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Thème

Symbolic dynamics

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بسم الله الرحمن الرحيم

To my eternal example, my moral support and source of joy and happiness

my father

Abd Elmadjid, may God have mercy on him

In the light of my days,

the source of my efforts,

my mother .

To my dear sisters

and To my brothers

and to my angel Aya

To my loyal friends Roumaissa Boudiba,

Hadjer, Rihab, Lamia, Sonia.

To the people who have always helped and encouraged me,

who were

always by my side.

Thanks

First of all thanks to ALLAH for giving me the strength to finish this work.

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I address my thanks to all the teachers in the Mathematics department.

الملخص

في هذه المذكرة تم معالجة "الحركية الرمزية" حيث تمحورت دراستنا على تقديم مدخل حول الديناميكية الرمزية وتقديم أبرز التعاريف والخصائص المهمة فيها. في الفصل الأول حاولنا التذكير بمعظم الخواص والتعاريف الرياضية المستعملة كما قمنا بتقديم الحركية الرمزية وكيفية نمذجة أنظمة أخرى باستعمال "تقسيم ماركوف" كما عرفنا "الأنتروبي الطوبولوجي" و"الأنتروبي في الأنظمة الرمزية". أما في الفصل الثاني فقد تناولنا الحركية الرمزية في التطبيقات الخطية بالأجزاء مع تقديم أمثلة لذلك ألا وهي :

Sigma-Delta modulators , Digital filters with overflow nonlinearity ،

و في الفصل الأخير قدمنا الحركية الرمزية لبعض سلاسل الأعداد الأولية وذلك بتقديم معارف تساهم في تطبيق الديناميكية الرمزية في الأعداد الأولية على الخصوص متتالية الفرق بين عددين أوليين متتاليين وكذلك استعمال مخطط **Ulam**.

Résumée

Dans cette mémoire, la «dynamisme symbolique» a été traité. Notre étude visait à fournir une introduction sur la dynamique symbolique et à présenter les définitions les plus importantes et les caractéristiques importantes qu'elle contient. Dans le premier chapitre, nous avons tenté de rappeler la plupart des propriétés mathématiques et des définitions utilisées. Nous avons également présenté la dynamique symbolique et comment modéliser d'autres systèmes en utilisant des «partitions de Markov» telles que nous avons défini «l'entropie topologique» et «l'entropie dans les systèmes symboliques». Dans le deuxième chapitre, nous avons traité la dynamique symbolique des cartes linéaires par morceaux avec des exemples:
Filtres numériques avec non-linéarité de débordement et **modulateurs Sigma-Delta.**

Dans le dernier chapitre, nous avons présenté la dynamique symbolique de certaines séquences de

nombres premiers en apportant des connaissances qui contribuent à l'application de la dynamique symbolique aux nombres premiers en particulier, la séquence des différences entre deux nombres premiers consécutifs, ainsi que l'utilisation du diagramme d'**Ulam** .

Abstract

In this thesis, "symbolic dynamics" was dealt with. Our study focused on providing an introduction on symbolic dynamics and presenting the most important definitions and important characteristics in it. In the first chapter, we tried to recall most of the mathematical properties and definitions used. We also presented symbolic dynamics and how to model other systems using “Markov partitions” as we defined “topological entropy” and “entropy in symbolic systems”. In the second chapter, we dealt with the symbolic dynamics of piecewise-linear maps with examples:
Digital filters with overflow nonlinearity and **Sigma-Delta modulators**.

In the last chapter, we have presented the symbolic dynamics of some prime number sequences by providing knowledge that contributes to the application of symbolic dynamics in prime numbers in particular, the difference sequence between two consecutive prime numbers, as well as the use of the **Ulam** diagram.

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General introduction

Dynamical systems and chaos have been a topic of great importance in last few years. we will study the sequence of the type $x_{n+1} = f(x_n)$, to analyse the nature of its dynamics. There are many methods and one of them is the *symbolic dynamics* which determine the nature of the points (periodic, chaotic,....).

Symbolic dynamics began when Jacques Hadamard applied this idea in 1898 to more complicated systems called *geodesic flows on surfaces of negative curvature*. So, it is one of the most powerful tool for understanding the chaotic behavior of a dynamical system. Symbolic dynamics arose as an attempt to study general dynamical systems by means of *discretizing space* as well as time. The basic idea is to divide the set of possible states into a finite number of pieces. Each piece is associated with a *symbol* and in this way the evolution of the system is described by an *infinite sequence of symbols*. This leads to a *symbolic* dynamical system that mirrors and helps us to understand the dynamical behavior of the original system.

Finally, we will give symbolic dynamics and some important elements to explain it.

My thesis is made up of three chapters:

1. The first chapter is devoted to the necessary tools and notions of dynamical systems and it is presented a specific type of dynamical systems which are symbolic dynamics and how they can model other systems by using Markov partitions.

2. In chapter two, we give Symbolic dynamics of piecewise-linear maps and some applications.

3. In the third chapter, we study the symbolic dynamics of some prime number sequences and show some its properties.

Chapter 1

Symbolic dynamics

In this chapter we will first present the notion of dynamical systems along some definitions to help us determine what properties to focus on some examples of dynamical systems. Then, we look at a specific type of dynamical systems which are symbolic dynamics and how they can model other systems using *Markov partitions*. We end with *topological entropy* and entropy on symbolic systems.

1.1 Prelimanery concepts

Definition 1.1 *A dynamical system is a system with a set of possible states which is the phase space X and a law that controls the changing of the system which is the time-evollution law f .*

Definition 1.2 *A (discrete-time) dynamical system is a pair (X, f) where X is a compact metric space and f is a map from X into itself.*

Example 1.1 *1-card shuffling is a discrete-time system because the states of this system change in discrete time steps.*
2-a swinging pendulum is a continuous-time system.

Rotations on the circle: We define the dynamical system (S^1, R_α) where S^1 is the unit circle in $\mathbb{C}$ and R_α is a map from S^1 into itself which rotates the circle by $2\pi\alpha$ so we can define as:

$$z \overset{R_\alpha}{\mapsto} e^{i2\pi\alpha}z.$$

Moreover, z belongs to the unit circle, so $R_\alpha(z) = e^{i2\pi(x+\alpha)}$, where $x \in [0, 1)$. Also, this system can defined on $\mathbb{R}/\mathbb{Z}$ with the map $S_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by:

$$[x] \overset{S_\alpha}{\mapsto} [x + \alpha],$$

Here $x+\alpha \pmod 1$. In this case, R_α is an *isometry* for a metric d which define on S^1 as the shortest arclenth between 2 points.

Definition 1.3 *Let (X, f) be a dynamical system and f invertible. The orbit of a point x is the set of images under the iterations of f or f^{-1} of x:*

$$\mathcal{O}(x) = \{y \in X \mid y = f^n(x) \text{ where } n \in \mathbb{Z}\}$$

Definition 1.4 *If $f(x) = x$, then x is a fixed point. If $f^n(x) = x$ for some $n \in \mathbb{Z}$, then x is a periodic point with period n. We define $Fix(f)$ to be the set of fixed points in X, and $P_n(f)$ the set of periodic points with period n. A subset $Y \subseteq X$ is invariant under f if $f(Y) = Y$.*

Definition 1.5 *A dynamical system (X, f) is topologically transitive if for all nonempty open subsets $U, V \subseteq X$, there exists some integer $n \geq 1$ such that $f^n(U) \cap V \neq \phi$.*

Example 1.2 *R_α is topologically transitive because it is clear that every open set will contain a point in the orbit of another open set because the orbit of any point is dense.*

Proposition 1.1 *If X is perfect (i.e., X has no isolated points) and there exists some x such that $\mathcal{O}(x)$ is dense on X, then f is topologically transitive.*

Proof. *Consider $U, V \subseteq X$ are two open sets. We have $\mathcal{O}(x)$ is dense, so, $\exists\ f^n(x) \in U$. Also, from beeing X perfect, so the set $V \setminus \{x, ..., f^n(x)\}$ is nonempty and open, so, there is some $f^m(x) \in V$, where $m > n$. We find $f^{m-n}(U) \cap V \neq \phi$.* ∎

Lemma 1.1 *(Baire Category Theorem). Let X be a compact metric space. The intersection of a countable collection of open, dense subsets of X is dense.*

Proposition 1.2 *Let X be a compact, second-countable metric space. If f is topologically transitive, then there exists an $x \in X$ with a dense orbit* [1].

Proof. *We consider $\{V_n\}_{n\in\mathbb{N}}$ a basis. $f^{-k}(V_n)$ is open because f is a continuous map, moreover the set*

$$W_n = \bigcup_{n\in\mathbb{N}} f^{-k}(V_n)$$

is open and dense in X, because f is continuous and topologically transitive. Care about $W_n \cap W_m$ where $n \neq m$, so, we will get a set of points whose orbit contain both V_n and V_m, this set of points is open and dense. Likewise, the set of points which their orbits are dense is $\bigcap_{n\in\mathbb{N}} W_n$. Through Lemma 1.1, this set is nonempty. So, it is dense. ∎

Expanding maps on the circle: Consider the dynamical system (S^1, F_n) with the map F_n is $z \overset{F_n}{\mapsto} z^n$ for $n \in \mathbb{N}$. Also, consider the same system on $\mathbb{R}/\mathbb{Z}$ with the map E_n which is defined as follows:

$$[x] \overset{E_n}{\mapsto} [nx \,(\mathrm{mod}\, 1)] \,.$$

where n is a natural number greater than 1 because of the determining the point 0 with 1); we should get $E_n(0) = E_n(1)$ and this condition satisfies only if $n \in \mathbb{N}$.

