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Thème

About Stable and Unstable Sets

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بسم الله الرحمن الرحیم

Dedicate

I dedicate this work...

To my father who supported me to continue my studies.

To my mother, who has been very patient and sacrificed more than me for this day.

To my older brother and the one who has the credit for all that I have accomplished: my brother Zakaria.

To my brothers:

Yousra, Rouaide, Ihebe, Abrare, and to the end of the cluster, my dear Safaa.

To my supporters who, despite the few, are loyal and a real gain.

Thanks

Thank God for his success for me.

Whoever does not thank God will not thank people.

Thanks to my Sheikh, for standing by me from

the first day and supporting me until the last moment.

Thank you Professor Elhadj Zeraoullia

It was a great honor to have you as a menotor

in this work.

I would like to thank Dr.Abdelhak Hafdhalla

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to review this brief study

I especially thank my mathematics departement, and I do not forget

every professor who taught me during my academic years.

ملخص

بادئا ذي بدء، تم تقديم دراسة مبسطة حول المجموعات المستقرة وغير المستقرة و نخصُ بالذكر المنوعات .

في الفصل الأوّل تعرضنا إلى تعريفات و خصائص أساسية و أهم ما جاء فيها تصنيفات المنوّعات، حيث صنّفناها إلى ثلاث أنوع: المنوعة المستقرة و المنوعة الغير مستقرة و المنوعة المركزية.

في الفصل الثاني درسنا ظاهرة الفوضى من خلال المحول في معادلات ذات شكل معين. قمنا بتحليل دقيق للسلوك الفوضوي في عدّة معلمات ثمّ توصلنا أخيرا للحالة التي تحدث فيها فوضى قويّة .

في الفصل الثالث أولينا أهميّة بالغة لنظريات الاستقرار بمفهوم ليابونوف ثمّ توسعنا إلى نظريات لاسال. كذلك مررنا بنظريات عدم الاستقرار .

نشير إلى أنّنا ذيّلنا أغلب التعريفات و النظريات بعدة أمثلة مكنَتنا من الوقوف على أهميّة هذه النظريات في حياتنا العمليّة.

Abstract

First of all, a simplified study of stable and unstable sets is presented, with special mention of manifolds.

In the first chapter, we dealt with definitions and basic characteristics and the most important thing it was the classification of the manifolds, where we classified them into three types: the stable manifold, the unstable manifold, and the central manifold.

In the second chapter, we examined the phenomenon of chaos through the boost converter in equations of a certain shape. We carefully analyzed complex behavior in several parameters and then finally arrived at the case in which robust chaos occurs.

In the third chapter, we attached great importance to stability theories with Lyapunov's concept, and then expanded to the LaSalle's principle. We also passed instability theories.

We point out that we have annotated most of the definitions and theories with several examples that enabled us to identify the importance of these theories in our practical life.

Résumé

Tout d'abord, une étude simplifiée des ensembles stables et instables est présentée, avec une mention spéciale des variétés.

Dans le premier chapitre, nous avons traité des définitions et des caractéristiques de base et le plus important était la classification des variétés, où nous les avons classées en trois types: la variété stable, la variété instable et la variété centrale.

Dans le deuxième chapitre, nous avons examiné le phénomène du chaos à travers le convertisseur boost dans des équations d'une certaine forme. Nous avons soigneusement analysé le comportement complexe dans plusieurs paramètres et sommes finalement arrivés au cas dans lequel un chaos robuste se produit.

Dans le troisième chapitre, nous avons attaché une grande importance aux théories de stabilité avec le concept de Lyapunov, puis nous avons élargi à celle du LaSalle. Nous avons également adopté les théories de l'instabilité.

Nous rappelons que nous avons annoté la plupart des définitions et théories avec plusieurs exemples qui nous ont permis d'identifier l'importance de ces théories dans notre vie pratique.

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Generale Introduction

The motion in dynamic systems around equilibrium is modeled by linear differential equations which are the best approximation of nonlinear differential equations under certain conditions. This notion is related to Taylor series expansion of the forces around the equilibrium point. So the behavior of the dynamic system is considered in the vicinity of this point/set. In this study, the three chapters are presented in a simple way.

The **first chapter** includes basic definitions regarding stable and unstable sets and manifolds. We also deal with the problem of attractors and periodic orbits in smooth systems and we will give two simple examples: The Lindemann mechanism and a simple AIDS model.

In **second chapter**, we use the mechanism *current-mode control* as a simple and easy *chaos generator*. The form represents the current mode controlled boost converter transformer. We'll be exposed to the behavior of a dynamic system under chaos and *robust chaos*. We will also give the causes of occururrence of *border collision bifurcations*.

In **third chapter**, we present Lyapunov's stability theory which depending on the construction of a function that meets the two conditions: The function is positive definite and its derivative is negative semi-definite or negative definite. We went through the classical *LaSalle invariance principle theory*. After that we concluded this chapter with a theory of unstability and some examples.

Chapter 1

Invariant Manifolds

We will study the local structure of a derivable function which is what we call mathematically *manifolds*. We trace the behavior of the dynamical system in the vicinity of the equilibrium point. For this purpose, we need to introduce the following definitions: Stable and unstable sets, stable, unstable and central manifolds[2]. These sets have the advantage of being invariant in the vicinity of the equilibrium point under the influence of flow. In this chapter, we offer some definitions and explain how is formed slow manifold and its role during the absence of the centre manifold and most important we will review the main results of centre manifold theory for finite dimensional systems through two simple examples: The Lindemann mechanism and a simple AIDS model.

1.1 The direction of movement in a dynamical system

The autonomous ordinary differential equation in the vicinity to the equilibrium point $x^* \in \mathbb{R}^n$ is given by:

$$x' = \frac{dx}{dt} = f(x), x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1.1}$$

The basic idea is to approximate this equation to a linear differential equation of the form:

$$\delta x' = J^* \delta x \tag{1.2}$$

where there no eigenvalue of J^* is equal to zero, $\delta x = x - x^*$, and J^* is a constant. The constant J^* represents the Jacobian at the equilibrium point x^*. The general solution to the equation (1.2) is:

$$\delta x(t) = \sum_{i>1} \alpha_i e^{\lambda_i t}, \alpha_i = c_i e_i$$

The λ_i represent the eigenvalues of J^* and e_i represent the corresponding eigenvectors. While c_i are coefficients chosen to meet the initial conditions.The importance of eigenvalues is that they give us an approximation of how fast eigenvalues converge towards the equilibrium point. Eigenvectors also give us an idea of the direction of convergence (do they converge towards the center of equilibrium or diverge). We consider the origin to be the point of equilibrium. All eigenvalues are negative, meaning that the equilibrium point is stable, this means that

$$0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_i| < \cdots < |\lambda_n|$$

hence

$$0 < e^{\lambda_n t} < \cdots < e^{\lambda_i t} < \cdots < e^{\lambda_2 t} < e^{\lambda_1 t}$$

so all terms $e^{\lambda_i t}$ for $i > 1$ will decay more quickly than $e^{\lambda_1 t}$, and on it $\delta x\,(t) \to c_1 e_1 e^{\lambda_1 t}$, and thus the limit of the sum $\delta x\,(t)$ devolves to $c_1 e^{\lambda_1 t} e_1$ when $t \to \infty$. The previous result, geometrically, means that there is convergence towards the equilibrium point along the eigenvector e_1 which we call the *slow* eigenvector.

1.2 Dynamic flow behavior around the equilibrium point

Assume that x_1 and x_2 are arbitrary points in the phase space, where x_2 are close to x_1. Now, we symbolize the evolution of movement from position x_1 to position x_2 it is the vector δx, and we write:

$$\delta x = x_2 - x_1$$

We obtain the evolution of motion in time of δx by the following formula:

$$\delta x' = x_2' - x_1' = f\,(x_2) - f\,(x_1)$$

but we know that

$$x_2 = \delta x + x_1$$

so

$$\delta x' = f\,(x_1 + \delta x) - f\,(x_1)$$

We extend the function $f\,(x_1 + \delta x)$ using Taylor series about the equilibrium point $\delta x = 0$:

$$\begin{aligned} f\,(x_1 + \delta x) &= f\,(x_1) + J_1 \delta x + \cdots \\ f\,(x_1 + \delta x) - f\,(x_1) &= J_1 \delta x + \cdots \\ \dot{\delta x} &= J_1 \delta x + \cdots \end{aligned}$$

We approximate the previous equation to the lowest order (we only take the first term), where J_1 represents the Jacobian value at x_1. Thus the differential equation becomes linear from the form:

$$(\delta x)' \approx J_1 \times \delta x \tag{1.3}$$

The geometric interpretation of the solutions $x = x(t)$ to the equation $x' = f(x)$ represents a *curve* in the space $X \subset \mathbb{R}$. If we take into account the variable t, a representative of time, then the solution $x(t)$ represents the law of motion of a moving point in space $X \subset \mathbb{R}$ with velocity J_1, so the set of solutions $x = x(t)$ is called a *local dynamic system* .These curves corresponding to solutions we call them system trajectories. Note that equation (1.3) above is almost like a linear differential equation about the equilibrium point x^*. The Jacobean matrix J_1 is a constant that does not change over time. In general, we will face a difficulty in explaining the decomposition of eigenvectors of J_1, but locally we will say that the trajectories converge or diverge from the equilibrium point according to the eigenvalues and the direction determined by the eigenvectors. But if we had two systems controlled by the first equation, the Jacobean matrix changes over time, and the same goes for eigenvalues and eigenvectors.

Definition 1.1 *We call a differentiable manifold if each point inside it, is identified by a unique identifier, in other words, it can be said that it is a geometric object that parameterizable continuously and smoothly.*

The word *smooth* indicates that the differentiable manifold has at least more than tow continuous derivatives. It is also possible to express the locally differentiable manifold in the vicinity of an equilibrium point, for example, by drawing a diagram of a certain function, because the differentiable manifolds accept extension continuously, that is,for each point in a manifold of d-dimension from a space with n-dimension we have $z = h(y)$. The variable y represents a set of d coordinates from the space on which the study is being conducted, and z represents the set of coordinates that remain $n - d$ also h represents a differentiable function for a set of variables.

Definition 1.2 *Consider C group of points in a dynamic system are mapped by the evolution operator into other points under the condition that the latter are from the same group. If this definition is met, we say that group C is an invariant set.*

Example 1.1 *The simplest example of an invariant set is the points of equilibrium, where they are mapped by the evolution operator to an other groups that apply to them (in fact they are itself).*

Example 1.2 *A trajectory is a set of points, each point that belongs to it that evolves under effect of the evolution operator to another point but still belongs to the same trajectory so a trajectory is an invariant set.*

Definition 1.3 *An invariant manifold is a topological group characterized by being invariant under the effect the dynamic system.*

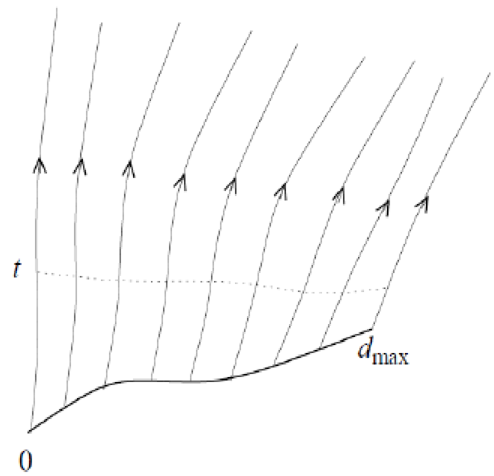

Figure 1.1: Diagram of the 2D invariant manifold.

Example 1.3 *The single equilibrium point, which is a simple example of a invariant manifold of zero dimension. But if we look at the set of equilibrium points, we will find that it lacks the condition of continuity, then it is not a invariant manifold.We know that invariant manifolds are distinguished by being differentiable manifolds, so we can write at every point of the two-dimensional manifold* $z = z(y)$ *and* $\frac{dz}{dt} = \sum_{i=1}^{d} \frac{\partial z}{\partial y_i} \frac{dy_i}{dt}$.

