$, then the equilibrium (a^*, h^*) changes from a stable focus to an unstable one as μ increases from μ_1 via μ_0 to μ_2; while at $\mu = \mu_0$ the Jacobian matrix evaluated at (a^*, h^*) has a pair of conjugate pure imaginary eigenvalues a Hopf bifurcation occurs and the Hopf bifurcation is supercritical and the bifurcated limit cycle is stable.

The positive equilibrium E^* and the periodic solution are spatially homogeneous solutions of the reaction-diffusion Gierer-Meinhardt model (3.24). For the homogeneous equilibrium solution E^*, by using Turing's technique, it was shown that diffusion-driven instability occurs when $m < \sqrt{\frac{-c_0+\gamma^2}{c_0+\gamma^2}\mu} \leq m+1$ and $r > r_m$. Then, if an appropriate small perturbation is added to the equilibrium solution E^*, there will appear spatial inhomogeneity with certain periodic spatial structure in the solution of system (3.24). For the homogeneous periodic solution, system (3.24) experiences a Turing instability for $c_0 < \gamma^2, \mu > \mu_0, \gamma > 1$ and $r > \bar{r}$. which accounts for the spot and stripe patterns in system.

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