An investigation of the Lengyel-Epstein R-D system

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CONTENTS

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Dedication

- We devote this modest work to all, who from near and far have given us their moral and physical support for the realization of this work.
- To our parents for their support during all our studies and who do not cease to lavish us with their love.
- To our brothers and sisters will find here the expression of our respect and love.
- To all of our colleagues in the Mathematics & Informatics department.
- To anyone we know and with whom we exchange feelings of friendship, love and respect.

Acknowledgment

In the name of Allah, the Most Merciful, the Most Beneficial, and all priase is due to Allah,

The work presented in this thesis has been carried out at the University of Cheikh Larbi TEBESSI, Institute for the exact sciences & the sciences of nature and life.

Departement of Mathematics & Informatics.

First of all, we would like to thank all our teachers for having given us the necessary knowledge in recent years.

We would also like to thank Mr. Dr Salem ABDELMALEK, our assets framed and supported during these five months to accomplish this project.

We would also like to thank Mr and M, for kindness with their students we wish to associate these thanks to our colleagues, Our friends from the university and all of us who are in our hearts, especially our dear parents, for his sacrifices, kindness and availability.

Abstract

The aim of the work is to study the stability **(Local and global)** and **the bifurcation** of the Lengyel-Epstein reaction deffusion system, which results from a certain chemical reaction. Our technique is based on **stability theory.** We used the Jacobian matrix to study the local stability and Lyapanuv functionals to study the global stability. As for the bifurcation, we take advantage of the Andronov theorem.

Abstract in Arabe

Introduction

Much of this work is devoted to the study of the Lengyl-Epstein system's **stability**. Lengyl-Epstein is a chemical reaction diffusion system[9]. To address this work, we must address the theories of stability (system stability) as well as the definition of reaction systems.

In Chapter 1 (Stability theory) This course deals with theories of stability of systems (systems with dimension 2.2), The study of systems of these dimensions to facilitate the study of the system of the Lengyel-Epstein system, which is also dimensions 2.2.[4]

The application of the theories of stability to the system Lotka-Volterra and study of its local and global stability in preparation for the study of our system.

Local stability is when the property of stability is related to the range or area of a particular sport outside which is exiled.

In this study, we rely mainly on the eigenvalues of the Jacobian matrix attached to the system.

Global stability The stability property is not related to a specific mathematical field.

In this study, we rely mainly on Lyapanuv's theorem.

In Chapter 2: (Reaction deffusionsystem) in this chapter we present the Definition of Reaction diffusion system with examples and how it occurs.

In Chapter 3: (An investigation of the Lengyel-Epstein R-D System) this is th important chpter, now we will study the stability of the Lengyel-Epstein system, stability and bifurcation.

First, we will work in the **ODE** model, the system whitout the laplacian operator, and study the local, global stability[10], then the bifurcation.[6]

Now we will work in the **PDE** model, the system whit the laplacian operator, and study the local, global stability[10], then the bifurcation.[6]

Preliminaries

This chapter is devoted to a reminder of useful preliminaries for the rest of the dissertation. We introduce some basic notation and notion.

0.1 Notation

- The set of the real numbers is denoted by $\mathbb{R}$.
- The set of the complex numbers is denoted by $\mathbb{C}$.
- Matrices are denoted by capital characters, e.g.A, B, C... etc.
- The trace of real and complex matrices is denoted by $\mathrm{tr}(A)$.
- The determinant of real and complex martices is denoted by $\det(A)$.
- The transpose of matrix A is denoted by $A^\top$.
- The norm is denoted by $\|.\|$.
- The Laplacian operator is denoted by Δx.
- The gradient is denoted by ∇.
- The real part of a complex number is denoted by $\mathrm{Re}(\lambda)$.
- The partial derivative of function $f(u, v)$ w.r.t. u is denoted by f_u or $\frac{\partial f}{\partial u}$.
- The derivative of function $f(x, t)$ w.r.t. t is denoted by $\dot{f}(t)$.
- The standard Sobolev space is denoted by $H^2[(0, \pi)]$.
- The conjugate of the operater L is denoted by L^*.

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0.2 General Notions

0.2.1 Green's formula

We recall now some of green's formulas which generalize to the multidimensional case the formula of integration by parts of dimension one. They are stated in the following:

Theorem 1 *[1]we suppose that Ω is an open domain of boundry $\partial\Omega$ continue with part. Then, if u and v are function of $H^1(\Omega)$, we have*

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int_{\Omega} u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uv\eta_i d\sigma \qquad , 1 \leqslant i \leqslant n, \tag{1}$$

we design by η_i the ith consinus director of normal η in $\partial\Omega$ directed towards the outside of Ω and we write $\eta_i = (\vec{\eta}.\vec{e}_i)$.

$d\sigma$ the superficial measure on $\partial\Omega$.

Corollary 2 *For all function u of $H^1(\Omega)$ and all function v of $H^1(\Omega)$, we have the Green formula*

$$\int_{\Omega} (\Delta u) v = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma - \int_{\Omega} \nabla u \nabla v. \tag{2}$$

Proof. Let given a consequence of Theorem precedent.

On suppose $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, the laplacian of a distribution u. Then, if u is a function of $H^1(\Omega)$, we have from (1) for all function v of $H^1(\Omega)$.

$$\begin{aligned}
-\int_{\Omega} (\Delta u) v &= -\sum_{i=1}^{n} \int_{\Omega} \frac{\partial^2 u}{\partial x_i^2} v dx \\
&= \sum_{i=1}^{n} \left\{ \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_{\Omega} \frac{\partial u}{\partial x_i} v \eta_i d\sigma \right\} \\
&= \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_{\Omega} \frac{\partial u}{\partial \eta} v d\sigma \\
&= \int_{\Omega} \nabla u \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma.
\end{aligned}$$

■

Remark 1 *The Green formula rest remains valid if $u, v \in C^1$ and the formula (2) remains valid if $u \in C^2$, $v \in C^1$.*

0.2.2 Quadratic Formula [1]

A quadratic formula is a homogeneous polynomial of the second degree with respect to n variables $u_1, u_2, ..., u_n$. A quadratic form always has the representation

$$\sum_{i,j=1}^{n} a_{ij} u_i u_j, \tag{3}$$

where

$$A = (a_{ij})_{1 \leq i,j \leq n},$$

is a symmetric matrix.

If we denote the matrix-column $(u_1, u_2, ..., u_n)$ with u and the quadratic formula with

$$A(u, u) = \sum_{i,j=1}^{n} a_{ij} u_i u_j, \tag{4}$$

we can write

$$A(u, u) = u^T A u = Au.u. \tag{5}$$

If

$$A = (a_{ij})_{1 \leq i,j \leq n},$$

is a real symmetric matrix, the form (4) is called the real quadratic formula. In this work, we are interested in real quadratic formula.

Definition 1 *a quadratic formula (4) is called defined non-negative if, for arbitrary real values of the variables*

$$A(u, u) \geq 0. \tag{6}$$

Definition 2 *a quadratic formula (4) is called defined positive if, for arbitrary values of non-zero variables $(u \neq 0)$, we have*

$$A(u, u) > 0. \tag{7}$$

Theorem 3 *a quadratic formula (4) is called defined positive if, and only if, all the succesifs principls determinants of her matrix a coefficients, are positives*

$$\det 1 > 0, \det 2 > 0, ..., \det n > 0. \tag{8}$$

Corollary 4 *In a positive quadratic formula (4) all the determinent principls of the matrix of coefficients, are positifs, when the successives principals determinants of a real sym-*

metric matrix are positive, all the remaining principals determinants are positives.

Remark 2 *if the principals successives determinants are non-negatives*

$$\det 1 \geq 0, \det 2 \geq 0, ..., \det n \geq 0, \tag{9}$$

it does not follow that $A(u,u)$ is defined as non-negative. Thus the forme

$$a_{11}u_1^2 + 2a_{12}u_1u_2 + a_{22}u_2^2, \tag{10}$$

in which $a_{11} = a_{12} = 0$, $a_{22} < 0$, satisfied (9) but is not defined non-negative.

We have, however, the following theorem:

Theorem 5 *a quadratic formula (4) is said defined as non- negative if, and only if, all the principal determinants of its matrix of coefficients are non-negative*

$$\det A \left[i_1, i_2, ..., i_n | i_1, i_2, ..., i_p\right] \geq 0,$$

where

$$1 \leq i_1 < i_2 < ... < i_p \leq n, \qquad \textit{and} \qquad p \leq n.$$

CHAPTER 1

Stability Theory

In this chapter we present an introduction to the theory of stability. Since this is a very broad area which includes not only many topics but also various notions of stability, we mainly focus on Lyapunov stability of equilibrium points and leave out topics such as the Poincaré–Bendixon theory, stability of periodic solutions, limit cycles, etc. Some of the proofs are omitted or carried out in special simple cases.

1.1 Stability of linear Systems

We call $(x, y) \in R^2$ the variable and write the system in the form

$$\begin{cases} x' = a_{11}x + a_{12}x \\ y' = a_{21}x + a_{22}y \end{cases}, \tag{1.1}$$

where the coefficients a_{ij} are real numbers. Letting $u = (x, y)$ and

$$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix},$$

the system can be written as u $u' = Au$. If A is nonsingular, which we always assume, the only equilibrium is $.(x, y) = (0, 0)$.We are going to study the qualitative properties of the solutions of (1.1), in particular their asymptotic behavior, as $t \to +\infty$.

To study the system

$$\begin{cases} x' = a_{11}x + a_{12}x, \\ y' = a_{21}x + a_{22}y, \end{cases}$$

we put

$$\begin{cases} f(x, y) = x' = a_{11}x + a_{12}x, \\ g(x, y) = y' = a_{21}x + a_{22}y, \end{cases}$$

we found the equilibrium point (x^*, y^*) such as $f(x^*, y^*) = (0, 0)$ and $g(x^*, y^*) = (0, 0)$, then the Jacobian matrix $J(x^*, y^*)$ and calculate the eigenvalues

$$J(x^*, y^*) = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix}.$$

Let us recall that the Jordan normal form of a nonsingular matrix A is a nonsingular matrix J, with the property that there exists an invertible matrix B such that $BA = JB$. The Jordan matrix J exists and has the same eigenvalues λ_1, λ_2 as A. Moreover, if λ_1, λ_2 are real numbers, then,

$$\lambda_1 \neq \lambda_2 \qquad \text{so} \qquad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{1.2}$$

If $\lambda_1 = \lambda_2$ then either

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \tag{1.3}$$

or

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \tag{1.4}$$

If the eigenvalues are complex, $\lambda = \alpha \pm i\beta$, then

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{1.5}$$

Theorem 6 *[4]Let $\lambda_{1,2}$ the eigenvalues of the Jacobian matrix then If*

$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 < \lambda_2 < 0$ *or* $\lambda_2 < \lambda_1 < 0$ *asymptotically stable is called a stable node, (See Figures 1.1 and 1.2)*

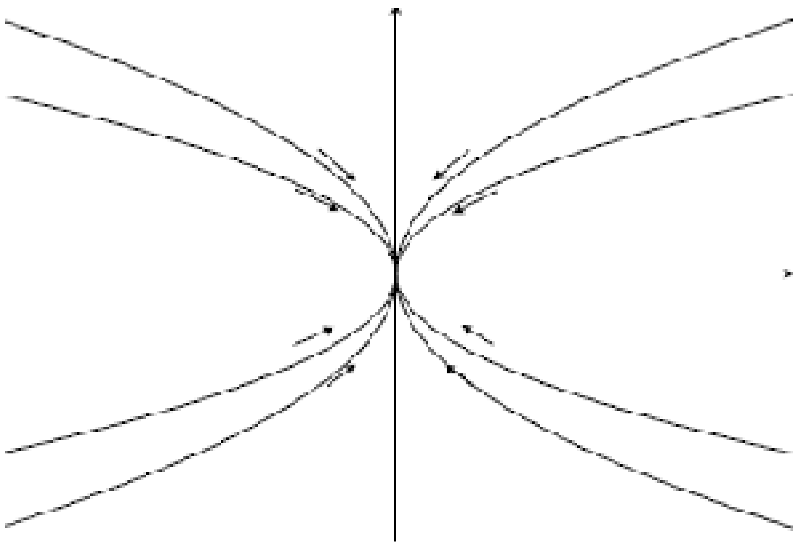


Fig 1.1

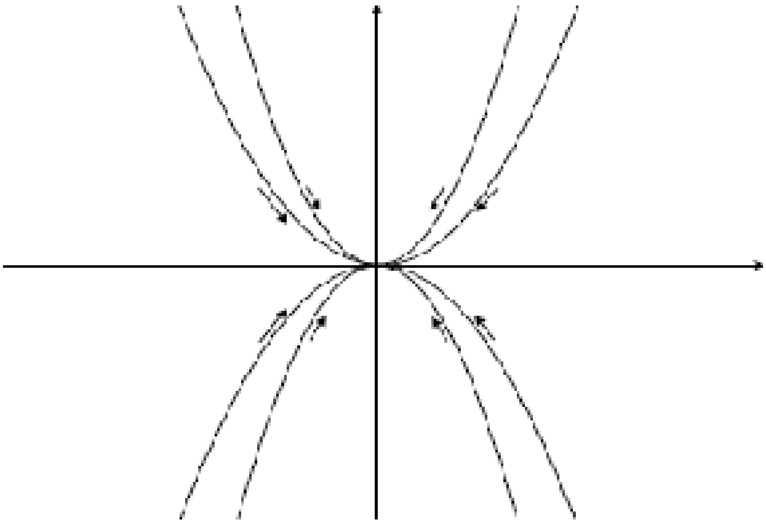


Fig 1.2

$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1.\lambda_2 > 0$ *We have an unstable node,(see Figures 1.3 and 1.4)*