Example 1.3 *For $n = 3$, E_3 has 2 fixed points 0 and $\frac{1}{2}$. Let $x = \frac{p}{q}$be a rational number, we find that all rational numbers are periodic points because there are a finite number of equivalence classes with denominator q. Moreover, if x is irrational, $E_3^n(x)$ cannot be equal to $\{x\}$. Hence, x is periodic and hence x is rational.*

Definition 1.6 *A dynamical system (X, f) is topologically mixing if for all nonempty open subsets $U, V \subseteq X$, there exists some $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \phi$ for all $n \geq N$.*

We will distinguish between transitivity and mixing as follows: If f is topologically transitive that is mean it will take a group of points *close* together and these points will *visit* every neighborhood of the space. But, If f is topologically mixing, it will eventually *spread* these points across the whole space.

Proposition 1.3 *If f is topologically mixing, then f is topologically transitive.*
Proof. *The proof is clear through the definitions.* ■

From the above we find that mixing is a stronger criterion than transitivity.

Example 1.4 *E_3 is named an expanding map: for any open interval $(a, b) \subseteq [0, 1)$, the length of the interval is $b - a$. When applying the map E_3 we find that the resulting interval length is three times, until the length exceeds 1. Also, when the length reaches 1,so, the interval covers the whole space. In adition, any subsequent iterations of E_3 will continue to cover the whole space. Every open set will come to cover the whole space because every open set contains an open interval, over the time. Hence, E_3 is topologically mixing. In general, all expanding maps E_n are topologically mixing.*

Proposition 1.4 *Isometries are not topologically mixing.*
Proof. *Consider V as some small open interval, and W_1, W_2 as open and small enough intervals so an interval of V'cannot intersect both W_1 and W_2 at the same time. Since f is an isometry, for every n, either $f^n(V) \cap W_1 = \phi$ or $f^n(V) \cap W_2 = \phi$. Thus, isometries are not topologically mixing.* ■

Example 1.5 *R_α is not topologically mixing because the map R_α is an isometry.*

Hyperbolic toral automorphisms: We will define a systems like expanding maps but with invertible functions (because expanding maps are noninvertible). Consider $A \in SL_n\mathbb{Z}$ be a matrix with integer entries and $\det(A) = 1$. We define the map $F_A : \mathbb{T}^n \to \mathbb{T}^n$ as:

$$F_A(x) = Ax \pmod 1$$

F_A is invertible because $\det(A) = 1$ (moreover the matrix $A^{-1} \in SL_n\mathbb{Z}$). F_A is a *toral automorphism* because it is an automorphism of the group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. F_A is hyperbolic if the magnitude of the eigenvalues of A are different from 1.

Example 1.6 *This common example of a hyperbolic toral automorphism on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (for $n = 2$) is defined by:*

$$L = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

the map is given by:

$$(x, y) \overset{F_L}{\mapsto} (2x + y, x + y) \pmod 1$$

When calculating the eigenvalues and associated eigenvectores we find $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$, $v_\pm = \begin{pmatrix} 2 \\ -1 \pm \sqrt{5} \end{pmatrix}$.

1.2 Symbolic dynamics

Definition 1.7 *We call a finite set $\mathcal{A}$ an alphabet and its elements symbols or letters. A finite sequence of letters $x_0 ... x_k$ is called a block or word. Infinite sequences $x : \mathbb{N}_0 \to \mathcal{A}$ built over these letters are denoted by $x = (x_i)_{i \in \mathbb{N}_0}$. Similarly, bi-infinite sequences $x : \mathbb{Z} \to \mathcal{A}$ by $x = (x_i)_{i \in \mathbb{Z}}$.*

We represent x by $...x_{-2}x_{-1}.x_0x_1x_2...$ where the point separates the negative indices from the others. The collection of all bi-infinite (or infinite) sequences is $\mathcal{A}^\mathbb{Z}$, the full $\mathcal{A}$-shift. As $\mathcal{A}$ has N letters, it is isomorphic to $\{0, 1, ..., N-1\}$. The full shift over this set is a full N-shift or a *Bernoulli shift*, which indicated by Ω_N. Also, symbolizing by Ω_N^R to the one-sided Bernoulli shift, the set of all infinite sequences. Giving $\Omega_N = \{0, ..., N-1\}^\mathbb{Z}$ with the product topology, where $\{0, ..., N-1\}$ has the discrete topology. The sets of sequences with a finite number of fixed coordinates cylinders are:

$$C_n^\alpha = C_{n_1,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k} = \{x \in \Omega_N \mid x_{n_i} = \alpha_i\}.$$

It is clear to confirm that cylinders form a basis for the topology on Ω_N, it is mean that, generally, open sets are unions of cylinders. Defining a metric d_λ on Ω_N that induces the same topology as the product topology for any $|\lambda| > 1$ by:

$$d_\lambda(x, y) = \sum_{n\in\mathbb{Z}} \frac{|x_n - y_n|}{\lambda^{|n|}}$$

Here d_λ is a metric on Ω_N because it satisfies all properties of metric (no negativity, symmetry, triangular inequality) [6].

Definition 1.8 *The shift map on Ω_N is the invertible map $\sigma_N : \Omega_N \rightarrow \Omega_N$ such that if $y = \sigma_N(x)$, then $y_i = x_{i+1}$. The shift map σ_N^R on Ω_N^R is defined similarly, but it is not invertible.*

Proposition 1.5 *Periodic points are dense.*

Proof. *Because cylinders are a basis on Ω_3^R, so, every open set is contained at least one cylinder C_n^α. Because a cylinder fixes a finite number of coordinates, choosing a block $w = w_0...w_k$ where if the cylinder fixes the i^{th} letter to be α_i, consequently $w_i = \alpha_i$. The infinite concatenation of w with itself is an infinite sequence in C_n^α. So, the periodic point $w_0...w_k w_0...w_k...$ is in the original open set ,hence, this proves that periodic points are dense* [6]. ■

In symbolic dynamics, the shift space act as phase spaces and the shift map as time-evolution laws. Let the one-sided Bernoulli shift, Ω_3^R and denoting the infinite sequences in Ω_3^R by $x = .x_0x_1x_2....$ through the one-sided shift transformation we find $\sigma_3^R(x) = .x_1x_2....$We aim to determine the relation between σ_3^R and E_3. We'll explain this through the following prpperties:

Proposition 1.6 *The map σ_3^R is topologically transitive and mixing.*

Proof. Consider V and W are two open sets, containing the cylinders C_n^α and C_m^β, respectively. Finding blocks $p_0...p_{k-1}$ and $q_0...q_{m-1}$ in V and W as we do in Proposition 1.6. Then, points x where $x_i = p_i$ for $0 \leq i \leq k-1$ and $x_i = q_i$ for $n \leq i \leq n+m-1$ satisfy $x \in V$ and $\sigma_3^R(x) \in W$ for any $n > k$. Hence, σ_3^R is topologically mixing, so, σ_3^R is topological transitive. ■

Definition 1.9 *A homomorphism $h : (X, f) \rightarrow (Y, g)$ from one dynamical system to another is a continuous function $h : X \rightarrow Y$ such that $h \circ f = g \circ h$ (i.e., the following diagram commutes):If h is onto, then h is a factor map, and we say that there is a topological semiconjugacy from X to Y. If h is one-to-one, then h is an embedding. If h is both onto and one-to-one, then it is a topological conjugacy.*

Example 1.7 *Consider the map $h : \Omega_3^R \rightarrow \mathbb{R}/\mathbb{Z}$ which take a sequence to the real number it represents in base 3. h is onto and $h \circ E_3 = \sigma_3^R \circ h$, h is a factor map. So, it is topological semiconjugacy from Ω_3^R to $\mathbb{R}/\mathbb{Z}$.*

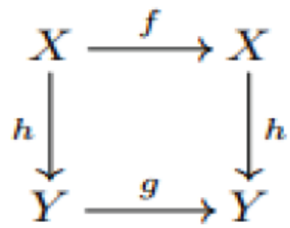

Figure 1.1: Diagram commutes.

Proposition 1.7 *Let h be a semiconjugacy from (X, f) to (Y, g). Fixed and periodic points are preserved under h. Topological transitivity and mixing in (X, f) imply topological transitivity and mixing in (Y, g), respectively.*
Proof. *From the fact that h is a continous function and commutes we have $h \circ f = g \circ h$. Obviously, if h is a conjugacy, the properties in (Y, g) extend to (X, f) too.* ■

1.2.1 Topological Markov shifts.

Symbolic systems is used to modelling other dynamical systems. We wiil see a more restricted space (Compared to the full Bernoulli shift) and a restricted shifts whose are named *topological Markov shifts*.