1.3 Special eigenspaces of equilibrium points

The extension of two eigenvectors (a pair) will generate a surface, and the extension of three eigenvectors is a three-dimensional hypersurface. In fact, this type of extension of the eigenvectors is very important. Because it defines the eigenvectors combinations that we need in this study, which are the invariant manifolds. In the following we will present three types of eigenspaces each with different characteristics:

Definition 1.4 *We say of an eigenspace* E^s *is stable if the eigenvector spanned by its corresponding eigenvalues has a completely negative real part.*

Definition 1.5 *We say of an eigenspace* E^u *is unstable if the eigenvector spanned by its corresponding eigenvalues has a completely positive real part.*

Definition 1.6 *We say of an eigenspace that it is a centre eigenspace* E^c*, if it is generated by eigenvectors whose corresponding eigenvalues are purely imaginary, then its true part is absent.*

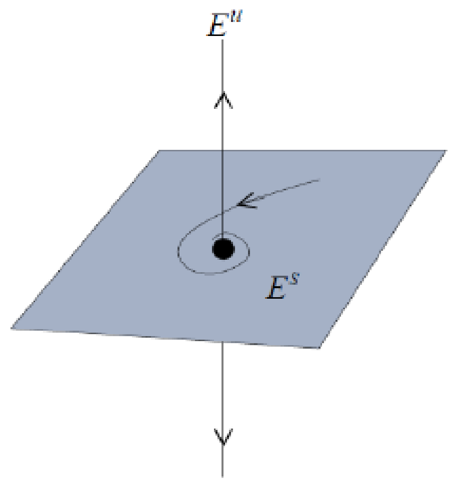

Figure 1.2: Flow diagram near the equilibrium point according.

In the case where the eigenvectors and their eigenvalues are complex conjugate pairs, the eigenspace is expanding in both the real and imaginary parts of the eigenvector. For more clarity, about flow near the equilibrium point we will give some examples[2].

Example 1.4 *In a three-dimensional system, assum we have a pair of complex conjugate eigenvalues where their real parts are negative and a positive eigenvalue. Figure 1.2 shows what this flow looks like precisely near the point of equilibrium.We observe two types of vectors, and thus there are at least two types of trajectories near the point of equilibrium.* E^s *represents a stable eigenspace, every trajectorie that started from this eigenspace is downing towards the equilibrium point (in the form of a spiral). But* E^u *represents an unstable eigenvector, so the systems escape away from the equilibrium point along this eigenvector. There is another trajectorie that combines the two previous trajectories, it concerns trajectories that do not start from eigenspace where the systems rotate spirally away from the equilibrium point along the* E^u *eigenvector. The movement corresponding to the stable spiral, is the reason for the system's bring onto unstable eigenspace. For the system, the latter represents a long-term development.*

Example 1.5 *In a three-dimensional system, we will assume that we have eigenvalues with negative real parts and two purely imaginary eigenvalues (the real part is zero). All paths starting from the centre manifold* E^c *are taken along a path approximately parallel to* E^s*. So the flow is concentrated in the center eigenspace. However, we could not know which directions the paths would follow on* E^c*. See Figure 1.3.*

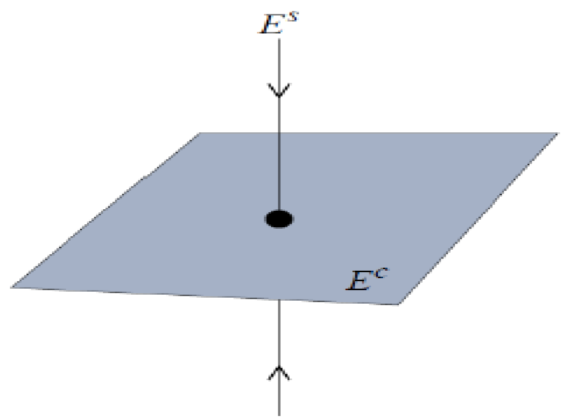

Figure 1.3: Schematic diagram of the strange dynamic behavior described.

1.4 Classification of invariant manifolds

We know from the previous definitions, that the extension of eigenspaces are only invariant manifolds, the following we present definitions of stable and unstable manifolds with some important notes:

Definition 1.7 *W^s is stable manifold of the equilibrium point p. It is a set of points in the phase space with two simple properties: (1) For $x \in W^s$, $\varphi^t(x) \to p$ as $t \longrightarrow \infty$. (2) W^s is tangent to E^s at p.*

Definition 1.8 *W^u is unstable manifold of the equilibrium point p. It is a set of points in the phase space with two simple properties: (1) For $x \in W^u$, $\varphi^t(x) \to p$ as $t \to -\infty$. (2) W^u is tangent to E^u at p.*

These definitions are not only specific to equilibria, we can generalize them to qualitative attractors. In general, these definitions concern the attractor of a dynamic system (i.e., equilibrium point) each of the attractors has its own set of manifolds. Often when we talk about a stable and unstable manifold, we must specify the attractor to which this manifold belongs.
We face difficulty in defining the central manifold, because we wanted the above centre manifold to be tangent to the centre eigenspace, but we are don't know the behavior of the system in the centre manifold and we are also ignorant of what is happening in the centre eigenspace near the equilibrium point, and accordingly, we cannot use stability to define this type of manifolds.

Definition 1.9 *The centre manifold of an equilibrium point p is an invariant manifold of the differential equations with the added property that the manifold is tangent to E^c at p.*

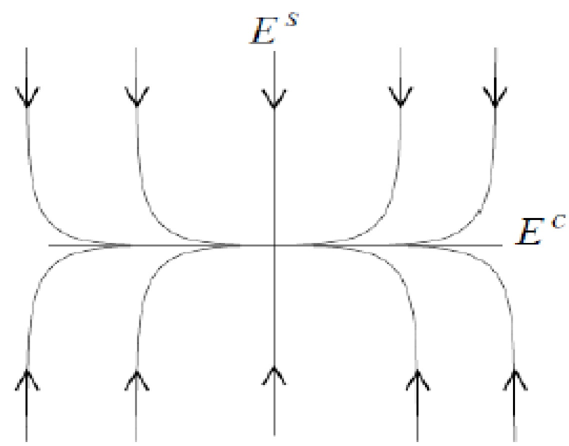

Figure 1.4: The flow for a planar system with a stable and a centre manifold like.

When we compare this definition with previous definitions of stable and unstable manifolds, we find that it is weaker[2]. So there is more than one central manifold. To understand why there are more than one central manifold. The Figure 1.4 shows the flow for a planar system with a stable and a centre manifold:Assume that none of the eigenvalues of the Jacobian matrix of an equilibrium point have positive real parts, and that some of the eigenvalues have zero real parts.

Theorem 1.1 *In some neighborhood U of the equilibrium point, there exists a unique centre manifold W^c such that, for any $x \in U, \varphi^t(x) \longmapsto W^c$ as $t \longrightarrow \infty$.*

In this case, $t \longmapsto \infty$ is just a sneaky way to avoidi making estimates of how long it takes for trajectories to collapse to the manifold. The importance of the centre manifold theory is to explain the behavior of systems that have a centre and no unstable manifold[2].

1.5 Applications of invariant manifolds

We will give two examples containing the central manifold. We can say that the principles of calculating local approximations for other types of invariant manifolds are often the same.

1.5.1 The Lindemann mechanism

The rate equations for the Lindemann mechanism are given by

$$\begin{cases} a' = -a^2 + \alpha ab \\ \\ b' = a^2 - \alpha ab - b \end{cases}$$

The value of the Jacobian matrix at the origin (equilibrium point according to hypothesis $(0,0)$) is equal to $J^\star = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. After a simple calculation, we find two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -1$. So we have a system with a centre manifold. we can use centre manifold theory to determine the stability of the equilibrium point. First, we determine the eigenvectors of $J^\star$. The eigenvectors satisfy $J^\star e_i = \lambda_i e_i$, or $(\lambda_i I - J^\star)\, e_i = 0$. For λ_0, we get

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_{01} \\ e_{02} \end{pmatrix} = 0$$

from which we conclude that $e_{02} = 0$, i.e., $e_0 = (1,0)$ that (or any multiple there of). Similarly, for λ_1, we get

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = 0$$

so $e_1 = (0,1)$. In this case, the two eigenvectors are the two coordinate axes. But this is generally not true. The centre manifold therefore approach the equilibrium point along the a axis. We conclude that it is possible to write an equation for the centre manifold of the form $b = b\,(a)$. We'll also have a problem if we wrote $a = a\,(b)$, because the manifold in this case will be vertical at the origen. We write the following equation:

$$a' = -a^2 + \alpha \times a \times b\,(a) \tag{1.4}$$

for the slow evolution along the centre manifold. Because the equilibrium point is at $(0,0)$ and the centre manifold enters the equilibrium point at a slope of zero, we know that the Taylor expansion of $b\,(a)$ is of the form

$$b = b_2 a^2 + b_3 a^3 + 0\left(a^4\right) \tag{1.5}$$

The symbol $O\,(a^4)$ indicates that the next term in the expansion, which we aren't writing down, would be proportional to a^4. This means that $\alpha \times a \times b\,(a) = 0\,(a^3)$. Thus, on the centre manifold near the equilibrium point, we have

$$a' = -a^2 + 0\,(a^3) \tag{1.6}$$

Here we have an equilibrium point having a stable and a centre manifold. As we said before the parallel movement of the stable manifold quickly leads to the central manifold at this rate $e^{\lambda_1 t} = e^{-t}$. Thus any movement in the system follows equation (1.4), without forgetting that it has been reduced to equation (1.6) near the equilibrium point. When $a > 0$ this indicates that the equilibrium point is stable. But here, the equilibrium point is semi-stable (stable only from the right), and even though equation (1.6) still controls in the system for small and negative (negligible) values of a, the trajectories move away from the equilibrium point, this is a strange behavior. When we use a centre manifold argument to determine the stability of an equilibrium, we generally need to do more work than we have done here. It is not typically the case that we can determine the behavior on the manifold by inspection. Usually in fact, we have to use the manifold equation to determine the coefficients of the Taylor expansion (1.5), then substitute the resultin equation into (1.4) and simplify to determine the leading-order behavior of the rate. As an exercise, let's work out the coefficients b_2 and b_3. The manifold equation for a planar system is

$$b' = \frac{db}{da}a'$$

We can evaluate $\frac{db}{da}$, the derivative on the manifold, directly:

$$\frac{db}{da} = 2b_2a + 3b_3a^2 + O\left(a^3\right)$$

Substituting in the rate equations and the derivative of b on the manifold is:

$$a^2 - \alpha a\left[b_2a^2 + b_3a^3 + 0\left(a^4\right)\right] - \left[b_2a^2 + b_3a^3 + 0\left(a^4\right)\right]$$

$$= \left[2b_2a + 3b_3a^2 + 0\left(a^3\right)\right]\left\{-a^2 + \alpha a\left[b_2a^2 + b_3a^3 + 0\left(a^4\right)\right]\right\}$$

We now collect on one side in powers of a.

$$a^2\left(1 - b_2\right) + a^3\left(2b_2 - \alpha b_2 - b_3\right) + 0\left(a^4\right) = 0$$

Since this equation must be valid for any value of a, the coefficients of each term must individually be equal to zero. Thus we get

$$b_2 = 1, b_3 = \left(2 - \alpha\right)b_2 = 2 - \alpha$$

1.5.2 A simple AIDS model

We will present a simple example of the spread of AIDS, through two groups of the population, the relationship between them is characterized by being infrequent and high risk:

$$c_1' = -\alpha c + p_1\left(c_1 + \beta_1 c_2\right) \tag{7a}$$

$$p_1' = p_1\left(1 - c_1 - \beta_1 c_2\right) \tag{7b}$$

$$c_2' = -\alpha c_2 + p_2\left(c_1 + \beta_2 c_1\right) \tag{7c}$$

$$p_2' = p_2\left(1 - c_2 - \beta_2 c_1\right) \tag{7d}$$

where p_j represents the number of healthy individuals in subpopulation j, while c_j represents the number of contagious individuals. Terms in αc_j represent increased mortality in the contagious group due to the disease. Terms of the form $p_j c_k$ (whether $j = k$ or not) represent the transmission of the disease from contagious individuals. Due to the assumptions of the model, $\alpha > 0$ and $0 < \beta_j < 1$. There are four steady states. The first one corresponds to extinction of the entire population, the second and third to extinction of one subpopulation or the other, and the last to coexistence of both subpopulations. We will now determine the stability of these four steady states.