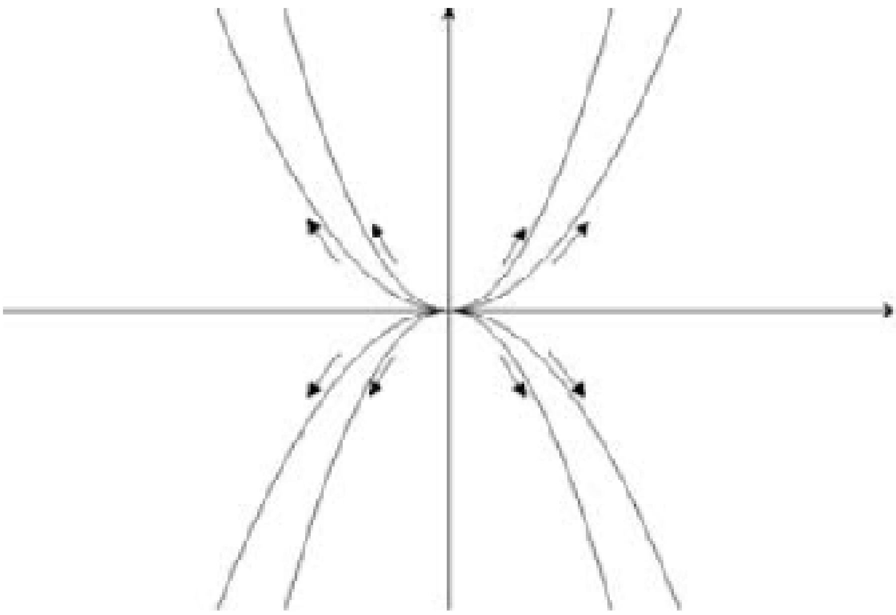


Fig 1.3

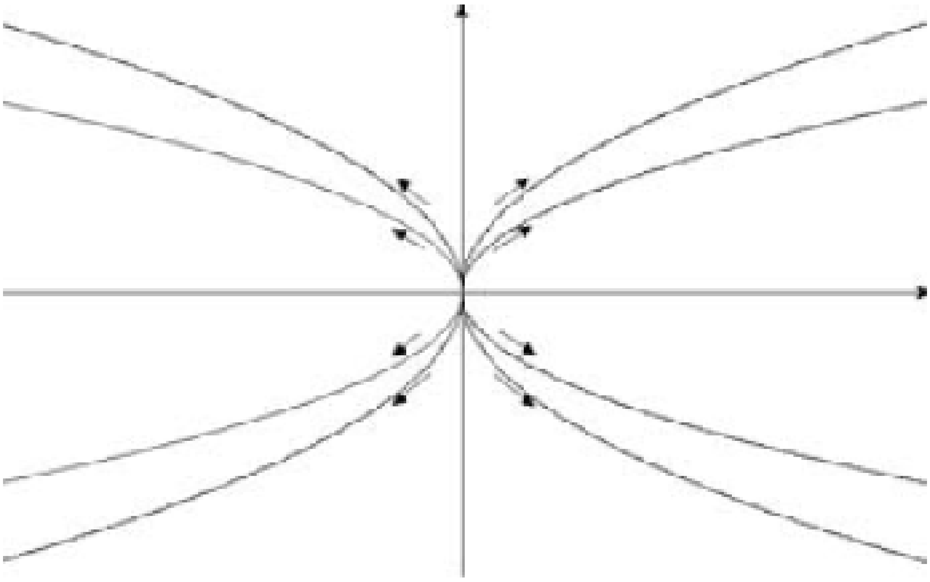


Fig 1.4

$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1.\lambda_2 < 0$ *the functions $y(x)$ are hyperbolas. The origin is unstable and is called a saddle, See (Figure 1.5)*

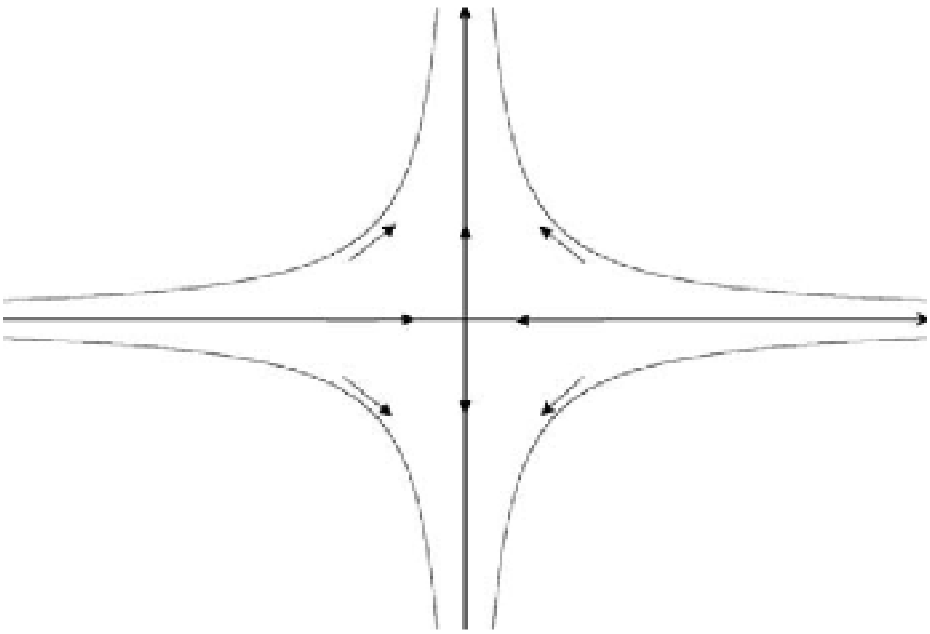


Fig 1.5

$\lambda \in \mathbb{C}, \lambda_{1,2} = \alpha \pm i\beta,\ \alpha < 0$ *asymptotically stable focus, (See Figure 1.6)*

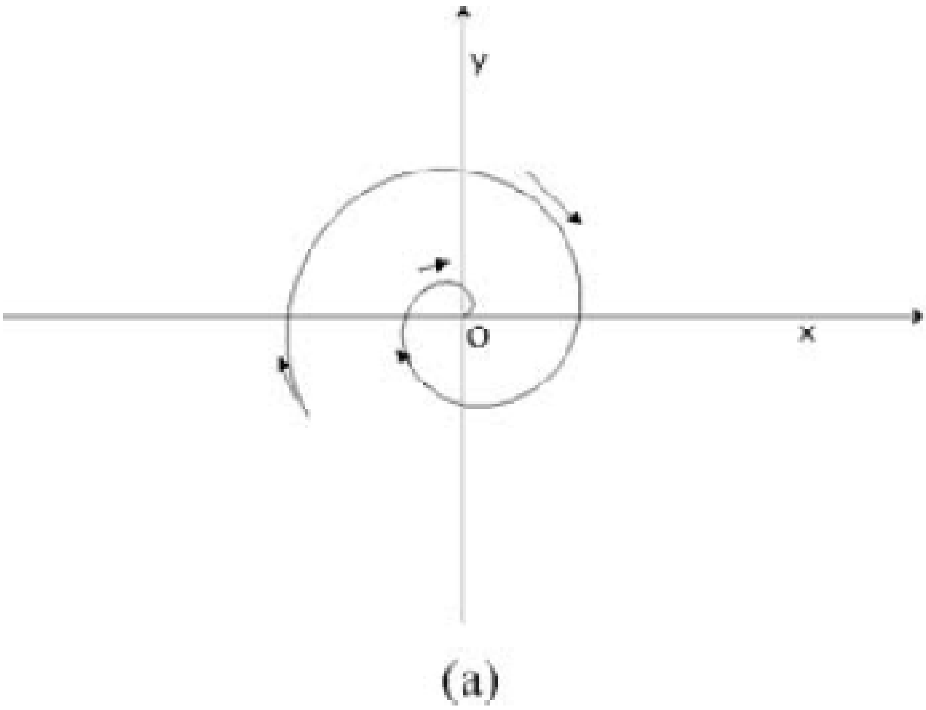


Fig 1.6

$\lambda_{1,2} = \alpha \pm i\beta,\ \alpha > 0$ *unstable focus, (see Figures 1.7)*

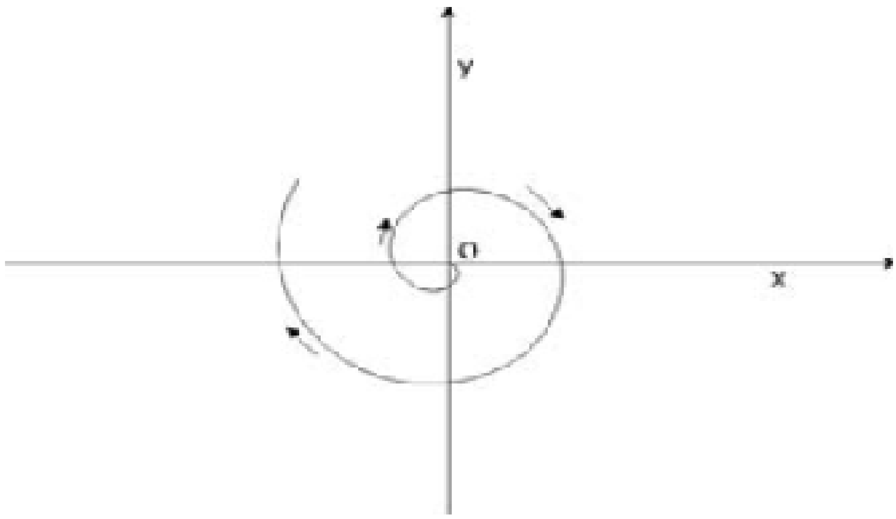


Fig 1.7

$\lambda_{1,2} = \pm i\beta$ *stable center, (see Figures 1.8)*

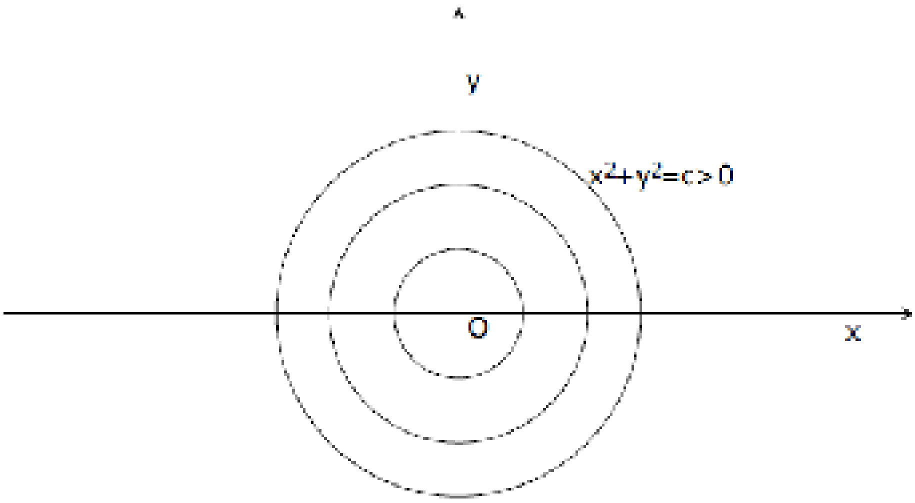


Fig 1.8

Theorem 7 *The area where the equilibrium point is asymptotically stable is that corresponding to:*

$$\begin{cases} \det(A) > 0, \\ \mathrm{Tr}(A) < 0. \end{cases}$$

In the particular case where $\begin{cases} \det(A) > 0, \\ \mathrm{tr}(A) = 0. \end{cases}$ *, the origin is a center.*

1.2 Lyapunov Stability Theory

1.2.1 Introduction

In this lecture we consider the stability of equilibrium points of autonomous nonlinear systems, both in continuous and discrete time. Lyapunov stability theory is a standard tool and one of the most important tools in the analysis of nonlinear systems. It may be extended relatively easily to cover nonautonomous systems and to provide a strategy for constructing stabilizing feedback controllers. In the sequel we present the results for time-invariant systems. They may be derived for time varying systems as well, but the essential idea is more accessible for the time invariant case. For further reading you can consult.

Definition 3 ***(Lyapunov functional)*** *We said a* ***lyapunov functional*** *every function associated to a reaction defusion system made of m equations, any function*

$$\mathbf{L} : \mathbb{R}^m \to \mathbb{R}^+. \tag{1.6}$$

Then

$$\frac{d}{dt}(\mathbf{L}(u_1(t,.), ..., u_m(t,.))) \leq 0. \tag{1.7}$$

For $t > 0$ and any solution $(u_1(.,x), ..., u_m(.,x))$ of the system.

Remark 3 *We can juste using a bounded functional to prove the global existence of solutions.*

1.2.2 Stability of an equilibrium point

Définition 1.2.1 ***(equilibrium point)*** *we say that $x_e$is an equilibrium point of a system*

$$\begin{cases} \dot{x}(t) = f\left(x\left(t\right)\right), \\ x\left(0\right) = x_0, \end{cases} \tag{1.8}$$

if x_e verify the equation

$$f\left(x\right) = 0,$$

for all $t \geq t_0$

Définition 1.2.2 *The equilibrium point x^* of the system (1.8) is stable in the sense of Lyapunov if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| \leq \delta \Rightarrow \|x\left(t\right) - x_e\| \leq \varepsilon, \forall t \geq t_0.$$

If not the equilibrium point is said instable.

1.2.3 Stability of Autonomous Systems ([12])

Consider the nonlinear autonomous (no forcing input) system

$$\dot{x} = f(x), \tag{1.9}$$

where $f : D \to R^n$ is a locally Lipschitz map from the domain $D \subseteq R^n$ to R^n. Suppose that the system (1.9) has an equilibrium point $x^* \in D$, i.e, $f(x^*) = 0$. We would like to characterize if the equilibrium point x^* is stable. In the sequel, we assume that x is the origin of state space. This can be done without any loss of generality since we can always apply a change of variables to $\xi = x - x^*$ to obtain

$$\dot{\xi} = f(\xi + x^*) = g(\xi),$$

and then study the stability of the new system with respect to $\xi = 0$, the origin. We have the following two types of stability.

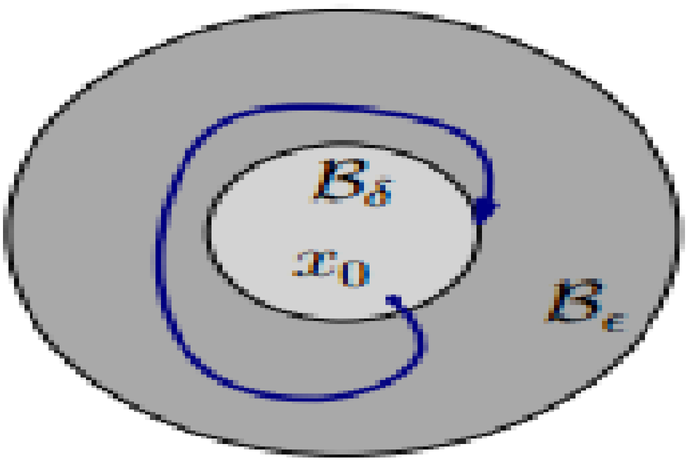


Figure 1.2.1: Illustration of a stable system

Definition 4 *The equilibrium $x = 0$ of (1.9) is*

1. *stable, if for each $\varepsilon > 0$ there exists a$\delta > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t > t_0. \tag{1.10}$$

2. asymptotically stable if it is stable and in addition δ can be chosen

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

1.2.4 Lyapunov's Direct Method

Let $L : D \to R$ be a continuously differentiable function de ned on the domain $D \subset R^n$ that contains the origin. The rate of change of V along the trajectories of (1.9) is given by

$$\begin{aligned}\dot{L} \triangleq \frac{d}{dt}L(x) &= \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}\frac{d}{dt}x_i \\ &= \left[\frac{\partial L}{\partial x_1}\frac{\partial L}{\partial x_2}...\frac{\partial L}{\partial x_n}\right]\dot{x} = \frac{\partial L}{\partial x}f(x).\end{aligned}$$

The main idea of Lyapunov's theory is that if $\dot{L}(x)$ is negative along the trajectories of the system, then $L(x)$ will decrease as time goes forward. Moreover, we do not really need to solve the nonlinear **ODE** (1.9) for every initial condition, but we just need some information about the drift $f(x)$.