Definition 1.10 *We can define a set of forbidden words $\mathcal{F}$ that determines whether certain sequences are admissible or not. That is, a sequence that contains a forbidden block is not admissible. We denote the set of all admissible sequences by $X_{\mathcal{F}}$. A shift space (or shift) is a subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some set of forbidden blocks $\mathcal{F}$. If $X \subseteq Y$ where Y is a shift, then X is a subshift. If $\mathcal{F}$ is finite, then $X_{\mathcal{F}}$ is called a subshift of finite type.*

While we can define a topological Markov shift by a set of forbidden blocks, we can also define it by the transition matrix:

Definition 1.11 *Let X be a topological Markov shift. The corresponding transition matrix T is the $0-1$ matrix where $T_{ij} = 1$ if the block ij is admissible, and $T_{ij} = 0$ if ij is forbidden.*

Example 1.8 *The transition matrix associated with the golden mean shift is defined as following:*

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Proposition 1.8 *Let T be the transition matrix for a topological Markov shift. The ij^{th} entry of T^k gives the number of admissible blocks of length $k+1$ that begins with i and ends with j.*

Proof. Let the block $x_1...x_k$. The following value can take values of 0 or 1, this depend on whether the block forbidden or not:

$$T_{x_1}.....T_{x_k} = \begin{cases} 0 & \text{if the block is forbidden} \\ 1 & \text{if the block is admissible.} \end{cases}$$

The ij^{th} entry of T^k is as follows:

$$\left[T^k\right]_{ij} = \sum_{x_2} \cdots \sum_{x_k} T_{ix_2}.....T_{x_k j}.$$

So, every admissible block that begins with i and ends with j participates 1 to the sum, while forbidden blocks participate nothing. ∎

Corollary 1.1 *The number of periodic points with period n is $P_n(\sigma) = trT^n$.*

Proof. The diagonal entries of T^n give the number of ways that a block of length $n+1$ can begin and end on the same number.the using of this is to build periodic sequences of period n. ∎

Definition 1.12 *A $0-1$ matrix T is called transitive if there exists an n such that T^n is positive (i.e., T^n has all positive entries).*

Lemma 1.2 *Let T be a $0-1$ matrix. If T^n is positive, then for all $m \geq n$, T^m is positive.*

Proof. *We can see that if the entries of the i^{th} row of T are all zero, then the i^{th} entry of any sensible matrix product TA also has an i^{th} row of all zeroes. So, every row of a transitive matrix T has at least one nonzero entry. We suppose that T^n is positive. Hence:*

$$T^{n+1}_{ik} = \sum_j T_{ij} T^n_{jk} > 0$$

Because there exists at least one $T_{ij} = 1$ and $T^n_{jk} > 0$. By induction, T^m is positive for $m \geq n$. ∎

Proposition 1.9 *If T is transitive, then the topological Markov shift Ω_T is topologically mixing.*

Proof. *We know that the basis for our topology on Ω_T are cylinder sets intersected with Ω_T. We choose the basis as sets of the form:*

$$C^{\alpha}_{k,T} = \Omega_T \cap \{x \in \Omega \mid x_i = \alpha_i \textit{ for all } -k \leq i \leq k\}$$

we have restricted our cylinder sets to be centered at the 0^{th} coordinate. We can see that for T transitive, $C^{\alpha}_{k,T}$ is nonempty if and only if α is an admissible sequence. Suppose that $V, W \subseteq \Omega_T$ are two open sets. So, there are cylinder sets contained in V and W:

$$C^{\alpha}_{k,T} \subseteq V \quad \textit{and} \quad C^{\beta}_{l,T} \subseteq W$$

where $\alpha = \alpha_{-k}...\alpha_k$ and $\beta = \beta_{-l}...\beta_l$. Let T^n be positive. Hence, there is an admissible block of length $m+1$ for all $m \geq n$ that begins with α_k and ends with β_{-l}. consequently, we find:

$$\sigma_T^{k+m+l}\left(C_{k,T}^{\alpha}\right) \cap C_{l,T}^{\beta} \neq \phi$$

for all $m \geq n$. Thus, T is topologically mixing, and it is topologically transitive. ■

Proposition 1.10 *If T is transitive, then periodic points in Ω_T are dense.*

Proof. *Let the cylinder $C_{k,T}^{\alpha}$. For some n, T^n is positive (because T is transitive), hence there is an admissible block of length $n+1$ that begins with α_k and ends with α_{-k}. It is obvious to construct a periodic point in all basis sets for Ω_T.* ■

1.2.2 Markov partitions

On a dynamical system (X, f) the main idea of Markov partitions is to divide the phase space into a finite number of closed subsets, $X_0, X_1, ..., X_N$. Then, we can associate each point $x \in X$ with its orbit through the partitions $x \mapsto \omega$, where $f^n(x) \in X_{\omega_n}$. Subsequently, we may be able producing a conjugacy between X and the shift space. but, we have 2 difficulties: the first one is that we can be coding a point by more than one sequence, and the second one is that a sequence may code more than one point.

In many time, the phase space X is a *connected manifold*; if the set $\{X_i\}$ covers X, and if X_i is closed, overlap of the *partition* is unavoidable. The set of overlapping points has measure zero, and it is unimportant to much of what we studing on (X, f). So, in general we disregard the first difficulty. Also, we would like for each sequence to code for at most one point, that is, the set $\bigcap_{n\in\mathbb{Z}} f^n(X_{\omega_n}) = \{x\}$. We can define a continuous map $h : \Lambda \subseteq \Omega_N \to X$ such that h is a factor map if we can find a partition of the phase space for which this is always true [1]-[4].

1.3 Topological entropy

Invariants are properties remain unchanged under conjugacy. They provide a way to determine whether 2 dynamical systems can be conjugate. For example topological transitivity is an invariant. We can tell that 2 dynamical systems are not conjugate if one is topologically transitive and the other is not. But, if 2 systems are both transitive this doesn't mean that they are conjugate. We define a complet invariant which is an invariant of a dynamical system that determined strictly when two systems are conjugate. In particular, topological entropy is an important invariant while not a complete invariance.

1.3.1 Entropy on symbolic systems

Conjugacy should be preserve the complexity of a dynamical system. Counting the number of admissible words of length n, and seeing how the number changes as n approaches infinity is a way to measure the complexity in symbolic dynamics.

Example 1.9 *The one -sided N-shift has 1 zero-length block, N one-length blocks, and N^n n-length blocks. But, a subshift Ω will have fewer. If ω_n is the number of admissible n-letter blocks of a subshift then $\omega_{n+m} \leq \omega_n.\omega_m$. So $\log(\omega_{n+m}) \leq \log(\omega_n) + \log(\omega_m)$.*

Lemma 1.3 *If $(a_n)_n$ is a subadditive sequence (that is, $a_{n+m} \leq a_n + a_m$), then the following limit $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists in $\mathbb{R} \cup \{\infty\}$.*

This lemma guarantees the existence of the limit which defined as follows:

$$h(\Omega) = \lim_{n\to\infty} \frac{1}{n} \log(\omega_n)$$

This is the topological entropy of a shift. In particular, the topological entropy of a full M-shift is $\log M$ and generally, it is not clear to count the topological entropy.

Lemma 1.4 *(Perron-Frobenius Theorem). Let T be a transitive $r \times r$ matrix. Then T has one eigenvector v (up to scalar) with positive coordinates. The eigenvalue λ corresponding to v is positive and is greater than the magnitude of all other eigenvalues [4].*

Proposition 1.11 *Let T be a transitive $r \times r$ matrix for the shift $\Omega_T \subseteq \Omega_r$. Let λ be the greatest eigenvalue of T. Then, the topological entropy is $h(\Omega_T) = \log \lambda$.*

Proof. *The number of admissile $(n+1)$-letter words of the shift Ω_T is the sum of all the entries of T^n. So we get*

$$\omega_{n+1} = \sum_{i=0}^{r-1} \sum_{j=0}^{r-1} T_{ij}^n.$$

*From the last lemma **(Perron-Frobenius Theorem)**, consider v is an eigenvector corresponding to λ, with positive coordinates bounded below by m and above by M. so:*

$$m \sum_j T_{ij}^n \leq \sum_j T_{ij}^n v_j = \lambda^n v_i \leq \lambda^n M.$$

moreover, we have:

$$m \sum_i \sum_j T_{ij}^n \leq r\lambda^n M,$$

From the last Inequality we can give an upper bound to $h(\Omega_T)$:

$$h(\Omega_T) = \lim_{n\to\infty} \frac{1}{n} \log(\omega_n) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(\lambda^n \times \frac{rM}{m}\right) = \log \lambda$$

In the same manner we can bound $h(\Omega_T)$ from below:

$$\lambda^n m \leq \lambda^n \times v_i = \sum_j T_{ij}^n v_j \geq M \sum_j T_{ij}^n.$$

From the last 3 equations we bound $h(\Omega_T)$ from below also by $\log \lambda$. so, $h(\Omega_T) = \log \lambda$. ■

Example 1.10 *We aim to calculate the topological entropy of Ω_T for the symbolic representation of the hyperbolic toral automorphism. The maximal eigenvalue for the 5×5 transition matrix is $\lambda = \frac{3+\sqrt{5}}{2}$. The entropy is $\log \lambda$.*

1.3.2 Topological entropy.

Let a dynamical system with finite resolution; we can not distinguish points within some $\epsilon > 0$ from each other. The topological entropy of the system is a measurement of the growth of the number of orbits that we can distinguish given our finite resolution. We consider dynamical systems with compact, second-countable metric space (with metric d). Suppose we can distinguish points with arbitrary, but finite, precision $\epsilon > 0$ and we cannot know which points are less than a distance ϵ apart. We define the metric d_n^f on (X, f) as follows:

$$d_n^f(x, y) = \max_{0 \leq k \leq n-1} d\left(f^k(x), f^k(y)\right).$$

On a topological Markov shift, if our resolution is so coarse that we can determine only the first coordinate of a sequence, then dividing the space so that indistinguishable points are grouped together. The number of admissible words of length 1 is the number of partitions. If we can track the system for $n-1$ iterations of the shift map, we can refining the partition (now, the number of partitions is ω_n). To do the same analysis on a general dynamic system, we need the following definition [2]:

Definition 1.13 *Given (X, f), the d_n^f-diameter of a subset $Y \subseteq X$ is the supremum of the d_n^f-distance between two points in Y.*

Definition 1.14 *Let $cov(n, \epsilon)$ be the cardinality of a minimal covering of X by sets with d_n^f-diameter less than ϵ .*

We can see that since d_n^f is a nondecreasing sequence of metrics, $cov(n, \epsilon)$ is nondecreasing with respect to n. Hence, if $n > m$, then $d_n^f(x, y) \geq d_m^f(x, y)$. Moreover, $cov(n, \epsilon)$ satisfies:

$$cov(n+m, \epsilon) \leq cov(n, \epsilon) . cov(m, \epsilon).$$

Consider that $\mathcal{U}$ and $\mathcal{V}$ as minimal covers of X with sets of d_n^f and d_m^f-diameters less than ϵ, respectively. Consider $U \in \mathcal{U}$ and $V \in \mathcal{V}$. So, sets of the form $U \cap f^{-n}(V)$ cover X because $\mathcal{U}$ and $\mathcal{V}$ are both covers of X. Moreover, these sets have d_{n+m}^f-diameters less than ϵ. This new cover has cardinality $cov(n,\epsilon).cov(m,\epsilon)$, confirming the inequality above. When applying the logarithm we find:

$$\log(cov(n+m,\epsilon)) \leq \log cov(n,\epsilon) + \log cov(m,\epsilon)$$

We define the topological entropy as:

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log cov(n,\epsilon).$$

It is not clear that the topological entropy does not depend on the metric on X. However, it is named the topological entropy because 2 metrics inducing the same topology on X will yield the same topological entropy.