$$J = \begin{pmatrix} -\alpha + p_1 & c_1 + \beta_1 c_2 & p_1\beta_1 & 0 \\ -p_1 & 1 - c_1 - \beta_1 c_2 & -p_1\beta_1 & 0 \\ p_2\beta_2 & 0 & -\alpha + p_2 & c_2 + \beta_2 c_1 \\ -p_2\beta_2 & 0 & -p_2 & 1 - c_2 - \beta_2 c_1 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of a diagonal matrix are just the diagonal elements. In this case, the eigenvalues are therefore $-\alpha$ (twice) and 1 (twice). The positive eigenvalues make the extinction fixed point unstable.

$$J_2 = \begin{pmatrix} -\alpha & \beta_1 & 0 & 0 \\ 0 & 1 - \beta_1 & 0 & 0 \\ \alpha\beta_2 & 0 & 0 & 1 \\ -\alpha\beta_2 & 0 & -\alpha & 0 \end{pmatrix}$$

The eigenvalues are

$$-\alpha, 1 - \beta_1, \sqrt{-\alpha}, -\sqrt{-\alpha}$$

Since $\beta_1 < 1$, the second eigenvalue is positive, so this steady state, in which population becomes extinct, it is also unstable. The third steady state is unstable too. The final steady state leads to a slightly more complex, but still tractable problem. We won't show the matrix J_4, the Jacobian evaluated at the fourth steady state. It's computed like the others, and its form doesn't suggest anything in particular. We'll just go straight to calculating the characteristic polynomial:

$$\frac{\left(\alpha + \lambda^2\right)\left(2\lambda\alpha\beta_2\beta_1 + \alpha\beta_1 - \lambda\alpha\beta_1 + \alpha\beta_2 - \lambda\alpha\beta_2 - \alpha\beta_2\beta_1 - \alpha - \lambda^2 + \lambda^2\beta_2\beta_1\right)}{-1 + \beta_2\beta_1} = 0$$

Since the eigenvalues are the solutions to this equation, it will follow that one of the terms in the numerator is equal to zero:

$$\lambda_{\pm}^{c} = \pm\sqrt{-\alpha}$$

Based on this we get a two-dimensional centre manifold associated with these eigenvalues. For simplicity, we arrange the powers of λ from largest to smallest in the second term:

$$\left(-1+\beta_2\beta_1\right)\lambda^2 + \left(2\alpha\beta_2\beta_1 - \alpha\beta_2 - \alpha\beta_1\right)\lambda + \alpha\beta_1 - \alpha + \alpha\beta_2 - \alpha\beta_2\beta_1 = 0$$

To find a value of λ we multiply the entire equation by -1 , then we rearrange the terms again according to the powers and get:

$$\lambda^2\left(1-\beta_1\beta_2\right) + \lambda\alpha\left[\beta_1\left(1-\beta_2\right) + \beta_2\left(1-\beta_1\right) + \alpha\left(1-\beta_1\right)\left(1-\beta_2\right)\right] = 0$$

We get positive coefficients for this equation. In this case, we will accept that the quadratic equation has roots where the real parts are negative. We conclude that the two eigenvalues and their corresponding eigenvectors are related to a stable manifold.
So the fourth equilibrium point therefore has a two-dimensional stable manifold and a two-dimensional centre manifold. We can apply the centre manifold theorem again, which says that, eventually, the system will collapse onto the centre manifold so that we need only concern ourselves with the dynamics in this plane. Let's compute the eigenvectors corresponding to the centre-manifold eigenvalues: This command calculates all the eigenvectors and eigenvalues. I'll only show the output corresponding to $\lambda_{\pm}^{c}$.

$$\left[\sqrt{-\alpha}, 1, \left\{\left[-\frac{\sqrt{-\alpha}\left(-1+\beta_1\right)}{\alpha\left(-1+\beta_2\right)}, \frac{-1+\beta_1}{-1+\beta_2}, -\frac{\sqrt{-\alpha}}{\alpha}, 1\right]\right\}\right],$$

$$\left[-\sqrt{-\alpha}, 1, \left\{\left[\frac{\sqrt{-\alpha}\left(-1+\beta_1\right)}{\alpha\left(-1+\beta_2\right)}, \frac{-1+\beta_1}{-1+\beta_2}, \frac{\sqrt{-\alpha}}{\alpha}, 1\right]\right\}\right]$$

In each line above, the first value is the eigenvalue, and the second is its multiplicity (how many different eigenvectors there are for this eigenvalue). The eigenvector itself appears after these two quantities. Note that the eigenvectors are of the form

$$e_{\pm}^{c} = \left(\mp\frac{i\sqrt{\alpha}\left(1-\beta_1\right)}{\alpha\left(1-\beta_2\right)}, \frac{1-\beta_1}{1-\beta_2}, \mp\frac{i\sqrt{\alpha}}{\alpha}, 1\right)$$

The basis of the centre eigenspace can be obtained simply by taking the real and imaginary parts of one of these vectors:

$$e_1 = \left(0, \frac{1-\beta_1}{1-\beta_2}, 0, 1\right) \text{ and } e_2 = \left(\frac{1-\beta_1}{\sqrt{\alpha}\left(1-\beta_2\right)}, 0, \frac{1}{\sqrt{\alpha}}, 0\right)$$

If we let $x = (c_1, p_1, c_2, p_2)$ and call the steady state we're currently analyzing x_4, then the centre eigenspace of this steady state can be written in the form:

$$x = x_4 + a_1 e_1 + a_2 e_2$$

We have

$$\begin{aligned}
c_1 &= \frac{1-\beta_1}{1-\beta_1\beta_2} + a_2 \frac{1-\beta_1}{\sqrt{\alpha}\,(1-\beta_2)} \\
p_1 &= \frac{\alpha\,(1-\beta_1)}{1-\beta_1\beta_2} + a_1 \frac{1-\beta_1}{1-\beta_2} \\
c_2 &= \frac{1-\beta_1}{1-\beta_1\beta_2} + \frac{a_2}{\sqrt{\alpha}} \\
p_2 &= \frac{\alpha\,(1-\beta_2)}{1-\beta_1\beta_2} + a_1
\end{aligned}$$

We normally like to express manifolds and eigenspaces as explicit rather than parametric functions. In other words, we would prefer to write $c_2 = g\,(c_1, p_1)$ and $p_2 = h\,(c_1, p_1)$. To do this, all we have to do is to eliminate a_1 and a_2 from the above equations. If we do, we get a very simple result:

$$c_1 = \frac{1-\beta_2}{1-\beta_1} c_1 \tag{8a}$$

$$\text{and } p_2 = \frac{1-\beta_2}{1-\beta_1} p_1 \tag{8b}$$

We could now think about expanding the centre manifold in a series by adding quadratic terms (in c_1^2, c_2^2 and $c_1 c^2$) to the above expressions, and then figuring out the coefficients using the manifold equation. For this particular model, the centre eigenspace turns out to be exactly the centre manifold, i.e., there are no quadratic or higher-order correction terms. We start by writing down the manifold equations:

$$c_2' = \frac{\partial c_2}{\partial c_1} c_1' + \frac{\partial c_2}{\partial p_1} p_1', \text{ and } p_2' = \frac{\partial p_2}{\partial c_1} c_1' + \frac{\partial p_2}{\partial p_1} p_1'$$

On the centre eigenspace, we have

$$\frac{\partial c_2}{\partial p_1} = \frac{\partial p_2}{\partial c_1} = 0,$$

which simplifies our manifold equations considerably. Now substitute the relevant equations into the remaining terms. For the c_2 equation, we get

$$-\alpha c_1 \frac{1-\beta_2}{1-\beta_1} + p_1 \frac{1-\beta_2}{1-\beta_1}\left(c_1 \frac{1-\beta_2}{1-\beta_1} + \beta_2 c_1\right) = \frac{1-\beta_2}{1-\beta_1}\left[-\alpha c_1 + p_1 \left(c_1 + \beta_1 c_1 \frac{1-\beta_2}{1-\beta_1}\right)\right].$$

$$-\alpha + p_1 \left(\frac{1-\beta_2}{1-\beta_1} + \beta_2\right) = -\alpha + p_1 \left(1 + \beta_1 \frac{1-\beta_2}{1-\beta_1}\right)$$

$$-\alpha + p_1 \frac{1-\beta_1\beta_2}{1-\beta_1} = -\alpha + p_1 \frac{1-\beta_1\beta_2}{1-\beta_1}.$$

The equation is identically satisfied by the centre eigenspace. It is easy to verify that the p_2 equation is also identically satisfied. This means that the centre eigenspace is actually an invariant manifold, and thus that it is the centre manifold for this problem.

If we now substitute our centre manifold equations (1.8) into equations $7a$ and $7b$, we get the ODEs governing motion on the manifold:

$$c' = -\alpha c_1 + c_1 p_1 \frac{1-\beta_1\beta_2}{1-\beta_1} \tag{9a}$$

$$p_1' = p_1 \left(1 - c_1 \frac{1-\beta_1\beta_2}{1-\beta_1}\right) \qquad 9b$$

This is the *Lotka-Volterra model*.

Chapter 2

Robust Chaos

Assume that $f(\hat{x}, \hat{y}, \rho)$ represent a two-dimensional piecewise smooth map with the parameter ρ and assume that Γ_ρ where $\hat{x} = h(\hat{y}, \rho)$ indicate a smooth curve, the latter separates the phase plane into two R_a and R_b.

$$f(\hat{x}, \hat{y}, \rho) = \begin{cases} f_1(\hat{x}, \hat{y}, \rho) \text{ for } \hat{x}, \hat{y} \in R_a \\ \\ f_2(\hat{x}, \hat{y}, \rho) \text{ for } \hat{x}, \hat{y} \in R_b \end{cases} \tag{2.1}$$

The two functions f_1 and f_2 must be continuous, and their derivatives are exist and continuous. The derivative of the continuous function f is not continuous at the border Γ_ρ. It is further assumed that the one-sided partial derivatives at the border are finite and the partial derivatives must be finite by one side. We will study the bifurcations of the system for different parameter ρ. One of the following standard types occurs: period doubling, saddle-node or Hopf bifurcation when the fixed point of the map is located in one of the smooth regions R_a or R_b and this is a kind of bifurcations. Also, a discontinuous change in the elements of the Jacobian matrix will occur where ρ is diverse and that is when the fixed point falls on the border and this is another type of bifurcations. *Robust chaos* (attractor is unique) occurs as a result of collision bifurcation under special conditions $[6, 7, 8]$.

2.1 Normal form of two-dimensional piecewise smooth map

In some areas neighborhood to the collision bifurcation, we change the coordinates [6] and notice the reduction of the piecewise smooth map to the normal shape[8, 9] equation (2.1):

$$G_\mu = \begin{cases} \begin{pmatrix} T_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for } x \leq 0 \\ \\ \begin{pmatrix} T_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for } x > 0 \end{cases} \tag{2.2}$$

Both x and y represent the new coordinates where the line $x = 0$ is the *border* that divides the phase space into L and R. The new parameter is obtained by scaling ρ. On the side L we have the trace τ_L and determinant δ_L of the Jacobian matrix. On the R side, we find their corresponding values τ_R and δ_R. The trace and determinant are the same in the map f which were calculated near the boundary collision point, because they are not affected by changing the coordinates.