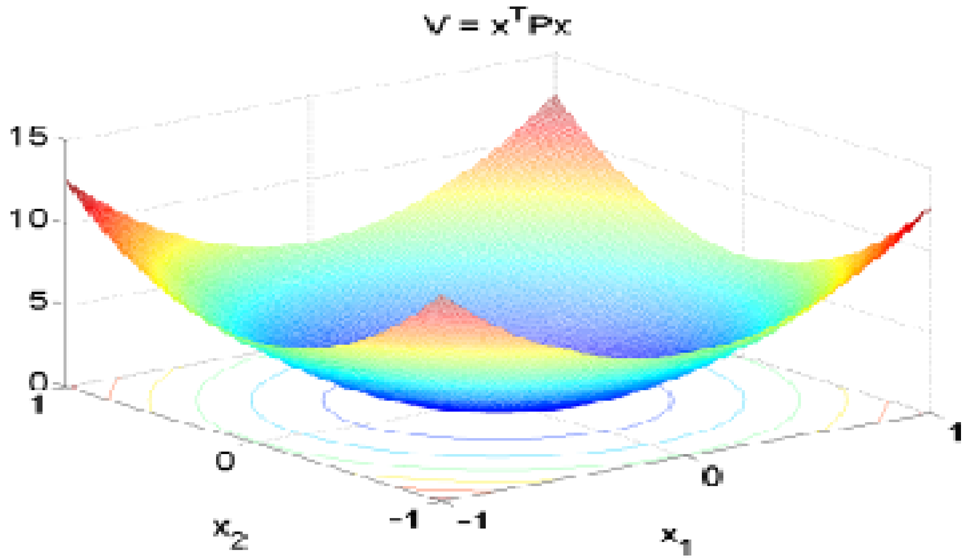


Figure 1.2.2: Lyapunov function in two states $L = x_1^2 + 1.5x_2^2$. The level sets are shown in the x_1x_2-plane.

(1.11)

Example 1 *[12]Consider the nonlinear system*

$$\begin{cases} \frac{\partial x_1}{\partial t} = -x_1 + 2x_1^2 x_2 \\ \frac{\partial x_2}{\partial t} = -x_2 \end{cases}$$

and the candidate Lyapunov function

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2,$$

with $\lambda_1, \lambda_2 > 0$*.If we plot the function* $V(x)$ *for some choice of '*λ*s we obtain the result in (1.11). This function has a unique minimum over all the state space at the origin. Moreover,* $V(x) \to \infty$ *as* $\|x\| \to \infty$.

Calculate the derivative of V *along the trajectories of the system*

$$\dot{V} = 2\lambda_1 x_1(-x_1 + 2x_1^2 x_2) + 2\lambda_2 x_2(-x_2) = -2\lambda_2 x_1^2 - 2\lambda_1 x_2^2 + 4\lambda_1 x_1^3 x_2.$$

Therefore, if $\dot{V}(x)$ *is negative,* V *will decrease along the solution of* $\dot{x} = f(x)$.

We are now ready to state Lyapunov's stability theorem.

Theorem 8 *[12]Let the origin* $x = 0 \in D \subset R^n$ *be an equilibrium point for* $\dot{x} = f(x)$. *Let* $V : D \to R$ *be a continuously di erentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in D \setminus \{0\} \tag{1.12}$$

$$\dot{V}(x) \leq 0, \forall x \in D \tag{1.13}$$

Then, $x = 0$ *is stable. Moreover, if*

$$\dot{V}(x) \leq 0, \forall x \in D \setminus \{0\}$$

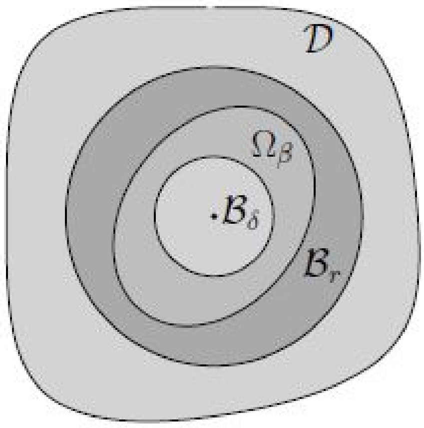
then $x = 0$ *is asymptotically stable*

Remark 4 *if* $V(x) > 0$, $\forall x \in D \setminus \{0\}$, *then* V *is called locally positive definite. If* $V(x) \geq 0$, $\forall x \in D \setminus \{0\}$, *then* V *is locally positive semi-de nite.If the conditions (1.12) are met, then* V *is called a Lyapunov function for the system* $\dot{x} = f(x)$.

Proof. Given any $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{x \in R^n, \|x\| \leq r\} \subset D$.

Let $\alpha = \min_{\|x\|=r} V(x)$. Choose $\beta \in (0, \alpha)$ and define $\Omega_\beta = \{x \in B_r, V(x) \leq \beta\}$.

domains in the proof



$_1$1.*jpg*

Various domains in the proof of Theorem 1

It holds that if $x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \ \forall t$

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta.$$

Further $\exists \delta > 0$ such that $\|x\| < \delta \Rightarrow V(x) < \beta$. Therefore, we have that

$$B_\delta \subset \Omega_\beta \subset B_r,$$

and furthermore

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in B_r,$$

Finally, it follows that

$$\|x\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \forall t > 0.$$

In order to show asymptotic stability, we need to to show that $x(t) \to 0$ as $t \to \infty$. In this case, it turns out that it is su cient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since V is monotonically decreasing and bounded from below by 0,then

$$V(x) \to c \geq 0, ast \to \infty.$$

■

Theorem 9 *Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : R^n \to R$ be a continuously di erentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \neq 0, \tag{1.14}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty, \tag{1.15}$$

$$\dot{V}(x) < 0, \forall x \neq 0, \tag{1.16}$$

then the origin is globally asymptotically stable.

Remark 5 *if the function V satis es the condition (1.15), then it is said to be radially unbounded.*

Example 2 *[12]Consider the system*

$$\begin{cases} \frac{\partial x_1}{\partial t} = x_2 \\ \frac{\partial x_2}{\partial t} = -h(x_1) - x_2 \end{cases}$$

where the function h is locally Lipschitz with $h(0) = 0$ and $x_1 h(x_1) > 0$, $\forall x_1 \neq 0$. Take the Lyapunov function candidate

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(s) ds.$$

The function V is positive definite for all $x \in R^2$ and is radially unbounded.
The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = -(1-k)x_2^2 - kx_1 h(x_1) < 0.$$

Hence the derivative of V is negative de nite for all $x \in R^2$, since $0 < k < 1$ (otherwise V is not positive definite!). Therefore, the origin is globally asymptotically stable.

1.3 Bifurcation

As an important dynamic bifurcation phenomenon in dynamical systems, Hopf bifurcation of periodic solutions has attracted great interest of many authors in the last several decades . In general, the study of Hopf bifurcation includes the existence and the properties such as the direction of bifurcation and the stability of bifurcating periodic solutions. In application, however, it is more difficult to determine the properties of Hopf bifurcation than to find the existence of a Hopf bifurcation. An approach applied to determine the properties of Hopf bifurcation is to derive the projected equation of original equations on the associated center manifold, that is, the so-called normal form. Then one may explore the local dynamical behaviors of a higher dimensional or even infinitely dimensional dynamical system near a certain nonhyperbolic steady state according to the normal form obtained. The normal form of Hopf bifurcation in ordinary differential equations **(ODEs)** with or without delays has been established well since in this case the equilibrium is always constant and there are also no effects of spatial diffusion.

Under some certain conditions, the reaction-diffusion equations under the homogeneous Neumann boundary condition may have the constant steady state and thus one can study the Hopf bifurcation of system at this constant steady state. Compared with the ODEs, it is more difficult to derive the normal form of Hopf bifurcation for reaction-diffusion equations at the constant steady state. Although Hassard et al. established the method computing the normal form of Hopf bifurcation in reaction-diffusion equations with the homogeneous Neumann boundary condition and also considered the Hopf bifurcation of spatially homogeneous periodic solutions in Brusselator system, using the same method, Jin et al. and Ruan as well as Yi et al. considered the Hopf bifurcation of spatially homogeneous periodic solutions for Gierer-Meinhardt system and CIMA reaction, respectively. There are few results regarding Hopf bifurcation of spatially nonhomogeneous periodic solutions for spatially homogeneous reaction-diffusion equations.

Based on the reason mentioned above, in this paper we consider the normal form of Hopf bifurcation of reaction-diffusion equations at the constant steady state following the idea in [[**?**]]. In order to have a clearer structure, we are concerned with the following general reaction-diffusion system coupled by two equations defined on one-dimensional spatial domain $(0, l\pi)$ with $l > 0$ and subject to Neumann boundary conditions, that is,

$$\begin{cases} u_t = d_1 u_{xx} + f_1(\lambda, u, v), x \in (0, l\pi), t > 0, \\ v_t = d_2 u_{xx} + f_2(\lambda, u, v), x \in (0, l\pi), t > 0, \end{cases} \tag{1.17}$$

$$u_x(0,t) = v_x(0,t) = u_x(l\pi,t) = v_x(l\pi,t) = 0, t > 0,$$

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x), x \in (0, l\pi), \end{aligned}$$

in which $d_1, d_2 > 0$ are the diffusion coefficients, $\lambda \in \mathbb{R}$ is the parameter, and $f_1, f_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are C^r $(r \geq 5)$ functions with $f_k(\lambda, 0, 0) = 0$; $k = 1, 2$, for any $\lambda \in \mathbb{R}$. Although Yi et al. described the algorithm determining the properties of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions for (1.17) at and also considered the Hopf bifurcation of a diffusive predator-prey system with Holling type-II functional response and subject to the homogeneous Neumann boundary condition, they did not give the normal form of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions of the general reaction-diffusion system (1.17) at $(0, 0)$.

This paper is organized as follows. In the next section, following the abstract method according to [8], we describe the algorithm determining the properties of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions for system (1.17) at the constant steady state .

1.3.1 Andronov-Hopf bifurcation

Andronov-Hopf bifurcation is the birth of a limit cycle from an equilibrium in dynamical systems generated by **ODE**s, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. The bifurcation can be supercritical or subcritical, resulting in stable or unstable (within an invariant two-dimensional manifold) limit cycle, respectively.

Definition 5 ([5]) *Consider an autonomous system of ordinary differential equations **(ODE)***

$$\dot{x} = f(x, \alpha), x \in R^n,$$

depending on a parameter $\alpha \in R$, where f is smooth.

- *Suppose that for all sufficiently small $|\alpha|$ the system has a family of equilibria $x^*(\alpha)$.*
- *Further assume that its Jacobian matrix $A(\alpha) = f_x(x^*(\alpha), \alpha)$ has one pair of complex eigenvalues*

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha),$$

that becomes purely imaginary when $\alpha = 0$, i.e., $\mu(0) = 0$ and $\omega(0) = \omega_0 > 0$. Then, generically, as α passes through $\alpha = 0$, the equilibrium changes stability and a unique limit cycle bifurcates from it. This bifurcation is characterized by a single bifurcation condition $\operatorname{Re} \lambda_{1,2} = 0$ (has codimension one) and appears generically in one-parameter families of smooth ODEs.

1.3.2 Two-dimensional Case

To describe the bifurcation analytically, consider the system above with $n = 2$,

$$\dot{x}_1 = f_1(x_1, x_2, \alpha),$$

$$\dot{x}_2 = f_2(x_1, x_2, \alpha),$$

If the following nondegeneracy conditions hold:

- **(AH.1)** $l_1(0) \neq 0$, where $l_1(\alpha)$ is the first Lyapunov coefficient (see below),

- **(AH.2)** $\mu'(0) \neq 0$,

then this system is locally topologically equivalent near the equilibrium to the normal form

$$\begin{cases} \dot{y}_1 = \beta y_1 - y_2 + \sigma y_1(y_1^2 + y_2^2), \\ \dot{y}_2 = y_1 - \beta y_2 + \sigma y_1(y_1^2 + y_2^2), \end{cases}$$

where $y = (y_1, y_2)T \in R^2$, $\beta \in R$, and $\sigma =$ sign $l_1(0) = \pm 1$.

- If $\sigma = -1$, the normal form has an equilibrium at the origin, which is asymptotically stable for $\beta \leq 0$ (weakly at $\beta = 0$) and unstable for $\beta > 0$. Moreover, there is a unique and stable circular limit cycle that exists for $\beta > 0$ and has radius $\sqrt{\beta}$. This is a supercritical Andronov-Hopf bifurcation (see Figure 1.12a),

- If $\sigma = +1$, the origin in the normal form is asymptotically stable for $\beta < 0$ and unstable for $\beta \geq 0$ (weakly at $\beta = 0$), while a unique and unstable limit cycle exists for $\beta < 0$. This is a subcritical Andronov-Hopf bifurcation (see Figure 1.12b).

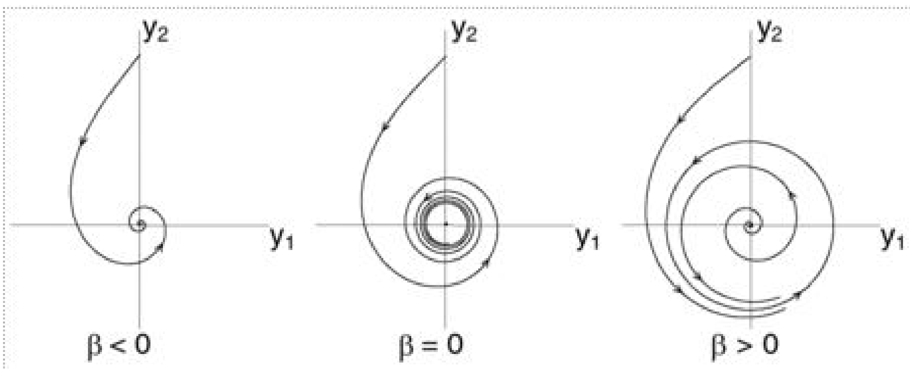


Figure 1.3.1a

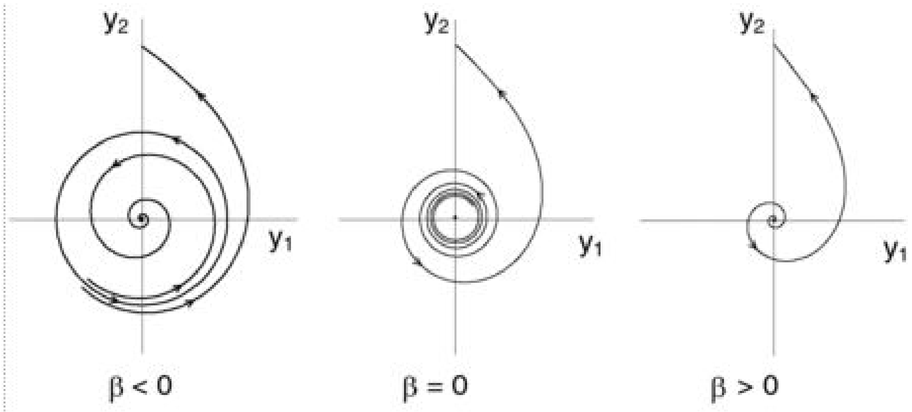


Figure 1.3.1b

1.3.3 Multi-dimensional Case

In the n-dimensional case with $n \geq 3$, the Jacobian matrix $A_0 = A(0)$ has:

- a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, as well as
- n_s eigenvalues with $\operatorname{Re}\lambda_j < 0$, and
- n_u eigenvalues with $\operatorname{Re}\lambda j > 0$,

with $n_s + n_u + 2 = n$. According to the Center Manifold Theorem, there is a family of smooth two-dimensional invariant manifolds W^c_α near the origin. The n-dimensional system restricted on W^c_α is two-dimensional, hence has the normal form above.