Proposition 1.12 *Let d and $d^{'}$be equivalent metrics on X. The topological entropies $h_d(f)$ and $h_{d'}(f)$ arising from the two metrics are equal.*

Proof. *Let the subset $Y \subseteq X \times X$ of points (x,y) such that $d(x,y) \geq \epsilon$. This is a compact subset of $X \times X$, with the inherited product topology. $d^{'}$ reaches a minimum $\delta(\epsilon)$ on some point $(x,y) \in Y$ because $d^{'}$ is continuous, by the extreme value theorem. morever, $\delta(\epsilon) > 0$ since $x \neq y$. So, points within ϵ of each other by d are within $\delta(\epsilon)$ by $d^{'}$. Hence,*

$$cov_d(n,\epsilon) \geq cov_{d'}(n,\delta(\epsilon))$$

This means that $h_d(f) \geq h_{d'}(f)$. When we switch d and $d^{'}$, we get the reverse inequality. So, topological entropy is independent of metric $h_d(f) = h_{d'}(f)$ [4]. ■

Corollary 1.2 *Topological entropy is an invariant of a topological conjugacy.*

Proof. Consider (X,f) and (Y,g) be conjugate with the homeomorphism $\phi : X \to Y$ and d is a metric on X. We can define $d^{'}$ a metric on Y such that:

$$d^{'}(y_1,y_2) = d\left(\phi^{-1}(y_1), \phi^{-1}(y_2)\right).$$

Then, ϕ is an isometry. Because topological entropy is independent of metric, we get $h(f) = h(g)$. ■

Proposition 1.13 *Let (X,f) and (Y,g) be semiconjugates by the factor map $\phi : X \to Y$. Then, $h(f) \geq h(g)$.*

Proof. ϕ is uniformly continuous because X is compact and ϕ is continuous. So, there exists some $\delta(\epsilon)$ satisfies [4]:

$$d_X(x,y) < \delta(\epsilon) \Longrightarrow d_Y(\phi(x), \phi(y)) < \epsilon$$

Thus, if two points are within $\delta(\epsilon)$ of each other on X, their images under ϕ are within ϵ on Y.This means:

$$cov_X(n, \delta(\epsilon)) \geq cov_Y(n, \epsilon)$$

Also,

$$h(f) \geq h(g).$$

■

Proposition 1.14 *If F_A is a hyperbolic toral automorphism defined by the 2×2 martix A with eigenvalues λ, λ^{-1}, where $|\lambda| > 1$, then the topological entropy is $h(F_A) = \log|\lambda|$.*

Proof. *The idea of proof is to be found a minimal cover for $\mathbb{T}^2$. From the topological entropy of a full n-shift we find that $h(E_n) \leq \log n$. The topological entropy of the expanding maps E_n are equal to $\log n$ .the details may be found in [2].* ■

Chapter 2

Symbolic dynamics of piecewise-linear maps

In this chapter we will present the theory of symbolic dynamics for piecewise-linear maps. We can see that a same symbolic sequence is generated by sets of initial conditions. We also show several properties of these sets. We will define piecewise-linear maps as: $F : O \to O$ for some set $O \subset R^n$ where O is partitioned into disjoint subsets $R_i, i \in L$, where L is an index set. and F is defined as:

$$F(x) = A_i x + b_i, \text{ if } x \in R_i$$

where A_i and b_i are n by n and n by 1 matrices respectively such that $A_i x + b_i \in O$ for all $x \in R_i$. So, we will also see that A_i is a map from R^n into R^n. Through composition of functions with O as the phase space the map F generates a discrete-time (semi) dynamical system. This definition of piecewise-linear maps is very general. actually any map can be represented in this way by choosing the index set L *large enough*. However, in general we will consider cases where L is countable or finite and where R_i's are *nice sets*. We denote $\underbrace{F \circ F ... \circ F}_{n \text{ times}}$ as F^n and $F^{-n} = (F^{-1})^n$ if F^{-1} exists.

We consider the index set L as the alphabet consisting of a set of symbols. the set of infinite sequences consisting of the symbols in L is denoted by $\sum$ which is called the symbolic sequences. We will also define $\sum^b$ as the set of bi-infinite sequences consisting of the symbols in L^2. We call the smallest positive integer n such that s repeats itself after n symbols as the period of a periodic sequence $s \in \sum$or $\sum^b$. We can give to every point $x \in O$ a symbolic sequence $(s_0, s_1, ...) = s \in \sum$ by a way of setting $s_k = i$ if $F^k(x) \in R_i$. We will denote the map which maps a point in O to its corresponding symbolic sequence by S. A sequence in $\sum$ is admissible if it is generated by some point in O. This means that the set of admissible sequences in $\sum$ is

$\sum_F = S(O)$. For $s \in \sum$, we define $S^{-1}(s) = \{x \in O : S(x) = s\}$. We define the shift operator σ as $\sigma(s_0, s_1, ...) = (s_1, s_2, ...)$. We devide $\sum$ into three subsets: $\sum_\sigma = \{s \in \sum : s \text{ is periodic}\}$ and $\sum_\beta = \{s \in \sum : s \text{ is eventually periodic,i.e., } s \notin \sum_\sigma \text{ and } \sigma^p(s) \in \sum_\sigma \text{ for some integer } p > 0\}$ and $\sum_\gamma = \sum \setminus \left(\sum_\sigma \cup \sum_\beta\right)$ which corresponds to the following partition of O:

$$\begin{cases} I_\alpha = S^{-1}\left(\sum_\sigma \cap \sum_F\right) \\ I_\beta = S^{-1}\left(\sum_\beta \cap \sum_F\right) \\ I_\gamma = S^{-1}\left(\sum_\gamma \cap \sum_F\right) \end{cases}$$

If F is a bijection, then we can assign a symbolic sequence $(..., s_{-1}, s_0, s_1, ...) = s^b \in \sum^b$ to each point in O through the map S^b in a similar way. Likewise the admissible sequences of F^{-1} in $\sum^b$ is $\sum_F^b = S^b(O)$. When F is invertible (F is bijective), and each A_i is also invertible, then by using the same alphabet L we can write F^{-1} as follows:

$$F^{-1}(x) = A_i^{-1}(x - b_i), \quad \text{if} \quad x \in A_i(R_i) + b_i$$

The periodic points of the map F are an important set of points, i.e., x such that $F^k(x) = x$ for some $k > 0$. The period of x is the least positive integer k such that $F^k(x) = x$.

Lemma 2.1 *A point z_0 is a periodic point of period k if and only if it satisfies*

$$\begin{cases} z_0 = A_{i_{k-1}} z_{k-1} + b_{i_{k-1}} \\ z_1 = A_{i_0} z_0 + b_{i_0} \\ \quad ..... \\ z_{k-1} = A_{i_{k-2}} z_{k-2} + b_{i_{k-2}} \end{cases}$$

for some $i_0 ..., i_{k-1}$ elements of the index set L and $z_j \in R_{i_j}$, for $j = 0, ..., k-1$ such that $z_l \neq z_j$ for $l \neq j$

Proof. The proof is obvious from the definitions. ■

Lemma 2.2 *Given $z_0 \in O$ and $(s_0, s_1, ...) = s \in \sum_F$ and define $z_{i+1} = A_{s_i} z_i + b_{s_i}$. Then for each $x \in S^{-1}(s)$, we have $A_{s_i}(y_i) = y_{i+1}$, where $y_i = F^i(x) - z_i$.*

Proof. We have $A_{s_i}(y_i) = A_{s_i} F^i(x) - A_{s_i} z_i = F^{i+1}(x) - b_{s_i} - A_{s_i} z_i = F^{i+1}(x) - z_{i+1} = y_{i+1}$. ■

Lemma 2.3 *Let z_0 be a point with associated symbolic sequence $s \in \sum_F$. Then for $x \in S^{-1}(s)$, $A_{s_i}(y_i) = y_{i+1}$, where $y_i = F^i(x) - F^i(z_0)$.*
Proof. *Using the Lemma 2.2 and we note that $F^{i+1}(z_0) = A_{s_i} z_i + b_{s_i} = z_{i+1}$.* ■

Lemma 2.4 *Let s be a k-periodic sequence $(i_0 i_1 i_2 ... i_{k-1} ...)$ and let z_0 satisfy the conditions of the Lemma 2.1 for some z_i's in O. Then for all $x \in S^{-1}(s)$, we have $\widetilde{A}(x - z_0) = F^k(x) - z_0$ where $\widetilde{A} = A_{i_{k-1}} A_{i_{k-2}} ... A_0$. This means, F^k acts as an affine map on $S^{-1}(s)$.*

Proof. We put $z_{k+l} = z_l$. By the Lemma 2.2, $A_{i_{k-1}} A_{i_{k-2}} ... A_0 y_0 = y_k$ and we obtain the result because $z_0 = z_k$. ■

Definition 2.1 *A point $x \in C$ is a root of unity if $x^k = 1$ for some positive integer k.*

Corollary 2.1 *Suppose $A_i = A$ for all $i \in L$. If no eigenvalue of A is a root of unity, then two distinct periodic points generate distinct periodic sequences, i.e., the map S restricted to periodic points is injective. In particular, a k-periodic periodic point must generate a k-periodic symbolic sequence.*
Proof. *Proving this Corollary 2.1 by contraposition. Let z_0, z_1 be 2 distinct periodic points, these points generate the same symbolic sequence s. There exist an integer k such that $F^k(z_0) = z_0$ and $F^k(z_1) = z_1$. By the Lemma 2.4, $A^k(z_1 - z_0) = F^k(z_1) - z_0 = z_1 - z_0$. Then a k-th root of unity is an eigenvalue of A Let a k-periodic periodic point generate an m-periodic symbolic sequence where $m < k$. Then the m^{th} iterate of the periodic point must generate the same symbolic sequence and this is a contradiction.* ■

When L has m elements, given the assumption of the Corollary 2.1 the number of k-periodic points is at most m^k.