2.2 Invariant manifolds and regions for robust chaos

To explore the boundary collision bifurcation in the piecewise smooth map (2.1)[5, 6, 7, 8, 13], it suffices to study the normal form in (2.1), because local bifurcations[2, 3] do not depend on μ (because it is a variable through zero) but rather they depend on each of τ_L, δ_L, τ_R and δ_R. We can write the fixed points of the system on both sides as follows:

$$\begin{cases} L^* = \left(\frac{\mu}{1-\tau_L+\delta_L}, \frac{-\delta_L\mu}{1-\tau_L+\delta_L}\right) \\ \\ R^* = \left(\frac{\mu}{1-\tau_R+\delta_R}, \frac{-\delta_R\mu}{1-\tau_R+\delta_R}\right) \end{cases}$$

Egenvalues $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\delta}\right)$ tell us about the stability or unstability of the system. If it is

$$\tau_L > (1 + \delta_L) \text{ and } \tau_R < (1 + \delta_R) \tag{2.3}$$

We have the following results:

1. There are no fixed points for $\mu < 0$.

2. For $\mu > 0$ there are two fixed points in each of L and R.

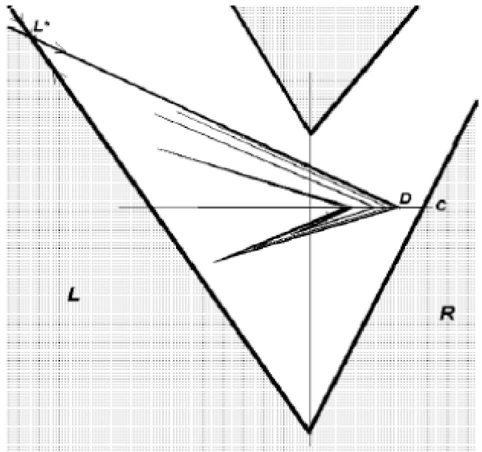

Figure 2.1: The stable and unstable manifolds of $L^*$for $\tau_L = 1.7, \delta_L = 0.5, \tau_R = -1.7, \delta_R = 0.5$.$R^*$is marked by the small cross in side the attractor.

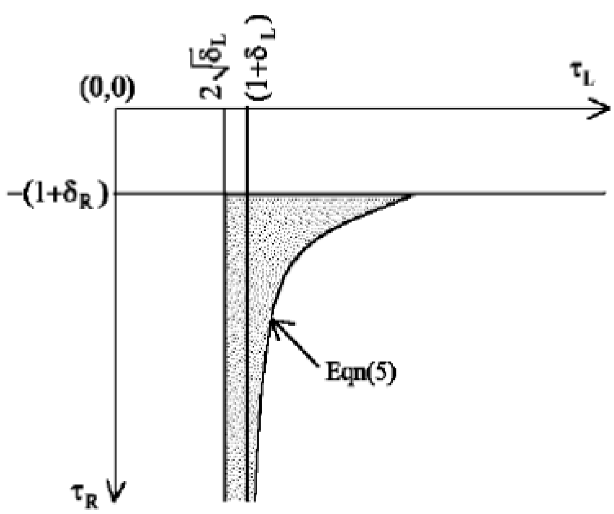

Figure 2.2: Schematic diagram of the parameter space region of the normal form eqution (2.8) where robust chaos is observed for $1 > \delta_L > 0, 1 > \delta_R > 0$ and $\mu > 0$.

3. For $\mu = 0$ the two border points will be born known as a *border collision pair bifurcation*. A similar state if $\tau_L < (1+\delta_L)$ and $\tau_R > (1+\delta_R)$ where μ is moving through zero. We notice that the two cases are similar, so we take the parameter region (2.2). We say about the fixed point that lies in L is *regular saddle* and we say about the other that lies in R is an *attractor* in the case of
$$(1+\delta_R) > \tau_R > -(1+\delta_R)$$
for $\mu > 0$. The previous result resembles the bifurcation of the saddle-node resulting on the border. We will exclude this region in the parameter space from our analysis related to chaotic behavior, due to the permanent existence of the periodic attractor in this region for $\mu > 0$.

4. For the condition $\tau_R = -(1+\delta_R)$ all fixed points on the line containing the points $\left(\frac{\mu}{1+\delta_R}, 0\right)$ and $\left(0, \frac{-\delta_R\mu}{1+\delta_R}\right)$ will be fixed points also in the next iteration. So we'll focus on this parameter from space region:
$$\tau_L > (1+\delta_L) \text{ and } \tau_R < -(1+\delta_R) \tag{2.4}$$
We'll check the property related to the attractor for $\mu > 0$.

5. If $1 > \tau_L \geq 0$ and $1 > \delta_R \geq 0$, then [7]we will have a flip saddle R^* and a regular saddle L^*.

6. Suppose S_L is the stable manifold and that $\mathbf{U}_L$ is the unstable manifold of L^* and $\mathbf{S}_R$ be the stable and $\mathbf{U}_R$ unstable manifold of R^*. For (2.1), all intersections of the unstable manifolds with $x = 0$ map to the line $y = 0$. Note that one linear map moves to another linear map through line $x = 0$, $\mathbf{U}_L$ and $\mathbf{U}_R$ will experience folds along the x-axis. Each image of fold point is a fold point as well. By a similar argument we conclude that $\mathbf{S}_L$ and $\mathbf{S}_R$ fold along the y-axis, and all pre-images of fold points are fold points.

7. Assume that $\lambda_{1R}, \lambda_{2R}$ are the eigenvalues at side R and $\lambda_{1L}, \lambda_{2L}$ are that the eigenvalues in side L. For condition (2.3), $\lambda_{1L} > \lambda_{2L} > 0$ and $0 > \lambda_{1R} > \lambda_{2R}$. The stable eigenvector at R^* has a slope $m_1 = (-\delta_R/\lambda_{1R})$ and the unstable eigenvector has a slope $m_2 = (-\lambda_R/\lambda_{2R})$.

8. Since points on an eigenvector map to points on the same eigenvector and since points on the y-axis map to the x-axis, we conclude that points of $\mathbf{U_R}$ to the left of y-axis map to points above x-axis. From this we find that $\mathbf{U_R}$ has an angle $m_3 = \frac{\delta_L\lambda_{2L}}{\delta_R - \tau_L\lambda_{2R}}$ after the first fold. Under condition (2.3) we have $m_1 > m_2 > 0$ and $m_3 < 0$. Therefore there must be a transverse homoclinic intersection in R. This *implies an infinity of homoclinic intersections and the existence of a chaotic orbit*.

9. We now investigate the stability of this orbit. The *basin boundary* consist by $\mathbf{S}_L$. $\mathbf{S}_L$ folds at the y-axis and intersects the x-axis at point C. The portion of $\mathbf{U}_L$ to the left of L^* goes to infinity and the portion to the right of L^* leads to the chaotic orbit. U_L meets the x-axis at point D, and then undergoes repeated foldings leading to an intricately folded compact structure as shown in Figure 2.3. The unstable eigenvector at L^* has a negative slope given by $-(\delta_L/\lambda_{1L})$. Therefore it must have a heteroclinic intersection with $\mathbf{S}_R$. Since both $\mathbf{U}_L$ and $\mathbf{U}_R$ have transverse intersections with $\mathbf{S}_R$, by the *Lambda Lemma* $[6, 10]$ we conclude that for each point q on $\mathbf{U}_R$ and for each $\in$-neighborhood $N_\epsilon(q)$, there exist points of $\mathbf{U}_L$ in $N_\epsilon(q)$. Since $\mathbf{U}_L$ comes arbitrarily close to $\mathbf{U}_R$, the *attractor must span* $\mathbf{U}_L$ in one side of the heteroclinic point. Since all initial conditions in L converge on $\mathbf{U}_L$ and all initial conditions in R converge on $\mathbf{U}_R$, and since there are points of U_L in every neighborhood of $\mathbf{U}_R$, *we conclude that the attractor is unique*.

10. Simple changes in parameters will cause minor changes to the Lyapunov exponents and *will not destroy the attractor*. Wherever the chaotic attractor is stable it is definitely robust. It is impossible for any point of the attractanr to be located to the right of the point D. The chaotic orbit is stable if the point D is to the left of the point C will be a chaotic saddle or an unstable chaotic orbit when D is located outside the basin of attraction. Here, we arrive at *the condition for the stability of the chaotic attractor* as follows:

$$\delta_L\tau_R\lambda_{1L} - \delta_R\lambda_{1L}\lambda_{2L} + \delta_R\lambda_{2L} - \delta_L\tau_R + \tau_L\delta_L - \delta_L^2 - \lambda_{2L}\delta_L > 0 \tag{2.5}$$

    If $\delta_L = \delta_R = \delta$ this condition reduces to $\tau_R\lambda_{1L} - \lambda_{1L}\lambda_2 + \tau_L - \tau_R - \delta > 0$.

11. The robust chaotic orbit continues to exist as τ_L is reduced below $(1+\delta_L)$.

12. With τ_L *slightly* below $(1+\delta_L)$, there is no fixed point in L for $\mu > 0$ but the invariant manifolds suffer only slight change. The invariant manifold of L associated with λ_{1L} still forms the attractor. The invariant manifolds in L, however, cease to exist for $\tau_L < 2\sqrt{\delta_L}$ since the eigenvalues become complex.

13. As τ_L is *reduced below* $2\sqrt{\delta_L}$ there is a sudden reduction in the size of the attractor as it spans only $\mathbf{U}_R$. So long as $\mathbf{U}_L$ exists, multiple attractors can not exist and therefore if the main attractor is chaotic, it is also robust. Therefore we see that for $1 > \delta_L > 0, 1 > \delta_R > 0$, the normal form equation (2.2) exhibits robust chaos in a portion of parameter space bounded by the conditions:

$$\tau_R = -(1+\delta_R), \tau_L > 2\sqrt{\delta_L}$$

    and (5) as shown in Figure 2.2.

14. There is a symmetric region of the parameter space with the roles of R and L interchanged, where the same phenomena are observed for $\mu < 0$. When the system approaches being one-dimensional, precisely for low values of the determinant, the principal attractor may not be able to remain chaotic even for $\tau_L > 2\sqrt{\delta_L}$ where the periodic orbits are stable.

15. Through the conditions for the existence of periodic windows in one-dimensional systems we determine the minimum for $\tau_L$where the the parameter range for robust chaos is limited by
$$\tau_R = 1, \tau_R > \frac{-\tau_L}{\tau_L - 1}$$
where τ represents the slopes of the piecewise linear function that line $x = 0$ divides into two halves.

16. Region for robust chaos will shrink in space τ_L-τ_R to zero region when the determinants on both sides are unity. For cases whose determinant is negative, we follow the same steps.

17. For $-1 < \delta_R < 0$, we have $1 > \lambda_{1R} > 0, \lambda_{2R} < -1$, and R^* is located above the x-axis. $\mathbf{U}_L$ converges on $\mathbf{U}_R$ from one side since the eigenvalue λ_{1R} is positive. If
$$\frac{\lambda_{1L} - 1}{\tau_L - 1 - \delta_L} > \frac{\lambda_{2R} - 1}{\tau_R - 1 - \delta_R} \tag{2.6}$$
then the intersection of $\mathbf{U}_L$ with the x-axis remains the rightmost point of the attractor and (5) still gives the parameter range for boundary crisis.

18. But if (2.5) is not satisfied, the intersection of $\mathbf{U}_R$ with the x-axis becomes the rightmost point of the attractor and the condition of existence of the chaotic attractor will change to
$$\frac{\lambda_{1L} - 1}{\tau_L - 1 - \delta_L} < \frac{\delta_L \left(\tau_L - \delta_L - \lambda_{2L}\right)}{\left(\tau_L - 1 - \delta_L\right)\left(\delta_R \lambda_{2L} - \delta_L \tau_R\right)} \tag{2.7}$$

19. For $\delta_L < 0$ and $\delta_R < 0$, L^* is below the x-axis and the same logic as above applies. $\mathbf{U}_L$ will not converge from $\mathbf{U}_R$ from one side because the stable manifold of R^* will have negative eigenvalue in the case of $\delta_L < 0$ and $\delta_R > 0$.

20. Where (2.7) is not satisfied we determine the boundary crisis numerically, because in this case there is no analytical condition. Since the eigenvalues are real for all τ_L, invariant manifolds $\mathbf{S}_L$ and $\mathbf{U}_L$ are always exist and this is in the case of $\delta_L < 0$.

21. For $\delta_L < 0$ *multiple attractors cannot exist.*

Under the condition that there is no more than one period-1 fixed point in both R_a and R_b, we would expect robust chaos in many piecewise smooth maps near of the border collision bifurcation because (2.2) is a normal form of the piecewise smooth map (2.1). There are also homoclinic intersections of the invariant manifolds linked to these fixed points and the same for heteroclinic and it is worth noting that both trace and determinant fulfill every condition mentioned.