Moreover, under the non-degeneracy conditions **(AH.1)** and (**AH.2**), the n-dimensional system is locally topologically equivalent near the origin to the suspension of the normal form by the standard saddle, i.e.

$$\begin{cases} \dot{y}_1 = \beta y_1 - y_2 + \sigma y_1(y_1^2 + y_2^2), \\ \dot{y}_2 = y_1 - \beta y_2 + \sigma y_1(y_1^2 + y_2^2), \end{cases}$$

$$\begin{cases} \dot{y}^s = -y^s, \\ \dot{y}^u = +y^u, \end{cases}$$

where $y = (y_1, y_2)T \in R^2$, $y^s \in R^{n_s}$,$y^u \in R^{n_u}$. Figure (1.13) shows the phase portraits of the normal form suspension when $n = 3$,$n_s = 1$,$n_u = 0$, and $\sigma = -1$.

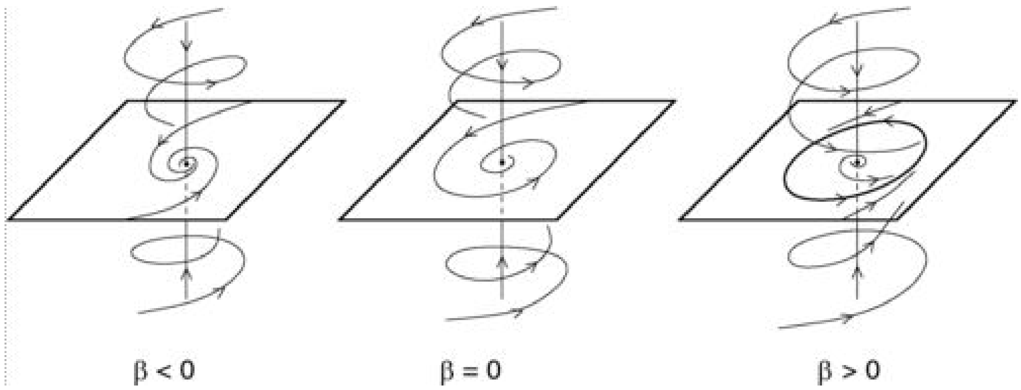


Figure 1.3.2

1.3.4 First Lyapunov Coefficient

Whether Andronov-Hopf bifurcation is subcritical or supercritical is determined by σ, which is the sign of the first Lyapunov coefficient $l1(0)$ of the dynamical system near the equilibrium. This coefficient can be computed at $\alpha = 0$ as follows. Write the Taylor expansion of $f(x, 0)$ at $x = 0$ as

$$f(x,0) = A_0 x + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + (\|x\|^4),$$

where $B(x, y)$ and $C(x, y, z)$ are the multilinear functions with components

$$B_j(x,y) = \sum_{k,l=1}^{n} \frac{\partial^2 f_j(\xi,0)}{\partial \xi_k \partial \xi_l} |_{\xi=0} x_k y_l, \ ,$$

$$C_j(x,y,z) = \sum_{k,l,m=1}^{n} \frac{\partial^3 f_j(\xi,0)}{\partial \xi_k \partial \xi_l \partial \xi_m} |_{\xi=0} x_k y_l z_m, \ ,$$

where $j = 1, 2, \ldots, n$. Let $q \in C^n$ be a complex eigenvector of A_0 corresponding to the eigenvalue $i\omega 0 : A_0 q = i\omega_0 q$. Introduce also the adjoint eigenvector $p \in C^n : AT_0 p = -i\omega_0 p$, $\langle p, q \rangle = 1$. Here $\langle p, q \rangle = p^{-T} q$ is the inner product in C^n.

$$l_1(0) = \frac{1}{2\omega_0} \operatorname{Re}\left[\langle p, C(q,q,\bar{q})\rangle - 2\left\langle p, B(q, A_0^{-1}B(q,\bar{q}))\right\rangle + \left\langle p, B(\bar{q}, B(2i\omega_0 I_n - A_O)^{-1}B(q,q))\right\rangle\right],$$

where In is the unit $n \times n$ matrix. Note that the value (but not the sign) of $l_1(0)$ depends on the scaling of the eigenvector q. The normalization $\langle q, q \rangle = 1$ is one of the options to remove this ambiguity. Standard bifurcation software computes $l_1(0)$ automatically.

For planar smooth ODEs with

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, f(x,0) = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} P(u,v) \\ Q(u,v) \end{pmatrix},$$

the setting $q = p = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ leads to the formula

$$l_1(0) = \frac{1}{8\omega_0}(P_{uuu}+P_{uvv}+Q_{vvv})+\frac{1}{8\omega_0^2}\left[P_{uv}(P_{uu}+P_{vv}) - Q_{uv}(Q_{uu}+Q_{vv})\right]-P_{uu}Q_{uu}+P_{vv}Q_{vv},$$

where the lower indices mean partial derivatives evaluated at $x = 0$.

1.4 The Lotka-Voltera Equation

In the 1920's Vito volterra asked whethere possible to explain the fluctuations thet had been observed in the fish population of the Adriactic sea fluctuations that that were of great concern to fishmen in times of low fish populations.Volterra (1926) construct the modelthat has become known as the Lotka-Volterra model (because A.J.Lotka (1925) construct a similar model in a different concontext about the same time), based on the assumptions thatfish and sharks were in a predator-prey relationship.

Here is a description of the model suggeted by Volterra. Let $x(t)$ be the number of fish and $y(t)$ the number of sharks at time t. We assume that the plankton ,which is the food supply for the fish, is unlimited, and thus that the per capita growth rate of the fish population in the absence of sharks would be constant. Thus, if there were no sharks the fish population would satisfy a differantial equation of the form $\frac{dx}{dt} = ax$. The sharks, on the other hand, depend on fish as their food supply, and we assume that if there were no fish, the sharks would have a constant per capita death rate; thus, in the absence of fish, the sharks would satisfy a differential equation of the form $\frac{dy}{dt} = -cy$.

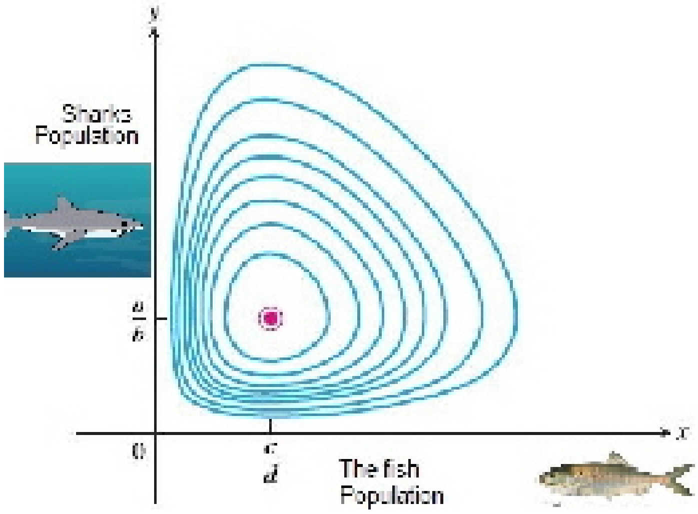


Figure 1.4.1

We assume that the presence of fish increases the shark growth rate, changing the per capita shark groth rate from $-d$ to $-d + c$. The presence of sharks reduces the fish populations, changing the per capita fish growth rate from a to $a - by$. This gives the Lotka-Volterra equations.

$$\begin{cases} \frac{dx}{dt} = x(a - by), \\ \frac{dy}{dt} = y(-d + cx), \end{cases} \tag{1.18}$$

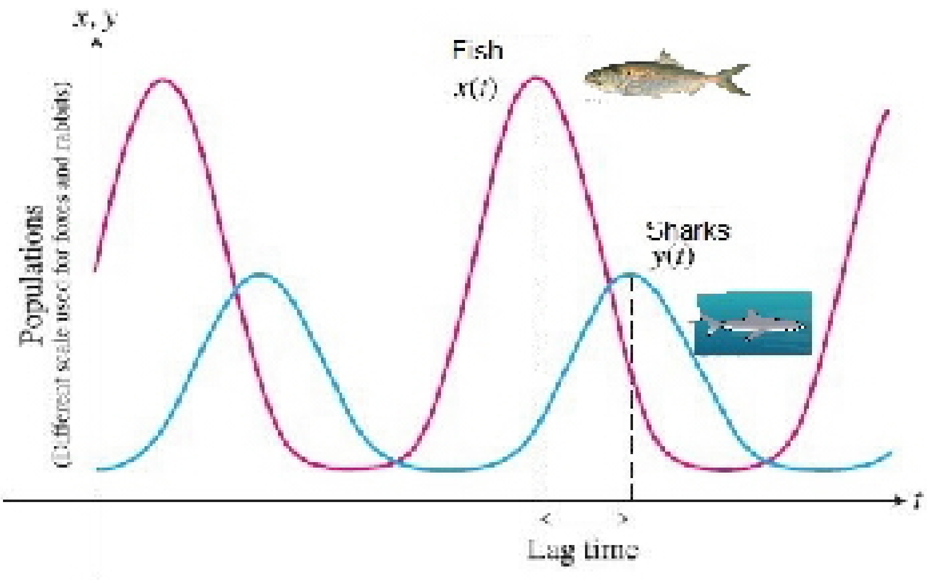


Figure 1.4.2

Now we will study the stability of Lotka-Volterra equation

$$\frac{dx}{dt} = 0,$$
$$\frac{dy}{dt} = 0,$$

which implies

$$\begin{cases} (a - by) = 0 & \text{or} & x = 0 \\ (dx - c) = 0 & \text{or} & y = 0 \end{cases}.$$

Then the equilibium points are :

$$(x^*, y^*) = (0,0)\ ,\ (x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right).$$

1.4.1 The Jacbian matrix

We have

$$\begin{aligned} J(x,y) &= \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \\ &= \begin{pmatrix} a - by & -bx \\ dy & dx - c \end{pmatrix}. \end{aligned}$$

For $(x^*, y^*) = (0,0)$

$$J(x^*, y^*) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}.$$

Then eigenvalues are : $\lambda_1 = a$ or $\lambda_2 = -c$, we see that

$$\lambda_1 \times \lambda_2 = -ac < 0,$$

then the system in the equilibium point $(0,0)$ is unstable, the fixed point at the origin is a saddle point.

For $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)$

$$\begin{aligned} J(\frac{c}{d}, \frac{a}{b}) &= \begin{pmatrix} a - b\left(\frac{a}{b}\right) & -b\left(\frac{c}{d}\right) \\ d\left(\frac{a}{b}\right) & -c + d\left(\frac{c}{d}\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}. \end{aligned}$$

The eigenvalues of this matrix are

$$\lambda_1 = i\sqrt{ac}, \lambda_2 = -i\sqrt{ac},$$

$$\lambda_1.\lambda_2 = \sqrt{ac},$$

then at the the equilibium point $\left(\frac{c}{d}, \frac{a}{b}\right)$ we have a stable center.

1.4.2 Lyapunov function

$$L(x, y) = dx + by - c\log x - a\log y,$$

we calculate the derivative of $L\left(x(t), y(t)\right)$ of t

$$\begin{aligned} \dot{L}(x, y) &= dx' + by' - c\frac{1}{x}x' - a\frac{1}{y}y' \\ &= d(ax - bxy) + b(dxy - cy) - c\frac{ax - bxy}{x} - a\frac{dxy - cy}{y} \\ &= dax - abxy + bdxy - bcy - ca + bcy - adx + ac \\ &= 0. \end{aligned}$$

As a first application we want to study the stability of the nontrivial equilibrium

$x^* = \frac{c}{d}, y^* = \frac{a}{b}$, we know that L has a strict local minimum at (x^*, y^*). we have $\dot{L}(x, y) = 0$, H is a Lyapunov function and one deduces that (x^*, y^*) is stable (but not asymptotically) We will see later on that the trivial equilibrium $(0, 0)$ is unstable.

Then according to Lyapunov the system is not stable globaly.

CHAPTER 2

Reaction–diffusion systems

2.1 Introduction to reaction-diffusion systems

In recent years, the reaction-diffusion systems have received a major treaty attention, motivated by their widespread incident in models of biological and chemical properties, and the rich structure of their solutions Given the many and varied applications of these systems; We will give the steps taken to modeling chemical problems such as reactions Oscillating chemicals (Brussélateur). Individuals dived from one problem to another: In chemistry, for example, they represent chemical substances. In biochemistry, they Can represent molecules. In metallurgy, atoms. In dynamics of Populations, they are humans. In population genetics, they represent characters. In biophysics, electrical charges or di erences of potential tial. In the environment, they can represent the animals or plants of a forest, Of a sea or of an eve ... For most of these problems, one shows finally to reaction-diffusion systems.

The conditions at the edges will be chosen according to the origin and the nature of the problem Studied: if there is no immigration of individuals across the boundary of the domain Ω on which the problem is posed, the conditions at the homogeneous edges of Neumann. If there are no individuals on the boundary, we take the conditions at the edges Homogeneous of Dirichlet. The unknown (the solution we are looking for) is a vector whose components are generally positive functions: in chemistry, for example, it is a vector of chemical concentrations. In biochemistry or metallurgy, it is a vector of concentrations in numbers of molecules or atoms, respectively. In population dynamics and in the environment, it is a vector of densities of Human, animal or plant populations ...

Initial conditions are generally positive; Since it is a concentration, densities, electrical charges,...etc.

All these problems are written in the form:

$$\frac{\partial u}{\partial t} - D\Delta u = f(u). \tag{2.1}$$

where $u(x,t) = (u_1(x,t), ..., u_m(x,t))$ is a vector of dependent variables, and it is the unknown, $f(x,t,u(x,t)) = (f_1(x,t,u(x,t)), ..., f_m(x,t,u(x,t)))$ it is the reaction (usually Nonlinear) and D is a square matrix $m.m$ definite positive and diagonalizable called the diffusion matrix. The terms of reaction are the result of any interaction between the components of u:

For example, in chemistry u is a chemical concentration vector and f represents the effect of the chemical reactions on these concentrations. The term $D\Delta u$ represents the Molecular diffusions across the reaction boundary.

The reaction-dissolution equations will be established in the case of a chemical and of molecular diffusion and indicate the necessary assumptions for their applications in various situations.

2.2 Hopf bifurcation

The term Hopf bifurcation (also sometimes called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongsto what is sometimes called dynamic (as opposed to static) bifurcation theory. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.

When the real parts of the eigenvalues are negative the fixed point is a stable focus; when they cross zero and become positive the fixed point becomes an unstable focus, with orbits spiralling out. But this change of stability is a local change and the phase portrait sufficiently far from the fixed point will be qualitatively unaffected: if the nonlinearity makes the far flow contracting then orbits will still be coming in and we expect a periodic orbit to appear where the near and far flow find a balance.