Definition 2.2 *The matrix A is called hyperbolic if all its eigenvalues are not on the unit circle.*

Lemma 2.5 *Suppose $A_i = A$ for all $i \in L$. If F is bijective and A is nonsingular and hyperbolic and all orbits (both forward and backward) of F are bounded, then the map S^b is injective.*

Proof. The proof is given by using the contradiction proof. Let z_0 and z_1 are distinct points generating the same bi-infinite symbolic sequence s^b. Through the Lemma 2.3 we find

$$F^i(z_1) - F^i(z_0) = A^i(z_1 - z_0).$$

Because A is hyperbolic, either $A^n(z_1 - z_0)$ becomes unbounded or $A^{-n}(z_1 - z_0)$ becomes unbounded as $n \to \infty$ which is contradition because we know that $F^n(z_1) - F^n(z_0)$ is bounded as $n \to \pm\infty$. ■

Lemma 2.6 *A periodic point z of period k satisfies the following equation:*

$$\left(I - A_{i_{k-1}}...A_{i_0}\right) z = A_{i_{k-1}}...A_{i_1} b_{i_0} + ... + A_{i_{k-1}} b_{i_{k-2}} + b_{i_{k-1}}$$

for some $i_0..., i_{k-1}$ elements of the index set (or alphabet) L, I is the identity matrix and $S(z)$ is the periodic sequence $(i_0 i_1 ... i_{k-1}...)$.

Proof. Consider $z_j = F^j(z)$. the z_j's should be satisfy the conditions of Lemma 2.1, because $z_k = z_0$. We rearrange the equations then follow. For each $s \in \sum\left(\sum^b\right)$, we have the set $S^{-1}(s)\left(\left(S^b\right)^{-1}(s)\right)$. We assign certain assumptions to R_i and A_i, enabling us to obtain certain results on these sets. ■

Theorem 2.1 *If the R_i's are convex sets, then for each $s \in \sum\left(s^b \in \sum^b\right)$, $S^{-1}(s)\left(\left(S^b\right)^{-1}\left(s^b\right)\right)$ is convex.*

Proof. *Let x and y be in $S^{-1}(s)$. We will prove this by induction that*
$F^k(x + \lambda(y - x)) = (1 - \lambda) F^k x + \lambda F^k y$ for all $k \geq 0, \lambda \in [0, 1]$. This is obviously true for $k = 0$.
Supposing that $F^k(x + \lambda(y - x)) = (1 - \lambda) F^k x + \lambda F^k y$.
So $F^k(x + \lambda(y - x)) \in R_{s_k}$ because $F^k x$ and $F^k y$ are in the convex set R_{s_k}. Hence,

$$F^{k+1}(x + \lambda(y - x)) = A_{s_k}\left((1 - \lambda) F^k x + \lambda F^k y\right) + b_{s_k} = (1 - \lambda) F^{k+1} x + \lambda F^{k+1} y$$

Because we have that $F^k x$ and $F^k y$ are in the same convex region, so is $F^k(x + \lambda(y - x))$ and therefore $x + \lambda(y - x) \in S^{-1}(s)$. If we have the case $s^b = (..., s^{-1}, s_0, s_1, ...) \in \sum^b$. Note that $\left(S^b\right)^{-1}\left(s^b\right) = \cap_{k=-\infty}^{0} F^{-k}\left(S^{-1}\left(s^k\right)\right)$, where $s^k = (s_k, s_{k+1}, ...)$ and intersections of convex sets are still convex. ■

The measure in the following is the Lebesque measure in R^n. denoting the Lebesque outer measure by m^*.

Lemma 2.7 *Suppose each R_i is measurable and L is countable. Then F is measurable in the sense that if D is a measurable set, then $F^{-1}(D)$ is also a measurable set, where F^{-1} is the set-theoretic inverse of F. moreover, if D is a measurable set, then $F(D)$ is also a measurable set.*

Proof. *Suppose L_i is the affine mappings $L_i(x) = A_i x + b_i$. Also, the Lemma follows from $F(D) = \cup_{i \in L} L_i(D \cap R_i)$ and $F^{-1}(D) = \cup_{i \in L}\left(R_i \cap L_i^{-1}(D \cap L_i(R_i))\right)$, where L_i^{-1} is the set-theoretic inverse of L_i. So, $F(D)$ and $F^{-1}(D)$ are both union of a countable number of measurable sets.* ■

Lemma 2.8 *Suppose each R_i is measurable and L is countable. For each $s \in \sum, \left(s^b \in \sum^b\right), S^{-1}(s)\left(S^b\right)^{-1}\left(s^b\right)$ is a measurable set.*

Proof. *Suppose* $s = (s_0, s_1, \ldots)$. *Then* $F^{-i}(R_{s_i})$ *is the set of points whose corresponding symbolic sequences have* s_i *as the* i*-th symbol. Through Lemma 2.7 we find that the set* $F^{-i}(R_{s_i})$ *is measurable and therefore* $S^{-1}(s) = \cap_{i=0}^{\infty} F^{-i}(R_{s_i})$ *is measurable. In similar way, we can proof for* $s^b \in \sum^b$. ■

Theorem 2.2 *Suppose that* O *is bounded. If* $s \in \sum_\gamma$ *and* $|\det A_i| \geq 1$ *for all* $i \in L$*, then* $S^{-1}(s)$ *has measure zero.*

Proof. Suppose $m^*(S^{-1}(s)) > 0$. Then $m^*(F^i(S^{-1}(s))) \geq m^*(S^{-1}(s)) > 0$, since each $F^i(S^{-1}(s))$ sits completely in some region R_k and F acts as an affine map which does not decrease the outer measure of a set. Because s is not periodic nor eventually periodic, $F^k(S^{-1}(s)) \cap F^j(S^{-1}(s)) = \phi$ for $k \neq j$. Hence the sets $F^k(S^{-1}(s))$ are disjoint, so $\cup_{k=1}^{n} F^k(S^{-1}(s))$ occupies a large area for large n. This implies that O is unbounded and this contradict with $m^*(S^{-1}(s)) = 0$, and then $S^{-1}(s)$ has Lebesque measure zero. ■

Theorem 2.3 *Suppose all region* R_i *are bounded by a fixed constant* M*. If* $|\det A_i| \geq 1 + \epsilon$ *for all* $i \in L$ *and some* $\epsilon > 0$*, then* $\left(S^{-1}(s) \left(S^b\right)^{-1}(s^b)\right)$ *has measure zero for all* $s \in \sum \left(s^b \in \sum^b\right)$.
Proof. *Let* $m^*(S^{-1}(s)) > 0$. *Each* $F^i(S^{-1}(s))$ *sits inside some* R_k*,via the definition of* $S^{-1}(s)$. *Because* $|\det A_i| \geq 1 + \epsilon$ *for all* $k \in L$*, the outer measure of* $F^i(S^{-1}(s))$ *goes to infinity as* $i \to \infty$*, and we find a contradition. Hence,* $m^*(S^{-1}(s)) = 0$. $\left(S^b\right)^{-1}(s^b)$ *also has measure zero because* $\left(S^b\right)^{-1}(s^b) \subset S^{-1}(s)$ *for* s *a truncation of* s^b *into a infinite sequence in* $\sum$. ■

Corollary 2.2 *If* L *is countable, all regions* R_i *are bounded by a fixed* M*, and* $|\det A_i| \geq 1 + \epsilon$ *for all* $i \in L$ *and some* $\epsilon > 0$*, then* $\sum_\gamma \cap \sum_F$ *is uncountable.*

Proof. We can prove it from the fact that $\sum_\alpha \cup \sum_\beta$ is countable and from the Theorem 2.3. ■

Definition 2.3 *A matrix* A *is said to be stable if all eigenvalues have norm less than or equal to one and all eigenvalues with norm equal to one correspond to a diagonal Jordan block (the corresponding invariant subspace is spanned by eigenvectors). A matrix* A *is said to be unstable if it is not stable.*

Theorem 2.4 *If any finite product of* A_i*'s is unstable and all trajectories are bounded, then* $S^{-1}(s)$ *for* s *periodic has measure zero. Therefore, if* L *is countable,* $I_\alpha \cup I_\beta$ *has measure zero and* $\sum_\gamma \cap \sum_F$ *is uncountable.*

Proof. Consider s with period k. From Lemma 2.4, it follows that F^k acts as a affine map on any point in $S^{-1}(s)$. The trajectory will be unbounded almost everywhere because the affine map is unstable. ■

Corollary 2.3 *If* $A_i = A, \forall i \in L$ *and* A *is unstable and all trajectories are bounded, then* $S^{-1}(s)$ *for* s *periodic has measure zero.*

We define the boundary ∂R of a set R as $\overline{R} \setminus R^0$. The following results give conditions when $S^{-1}(s)$ for a periodic sequence s is not a set of measure zero.