Chapter 3

Lyapunov stability theory

From the seventeenth century to the beginning of the twentieth century, the concepts of *classical stability* was developed in an impressive way until the emergence of Lyapunov's theory[14, 15]. On 12 October 1892 Lyapunov presented his thesis on the stability of the movement, which is one of the greatest ideas in the development of modern technology. He has performed the unstable of steam engines led to a significant rise in the number of accidents and is the main reason for the development of stability theories. The theory of Lyapunov is considered a cornerstone in the theories of system control almost, as it describes the stability of a dynamic system. Then the *Lyapunov's second stability theory* is the most popular[15]. Also, Lyapunov's second method is applicable to both linear and nonlinear systems: We construct a function with two basic conditions: The function is positive definite and its derivative is negative semi-definite or negative definite. The method for constructing the Lyapunov function is extremely difficult, since it is not easy to find a function that satisfies both conditions. Notice that the *classical LaSalle invariance principle* is used in the analysis and stability of autonomous systems, it is based on *invariant set*. So the principle of LaSalle also faces another difficulty if the set is not invariant, that is when the systems are no autonomous. In this chapter we present Lyapunov's stability theorem, then the LaSalle's invariance principle and its generalization, and finally we dealt with the *unstability theory* with the inclusion of illustrative examples.

3.1 Some definitions and consequences

In this chapter, we focus on the differential equations from the formula

$$x' = f\left(x,t\right), \quad x\left(t_o\right) = x_o, x \in \mathbb{R}^n \tag{3.1}$$

where $t \geq 0$. The system defined by (3.1) is said to be *autonomous* or time-invariant if f does not depend explicitly on t. It is said to be *linear* if $f(x,t) = A(t)x$ for some $A(.) : \mathbb{R}_+ \longmapsto \mathbb{R}^{n\times n}$ and *nonlinear* otherwise. In this chapter, we will assurne that $f(x,t)$ is *piecewise continuous* with respect to t, that is, there are only finitely many *discontinuity* points in any compact set. The notation B_h will be short-hand for $B(0,h)$, the ball of radius h centered at 0. Properties will be said to be true

- Locally if they are true for all x_0 in some ball B_h,
- Globally if they are true for all $x_0 \in \mathbb{R}^n$.
- Semi-globally if they are true for all $x_0 \in B_h$ with h arbitrary.
- Uniformly if they are true for all $t_0 \geq 0$.

The properties are considered locally true.

3.1.1 Lipschitz's condition and its consequences

Definition 3.1 *We can say that the function f is a Lipschitz local function continuous in x if for some $h > 0$ there exists $l \geq 0$ such that*

$$|f(x_1,t) - f(x_2,t)| \leq l\,|x_1 - x_2|\,, \text{ for all } x_1, x_2 \in B_h, t \geq 0 \tag{3.2}$$

The constant l is the *Lipschitz's constant*. A definition for globally Lipschitz continuous functions follows by requiring equation (3.1) to hold for $x_1, x_2 \in \mathbb{R}^n$. The definition [9]of semi-globally Lipschitz continuous functions holds as well by requiring that equation (3.2) hold in B_h for arbitrary h but with l possibly a function of h. The Lipschitz property is by default assumed to be uniform in t. If f is Lipschitz continuous in x, it is continuous in x. On the other hand, if f has *bounded partial derivatives* in x, then it is Lipschitz. Formally, if $D_1 f(x,t) = \left(\frac{\partial f_i}{\partial x_j}\right)$ indicates the partial derivative matrix of f with respect to x (the symbol 1 stands for the first argument of $f(x,t)$), then $|D_1 f(x,t)| \leq l$: implies that f is Lipschitz continuous with Lipschitz constant l (again locally, globally or semi-globally depending on the region in x that the bound on $|D_2 f(x,t)|$ is valid). If f is locally bounded and Lipschitz continuous in x, then the differential equation (3.1) has a unique solution on some time interval (so long as $x \in B_h$).

Definition 3.2 (Equilibrium point) *We say about x^* an equilibrium point to the equation (3.1) if $f(x^*,t) \equiv 0$ for all $t \geq 0$. If $f(x,t)$ is Lipschitz continuous in x, then the solution $x(t) \equiv x^*$ for all t is called an equilibrium solution.*

By translating the origin to the equilibrium point x^*we can make 0 an equilibrium point of (3.1). The Lipschitz's hypothesis gives us bounds on the rates of convergence or divergence of solutions from the origin.

Proposition 3.1 (Rate of Growth/Decay) *Under the assumption that the origin $x = 0$ is an equilibrium point of (3.2) and f is Lipschitz in x with Lipschitz constant l and piecewise constant with respect to t, so the solution $x(t)$ will satisfie the following*

$$|x_0| e^{l(t-t_0)} \geq |x(t)| \geq |x_0| e^{-l(t-t_0)} \tag{3.3}$$

as lang as $x(t)$ remains in B_h.

Proof. Because $|x|^2 = x^T x$, it follows that

$$\begin{aligned} \left|\frac{d}{dt}|x|^2\right| &= 2|x|\left|\frac{d}{dt}|x|\right| \\ &= 2\left|x^T\frac{d}{dt}x\right| \leq 2|x|\left(\frac{d}{dt}|x|\right) \end{aligned} \tag{3.4}$$

and for that

$$\left|\frac{d}{dt}|x|\right| \leq \frac{d}{dt}|x|$$

Since the function $f(x,t)$ is Lipschitz continuous and satisfies $f(x,0) = 0$, then

$$-l|x| \leq \frac{d}{dt}|x| \leq l|x| \tag{3.5}$$

This includes that every trajectory that begins inside the ball B_h will not leave it for at least a finite period of time. Also, if $f(x,t)$ is globally Lipschitz, it guarantees that the solution has no finite escape time, that is, it is finite at every finite instant. This proposition also proves that the exponential convergence to zero is faster than the convergence of the solutions $x(t)$. ■

Based on that, we will present stability definitions. Informally $x = 0$ is stable equilibrium point if trajectories $x(t)$ of (3.1) remain close to the origin if the initial condition x_0 is close to the origin. For more clarity, we follow the next definitions:

Definition 3.3 (Stability in the sense of Lyapunov) *The equilibrium point $x = 0$ is called a stable equilibrium point of (3.1) if for all $t_0 \geq 0$ and $\epsilon < 0$, there exists $\delta(t_0, \epsilon)$ such that*

$$|x_0| < \delta(t_0, \epsilon) \implies |x(t)| < \epsilon, \ \forall t \geq t_0 \tag{3.6}$$

The solution $x(t)$ of (3.1) starts from x_0 at to t_0. Figure 3.1 illustrates the trajectories that start in the ball B_δ and not leave the ball B_ϵ. We call this definition *stability in the sense of Lyapunov* $[11, 12]$at time to t_0.

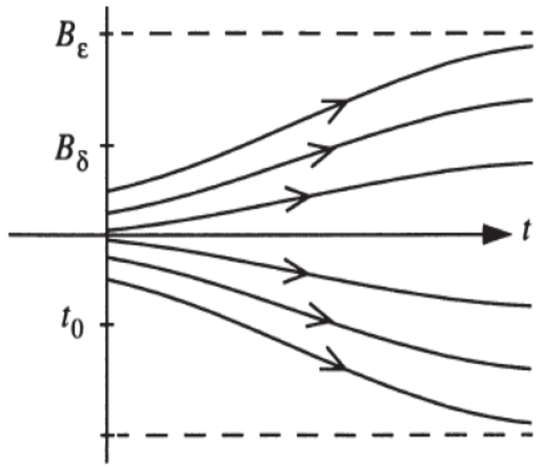

Figure 3.1: Diagram of stability definition.

Definition 3.4 (Uniform Stability) *The equilibrium point $x = 0$ is called a uniformly stable equilibrium point of (3.1) if in the preceding definition δ can be chosen independent of t_0.*

The definition of uniform stability captures the notion that the equilibrium point is not getting progressively less stable wilh time [15]. Then, it prohibits a situation in which given an $\epsilon > 0$, the ball of initial conditions of radius $\delta(t_0, \epsilon)$ in the definition of stability required to hold trajectories in the ϵ ball tends to zero as $t_0 \longrightarrow \infty$. There is a weakness in the definition of stability, as it does not require the trajectories to begin near the origin to tend to the origin asymptotically. The next definition includes this property.

Definition 3.5 (Asymptotic Stability) *The equilibrium point $x = 0$ is an asymptotically stable equilibrium point of (3.1) if (1) $x = 0$ is a stable equilibrium point of (3.1). (2) $x = 0$ is attractive[1], that is for all $t_0 \geq 0$ there exists a $\delta(t_0)$ such that*

$$|x_0| < \delta \implies \lim_{t \longrightarrow \infty} |x(t)| = 0$$

Asymptotic stability indicates that the convergence of paths with the origin does not necessarily mean that the equilibrium point is stable.The example below illustrates the concept of asymptotic stability.

$$\begin{cases} x_1' = x_1^2 - x_2^2 \\ \\ x_2' = 2x_1x_2 \end{cases} \tag{3.7}$$

We notice that all trajectories tend to the origin when $t \longrightarrow \infty$, except for the trajectory that follows the positive x_1 axis to $+\infty$. If we assume that this *point at infinity* is the same as the point

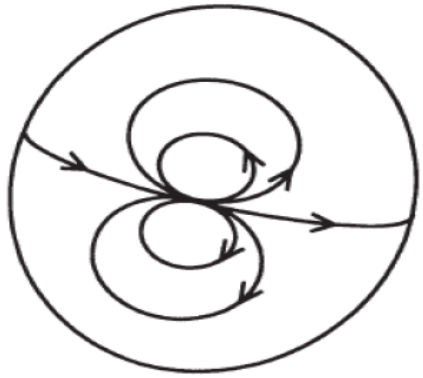

Figure 3.2: Equilibrium point whichis not stable and their trajectories.

at $x_1 = -\infty$. In this case, all trajectories go to the origin and the equilibrium point is not stable in the sense of Lyapunov: Given any $\epsilon > 0$, no matter how small a δ we choose for the ball of initial condition, there are always some initial conditions close to the x_1 axis which will exit the ϵ ball before converging to the origin. The trajectory starting from the x_1 axis gives us a prediction about this behavior.

Definition 3.6 (Uniform Asymptotic Stability) *The equilibrium point $x = 0$ is a uniformly asymptotically stable equilibrium point of* (3.1) *if (1) $x = 0$ is a uniformly stable equilibrium point of* (3.1). *(2) The trajectory $x(t)$ converges uniformly to 0, that is, there exists $\delta > 0$ and a function $\gamma(\tau, x_0) : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that $\lim_{\tau\longrightarrow\infty} \gamma(\tau, x_0) = 0$ for all $x_0 \in B_\delta$ and*

$$|x_0| < \delta \Longrightarrow |x(t)| \leq \gamma(t - t_0, x_0), \ \forall\, t \geq t_0.$$

The previous definitions are local, since they concern neighborhoods of the equilibrium point. Global asymptotic stability and *global uniform asymptotic stability* are defined as follows:

Definition 3.7 (Global asymptotic stability) *The equilibrium point $x = 0$ is a globally asymptotically stable equilibrium point of* (3.1) *if it is stable and $\lim_{t\longrightarrow\infty} x(t) = 0$ for all $x_0 \in \mathbb{R}^n$.*

Definition 3.8 (Global uniform asymptotic stability.) *The equilibrium point $x = 0$ is a globally, uniformly, asymptotically stable equilibrium point of* (3.1) *if it is globally asymptotically stable and if in addition, the convergence to the origin of trajectories is uniform in time, that is to say that there is a function $\gamma : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ such that*

$$|x(t)| \leq \gamma(x_0, t - t_0), \ \forall\, t \geq 0$$

In the definitions of asymptotic stability we cannot know how fast the trajectories converge to the origin. For time varying and nonlinear systems, the rate of convergence can be of many different types. Otherwise, time-invariant linear systems, the speed of convergence of trajectories either to or from the origin is exponential. In the following we will present the strongest stability types, which depend on an exponential rate of convergence.