(2.2)

The Hopf bifurcation theorem makes the above precise. Consider the planar system where μ is a parameter. Suppose it has a fixed point $(x, y) = (x_0, y_0)$, which may depend on μ. Let the eigenvalues of the linearised system about this fixed point be given by $\lambda(\mu)$, $\bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$.
Suppose further that for a certain value of μ, say $\mu = \mu_0$, the following conditions are satisfied (As mentioned in [11] and [14]):

1.$\alpha(\mu_0) = 0, \beta(\mu_0) = \omega \neq 0$, where $sgn(\omega) = sgn[(\partial g_\mu/\partial x)|_{\mu=\mu_0}(x_0, y_0)]$(non-hyperbolicity condition: conjugate pair of imaginary eigenvalues)

2.$\frac{d\alpha(\mu)}{d\mu}\big|_{\mu=\mu_0} = d \neq 0$(transversality condition: the eigenvalues cross the imaginary axis with non-zero speed)

3. $a \neq 0$, where

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16w}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}),$$

with $f_{xy} = (\partial^2 f_\mu/\partial x\partial y)\left|_{\mu=\mu_0}(x_0, y_0)\right.$,etc.(genericity condition)

Then a unique curve of periodic solutions bifurcates from the fixed point into the region $\mu > \mu_0$ if $ad < 0$ or $\mu < \mu_0$ if $ad > 0$. The fixed point is stable for $\mu > \mu_0$ (resp. $\mu < \mu_0$) and unstable for $\mu < \mu_0$ (resp. $\mu > \mu_0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable)if the fixed point is unstable (resp. stable) on the side of $\mu = \mu_0$ where the periodic solutions exist.

The amplitude of the periodic orbits grows like $\sqrt{|\mu - \mu_0|}$ whilst their periods tend to $2\pi/|w|$ as μ tends to μ_0. The bifurcation is called supercritical if the bifurcating periodic solutions are stable, and subcritical if they are unstable.

This $2D$ version of the Hopf bifurcation theorem was known to Andronov and his co-workers from around 1930, and had been suggested by Poincar´e in the early 1890s. Hopf , in 1942, proved the result for arbitrary (finite) dimensions. Through centre manifold reduction the higher-dimensional version essentially reduces to the planar one provided that apart from the two purely imaginary eigenvalues no other eigenvalues have zero real part. In his proof (which predates the centre manifold theorem), Hopf assumes the functions f_μ and g_μ to be analytic, but C^5differentiability is sufficient. Extensions exist to infinite-dimensional problems such as differential delay equations and certain classes of partial differential equations (including the Navier-Stokes equations).

2.3 Examples of Reaction–diffusion systems

2.3.1 The Gierer-Meinhardt System

The G-M Model is a reaction-diffusion system of the activator-inhibitor type that appears to account for many important types of pattern formation and morphogenesis observed in development, in their seminal paper, Gierer and Meinhardt ([16]) proposed the model

$$\begin{cases} \frac{\partial a}{\partial t} = -\mu a + c\rho\frac{a^r}{h^s} + \rho\rho_0 + D_a\frac{\partial^2 a}{\partial x^2} \\ \frac{\partial h}{\partial t} = c'\rho'\frac{a^T}{h^u} - \gamma h + D_h\frac{\partial^2 h}{\partial x^2} \end{cases}, \tag{2.3}$$

where $a(x,t)$ represents the population density of the activator and $h(x,t)$ the inhibitor, and $\rho_0, \rho, \rho', c', c, \mu, \gamma, r, s, T, u, D_a, D_h$ are all positive constants. The activator a and the inhibitor h act on the sources with density $\rho(x)$ and $\rho'(x)$, respectively.For simplicity, we assume the sources are evenly distributed in space, i.e. $\rho(x) = \rho$, $\rho'(x) = \rho'$ and the basal production of the activator is proportional to ρ. The terms $-\mu a$ and $-\gamma h$ represent the rates that a and h are removed by either enzyme degradation, or leakage, or reuptake by the source, or by any combination of these mechanisms. D_a and D_h are the diffusion coefficients of the activator and inhibitor, respectively.

2.3.2 System of Lengyel-Epstien

The Lengyel–Epstein System is the following reaction-diffusion equations

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2} \\ \frac{\partial v}{\partial t} = (\sigma b)\Delta v + (\sigma b)\left(u - \frac{uv}{1+u^2}\right) \end{cases}, \qquad x \in \Omega, t > 0, \tag{2.4}$$

which was derived from the chlorite iodide malonic acid (CIMA) chemical reaction introduced by Lengyel and Epstein and can be used to design chemical systems capable of displaying stationary, symmetry breaking reaction diffusion patterns (Turing structures). Here u and v are the concentrations of the active iodide I^- and inhibitor (ClO_2^-) at time t, respectively, a and b are positive parameters related to the feed concentrations, $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch.

A closely related system for this chemical reaction mechanism is the chlorine dioxide-iodine-malonic acid (CDIMA) reaction shown below.

$$\begin{cases} MA + I_2 \to IMA + I + H^+ \\ ClO_2 + I^- \to ClO_2 + \frac{1}{2}I_2 \\ CIO_2^- + 4I^- + 4H^+ \to Cl + 2I_2 + 2H_2O \\ S + I_2 + I_- \to SI^{-3} \end{cases}$$

The first reaction serves as a source of the activator I^-, the second produces the inhibitor chlorite ion, the third shows regeneration of iodine, and the last reaction shows the complex formation between the activator iodide (I^-) and the indicator starch.

In the CDIMA system of reaction, the concentration of Malonic acid (MA), Chloride Dioxide (ClO_2) and Iodine (I_2) displays very little variation and essentially they can be considered constant. Since only the activator iodine ion (I^-) and the inhibitor chlorite ion (ClO_2^-) show wide concentration variation, the system can be approximated by two variables model.

In the presence of starch which is used as indictor, the diffusion rate of the activator (I^-) is slower than that of the inhibitor (ClO_2^-). The starch which is much bigger molecule forms a chemical complex with I^- effectively reducing the diffusion rate of I^-. This allows the inhibitor to diffuse faster creating a condition that leads to oscillatory phenomenon. In laboratory conditions, a sample of parameters is taken in the range $0 < a < 35, 0 < b < 8, \sigma = 8$.

In Ni and Tang, studied the initial value problem of the corresponding reactiondiffusion model with the no-flux boundary condition. "Yi" used b as the bifurcation parameter and obtained a critical value b^* of b such that both the **ODE** and **PDE** models exhibit a Hopf bifurcation as b crosses b^*. They calculated the first Lyapunov coefficient which determines the stability and and direction of the periodic solution bifurcating from the equilibrium point for the **ODE**. From the view point of Chemistry and Physics, periodic solutions represent the oscilations of the concentrations of I^- and ClO_2^- .

However the first Lyapunov coefficient can be zero for certain value of $\alpha = \frac{a}{5}$.

In this situation, the criteria of the stability of the bifurcating periodic solution.

CHAPTER 3

An investigation of the Lengyel-Epstein R-D System

As we see before about lengyl eptein now we will investigate this system by studying the stability in all case (**EDO** and **PDE** models) locally and globally, How changes occur in stability, Bifurcations.

We will remind that system it's an **Antivitor-Inhibitor** system, But at first we must find the equilibrium point (steady state) of the system.

3.1 Equilibrium point

To find the equilibrium point of the system (2.4) we calculate:

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases} \quad so \quad \begin{cases} a-u-\frac{4uv}{1+u^2}=0.........(1) \\ (\delta b)\left(u-\frac{uv}{1+u^2}\right)=0......(2) \end{cases},$$

$$(\sigma b)u\left(1-\frac{v}{1+u^2}\right)=0.$$

Then

$$\begin{cases} u=0, \\ \text{or} \\ 1-\frac{v}{1+u^2}, \end{cases}$$

because

$$\text{if } u=0 \qquad \text{then} \qquad a=0, \text{refused}$$

$$\text{if } u\neq 0 \qquad \text{then} \qquad v=1+u^2,$$

the equation (1) become

$$a-u-\frac{4u(1+u^2)}{1+u^2}=0,$$

then

$$a-u-4uv=0,$$

which implies

$$a-5u=0,$$

so

$$\begin{cases} u=\frac{a}{5} \\ v=1+u^2=1+\frac{a^2}{25} \end{cases},$$

if $\frac{3\alpha^2-5}{\alpha}<(\sigma b)$ the system 2.4 has a unique constant steady state solution

$$(u^*,v^*)=\left(\frac{a}{5},1+\frac{a^2}{25}\right).$$

We put

$$\alpha=\frac{a}{5},$$

then the equilibrium point of the system

$$(u^*,v^*)=\left(\alpha,1+\alpha^2\right).$$

Definition 6 (Invariant regions) *[3]*

In this subsection, we examine the invariant regions for the system (2.4)

$$\Re = (0, \alpha) \times \left(0, 1 + \alpha^2\right)$$

in the phase plane which actually attracts all solutions of this system, regardless of the initial values u_0 and v_0.

To begin with, we first show that the initial-boundary value problem (**??**)–(**??**) has a unique solution $(u(x,t), v(x,t))$ that is defined for all $t > 0$ and is bounded by some positive constants depending on a, u_0 and v_0.

$$f_u(u^*, v^*) = \frac{(3\alpha^2 - 5)}{(1 + \alpha^2)} > 0,$$

then we call u an activator, v an inhibitor, and the system (2.4) an activator inhibitor system. Clearly this is fulfilled if $3\alpha^2 - 5 > 0$.

3.2 The ODE model

In the **ODE** we will stady the stability of the system without using the laplacian operator Δ.

3.2.1 Local stability

$$\begin{cases} \frac{\partial u}{\partial t} = a - u - \frac{4uv}{1+u^2} \\ \frac{\partial v}{\partial t} = (\sigma b)\left(u - \frac{uv}{1+u^2}\right) \end{cases} ; \qquad x \in \Omega, t > 0. \tag{3.1}$$

The Jacobian matrix

We have

$$J(u,v) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.$$

Such as

$$\begin{aligned} f_u(u,v) &= -1 - \left(\frac{4v(1+u^2) - 2u(4uv)}{(1+u^2)^2}\right) \\ &= -1 - \frac{4u^2v + 4v - 8u^2v}{(1+u^2)^2} \\ &= -1 - \frac{-4u^2v + 4v}{(1+u^2)^2} \\ &= \frac{4v + 4u^2v}{(1+u^2)^2} - 1. \end{aligned}$$

$$f_v(u,v) = -\frac{4u}{1+u^2}.$$

$$\begin{aligned} g_u(u,v) &= \sigma b - \sigma b \frac{v(1+u^2) - 2u(uv)}{(1+u^2)^2} \\ &= \sigma b \left(1 - \frac{u^2 v - uv}{(1+u^2)^2}\right), \end{aligned}$$

$$g_v(u,v) = -\sigma b \frac{u}{1+u^2}.$$

Then

$$J(u,v) = \begin{pmatrix} \frac{4v+4u^2v}{(1+u^2)^2} - 1 & -\frac{4u}{1+u^2} \\ \sigma b \left(1 - \frac{u^2v-uv}{(1+u^2)^2}\right) & -\sigma b \frac{u}{1+u^2} \end{pmatrix}$$

$$J(u^*,v^*) = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & \frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix}. \tag{3.2}$$

The caracteristic equation

$$\begin{aligned} \det(J - \lambda I) &= \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} - \lambda & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} - \lambda \end{pmatrix} \\ &= \left(\frac{3\alpha^2-5}{1+\alpha^2} - \lambda\right)\left(\frac{2\sigma\alpha^2 b}{1+\alpha^2} - \lambda\right) - \left(-\frac{4\alpha}{1+\alpha^2}\right)\left(\frac{2\sigma\alpha^2 b}{1+\alpha^2}\right). \end{aligned} \tag{3.3}$$

The caracteristic equation is given by $\lambda^2 - \lambda T + D = 0$ where

$$\begin{aligned} T \; &: = \operatorname{Tr} J(u^*,v^*) = \left(\frac{3\alpha^2-5}{1+\alpha^2}\right) + \left(\frac{2\sigma\alpha^2 b}{1+\alpha^2}\right) = \frac{3\alpha^2 - 5 + \sigma\alpha b}{1+\alpha^2}, \\ D \; &: = \det J(u^*,v^*) = \left(\frac{3\alpha^2-5}{1+\alpha^2}\right)\left(\frac{2\sigma\alpha^2 b}{1+\alpha^2}\right) - \left(-\frac{4\alpha}{1+\alpha^2}\right)\left(\frac{2\sigma\alpha^2 b}{1+\alpha^2}\right) = \frac{5\sigma\alpha b}{1+\alpha^2}. \end{aligned}$$

- It easy to see that $\det J(u^*,v^*) > 0$ and, if we have $\frac{3\alpha^2-5}{\alpha} < \sigma b$, the Jacobian matrix $J(u^*,v^*)$ has eigenvalues with negative real parts since $\operatorname{Tr} J(u^*,v^*) < 0$. Then The equilibrium point is stable focus if $\frac{3\alpha^2-5}{\alpha} < \sigma b$, if not we have unstable focus

- If $0 < \frac{3\alpha^2-5}{\alpha} < \sigma b$, The stready state (u^*,v^*) is asymptotically stable if $\lambda_1 \geq a_{11}$ or $\lambda_1 < a_{11}$ and $0 < d = \frac{c}{b} < \bar{d}$.

- If $\sigma b > 13 + \frac{5}{\alpha} + 4\sqrt{10\alpha^2 + 10}$, (u^*,v^*) is a stable node.

- or $0 < \sigma b < 13 + \frac{5}{\alpha} - 4\sqrt{10\alpha^2 + 10}$ and $\alpha^2 < \frac{5}{3}$, (u^*, v^*) is a unstable node.