Theorem 2.5 *If $A_i = A, \forall i \in L$ and A is stable, and there exists a point z with associated symbolic sequence $s \in \sum$ such that the distance of $F^i z$ to the boundary ∂R_{s_i} of region R_{s_i} is larger than $e > 0$ for all i. Then z is an interior point of $S^{-1}(s)$. In other words, $S^{-1}(s)$ is not a set of measure zero.*

Proof. *Using the Euclidean norm on $\mathbb{R}^n$ and the induced norm on $n \times n$ matrices. Consider $z_i = F^i(z)$, $i = 0, ...k-1$ and let $A = TJT^{-1}$ which is a Jordan form decomposition of A, where J is a Jordan form matrix with Jordan blocks, it has the following form:*

$$\begin{pmatrix} \lambda & \delta & & \\ & \lambda & \delta & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

We have $\|J\| \leq 1$ for sufficiently small $\delta > 0$ because we know that A is stable. Consider the open sets centered at z_i and defined as the following

$$D_i(\rho) = \left\{x : \left\|T^{-1}(x - z_i)\right\| < \rho\right\}$$

We suppose that $\rho > 0$ is such that each $D_i(\rho)$ is inside R_{s_i}. For $x \in D_i(\rho)$, $F(x) - z_{i+1} = A(x - z_i)$ by Lemma 2.3. Hence

$$\left\|T^{-1}(F(x) - z_{i+1})\right\| = \left\|JT^{-1}(x - z_i)\right\| \leq \left\|T^{-1}(x - z_i)\right\| < \rho$$

So $F(x) \in D_{i+1}(\rho)$. Therefore $F(D_i(\rho)) \subseteq D_{i+1}(\rho)$ and $D_0(\rho)$ is in $S^{-1}(s)$. Such a ρ can be found because each z_i is an interior point of R_{s_i} and it is bounded away from the boundary of R_{s_i}. ■

Corollary 2.4 *If $A_i = A, \forall i \in L$ and A is stable, and there exists a k-periodic point z with associated symbolic sequence $s \in \sum_\alpha$ such that $F^i z$ is an interior point of R_{s_i} for $i = 0, ...k-1$, then z is an interior point of $S^{-1}(s)$. In other words, $S^{-1}(s)$ is not a set of measure zero.*

Proof. Because there are only a finite number of distinct $F^i z$, we can find $\epsilon > 0$ and it satisfies the hypothesis of Theorem 2.5. ■

Corollary 2.5 *Suppose that L is finite, $A_i = A, \forall i \in L$, A is stable, $|\det A| = 1$ and O is bounded. If $z \in I_\gamma$, then the orbit $F^i(z)$ must approach the boundary of one of the regions R_i, i.e., the distance between the boundary ∂R_i and the set $\cup F^j(z)$ is zero for some $i \in L$.*

Proof. Because there are only a finite number of regions, if the orbit of z does not approach any of the boundaries of the regions, so we can find $\epsilon > 0$ and it satisfies the hypothesis of theorem 2.5. Hence, $S^{-1}(S(z))$ is not a measure zero set and this is contradiction with Theorem 2.2. ■

2.1 Some applications

We will present an important theory which it can be used in our applications in which the system can be written as a piecewise-linear map through an input signal. The system can be thought of as operating on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ in some cases.

Theorem 2.6 *Let F be defined on the torus $\mathbb{T}^n$ as follows:*

$$F(x) = Ax + c, \mod 1$$

If A is an integer-valued matrix and $(e - c)$ is in the range space of $A - I$, then the dynamics of this system is topologically conjugate to that of the map:

$$F(x) = Ax + e, \mod 1$$

Proof. We represent $\mathbb{T}^n$ as the hypercube $[d_1, 1 + d_1] \times [d_2, 1 + d_2] \times \ldots \times [d_n, 1 + d_n]$ for some $d = (d_1, \ldots d_n)^T$, instead of representing it as the hypercube $[0, 1]^n$. Since A is integer-valued, for any integer-valued b, $A(x + b) + c$ and $Ax + c$ differs by a integer valued vector Ab. So, the map $F(x)$ has the same representation on the hypercube $[d_1, 1 + d_1] \times [d_2, 1 + d_2] \times \ldots \times [d_n, 1 + d_n]$, and seting d such that $(A - I)d = e - c$ and $x' = x - d$. Then $x' \in [0, 1]^n$ for $x \in [d_1, 1 + d_1] \times [d_2, 1 + d_2] \times \ldots \times [d_n, 1 + d_n]$. Let $F'(x') = F(x) - d$, also $F'(x') = F(x) - d = A(x' + d) + c - d = Ax' + e$. ∎

Corollary 2.6 *If $A - I$ is invertible (or equivalently, 1 is not an eigenvalue of A) and integer-valued, then*

$$F_1(x) = Ax + c, \mod 1$$

is topologically conjugate to

$$F_2(x) = Ax, \mod 1$$

for all c.

Proof. We can prove it from the Theorem 2.6. ∎

If we regard c as the constant input to the system, this Corollary says that any constant input to the system will not change the structure of dynamics of the system if $A - I$ is invertible. But, if $A - I$ is not invertible, then the input to the system can have an effect on the dynamics. Consider the rotation on a circle:

$$F(x) = x + c \quad \mod 1$$

where z and c are real numbers that depend on whether c is rational or not. The system will have periodic points or no periodic points. So, to obtain a system of the form

$$F(x) = Ax + c, \mod 1$$

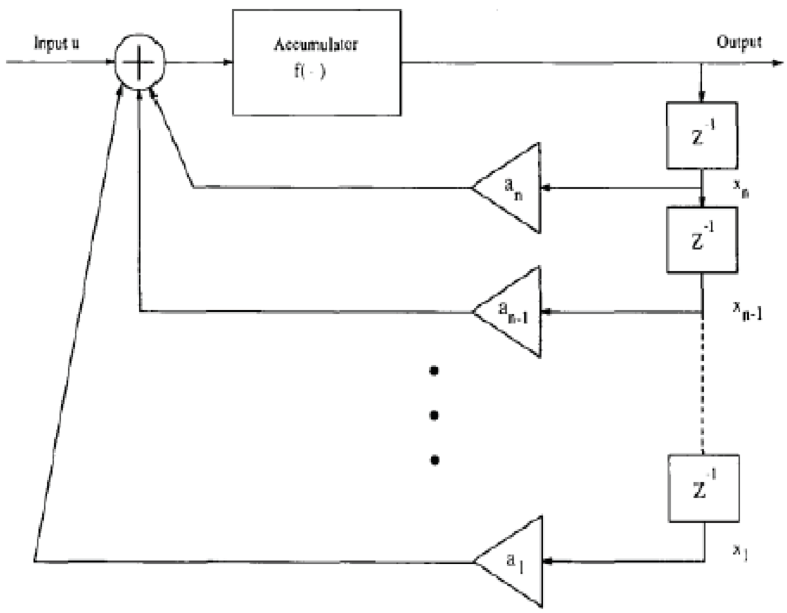

Figure 2.1: Direct-form realization of a digital filter with overflow nonlinearity.

whose dynamical behavior changes with the input vector c, either the matrix A has non-integer coefficients, such as the filter of A has 1 as an eigenvalue [9], such as in ideal single-loop sigma-delta modulators.

2.1.1 Digital filters with overflow nonlinearity

A digital filter with overflow nonlinearity can be implemented in direct form [9]-[14] as shown in the following Figure 2.1
We define the overflow nonlinearity: We consider is the 2's complement overflow nonlinearity as: $f(x) = x - 2t \quad$ for $\ -1 + 2t \leq x < 1 + 2t$, t an integer. A graph of $f(.)$ is shown in the Figure 2.2:The state equations for this system are:

$$
\begin{aligned}
x(k+1) &= \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & & & 1 \\ a_1 & a_2 & \dots & & a_n \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u \\
&= Ax(k) + b(2s + u)
\end{aligned}
$$

where $s = Int(a_1 x_1(k) + ... + a_n x_n(k) + u)$ and we define $Int : R \rightarrow R$ as: $Int(x) = -t$ if

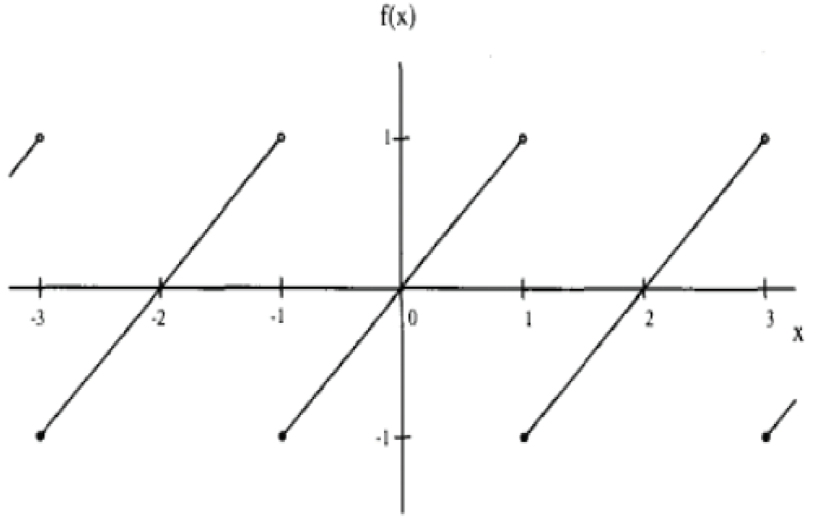

Figure 2.2: The complement overflow nonlinearity $f(x)$.