Definition 3.9 (Exponential stability, rate of convergence) *The equilibrium point* $x = 0$ *is an exponentially stable equilibrium point of* (3.1) *if there exist* $m, \alpha > 0$ *such that*

$$|x(t)| \leq me^{-\alpha(t-t_0)}|x_0| \tag{3.8}$$

for all $x_0 \in B_h, t \geq t_0 \geq 0$. *The constant* α *is called (an estimate of) the rate of convergence.*

For *global exponential stability*, we define it by requiring equation (3.8) to hold for aII $x_0 \in \mathbb{R}^n$. Also, *semi-global exponential stability* is defined similarly except that the constants m, α become a functions to h. For linear systems, the exponential stability appears equivalent to the asymptotic stability. The exponential stability is considered stronger than the asymptotic stability.

3.2 Lyapunov stability theorems

Lyapunov's second method or *direct method* is based on this concept: If we have a system, and this system has a point or equilibrium state, then the total energy stored in that system will reduce over time until it reaches its lowest value at a point or state of equilibrium. Therefore, to determine the stability of the systems, the Lyapunov function must be defined. The determination of this function is related to the concept of energy dissipation. So, we will provide some definitions and characteristics that will help us to understand Lyapunov's mechanism[14].

Definition 3.10 (Class K, KR Functions) *A function* $\alpha(.) : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ *belongs to class* K *if it is continuous, strictly increasing and* $\alpha(0) = 0$. *The function* $\alpha(.)$ *is said to belong to class* KR *if* α *is of class* K *and in addition,* $\alpha(p) \longrightarrow \infty$ *as* $p \longrightarrow \infty$.*(denoted by* $\alpha(.) \in K$*).*

Definition 3.11 (Locally positive definite functions) *A continuous function* $v(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ *is called a locally positive definite function* $(l.p.d.f)$ *if, for some* $h > 0$ *and some* $\alpha(.)$ *of class* K*,*

$$v(0,t) = 0 \text{ and } v(x,t) \geq \alpha(|x|), \ \forall\, x \in B_h, \ \ t \geq 0. \tag{3.9}$$

Definition 3.12 (Positive definite functions) *A continuous function* $v(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ *is called a positive definite function* $(p.d.f.)$ *if for some* $\alpha(.)$ *of class* KR

$$v(0,t) = 0 \text{ and } v(x,t) \geq \alpha(|x|), \ \forall\, x \in \mathbb{R}^n, \ \ t \geq 0. \tag{3.10}$$

and $\alpha(p) \longrightarrow \infty$ *as* $p \longrightarrow \infty$.

In the previous definitions of $l.p.d.f.s$ and $p.d.f.s$, the energy was not bounded from the top due to variety of t.

Definition 3.13 (Decrescent functions) *A continuous function $v(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ is called a decrescent function if, there exists a function $\beta(.)$ class K, such that*

$$v(x,t) \leq \beta(|x|), \ \forall \, x \in B_h, \ \ t \geq 0 \tag{3.11}$$

Example 3.1 (Examples of energy-like functions) *In the following we will see examples of K and KR-class energy-like function. Which achieve these properties:*

1. $v(x,t) = |x|^2$: $p.d.f.$, decrescent.
2. $v(x,t) = x^T P x$, with $P \in \mathbb{R}^{n \times n} > 0$: $p.d.f$, decrescent.
3. $v(x,t) = (t+1)|x|^2 : p.d.f.$
4. $v(x,t) = e^{-t}|x|^2$: decrescent.
5. $v(x,t) = \sin^2\left(|x|^2\right)$: $l.p.d.f$, decrescent.
6. $v(x,t) = e^t x^T P x$ with P not positive definite: not in any of the above classes.
7. $v(x,t)$ not explicitly depending on time t: decrescent.

Generally speaking, the basic theorem of Lyapunov states that when $v(x,t)$ is a $p.d.f.$ or an $l.p.d.f.$ and $\frac{dv(x,t)}{dt} \leq 0$ then we can conclude stability of the equilibrium point. The time derivative is taken along the trajectories of (3.1), i.e.,

$$\frac{dv(x,t)}{dt}\Big|_{(3.1)} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x} f(x,t) \tag{3.12}$$

The rate of change of $v(x,t)$ along the trajectories of the vector field (3.1) is also called the *Lie derivative* of $v(x,t)$ along $f(x,t)$. The following table is based on translated the origin to the equilibrium point.

	Conditions on $v(x,t)$	Conditions on $-\dot{v}(x,t)$	Conclusions
1.	l.p.d.f.	≥ 0 locally	stable
2.	l.p.d.f., decrescent	≥ 0 locally	uniformly stable
3.	l.p.d.f., decrescent	l.p.d.f.	uniformly asymptotically stable
4.	p.d.f., decrescent	p.d.f.	globally unif. asymp. stable

Proof. 1. Since v is an $l.p.d.f.$, we have that for some $\alpha(.) \in K$,

$$v(x,t) \geq \alpha(|x|), \forall\, x \in B_s, \tag{3.13}$$

also, the hypothesis is that

$$v'(x,t) \leq 0\ ,\ \forall\, t \geq t_0, \forall\, x \in B_r \tag{3.14}$$

Given $\in > 0$, define $\epsilon_1 = \min(\epsilon, r, s)$. Choose $\delta > 0$ such that

$$\beta(t_0, \delta) = \sup_{|x| \leq \delta} v(x, t_0) < \alpha(\epsilon_1)$$

Such a δ always exists, since $\beta(t_0, \delta)$ is a continuous function of δ and $\alpha(\epsilon_1) > 0$. We now claim tha $|x(t_0)| \leq \delta$ indicates that $|x(t)| \leq \epsilon_1$, $\forall t \geq t_0$. The proof is by contradiction. Clearly since

$$\alpha(|x(t_0)|) \leq v(x(t_0), t_0) < \alpha(\epsilon_1)$$

it follows that $|x(t_0)| < \epsilon_1$. Now, if it is not true that $|x(t)| < \epsilon_1$ for all t, let $t_1 > t_0$ be the first instant such that $|x(t)| \geq \epsilon_1$. Then

$$v(x(t_1), t_1) \geq \alpha(\epsilon_1) > v(x(t_0), t_0) \tag{3.15}$$

But this is a contradiction, since $v'(x(t), t) \leq 0$ for all $|x| < \epsilon_1$. Thus,

$$|x(t)| \ < \epsilon_1, \forall\, t \geq t_0$$

2. Since v is decrescent,

$$\beta(\delta) = \sup_{|x| \leq \delta} \sup_{t \geq t_0} v(x,t) \tag{3.16}$$

is nondecreasing and satisfies for some d

$$\beta(\delta) < \infty \ \text{ for } \ 0 \leq \delta \leq d$$

Now choose δ such that $\beta(\delta) < \alpha(\epsilon_1)$ .If $-\dot{v}(x,t)$ is an $l.p.d.f.$, then $v'(x,t)$ satisfies the conditions of the previous proof so that 0 is a uniformly stable equilibrium point. We need to show the existence of $\delta_1 > 0$ such that for $\epsilon > 0$ there exists $T(\epsilon) < \infty$ such that

$$|x_0| < \delta_1 \implies |\phi(t_1 + t, x_0, t_1)| < \epsilon \text{ when } \ t > T(\epsilon)$$

The hypotheses guarantee that there exist functions $\alpha(.), \beta(.), \gamma(.) \in K$ such that $\forall t \geq t_0, \forall x \epsilon B_r$, such that

$$\begin{cases} \alpha(|x|) \leq v(x,t) \leq \beta(|x|), \\ \\ \quad v'(x,t) \leq -\gamma(|x|) \end{cases}$$

Given $\epsilon > 0$, define δ_1, δ_2 and T by

$$\begin{aligned} \beta(\delta_1) &< \alpha(r), \\ \beta(\delta_2) &< \min(\alpha(\epsilon), \beta(\delta_1)), \\ T &= \alpha(r)/\gamma(\delta_2). \end{aligned}$$

This choice is explained in Figure 3.3. We now show that there exists at least one instant $t_2 \in [t_1, t_1+T]$ when $|x_0| < \delta_2$. The proof is by contradiction. The notation $\phi(t, x_0, t_0)$ stands for the trajectory of (3.1) starting from x_0 at time t_0. Indeed, if

$$|\phi(t, x_0, t_1)| \geq \delta_2,\ \forall t \in [t_1, t_1+T],$$

then it follows that

$$\begin{aligned} 0 &\leq \alpha(\delta_2) \leq v(s(t_1+T, x_0, t_1), t_1+T) \\ &= v(t_1, x_0) + \int_{t_1}^{t_1+T} \dot{v}(\tau, \phi(\tau, x_0, t_1))\, d\tau \\ &\leq \beta(\delta_1) - T\gamma(\delta_2) \\ &\leq \beta(\delta_1) - \alpha(r) \\ &< 0 \end{aligned}$$

To create a contradiction, we compare the ends of previous chain of inequalities. Now, if $t \geq t_1+T$, then

$$\alpha(|\phi(t, x_0, t_1)|) \leq v(t, \phi(t, x_0, t_1)) \leq v(t_2, \phi(t_2, x_0, t_1)),$$

since $v'(x,t) \leq 0$ (the definition of δ_1 guarantees that the trajectory stays in B_r, so that $v'(x,t) \leq 0$). Thus, we get

$$\begin{aligned} \alpha(|\phi(t, x_0, t_1)|) &\leq v(t_2, \phi(t_2, x_0, t_1)) \leq \beta(|\phi(t_2, x_0, t_1)|) \\ &\leq \beta(\delta_2) \\ &< \alpha(\epsilon) \end{aligned}$$

so that $\phi(t_2, x_0, t_1) < \epsilon$ for $t \geq t_1+T$. ■

The tabular version of Lyapunov's theorem is meant to focus the following correlations between the assumptions on $v(x,t), v'(x,t)$ and the conclusions: Decrescence of $v(x,t)$ is related with uniform stability and the local positive definite character of $v'(x,t)$ being associated with asymptotic stability.

(a) $-v'(x,t)$ is required to be an $I.p.d.f$ for asymptotic stability.

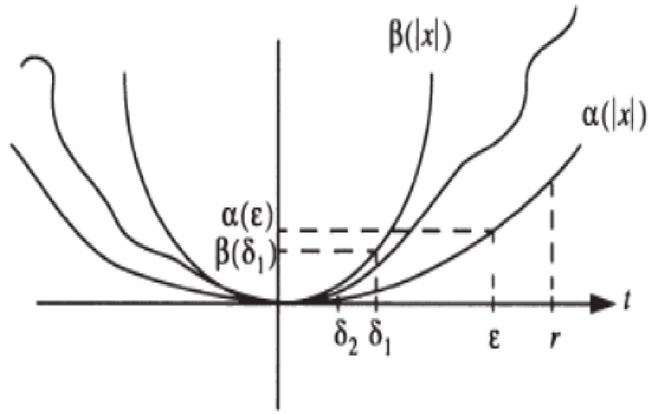

Figure 3.3: Lyapunov/s theorem and their constants.

(b) $v(x,t)$ being a $p.d.f$ is associated with global stability. However, this correlation is not perfect, since $v(x,t)$ being $I.p.d.f.$ and $-v'(x,t)$ being $l.p.d.f$ does not guarantee local asymptotic stability.