(These results are proved in ([10]))

3.2.2 Global stability

Let define

$$L(u,v) = (\delta b)(u-u^*)^2\frac{(u-2u^*)}{3} + 2(v-v^*)^2,$$

$$\dot{L}(u,v) = \frac{\partial L}{\partial u}(u,v).\frac{\sigma u}{\sigma t} + \frac{\partial L}{\partial v}(u,v).\frac{\partial v}{\partial t},$$

We note that

$$\frac{d}{du}\left[(u-u^*)^2\frac{(u-2u^*)}{3}\right] = u^2 - (u^*)^2. \tag{3.4}$$

For $(u(x,t), v(x,t))$ solution of (2.4) in $\Re$, take the following Lyapunov function

$$L(u(x,t), v(x,t)) = (\delta b)(u(x,t) - u^*(x,t))^2\frac{(u(x,t) - 2u^*(x,t))}{3} + 2(v(x,t) - v^*(x,t))^2.$$

Using equality (3.4) and the boundary condition for (u, v) the time derivative of $L(t)$ along the solutions of system (2.4) can be written as

$$\begin{aligned} L'(t) &= \left[(\sigma b)(u^2 - (u^*)^2)\frac{\partial u}{\partial t} + 4(v-v^*)\frac{\partial v}{\partial t}\right] \\ &= (\sigma b)\phi(u)[(u^2 - (u^*)^2)(f_a(u) - f_a(u^*)) - 4(v-v^*)], \end{aligned}$$

if $a^2 < 27$ $f_a(u)$ is a strictly decreasing function in $]0, a]$, so that, by the mean value theorem, for an appropriate γ between u and u^*

$$(u^2 - (u^*)^2(f_a(u) - f_a(u^*)) = (u+u^*)(u-u^*)^2 f_a'(\gamma) < 0, u \neq u^*.$$

Now suppose

$$27 < a^2 \leq \frac{125}{4}.$$

Easy calculations yield

$$f_a'(u) = -\frac{a}{u^2} + a - 2u, f_a''(u) = \frac{2a}{u^3} - 2.$$

It turns out that $f_a(u)$ is strictly decreasing in $]0, u^*]$ and in $[\frac{a}{2}, a]$. Moreover $\sqrt[3]{a}$ is the unique saddle point for $f_a(u)$ and $f_a'(\sqrt[3]{a}) = \sqrt[3]{a}, \sqrt[3]{a}\left(\sqrt[3]{a^2} - 3\right)$ increases with a. A comparison between the curves $v = \frac{a}{u^2}$ and $v = a - 2u$ shows that $f_a(u)$ has a local minimum point $\mathring{u}, u^* < u < \sqrt[3]{a}$, a local maximum point $\bar{u}$, $\sqrt[3]{a} < \bar{u} < \frac{a}{2}$, So that $f_a(u)$ is strictly decreasing in$]0, \mathring{u}]$ and in$]0, a]$ strictly increasing in$]\mathring{u}, \bar{u}[$. In addition $\bar{u}$ and $f_a(\bar{u})$ increase with respect to a. When $a^2 = \frac{125}{4}$, we get $\bar{u} = 2u^*$ and

$$f_a(\bar{u}) = f_a(2u^*) = \frac{3}{2}\left(1 + \frac{4a^2}{25}\right) = 9 = 4\left(1 + \frac{a^2}{25}\right) = f_a(u^*).$$

Inequality $u < u^*$ implies $(u - u^*)(f_a(u) - f_a(u^*)) < 0$.
Inequality $u > u^*$ implies $(u - u^*)(f_a(u) - f_a(u^*)) \leq 0$.

3.2.3 Bifurcation of the ODE model

For the reaction–diffusion Lengyel–Epstein system (2.4), the local system is an ordinary differential equation in form of

$$\begin{cases} \frac{\partial u}{\partial t} = a - u - \frac{4uv}{1+u^2} := F(u, v), \\ \frac{\partial v}{\partial t} = \sigma b(u - \frac{uv}{1+u^2}) := G(u, v). \end{cases} \tag{3.5}$$

The system (3.5) has a unique equilibrium point $(u^*, v^*) = (\alpha, 1 + \alpha^2)$, where $\alpha = \frac{a}{5}$. The Jacobian matrix of the system of (3.5) at (u^*, v^*) is

$$J = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det(J - \lambda I) &= \begin{vmatrix} \frac{3\alpha^2-5}{1+\alpha^2} - \lambda & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} - \lambda \end{vmatrix} \\ &= \left(\frac{3\alpha^2 - 5}{1 + \alpha^2} - \lambda\right) - \left(\frac{\sigma\alpha b}{1 + \alpha^2} - \lambda\right) - \left(\frac{2\sigma\alpha^2 b}{1 + \alpha^2}\right)\left(-\frac{4\alpha}{1 + \alpha^2}\right) = 0, \end{aligned}$$

it is given by $\lambda^2 - \lambda T + D = 0$, where

$$T := \operatorname{Tr} J = \frac{3\alpha^2 - 5 - \sigma\alpha b}{1 + \alpha^2}, \qquad D := \det J = \frac{5\sigma\alpha b}{1 + \alpha^2}.$$

As we know the system (3.5) is an activator–inhibitor system under the condition

$$3\alpha^2 - 5 > 0, \tag{H1}$$

since $F_u(u^*, v^*) > 0, G_v(u^*, v^*) < 0, F_v(u^*, v^*) < 0$ and $G_u(u^*, v^*) > 0$. It is clear that if

$$0 < 3\alpha^2 - 5 < \sigma\alpha b,$$

holds, then the equilibrium (u^*, v^*) of system (3.5) is locally asymptotically stable.Next we analyze the Hopf bifurcation occurring at (u^*, v^*) by choosing b as the bifurcation parameter. Denote

$$b_0 := \frac{3\alpha^2 - 5}{\sigma\alpha}.$$

Then when $b = b_0$, the Jacobian matrix J has a pair of imaginary eigenvalues

$$\lambda = \pm i\sqrt{\frac{5\sigma\alpha b_0}{1+\alpha^2}}.$$

Let $\lambda = \beta(b) \pm i\omega(b)$ be the roots of $\lambda^2 - \lambda T + D = 0$,then

$$\beta(b) = \frac{3\alpha^2 - 5 - \sigma\alpha b}{2(1+\alpha^2)}, \qquad \omega(b) = \frac{1}{2}\sqrt{\frac{20\sigma\alpha b}{1+\alpha^2} - \left(\frac{3\alpha^2 - 5 - \sigma\alpha b}{1+\alpha^2}\right)^2},$$

and

$$\beta'(b)\mid_{b=b_0} = \frac{\sigma\alpha}{2(1+\alpha^2)} < 0.$$

By the Poincaré–Andronov–Hopf Bifurcation Theorem, we know that system (3.5) undergoes a Hopf bifurcation at (u^*, v^*) when $b = b_0$. However, the detailed nature of the Hopf bifurcation needs further analysis of the normal form of the system. To that end we translate the equilibrium (u^*, v^*) to the origin bythe translation $u = u - u^*, v = v - v^*$. For the sake of convenience, we still denote u and v by u and v, respectively

Thus, the local system (3.5) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = 4\alpha - u - \frac{4(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}, \\ \frac{\partial v}{\partial t} = \sigma b(u + \alpha - \frac{(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}), \end{cases} \tag{3.6}$$

Rewrite system (3.6) to

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = J\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,b) \\ g(u,v,b) \end{pmatrix}, \tag{3.7}$$

By taylor's expansion

$$f(u,v,b) = \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2}u^2 + \frac{4(\alpha^2-1)}{(1+\alpha^2)^2}uv + \frac{4(\alpha^4 - 6\alpha^2 + 1)}{(1+\alpha^2)^3}u^3 + \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^3}u^2v + O\left(|u|^4, |u|^3|v|\right),$$

$$g(u,v,b) = \frac{\alpha b}{4} f(u,v,b).$$

Set matrix

$$P = \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix},$$

where

$$M = \frac{(1+\alpha^2)\sqrt{4D-T^2}}{8\alpha} \text{ and } N = \frac{3\alpha^2 - 5 + \sigma\alpha b}{8\alpha}.$$

Clearly

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{N}{M} & \frac{1}{M} \end{pmatrix},$$

and when

$$b = b_0, N_0 = N\mid_{b=b_0} = \frac{1}{4}\sigma b_0, M_0 = M\mid_{b=b_0} = \frac{\sqrt{5(1+\alpha^2)(3\alpha^2-5)}}{4\alpha} \qquad \omega(b_0) = \sqrt{\frac{5(3\alpha^2-5)}{1+\alpha^2}}.$$

By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix},$$

system (3.7) becomes

$$\begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = J(b) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x,y,b) \\ g^2(x,y,b) \end{pmatrix}, \tag{3.8}$$

here

$$J(b) = \begin{pmatrix} \beta(b) & -\omega(b) \\ \omega(b) & \beta(b) \end{pmatrix}.$$

$$F^1(x,y,b) = A_{20}x^2 + A_{11}xy + A_{21}x^2y + A_{30}x^3 + \varphi\left(|x|^4, |x|^3|y|\right),$$

where,

$$A_{20} = \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2}, \qquad A_{11} = \frac{4M(\alpha^2-1)}{(1+\alpha^2)^2},$$

$$A_{21} = \frac{4M\alpha(3-\alpha^2)}{(1+\alpha^2)^3}, \qquad A_{30} = \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2} + \frac{4N(\alpha^2-1)}{(1+\alpha^2)^2},$$

and

$$F^2(x,y,b) = \left(-\frac{N}{M} + \frac{1}{4M\sigma b}\right) F^1(x,y,b).$$

Rewrite (3.8) in the following polar coordinates form:

$$\dot{r} = \beta(b)r + a(b)r^3 + ...,$$

$$\dot{\theta} = \omega(b) + c(b)r^2 + ..., \tag{3.9}$$

then the Taylor expansion of (3.9) at $b = b_0$ yields

$$\dot{r} = \beta'(b_0)(b - b_0)r + a(b_0)r^3 + \varphi\left((b - b_0)^2 r, (b - b_0) r^3, r^5\right),$$
$$\dot{\varphi} = \omega(b_0) + \omega'(b_0)(b - b_0) + c(b_0)r^2 + \varphi((b - b_0)^2, (b - b_0)r^2, r^4).$$

In order to determine the stability of the periodic solution, we need to calculate the sign of the coefficient $a(b_0)$, which is given by

$$a(b_0) = \frac{1}{16}\left[F^1_{xxx} + F^1_{xyy} + F^2_{xxy} + F^2_{yyy}\right] + \frac{1}{16\omega(b_0)}[F^1_{xy}(F^1_{xx} + F^1_{yy}) - F^2_{xy}(F^2_{xx} + F^2_{yy}) - F^1_{xx}F^2_{xx} + F^1_{yy}F^2_{yy}],$$

where all partial derivatives are evaluated at the bifurcation point, i.e., $(x, y, b) = (0, 0, b_0)$.

Since the highest order of y in both $F^1(x, y, b)$ and $F^2(x, y, b)$ is less than 2, we have that

$$F^1_{xyy}(0, 0, b_0) = F^2_{yyy}(0, 0, b_0) = F^1_{yy}(0, 0, b_0) = F^2_{yy}(0, 0, b_0) \equiv 0.$$

On the other hand, it is easy to calculate that

$$F^2_{xxy}(0, 0, b_0) = F^2_{xx}(0, 0, b_0) = F^2_{xy}(0, 0, b_0) = 0.$$

Thus,

$$a(b_0) = \frac{1}{16}F^1_{xxx}(0, 0, b_0) + \frac{1}{16\omega(b_0)}F^1_{xy}(0, 0, b_0)F^1_{xx}(0, 0, b_0).$$

By tedious but simple calculations, we can obtain

$$a(b_0) = \frac{2\alpha^4 - 27\alpha^2 - 5}{2\alpha^2(1 + \alpha^2)},$$

which implies that $a(b_0) < 0$ if and only if $0 < \alpha^2 < \frac{(27+\sqrt{769})}{4}$.

Now from Poincaré–Andronov–Hopf Bifurcation Theorem, $\beta'(b_0) < 0$ and the above calculation of $a(b_0)$, we summarize our results as follows.

Theorem 10 *[6]Suppose that $\sigma, \alpha > 0$ so that (**H1**) is satisfied, and let $b_0 = \frac{3\alpha^2 - 5}{\sigma\alpha}$.*

1. *The equilibrium (u^*, v^*) of system (3.5) is locally asymptotically stable when $b > b_0$, and unstable when $b < b_0$;*
2. *The system (3.5) undergoes a Hopf bifurcation at (u^*, v^*) when $b = b_0$; the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are*

orbitally asymptotically stable if

$$\frac{5}{3} < \alpha^2 < \frac{27+\sqrt{769}}{4}; \tag{H2}$$

and the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable if

$$\alpha^2 > \frac{27+\sqrt{769}}{4}, \tag{H'2}$$

The discussion above shows that the local system (3.5) has periodic solutions arising from Hopf bifurcation. In the following we employ the Poincaré–Bendixson theorem to verify that the system (3.5) has periodic solutions.

Theorem 11 *Suppose that* $\sigma, \alpha > 0$ *so that* ***(H1)*** *is satisfied, and* $b < \frac{(3\alpha^2-5)}{\sigma\alpha}$. *Then system (3.5) has at least one stable periodic solution satisfying* $0 < u(t) < a$ *and* $0 < v(t) \leq 1+\varepsilon$ *for some* $\varepsilon > \frac{\alpha^2}{25}$.

Proof. Set $l_1 = \{(u, v : 0 \leq u \leq a, v = 0)\}$, $l_2 = \{(u,v) : u = a, 0 \leq v \leq 1+\varepsilon\}$, $l_3 = \{(u,v) : 0 \leq u \leq a, v = 1+\varepsilon\}$, and $l_4 = \{(u,v) : u = 0, 0 \leq v \leq 1+\varepsilon\}$,where $\varepsilon > \frac{\alpha^2}{25}$ is a constant. Let C denote the Jordan curve consisting of the linesegments l_1, l_2, l_3 and l_4, and D denote the interior of C. Then we have that $\frac{dv}{dt}|_{(u,v)\in l_1} > 0$, $\frac{du}{dt}|_{(u,v)\in l_2} < 0$, $\frac{dv}{dt}|_{(u,v)\in l_3} < 0$ and $\frac{du}{dt}|_{(u,v)\in l_4} > 0$. This implies that the trajectories starting at the boundary of D point inwards. Meanwhile, the unique positive equilibrium (u^*, v^*) is in the domain D, and is unstable. Hence, there exists at least a stable periodic solution which belongs to D from the Poincaré–Bendixson theorem. This completes the proof. ■

3.3 The PDE model

3.3.1 Local stability

We have the Lengyl Epstein system 2.4

The eigenvalues of Δ are λ_i such as $\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1}$.

We have

$$\begin{cases} f(u,v) = a - u - \frac{4uv}{1+u^2}, \\ g(u,v) = (\sigma b)\left(u - \frac{uv}{1+u^2}\right). \end{cases}$$

Then

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u,v), \\ \frac{\partial v}{\partial t} = (\sigma b)\Delta v + g(u,v). \end{cases}$$

The system had an equilibrium point $(u^*, v^*) = (\alpha, 1+\alpha^2), \alpha = \frac{a}{5}$.

$$J(u^*, v^*) = \begin{pmatrix} a_{11} & -a_{21} \\ a_{12} & -a_{22} \end{pmatrix}.$$

We say that this constant solution is Turing unstable if it is stable in the absence of diffusion, and it becomes unstable when diffusion is present. More precisely, this requires the following two conditions:

(i) It is stable as an equilibrium of the system of ordinary differential equations

$$\frac{d_u}{d_t} = f(u,v), \frac{d_v}{d_t} = (\sigma b)g(u,v). \tag{3.10}$$

(ii) It is unstable as a steady state of the reaction-diffusion equations (2.4) subject to the homogeneous Neumann boundary conditions. Using the ODE theory we can easily find a sufficient and necessary condition for.