$2t-1 \leq x \leq 2t+1$, $t \in \mathbb{Z}$. We suppose that the input u is constant. The set

$$O = \{x \in R^n : -1 \leq x_i \leq 1\}$$

is the phase space. The alphabet L is $\{-m, ..., -1, 0, 1, ..., m\}$, where m be the least integer such that for all $x \in O$, $-2m-1 \leq a_1x_1 + ... + a_nx_n + u < 2m+1$. The regions R_i will be $R_i = \{x : Int\,(a_1x_1 + ... + a_nx_n + u) = -i\}$. The determinant of A is $(-1)^{n+1} a_1$ and from the previous results we can conclude that:

1. (a) Each $S^{-1}(s)$ is convex.

   (b) If $|a_1| \geq 1$ then $S^{-1}(s)$ for $s \in \sum_\gamma$ has Lebesque measure zero.

   (c) If A is stable and there exists a periodic point x whose orbit does not intersect the boundary plane $x_n = -1$, then $S^{-1}(S(s))$ has nonzero measure.

   (d) If A is stable and $|a_1| = 1$ then for all $x \in I_\gamma$, the orbit of x approaches the boundary planes $x_n = 1$ or $x_n = -1$.

   (e) If $|a_1| \geq 1 + e$ for $e > 0$, then $S^{-1}(s)$ has Lebesque measure zero for all $s \in \sum$ and $\sum_\gamma \cap \sum_F$ is uncountable.

   (f) If the a_i's are integers and 1 is not an eigenvalue of A, then the constant input u does not change the qualitative behavior of the system. Since $S^{-1}(s)$ is convex, $S^{-1}(s)$ being of measure zero implies that $S^{-1}(s)$ is a convex set lying in a proper affine subspace of the phase space.

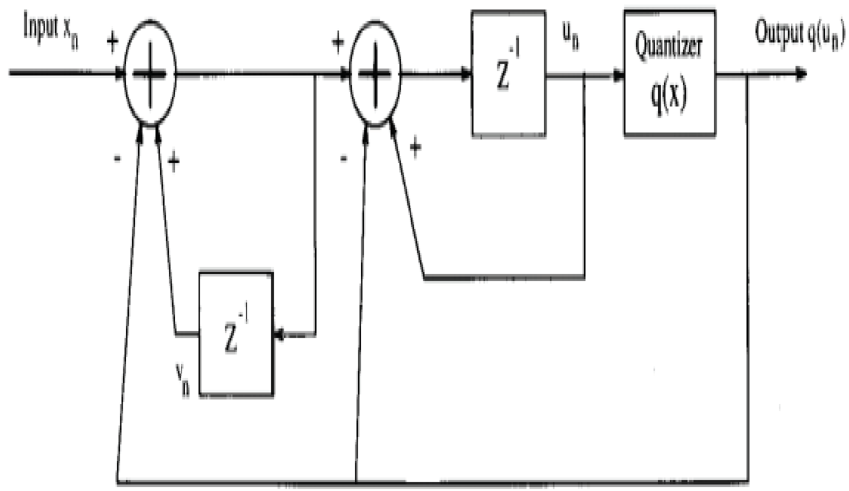

Figure 2.3: Double loop sigma-delta modulator.

2.1.2 Sigma-Delta modulators

The output of sigma-delta modulators is in fact the symbolic sequence s. So, $S^{-1}(s)$ can be considered as the *basin of attraction* of the output sequence s. In an ideal single-loop sigma-delta modulator [17], all initial conditions generate periodic output or aperiodic output this depend on whether the input is rational or irrational, respectively. But, this is not the case for a double-loop Sigma-Delta modulator. Particularly, we will see that the set of points which generate periodic output is of measure zero, for any input.

Consider the double-loop sigma-delta modulator [18] with a four-level quantizer which is shown in the following Figure 2.3 :

$$\begin{cases} u_{n+1} = x_n + v_n - q(u_n) \\ \\ v_{n+1} = v_{n+1} + u_n - q(u_n) \end{cases}$$

where x_n is the input and $q(u_n)$ is defined as

$$q(u_n) = \begin{cases} 3 & u_n \geq 2 \\ 1 & 0 \leq u_n < 2 \\ -1 & -2 \leq u_n < 0 \\ -3 & u_n < -2 \end{cases}$$

It is clear that for $u_n \in [-4, 4)$, that $|u_n - q(u_n)| \leq 1$. If $|x_n| < 1$, $u_n \in [-4, 4)$ and $|v_n - q(u_n)| \leq 2$, then $|u_{n+1}| \leq |x_n| + |u_n - q(u_n)| + |v_n - q(u_n)| < 4$ and
$|v_{n+1} - q(u_{n+1})| = |u_{n+1} - q(u_{n+1}) - (u_n - q(u_n))| \leq 2$.

So we can take $\{(u,v) : u \in [-4,4), |v - q(u)| \leq 2\}$ to be the phase space when the input x_n has magnitude less than 1. We assume that the input is constant. We can cast this system into the form of a piecewise-linear map:

$$\begin{pmatrix} v(n+1) \\ u(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v(n) \\ u(n) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} -1 \\ -2 \end{pmatrix} s$$

where x is the input and $s = q(u_n)$ is the output of the system. Here, the alphabet L is $\{-3,-1,1,3\}$, the regions are defined as $R_i = \{(u,v) : q(u) = i\}$ for $i \in L$ and $A_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for all $i \in L$. Since the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is unstable, from the Corollary 2.3 we find that for a constant input, almost all initial conditions in the phase space will result in aperiodic output. Moreover, through Theorem 2.2 there exists uncountably many aperiodic output sequences for a given input.

These results are also true for any ideal n-loop $(n \geq 2)$ sigma-delta modulator where a bounded phase space can be defined (for example, a bounded invariant set).

Chapter 3

Symbolic dynamics of some prime number sequences

Number theory is the study of the integer numbers with a vast technological applications. There are a relation between the dynamical systems concepts and number-theoretic ideas, namely arithmetic dynamics, dynamics over finite fields, or symbolic dynamics.

3.1 Prime spirals

For example, the *Ulam spiral* [36] is formed through writing integers in a square spiral and marking the particular location of the primes (As shown in the Figure 3.1). When we use the representation of the Ulam spiral, we find that the prime numbers tend to distribute and appears often on the diagonals of the spiral. To distinguish the prime numbers, we marke them in a different color

as we see in the last Figure 3.1. We can explain the diagonal-like accumulation patterns of prime numbers through seeing that these correspond to zeros of prime-generating polynomial equations like the *Eulerian form* $f(n) = 4n^2 + bn + c$.
In the Figure 3.2 we see *Gaussian primes* which are the upper-left spiral primes of the form $4n+3$, and *Pythagorean primes* which are the bottom-right the rest of the primes. We see that we can be performed this initial mapping with arbitrarily different modulus k this means that $kn + b$ with $b = 0, 1, 2, ..., k-1$. Let we construct the residue classes formed through taking the primes modulo k. All but a finite number of primes are congruent with $\phi(k)$ different residue classes, where $\phi(.)$ is the Euler totient function [20] this is based on Dirichlet's theorem on arithmetic progressions. For example, if $\phi(3) = \phi(4) = \phi(6) = 2$, its meaning that for $k = 3, 4$ and 6 the symbolic sequence constructed from taking the residue classes of primes modulo k is essentially formed

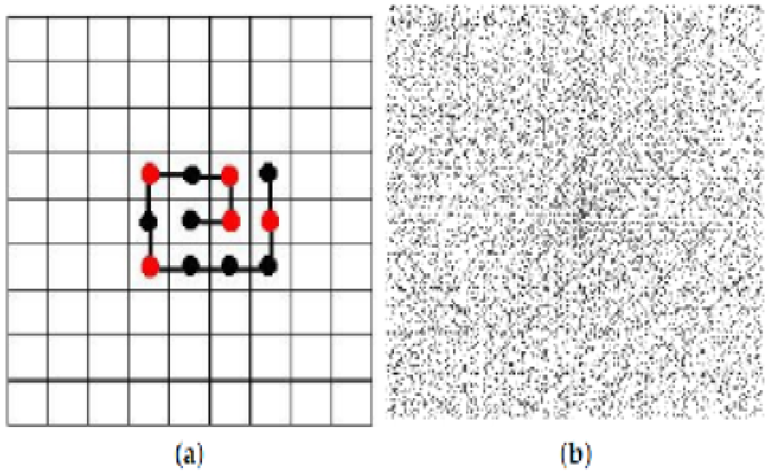

Figure 3.1: **(a)** The Ulam spiral for the sequence $1, 2...12$, with primes in red. **(b)** A full 200×200 Ulam spiral showing the primes as individual black dots.