3.2.1 Example of Lyapunov's theory

The next figure represent an RLC circuit wich consisting of the following elements linear inductor, nonlinear capacitor and inductor. Also, we have a model for a mechanical system with a mass coupled to a nonlinear spring and nonlinear damper. Using as state variables x_1, the charge on the capacitor (respectively, the position of the block) and x_2, the current through the inductor (respectively, the velocity of the block). The following equations describe this system:

$$\begin{cases} x_1' = x_2 \\ x_2' = -f(x_2) - g(x_1) \end{cases} \tag{3.17}$$

$f(.)$ represent a continuous function modeling the resistor current-voltage characteristic, and $g(.)$ the capacitor charge-voltage characteristic (respectively the friction and restoring force models in the mechanical analog). We suppose that f, g both modellocally passive elements, i.e., there exists a σ_0 such that

$$\sigma f(\sigma) \geq 0 \ \forall \sigma \epsilon [-\sigma_0, \sigma_0]$$

$$\sigma g(\sigma) \geq 0 \ \forall \sigma \epsilon [-\sigma_0, \sigma_0]$$

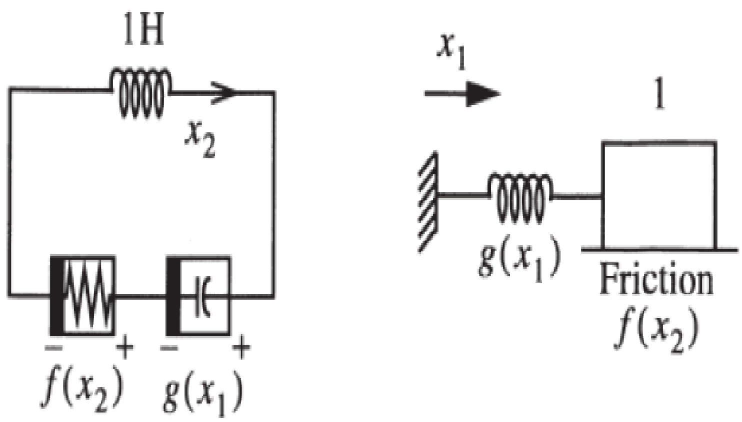

Figure 3.4: RLC circuit simple application of Lyapunov therem.

The Lyapunov function candidate is the total energy of the system, namely:

$$v(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)\, d\sigma$$

The first term is the energy stored in the inductor (kinetic energy of the body) and the second term the energy stored in the capacitor (potential energy stored in the spring).The function $v(x)$ is an $l.p.d.f.$, provided that $g(x_1)$ is not identically zero on any interval. Also, we have

$$v'(x) = x_2\left[-f(x_2) - g(x_1)\right] + g(x_1)\, x_2 = -x_2 f(x_2) \leq 0$$

while $|x_2|$ is less than σ_0. This establishes the stability but not asymptotic stability of the origin. In effect, the origin is actually asymptotically stable, but this needs the *LaSalle principle*. We will return to this point later.

3.2.2 Exponential Stability Theorems

Lyapunov's basic theorems depend on giving explicit rates of convergence towards equilibrium. We can modify these theorems to accommodate exponentially stable equilibria. Since a stable equilibrium is *robust* for perturbation, all practical applications seek it.

Theorem 3.1 (Exponential stability theorem and its converse) *Suppose that $f(x,t) : \mathbb{R}_+ \times \mathbb{R}^n \longmapsto \mathbb{R}^n$ has a continuous first partial derivatives in x and is piecewise continuous in t , Then the two statements below are equivalent*[15]*:*
(1) $x = 0$ is a locally exponentially stable equilibrium point of $x' = f(x,t)$, i.e., if $x \in B_h$ for h small enough, there exist $m, \alpha > 0$ such that

$$|\phi(\tau, x, t)| \leq m e^{-\alpha(\tau - t)}$$

(2) There exists a function $v(x,t)$ and some constants $h, \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ such that for all $x \in B_h$,$t \geq 0$ we have

$$\begin{cases} \alpha_1 |x|^2 \leq v(x,t) \leq \alpha_2 |x|^2 \\ \frac{dv(x,t)}{dt} \mid_{(5.1)} \leq -\alpha_3 |x|^2 \\ \left| \frac{\partial v(x,t)}{\partial t} \right| \leq \alpha_4 |x|^2 \end{cases} \tag{3.18}$$

Proof. $(1) \implies (2)$: We will prove the inequalities of (3.18) in turn, starting from the definition of $v(x,t)$: indicate by $\phi(\tau, x, t)$ the solution of (3.1) at time r starting from x at time t, and define

$$v(x,t) = \int_t^{t+T} |\phi(\tau, x, t)|^2 \, d\tau \tag{3.19}$$

where T will be defined later. From the exponential stability of the system at rate α and the lower bound on the rate of growth, we have

$$m|x| e^{-\alpha(\tau - t)} \geq |\phi(\tau, x, t)| \geq |x| e^{-l(\tau - t)} \tag{3.20}$$

for $x \in B_h$ for some h. Also l the Lipschitz constant of $f(x,t)$ exists because of the assumption that $f(x,t)$ has continuous first partial derivatives with respect to x. This, when used in (3.19), yields the first inequality of (3.18) for $x \in B_h$ (where h' is chosen to be h/m) with

$$\alpha_1 = \frac{\left(1 - e^{-2lT}\right)}{2l}, \quad \alpha_2 = m^2 \frac{\left(1 - e^{-2\alpha T}\right)}{2\alpha} \tag{3.21}$$

Differentiating (3.19) with respect to t yields

$$\frac{dv(x,t)}{dt} = |\phi(t+T, x, t)|^2 - |\phi(t, x, t)|^2 + \int_t^{t+T} \frac{d}{dt} \left(|\phi(\tau, x(t), t)|^2 \right) d\tau \tag{3.22}$$

Note that d/dt is the derivative with respect to the initial time t along the trajectories of (3.1). However, since for all Δt the solution satisfies

$$\phi(\tau, x(t + \Delta t), t + \Delta t), t + \Delta t = \phi(\tau, x(t), t)$$

we have that that $\frac{d}{dt}\left(|\phi(\tau, x(t), t)|^2\right) \equiv 0$. Using the fact that $\phi(t, x(t), t) = x$ and the exponential bound on the solution, we have that

$$\frac{dv(x,t)}{dt} \leq -\left(1 - m^2 e^{-2\alpha T}\right) |x|^2$$

The second inequality of (3.18) now follows, provided that $T > \left(\frac{1}{\alpha}\right)$ In m and

$$\alpha_3 = 1 - m^2 e^{-2\alpha T}$$

Differentiating (3.19) with respect to x_j, we have

$$\frac{\partial v(x,t)}{\partial x_i} = 2\int_t^{t+T} \sum_{j=1}^{n} \phi_j(\tau,x,t)\frac{\partial \phi_j(\tau,x,t)}{\partial x_i} d\tau \tag{3.23}$$

By way of notation define

$$Q_{ij}(\tau,x,t) = \frac{\partial \phi_j(\tau,x,t)}{\partial x_i}$$

and

$$A_{ij}(x,t) = \frac{\partial f_i(x,t)}{\partial x_j}$$

Interchanging the order of differentiation by τ, with differentiation by x_j yields that

$$\frac{d}{d\tau}Q(\tau,x,t) = A(\varphi(\tau,x,t),t).Q(\tau,x,t) \tag{3.24}$$

Thus $Q(\tau,x,t)$ is the state transition matrix associated with the matrix $A(\phi(\tau,x,t),t)$. By the assumption on boundedness of the partials of f with respect to x, it follows that $|A(.,.)| \leq k$ for some k, so that

$$|Q(\tau,x,t)| \leq e^{k(\tau-t)}$$

using this and the bound for exponential convergence in (3.23) yields

$$\left|\frac{\partial v(x,t)}{\partial x}\right| \leq 2\int_t^{t+T} m|x|\, e^{(k-\alpha)(\tau-t)}d\tau$$

which is the last equation of (3.18) if we define

$$\alpha_4 = \frac{2m\left(e^{(k-\alpha)T}-1\right)}{(k-\alpha)}$$

This completes the proof, but note that $v(x,t)$ is only defined for $x\epsilon B_{h'}$ with $h' = h/m$, to guarantee that $\phi(\tau,x,t)\,\epsilon B_h$ for all $\tau \geq t$.

$(2) \implies (1)$: This is simple, as may be verified by noting that equation (3.18) reveals that

$$v'(x,t) \leq -\frac{\alpha_3}{\alpha_2}v(x,t) \tag{3.25}$$

This in turn indicates that

$$v(t,x(t)) \leq v(t_0,x(t_0))\, e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)} \tag{3.26}$$

Using the lower bound for $v(t,x(t))$ and the upper bound for $v(t_0,x(t_0))$ we get

$$\alpha_1|x(t)|^2 \leq \alpha_2|x(t_0)|^2 e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}. \tag{3.27}$$

Using the estimate of (3.27) it follows that

$$|x(t)| \leq m|x(t_0)|\, e^{-\alpha(t-t_0)}.$$

with $m = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}}$, $\alpha = \frac{\alpha_3}{2\alpha_2}$. ■

3.3 LaSalle's invariance principle

Lyapunov's function does not guarantee that the solution will be attracted to the equilibrium point/set, so we need to extend the theory of Lyapunov. We will now turn to the famous LaSalle principle[12], which assures that a solution will always gravitate towards the point of equilibrium even when $dv \leq 0$. The importance of the Lassalle principle lies in two main points:

1. Conclusion of asymptotic stability even when $-v'(x,t)$ is not an $l.p.d.f.$
2. Prove that the trajectories that start in a certain region will converge with one of the points of equilibrium in that region.

Definition 3.14 (ω-limit set) *A set $S \subset \mathbb{R}^n$ is the ω-limit set of a trajectory $\phi(., x_0, t_0)$ if for every, $y \in$ there exists a sequence of times $t_n \longrightarrow \infty$ such that $\phi(t_n, x_0, t_0) \longrightarrow y$.*

Definition 3.15 (Invariant set) *A set $M \subset \mathbb{R}^n$ is said to be an invariant set if whenever $y \in M$ and $t_0 \geq 0$, we have*

$$\phi(t, y, t_0) \in M, \ \forall t \geq t_0$$

The following propositions gives some properties of invariant sets and ω-limit sets:

Proposition 3.2 *If $\phi(., x_0, t_0)$ is a bounded trajectory, its ω-limit set is compact. Further, $\phi(t, x_0, t_0)$ approaches its ω-limit set as $t \longrightarrow \infty$.*

Proposition 3.3 *Assume that the system (3.1) is autonomous and let S be the ω-limit set of any trajectory. Then S is invariant.*

Proof. Let $y \in S$ and $t_1 \geq 0$ be arbitrary. We need to show that $\phi(t, y, t_1) \in S$ for all $t \geq t_1$. Now $y \in S \implies \exists t_n \longrightarrow \infty$ such that $\phi(t_n, x_0, t_0) \longrightarrow y$ as $n \longrightarrow \infty$. Since trajectories are continuous in initial conditions, it follows that

$$\phi(t, y, t_1) = \lim_{n \longrightarrow \infty} \phi(t, \phi(t_n, x_0, t_0), t_1) = \lim_{n \longrightarrow \infty} \phi(t + t_n - t_1, x_0, t_0)$$

since the system is autonomous. Now, $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ so that the right hand side converges to an element of S. ■

Proposition 3.4 (LaSalle's Principle) *Let $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously differentiable and suppose that*

$$\Omega_c = \{x \in \mathbb{R}^n : v(x) \leq c\}$$

is bounded and that $v' \leq 0$ for all $x \in \Omega_c$. Define $S \subset \Omega_c$ by

$$S = \{x \in \Omega_c \ : v'(x) = 0\}$$

and let M be the largest invariant set in$S$. Then, whenever $x_0 \in \Omega_c$, $\phi(t, x_0, 0)$ approaches M as $t \longrightarrow \infty$.

Proof. Let $x_0 \in \Omega_c$. Since $v(\phi(t, x_0, 0))$ is a nonincreasing function of time we see that $\phi(t, x_0, 0) \in \Omega_c$, $\forall t$. Further, since Ω_c is bounded $v(\phi(t, x_0, 0))$ is also bounded below. Let

$$c_0 = \lim_{t \longrightarrow \infty} v(\phi(t, x_0, 0))$$

and let L be the ω limit set of the trajectory. Then $v(y) = c_0$ for $y \in L$. Since L is invariant we have that $v'(y) = 0, \forall y \in L$ so that $L \subset S$. Since M is the largest invariant set inside S, we have that $L \subset M$. Since $s(t, x_0, 0)$ approaches L as $t \longrightarrow \infty$, we have that $s(t, x_0, 0)$ approaches M as $t \longrightarrow \infty$. ■

Theorem 3.2 (LaSalle's principle to establish asymptotic stability) *Let $v : \mathbb{R}^n \longmapsto \mathbb{R}$ be such that on $\Omega_c = \{x \in \mathbb{R}^n : v(x) \leq c\}$, a compact set we have $v'(x) \leq 0$. Define*

$$S = \{x \in \Omega_c : v'(x) = 0\}$$

Then, if S contains no trajectories other than $x = 0$ then 0 is asymptotically stable.