The Jacobian matrix of (3.10) evaluated at (u^*, v^*) is

$$J = \begin{bmatrix} f_0 & f_1 \\ (\sigma b)g_0 & (\sigma b)g_1 \end{bmatrix},$$

with:

$$f_0 = \frac{3\alpha^2 - 5}{1+\alpha^2}, f_1 = -\frac{4\alpha}{1+\alpha^2}, g_0 = \frac{2\alpha}{1+\alpha^2}, g_1 = -\frac{\alpha}{1+\alpha^2}. \tag{3.11}$$

Then

$$\frac{\det(j)}{(\sigma b)} = \frac{3\alpha^2 - 5}{1+\alpha^2}.\frac{2\alpha}{1+\alpha^2} - (-\frac{4\alpha}{1+\alpha^2}.(-\frac{\alpha}{1+\alpha^2})) = \frac{5\alpha}{1+\alpha^2} > 0.$$

(we know $(\sigma b) > 0$)

$$\mathrm{Tr}(J) = f_0 + (\sigma b)g_1 = 3\alpha^2 - 5 < 0.$$

Lemma 12 *[15]Assume (H) holds. If $\lambda_1 \geqslant f_0$, or $\lambda_1 < f_0$ and $0 < d = \frac{c}{b} < \tilde{d}$, then the constant steady state (u^*, v^*) is asymptotically stable. If $\lambda_1 < f_0$ and $d > \tilde{d}$, then (u^*, v^*) is unstable.*

Proof. Consider the linearization operator

$$L_\sigma = \begin{pmatrix} \Delta + f_0 & f_1 \\ \sigma b g_0 & \sigma c\Delta + \sigma b g_1 \end{pmatrix},$$

where f_0, f_1, g_0 and g_1 are given in (3.11). From the standard linear operator theory, it is known that if all the eigenvalues of this operator have negative real parts, then (u^*, v^*)

is asymptotically stable, and if some eigenvalues have positive real parts, the (u^*, v^*) is unstable.

Suppose $(\Phi(x), \Psi(x))$ is an eigenfunction of L_σ corresponding to an eigenvalue μ. Then

$$(\Delta\Phi + (f_0 - \mu)\,\Phi + f_1\Psi, \sigma c\Delta\Psi + \sigma b g_0 \Phi + (\sigma b g_1 - \mu)\,\Psi) = (0,0)\,.$$

Letting $\Phi = \sum_{0\leqslant i\leqslant\infty,\ 1\leqslant j\leqslant m} a_{ij}\Phi_{ij}$ and $\Psi = \sum_{0\leqslant i\leqslant\infty,\ 1\leqslant j\leqslant m} b_{ij}\Phi_{ij}$,
we find that

$$\sum_{0\leqslant i\leqslant\infty,\ 1\leqslant j\leqslant m} \begin{pmatrix} f_0 - \lambda_i - \mu & f_1 \\ \sigma b g_0 & \sigma b g_1 - \sigma c\lambda_i - \mu \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \Phi_{ij} = 0.$$

It readily follows that μ is an eigenvalue of L_σ if and only if for some $i \geqslant 0$ the determinant of the matrix is zero, that is,

$$\mu^2 + P_i\mu + \sigma b Q_i = 0,$$

where

$$P_i = (1+\sigma c)\,\lambda_i - (f_0 + \sigma b g_1) = (1+\sigma c)\,\lambda_i - \frac{3\alpha^2 - 5 - \sigma\alpha b}{1+\alpha^2} > 0,$$

by (3.11) and condition (H), and

$$Q_i = (f_0 - \lambda_i)\,(g_1 - d\lambda_i) - f_1 g_0 = d\lambda_i\,(\lambda_i - f_0) + \frac{\alpha}{1+\alpha^2}\,(\lambda_i + 5)\,, \tag{3.12}$$

since $c = bd$ and $f_1 g_0 = \frac{5\alpha}{1+\alpha^2}$. Clearly, $Q_0 > 0$ for $\lambda_0 = 0$. Now, if $\lambda_1 \geqslant f_0$, then (5.5) $Q_i > 0$ for all $i \geqslant 1$. This implies that $\operatorname{Re}\mu < 0$ for all eigenvalues μ, and so the steady-state (u^*, v^*) is asymptotically stable.

If $\lambda_1 < f_0$ and $0 < d < \tilde{d}$, then

$$\lambda_i < f_0 \qquad and \qquad d < d_i, i \in [1, i_\alpha]\,.$$

It follows that $Q_i > 0$ for all $i \in [1, i_\alpha]$. Furthermore, if $i > i_\alpha$, then $\lambda_1 \geqslant f_0$ and $Q_i > 0$ by (5.5). The argument leads to the asymptotical stability of (u^*, v^*) again.

Finally, if $\lambda_1 < f_0$, and $d > \tilde{d}$ then we may assume that the minimum in (5.4) is attained by $k \in [1, i_\alpha]$. Thus

$$d > d_k,$$

which implies $Q_i < 0$, and so the instability of (u^*, v^*) follows. ■

Theorem 13 *[15]If $3\alpha^2 - 5 \leq 0$, then (u^*, v^*) is asymptotically stable for system (2.4). Assuming*

$$0 < \frac{3\alpha^2 - 5}{\alpha} \leq \sigma b,$$

then (u^*, v^*) is asymptotically stable if
(a) $\lambda_1 \geq \frac{3\alpha^2-5}{1+\alpha^2}$,
(b) $\lambda_1 < \frac{3\alpha^2-5}{1+\alpha^2}$ or $\frac{c}{b} < D$.
where D is the solution of the equation

$$((3\alpha^2 - 5)x + \alpha)^2 = 32\alpha^3 x$$

such that

$$D > \frac{\alpha}{3\alpha^2 - 5},$$

Proof. Rewrite system (1.1) in the vectorial form

$$\frac{\partial z}{\partial t} = D\Delta z + F(z),$$

where

$$z = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & (\sigma b) \end{pmatrix}, \qquad F(z) = \begin{pmatrix} a - u - \frac{4uv}{1+u^2} \\ (\sigma b)(u - \frac{uv}{1+u^2}) \end{pmatrix}.$$

As proved in [4], the solution (u^*, v^*) is asymptotically stable for (3.7) if $z = 0$ is asymptotically stable for the linearized system

$$\frac{\partial z}{\partial t} = D\Delta z + Az,$$

where

$$A = J(u^*, v^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

is defined by (2.3). By Theorem 1 [4], the zero solution is globally asymptotically stable for (3.8) if, for each non-negative integer n, the eigenvalues of matrix $A - \lambda_n D$ have negative real parts.

If $3\alpha^2 - 5 \leq 0$, it easy to see that, for each n, the matrix $A - \lambda_n D$ has positive determinant and negative trace. As a consequence, all the eigenvalues of $A - \lambda_n D$ have negative real parts. Now suppose

$$0 < \frac{3\alpha^2 - 5}{\alpha} \leq \sigma b,$$

so that $a_{11} > 0$. For eigenvalue $\lambda_0 = 0.A - \lambda_n D$ and it is known that

$$\det A > 0, trA > 0.$$

Next suppose $\lambda_1 \geq a_{11}$.Then again

$$\det(A - \lambda_n D) > 0 \quad \text{and} \quad \text{Tr}(A - \lambda_n D) < 0 \quad \text{for each} \quad n \geq 1,$$

because $\lambda_n \geq a_{11}$, too. Finnaly assume
$\lambda_1 < a_{11}$

For each eigenvalue $\lambda_n, n > 1$, such that $\lambda_n \geq a_{11}$, as before, $A - \lambda_n D$ has eigenvalues with negative real parts. Let λ be one of the eigenvalues less than a_{11}. We see that

$$\text{Tr}(A - \lambda D) = (a_{11} - \lambda) + (-a_{22} - (\sigma c)\lambda) < a_{11} - a_{22} = \text{Tr}\, A < 0$$

and

$$\det(A - \lambda D) = (\sigma c)\lambda^2 - ((\sigma c)a_{11} - a_{22})\lambda + (a_{12}a_{21} - a_{11}a_{22}).$$

Since $a_{12}a_{21} - a_{11}a_{22} - \det A > 0$,if

$$\frac{c}{b} \leq \frac{\alpha}{3\alpha^2 - 5},$$

one has $(\sigma c)a_{11} \leq a_{22}$ so that $\det(A - \lambda_n D)$ is positive, next assume

$$\frac{c}{b} \leq \frac{\alpha}{3\alpha^2 - 5}.$$

Notice that the trinomial

$$(\sigma c)\lambda^2 - ((\sigma c)a_{11} - a_{22})\lambda + (a_{12}a_{21} - a_{11}a_{22}).$$

is positive if its discriminant is negative, that is

$$((\sigma c)a_{11} - a_{22})^2 < 4(\sigma c)a_{12} - a_{21}.$$

Dividing both sides by $\frac{\sigma^2}{b(1+\alpha^2)^2}$, previous inequality turns into

$$((3\alpha^2 - 5)\frac{c}{b} + \alpha)^2 < 32\frac{c}{b}\alpha^3.$$

By a comparison, in $[0; +\infty[$, between the parabola $y = ((3\alpha^2 - 5)x + \alpha)^2$ and the line $y = 32\alpha^3 x$, it easy to see that, at point $\bar{x} = \frac{\alpha}{3\alpha^2 - 5}$, we have $((3\alpha^2 - 5)\bar{x} + \alpha)^2 < 32\alpha^3\bar{x}$

and the line intersects the parabolic curve at two pointsx_1, x_2 such that $0 < x_1 < \bar{x} < x_2$. Setting $D = x_2$, we obtain that D is the solution of Eq. (3.6) satisfying$D > \frac{\alpha}{3\alpha^2-5}$.In addition, the inequality

$$((3\alpha^2 - 5)x - \alpha)^2 < 32\alpha^3 x,$$

holds for $\frac{\alpha}{3\alpha^2-5} < x < D$. We conclude that $\det(A - \lambda D)$ is positive if

$$\frac{c}{b} \leq \frac{\alpha}{3\alpha^2 - 5} \quad \text{or} \quad \frac{\alpha}{3\alpha^2 - 5} < \frac{c}{b} < D.$$

The poof is complete. ■

3.3.2 Global stability

The systeme 2.4 can be written in the form

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \phi(u) \left[\left(f_a\left(u\right) - f_a\left(u^*\right) \right) - 4\left(v - v^*\right) \right] x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = \left(\sigma c\right) \Delta v + \left(\sigma b\right) \phi\left(u\right) \left[\left(u + u^*\right)\left(u - u^*\right) - \left(v + v^*\right) \right], \end{cases} \tag{3.13}$$

where

$$\phi(u) = \frac{u}{1+u^2}, f_a(u) = \frac{a-u}{\phi(u)}, f_a(u^*) = 4v^*.$$

Now as we have the Lyapanuv's fuction

$$E(u,v) = (\sigma b)(u-u^*)^2\frac{(u-2u^*)}{3} + 2(v-v^*)^2,$$

then

$$\frac{dE}{dt}(u,v) = \frac{\partial E}{\partial u}(u,v).\frac{\sigma u}{\sigma t} + \frac{\partial E}{\partial v}(u,v).\frac{\partial v}{\partial t},$$

we have

$$\frac{d}{du}\left[(u-u^*)^2\frac{(u-2u^*)}{3}\right] = u^2 - (u^*)^2, \tag{3.14}$$

then

$$\frac{dE}{dt}(u,v) = (\sigma b)(u^2 - (u^*)^2)\frac{\partial u}{\partial t} + 4(v-v^*)^2\frac{\partial v}{\partial t}.$$

For $(u(x,t), v(x,t))$ solution of (2.4) in R_a, take the following Lyapunov function

$$L(t) = \int\limits_{\Omega} E(u(x,t), v(x,t))dx.$$

Using equality (3.14) and the boundary condition for (u,v) the time derivative of $L(t)$

along the solutions of system (3.13) can be written as

$$\begin{aligned} L'(t) &= \int_\Omega \left[(\sigma b)(u^2-(u^*)^2)\frac{\partial u}{\partial t}+4(v-v^*)\frac{\partial v}{\partial t}\right]dx. \\ L'(t) &= \int_\Omega [\left((\sigma b)(u^2-(u^*)^2\right)(\Delta u+\phi(u)\left[(f_a(u)-f_a(u^*))-4(v-v^*)\right]\right)) \\ &\quad +(4(v-v^*)\left((\sigma c)\Delta v+(\sigma b)\phi(u)\left[(u+u^*)(u-u^*)-(v+v^*)\right]\right))]dx. \end{aligned}$$

According to Green formula we find

$$L'(t)=(\sigma b)\int_\Omega \phi(u)[(u^2-(u^*)^2)(f_a(u)-f_a(u^*))-4(v-v^*)]dx-(\sigma b)\int_\Omega 2u\left|\nabla u\right|^2 dx-4(\sigma c)\int_\Omega \left|\nabla v\right|^2 dx,$$

where ∇ is the gradient with respect to the spatial variable x. If $a^2<27$ $f_a(u)$ is a strictly decreasing function in $]0,a]$, so that, by the mean value theorem, for an appropriate γ between u and u^*

$$(u^2-(u^*)^2(f_a(u)-f_a(u^*))=(u+u^*)(u-u^*)^2 f_a'(\gamma)<0, u\neq u^*.$$

Now suppose

$$27<a^2\leq\frac{125}{4}.$$

Easy calculations yield

$$f'(u)=-\frac{a}{u^2}+a-2u, f''(u)=\frac{2a}{u^3}-2.$$

It turns out that $f_a(u)$ is strictly decreasing in $]0,u^*]$ and in $\left[\frac{a}{2},a\right]$. Moreover $\sqrt[3]{a}$ is the unique saddle point for $f_a(u)$ and $f_a'(\sqrt[3]{a})=\sqrt[3]{a}, \sqrt[3]{a}\left(\sqrt[3]{a^2}-3\right)$ increases with a. A comparison between the curves $v=\frac{a}{u^2}$ and $v=a-2u$ shows that $f_a(u)$ has a local minimum point $\mathring{u}, u^*<u<\sqrt[3]{a}$, a local maximum point $\bar{u}, \sqrt[3]{a}<\bar{u}<\frac{a}{2}$, So that $f_a(u)$ is strictly decreasing in$]0,\mathring{u}]$ and in$]0,a]$ strictly increasing in$]\mathring{u},\bar{u}[$. In addition $\bar{u}$ and $f_a(\bar{u})$ increase with respect to a. When $a^2=\frac{125}{4}$, we get $\bar{u}=2u^*$ and

$$f_a(\bar{u})=f_a(2u^*)=\frac{3}{2}\left(1+\frac{4a^2}{25}\right)=9=4\left(1+\frac{a^2}{25}\right)=f_a(u^*).$$

Inequality $u<u^*$ implies $(u-u^*)(f_a(u)-f_a(u^*))<0$.

Inequality $u>u^*$ implies $(u-u^*)(f_a(u)-f_a(u^*))\leq 0$.

3.3.3 Bifurcation of the PDE model

In this part, we will derive conditions for the diffusion-driven instability with respect to the equilibrium solution, the spatially homogenous solution of the reaction–diffusion Lengyel–Epstein system. Such diffusion-driven instability for the equilibrium solution (u^*, v^*) has been investigated in [15], and here we derive only the special case when $\Omega = (0, \pi)$ for completeness of our analysis.