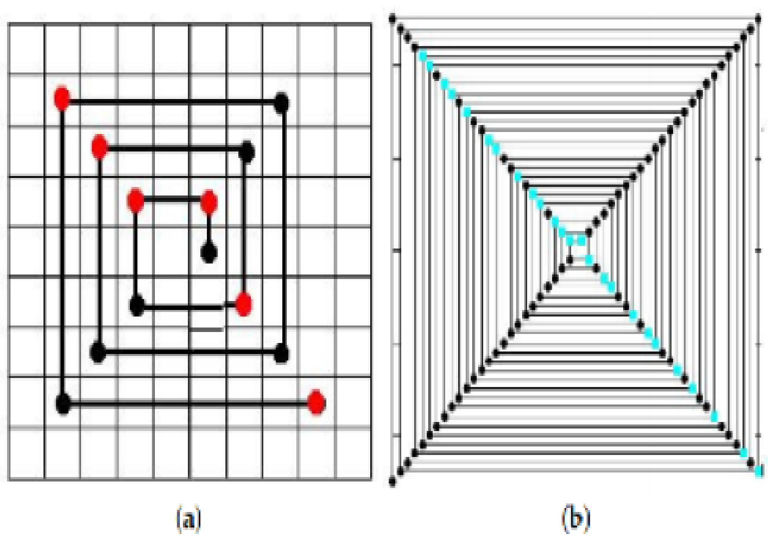

Figure 3.2: **(a)** The diagonal spiral for the sequence $1, 2...13$, with primes in red.**(b)** The diagonal spiral for $N = \{2...100\}$, with primes in blue.

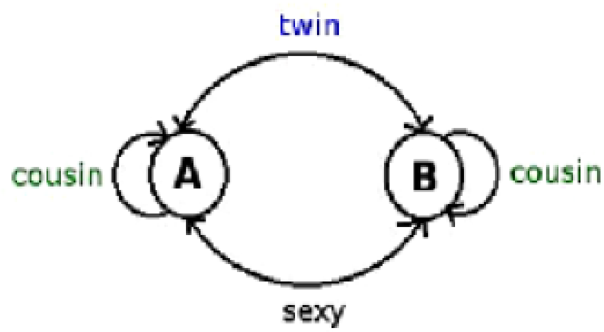

Figure 3.3: A network representation of the prime numbers sequence and state transitions.

by just 2 symbols. For clarification, the sequence formed by Pythagorean and Gaussian primes is an appropriate particular case to distinguish the two types of prime numbers. For example, for modulus $k = 5, 8, 10, 12$ the totient function is 4, hence, in these cases, the resulting sequence will have 4 symbols, and so on.

3.2 The transitions between Pythagorean and Gaussian primes

When the sequence of Gaussian and Pythagorean primes (modulus $k = 4$) has been extracted, we can see the sequence of transitions between these 2 classes. This new symbolic sequence now has 4 symbols, one per transition, in another meaning, the set $\{AA, AB, BA, BB\}$. We can construct this sequence by sliding (moving) a block-2 window and assigning to each pair of consecutive symbols a new symbol, for example the consecutive primes 3 and 5 map into the symbol AB, because 3 is a Gaussian prime (A) and 5 is a Pythagorean prime (B). This new sequence will be named the *transition sequence*. See Figure 3.3.We represent an illustration of such a Markov Chain process in the Figure 3.4. Jumps from "A" to "B" are associated to either twin or sexy pairs, whereas self-loops are associated to cousin pairs.

3.3 Residue classes of gaps mod k: twins, cousins and sexies

Two consecutive primes are separated by a gap $g = 2$ are known as twin primes (for instance 3, 5 are twins). Consecutive primes on the integer line separated by a gap $g = 4$ (cousin primes) or 6 (sexy primes) (like 7, 11 are cousins and 23, 29 are sexies). The Figure 3.4 represent an illustration of the manifestation of twin, cousin and sexy prime pairs.

Gaps between primes can grow arbitrarily large, hence, to study the properties of gaps with

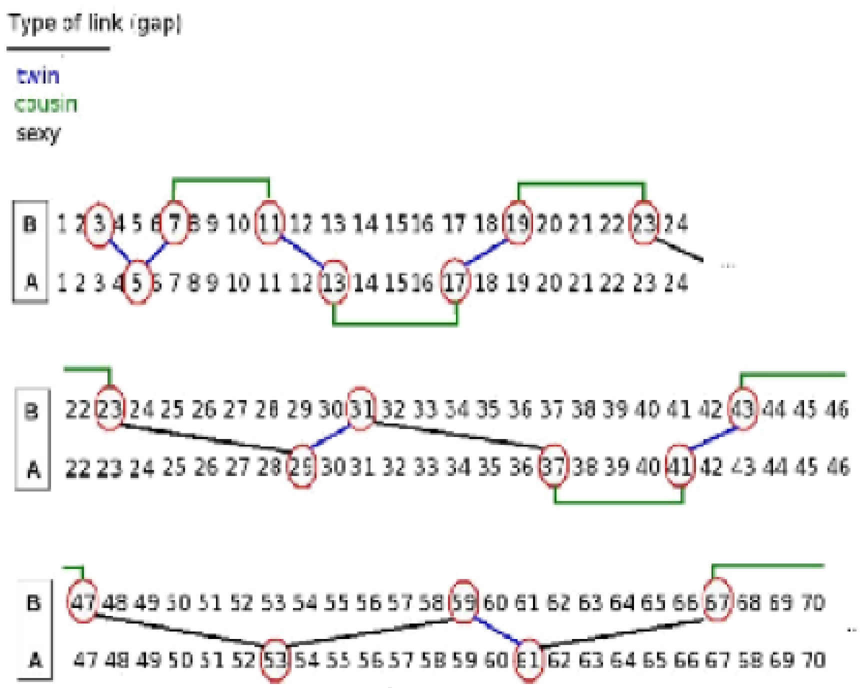

Figure 3.4: Transitions between consecutive primes labelled in terms of twins,cousins and sexies.

the tools of dynamical systems theory we need to make the sequence stationary. Also, we will do this through taking modulo k and coding the gaps according to the residue class to which they correspond. For $k = 6$, we can classify the gaps into 3 families: we name sexy-like to those congruent with $0 \bmod 6$, those congruent with $2 \bmod 6$ will be named twin-like, and those congruent with $4 \bmod 6$ will be named cousin-like. Incidentally, note that transitions between the Pythagorean and Gaussian branches are only due to twin-like and sexy-like gaps.

Conjecture 3.1 *The k-tuple conjecture by Hardy and Littlewood indeed predicts that all admissible k-tuple of primes occur infinitely often.*

A special type of sequence is the sequence of gap residues

$$g(n) = (p(n+1) - p(n)) \bmod 6$$

3.4 Enumerating Blocks: Spectrum of Renyi Entropies

A block of size m constitutes a string of m consecutive symbols in the sequence: is every single possible block appearing in the sequence?, if the answer if yes, then with what frequency?
Let $x_{+1} = F(x_t)$, $F : \mathcal{X} \subset \mathbb{R} \longrightarrow \mathcal{X} \subset \mathbb{R}$ be a dynamical systemand and assume a symbolic sequence $\mathcal{S} = (s_0, s_1, ...)$ extracted from a trajectory x_0, $F(x_0)$, $F^2(x_0)$, ... of this map by a certain

partition of the phase space $\mathcal{P}$ into p symbols. In this infinite sequence we will focus on the statistical properties of a generic block of m consecutive symbols $s = [s_1 ... s_m]$. A complet describtion to the statistics of these blocks needs the whole spectrum named *Renyi dynamical entropies* $h(\beta)$, with $\beta \in \mathbb{R}$ [44] that weight the measure, the frequency of each block in different ways. For $\beta = 0$, $h(0)$ is the topological entropy. If $\mathcal{A}(m)$ is the set of all admissible blocks which present in the dynamics of length m, also the topological entropy is defined in terms of $|\mathcal{A}(m)|$ as follows:

$$|\mathcal{A}(m)| \sim e^{mh(0)}$$

Also, $h(0)$ describes the number of new, different admissible blocks that can appear in the symbolic sequence as we increase its length m. We have

$$h(0) = \sup_{\mathcal{P}} \lim_{m \longrightarrow \infty} \frac{1}{m} \log |\mathcal{A}(m)|$$

There is no metric for this quantity, we only calculate admissible blocks. This is the symbolic analog of the rate of new trajectories that are admissible in a dynamical system as time increases. For $\beta = 1$, $h(1)$ minimizes to the symbolic version of the well known *Kolmogorov–Sinai entropy*:

$$h(1) = \sup_{\mathcal{P}} \lim_{m \longrightarrow \infty} \frac{-1}{m} \sum_{\mathcal{A}(m)} P(s) \log P(s)$$

Here, the summation goes by all the admissible configurations s and $P(s)$ is the probability of each one of them. This entropy weights logarithmically the measure of each type of block, hence it is a *metric entropy*. A traditional definition of a chaotic process is related with a finite and positive value of $h(1)$.

For $\beta > 1$, a spectrum of Renyi entropies is defined by:

$$h(\beta > 1) = \sup_{\mathcal{P}} \lim_{m \longrightarrow \infty} \frac{1}{1-\beta} \frac{1}{m} \log \sum_{\mathcal{A}(m)} P(s)^{\beta}$$

where $h(\beta)$ is a monotonically decreasing function of β and $h(\beta+1) \leq h(\beta)$. For some chaotic processes like the binary shift map [20, 37], we can see the independence of $h(\beta)$ from β. If $h(\beta)$ truly depends on β we say that the system has a non-trivial spectrum of Renyi entropies. We note that $h(\beta)$ are strictly speaking not simple entropies but entropy rates, that is, they are intensive quantities in the statistical mechanics sense. Also, we interest to using this notation

$$h(\beta) = \lim_{m \longrightarrow \infty} \frac{H_m(\beta)}{m}$$

here, $H_m(\beta)$ is the *Renyi block entropies*.

3.5 General conclusion

Symbolic dynamics is one of the most powerful tool for understanding the chaotic behavior of a dynamical system. Symbolic dynamics help us to study and to analyse the nature of general dynamical system and now its ideas have important applications in many fields.

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