Theorem 3.3 (Application of LaSalle's principle to prove global asymptotic stability) *Let $v(x) : \mathbb{R}^n \longmapsto \mathbb{R}$ be a $p.d.f.$ and $v'(x) \leq 0$ for all $x \in \mathbb{R}^n$. Also, let the set*

$$S = \{x \in \mathbb{R}^n : v'(x) = 0\}$$

contain no nontrivial trajectories. Then 0 is globally, asymptvtically stable.

Example 3.2 *Spring-mass system with damper: This system is described by*

$$\begin{cases} x_1' = x_2 \\ x_2' = -f(x_2) - g(x_1) \end{cases} \tag{3.28}$$

If $f.g$ are locally passive. i.e.,

$$\sigma f(\sigma) \geq 0, \ \forall \sigma \in [-\sigma_0, \sigma_0]$$

A suitable Lyapunov function $(I.p.d.f)$ is

$$v(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)\, d\sigma$$

We have

$$v'(x_1, x_2) = -x_2 f(x_2) \leq 0 \text{ for } x_2 \in [-\sigma_0, \sigma_0]$$

Now choose

$$c = \min(v(-\sigma_0, 0), v(\sigma_0, 0))$$

Then $v' \leq 0$ *for*

$$x \in \Omega_c = \{(x_1, x_2) : v(x_1, x_2) \leq c\}$$

As a consequence of LaSalle's principle, the trajectory enters the largest invariant set in

$$\Omega_c \cap \{(x_1, x_2) : v' = 0\} = \Omega_c \cap \{x_1, 0\}$$

To obtain the largest invariant set in this region note that $x_2(t) \equiv 0 \implies x_1(t) \equiv x_{10} \implies x_1'(t) \equiv 0 = -f(0) - g(x_{10})$, *then* $g(x_{10}) = 0 \implies x_{10} = 0$. *Thus, the largest invariant set inside* $\Omega_c \cap \{(x_1, x_2) : v' = 0\}$ *is the origin. Thus, the origin is locally asymptotically stable. The application of LaSalle's principle shows that one can give interesting conditions for the convergence of trajectories of the system of* (3.28) *even when* $g(.)$ *is not passive. It is easy to see that the arguments given above can be easily modified to obtain convergence results for the systern* (3.28), *provided that* $\int_0^{x_1} g(\sigma)\, d\sigma$ *is merely bounded below.*

Now, we are introducing a generalization of LaSalle's principle:

Theorem 3.4 (Global LaSalle's principle) *Consider the system of* (3.1). *Let* $v(x)$ *be a* $p.d.f$ *with* $v' \leq 0, \forall x \in \mathbb{R}^n$. *If the set*

$$s = \{x \in \mathbb{R}^n, v'(x) = 0\}$$

contains no trajectories other than $x = 0$ *then* 0 *is globally asymptotically stable.*

There is a remarkable aversion to LaSalle's theory that holds for periodic systems:

Theorem 3.5 (LaSalle's principle for periodic systems) *Suppose that the system of* (3.1) *is periodic, i.e.,*

$$f(x, t) = f(x, t + T), \ \forall t, \ \forall x \in \mathbb{R}^n$$

Moreover, let $v(x, t)$ *be a* $p.d.f$, *which is periodic in t also with period* T. *Define*

$$s = \{x \in \mathbb{R}^n, v'(x, t) = 0, \ \forall t \ \geq 0\}$$

Then if $v'(x, t) \leq 0, \forall t \geq 0, \forall x \in \mathbb{R}^n$ *and the largest invariant set in* S *is the origin, then the origin is globally (uniformly) asymptotically stable.*

The Lassalle invariance principle is limited in its applications due to the fact that it applies to periodic and time-invariant systems. Therefore, we face some difficulties in extending the result of this principle to arbitrary time-varying systems:

1. $\{x : v'(x,t) = 0\}$ may be a time-varying set.
2. Tbe ω-limit set of a trajectory is itself not invariant. However, if we have the hypothesis that

$$v'(x,t) \leq -\omega(x) \leq 0$$

then the set S may bedefined to be

$$\{x : \omega(x) = 0\}$$

We may state the following gencralization of LaSalle's theorem:

Theorem 3.6 (Generalization of LaSalle's theorem) *Suppose that the vector field $f(x,t)$ of (3.1) is locally Lipschitz continuous in x, uniformly in t, in a ball ofradius r. Let $v(x,t)$ satisfy for functions α_1, α_2 of class K*

$$\alpha_1(|x|) \leq v(x,t) \leq \alpha_2(|x|) \tag{3.29}$$

Moreover, for some non-negative function $\omega(x)$, assume that

$$v'(x,t) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(x,t) \leq -\omega(x) \leq 0 \tag{3.30}$$

Then for all $|x(t_0)| \leq \alpha_2^{-1}(\alpha_1(r))$, the trajectories $x(.)$ are bounded and

$$\lim_{t \longrightarrow \infty} \omega(x(t)) = 0 \tag{3.31}$$

Proof. The proof of this theorem needs a fact from analysis called *Barbalat's lemma*, which states that if if $\phi(.) : \mathbb{R} \longmapsto \mathbb{R}$ is a uniformly continuous integrable function with $\int_0^\infty \phi(t) < \infty$ then $\lim_{t \longrightarrow \infty} \phi(t) = 0$. The requirement of uniform continuity of if $\phi(.)$ is necessary for this lemma, as easy counterexamples will show. We will use this lemma in what follows. First, note that a simple contradiction argument shows that for any $p < r$,

$$|x(t_0)| \leq \alpha_2^{-1}(\alpha_1(p)) \implies |x(t)| \leq p, \ \forall t \geq t_0$$

Thus $|x(t)| < r$ for all $t \geq t_0$, so that $v(x(t), t)$ is monotone decreasing. This yields that

$$\int_{t_0}^{t} \omega(x(\tau))\, d\tau \leq \int_{t}^{t_0} \dot{v}(x(\tau), \tau)\, d\tau = v(x(t_0), t_0) - v(x(t)\, t) \tag{3.32}$$

Since $v(x,t)$ is bounded below by 0, it follows that $\int_{t_0}^{t}\omega(x(\tau))\,d\tau<\infty$. By the continuity of $f(x,t)$ (Lipschitz in x, uniformly in t) and the boundedness of $x(t)$, it follows that $x(t)$ is uniformly continuous, and so is $\omega(x(t))$. Using Barbalat's lemma it follows that $\lim_{t\longrightarrow\infty}\omega(x(t))=0$. ■
The theorem indicates that $x(t)$ approaches a set E defined by

$$E=\{x\in B_r\ :\omega(x)=0\}$$

In fact, it is very difficult to show that the set is invariant. However, this becomes possible if the function $f(x,t)$ is: Autonomous, T-periodic or asymptotically autonomous.

3.4 Unstability theorems

In previous definitions, we discussed Lyapunov's theorems of stability and were able to prove the stability of the equilibrium point. Now we will deal with the theorems of *unstability* of the equilibrium point.

Definition 3.16 (Unstable equilibrium) *The equilibrium point 0 is unstable at t_0 if it is not stable.*

We may parse the definition, by systematically negating the definition of stability: There exists an $\epsilon>0$, such that for all δ balls of initial conditions (no matter how small the ball), there exists at least one initial condition, such that the trajectory is not confined to the ϵ ball; that is to say:$\forall\delta\exists x_0\epsilon B_\delta$ such that $\exists t_\delta$ with $|x_{t_\delta}|\geq\epsilon$.
Note that unstability is a local concept. However, there are not many who can provide a definition of unstability. The definition of unstability does not require every initial condition starting arbitrarily near the origin to be expelied from a neighborhood of the origin, it just requires one from each arbitrarily small neighborhood of the origin to be expelled away. We'll see this in the static linear systems $\dot{x}=Ax$ is unstable, if just one eigenvalue of A lies in $\mathbb{C}_+^{\circ}$. The unstability theorems have the same mechanism: They insist on v' being an $l.p.d.f$, so as to have a mechanism for the increase of v. However, since we do not need to guarantee that every initial condition close to the origin is *repelled* from the origin, we do not need to assurne that v is an $l.p.d.f.$
We state and prove two examples of unstability theorems:

Theorem 3.7 (Unstability theorem) *The equilibrium point 0 is unstable at time t_0 if there exists a decrescent function $\phi(.):\mathbb{R}^n\times\mathbb{R}_+\longmapsto\mathbb{R}$ such that (1) $\dot{v}(x,t)$ is an $l.p.d.f.$ (2) $v(0,t)=0$ and there exist po ints x arbitrarily close to 0 such that $v(x,t_0)>0$.*

Proof. We are given that there exists a function $v(x,t)$ such that

$$\begin{aligned} v(x,t) &\leq \beta(|x|), \quad x \in B_r \\ \dot{v}(x,t) &\geq \alpha(|x|), \quad x \in B_\delta \end{aligned}$$

We need to show that for some $\epsilon > 0$, there is no δ such that

$$|x_0| < \delta \implies |x(t)| \leq \epsilon, \forall t \geq t_0$$

Now choose $\epsilon = \min(r,s)$. Given $\delta > 0$ choose x_0 with $|x_0| < \delta$ and $v(x_0,t_0) > 0$. Such a choice is possible by the hypothesis on $v(x,t_0)$. So long as $\phi(t,x,t_0)$ lies in B_ϵ, we have $v'(x(t),t) \geq 0$, which shows that

$$v(x(t),t) \geq v(x_0,t_0) > 0$$

This implies that $|x(t)|$ is bounded away from 0. Thus $v'(x(t),t)$ is bounded away from zero. Thus, $v(x(t),t)$ will exceed $\beta(\epsilon)$ in finite time.then $|x(t)|$ will exceed ϵ in finite time. ■

Theorem 3.8 (Chetaev's theorem.) *The equilibrium point* 0 *is unstable at time if there is a decrescent function* $v : \mathbb{R}_+ \times \mathbb{R}^n \longmapsto \mathbb{R}$ *such that: (1)* $v'(x,t) = \lambda v(x,t) + v_1(x,t)$ *where* $\lambda > 0$ *and* $v_1(x,t) \geq 0, \forall t \geq 0, \forall\, x \in B_r$. *(2)* $v(0,t) = 0$ *and there exist points* x *arbitrarily close to* 0 *such that* $v(x,t_0) > 0$.

Proof. Choose $\epsilon = r$ and given $\delta > 0$ pick x_0 such that $|x_0| < \delta$ and $v(x_0,t_0) > 0$. When $|x(t)| \leq r$, we have

$$v'(x,t) = \lambda v(x,t) + v_1(x,t) \geq \lambda v(x,t)$$

If we multiply the inequality above by integrating factor $e^{-\lambda t}$, it follows that

$$\frac{dv(x,t)\, e^{-\lambda t}}{dt} \geq 0$$

Integrating this inequality from t_0 to t yields

$$v(x(t),t) \geq e^{\lambda(t-t_0)} v(x_0,t_0)$$

Thus $v(x(t),t)$ grows without bound. Since $v(x,t)$ is decrescent

$$v(x,t) \geq \beta(|x|)$$

for some function of dass K, so that for some t_δ, $v(x(t),t) > \beta(\epsilon)$ establishing that $|x(t_\delta)| \leq \epsilon$. ■

3.5 General conclusion

First of all, a simplified study of stable and unstable sets is presented, with special mention of manifolds.
In the first chapter, we dealt with definitions and basic characteristics and the most important thing it was the classification of the manifolds, where we classified them into three types: the stable manifold, the unstable manifold, and the central manifold.
In the second chapter, we examined the phenomenon of chaos through the boost converter in equations of a certain shape. We carefully analyzed complex behavior in several parameters and then finally arrived at the case in which robust chaos occurs.
In the third chapter, we attached great importance to stability theories with Lyapunov's concept, and then expanded to the LaSalle's principle. We also studied unstability theories.
We point out that we have annotated most of the definitions and theories with several examples that enabled us to identify the importance of these theories in our practical life.

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