Consider the following system with the no-flux boundary condition in a one-dimensional "cube" $(0, \pi)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a - u - \frac{4uv}{1+u^2}, \\ \frac{\partial v}{\partial t} = \sigma(c\frac{\partial^2 u}{\partial x^2} + b(u - \frac{uv}{1+u^2})), \\ u_x(0,t) = u_x(\pi,t) = 0, \\ v_x(0,t) = v_x(\pi,t) = 0. \end{cases} \tag{3.15}$$

It is well known that the operator $u \to -u_{xx}$ with the above no-flux boundary condition has eigenvalues and eigenfunctions as follows:

$$\mu_0 = 0, \phi_0(x) = \sqrt{\frac{1}{\pi}}, \mu_k = k^2, \phi_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx), k = 1, 2, 3...$$

The linearized system of Eq. (3.15) at $(\alpha, 1+\alpha^2)$ has the form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} := D \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + J \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.16}$$

where

$$D := \begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix}, J := \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix},$$

with domain

$$\left\{(u,v) \in H^2\left[(0,\pi)\right] \times H^2\left[(0,\pi)\right] : u_x(0,t) = u_x(\pi,t), v_x(0,t) = v_x(\pi,t) = 0\right\},$$

where the $H^2[(0,\pi)]$ is the standard Sobolev space.

From the standard linear operator theory, it is known that if all the eigenvalues of the operator L have negative real parts, then (u^*, v^*) is asymptotically stable, and if some eigenvalues have positive real parts, the (u^*, v^*) is unstable.

We consider the following characteristic equation of the operator L:

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Let $(\phi(x), \psi(x))^T$ be an eigenfunction of L corresponding to the eigenvalue μ, and let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx, \tag{3.17}$$

where a_k and b_k are coefficients, we obtain that

$$-k^2 D \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx + J \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx = \mu \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx.$$

Hence,

$$(J - k^2 D) \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \mu \begin{pmatrix} a_k \\ b_k \end{pmatrix} (k = 1, 2, 3...).$$

Denote

$$J_k = J - k^2 D = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} - k^2 & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} - k^2\sigma\alpha \end{pmatrix} (k = 1, 2, 3...).$$

It follows from this, that the eigenvalues of L are given by the eigenvalues of J_k for $k = 0, 1, 2, ...$ The characteristic equation of J_k obtain from

$$\det(J_k - \mu I) = \begin{vmatrix} \frac{3\alpha^2-5}{1+\alpha^2} - k^2 - \mu & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} - k^2\sigma\alpha - \mu \end{vmatrix},$$

we obtain

$$\mu^2 - \mu T_k + D_k = 0, k = 1, 2, ..., \tag{3.18}$$

where

$$\begin{aligned} T_k &= \text{Tr}\, J_k = -k^2(1 + \sigma c) + \frac{3\alpha^2 - 5 - \sigma\alpha b}{1 + \alpha^2}, \\ D_k &= \det J_k = \sigma c k^4 + \sigma \left(\frac{\alpha b}{1 + \alpha^2} - \frac{(3\alpha^2 - 5)c}{1 + \alpha^2} \right) k^2 + \frac{5\sigma\alpha b}{1 + \alpha^2}. \end{aligned}$$

By analyzing the distribution of the roots of Eq. (3.18), we can obtain the following conclusions.

Theorem 14 *Suppose that* $b > b_0 = \frac{3\alpha^2-5}{\sigma\alpha}$ *so that* (u^*, v^*) *is a locally asymptotically stable equilibrium for (3.5). Then* (u^*, v^*) *is an unstable equilibrium solution of (3.15) if*

$$\alpha^3 > 3 \text{ and } c > \frac{3\alpha b}{\alpha^2 - 3}; \tag{H3}$$

and (u^*, v^*) *is a locally asymptotically stable equilibrium solution of (3.15) if*

$$\frac{5}{3} < \alpha^2 \le 3, \tag{H4}$$

or

$$\alpha^2 > 3 \text{ and } 0 < c < \frac{3\alpha b}{\alpha^2 - 3}. \tag{H5}$$

Proof. For convenience, we rewrite D_k as

$$D_k = \sigma c k^2 \left(k^2 - \frac{3\alpha^5 - 5}{1 + \alpha^2} \right) + \frac{\sigma \alpha b}{1 + \alpha^2} k^2 + \frac{5\sigma \alpha b}{1 + \alpha^2}.$$

Clearly, $D_1 < 0$ follows from (H3). This implies that Eq (3.18) has at least one root with positive real part. Hence (u^*, v^*) is an unstable equilibrium solution of (3.15). We have $D_{k+1} > D_k$ for $k \ge 0$. (H4) implies that $\frac{1-(3\alpha^2-5)}{(1+\alpha^2)} \ge 0$, and hence $D_1 > 0$. Thus $D_k > 0$ for $k \ge 1$. Meanwhile, we know that $T_{k+1} < T_k$ for $k \ge 0$ from the definition of T_k. This and $T_0 < 0$ leads to that all the roots of Eq. (3.18) have negative real parts. Therefore, (u^*, v^*) is a locally asymptotically stable equilibrium solution of (3.15). Similarly, we can obtain that (H5) ensure that all the roots of Eq. (3.18) have negative real parts, and hence the conclusion follows. This completes the proof. ■

Stability of spatial homogeneous periodic solution: bounded spatial domain

A periodic solution $\phi(t)$ of (2.4) is also a (spatially homogeneous) periodic solution of (3.15). Thus, (3.15) also possess any periodic solution as (2.4), including the ones from Hopf bifurcation in Theorem. A Hopf bifurcation analysis can also be performed for the partial differential equation (3.15) at the same bifurcation point, but from the local uniqueness of periodic solutions near Hopf bifurcation point, only spatial homogeneous periodic solutions exist near $b = b_0 = \frac{(3\alpha^2-5)}{\sigma\alpha}$. However, the stability of these periodic solutions with respect to (3.15) could be different from that for (2.4). First if $\phi(t)$ is an unstable periodic solution of (2.4), then it is clearly also unstable for (3.15); second if (u^*, v^*) is an unstable equilibrium solution of (3.15) but stable for (2.4), then the nearby bifurcating periodic solutions through Hopf bifurcation are also unstable. The latter case illustrates the interaction of Hopf instability and Turing instability. For fixed $b > b_0, (u^*, v^*)$ is a stable equilibrium point for the **ODE** (2.4), but for $\alpha^2 > 3$ and the diffusion coefficient $c > \frac{3\alpha b}{\alpha^2-3}$, it becomes unstable for (3.15) through Turing instability, now we decreases b so that $b < b_0$, a Hopf bifurcation occurs, but it is not destabilizing but causes additional instability.

Our main result in this section is that if (H4) or (H5) holds, and the bifurcating periodic solution is stable with respect to (2.4), then it is also stable with respect to (3.15).

Theorem 15 ***(Th6)****Suppose that $\sigma, \alpha > 0$ so that (H1) is satisfied, and let $b_0 = \frac{3\alpha^2-5}{\sigma\alpha}$. Then the system (3.15) undergoes a Hopf bifurcation at (u^*, v^*) when $b = b_0$.*

1. *If (H'2) is satisfied, then the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable.*
2. *If (H3) is satisfied, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable.*
3. *If (H4) or (H5) is satisfied, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable.*

We only need to prove part 3 of the theorem.We use the normal form method and center manifold to

study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. Let L^* be the conjugate operator of L:

$$L^* \begin{pmatrix} u \\ v \end{pmatrix} = D \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + J^* \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$J^* = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & \frac{2\sigma\alpha^2 b}{1+\alpha^2} \\ -\frac{4\alpha}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix},$$

whith domain

$$\left\{(u,v) \in H^2[(0,\pi)] \times H^2\left[(0,\pi)\right] | u_x(0,t) = u_x(\pi,t) = 0, v_x(0,t) = (\pi,t) = 0\right\}.$$

Let

$$q = \begin{pmatrix} 1 \\ \frac{3\alpha^2-5}{4\alpha} - \frac{\omega(b_0)(1+\alpha^2)}{4\alpha} i \end{pmatrix}, q^* = \frac{2\alpha}{\omega(b_0)(1+\alpha^2)\pi} \begin{pmatrix} \frac{\omega(b_0)(1+\alpha^2)}{4\alpha} + \frac{3\alpha^2-5}{4\alpha} & i \\ -i \end{pmatrix}.$$

It is easy to see that $\langle L^* a, b\rangle = \langle a, Lb\rangle$ for any $a \in D_{L^*}, b \in D_L$, and $L^* q^* = -i\omega_0 q^*, Lq = i\omega_0 q, \langle q^*, q\rangle = 1, \langle q^*, \bar{q}\rangle = 0$. Here $\langle a, b\rangle = \int_{(0,\pi)} \bar{a}^T b dx$ denotes the inner product in $L^2[(0,\pi)] \times L^2[(0,\pi)]$. Write

$$(u,v)^T = zq + \overline{zq} + w; z = \left\langle q^*, (u,v)^T \right\rangle.$$

Thus,

$$\begin{aligned} u &= z+\bar{z}+w_1, \\ v &= z\left(\frac{3\alpha^2-5}{4\alpha}-\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}i\right)+\bar{z}\left(\frac{3\alpha^2-5}{4\alpha}+\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}i\right)+w_2. \end{aligned}$$

Our system in (z,w) coordinates becomes

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z+\left\langle q^*,\tilde{f}\right\rangle, \\ \frac{dw}{dt} &= Lw+\left[\tilde{f}-\left\langle q^*,\tilde{f}\right\rangle q-\left\langle \bar{q}^*,\tilde{f}\right\rangle \bar{q}\right], \end{aligned}$$

with $\tilde{f}=(f,g)^T$, where f and g are defined as (3.16). Straightforward but tedious calculations show that

$$\begin{aligned} \left\langle q^*,\tilde{f}\right\rangle &= \frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\left\{\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f-\frac{3\alpha^2-5}{4\alpha}fi-gi\right\}=\frac{f}{2}, \\ \left\langle \bar{q}^*,\tilde{f}\right\rangle &= \frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\left\{\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f+\frac{3\alpha^2-5}{4\alpha}fi-gi\right\}=\frac{f}{2}, \\ \left\langle q^*,\tilde{f}\right\rangle q &= \frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix} \frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f \\ \left(\frac{3\alpha^2-5}{4\alpha}-\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}i\right)\left(\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f\right) \end{pmatrix}, \\ \left\langle \bar{q}^*,\tilde{f}\right\rangle \bar{q} &= \frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix} \frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f \\ \left(\frac{3\alpha^2-5}{4\alpha}+\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}i\right)\left(\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f\right) \end{pmatrix}, \\ \left\langle q^*,\tilde{f}\right\rangle q+\left\langle \bar{q}^*,\tilde{f}\right\rangle \bar{q} &= \frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix} \frac{\omega(b_0)(1+\alpha^2)}{2\alpha}f \\ \frac{\omega(b_0)(1+\alpha^2)}{2\alpha}g \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \end{aligned}$$

$$H(z,\bar{z},w)=\bar{f}-\left\langle q^*,\tilde{f}\right\rangle q-\left\langle \bar{q}^*,\tilde{f}\right\rangle \bar{q}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Write $w=(\frac{w_{20}}{2})z^2+w_{11}z\bar{z}+(\frac{w_{02}}{2})\bar{z}^2+\theta(|z|^3)$ for the equation of the center manifold, we can obtain:

$$(2i\omega_0-L)\omega_{20}=0,(-L)\omega_{11}=0 \text{ and } \omega_{02}=\bar{\omega}_{20}.$$

This implies that $\omega_{20}=\omega_{02}=\omega_{11}=0$. Thus, the equation on the center manifold in $z,\bar{z}$ coordinates now is

$$\frac{dz}{dt}=i\omega_0 z+\frac{1}{2}g_{20}z^2+g_{11}z\bar{z}+\frac{1}{2}g_{02}\bar{z}^2+\frac{1}{2}g_{21}z^2\bar{z}+\theta\left(|z|^4\right),$$

among of which,

$$
\begin{aligned}
g_{20} &= \frac{1}{2}\left[B_{20} + 2B_{11}q_2\right], \\
g_{11} &= \frac{1}{2}\left[B_{20} + B_{11}\bar{q}_2 + B_{11}q_2\right], \\
g_{02} &= \frac{1}{2}\left[B_{20} + 2B_{11}\bar{q}_2\right], \\
g_{21} &= \frac{1}{2}\left[B_{30} + B_{21}\bar{q}_2 + 2B_{21}q_2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
B_{20} &= \frac{\partial^2 f}{\partial u^2}(0,0) = \frac{8\alpha\left(3-\alpha^2\right)}{\left(1+\alpha^2\right)^2}, \\
B_{11} &= \frac{\partial^2 f}{\partial u \partial v}(0,0) = \frac{4(\alpha^2-1)}{\left(1+\alpha^2\right)^2}, \\
B_{30} &= \frac{\partial^3 f}{\partial u^3}(0,0) = \frac{24(\alpha^4-6\alpha^2+1)}{\left(1+\alpha^2\right)^3}, \\
B_{21} &= \frac{\partial^3 f}{\partial u^2 \partial v}(0,0) = \frac{8\alpha(3-\alpha^2)}{\left(1+\alpha^2\right)^3}, \\
q_2 &= \tau + \rho i = \frac{3\alpha^2-5}{4\alpha} - \frac{\omega(b_0)\left(1+\alpha^2\right)}{4\alpha} i.
\end{aligned}
$$

Accordin to [8],

$$
c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{02}\right|^2\right) + \frac{g_{21}}{2}.
$$

Then,

$$
\begin{aligned}
\operatorname{Re} c_1(0) &= \operatorname{Re}\left\{\frac{i}{2\omega_0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{02}\right|^2\right) + \frac{g_{21}}{2}\right\} \\
&= \operatorname{Re}\left\{\frac{i}{2\omega_0}g_{20}g_{11} + \frac{g_{21}}{2}\right\}.
\end{aligned}
$$

Since $g_{20}g_{11} = \frac{1}{4}\left(B_{20} + 2B_{11}\tau\right)^2 + \frac{1}{2}B_{11}\rho\left(B_{20} + 2B_{11}\tau\right)i$, we can get that

$$
\begin{aligned}
\operatorname{Re} c_1(0) &= \frac{B_{11}\left(B_{20} + 2B_{11}\tau\right)\left(1+\alpha^2\right)}{16\alpha} + \frac{1}{4}\left(B_{30} + 3B_{21}\tau\right) \\
&= \frac{2\alpha^4 - 27\alpha^2 - 5}{2\alpha^2\left(1+\alpha^2\right)^2}.
\end{aligned}
$$

Obvious $\operatorname{Re} c_1(0) < 0$ if and only if (H2) holds. When (H2) holds, either (H3), (H4) or (H5) is satisfied, but when (H3) is satisfied, the equilibrium is unstable with respect to (3.15), thus the bifurcating periodic solutions are also unstable; when (H4) or (H5) is satisfied, (u^*, v^*) is stable with respect to (3.15) thus the above analysis implies the stability of the periodic solutions.

3.4 Conclusion

The Lengyel-Epstien's system **(CIMA)** is an interactive chemical system in whitch reduce 4 interactive equations of 4 reactors **(chlorite–iodide–malonic acid)** to only 2 interactive equations of 2 reactors (the activitor iodide(I^-) and the inhibitor chlorite (ClO_2^-) with 2 parameters (a and b). We were exposed to the local and global stability and the bifurcation of this system. This investigation allows us to define behavior of the reactors in a certain time t and a certain spot x.

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